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EXTENSIONS OF 2-POINT SELECTIONS

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Abstract. We consider a special order-like relation on the subsets of a given space X, which is generated by a continuous selection f for at most 2-point subsets of X. The relation allows to define a "minimal" set of any non-empty compact subset of X, which is then used to construct continuous extensions of f over families of non-empty finite subsets of X. For instance, we show that f can be extended to a continuous selection for at most 3-point subsets if and only if the hyperspace of at most 3-point subsets has a continuous selection. Other possible applications are demonstrated as well.

Dedicated to Professor Takao Hoshina on the occasion of his 60$^{\text{th}}$ birthday

1. Introduction

Let X be a topological space, and let $\mathcal{F}(X)$ be the set of all non-empty closed subsets of X. Also, let $\mathcal{D} \subset \mathcal{F}(X)$. A map $f : \mathcal{D} \to X$ is a *selection* for $\mathcal{D}$ if $f(S) \in S$ for every $S \in \mathcal{D}$. A selection $f : \mathcal{D} \to X$ is *continuous* if it is continuous with respect to the relative Vietoris topology τ_V on $\mathcal{D}$. Let us recall that τ_V is generated by all collections of the form

$$\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{F}(X) : S \subset \bigcup \mathcal{V} \ \text{ and } \ S \cap V \neq \emptyset, \text{ whenever } V \in \mathcal{V} \right\},$$

where $\mathcal{V}$ runs over the finite families of open subsets of X. Sometimes, for reasons of convenience, we will also say that f is *Vietoris continuous* to stress the attention that f is continuous with respect to the topology τ_V.

In the sequel, all spaces are assumed to be at least Hausdorff. In this note, we are interested of continuous selections for $\mathcal{D}$, where $\mathcal{D}$ is a family of finite subsets of X. To this end, let

$$\mathcal{F}_n(X) = \{S \in \mathcal{F}(X) : |S| \leq n\}, \quad n \geq 1.$$

Note that we may identify X with the set $\mathcal{F}_1(X)$, and, in fact, X is homeomorphic to the space $(\mathcal{F}_1(X), \tau_V)$. The latter means that the Vietoris topology is *admissible*, see [**4**].

It should be mentioned that there are spaces X (for instance, one can take X to be the real line $\mathbb{R}$), which have a continuous selection for $\mathcal{F}_n(X)$ for every $n \geq 2$, but they have no continuous selection for $\mathcal{F}(X)$, see [**1**]. On the other hand, we don't know if there exists a space X which has a continuous selection for $\mathcal{F}_n(X)$ for some $n \geq 2$, but it has no continuous selection for $\mathcal{F}_{n+1}(X)$, see [**3**].

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In the present paper we are mainly interested in the above problem when $n = 2$. Briefly, we show that a continuous selection f for $\mathcal{F}_2(X)$ can be continuously extended to a selection for $\mathcal{F}_3(X)$ if and only if $\mathcal{F}_3(X)$ has a continuous selection, see Corollary 4.2. We also demonstrate that, for a space X with only one non-isolated point, the hyperspace $\mathcal{F}_2(X)$ has a continuous selection if and only if $\mathcal{F}_3(X)$ has a continuous selection, see Corollary 5.4. The technique developed to achieve these results is based on an order-like relation on the subsets of a given space X that is generated by a continuous selection for $\mathcal{F}_2(X)$, see the next section. In particular, it culminates in an extension result (Theorem 3.2) that may have an independent interest. Finally, we also consider a local version of this selection-extension problem for hyperspaces, see Section 5.

2. An Order-like Relation on Subsets

Suppose that $f : \mathcal{F}_2(X) \to X$ is a selection. Then, it defines a natural order-like relation $\preceq$ on X by letting $x \preceq y$ if and only if $f(\{x, y\}) = x$, see [**4**]. For convenience, we will write that $x \prec y$ if $x \preceq y$ and $x \neq y$.

The relation is very similar to a linear order on X in that it is both reflexive and antisymmetric, but, unfortunately, it may fail to be transitive. In the present paper, we extend this relation to all subsets of X. Namely, if B and C are subsets of X (not necessarily non-empty), then we shall write that $B \preceq C$ if $y \preceq z$ for every $y \in B$ and $z \in C$. As before, we will write that $B \prec C$ if $y \prec z$ for every $y \in B$ and $z \in C$, equivalently, if $B \preceq C$ and $B \cap C = \emptyset$.

