NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 38 (2008), 45–55

ON THE PRE-BÉZOUT PROPERTY OF WIENER ALGEBRAS ON THE DISC AND THE HALF-PLANE

Raymond Mortini and Amol Sasane

(Received June 2008)

Abstract. Let $\mathbb{D}$ denote the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$, and $\mathbb{C}_+$ denote the closed right half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$.

(1) Let $W^+(\mathbb{D})$ be the Wiener algebra of the disc, that is the set of all absolutely convergent Taylor series in the open unit disk $\mathbb{D}$, with pointwise operations.
(2) Let $W^+(\mathbb{C}_+)$ be the set of all functions defined in the right half-plane $\mathbb{C}_+$ that differ from the Laplace transform of a function $f_a \in L^1(0,\infty)$ by a constant. Equipped with pointwise operations, $W^+(\mathbb{C}_+)$ forms a ring.

We show that the rings $W^+(\mathbb{D})$ and $W^+(\mathbb{C}_+)$ are pre-Bézout rings.

1. Introduction

The aim of this paper is to show that the rings $W^+(\mathbb{D})$ and $W^+(\mathbb{C}_+)$ (defined below) are pre-Bézout.

We first recall the notion of a pre-Bézout ring.

Definition 1.1. Let R be a commutative, unital ring.
(1) An element $d \in R$ is called a *greatest common divisor* of $a, b \in R$ if it is a divisor of a and b and if k is another divisor, then k divides d.
(2) The ring R is said to be *pre-Bézout* if for every $a, b \in R$ for which there exists a greatest common divisor d, there exist $x, y \in R$ such that $d = xa + yb$.

Michael von Renteln [**12**, Theorem 2.4, p. 54] proved that the disc algebra $A(\mathbb{D})$ (the ring of continuous functions on the closed unit disc $\overline{\mathbb{D}}$, which are holomorphic in the open unit disc $\mathbb{D}$, with the usual pointwise operations) is pre-Bézout. The first author of the present paper [**8**] showed the pre-Bézout property for the Sarason algebra $QA = (C(\mathbb{T}) + \widetilde{C(\mathbb{T})}) \cap H^\infty(\mathbb{D})$ of bounded holomorphic functions having quasicontinuous boundary values. (Here $\widetilde{C(\mathbb{T})}$ denotes the set of harmonic conjugates of continuous functions on $\mathbb{T}$.) Note that the algebra $H^\infty(\mathbb{D})$ (of all bounded and holomorphic functions in the open unit disc, with pointwise operations) is not pre-Bézout [**12**, Remark, p. 54]. In this article, we will show that the rings $W^+(\mathbb{D})$ and $W^+(\mathbb{C}_+)$ (defined below) are pre-Bézout.

2000 *AMS Mathematics Subject Classification:* Primary 30H05 ; Secondary 42A99, 93D15.
Key words and phrases: pre-Bézout ring, Wiener algebra.

Throughout the article, we will use the following notation:

$$\begin{aligned} \mathbb{D} &:= \{z \in \mathbb{C} \mid |z| < 1\} \\ \overline{\mathbb{D}} &:= \{z \in \mathbb{C} \mid |z| \leq 1\} \\ \mathbb{C}_+ &:= \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}. \end{aligned}$$

Definition 1.2.

(1) The *Wiener algebra of the disc*, $W^+(\mathbb{D})$, is the set of all functions $f : \overline{\mathbb{D}} \to \mathbb{C}$ such that f is analytic in $\mathbb{D}$ and $\sum_{n=0}^{\infty} |a_n| < \infty$ for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$. Equipped with pointwise operations and the norm $\|f\|_{W^+} := \sum_{n=0}^{\infty} |a_n|$, $W^+(\mathbb{D})$ is a Banach algebra.

(2) Let $W^+(\mathbb{C}_+)$ denote the set of all functions $F : \mathbb{C}_+ \to \mathbb{C}$ such that $F(s) = \widehat{f_a}(s) + f_0$ $(s \in \mathbb{C}_+)$, where $f_a \in L^1(0, \infty)$, $f_0 \in \mathbb{C}$, and $\widehat{f_a}$ denotes the Laplace transform of f_a given by

$$\widehat{f_a}(s) = \int_0^{\infty} e^{-st} f_a(t)dt, \quad s \in \mathbb{C}_+.$$

Equipped with pointwise operations and the norm

$$\|F\|_{W^+} = \|f_a\|_{L^1} + |f_0|,$$

$W^+(\mathbb{C}_+)$ is a Banach algebra.

We note that $W^+(\mathbb{C}_+)$ is contained in the set of all holomorphic functions on the (open) right half-plane that admit continuous extensions to the imaginary axis and have a limit at infinity. We will call $W^+(\mathbb{C}_+)$ the *Wiener-Laplace algebra.*

Remark 1.3.

