

## OSCILLATORY BEHAVIOR OF FOURTH ORDER NONLINEAR DIFFERENCE EQUATIONS

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Abstract. Some new criteria for the bounded oscillation of nonlinear fourth order difference equation

$$\Delta \left( \frac{1}{a_3(n)} \left( \Delta \left( \frac{1}{a_2(n)} \left( \Delta \left( \frac{1}{a_1(n)} (\Delta x(n))^{\alpha_1} \right)^{\alpha_2} \right)^{\alpha_3} \right) \right) + q(n)f(x[g(n)]) \right) = 0$$

are established.

### 1. Introduction

Consider the fourth order nonlinear difference equation

$$L_4 x(n) + q(n)f(x[g(n)]) = 0, \quad n \in \mathbb{N} = \{0, 1, 2, \dots\}, \quad (1.1)$$

where

$$L_0 x(n) = x(n), \quad L_k x(n) = \frac{1}{a_k(n)} (\Delta L_{k-1} x(n))^{\alpha_k}, \quad k = 1, 2, 3, \quad L_4 x(n) = \Delta L_3 x(n) \quad (1.2)$$

and  $\Delta$  is the forward difference operator defined by  $\Delta x(n) = x(n+1) - x(n)$ .

In what follows, we shall assume that

(i).  $a_i(n), q(n) : \mathbb{N} \rightarrow \mathbb{R}^+ = (0, \infty)$ ,  $i = 1, 2, 3$  and

$$\sum_{j=n_0 \geq 0}^{\infty} a_i^{1/\alpha_i}(j) = \infty, \quad i = 1, 2, 3; \quad (1.3)$$

(ii).  $g(n) : \mathbb{N} \rightarrow \mathbb{R}$ ,  $\{g(n)\}$  is nondecreasing, and  $\lim_{n \rightarrow \infty} g(n) = \infty$ ;

(iii).  $f \in C(\mathbb{R}, \mathbb{R})$ ,  $xf(x) > 0$  and  $f'(x) \geq 0$  for  $x \neq 0$ ;

(iv).  $\alpha_i$ ,  $i = 1, 2, 3$  are the ratios of positive odd integers.

The domain  $\mathcal{D}(L_4)$  of  $L_4$  is defined to be the set of all functions  $x(n) : [n_x, \infty) \rightarrow \mathbb{R}$ ,  $n_x \geq n_0 \geq 0$  such that  $L_j x(n)$ ,  $0 \leq j \leq 4$  exist on  $[n_x, \infty)$ . Our attention is restricted to those solutions  $\{x(n)\} \in \mathcal{D}(L_4)$  which satisfy  $\sup_{n \geq N} |x(n)| > 0$  for  $N \geq n_x$ .

A solution  $\{x(n)\}$  of equation (1.1) is called oscillatory if for any  $m \in \mathbb{N}$  there exist  $m_1, m_2 \geq m$  such that  $x(m_1)x(m_2) < 0$ , otherwise it is nonoscillatory. Equation (1.1) is called  $B$ -oscillatory if all its bounded solutions are oscillatory.

Determining oscillation and nonoscillation criteria for difference equations has received a great deal of attention in the last few years. See, for example the monographs [1–6,9,11] and the references cited therein. This interest is motivated by the importance of difference equations in the numerical solutions of differential equations. Compared to first and second order difference equations, the study of higher order equations and in particular fourth order equations of type (1.1) with  $\alpha_i = 1$ ,  $i = 1, 2, 3$  and some  $a_i(n) \equiv 1$ ,  $i = 1, 2, 3$  has received considerably less attention (see [7,10,12–16]) and the references cited therein).

Our purpose of this paper is to obtain some sufficient conditions for the oscillation of all bounded solutions of equation (1.1). Also, establish some necessary and sufficient conditions for the bounded oscillation and nonoscillation of equation (1.1). Further, our equation is quite general and therefore, the results of this paper even in some special cases complement and extend some well-known results appeared in the literature (see [7,10,12–16]).

## 2. Main Results

Consider the following inequalities

$$\Delta \left( \frac{1}{a_1(n)} (\Delta x(n))^{\alpha_1} \right) + q(n)f(x[g(n)]) \leq 0, \quad (2.1)$$

$$\Delta \left( \frac{1}{a_1(n)} (\Delta x(n))^{\alpha_1} \right) + q(n)f(x[g(n)]) \geq 0 \quad (2.2)$$

and the equation

$$\Delta \left( \frac{1}{a_1(n)} (\Delta x(n))^{\alpha_1} \right) + q(n)f(x[g(n)]) = 0, \quad (2.3)$$

where  $a_1(n)$ ,  $q(n)$ ,  $g(n)$ ,  $\alpha_1$  and  $f$  are as in equation (1.1) satisfying the conditions (i) – (iv).