Here are some basic properties of this relation.

Proposition 2.1. *Let X be a space, $f : \mathcal{F}_2(X) \to X$ be a selection, and let "$\preceq$" be the order-like relation generated by f. Also, let $B, C \in \mathcal{F}(X)$ be such that $B \preceq C$ and $C \preceq B$. Then, both B and C are singletons, and $B = C$.*

Proof. The observation is almost obvious. Namely, take points $y \in B$ and $z \in C$. Then, by definition, $y \preceq z$ and $z \preceq y$, so $y = z$. That is, $C = \{y\} = \{z\} = B$. $\square$

Proposition 2.2. *Let X be a space, $f : \mathcal{F}_2(X) \to X$ be a selection, and let "$\preceq$" be the order-like relation generated by f. Also, let $B, C \in \mathcal{F}(X)$ be such that $B \preceq C$ and $B \cap C \neq \emptyset$. Then, $B \cap C$ is a singleton.*

Proof. Suppose that $y, z \in B \cap C$. Then, by definition, $y \preceq z$ and $z \preceq y$, so $y = z$. $\square$

Proposition 2.3. *Let X be a space, $f : \mathcal{F}_2(X) \to X$ be a selection, and let "$\preceq$" be the order-like relation generated by f. Also, let $S \in \mathcal{F}(X)$, and let $B, C \subset S$ be such that $B \preceq S \setminus B$ and $C \preceq S \setminus C$. Then, either $B \subset C$ or $C \subset B$.*

Proof. Suppose, if possible, that this fails. Then, $B \setminus C \neq \emptyset$ and $C \setminus B \neq \emptyset$, so there is a point $y \in B \setminus C$ and a point $z \in C \setminus B$. However, this implies that $z \preceq y$ because $y \in S \setminus C$, and $y \preceq z$ because $z \in S \setminus B$. Hence, $y = z$, but $y \neq z$. This is a contradiction, which completes the proof. $\square$

We are now ready for our main result concerning this relation. Towards this end, we introduce the following concept. Let $f : \mathcal{F}_2(X) \to X$ be a selection, "$\preceq$" be the corresponding order-like relation generated by f, and let $S \in \mathcal{F}(X)$. We shall say that a subset $B \subset S$, $B \in \mathcal{F}(X)$, is an *f-minimum* of S if

(1) $B \preceq S \setminus B$,

(2) If $C \subset S$, $C \in \mathcal{F}(X)$, and $C \preceq S \setminus C$, then $B \subset C$.

In this case we will write that $B = \min_f S$.

Lemma 2.4. *Let X be a space, $f : \mathcal{F}_2(X) \to X$ be a selection, and let "$\preceq$" be the order-like relation generated by f. Then, every non-empty compact subset $S \subset X$ has a unique f-minimum.*

Proof. Let $S \in \mathcal{F}(X)$ be compact, and let $B = \min_f S$ and $C = \min_f S$. Then, by definition, $B \preceq S \setminus B$ and $C \preceq S \setminus C$. Hence, by Proposition 2.3, either $B \subset C$ or $C \subset B$. According once again to the definition of an f-minimal set, we get that $B = C$.

Turning to the existence of f-minimal sets, consider the family

$$\mathcal{D} = \{B \in \mathcal{F}(S) : B \preceq S \setminus B\}.$$

Note that $S \in \mathcal{D}$ because $S \preceq S \setminus S = \emptyset$, so $\mathcal{D} \neq \emptyset$. On the other hand, $\mathcal{D}$ consists of compact sets, and, by Proposition 2.3, it has the finite intersection property. Hence, $D = \bigcap \mathcal{D} \in \mathcal{F}(X)$. In fact, $D \in \mathcal{D}$. Indeed, if $D = S$, this was mentioned before. If $D \neq S$, take points $y \in D$ and $z \in S \setminus D$. Then, there exists $B \in \mathcal{D}$, with $z \notin B$. However, $y \in D \subset B$, and therefore $y \preceq z$. That is, $D \preceq S \setminus D$, which completes the proof. $\square$