(1) From the application point of view, the above algebras also arise as natural classes of transfer functions of stable distributed parameter systems in control theory; see [**11**].

(2) We use the notation $W^+(\mathbb{C}_+)$ in order to highlight the similarity with $W^+(\mathbb{D})$. Indeed, $W^+(\mathbb{D})$ is isomorphic to the algebra of summable sequences $\ell^1(\mathbb{N})$ with convolution, pointwise addition, and the $\ell^1(\mathbb{N})$ norm. Now instead of this "discrete" convolution algebra, we consider "distributed" summable functions $L^1(0, \infty)$, again with convolution, pointwise addition, and the $L^1(0, \infty)$-norm, and attach the identity element δ (=Dirac distribution) to it, we obtain the convolution algebra $L^1(0, \infty) + \mathbb{C}\delta$. Then $W^+(\mathbb{C}_+)$ is isomorphic to the algebra $L^1(0, \infty) + \mathbb{C}\delta$ via Laplace transformation.

Our main results are the following:

Theorem 1.4. *The ring $W^+(\mathbb{D})$ is pre-Bézout.*

Theorem 1.5. *The ring $W^+(\mathbb{C}_+)$ is pre-Bézout.*

Remark 1.6.

(1) The relevance of the pre-Bézout property in control theory is the following: Suppose R is a pre-Bézout ring and we have a plant whose transfer function p belongs to the field of fractions of R. Then p has a weakly coprime factorization if and only if p has a coprime factorization; see [**10**, Proposition, p. 54].

(2) We recall that a commutative ring R is called *Bézout* if every finitely generated ideal in R is principal.
Neither of our algebras $W^+(\mathbb{D})$ nor $W^+(\mathbb{C}_+)$ are Bézout. That $W^+(\mathbb{D})$ is not Bézout can be shown by considering the ideal (f, g), where
$$f = (1-z)^3 \quad \text{and} \quad g = (1-z)^3 e^{-\frac{1+z}{1-z}};$$
see [**9**, Remark after Theorem 1, p. 224]. On the other hand the fact that $W^+(\mathbb{C}_+)$ is not Bézout follows from a general result which says that if R is any subring of the ring H^∞ (of bounded analytic functions in the open right half-plane Re$(s)> 0$, with pointwise addition and multiplication), such that R contains the Laplace transforms of functions from $L^1(0,\infty)$, then R has a finitely generated ideal which is not principal; see [**6**, Theorem].

In Sections 3 and 4 we will give the proofs of Theorems 1.4 and 1.5, respectively, but before doing that, in Section 2, we first give a few preliminaries.

2. Preliminaries

It is well known that the maximal ideals (or kernels of multiplicative linear functionals) of $W^+(\mathbb{D})$ have the form
$$\mathfrak{m}_a = \{f \in W^+(\mathbb{D}) \mid f(a) = 0\}$$
for some $a \in \overline{\mathbb{D}}$. Similarly, the set of maximal ideals in $W^+(\mathbb{C}_+)$ coincides with the set of ideals of the form $\mathfrak{M}_{s_0}$ and $\mathfrak{M}_\infty$, where
$$\mathfrak{M}_{s_0} = \{F \in W^+(\mathbb{C}_+) \mid F(s_0) = 0\}, \quad s_0 \in \mathbb{C}_+,$$
and $\mathfrak{M}_\infty$ is given by the kernel of the homomorphism $\varphi : W^+(\mathbb{C}_+) \to \mathbb{C}$ defined by
$$F = \widehat{f_a} + f_0 \overset{\varphi}{\mapsto} f_0 \quad (f_a \in L^1(0,\infty),\ f_0 \in \mathbb{C}).$$
That is,
$$\mathfrak{M}_\infty = \{F \in W^+(\mathbb{C}_+) \mid \exists f_a \in L^1(0,\infty) \text{ such that } F = \widehat{f_a}\} = \widehat{L^1(0,\infty)}.$$

Since every maximal ideal is closed, all the sets $\mathfrak{m}_\alpha$, $|\alpha| = 1$, are commutative Banach subalgebras of $W^+(\mathbb{D})$. Similarly, $\mathfrak{M}_{i\beta}$, $\beta \in \mathbb{R}$, and $\mathfrak{M}_\infty$ are commutative Banach subalgebras of $W^+(\mathbb{C}_+)$. Obviously these algebras have no identity element. But there is a substitute, namely the notion of the bounded approximate identity, which will be useful in the sequel.

