

GAMES AND METRISABILITY OF MANIFOLDS

JILING CAO¹, DAVID GAULD², SINA GREENWOOD³, AND ABDUL MOHAMAD⁴

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Abstract. Using topological games we investigate connections between properties of topological spaces and their spaces of continuous functions with the compact-open topology. This leads to new criteria for metrisability of a manifold. We show that a manifold M is metrisable if and only if a winning strategy applies to certain topological games played on $C_k(M)$. We also show that M is metrisable if and only if $C_k(M)$ is Baire, and even if and only if it is Volterra.

1. Introduction and Topological Games

Topological games have become a valuable tool in the study of topological properties and many games are now well-studied. Hitherto we are unaware of any connections between topological games, particularly played on function spaces, and the problem of metrisability of topological manifolds. After investigating games and the relationship between topological properties of function spaces we describe a number of conditions equivalent to metrisability of a manifold. In particular we show that a manifold is metrisable if and only if the corresponding space of real-valued functions with the compact-open topology is a Baire space (indeed, we can even weaken the latter to Volterra as defined in [7]).

Unfortunately there seems to be some inconsistency with the naming of topological games so we shall describe explicitly the games we are discussing.

Definition 1.1. Suppose that players α and β play a game on a topological space X which involves them taking turns at choosing points and/or subsets of X . A *strategy* for α is a function which tells α what points or sets to select given all the previous points and sets chosen by β . A *stationary strategy* for α is a function which tells α what points or sets to select given only the most recent choice of points and sets chosen by β . A *winning (stationary) strategy* for α is a (stationary) strategy which guarantees that α will win whatever moves β might make.

One of the most basic games is the following.

Definition 1.2. Two players α and β play alternately on a topological space X . Player β begins by choosing a non-empty open $V_0 \subset X$. After that the players choose successive non-empty open subsets of their opponent's previous move; denote

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by U_n (respectively V_n) the n^{th} choice of player α (respectively β). Player α *wins* if the intersection of the sets is non-empty; otherwise player β *wins*. This game is called the *Banach-Mazur game* in [5, p.1], [16, p.204] and [21, p. 200], and the *Choquet game* in [9, p.19] and [12, p. 43]. We shall call this the *Banach-Mazur game*.

Definition 1.3. Again two players α and β play alternately on a topological space X . Player β begins by choosing a point x_0 and an open set V_0 with $x_0 \in V_0 \subset X$. When β has chosen point x_n and open set V_n with $x_n \in V_n \subset X$, player α chooses an open $U_n \subset V_n$ with $x_n \in U_n$. Then β chooses x_{n+1} and V_{n+1} with $x_{n+1} \in V_{n+1} \subset U_n$ and V_{n+1} open. Player α *wins* if the intersection of the sets is non-empty; otherwise player β *wins*. This game is called the *Choquet game* in [5, p.1] and the *strong Choquet game* in [12, p. 44] and [21, p. 200]. We shall call this the *Choquet game*.

At least the following terms seem to have general agreement.

Definition 1.4. A space X is *weakly α -favourable* (also called *Choquet* in [9, p.19]) if player α has a winning strategy in any Banach-Mazur game played on X . The space X is *α -favourable* if player α has a stationary winning strategy in any Banach-Mazur game played on X . The space X is *strongly α -favourable* if player α has a stationary winning strategy in any Choquet game played on X . The fourth possibility for X , that player α has a winning strategy in any Choquet game played on X , is called *strongly Choquet* in [12, Definition 8.14].

In [9] the following game is also discussed.

Definition 1.5. Let X be a topological space. The game $G_{K,L}^o(X)$ has, at the n^{th} stage, player K choose a compactum $K_n \subset X$ after which player L chooses another compactum $L_n \subset X$ so that $L_n \cap K_i = \emptyset$ for each $i \leq n$. Player K *wins* if $\{L_n\}_{n < \omega}$ has a discrete open expansion.

Recall that an *open expansion* of a family $\{S_\alpha / \alpha \in A\}$ of subsets of X consists of a family $\{U_\alpha / \alpha \in A\}$ of open sets such that $S_\alpha \subset U_\alpha$ for each $\alpha \in A$, and that a family \mathcal{S} of sets is *discrete* if each point of X has a neighbourhood which meets at most one member of \mathcal{S} .

The following game was introduced in [13] in order to study automatic continuity of group operations of a semi-topological group.