3. Selection-regular Selections

Let X be a space, and let $\mathcal{K}(X) = \{S \in \mathcal{F}(X) : |S| < \omega\}$. Also, let $f : \mathcal{F}_2(X) \to X$ be a continuous selection, and let $\mathcal{D} \subset \mathcal{K}(X)$ be such that $\min_f S \in \mathcal{D}$ for every $S \in \mathcal{D}$. We shall say that a selection $h : \mathcal{D} \to X$ is *f-regular* if $h(S) = h(\min_f S)$ for every $S \in \mathcal{D}$.

Let us observe that if $\mathcal{D} = \mathcal{F}_2(X)$, then h is f-regular if and only if $h = f$. That is, any f-regular selection h provides an extension of f to the elements of $\mathcal{D}$ in sense that $h(S) = f(S)$ for every $S \in \mathcal{D} \cap \mathcal{F}_2(X)$. In particular, this also implies that there are continuous selections for $\mathcal{F}_2(X)$ which are not f-regular. On the other hand, we have the following general example of continuous selections $g : \mathcal{F}_3(X) \to X$ which are not $g \restriction \mathcal{F}_2(X)$-regular.

Example 3.1. Let X be a space, $\mathcal{C}$ be a disjoint cover of X consisting of non-empty clopen subsets of X, with $|\mathcal{C}| \geq 3$, and let $h : \mathcal{F}_3(X) \to X$ be a continuous selection such that $|\min_f S| = 1$ for every $S \in \mathcal{F}_3(X)$, where $f = h \restriction \mathcal{F}_2(X)$. Then, there exists a continuous selection $g : \mathcal{F}_3(X) \to X$ which is not f-regular, but $f = g \restriction \mathcal{F}_2(X)$.

Proof. By hypothesis, X has a cover of pairwise disjoint non-empty clopen sets $C_1, C_2, C_3 \subset X$ and points $x_i \in C_i$, $1 \leq i \leq 3$, such that $\min_f\{x_1, x_2, x_3\} = x_1$. Let $\mathcal{U} = \langle\{C_1, C_2, C_3\}\rangle \cap \mathcal{F}_3(X)$, which is a τ_V-clopen subset of $\mathcal{F}_3(X)$, with $\mathcal{U} \cap \mathcal{F}_2(X) = \emptyset$. Next, define a continuous selection $g : \mathcal{F}_3(X) \to X$ by letting that $g(S) \in S \cap C_3$ if $S \in \mathcal{U}$, and $g(S) = h(S)$ otherwise. Then, $g \restriction \mathcal{F}_2(X) = f$ because $\mathcal{U} \cap \mathcal{F}_2(X) = \emptyset$. However, g is not f-regular because $g(\{x_1, x_2, x_3\}) = x_3$, while $g(\min_f\{x_1, x_2, x_3\}) = g(x_1) = x_1$. $\square$

In our next considerations, to any family $\mathcal{D} \subset \mathcal{K}(X)$ we associate the family

$$\min_f(\mathcal{D}) = \{\min_f S : S \in \mathcal{D}\}.$$

Note that $|\min_f S| = 1$ or $|\min_f S| \geq 3$, but $|\min_f S| = 2$ is impossible. On the other hand, with respect to selections, it suffices to consider at least 2-point sets. Namely, any selection is continuous on the singletons of X. Thus, the substantial part of $\min_f(\mathcal{D})$ are the non-singletons, i.e. the following family

$$\min_f^*(\mathcal{D}) = \{B \in \min_f(\mathcal{D}) : |B| > 1\}.$$

Theorem 3.2. *Let X be a space, $f : \mathcal{F}_2(X) \to X$ be a continuous selection, and let $\mathcal{D} \subset \mathcal{K}(X)$ be such that $\min_f(\mathcal{D}) \subset \mathcal{D}$. Then, the following are equivalent.*
(a) *$\mathcal{D}$ has a continuous f-regular selection,*
(b) *$\mathcal{D}$ has a continuous selection,*
(c) *$\min_f^*(\mathcal{D})$ has a continuous selection.*

To prepare for the proof of Theorem 3.2, we need the following simple criterion for continuity in $\mathcal{F}_2(X)$, see [**2**, Theorem 3.1].

