

CLIQUEISHNESS AND CLUSTER MULTIFUNCTIONS

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Abstract. In this paper, we introduce the cluster operation acting on multifunctions as a base tool to investigate continuity properties of multifunctions and their selections. The main result of the paper establishes the relationship between cliquish multifunction and the set of all points at which cluster multifunction is non-empty valued. We also concern how to find special types of selections for a multifunction.

1. Terminology, Basic Definitions and Preliminary Properties of Cluster Multifunction

The main goal of the present paper is to introduce a unified concept for studying some problems concerning continuity properties of multifunctions and selection theorems. The crucial notion of the paper is the cluster operator acting on multifunctions $F : X \rightarrow Y$ with respect to a cluster system $\mathcal{E} \subset 2^X$. Our investigation is focused on the relationship between original multifunction F and resultant cluster multifunction $\mathcal{E}_F$.

In the sequel the symbols X, Y are topological spaces, $\bar{A}, \mathbb{N}, \mathbb{R}$ denote the closure of A , natural numbers $\{0, 1, 2, \dots\}$ and reals with usual topology, respectively. A set S is quasi open, if for any open set H intersecting S there is non-empty open set $H_0 \subset H$ such that $H_0 \subset S$ or equivalently, if S is of the form $G \cup A$, where G is open and A is a nowhere dense subset of $\bar{G}$. A point x is an accumulation point of a sequence $\{S_n\}_{n \in \mathbb{N}}$, if for any open set V containing x , intersection $V \cap S_n$ is non-empty frequently. The set of all accumulation points of $\{S_n\}_{n \in \mathbb{N}}$ is called upper Kuratowski limit of $\{S_n\}_{n \in \mathbb{N}}$, denoted by $\overline{\lim} S_n$.

A multifunction F is a non-empty subset of the Cartesian product $X \times Y$ with the values $\{y \in Y : [x, y] \in F\} =: F(x)$ (it can be empty valued at some points). By $\text{Dom}(F)$, we denote the domain of F , i.e. the set of all arguments x in which $F(x)$ is non-empty. For a multifunction with domain A we will use the notation $F : A \rightarrow Y$. If $\text{Dom}(F)$ is a dense set, F is said to be densely defined. $\text{Gr}(F) = \{[x, y] : x \in \text{Dom}(F), y \in F(x)\}$ is the graph of F and $\bar{F}$ is the multifunction defined by $\bar{F}(x) = \{y : [x, y] \in \overline{\text{Gr}(F)}\}$. By $F(A)$ we denote the set $\bigcup_{a \in A} F(a)$. A multifunction F is *locally bounded at* x , if there is an open set G containing x and a compact set C such that $F(G) \subset C$ and F is *locally bounded*, if it is so at any point. A function $f : A \rightarrow Y$ is understood as a strictly single valued multifunction with values $\{f(x)\}$ for any $x \in A \subset X$. A multifunction $S : A \rightarrow Y$ is *submultifunction* of F , $S \subset F$, if

$$S(x) \subset F(x), \quad \text{for all } x \in A,$$

and a function $f : A \rightarrow Y$ is a selection of a multifunction F on A , if

$$f(x) \in F(x), \quad \text{for all } x \in A.$$

For any set $W \subset Y$ the upper, resp. lower inverse image is defined as $F^+(W) = \{x \in \text{Dom}(F) : F(x) \subset W\}$, $F^-(W) = \{x \in \text{Dom}(F) : F(x) \cap W \neq \emptyset\}$.

The basic types of continuity are lower and upper semi continuity, briefly *lsc* and *usc*.

Definition 1. A multifunction F is *lsc* (*usc*) at $x \in \text{Dom}(F)$, if for any open set V for which $F(x) \cap V \neq \emptyset$ ($F(x) \subset V$) there is open set $U \ni x$ such that $F(u) \cap V \neq \emptyset$ ($F(u) \subset V$) for any $u \in U \cap \text{Dom}(F)$. Multifunction F is *lsc*(*usc*), if it is so at any point $x \in \text{Dom}(F)$. Which means $F^-(V)$ ($F^+(V)$) is relatively open in $\text{Dom}(F)$ for any open set $V \subset Y$. Finally, F is *usco* at x , if $F(x)$ is non-empty compact and F is *usc* at x .