Definition 2.1. Let R be a commutative Banach algebra (without identity element). We say that R has a *bounded approximate identity* if there exists a bounded sequence $(e_n)_n$ of elements e_n in R such that for any $f \in R$,
$$\lim_n \|e_n f - f\| = 0.$$

We will also need the following technical result:

Proposition 2.2 (Varopoulos, [**16**])**.** *Let R be a Banach algebra with a bounded left approximate identity. Then for every sequence $(a_n)_{n\geq 1}$ in R converging to 0, there exists a sequence $(b_n)_{n\geq 1}$ in R converging to 0, as well as an element $c \in R$ such that for all $n \geq 1$, $a_n = cb_n$.*

Lemma 2.3. *Let R be a commutative integral domain with identity 1. If d ($\neq 0$) is a greatest common divisor of $f_1, \dots, f_n$, then 1 is a greatest common divisor of $\frac{f_1}{d}, \dots, \frac{f_n}{d}$.*

Proof. Clearly 1 divides $\frac{f_1}{d}, \dots, \frac{f_n}{d}$. If h is a divisor of $\frac{f_1}{d}, \dots, \frac{f_n}{d}$, then $\frac{f_k}{d} = hg_k$, for some $g_k \in R$, $k = 1, \dots, n$. So dh is common divisor of $f_1, \dots, f_n$, and as d is the greatest common divisor of $f_1, \dots, f_n$, dh divides d, that is, $dhk = d$ for some $k \in R$. Since R is an integral domain and $d \neq 0$, we obtain $hk = 1$, that is, h divides 1, proving the claim. □

3. $W^+(\mathbb{D})$ is pre-Bézout

Let $z_0 \in \mathbb{T} := \{z \in \mathbb{D} \mid |z| = 1\}$. Consider the maximal ideal

$$\mathfrak{m}_{z_0} := \{f \in W^+(\mathbb{D}) \mid f(z_0) = 0\}.$$

We will use the following result on the existence of a bounded approximate identity for $\mathfrak{m}_{z_0}$. Without loss of generality, we take $z_0 = 1$.

Proposition 3.1 (Faivyševskij, [**2**]). *Let $(r_n)_{n\in\mathbb{N}}$ be any sequence of positive numbers such that $r_n \searrow 1$, and let*

$$e_n(z) := \frac{z-1}{z-r_n}.$$

Then $(e_n)_{n\in\mathbb{N}}$ is a bounded approximate identity for $\mathfrak{m}_1$.

A rather lengthy proof of the above result in the case when $r_n = 1 + \frac{1}{n}$ can be found in [**5**, Lemma 1]. For the reader's convenience we present a short proof here.

Proof. A simple calculation gives that $||\frac{z-1}{z-1-\epsilon}||_{W^+} \leq 2$. Since the partial sums $S_n - S_n(1)$ for f approximate $f \in \mathfrak{m}_1$, it suffices to consider $q(z) = (z-1)p(z)$, where $p \in \mathbb{C}[z]$. But

$$\left\| \frac{z-1}{z-1-\epsilon}\, q - q \right\|_{W^+} = \left\| \epsilon\, \frac{q}{z-1-\epsilon} \right\|_{W^+} = \epsilon \left\| \frac{z-1}{z-1-\epsilon}\, p \right\|_{W^+} \leq 2\epsilon\, ||p||_{W^+}.$$

□

We will also use the following fact proved on page 301 of the Proof of the Theorem in [**13**].

Proposition 3.2 (M. von Renteln)**.** *Let $f \in W^+(\mathbb{D})$ and $z_0 \in \mathbb{D}$ be such that $f(z_0) = 0$. Then $\frac{f}{z-z_0} \in W^+(\mathbb{D})$.*

We will also need the corona theorem for $W^+(\mathbb{D})$; see for example [**13**, Theorem]:

Proposition 3.3. *If $f_1, \dots, f_n \in W^+(\mathbb{D})$ are such that*

$$\text{for all } z \in \overline{\mathbb{D}}, \;\; |f_1(z)| + \cdots + |f_n(z)| > 0,$$

then there exist $g_1, \dots, g_n \in W^+(\mathbb{D})$ such that

$$\text{for all } z \in \overline{\mathbb{D}}, \;\; g_1(z)f_1(z) + \cdots + g_n(z)f_n(z) = 1.$$

Lemma 3.4. *Suppose that $f_1, \dots, f_n \in W^+(\mathbb{D})$ and d is a greatest common divisor of $f_1, \dots, f_n$. If $z_0 \in \overline{\mathbb{D}}$ is a common zero of $f_1, \dots, f_n$, then $d(z_0) = 0$ as well.*

Proof. If $z_0 \in \mathbb{D}$, then let m be the least integer among the multiplicities of z_0 as a zero respectively of $f_1, \dots, f_n$. By Proposition 3.2, $(z - z_0)^m$ is a divisor of $f_1, \dots, f_n$. But since d is the greatest common divisor of $f_1, \dots, f_n$, it follows that $(z - z_0)^m$ divides d.