Proposition 3.3. *Let X be a space, $f : \mathcal{F}_2(X) \to X$ be a selection, and let "$\preceq$" be the order-like relation generated by f. Also, and let $x, y \in X$ be such that $x \prec y$. Then, f is continuous at $\{x, y\}$ if and only if there are open sets U and V such that $x \in U$, $y \in V$, and $U \prec V$.*

In fact, relying on this criterion, we have the following crucial result concerning the proof of Theorem 3.2.

Lemma 3.4. *Let X be a space, $f : \mathcal{F}_2(X) \to X$ be a continuous selection, and let "$\preceq$" be the order-like relation generated by f. Then, whenever $S \in \mathcal{K}(X)$, there is a disjoint family $\{V_x : x \in S\}$ of open subsets of X such that*
(a) *$x \in V_x$, for every $x \in S$,*
(b) *if $T \in \langle\{V_x : x \in S\}\rangle$, then $\min_f T \in \langle\{V_x : x \in \min_f S\}\rangle$.*

Proof. By Proposition 3.3, there exists a disjoint family $\{V_x : x \in S\}$ of open subsets of X such that $x \in V_x$, $x \in S$, and if $x, y \in S$ and $x \prec y$, then $V_x \prec V_y$. This family is as required. Indeed, take a $T \in \langle\{V_x : x \in S\}\rangle$, and let $B = \bigcup\{V_x \cap T : x \in \min_f S\}$. Then,

$$T \setminus B = \bigcup \{T \cap V_y : y \in S \setminus \min_f S\},$$

and therefore $B \preceq T \setminus B$ because $V_x \prec V_y$, for every $x \in \min_f S$ and $y \in S \setminus \min_f S$. Hence, by definition,

$$\min_f T \subset B \subset \bigcup\{V_x : x \in \min_f S\}. \tag{1}$$

Take now points $x, y \in S$, with $\min_f T \cap V_x \neq \emptyset = \min_f T \cap V_y$. Then, $z \prec t$ for every $z \in \min_f T \cap V_x$ and $t \in V_y \cap T$. So, according to the properties of the family $\{V_x : x \in S\}$, we have that $V_x \prec V_y$, i.e. that $x \prec y$. In particular, this implies that

$$\min_f S \subset \{x \in S : V_x \cap \min_f T \neq \emptyset\}. \tag{2}$$

Thus, according to (1) and (2), we finally get that $x \in \min_f S$ if and only if $\min_f T \cap V_x \neq \emptyset$, i.e. that $\min_f T \in \langle\{V_x : x \in \min_f S\}\rangle$. $\square$

Proof of Theorem 3.2. Since $\min_f^*(\mathcal{D}) \subset \min_f(\mathcal{D}) \subset \mathcal{D}$, the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are obvious. So, we are going to prove only that (c) $\Rightarrow$ (a). Suppose that $g^* : \min_f^*(\mathcal{D}) \to X$ is a continuous selection. Then, g^* defines a continuous selection $g : \min_f(\mathcal{D}) \to X$ by letting $g(B) = g^*(B)$ if $B \in \min_f^*(\mathcal{D})$, and $g(B) \in B$ otherwise. Next, we define an f-regular selection h for $\mathcal{D}$ by $h(S) = g(\min_f S)$ for every $S \in \mathcal{D}$. It remains to show that h is continuous. To this end, take an $S \in \mathcal{D}$, and a neighbourhood U of $h(S)$. Since $g(\min_f S) \in U$ and g is continuous, there exists a finite family $\{W_x : x \in \min_f S\}$ of disjoint open subset of X such that $x \in W_x$, $x \in \min_f S$, and $g\left(\langle\{W_x : x \in \min_f S\}\rangle\right) \subset U$. On the other hand, by Lemma 3.4, there exists a disjoint family $\{V_x : x \in S\}$ of open subset of X such that $x \in V_x$, $x \in S$, and $\min_f T \in \langle\{V_x : x \in \min_f S\}\rangle$ for every $T \in \langle\{V_x : x \in S\}\rangle$. Take $U_x = W_x \cap V_x$ if $x \in \min_f S$, and $U_x = V_x$ otherwise. Then, $\min_f T \in \langle\{U_x : x \in \min_f S\}\rangle$ provided $T \in \langle\{U_x : x \in S\}\rangle \cap \mathcal{D}$, so $h(T) = g(\min_f T) \in U$. □