In next two definitions, we introduce notions of an $\mathcal{E}$ -cluster point and lower and upper $\mathcal{E}$ -continuity, as a basic tool to investigate many properties of multifunctions. These notions were firstly studied in [13], next in [7], [10], [15], [16]. As it was said, a cluster system is any non-empty system $\mathcal{E}$ of non-empty subsets of X . For some special cluster systems we will use special notation. For example,

$$\mathcal{O} = \{O : O \text{ is open non-empty}\},$$

$$\mathcal{B}r = \{B : B \text{ is of second category with the Baire property}\},$$

$\mathcal{B}$ denotes a system such that

$$\mathcal{O} \subset \mathcal{B} \subset \mathcal{O} \cup \mathcal{B}r,$$

$$\mathcal{A} = \{A : A \text{ is not nowhere dense}\},$$

$$\mathcal{E}^\circ = 2^X \setminus \{\emptyset\},$$

$$\mathcal{D} = \{A : A \text{ is of second category}\},$$

$$\mathcal{B}^* = \{A : A \text{ is not nowhere dense with the Baire property}\}$$

and last but not least $\mathcal{E} = \mathcal{F}$, where $\mathcal{F}$ is a filter in X . Let as stress, $\mathcal{E}$ is always a non-empty system and any set $E \in \mathcal{E}$ is non-empty.

Definition 2. A point $y \in Y$ is an $\mathcal{E}$ -cluster point of F at a point x , if for any open sets $V \ni y$ and $U \ni x$ there is a set $E \in \mathcal{E}$, $E \subset U$ such that $F(e) \cap V \neq \emptyset$ for any $e \in E$. The set of all $\mathcal{E}$ -cluster points of F at x is denoted by $\mathcal{E}_F(x)$. A multifunction $\mathcal{E}_F$ with the values $\mathcal{E}_F(x)$ is called the $\mathcal{E}$ -cluster multifunction of F .

Remark 3. Note $\mathcal{E}_F \subset \overline{F}$ and $\mathcal{E}_F$ is empty valued outside of the closure of $\text{Dom}(F)$. At the points in $\text{Dom}(F)$, it can be also empty valued. For example, if $F \subset \mathbb{R} \times \mathbb{R}$ is defined $F(x) = \{1\}$ for $x > 0$, F is identical with the Dirichlet function for $x < -1$ and it is empty valued otherwise, then $\mathcal{O}_F(x) = \{1\}$ for $x \geq 0$ and empty valued otherwise, where $\mathcal{O}$ is the cluster system of all non-empty open sets in $\mathbb{R}$.

Formally next definition is motivated by the notion of lower/upper quasi continuity.

Definition 4 (Semi- $\mathcal{E}$ -continuity). A multifunction F is $l - \mathcal{E}$ -continuous ($u - \mathcal{E}$ -continuous) at $x \in \text{Dom}(F)$, if for any open sets V, U such that $V \cap F(x) \neq \emptyset$ ($F(x) \subset V$) and $x \in U$ there is a set $E \in \mathcal{E}$, $E \subset U \cap \text{Dom}(F)$ such that $F(e) \cap V \neq \emptyset$ ($F(e) \subset V$) for any $e \in E$. The global definitions are given by local ones at any point of $\text{Dom}(F)$.

Using cluster concept, $l - \mathcal{E}$ -continuity at x can be characterized by inclusion $\mathcal{E}_F(x) \supset F(x) \neq \emptyset$ and global $l - \mathcal{E}$ -continuity by inclusion $\mathcal{E}_F \supset F$. For a system $\mathcal{B}r$, $l - \mathcal{B}r$ -continuity ($u - \mathcal{B}r$ -continuity) will be called *lower* (*upper*) *Baire continuity*, respectively and for $\mathcal{O}$ it is *lower* (*upper*) *quasi continuity*. Note, the sets in $\mathcal{O}$ ($\mathcal{B}r$) are open (second category with the Baire property) in X , not relatively open (relatively second category with the Baire property) in domain of F . Any multifunction F with respect to cluster system $\mathcal{E}^\circ = 2^X \setminus \{\emptyset\}$ is $l - \mathcal{E}^\circ$ -continuous, since $F \subset \mathcal{E}_F^\circ = \overline{F}$.

Considering $\mathcal{E}^\circ = 2^X \setminus \{\emptyset\}$, $\mathcal{E}^\circ$ -cluster point is the classical one. Apart from this, many types of cluster points for functions have been defined and the results achieved have many applications in topology and measure theory. (see [20]).

Definition of semi- $\mathcal{B}r$ -continuity and the basic properties of $\mathcal{B}r$ -cluster multifunctions were studied in [13] and [14], where the structure of the set of all *usc/lsc* points of lower/upper Baire continuous multifunctions and the existence of quasi continuous selections of upper Baire continuous multifunctions were investigated. Recently, existence of quasi continuous selection has been generalized in [2]. For the systems $\mathcal{O}$ and $\mathcal{B}r$, the cluster multifunctions of functions are minimal, which are nice alternate of selections [3], [4], [8], [9], [16], [18]. A system B^* was investigated in [7]. For system $\mathcal{A}$, lower- $\mathcal{A}$ -continuity is called *lower demicontinuity* [18] and if F is $l - \mathcal{A}$ -continuous (Y is supposed to be T_1 regular locally compact), then $\mathcal{A}_F$ is lower quasi continuous with quasi open domain what has significant consequences for further investigation [16].