We shall employ the following lemma, which is a special case of Lemma 2.3 in [13].

**Lemma 1.** If inequality (2.1) (inequality (2.2)) has an eventually positive (negative) solution, then equation (2.3) also has eventually positive (negative) solution.

We set

$$Q(n) = a_2^{1/\alpha_2}(n) \left( \sum_{s=n}^{\infty} a_3^{1/\alpha_3}(s) \left( \sum_{j=s}^{\infty} q(j) \right)^{1/\alpha_3} \right)^{1/\alpha_2}, \quad n \geq n_0 \in \mathbb{N}$$

and

$$F(x) = f^{1/(\alpha_2\alpha_3)}(x), \quad x \in \mathbb{R}.$$

Now, we present the following comparison result.

**Theorem 2.** Let conditions (i) – (iv) hold. If the equation

$$\Delta \left( \frac{1}{a_1(n)} (\Delta x(n))^{\alpha_1} \right) + Q(n)F(x[g(n)]) = 0 \quad (2.4)$$

is oscillatory, then equation (1.1) is  $B$ -oscillatory.

**Proof.** Let  $\{x(n)\}$  be a bounded nonoscillatory solution of equation (1.1), say,  $x(n) > 0$  for  $n \geq n_0 \in \mathbb{N}$ . By condition (1.3), we can easily see that  $x(n)$  satisfies the inequalities

$$\Delta x(n) > 0, L_2 x(n) < 0, L_3 x(n) > 0 \text{ and } L_4 x(n) \leq 0 \text{ for } n \geq n_1 \geq n_0. \quad (2.5)$$

Summing equation (1.1) from  $n$  to  $u \geq n \geq n_1$  and letting  $u \rightarrow \infty$ , we find

$$L_3 x(n) \geq \sum_{j=n}^{\infty} q(j) f(x[g(j)]).$$

One can easily see that

$$\Delta L_2 x(n) \geq a_3^{1/\alpha_3}(n) \left( \sum_{j=n}^{\infty} q(j) \right)^{1/\alpha_3} f^{1/\alpha_3}(x[g(n)]). \quad (2.6)$$

Summing (2.6) from  $n \geq n_1$  to  $v \geq n$  and letting  $v \rightarrow \infty$ , we have

$$-L_2 x(n) \geq \left( \sum_{s=n}^{\infty} a_3^{1/\alpha_3}(s) \left( \sum_{j=s}^{\infty} q(j) \right)^{1/\alpha_3} \right) f^{1/\alpha_3}(x[g(n)]),$$

or

$$\begin{aligned} -\Delta L_1 x(n) &\geq a_2^{1/\alpha_2}(n) \left( \sum_{s=n}^{\infty} a_3^{1/\alpha_3}(s) \left( \sum_{j=s}^{\infty} q(j) \right)^{1/\alpha_3} \right)^{1/\alpha_2} f^{1/(\alpha_2 \alpha_3)}(x[g(n)]) \\ &:= Q(n)F(x[g(n)]) \end{aligned} \quad (2.7)$$

for  $n \geq n_1$ . By applying Lemma 1, we see that equation (2.4) has a positive solution, a contradiction. This completes the proof. ■

We assume that the function  $F(x) = f^{1/(\alpha_2 \alpha_3)}(x)$ ,  $x \in \mathbb{R}$  satisfies

$$-F(-xy) \geq F(xy) \geq F(x)F(y) \text{ for } xy > 0. \quad (2.8)$$

Also, we let

$$\eta[n, n_0] = \sum_{i=n_0}^n a_1^{1/\alpha_1}(i) \text{ for } n \geq n_0 \in \mathbb{N}$$

and

$$\bar{Q}(n) = Q(n)F(\eta[g(n), N]) \text{ for } n \geq N \geq n_0.$$

Now, we present the following comparison result.

**Theorem 3.** Let conditions (i) – (iv) and (2.8) hold,

$$g(n) = n - \tau + 1, \tau \geq 1 \text{ is a real number.} \quad (2.9)$$

If the first order equation

$$\Delta y(n) + \overline{Q}(n)F\left(y^{1/\alpha_1}[n - \tau]\right) = 0 \quad (2.10)$$

is oscillatory, then equation (1.1) is  $B$ -oscillatory.