If on the other hand $z_0 \in \mathbb{T}$, then $f_1, \dots, f_n \in \mathfrak{m}_{z_0}$, where

$$\mathfrak{m}_{z_0} := \{f \in W^+(\mathbb{D}) \mid f(z_0) = 0\}.$$

By Proposition 3.1, $\mathfrak{m}_{z_0}$ has a bounded approximate identity. Applying Proposition 2.2, with $(a_n)_{n\geq 1} := (f_1, \dots, f_n, 0, 0, 0, \dots)$, we get the existence of an element $c \in \mathfrak{m}_{z_0}$, and $g_1, \dots, g_n \in \mathfrak{m}_{z_0}$ such that $f_k = cg_k$, $k = 1, \dots, n$. So we have a common divisor c of $f_1, \dots, f_n$. Since $c(z_0) = 0$, and d is a greatest common divisor, we have that c divides d and hence $d(z_0) = 0$, too. □

Proof of Theorem 1.4. Let $f_1, \dots, f_n \in W^+(\mathbb{D})$ have a greatest common divisor d $(\neq 0)$. By the algebraic result in Lemma 2.3, it follows that 1 is a greatest common divisor of $\frac{f_1}{d}, \dots, \frac{f_n}{d}$. Lemma 3.4 gives

$$\left|\frac{f_1}{d}\right| + \dots + \left|\frac{f_n}{d}\right| > 0 \quad \text{in } \overline{\mathbb{D}}.$$

By Proposition 3.3, it follows that there exist $g_1, \dots, g_n \in W^+(\mathbb{D})$ such that

$$g_1 \frac{f_1}{d} + \dots + g_n \frac{f_n}{d} = 1,$$

and so $g_1 f_1 + \dots + g_n f_n = d$, completing the proof of the theorem. □

4. $W^+(\mathbb{C}_+)$ is pre-Bézout

We will first prove that the maximal ideals $\mathfrak{M}_{i\beta}$, $\beta \in \mathbb{R}$, and $\mathfrak{M}_\infty$ in $W^+(\mathbb{C}_+)$ have a bounded approximate identity.

To this end, we need the following lemma. Let us point out that the proofs of Lemma 4.1 and Theorem 4.2 follow closely those in [**14**] for the algebra

$$A = \left\{ \widehat{f_a}(s) + \sum_{k=0}^{\infty} a_k e^{-st_k} \;\middle|\; f_a \in L^1(0,\infty), (a_k) \in \ell^1, 0 = t_0 < t_1, t_2, \cdots \right\},$$

$\mathrm{Re}(s) \geq 0$.

Lemma 4.1. *Suppose $F \in \mathfrak{M}_0$. Then for each $\epsilon > 0$, there exists $P \in \mathfrak{M}_0$ such that $P = \widehat{p_a} + p_0$, where $p_a \in L^1(0,\infty)$ has compact support, $p_0 \in \mathbb{C}$, and $\|F - P\|_{W^+} < \epsilon$.*

Proof. Let $\epsilon > 0$ be given. Let $F = \widehat{f_a} + f_0$, where $f_a \in L^1(0,\infty)$ and $f_0 \in \mathbb{C}$. Choose a compactly supported $p_a \in L^1(0,\infty)$ such that

$$\|p_a - f_a\|_{L^1} < \frac{\epsilon}{2}.$$

Set

$$P := \widehat{p_a} + \underbrace{\left(- \int_0^\infty p_a(t)dt \right)}_{=:p_0}.$$

Then $P \in W^+(\mathbb{C}_+)$ and

$$P(0) = \widehat{p_a}(0) + p_0 = \int_0^\infty p_a(t)dt - \int_0^\infty p_a(t)dt = 0.$$

So $P \in \mathfrak{M}_0$. We have

$$\begin{aligned}\left|f_0 + \int_0^\infty p_a(t)dt\right| &= \left|f_0 + \int_0^\infty f_a(t)dt + \int_0^\infty \big(p_a(t) - f_a(t)\big)dt\right| \\ &= \left|f_0 + \widehat{f_a}(0) + \int_0^\infty \big(p_a(t) - f_a(t)\big)dt\right| \\ &\leq |F(0)| + \|p_a - f_a\|_{L^1} < 0 + \frac{\epsilon}{2} = \frac{\epsilon}{2}.\end{aligned}$$

Thus

$$\|F - P\|_{W^+} = \|f_a - p_a\|_{L^1} + \left|f_0 + \int_0^\infty p_a(t)dt\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This completes the proof. □

Theorem 4.2.