Lemma 5. For any net $\{x_t\}$ converging to x and $y_t \in \mathcal{E}_F(x_t)$, $\mathcal{E}_F(x)$ contains all accumulation points of the net $\{y_t\}$.

Proof. Let y be an accumulation point of $\{y_t\}$. Then for any open sets $V \ni y$ and $U \ni x$ there are frequently given indexes t' such that $x_{t'} \in U$ and $y_{t'} \in V \cap \mathcal{E}_F(x_{t'})$. Hence there is $E \in \mathcal{E}$, $E \subset U$ such that $F(e) \cap V \neq \emptyset$ for any $e \in E$. That means $y \in \mathcal{E}_F(x)$. $\square$

Remark 6. From Lemma 5, the multifunction $\mathcal{E}_F$ has a closed graph. Hence it has closed values, too. This means $\mathcal{E}_F^-(K)$ is closed for any compact set K or equivalently, $\mathcal{E}_F^+(G)$ is open for any open G with compact complement. Hence $\mathcal{E}_F$ is *usc* with respect to the Fell topology given on Y by open sets with compact complement. Further continuity properties of cluster multifunctions and selection theorems are studied in [16].

Definition 7. A multifunction F has $\mathcal{E}$ -closed graph at x , if $\mathcal{E}_F(x) \subset F(x)$ and F has $\mathcal{E}$ -closed graph, if it has $\mathcal{E}$ -closed graph at any point from X . If F has $\mathcal{E}_F^\circ$ -closed graph / $\mathcal{E}_F^\circ$ -closed graph at x (i.e., $\overline{F} = \mathcal{E}_F^\circ \subset F$ / $\overline{F}(x) = \mathcal{E}_F^\circ(x) \subset F(x)$), then we say only F has closed graph / closed graph at x .

Remark 8. Notion of $\mathcal{E}$ -closedness of graph is more general than closedness of graph, because if F has closed graph, then $\mathcal{E}_F \subset \overline{F} = F$. On the other hand,

multifunction G from $\mathbb{R}$ to $\mathbb{R}$ defined as $G(x) = \{0, 1\}$ for x rational and $G(x) = \{0\}$ otherwise is $u\mathcal{B}r$ -continuous with $\mathcal{B}r$ -closed graph ($\mathcal{B}r_G(x) = \{0\}$ for all x), but its graph is not closed.

2. Cliquishness of Multifunction in Cluster Setting

In this paragraph we present the main results concerning selection theorems. Studying notion of fragmentability (of topological spaces and multifunctions), it seems to have close connection to cliquishness, which was introduced for function with values in metric spaces [5]. Roughly speaking, for any $\varepsilon > 0$ and any non-empty open set $H \subset X$ there is non-empty open sets $G \subset H$ such that, distance between any two values $f(a)$ and $f(b)$ ($a, b \in G$) is less then ε , or any value $f(g)$ ($g \in G$) belongs to an open set with diameter less then ε . If range is topological space, definition could be introduced by following way: For any open cover $\mathcal{V}$ of Y and any non-empty open set $H \subset X$ there are non-empty open sets $G \subset H$ and $V \in \mathcal{V}$ such that for any $g \in G$ the values $f(g)$ belongs to V . These definition for metric spaces are not equivalent, but topological variant seems to be useful tool for further investigation. For multifunction we propose next formulation of cliquishness: For any open cover $\mathcal{V}$ of Y and non-empty open set $H \subset X$ there are non-empty open sets $G \subset H$ and $V \in \mathcal{V}$ such that $F(g) \cap V \neq \emptyset$ for any $g \in G$. Now let us try to find connection with cliquishness and fragmentability.