**Proof.** Let  $\{x(n)\}$  be a bounded nonoscillatory solution of equation (1.1), say,  $x(n) > 0$  for  $n \geq n_0 \in \mathbb{N}$ . As in the proof of Theorem 2, we obtain (2.5) and (2.7) for  $n \geq n_1 \geq n_0$ . Now

$$x(n) - x(n_1) = \sum_{i=n_1}^{n-1} \Delta x(i) = \sum_{i=n_1}^{n-1} \left(a_1^{1/\alpha_1}(i)\right) \left(a_1^{-1/\alpha_1}(i) \Delta x(i)\right).$$

Using the fact that  $\{a_1^{-1/\alpha_1}(n) \Delta x(n)\}$  is nonincreasing for  $n \geq n_1$ , we obtain

$$x(n) \geq \eta[n - 1, n_1] a_1^{-1/\alpha_1}(n - 1) \Delta x(n - 1), \quad n \geq n_1 + 1. \quad (2.11)$$

There exists  $n_2 \in \mathbb{N}$ ,  $n_2 \geq n_1 + \tau + 1$  such that

$$x[n - \tau + 1] \geq \eta[n - \tau, n_1] a_1^{-1/\alpha_1}[n - \tau] \Delta x[n - \tau] \text{ for } n \geq n_2. \quad (2.12)$$

$$:= \eta[n - \tau, n_1] Z^{1/\alpha_1}[n - \tau], \quad n \geq n_2 \quad (2.13)$$

where  $Z(n) = (\Delta x(n))^{\alpha_1} / a_1(n)$ ,  $n \geq n_2$ . Using (2.13) in (2.7) we get

$$\Delta Z(n) + \overline{Q}(n)F\left(Z^{1/\alpha_1}[n - \tau]\right) \leq 0 \text{ for } n \geq n_2. \quad (2.14)$$

But in view of a result in [9, Corollary 7.6.1], equation (2.10) has an eventually positive solution, which is a contradiction. This completes the proof. ■

Using a well known oscillation result for first order equations [9, Theorem 7.2.1], the following corollary is immediate.

**Corollary 4.** Let conditions (i) – (iv), (2.8) and (2.9) hold. Then, equation (1.1) is  $B$ -oscillatory if one of the following conditions holds:

(I<sub>1</sub>).  $F(y^{1/\alpha_1})/y \geq 1$  for  $y \neq 0$  and

$$\liminf_{n \rightarrow \infty} \sum_{j=n-\tau}^{n-1} \overline{Q}(j)F(\eta[j - \tau, n_0]) > \left(\frac{\tau}{\tau + 1}\right)^{\tau+1}$$

(I<sub>2</sub>).  $\int_{\pm 0} \frac{du}{F(u^{1/\alpha_1})} < \infty$  and

$$\sum_{j=n-\tau}^{\infty} \overline{Q}(j)F(\eta[j - \tau, n_0]) = \infty.$$

Next, we let  $\overline{F}(x) = f^{1/(\alpha_1 \alpha_2 \alpha_3)}(x)$ ,  $x \in \mathbb{R}$  and assume that

$$\int^{\pm \infty} \frac{du}{\overline{F}(u)} < \infty. \quad (2.15)$$

Now, we prove the following oscillation result.

**Theorem 5.** Let conditions (i) – (iv), (2.9) and (2.15) hold. If

$$\sum_{s=n_2}^{\infty} \left( a_1[s-\tau] \sum_{j=s}^{\infty} Q(j) \right)^{1/\alpha_1} = \infty, \quad (2.16)$$

then equation (1.1) is  $B$ -oscillatory.