(a) *Let* $\mathfrak{M}_0 := \{F \in W^+(\mathbb{C}_+) \mid F(0) = 0\}$ *and*

$$E_n := \frac{s}{s + \frac{1}{n}}, \quad n \in \mathbb{N}.$$

Then $(E_n)_{n\in\mathbb{N}}$ *is a bounded approximate identity for* $\mathfrak{M}_0$.

(b) *Let* $\mathfrak{M}_\infty = \widehat{L_1(0, \infty)}$ *and*

$$U_n = \widehat{n\mathbf{1}_{[0,\frac{1}{n}]}}, \quad n \in \mathbb{N},$$

where $\mathbf{1}_{[0,\frac{1}{n}]}(t)$ *is* 1 *if* $t \in [0, \frac{1}{n}]$, *and* 0 *otherwise. Then* $(U_n)_{n\geq 1}$ *is a bounded approximate identity for* $\mathfrak{M}_\infty$.

Proof. (b) The existence of a bounded approximate identity for $\mathfrak{M}_\infty$ follows [**1**, Theorem 6.5, p. 105]. The above example is easy to check.

(a) We note that

$$\|E_n\|_{W^+} = \left\|1 + \widehat{\big(-\frac{1}{n}e^{-\frac{t}{n}}\big)}\right\|_{W^+} = |1| + \left\|-\frac{1}{n}e^{-\frac{t}{n}}\right\|_{L^1} = 1 + 1 = 2,$$

and so the sequence is bounded.

Given $F \in \mathfrak{M}_0$, and $\epsilon > 0$ arbitrarily small, in view of Lemma 4.1, we can find a $P \in \mathfrak{M}_0$ such that $P = \widehat{p_a} + p_0$, where $p_a \in L^1(0, \infty)$ has compact support, $p_0 \in \mathbb{C}$, and $\|F - P\|_{W^+} < \epsilon$. Then

$$\|E_nF - F\|_{W^+} \leq \|E_nP - P\|_{W^+} + \|E_n\|_{W^+}\|F - P\|_{W^+} + \|F - P\|_{W^+}.$$

So it is enough to prove that

$$\lim_{n\to\infty} \|E_nP - P\|_{W^+} = 0$$

for all $P \in \mathfrak{M}_0$ such that $P = \widehat{p_a} + p_0$, where $p_a \in L^1(0, \infty)$ has compact support, and $p_0 \in \mathbb{C}$. We do this below.

We have

$$E_nP - P = \frac{s + \frac{1}{n} - \frac{1}{n}}{s + \frac{1}{n}}P - P = -\frac{1}{n}\frac{1}{s + \frac{1}{n}}P = -\frac{1}{n}\left(\widehat{(e^{-t/n} * p_a)} + p_0\widehat{e^{-t/n}}\right).$$

Let c be given by

$$c(t) := \int_0^t e^{-\frac{t-\tau}{n}}p_a(\tau)d\tau + p_0e^{-\frac{t}{n}}.$$

Then $c \in L^1(0,\infty)$. Let $T>0$ be such that $\operatorname{supp}(p_a) \subset [0,T]$. We have

$$\|E_nP - P\|_{W^+} = \frac{1}{n}\|c\|_{L^1} = \frac{1}{n}\int_0^\infty |c(t)|dt = \underbrace{\frac{1}{n}\int_0^T |c(t)|dt}_{(I)} + \underbrace{\frac{1}{n}\int_T^\infty |c(t)|dt}_{(II)}.$$

We estimate (I) as follows:

$$\begin{aligned}(I) = \frac{1}{n}\int_0^T |c(t)|dt &= \frac{1}{n}\int_0^T \left|\int_0^t e^{-\frac{t-\tau}{n}} p_a(\tau)d\tau + p_0 e^{-\frac{t}{n}}\right| dt \\ &\leq \frac{1}{n}\int_0^T \left[\int_0^t e^{-\frac{t-\tau}{n}} |p_a(\tau)|d\tau + |p_0|e^{-\frac{t}{n}}\right] dt \\ &\leq \frac{1}{n}\underbrace{\int_0^T \big[\int_0^t 1\cdot |p_a(\tau)|d\tau + |p_0|\cdot 1\big] dt}_{(III)}.\end{aligned}$$