4. Extensions of 2-point Selections

In this section we provide some possible applications of the extension theorem in the previous section. To this end, let us recall that

$$\mathcal{F}_n(X) = \{S \in \mathcal{F}(X) : |S| \leq n\}.$$

Also, for $1 \leq m \leq n$, we let $\mathcal{F}_{(m,n)}(X) = \{S \in \mathcal{F}(X) : m \leq |S| \leq n\}$.

Corollary 4.1. *Let X be a space, and let $f : \mathcal{F}_2(X) \to X$ and $g : \mathcal{F}_{(3,3)}(X) \to X$ be continuous selections. Then, there exists a continuous selection h for $\mathcal{F}_3(X)$ such that $h \restriction \mathcal{F}_2(X) = f$, i.e. f can be extended to a continuous selection for $\mathcal{F}_3(X)$.*

Proof. Let $\mathcal{D} = \mathcal{F}_3(X)$. As it was mentioned before, $\min_f S \notin \mathcal{F}_{(2,2)}(X)$ for every $S \in \mathcal{D}$. So, in this case, we have that $B \in \mathcal{F}_{(3,3)}(X)$ for every $B \in \min_f^*(\mathcal{D})$, i.e. that $\min_f^*(\mathcal{D}) \subset \mathcal{F}_{(3,3)}(X)$. Hence, by hypothesis, $\min_f^*(\mathcal{D})$ has a continuous selection, take, for instance, $g \restriction \min_f^*(\mathcal{D})$. Therefore, by Theorem 3.2, $\mathcal{D} = \mathcal{F}_3(X)$ has a continuous f-regular selection h. In particular, h is an extension of f, which completes the proof. □

Related to this consequence, we have the following natural question.

Question 1. Let X be a space such that, for some $n \geq 2$, both families $\mathcal{F}_n(X)$ and $\mathcal{F}_{(n+1,n+1)}(X)$ have continuous selections. Is it true that $\mathcal{F}_{n+1}(X)$ has also a continuous selection?

Corollary 4.1 is interesting also from another point of view. Namely, for a space X and a continuous selection $f : \mathcal{F}_2(X) \to X$, we may ask if f can be extended to a continuous selection for $\mathcal{F}_3(X)$. On the other hand, we may only be interested if there exists a continuous selection for $\mathcal{F}_3(X)$. It turns out that these two properties are equivalent.

Corollary 4.2. *Let X be a space, and let $f : \mathcal{F}_2(X) \to X$ be a continuous selection. Then, f can be extended to a continuous selection for $\mathcal{F}_3(X)$ if and only if $\mathcal{F}_3(X)$ has a continuous selection.*

Proof. Note that, by Corollary 4.1, f can be extended to a continuous selection for $\mathcal{F}_3(X)$ if and only if $\mathcal{F}_{(3,3)}(X)$ has a continuous selection. So, it is an immediate consequence of Corollary 4.1. □

5. One-point Extensions

In this section we consider a local version of the previous extension problem. Namely, let $f : \mathcal{F}_2(X) \to X$ be a continuous selection, and let $p \in X$ be a fixed point. Now, we become interested if f can be extended to a continuous selection for $\mathcal{F}_3(X)$ provided $\mathcal{F}_3(X \setminus \{p\})$ has a continuous selection.