**Proof.** Let  $\{x(n)\}$  be a bounded nonoscillatory solution of equation (1.1), say,  $x(n) > 0$  for  $n \geq n_0 \in \mathbb{N}$ . As in the proof of Theorem 2, we obtain (2.7) for  $n \geq n_1$ . One can easily see that

$$L_1 x(n) \geq \left( \sum_{j=n}^{\infty} Q(j) \right) F(x[n-\tau+1]). \quad (2.17)$$

Using the fact that  $L_1 x$  is nonincreasing, it follows from (2.17) that

$$\Delta x[n-\tau] \geq a_1^{1/\alpha_1}[n-\tau] \left( \sum_{j=n}^{\infty} Q(j) \right)^{1/\alpha_1} \bar{F}(x[n-\tau+1]),$$

or

$$\frac{\Delta x[n-\tau]}{\bar{F}(x[n-\tau+1])} \geq a_1^{1/\alpha_1}[n-\tau] \left( \sum_{j=n}^{\infty} Q(j) \right)^{1/\alpha_1}, \quad n \geq n_2 \geq n_1. \quad (2.18)$$

Summing (2.18) from  $n_2$  to  $n \geq n_2$ , we have

$$\sum_{s=n_2}^n \left( a_1[s-\tau] \sum_{j=s}^{\infty} Q(j) \right)^{1/\alpha_1} \leq \sum_{s=n_2}^n \frac{\Delta x[s-\tau]}{\bar{F}(x[s-\tau+1])} \leq \int_{x[n_2-\tau]}^{x[n-\tau+1]} \frac{dy}{\bar{F}(y)} < \infty,$$

which contradicts (2.16). This completes the proof. ■

**Theorem 6.** Let conditions (i) – (iv) hold. If

$$\sum_{s=n_2}^{\infty} \left( a_1(s) \sum_{j=s}^{\infty} Q(j) \right)^{1/\alpha_1} = \infty, \quad (2.19)$$

then equation (1.1) is  $B$ -oscillatory.

**Proof.** Let  $\{x(n)\}$  be a bounded nonoscillatory solution of equation (1.1), say,  $x(n) > 0$  for  $n \geq n_0 \in \mathbb{N}$ . As in the proof of Theorem 5, we obtain (2.17) which takes the form

$$\Delta x(n) \geq \left( a_1(n) \sum_{j=n}^{\infty} Q(j) \right)^{1/\alpha_1} \bar{F}(x[g(n)]) \text{ for } n \geq n_2 \geq n_1.$$

Since  $\{x(n)\}$  is an increasing sequence for  $n \geq n_2$ , there exists  $n_3 \geq n_2$  and a constant  $C > 0$  such that

$$x[g(n)] \geq C \text{ for } n \geq n_3.$$

Thus,

$$\Delta x(n) \geq \left( a_1(n) \sum_{j=n}^{\infty} Q(j) \right)^{1/\alpha_1} \bar{F}(C), \quad n \geq n_3. \quad (2.20)$$

Summing (2.20) from  $n_3$  to  $n-1$  and using condition (2.19), we arrive at a contradiction. This completes the proof. ■

Next, we will present some necessary and sufficient conditions for all bounded solutions of equation (1.1) to be oscillatory, or nonoscillatory.

**Theorem 7.** Let conditions (i) – (iv) hold. Then, equation (1.1) is  $B$ -oscillatory if and only if condition (2.19) is satisfied.

**Proof.** Suppose that (2.19) holds and assume that equation (1.1) has a bounded nonoscillatory solution  $\{x(n)\}$ . The proof is similar to that of Theorem 6 and hence omitted.

Assume that (2.19) does not hold. We may suppose that

$$\sum_{j_1=n_0 \in \mathbb{N}}^{\infty} \left( a_1(j_1) \sum_{j_2=j_1}^{\infty} \left( a_2(j_2) \sum_{j_3=j_2}^{\infty} \left( a_3(j_3) \sum_{j=j_3}^{\infty} q(j) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1} < \infty. \quad (2.21)$$

Then, we can choose  $n_1 \geq n_0$  sufficiently large such that for  $n \geq n_1$ ,

$$\sum_{j_1=n_1}^{\infty} \left( a_1(j_1) \sum_{j_2=j_1}^{\infty} \left( a_2(j_2) \sum_{j_3=j_2}^{\infty} \left( a_3(j_3) \sum_{j=j_3}^{\infty} f(\gamma)q(j) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1} < \frac{\gamma}{2} \quad (2.22)$$

for some constant  $\gamma > 0$ .

Let  $\{x(n)\}$  be a solution of the following equation  
 $x(n) =$

$$\gamma - \sum_{j_1=n}^{\infty} \left( a_1(j_1) \sum_{j_2=j_1}^{\infty} \left( a_2(j_2) \sum_{j_3=j_2}^{\infty} \left( a_3(j_3) \sum_{j=j_3}^{\infty} q(j)f(x[g(j)]) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1}. \quad (2.23)$$

Then we easily see that  $\{x(n)\}$  is a solution of equation (1.1). Next, we shall show that equation (2.23) has a bounded nonoscillatory solution  $\{x(n)\}$  by using the fixed point theorem of Schauder.