Since the integral (III) does not depend on n, we obtain that

$$\lim_{n\to\infty} \frac{1}{n}\int_0^T |c(t)|dt = 0.$$

Furthermore,

$$\begin{aligned}(II) &= \frac{1}{n}\int_T^\infty |c(t)|dt = \frac{1}{n}\int_T^\infty e^{-\frac{t}{n}}\left|\int_0^t e^{\frac{\tau}{n}} p_a(\tau)d\tau + p_0\right| dt \\ &= \frac{1}{n}\int_T^\infty e^{-\frac{t}{n}}\left|\int_0^\infty e^{\frac{\tau}{n}} p_a(\tau)d\tau + p_0\right| dt \qquad (\text{since } \operatorname{supp}(p_a)\subset[0,T]) \\ &= \frac{1}{n}\int_T^\infty e^{-\frac{t}{n}}\left|\widehat{p_a}\Big(-\frac{1}{n}\Big) + p_0\right| dt\end{aligned}$$

Since p_a has compact support in $[0,T]$, $\widehat{p_a}$ is an entire function by the Payley-Wiener theorem; see for instance [**15**, Theorem 7.2.3, p. 122]. Consequently,

$$\begin{aligned}(II) &= \frac{1}{n}\int_T^\infty e^{-\frac{t}{n}}\left|\widehat{p_a}\Big(-\frac{1}{n}\Big) + p_0\right| dt \\ &= \frac{1}{n}\int_T^\infty e^{-\frac{t}{n}} dt \cdot \left|\widehat{p_a}\Big(-\frac{1}{n}\Big) + p_0\right| \\ &= e^{-\frac{T}{n}}\left|\widehat{p_a}\Big(-\frac{1}{n}\Big) + p_0\right| \overset{n\to\infty}{\longrightarrow} 1\cdot|\widehat{p_a}(0)+p_0| = |P(0)| = 0.\end{aligned}$$

This completes the proof of the case (a). $\square$

Remark 4.3. The case of $\mathfrak{M}_{i\beta}$ works in a similar manner.

Theorem 4.4. *Let $F \in W^+(\mathbb{C}_+)$, and let $s_0 \in \mathbb{C}$ be such that* $\operatorname{Re}(s_0) > 0$ *and* $F(s_0) = 0$. *Then* $\frac{F}{s-s_0} \in W^+(\mathbb{C}_+)$.

Proof. Let $F = \widehat{f_a} + f_0$, where $f_a \in L^1(0,\infty)$ and $f_0 \in \mathbb{C}$. Since $F(s_0) = 0$, we have

$$F(s_0) = \int_0^\infty e^{-s_0\tau} f_a(\tau)d\tau + f_0 = 0. \tag{1}$$

Let c be defined by

$$c(t)=\begin{cases} f_0 e^{s_0 t}+\displaystyle\int_0^t e^{s_0(t-\tau)} f_a(\tau)d\tau & \text{for } t>0,\\ 0 & \text{for } t<0.\end{cases} \tag{2}$$

We have

$$\begin{aligned}
\int_0^\infty |c(t)|dt &= \int_0^\infty e^{\mathrm{Re}(s_0)t}\left|f_0+\int_0^t e^{-s_0\tau} f_a(\tau)d\tau\right|dt\\
&= \int_0^\infty e^{\mathrm{Re}(s_0)t}\left|-\int_t^\infty e^{-s_0\tau} f_a(\tau)d\tau\right|dt \quad \text{(using (1))}\\
&\le \int_0^\infty e^{\mathrm{Re}(s_0)t}\int_t^\infty e^{-\mathrm{Re}(s_0)\tau}|f_a(\tau)|d\tau dt\\
&= \int_0^\infty \int_t^\infty e^{\mathrm{Re}(s_0)t}e^{-\mathrm{Re}(s_0)\tau}|f_a(\tau)|d\tau dt\\
&= \int_0^\infty \int_0^\tau e^{\mathrm{Re}(s_0)t}e^{-\mathrm{Re}(s_0)\tau}|f_a(\tau)|dt d\tau\\
&= \int_0^\infty e^{-\mathrm{Re}(s_0)\tau}|f_a(\tau)|\int_0^\tau e^{\mathrm{Re}(s_0)t}dt d\tau\\
&= \int_0^\infty e^{-\mathrm{Re}(s_0)\tau}|f_a(\tau)|\frac{e^{\mathrm{Re}(s_0)\tau}-1}{\mathrm{Re}(s_0)}d\tau\\
&\le \frac{1}{\mathrm{Re}(s_0)}\int_0^\infty |f_a(\tau)|d\tau<\infty.
\end{aligned}$$

So $c\in L^1(0,\infty)$.

Let $\beta\in\mathbb{C}$ be such that $\mathrm{Re}(\beta)>\mathrm{Re}(s_0)$. Then, by denoting functions of the form $x\mapsto e^{\alpha x}g(x)$ by $e^{\alpha x}g$, we have from (2) that

$$e^{-\beta t}c(t)=\begin{cases} f_0 e^{(s_0-\beta)t}+\Big(\big[e^{(s_0-\beta)x}u\big]*\big[e^{-\beta x}f_a\big]\Big)(t) & \text{for } t>0\\ 0 & \text{for } t<0\end{cases} \tag{3}$$

where u denotes the step function, given by $u(t)=1$ for $t>0$ and $u(t)=0$ otherwise.