Topological space (Y, τ) is fragmented by metric d , if for any $\varepsilon > 0$ and any non-empty subset A of Y there is an open set $G \in \tau$ such that $A \cap G \neq \emptyset$ and $d\text{-diam}(A \cap G) < \varepsilon$, ([12]). Multifunction $F : X \rightarrow (Y, \tau)$ is fragmented by d , if for any $\varepsilon > 0$ and any non-empty open set $H \subset X$ there is non-empty open set $G \subset H$ and $V \in \tau$, such that $F(g) \cap V \neq \emptyset$ for any $g \in G$ and $F(G) \cap V \subset B_\varepsilon$, where B_ε is ε -ball (see similar but deferent notion in [11]). Both conditions “ $F(g) \cap V \neq \emptyset$ ” and “ $F(G) \cap V \subset B_\varepsilon$ ” could be joined, if we stated “smallness” of covers of Y . Namely, if V from some cover $\mathcal{V}$ of Y is “small”, then condition “ $F(g) \cap V \neq \emptyset$ ” says, that $F(G) \cap V$ (as a subset of V) is bounded by V , i.e. “small”. Comparing with definition of cliquishness of multifunction above, concepts of fragmentability of multifunction and cliquishness of multifunction are very similar. Difference appears only as for a choice of covers. General notion of cliquishness of multifunctions will be defined in cluster setting below as alternative notion of fragmentability and we will study its closed connection to problems concerning selection, *usco* submultifunction, submultifunction in Baire class one with respect to upper Kuratowsky limit and quasi continuous selection.

Definition 9 (for function see [6]). A multifunction F is sub-continuous at a point x , if for any net $\{x_\alpha \in \text{Dom}(F)\}$ converging to x , any net $\{y_\alpha \in F(x_\alpha)\}$ has an accumulation point. Global definition is given by local one at any point $x \in \overline{\text{Dom}(F)}$. Note, if F is sub-continuous at x and $F(x)$ is closed, then $F(x)$ is compact and if F is locally bounded, then F is sub-continuous.

Theorem 10. *Let Y be completely regular and F be sub-continuous. Then $\mathcal{E}_F$ is sub-continuous.*

Proof. Let $\{x_\alpha\}$ converge to x and $\{y_\alpha \in \mathcal{E}_F(x_\alpha)\}$. Let $\mathcal{B}$ be uniformity for Y . Since $y_\alpha \in \mathcal{E}_F(x_\alpha)$, for any open U containing x and any $B \in \mathcal{B}$ pick up $x_{\alpha(U)} \in U$ and $z_{\alpha(U, B)}$ from $y_\alpha[B] \cap F(x_{\alpha(U)})$. It is clear that $\{x_{\alpha(U)}\}$ converges to x . Since

F is sub-continuous at x there is subnet $\{z_{\alpha'(U,B)}\}$ of $\{z_{\alpha(U,B)}\}$ converging to y . We will show that y is a accumulation point of $\{y_\alpha\}$. Let $B \in \mathcal{B}$ and $B_1 \in \mathcal{B}$ be a symmetric such that $B_1 \circ B_1 \subset B$. Then $(y, z_{\alpha'(U,B)}) \in B_1$ and $(z_{\alpha'(U,B)}, y_{\alpha'}) \in B_1$. Hence $(y, y_{\alpha'}) \in B$. $\square$

Corollary 11. *Let Y be completely regular and F be sub-continuous. Then $\mathcal{E}_F$ is usc.*

Proof. If not, there is a net $\{x_\alpha\}$ converging to x and a net $\{y_\alpha \in \mathcal{E}_F(x_\alpha) \setminus V\}$ for some open set $V \supset \mathcal{E}_F(x)$. From theorem above, there is an accumulation point $y \in Y \setminus V$ of net $\{y_\alpha \in \mathcal{E}_F(x_\alpha)\}$. But $y \in \mathcal{E}_F(x) \subset V$ (see Lemma 5), contradiction with $y \in Y \setminus V$. $\square$

Next definition generalizes fragmentation of topological space by cover (not necessary open) [1].

Definition 12. We will say that a topological space Y is fragmented by a cover $\mathcal{V}$ (briefly Y is $\mathcal{V}$ -fragmented), if each non-empty subset of Y has a non-empty relatively open subset contained in some member of $\mathcal{V}$.

Lemma 13. *Let Y be $\mathcal{V}$ -fragmented and F be a l - $\mathcal{E}$ -continuous multifunction. Then for any open set G intersecting $\text{Dom}(F)$ there are $E \in \mathcal{E}$, $E \subset G$, $V \in \mathcal{V}$ and non-empty open set $H \subset Y$ such that*

- (i) $F(e) \cap H \neq \emptyset$ and $F(e) \cap V \neq \emptyset$ for any $e \in E$,
- (ii) $F(E) \cap H \subset V$.