We introduce the Banach space  $\ell^\infty$  of all bounded real sequences  $\{x(n)\}$ , ( $n \geq n_0$ ) with norm  $\|x\| = \sup_n |x(n)|$ . We define a bounded, convex and closed subset  $\mathcal{B}$  of  $\ell^\infty$  as

$$\mathcal{B} = \left\{ x \in \ell^\infty : \frac{\gamma}{2} \leq x(n) \leq \gamma, \quad n \geq n_0 \right\}.$$

Next, let  $T$  be a mapping defined on  $\mathcal{B}$  as follows: For  $x = \{x(n)\}$  in  $\mathcal{B}$ ,  $(Tx)(n) =$

$$\gamma - \sum_{j_1=n}^{\infty} \left( a_1(j_1) \sum_{j_2=j_1}^{\infty} \left( a_2(j_2) \sum_{j_3=j_2}^{\infty} \left( a_3(j_3) \sum_{j=j_3}^{\infty} q(j) f(x[g(j)]) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1}. \quad (2.24)$$

Then the mapping  $T$  satisfies the following:

(i<sub>1</sub>).  $T$  maps  $\mathcal{B}$  into  $\mathcal{B}$ . In fact, for any  $x \in \mathcal{B}$ , from (2.22) and (2.24) we have that

$$\gamma \geq (Tx)(n) \geq \gamma - \frac{\gamma}{2} = \frac{\gamma}{2}, \text{ so } Tx \in \mathcal{B}.$$

(i<sub>2</sub>). The mapping  $T$  is continuous on  $\mathcal{B}$ . Let  $\{x^{(m)}(n)\}$  be a sequence in  $\mathcal{B}$  converging to  $x$ . We must show that  $Tx^{(m)}$  converges to  $Tx$ . By (2.22), for any  $\epsilon > 0$ , we can choose  $n_2 \in \mathbb{N}$ ,  $n_2 \geq n_1$  such that

$$\sum_{j_1=n_2}^{\infty} \left( a_1(j_1) \sum_{j_2=j_1}^{\infty} \left( a_2(j_2) \sum_{j_3=j_2}^{\infty} \left( a_3(j_3) \sum_{j=j_3}^{\infty} q(j) f(\gamma) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1} < \frac{\epsilon}{3}. \quad (2.25)$$

Furthermore, we can see that the series  $\sum_{j=n}^{\infty} q(j) f(x^{(m)}[g(j)])$  converges to the series  $\sum_{j=n}^{\infty} q(j) \times f(x[g(j)])$  uniformly with respect to  $m$ . So, we can choose  $N$  such that for all  $m \geq N$ ,

$$\begin{aligned} & \left| \sum_{j_1=0}^{n_2-1} \left( a_1(j_1) \sum_{j_2=j_1}^{\infty} \left( a_2(j_2) \sum_{j_3=j_2}^{\infty} \left( a_3(j_3) \sum_{j=j_3}^{\infty} q(j) f(x^{(m)}[g(j)]) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1} \right. \\ & \quad \left. - \sum_{j_1=0}^{n_2-1} \left( a_1(j_1) \sum_{j_2=j_1}^{\infty} \left( a_2(j_2) \sum_{j_3=j_2}^{\infty} \left( a_3(j_3) \sum_{j=j_3}^{\infty} q(j) f(x[g(j)]) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1} \right| \\ & < \frac{\epsilon}{3}. \end{aligned} \quad (2.26)$$

In the following, we shall show that  $|T(x^{(m)}(n) - T(x(n))| < \epsilon$  for any  $n$  and  $m \geq N$ .

(I). If  $n \geq n_2$ , then from (2.24) and (2.25) we find

$$\begin{aligned}
 & |T(x^{(m)}(n)) - T(x(n))| \\
 & \leq 2 \left| \sum_{j_1=n}^{\infty} \left( a_1(j_1) \sum_{j_2=j_1}^{\infty} \left( a_2(j_2) \sum_{j_3=j_2}^{\infty} \left( a_3(j_3) \sum_{j=j_3}^{\infty} f(\gamma)q(j) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1} \right| \\
 & < \frac{2\epsilon}{3} < \epsilon.
 \end{aligned}$$