Definition 1.6. Let X be a topological space, and $D \subset X$ a dense subset. The game $\mathcal{G}_S(D)$, involves two players α and β . Players β and α choose alternately non-empty open subsets V_n and U_n in X just as in the Banach-Mazur game. Player α *wins* a game if $\bigcap_{n < \omega} U_n$ is non-empty and each sequence $\langle x_n \rangle_{n < \omega}$ with $x_n \in U_n \cap D$ for all $n < \omega$ has a cluster point in X . The space X is *strongly Baire* if X is regular and there is a dense subset $D \subset X$ such that β does not have a winning strategy in the game $\mathcal{G}_S(D)$ played in X .

Theorem 1.7 ([13]). *Every strongly Baire semi-topological group is a topological group.*

Remark 1.8. It is a well-known and standard result that X is Baire if and only if player β does not have a (stationary) winning strategy in the Banach-Mazur game, see [12, p.43] or [20].

It is clear that Čech complete spaces are strongly Baire, and strongly Baire spaces are Baire. Note that a metric space is Baire if and only if it is strongly Baire. Thus, strong Baireness and (weak, strong) α -favourability are distinct properties, even in the class of metric spaces. In general, there is a Baire space which fails to be strongly Baire, e.g., the Sorgenfrey line.

Remark 1.9 ([18]). Every completely metrisable space both is strongly α -favourable and provides player α a winning strategy in the game $\mathcal{G}_S(D)$ played on X for any dense subset $D \subset X$.

2. Properties of a Space and its Function Space

In this section we find connections between properties of a topological space and properties of its space of real-valued functions with the compact-open topology which, for a given space X , we denote by $C_k(X)$. Recall that sets of the form $\mathcal{N}(f, C, \varepsilon) = \{g \in C_k(X) / |g(x) - f(x)| < \varepsilon \text{ for each } x \in C\}$ form a neighbourhood basis for $f \in C_k(X)$ as C ranges through compacta in X and $\varepsilon > 0$.

Recall that a space X is a k -space if any subset $A \subset X$ is closed if and only if $A \cap K$ is closed for each compact $K \subset X$ and is *hemicompact*, [1, p.486], if there is a sequence $\langle K_n \rangle$ of compact subsets so that each compact $K \subset X$ is contained in some K_n . The definition of *cosmic* is found, for example, in [8].

Proposition 2.1 ([17, Corollary 5.2.5(a)]). *Let X be any space. Then $C_k(X)$ is Polish if and only if X is a hemicompact, cosmic k -space.*

Proposition 2.2. *Suppose that X is locally compact, Hausdorff and path-connected and that $C_k(X)$ is a space of second category. Then X is hemicompact.*

Proof. We may assume that X is non-compact; choose a point $x_0 \in X$. Set

$$\mathcal{K} = \{K / x_0 \in K \subset X \text{ and } K \text{ is compact}\}.$$

Because X is connected and non-compact it follows that each member of \mathcal{K} has non-empty boundary. For each $n < \omega$ set

$$U_n = \{f \in C_k(X) / f(\partial K) > n \text{ for some } K \in \mathcal{K}\}.$$

Clearly each set U_n is open. Each U_n is also dense in $C_k(X)$. Indeed, suppose that U is a non-empty open subset of $C_k(X)$: in order to show that $U \cap U_n \neq \emptyset$ we assume that $U = \mathcal{N}(f, C, \varepsilon)$ for some $f \in C_k(X)$, compact $C \subset X$ and $\varepsilon > 0$. Using local compactness of X and compactness of C we may find a compact subset $K \subset X$ containing x_0 and C in its interior. Then $K \in \mathcal{K}$. Applying Tietze's Extension Theorem to the normal space K we may find a function $g_n \in C_k(X)$ which agrees with f on C and is $n+1$ on $\partial K \cup (X \setminus K)$. Then $g_n \in \mathcal{N}(f, C, \varepsilon) \cap U_n$, so U_n is dense. As $C_k(X)$ is of second category it follows that $\bigcap_{n < \omega} U_n \neq \emptyset$; choose $g \in \bigcap_{n < \omega} U_n$. Now choose $K_n \in \mathcal{K}$ so that $g(\partial K_n) > n$. It is claimed that $X = \bigcup_{n < \omega} K_n$. This will show that X is σ -compact and hence hemicompact because X is also locally compact.