Proof. Since Y is fragmented by cover $\mathcal{V}$, for set $F(G) \neq \emptyset$ there is an open set H , $F(G) \cap H \neq \emptyset$ and there is $V \in \mathcal{V}$ such that $F(G) \cap H \subset V$. From l - $\mathcal{E}$ -continuity there is a set $E \subset G$, $E \in \mathcal{E}$, such that for any $e \in E$, $\emptyset \neq F(e) \cap H \subset F(G) \cap H \subset V$. Hence $F(e) \cap V \neq \emptyset$. $\square$

The properties “ $F(e) \cap V \neq \emptyset$ ” and “ $F(E) \cap H \subset V$ ” from Lemma 13 are motivation for introduction of topological cliquishness of multifunction. As it was said, if we state smallness of the covers, the property (ii) can be omitted. For multifunction in cluster setting we propose next definition.

Definition 14. Let $\mathcal{V}$ be cover (not necessary open) of (Y, τ) , τ topology on Y . A multifunction F is $\mathcal{V}$ -topologically $\mathcal{E}$ -cliquish at a point x (not necessary from domain of F), if for any open set G containing x there are $V \in \mathcal{V}$, $E \in \mathcal{E}$, $E \subset G$ such that $F(e) \cap V \neq \emptyset$ for any $e \in E$. A multifunction F is $\mathcal{V}$ -topologically $\mathcal{E}$ -cliquish, if it is so at any point $x \in X$. A multifunction F is τ -topologically $\mathcal{E}$ -cliquish (at x), if it is $\mathcal{V}$ -topologically $\mathcal{E}$ -cliquish (at x) for any open cover $\mathcal{V}$ of Y . It is clear, if F is $\mathcal{V}$ -topologically $\mathcal{E}$ -cliquish, then F is densely defined.

Next lemma follows immediately from definition.

Lemma 15.

- (1) *If $\mathcal{E}_F(x) \neq \emptyset$ (spacial case, if F is l - $\mathcal{E}$ -continuous at x), then F is τ -topologically $\mathcal{E}$ -cliquish at x ,*
- (2) *The set of all points at which F is $\mathcal{V}$ -topologically $\mathcal{E}$ -cliquish is closed,*

- (3) If F is $\mathcal{V}$ -topologically $\mathcal{E}$ -cliquish on a dense set, then F is $\mathcal{V}$ -topologically $\mathcal{E}$ -cliquish at any point,
- (4) If Y is $\mathcal{V}$ -fragmented and F is densely defined l - $\mathcal{E}$ -continuous, then F is $\mathcal{V}$ -topologically $\mathcal{E}$ -cliquish (see Lemma 13).

The question is, what is a sufficient number of covers $\{\mathcal{V}_t\}_{t \in T}$ of (Y, τ) under which a multifunction F is $\mathcal{V}_t$ -topologically $\mathcal{E}$ -cliquish, to be τ -topologically $\mathcal{E}$ -cliquish. The further progress is based on a sequence $\mathcal{V}_n$ of "small" covers and assumption that multifunction F is $\mathcal{V}_n$ -topologically $\mathcal{E}$ -cliquish for any n . Hence for any n and any non-empty open set G , there is at least one pair (E, V_n) , $E \subset G$, $E \in \mathcal{E}$, $V_n \in \mathcal{V}_n$ such that $F(e) \cap V_n \neq \emptyset$ for any $e \in E$ and simultaneously $F(E) \cap V_n \subset V_n$ is bounded by "small" set V_n .

Definition 16 ([19]). A base $\mathcal{B}$ for a space X is called a base of countable order or BCO, if there exists a sequence $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ of bases of X consisted of subsets of $\mathcal{B}$ satisfying: for each $x \in X$, if $x \in B_n \in \mathcal{B}_n$ and $B_{n+1} \subset B_n$ for every $n \in \mathbb{N}$, then $\{B_n\}_{n \in \mathbb{N}}$ is a base of x .

Definition 17. A sequence $\{\mathcal{S}_n\}_{n \in \mathbb{N}}$ of covers of Y is τ -frequently small, where τ is topology on Y , if whenever y_0 is an accumulation point of a sequence $\{S_n \in \mathcal{S}_n\}_{n \in \mathbb{N}}$ and $V \in \tau$ is open containing y , then $S_n \subset V$ frequently.

For example, if (Y, τ) is fragmented by metric d , then $\mathcal{S}_n = \{A \in \tau : d\text{-diameter of } A \text{ is less than } \frac{1}{n}\}$ is d -frequently small sequence and moreover, if F densely defined lower demicontinuous, then F is $\mathcal{S}_n$ -topologically $\mathcal{A}$ -cliquish.