Turning to this problem, we consider the order-like relation "$\preceq$" generated by f, and we let

$$L_p = \{x \in X : x \prec p\} \quad \text{and} \quad R_p = \{y \in X : p \prec y\}.$$

Also, we consider the families

$$\mathcal{P} = \{S \in \mathcal{F}_{(3,3)}(X) : S \cap L_p \neq \emptyset \neq R_p \cap S\},$$

and

$$\mathcal{Q} = \{S \in \mathcal{F}_{(3,3)}(X) : S \cap L_p = \emptyset \text{ or } S \cap R_p = \emptyset\}.$$

Proposition 5.1. *Let X be a space, $p \in X$, and let $f : \mathcal{F}_2(X) \to X$ be a continuous selection. Then, $\min_f S$ is a singleton for every $S \in \mathcal{Q}$, with $p \in S$.*

Proof. Suppose that $S \in \mathcal{Q}$, and let $S = \{x, y, p\}$ for some points $x, y \in X \setminus \{p\}$, with $x \prec y$. If $\{x, y\} \subset L_p$, then $x \prec p$ and $y \prec p$, so $\min_f\{x, y, p\} = \{x\}$. In the same way, if $\{x, y\} \subset R_p$, then $p \prec x$ and $p \prec y$, so $\min_f\{x, y, p\} = \{p\}$. This complete the proof. $\square$

Proposition 5.2. *Let X be a space, $p \in X$, and let $f : \mathcal{F}_2(X) \to X$ be a continuous selection. Then, there exists a continuous selection $g : \mathcal{P} \to X$.*

Proof. Whenever $S \in \mathcal{P}$, let us observe that $1 \leq |S \cap L_p| \leq 2$ and $1 \leq |S \cap R_p| \leq 2$. Then, for every $S \in \mathcal{P}$, let ℓ_S be the minimal element of $S \cap L_p$ with respect to the order-like relation "$\preceq$" generated by f, and let r_S be the corresponding maximal element of $S \cap R_p$. Now, define a selection $g : \mathcal{P} \to X$ by $g(S) = f(\{\ell_S, r_S\})$, $S \in \mathcal{P}$. To check the continuity of g, take an $S \in \mathcal{P}$, and a neighbourhood U of $g(S)$. Since f is continuous, there are open subsets $V, W \subset X$ such that $\ell_S \in V \subset L_p$, $r_S \in W \subset R_p$, and $f(\langle\{V, W\}\rangle) \subset U$. We distinguish the following cases for $x \in S \setminus \{\ell_S, r_S\}$. If $p \neq x$, then either $x \in L_p$ or $x \in R_p$, say $x \in L_p$. In this case, $\ell_S \prec x$ and, by Proposition 3.3, there are open subsets $V_1, V_2 \subset L_p$ such that $\ell_S \in V_1 \subset V$, $x \in V_2$, and $V_1 \prec V_2$. Thus, we get a τ_V-neighbourhood $\langle\{V_1, V_2, W\}\rangle$ of S such that $T \in \langle\{V_1, V_2, W\}\rangle$ implies $\ell_T \in V_1 \subset V$ and $r_T \in W$. Hence, $\{\ell_T, r_T\} \in \langle\{V, W\}\rangle$, and therefore $g(T) = f(\{\ell_T, r_T\}) \in U$. The case $x \in R_p$ is symmetric. Suppose finally that $x = p$. Then $S = \{\ell_S, p, r_S\}$ and $\ell_S \prec p \prec r_S$. Hence, there are open sets V_0, O, W_0 such that $\ell_S \in V_0 \subset V$, $p \in O$, $r_S \in W_0 \subset W$, and $V_0 \prec O \prec W_0$. Thus, just like before, we get a τ_V-neighbourhood $\langle\{V_0, O, W_0\}\rangle$ of S such that $T \in \langle\{V_0, O, W_0\}\rangle$ implies $\ell_T \in V_0 \subset V$ and $r_T \in W_0 \subset W$. Hence, $g(T) = f(\{\ell_T, r_T\}) \in U$ which completes the proof. $\square$

We are now ready for our main result in this section.

Theorem 5.3. *Let X be a space, $p \in X$, and let $f : \mathcal{F}_2(X) \to X$ and $g : \mathcal{F}_3(X \setminus \{p\}) \to X$ be continuous selections. Then, f can be extended to a continuous selection for $\mathcal{F}_3(X)$.*

Proof. Let $\mathcal{D} = \mathcal{F}_3(X)$, and let $L_p, R_p \subset X$ and $\mathcal{P}, \mathcal{Q} \subset \mathcal{F}_{(3,3)}(X)$ be defined as at the beginning of this section. By Theorem 3.2, it suffices to show that $\min_f^*(\mathcal{D})$ has a continuous selection. To this end, note that $S \in \min_f^*(\mathcal{D})$ if and only if S has no minimal element. In particular, if $S \in \min_f^*(\mathcal{D})$, then $|S| = 3$.