Suppose that $x \in X$ but $x \notin \bigcup_{n < \omega} K_n$. Choose a path $\pi : [0, 1] \rightarrow X$ from x_0 to x . For each n , as $x_0 \in K_n$ while $x \notin K_n$, it follows that there is $t_n \in [0, 1]$ such that $\pi(t_n) \in \partial K_n$. The sequence $\langle t_n \rangle$ has a convergent subsequence; by deleting some of the sets K_n if necessary we may assume that $\langle t_n \rangle$ converges, say to t . As

$t_n \rightarrow t$, it follows that $g\pi(t_n) \rightarrow g\pi(t)$. This gives a contradiction as $g\pi(t_n) > n$ for each n . \square

We observe that the requirement that X be path-connected in Proposition 2.2 can be weakened to requiring that each pair of points of X should lie in a sequentially compact, connected subset of X .

Related to this result is the following. We require the following concept, which reduces to a q -point when $D = X$.

Definition 2.3. Let X be a topological space and $D \subset X$ a dense subset. We shall call a point $x \in X$ a q_D -point if there is a sequence $\langle U_n \rangle_{n < \omega}$ of open neighbourhoods of x such that if $x_n \in U_n \cap D$ for each $n < \omega$, then the sequence $\langle x_n \rangle_{n < \omega}$ has a cluster point in X .

Proposition 2.4. For a Tychonoff space X , the following are equivalent:

- (a) $C_k(X)$ is strongly Baire.
- (b) $C_k(X)$ is Baire and X is hemicompact.

Proof. (b) \Rightarrow (a). It is a classical result of Arens in [1] that if X is hemicompact, then $C_k(X)$ is metrisable. In addition, Baireness and strong Baireness are equivalent for any metrisable space.

(a) \Rightarrow (b). Suppose that $C_k(X)$ is strongly Baire. Since any strongly Baire space is Baire, we only need to show that X is hemicompact. Let $D \subset C_k(X)$ be a dense subset such that β does not have a winning strategy in the game $\mathcal{G}_S(D)$ played in $C_k(X)$. This means that for any strategy t that β applies, there will be a sequence $\langle U_n \rangle_{n < \omega}$ of non-empty open subsets in $C_k(X)$ such that $\bigcap_{n < \omega} U_n \neq \emptyset$ and any sequence $\langle f_n \rangle_{n < \omega}$ with $f_n \in U_n \cap D$ has a cluster point in $C_k(X)$. Thus each point of $\bigcap_{n < \omega} U_n$ is a q_D -point. Let $g \in \bigcap_{n < \omega} U_n$ and let $E = D - g$, i.e. a translation of D . Then E is dense in $C_k(X)$, and the zero function f_0 is a q_E -point in $C_k(X)$. For convenience, let $\langle \mathcal{N}(f_0, K_n, \varepsilon_n) \rangle_{n < \omega}$ be a sequence of non-empty basic open neighbourhoods of f_0 in $C_k(X)$ such that if $g_n \in \mathcal{N}(f_0, K_n, \varepsilon_n) \cap E$ for each $n < \omega$, then $\langle g_n \rangle_{n < \omega}$ clusters in $C_k(X)$, where each $K_n \subset X$ is compact and each $\varepsilon_n > 0$.

We first claim that $X = \bigcup_{n < \omega} K_n$. If not, there will be a point $x_0 \in X \setminus \bigcup_{n < \omega} K_n$. For each $n < \omega$, we can pick a $g_n \in C_k(X)$ such that $g_n(K_n) = 0$ and $g_n(x_0) = n$. Furthermore, since E is dense in $C_k(X)$, for each $n < \omega$, we can choose an $h_n \in C_k(X)$ such that

$$h_n \in \mathcal{N}(g_n, K_n, \varepsilon_n) \cap \mathcal{N}(g_n, \{x_0\}, 1/3) \cap E.$$

It is clear that $h_n \in \mathcal{N}(f_0, K_n, \varepsilon_n) \cap E$ for each $n < \omega$, as $f_0 \upharpoonright K_n = g_n \upharpoonright K_n$ for each $n < \omega$. But $\langle h_n \rangle_{n < \omega}$ cannot have any cluster point in $C_k(X)$, simply because $h_n(x_0) \in (n - 1/3, n + 1/3)$ for all $n < \omega$. This is a contradiction.