Theorem 18. (as for technics of proof see [18]) Let X be BCO Baire, (Y, τ) be completely regular, $\{\mathcal{S}_n\}_{n \in \mathbb{N}}$ be a τ -frequently small sequence of covers of Y and F be sub-continuous. If multifunction F is $\mathcal{S}_n$ -topologically $\mathcal{E}$ -cliquish for any n , then there is a dense G_δ set H (i.e., residual) such that

- 1. $\mathcal{E}_F$ is usco and $H \subset \text{Dom}(\mathcal{E}_F)$, so F is τ -topologically $\mathcal{E}$ -cliquish,
- 2. $\mathcal{E}_F$ has a non-empty compact valued sub-continuous submultifunction $S : H \rightarrow Y$ which is upper Kuratowsky limit of sequence of continuous functions $f_n : H \rightarrow Y$, i.e., $S(x) = \overline{\lim} f_n(x)$, $x \in H$.

Proof. A pair (U, S) is n -admissible, if $S \in \mathcal{S}_n$, $U \in \mathcal{B}_n$ ($\mathcal{B}_n$ associated with BCO) and $U \cap F^-(S)$ contains a set from $\mathcal{E}$. Such pairs exist for any n , because F is $\mathcal{S}_n$ -topologically $\mathcal{E}$ -cliquish, hence for any n there are $U \in \mathcal{B}_n$ and $S \in \mathcal{S}_n$ such that $U \cap F^-(S)$ contains a set from $\mathcal{E}$. Using induction we can define a sequence α_n of families of n -admissible pairs such that

- (a) for any n the family $\{U : (U, S) \in \alpha_n \text{ for some } S\}$ is pair-wise disjoint and its sum $H_n = \bigcup \{U : (U, S) \in \alpha_n \text{ for some } S\}$ is dense in X ,
- (b) for any $(U, S) \in \alpha_{n+1}$ there is $(U', S') \in \alpha_n$ such that $U \subset U'$.

- (1) Put $H = \bigcap_{n \in \mathbb{N}} H_n$. Since X is Baire, H is a dense G_δ set. If $x \in H$, then there is uniquely determined decreasing sequence $\{U_n(x)\}_{n \in \mathbb{N}}$, which is base of x , and for any n there is a set $S_n(x) \in \mathcal{S}_n$ such that $(U_n(x), S_n(x))$ is n -admissible pair. If $x \in H_n$, then there is only one set $U_n(x)$ containing x from the family $\{U : (U, S) \in \alpha_n \text{ for some } S\}$. Define a sequence $\{f_n\}_{n \in \mathbb{N}}$,

where $f_n : H_n \rightarrow Y$ is constance function on $U_n(x)$ and $f_n|_{U_n(x)} = s_n$ where s_n is arbitrary constant form $S_n(x) \cap F(e_n)$ for some $e_n \in U_n(x) \cap F^-(S_n(x))$. Hence f_n is constant function on the open sets from $\{U : (U, S) \in \alpha_n \text{ for some } S\}$. So f_n is continuous.

- (2) Let $x \in H$, $x \in \bigcap_{n \in \mathbb{N}} U_n(x)$ and let y_0 be an accumulation point of $\{f_n(x)\}_{n \in \mathbb{N}} = \{s_n\}_{n \in \mathbb{N}}$. Such accumulation points exist because if a sequence $\{z_k\}$ converges to x , then $z_k \in U_n(x)$ eventually. Since function $f_n = s_n \in F(e_n)$ is constant on $U_n(x)$, then from sub-continuity of F , $\{s_n\}_{n \in \mathbb{N}}$ has an accumulation point. We will shaw that $y_0 \in \mathcal{E}_F(x)$. Let $V \ni y_0$, $W \ni x$ be open. From BCO property, $U_n(x) \subset W$ eventually. Since $\{S_n\}_{n \in \mathbb{N}}$ is a τ -frequently small sequence and y_0 is an accumulation point of $\{f_n(x) = s_n \in S_n\}_{n \in \mathbb{N}}$, $S_n \subset V$ frequently. A pair $(U_n(x), S_n)$ is n -admissible, hence $U_n(x) \cap F^-(S_n) \subset W \cap F^-(V)$ contains a set from $\mathcal{E}$ frequently. That means $y_0 \in \mathcal{E}_F(x)$. Consequently, the multifunction $\mathcal{E}_F$ is non-empty valued on H . Since F is sub-continuous, $\mathcal{E}_F$ is sub-continuous as well (Theorem 10) and $\mathcal{E}_F(x)$ is compact. By Corollary 11, $\mathcal{E}_F$ is usco. So, it is sufficient to put $S(x) := \overline{\lim} f_n(x)$ for $x \in H$. Since $S(x) \subset \mathcal{E}_F(x)$, S is compact valued and sub-continuous.