Now, from one hand, we have that $\mathcal{P} \cap \mathcal{Q} = \emptyset$ and $\mathcal{P} \cup \mathcal{Q} = \mathcal{F}_{(3,3)}(X)$, hence $\min_f^*(\mathcal{D}) \subset \mathcal{P} \cup \mathcal{Q}$. From another hand, by Proposition 5.1, $S \notin \min_f^*(\mathcal{D})$ provided $p \in S \in \mathcal{Q}$. Hence, $\mathcal{Q} \cap \min_f^*(\mathcal{D}) \subset \mathcal{H}$, where

$$\mathcal{H} = \{S \in \mathcal{F}_{(3,3)}(X) : S \subset L_p \text{ or } S \subset R_p\}.$$

Finally, let us observe that both $\mathcal{P}$ and $\mathcal{H}$ are τ_V-open, and clearly $\mathcal{P} \cap \mathcal{H} = \emptyset$. Therefore, $\mathcal{P} \cap \min_f^*(\mathcal{D})$ and $\mathcal{Q} \cap \min_f^*(\mathcal{D}) = \mathcal{H} \cap \min_f^*(\mathcal{D})$ define a τ_V-clopen partition for $\min_f^*(\mathcal{D})$. Thus, we can define a continuous selection for $\min_f^*(\mathcal{D})$ by pointing out how to define continuous selections for $\mathcal{P}$ and $\mathcal{H}$. To this end, let us observe that $\mathcal{H} \subset \mathcal{F}_3(X \setminus \{p\})$, hence $g \restriction \mathcal{H}$ is a continuous selection for $\mathcal{H}$. On the other hand, by Proposition 5.2, $\mathcal{P}$ has also a continuous selection. This, in fact, completes the proof. □

Corollary 5.4. *Let X be a space with only one non-isolated point $p \in X$, and let $f : \mathcal{F}_2(X) \to X$ be a continuous selection. Then, f can be extended to a continuous selection for $\mathcal{F}_3(X)$.*

Proof. By Theorem 5.3, it suffices to show that $\mathcal{F}_3(X \setminus \{p\})$ has a continuous selection. However, $X \setminus \{p\}$ is a discrete space, and any selection for $\mathcal{F}_3(X \setminus \{p\})$ will be continuous. □

Note that, in Corollary 5.4, the space X has a dense set of isolated points. Related to this, we were recently informed by M. Hrusak that he and J. Steprans proved that if a space X has a countable dense set of isolated points and a continuous selection for $\mathcal{F}_2(X)$, then it has a continuous selection for $\mathcal{F}_3(X)$ as well; their paper is in process.

Corollary 5.5. *Let X be a collectionwise normal zero-dimensional space which has a continuous selection $f : \mathcal{F}_2(X) \to X$, $P \subset X$ be a closed discrete set, and let g be a continuous selection for $\mathcal{F}_3(X \setminus P)$. Then, f can be extended to a continuous selection for $\mathcal{F}_3(X)$.*

Proof. By Corollary 4.2, it suffices to show that $\mathcal{F}_3(X)$ has a continuous selection. Since X is collectionwise normal and zero-dimensional, there exists a clopen discrete cover $\{X_p : p \in P\}$ of X such that $p \in X_p$, $p \in P$. Then, by hypothesis, each $\mathcal{F}_3(X_p \setminus \{p\})$, $p \in P$, has a continuous selection. Hence, by Theorem 5.3, $f \restriction \mathcal{F}_2(X_p)$ can be extended to a continuous selection $f_p : \mathcal{F}_3(X_p) \to X_p$ for every $p \in P$. Now, we consider a well-ordering $\preceq$ on P, and then, for every $S \in \mathcal{F}_3(X)$, we let $p(S) = \min\{p \in P : S \cap X_p \neq \emptyset\}$. Finally, we define a continuous selection h for $\mathcal{F}_3(X)$ by letting $h(S) = f_{p(S)}\left(S \cap X_{p(S)}\right)$ for every $S \in \mathcal{F}_3(X)$. □

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