Recall the fact that if $g_a\in L^1(0,\infty)$, then for a complex number α such that $\mathrm{Re}(\alpha)>0$, $(\widehat{e^{-\alpha t}g_a})(s)=\widehat{g_a}(s+\alpha)$ (for $s\in\mathbb{C}_+$). Using this, we obtain

$$\big(\widehat{e^{-\beta t}c}\big)(s)=\widehat{c}(s+\beta) \quad\text{and}\quad \big(\widehat{e^{-\beta t}f_a}\big)(s)=\widehat{f}_a(s+\beta) \quad (s\in\mathbb{C}_+).$$

Since the Laplace transform of a convolution is the product of the Laplace transforms (see for instance [**4**, Proposition 14.1]), we have

$$\Big(\widehat{\big[e^{(s_0-\beta)x}u\big]*\big[e^{-\beta x}f_a\big]}\Big)(s)=\frac{1}{s+\beta-s_0}\cdot\widehat{f}_a(s+\beta) \quad (s\in\mathbb{C}_+).$$

Using these facts, we see by taking Laplace transform on both sides of (3) that

$$\widehat{c}(s+\beta)=\frac{f_0}{s+\beta-s_0}+\frac{1}{s+\beta-s_0}\cdot\widehat{f}_a(s+\beta)=\frac{F(s+\beta)}{s+\beta-s_0} \quad (s\in\mathbb{C}_+).$$

So for all s such that $\mathrm{Re}(s) > \mathrm{Re}(s_0)$, we have

$$\widehat{c}(s) = \frac{F(s)}{s - s_0}.$$

By the identity principle, the above holds in $\mathbb{C}_+$. So $\frac{F}{s-s_0} = \widehat{c} \in W^+(\mathbb{C}_+)$. $\square$

In our proof of Theorem 1.5 we will need the known corona theorem for $W^+(\mathbb{C}_+)$ below (see [**7**, Theorem 4.3.(a2)]), and we include its short proof here:

Proposition 4.5. *If $F_1, \dots, F_n \in W^+(\mathbb{C}_+)$ are such that there exists a $\delta > 0$ such that*

$$\textit{for all } s \in \mathbb{C}_+, \ \ |F_1(s)| + \cdots + |F_n(s)| > \delta > 0, \tag{4}$$

then there exist $G_1, \dots, G_n \in W^+(\mathbb{C}_+)$ such that

$$\textit{for all } s \in \mathbb{C}_+, \ \ G_1(s)F_1(s) + \cdots + G_n(s)F_n(s) = 1. \tag{5}$$

Proof. That (5) implies (4) is easy to see. The reverse implication follows from the classical result (see [**3**, p.112]) that the maximal ideals of $W^+(\mathbb{C}_+)$ are given by $\mathfrak{M}_\infty$ and $\mathfrak{M}_{s_0}$, where $s_0 \in \mathbb{C}_+$. Indeed, suppose that $F_1, \dots, F_n \in W^+(\mathbb{C}_+)$ satisfy (4), but that the ideal $(F_1, \dots, F_n) \neq (1)$. Then there exists a maximal ideal $\mathfrak{M}$ that contains $(F_1, \dots, F_n)$. We now consider the two possible cases:

(i) If $\mathfrak{M} = \mathfrak{M}_{s_0}$ for some $s_0 \in \mathbb{C}_+$, then $(F_1, \dots, F_n) \subset \mathfrak{M}_{s_0}$ yields that $F_1(s_0) = \cdots = F_n(s_0) = 0$, which contradicts (4).

(ii) Now suppose that $\mathfrak{M} = \mathfrak{M}_{s_0}$. Let $F_k = \widehat{f_{k,a}} + f_{k,0}$, where $f_{k,a} \in L^1(0, \infty)$ and $f_{k,0} \in \mathbb{C}$, $k = 1, \dots, n$. Since $(F_1, \dots, F_n) \subset \mathfrak{M}_\infty$, we have $f_{1,0} = \cdots = f_{n,0} = 0$. Hence $F_k = \widehat{f_{k,a}}$, $k = 1, \dots, n$. Passing to the limit $s \to \infty$ in (4), we obtain the contradiction that $0 \geq \delta$.