□

Corollary 19. *Let X be BCO Baire, (Y, τ) be completely regular locally compact, S_n be τ -frequently small and F be sub-continuous. If a multifunction F is S_n -topologically $\mathcal{E}$ -cliquish for any n , then*

- (1) $\mathcal{E}_F$ is usco defined on an open set W with nowhere dense complement,
- (2) $\mathcal{E}_F$ has a non-empty compact valued sub-continuous submultifunction $S : H \rightarrow Y$ which is upper Kuratowsky limit of sequence of continuous functions $f_n : H \rightarrow Y$, i.e., $S(x) = \overline{\lim} f_n(x)$, $x \in H$.
- (3) there is quasi continuous selection $f : W \rightarrow Y$ of $\mathcal{E}_F$ on W . So, if F has $\mathcal{E}$ -closed graph, S and $\mathcal{E}_F$ are submultifunctions of F and f is selection of F on W (so F is defined at least on W).

Proof. (1) By Theorem 18, $\mathcal{E}_F$ is usco, so it is locally bounded at any point form $\text{Dom}(\mathcal{E}_F)$, since Y is locally compact. Hence, for any $x \in \text{Dom}(\mathcal{E}_F)$ there is open set $G \ni x$ and compact set C such that $\mathcal{E}_F(G) \subset C$. Since $\text{Dom}(\mathcal{E}_F)$ is dense, by Remark 6, $\mathcal{E}_F$ is defined at any point from G . That means $\mathcal{E}_F$ is usco defined on open residual set W with nowhere dense complement.

Item (2) is same as in Theorem 18 and existence of quasi continuous selection in item (3) follows from [2]. □

We would like to make a few remarks and comparisons with main result from [18]. As for assumptions in [18, Theorem 1], F is supposed to be densely defined and lower demicontinuous (i.e., $l\mathcal{A}$ -continuous) and result concerns existence of usco submultifunction of $\mathcal{E}_F^\circ$ defined on a dense G_δ set. Theorem 18 deals with existence of usco multifunction, Baire one submultifunction with respect upper Kuratowsky limit and lower demicontinuity is generalized by more general notion of cliquishness. On the other hand, quality of Y from [18] (partition completeness)

is substituted by sub-continuity of F . In next remark, let us try to give the more discussed cases.

Remark 20.

- (1) Consider two completely regular topological spaces (Y, τ) and (Y, σ) , X BCO Baire and suppose there is a τ -frequently small system $\mathcal{S}_n = \{S_n : S_n \in \sigma\}$. If F is S_n -topologically $\mathcal{E}$ -cliquish for any n and τ -sub-continuous, then both items (1), (2) from Theorem 18 hold with respect to topology τ . Moreover, if $\sigma \subset \tau$, then $\mathcal{E}_F^\tau \subset \mathcal{E}_F^\sigma$, hence by Lemma 15(1), F is σ -topologically $\mathcal{E}$ -cliquish (also σ -sub-continuous) and both items (1), (2) from Theorem 18 hold with respect to topology σ .
- (2) Specially, suppose (Y, τ) is metrizable by metric d and any non-empty set $G \in \tau$ contains a non-empty set $H \in \sigma$. Then $\mathcal{S}_n = \{S_n : S_n \in \sigma, d - \text{diam}(S_n) < \frac{1}{n}\}$ is d -frequently small. If F is S_n -topologically $\mathcal{E}$ -cliquish (specially, if F is σ -topological $\mathcal{E}$ -cliquish) and F is d -sub-continuous, then (1), (2) from Theorem 18 hold with respect to metric space d .
- (3) Let Y be completely regular and F be locally bounded. Then F is sub-continuous and using Lemma 15(1), item (1) from Theorem 18 is characterization of τ -topological $\mathcal{E}$ -cliquishness. Namely, F is τ -topologically $\mathcal{E}$ -cliquish if and only if $\mathcal{E}_F$ is non-empty valued on a dense set.

For T_1 regular second countable space (i.e, metrizable), S from Theorem 18 is from lower Borel class one and usc on G_δ residual set, as next lemma shows.