Next, choose a sequence $\langle \mathcal{N}(f_0, C_n, \delta_n) \rangle$ of open neighbourhoods of f_0 , where C_n is compact with $K_n \subset C_n \subset C_{n+1}$ and $0 < \delta_{n+1} < \delta_n \leq \varepsilon_n$, such that

$$\{f_0\} = \bigcap_{n < \omega} \mathcal{N}(f_0, C_n, \delta_n) = \bigcap_{n < \omega} \overline{\mathcal{N}(f_0, C_n, \delta_n)}.$$

This may be done as follows. Define $O_{ij} = \mathcal{N}(f_0, K_i, 1/j)$ for $i, j < \omega$ and relabel as sets G_n so that $\{G_n / n < \omega\} = \{O_{ij} / i, j < \omega\}$. Note that $\{f_0\} = \bigcap_{n < \omega} G_n$.

Regularity of $C_k(X)$ allows us to shrink the sets G_n to open sets H_n so that $f_0 \in H_n \subset \overline{H_n} \subset G_n$. Moreover, by shrinking further if necessary, we may assume that each set H_n is of the required form $\mathcal{N}(f_0, C_n, \delta_n)$.

If X is not hemicompact, then there will be some compact subset $K \subset X$ such that for each $n < \omega$, $K \not\subset C_n$. For each $n < \omega$, we can pick a point $x_n \in K \setminus C_n$, and a function $p_n \in C_k(X)$ such that $p_n(C_n) = \{0\}$ and $p_n(x_n) = 2$. For each $n < \omega$, we have $\mathcal{N}(f_0, C_n, \delta_n) \cap \mathcal{N}(p_n, \{x_n\}, 1) \neq \emptyset$ so we may choose $q_n \in C_k(X)$ such that $q_n \in \mathcal{N}(f_0, C_n, \delta_n) \cap \mathcal{N}(p_n, \{x_n\}, 1) \cap E$. As $q_n \in \mathcal{N}(f_0, C_n, \delta_n) \cap E$ for each $n < \omega$, f_0 must be a cluster point of $\langle q_n \rangle_{n < \omega}$. However, we have $q_n \notin \mathcal{N}(f_0, K, 1)$ for all $n < \omega$. This is a contradiction, which proves that X is hemicompact. \square

3. Applications to Manifolds

The major result in this section explores metrisability of manifolds in terms of games. Here a manifold is assumed to be a connected Hausdorff space in which each point has a neighbourhood homeomorphic to euclidean space \mathbb{R}^n for some n (which is unique).

Note that all manifolds are locally compact, path-connected k -spaces. A manifold M is metrisable if and only if any one of the following conditions holds for M : paracompact, σ -compact, hemicompact, cosmic. A fuller list may be found in [6, Theorem 2].

Definition 3.1 ([9]). Let X be a topological space. A family \mathcal{L} of non-empty compact subsets of X *moves off* compacta of X provided that for each compact $K \subset X$ there is $L \in \mathcal{L}$ such that $K \cap L = \emptyset$. A space X *has the Moving Off Property (MOP)* provided that for each family \mathcal{L} of compact subsets of X which moves off compact subsets of X there is an infinite subset $\mathcal{L}' \subset \mathcal{L}$ which has a discrete open expansion.

In [9, Section 5] the author notes that in a normal or locally compact space the Moving Off Property is equivalent to the Weak Moving Off Property, which merely requires that the infinite subcollection should be discrete rather than requiring an open expansion to be discrete. It is proved in [11] that for a locally compact space X , $C_k(X)$ is Baire if and only if X has the MOP.

Definition 3.2. A topological space X is *Volterra*, [7], provided that the intersection of any two dense G_δ -subsets is dense.

Of course every Baire space is Volterra but the converse is false in general. Nevertheless situations under which the converse is true have been explored by various authors, see [7], [10], [3] and [4] for example.

Theorem 3.3. *For a manifold M , the following are equivalent:*

- (1) M is metrisable;
- (2) $C_k(M)$ is strongly α -favourable;
- (3) $C_k(M)$ is strongly Choquet;
- (4) $C_k(M)$ is α -favourable;
- (5) $C_k(M)$ is weakly α -favourable;

- (6) K has a winning strategy in $G_{K,L}^0(M)$;
- (7) Player α has a stationary winning strategy in $\mathcal{G}_S(D)$ played in $C_k(M)$ for any dense subset $D \subset C_k(M)$;
- (8) Player α has a winning strategy in $\mathcal{G}_S(D)$ played in $C_k(M)$ for any dense subset $D \subset C_k(M)$;
- (9) $C_k(M)$ is a strongly Baire space;
- (10) $C_k(M)$ is a Baire space;
- (11) $C_k(M)$ is a Volterra space;
- (12) M has the Moving Off Property.