Consequently $(F_1, \dots, F_n) = (1)$, and so (5) holds for some $G_1, \dots, G_n \in W^+(\mathbb{C}_+)$. $\square$

Lemma 4.6. *Suppose that $F_1, \dots, F_n \in W^+(\mathbb{C}_+)$ and that D is a greatest common divisor of $F_1, \dots, F_n$. If $s_0 \in \mathbb{C}_+$ is a common zero of $F_1, \dots, F_n$, then $D(s_0) = 0$ as well.*

Proof. If $\mathrm{Re}(s_0) > 0$, then let m be the least integer among the multiplicities of s_0 as a zero respectively of $F_1, \dots, F_n$. By Theorem 4.4, $(s - s_0)^m$ is a divisor of $F_1, \dots, F_n$. But since D is the greatest common divisor of $F_1, \dots, F_n$, it follows that $(s - s_0)^m$ divides D.

If on the other hand $\mathrm{Re}(s_0) = 0$, then $F_1, \dots, F_n \in \mathfrak{M}_{s_0}$, where

$$\mathfrak{M}_{s_0} := \{F \in W^+(\mathbb{C}_+) \mid F(s_0) = 0\}.$$

By Theorem 4.2, $\mathfrak{M}_{s_0}$ has a bounded approximate identity. Applying Proposition 2.2, with $(a_n)_{n \geq 1} := (F_1, \dots, F_n, 0, 0, 0, \dots)$, we get the existence of an element $C \in \mathfrak{M}_{s_0}$, and $G_1, \dots, G_n \in \mathfrak{M}_{s_0}$ such that $F_k = CG_k$, $k = 1, \dots, n$. So we have a common divisor C of $F_1, \dots, F_n$. Since D is a greatest common divisor, C divides D and so $D(s_0) = 0$, too. $\square$

Lemma 4.7. *Suppose that $F_1, \dots, F_n \in \mathfrak{M}_\infty$ and that $D \in W^+(\mathbb{C}_+)$ is a greatest common divisor of $F_1, \dots, F_n$. Then $D \in \mathfrak{M}_\infty$ as well.*

Proof. Applying Theorem 4.2 and Proposition 2.2 to the sequence $(a_n) = (F_1, F_2, \dots, F_n, 0, 0, 0 \dots)$ we get a common divisor $C \in \mathfrak{M}_\infty$ of the F_j's, $j = 1, \dots, n$. Hence C divides D; that is $D = QC$ for some $Q \in W^+(\mathbb{C}_+)$. Therefore $D \in \mathfrak{M}_\infty$. □

Proof of Theorem 1.5. Let $F_1, \dots, F_n \in W^+(\mathbb{C}_+)$ have a greatest common divisor D. By the algebraic result in Lemma 2.3, it follows that 1 is a greatest common divisor of $\frac{F_1}{D}, \dots, \frac{F_n}{D}$.

Lemma 4.6 gives

$$\left|\frac{F_1}{D}\right| + \dots + \left|\frac{F_n}{D}\right| > 0 \text{ in } \mathbb{C}_+. \tag{6}$$

Since $F_k/D \in W^+(\mathbb{C}_+)$ for each k, $F_k/D = \widehat{h_k} + \alpha_k$, where $h_k \in L^1(0,\infty)$ and $\alpha_k \in \mathbb{C}$. Since $h_k \in L^1(0,\infty)$, we have

$$\lim_{\substack{s\to\infty \\ s\in\mathbb{C}_+}} \widehat{h_k}(s) = 0. \tag{7}$$

We consider the two possible cases:

(i) All the α_k's are zero. Then by Lemma 4.7, it follows that 1 is the Laplace transform of an element in $L^1(0,\infty)$, which is a contradiction.

(ii) At least one of the α_k's is not zero. Then $|\alpha_1| + \dots + |\alpha_n| > 0$. So for $s \in \mathbb{C}_+$ such that $|s| > R$ with a large enough R, (7) gives the existence of a $\delta > 0$ such that

$$\left|\frac{F_1}{D}\right| + \dots + \left|\frac{F_n}{D}\right| > \delta > 0,$$

while on the compact set K consisting of $s \in \mathbb{C}_+$ with $|s| \le R$, $|\frac{F_1}{D}| + \dots + |\frac{F_n}{D}|$ is at least as large as its minimum value on K, which is positive by (6). So by Proposition 4.5, it follows that there exist $G_1, \dots, G_n \in W^+(\mathbb{C}_+)$ such that

$$G_1\frac{F_1}{D} + \dots + G_n\frac{F_n}{D} = 1,$$

and so $G_1F_1 + \dots + G_nF_n = D$.

This completes the proof of the theorem. □

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Raymond Mortini
Université Paul Verlaine - Metz
LMAM et Département de Mathématiques
Ile du Saulcy
F-57045 METZ
FRANCE
mortini@univ-metz.fr

Amol Sasane
Mathematics Department
London School of Economics
Houghton Street, London WC2A 2AE
UNITED KINGDOM
A.J.Sasane@lse.ac.uk