Lemma 21. *Let X be BCO Baire, (Y, d) be a separable metric space and F be sub-continuous. If a multifunction F is d -topologically $\mathcal{E}$ -cliquish, then multifunction $S : H \rightarrow Y$ defined on G_δ residual set H from Theorem 18 is usc except for a set of the first category and $S^-(Q)$ is G_δ for any closed set Q .*

Proof. If F is d -topological $\mathcal{E}$ -cliquish, then it is $\mathcal{S}_n^d$ -topological $\mathcal{E}$ -cliquish for any n , where $\mathcal{S}_n^d = \{S_n^d(y) : y \in Y\}$ and $S_n^d(y) = \{z : d(z, y) < \frac{1}{n}\}$. Let S be from Theorem 18 and let $\{V_l\}_{l \in \mathbb{N}}$ be a sequence of all finite unions of sets from a base $\{G_n\}_{n \in \mathbb{N}}$ of Y . Since any closed set in Y is G_δ and F is sub-continuous, for any closed set $Q = \bigcap_{n \in \mathbb{N}} Q_n$ (where Q_n are open), $S^-(Q) = \bigcap_{k \in \mathbb{N}} \bigcup_{n > k} f_n^-(Q_k)$ is G_δ . Then $S^+(V_l) = \bigcup_{k \in \mathbb{N}} A_k^l$, where A_k^l are closed (f_n are continuous). Since S is compact valued, the set D_u of all upper semi discontinuity points of S can be expressed by following way (A° is interior of A)

$$\begin{aligned} D_u &= \bigcup_{l \in \mathbb{N}} \left[S^+(V_l) \setminus (S^+(V_l))^\circ \right] \\ &= \bigcup_{l \in \mathbb{N}} \left[\left(\bigcup_{k \in \mathbb{N}} A_k^l \right) \setminus \left(\bigcup_{k \in \mathbb{N}} A_k^l \right)^\circ \right] \subset \bigcup_{l \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \left(A_k^l \setminus (A_k^l)^\circ \right), \end{aligned}$$

what is a set of the first category. $\square$

Lemma 22. *Let (Y, d) be metric and G be lower quasi continuous submultifunction of $\mathcal{E}_F$. Let $\mathcal{E}(G, F, \varepsilon) = \{x \in \text{Dom}(G) \cap \text{Dom}(F) : d(G(x), F(x)) < \varepsilon\}$, $\varepsilon > 0$. If $H \cap \text{Dom}(G) \neq \emptyset$ for some open set H , then there is $E \in \mathcal{E}$ such that*

$E \subset H \cap \mathcal{E}(G, F, \varepsilon)$. Consequently, $\text{Dom}(G) \setminus \mathcal{O}(G, F, \varepsilon)$ is nowhere dense and $\text{Dom}(G) \setminus \mathcal{B}(G, F, \varepsilon)$ is a set of the first category.

Proof. Let H be open and $H \cap \text{Dom}(G) \neq \emptyset$. For fixed $a \in H \cap \text{Dom}(G)$ pick up $y \in G(a)$. Multifunction G is lower quasi continuous at a , then there is non-empty set $H_1 \subset H$ such that $d(y, G(h_1)) < \varepsilon/2$ for any $h_1 \in H_1$. For fixed $h_1 \in H_1$, choice $g \in G(h_1)$ and $\delta > 0$ such that $S(g, \delta) \subset S(y, \varepsilon/2)$. Since $g \in G(h_1) \subset \mathcal{E}_F(h_1)$, there is $E \in \mathcal{E}$, $E \subset H_1$ such that $S(g, \delta) \cap F(e) \neq \emptyset$ for any $e \in E \subset H_1 \subset H$, so $d(G(e), F(e)) < \varepsilon$ for any $e \in E$. Hence $E \subset H \cap \mathcal{E}(F, G, \varepsilon)$. $\square$

Theorem 23. Let X be BCO Baire, (Y, d) metric locally compact and let F be sub-continuous. Let $\mathcal{E} \in \{\mathcal{O}, \mathcal{B}\}$. If F is d -topologically $\mathcal{E}$ -cliquish/ $\mathcal{E}_F$ is non-empty valued on a dense set, then there is quasi continuous selection $f : W \rightarrow Y$ of $\mathcal{E}_F$ on open set W with nowhere dense complement and $f(x) \in \overline{F(x)}$ on a residual set. So, if F has closed values, then f is selection of F on a residual set.

Proof. Multifunction F is $\mathcal{S}_n^d$ -topological $\mathcal{E}$ -cliquish for any n , where $\mathcal{S}_n^d$ is from proof of Lemma 21. By Corollary 19, there is quasi continuous selection $f : W \rightarrow Y$ of $\mathcal{E}_F$ on open set W , where complement of W is nowhere dense. Then, by Lemma 5, for any n , set $\text{Dom}(f) \setminus \mathcal{E}(f, F, \frac{1}{n}) = W \setminus \mathcal{E}(f, F, \frac{1}{n})$ is of first category, so

$$W \setminus \bigcap_{n \in \mathbb{N}} \mathcal{E}\left(f, F, \frac{1}{n}\right) = W \setminus \{x : d(f(x), F(x)) = 0\} = W \setminus \{x : f(x) \in \overline{F(x)}\}$$

is of first category. $\square$

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