Proof. (1) \Rightarrow (2) follows from Proposition 2.1 and Remark 1.9.

(2) \Rightarrow (3) \Rightarrow (5) and (2) \Rightarrow (4) \Rightarrow (5), as well as (7) \Rightarrow (8), (8) \Rightarrow (9) and (9) \Rightarrow (10) \Rightarrow (11) are trivial.

By [9, Theorem 4.1], for a locally compact space X , K has a winning strategy in $G_{K,L}^0(X)$ if and only if X is paracompact. Also, by [15, Theorem 1.2], a locally compact space X is paracompact if and only if $C_k(X)$ is weakly α -favourable. Since a manifold is paracompact if and only if it is metrisable, we have (5) \Leftrightarrow (1) \Leftrightarrow (6).

(1) \Rightarrow (7). If M is metrisable, then $C_k(M)$ is completely metrisable and thus player α has a stationary winning strategy in the game $\mathcal{G}_S(D)$ played in $C_k(M)$ for any dense set $D \subset C_k(M)$.

(11) \Rightarrow (10) follows from [4, Theorem 3.4].

(10) \Leftrightarrow (12) follows from [11, Theorem 2.1].

(10) \Rightarrow (1) follows from Proposition 2.2. \square

Most completeness properties in the literature relate complete metrisability and α -favourability. As an example, consider pseudo-completeness. Thus, the following corollary improves equivalence condition 6 of [8, Theorem 2].

Corollary 3.4. *Let M be a manifold. Then M is metrisable if and only if $C_k(M)$ is pseudo-complete.*

Note that every manifold is locally compact, and so any manifold is strongly α -favourable, and hence α -favourable, strongly Choquet and weakly α -favourable. See, for example, [18].

Remark 3.5. A very similar argument to that in the proof of Proposition 2.4 shows that if X is a Tychonoff space then $C_p(X)$ is strongly Baire if and only if $C_p(X)$ is Baire and X is countable. Hence there is no analogue of Theorem 3.3 for $C_p(M)$.

4. Open Questions

Propositions 2.2 and 2.4 motivate the following question.

Question 4.1. *For a Tychonoff space X is there any relation between the Bairness of $C_k(X)$ and hemicompactness of X ?*

The following questions also seem to be interesting in the light of Theorem 3.3. The intention is that the property \mathcal{P} should be similar in nature to the Moving Off Property.

Question 4.2. *Let X be a Tychonoff space. Is there a property \mathcal{P} such that X has \mathcal{P} if and only if $C_k(X)$ is strongly Baire?*

Question 4.3. *Are the following equivalent for a Tychonoff space X ?*

- (1) $C_k(X)$ is strongly α -favourable.
- (2) $C_k(X)$ is strongly Choquet.
- (3) $C_k(X)$ is α -favourable.
- (4) $C_k(X)$ is weakly α -favourable.

Remark 4.4. The answer to Question 4.3 is affirmative when X is locally compact and paracompact, [19, Theorem 2.3]. More precisely, for a locally compact space X , each of conditions (1), (2), (3) and (4) in Question 4.3 is equivalent to X being paracompact. Further, there is a locally compact space X such that $C_k(X)$ is Baire but not weakly α -favourable, see [11, Example 4.1].

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Jiling Cao
School of Computing and Mathematical
Sciences
Auckland University of Technology
Private Bag 92006
Auckland 1020
NEW ZEALAND
jiling.cao@aut.ac.nz

Sina Greenwood
Department of Mathematics
The University of Auckland
Private Bag 92019
Auckland 1020
NEW ZEALAND
sina@math.auckland.ac.nz

David Gauld
Department of Mathematics
The University of Auckland
Private Bag 92019
Auckland 1020
NEW ZEALAND
d.gauld@auckland.ac.nz

Abdul Mohamad
Department of Mathematics and Statistics
Sultan Qaboos University
P.O. Box 36
Muscat
SULTANANTE OF OMAN
mohamad@squ.edu.om