(II). If  $n \leq n_2$ , from (2.24), (2.25) and (2.26), we have

$$\begin{aligned}
 & |T(x^{(m)}(n)) - T(x(n))| \\
 & \leq \left| \sum_{j_1=n}^{n_2-1} \left( a_1(j_1) \sum_{j_2=j_1}^{\infty} \left( a_2(j_2) \sum_{j_3=j_2}^{\infty} \left( a_3(j_3) \sum_{j=j_3}^{\infty} q(j)f(x^{(m)}[g(j)]) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1} \right. \\
 & \quad \left. - \sum_{j_1=n}^{n_2-1} \left( a_1(j_1) \sum_{j_2=j_1}^{\infty} \left( a_2(j_2) \sum_{j_3=j_2}^{\infty} \left( a_3(j_3) \sum_{j=j_3}^{\infty} q(j)f(x[g(j)]) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1} \right| \\
 & \quad + \left| \sum_{j_1=n_2}^{\infty} \left( a_1(j_1) \sum_{j_2=j_1}^{\infty} \left( a_2(j_2) \sum_{j_3=j_2}^{\infty} \left( a_3(j_3) \sum_{j=j_3}^{\infty} q(j)f(x^{(m)}[g(j)]) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1} \right| \\
 & \quad + \left| \sum_{j_1=n_2}^{\infty} \left( a_1(j_1) \sum_{j_2=j_1}^{\infty} \left( a_2(j_2) \sum_{j_3=j_2}^{\infty} \left( a_3(j_3) \sum_{j=j_3}^{\infty} q(j)f(x[g(j)]) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1} \right| \\
 & < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon \text{ for } m \geq N.
 \end{aligned}$$

Clearly, (I) and (II) together yield that  $|Tx^{(m)} - Tx| < \epsilon$  for any  $n \in \mathbb{N}$  and  $m \in \mathbb{N}$  which completes the proof that the mapping  $T$  is continuous on  $\mathcal{B}$ .

(i<sub>3</sub>). The set  $T(\mathcal{B})$  is uniformly Cauchy. Let  $\{x(n)\}$  be any sequence in  $\mathcal{B}$ . Then, for any given  $\epsilon > 0$ , there exists an  $n_2 \in \mathbb{N}$  such that (2.25) holds, and

for any  $n, m \geq n_2$ ,

$$\begin{aligned}
& |T(x(m)) - T(x(n))| \\
& \leq \left| \sum_{j_1=m}^{\infty} \left( a_1(j_1) \sum_{j_2=j_1}^{\infty} \left( a_2(j_2) \sum_{j_3=j_2}^{\infty} \left( a_3(j_3) \sum_{j=j_3}^{\infty} q(j) f(x[g(j)]) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1} \right| \\
& + \left| \sum_{j_1=n}^{\infty} \left( a_1(j_1) \sum_{j_2=j_1}^{\infty} \left( a_2(j_2) \sum_{j_3=j_2}^{\infty} \left( a_3(j_3) \sum_{j=j_3}^{\infty} q(j) f(x[g(j)]) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1} \right| \\
& < \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon.
\end{aligned}$$

Thus, the set  $T(\mathcal{B})$  is uniformly Cauchy.

Now by Theorem 3.5 in [8], it follows that (2.23) has a positive solution  $\{x(n)\}$ . This proves the necessity. ■

The following theorem presents a necessary and sufficient condition for the existence of a bounded solution of equation (1.1).

**Theorem 8.** Assume that (i) – (iv) except (1.3) hold, and

$$\int_0^{\infty} q(s) ds = \infty. \quad (2.27)$$

Then a necessary and sufficient condition for equation (1.1) to have positive solution  $\{x(n)\}$  satisfying  $\beta \geq x(n) \geq \gamma > 0$  ( $\beta$  and  $\gamma$  are constants) for all  $n \in \mathbb{N}$  is that

$$\sum_{j_1=n_0}^{\infty} \left( a_1(j_1) \sum_{j_2=n_0}^{j_1-1} \left( a_2(j_2) \sum_{j_3=n_0}^{j_2-1} \left( a_3(j_3) \sum_{j=n_0}^{j_3-1} q(j) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1} < \infty. \quad (2.28)$$

**Proof.** Necessity. If  $\{x(n)\}$  is a positive solution of equation (1.1) and the condition  $\beta \geq x(n) \geq \gamma > 0$  is satisfied, then we have in view of equation (1.1),

$$L_3 x(n) = L_3 x(n_0) - \sum_{j=n_0}^{n-1} q(j) f(x[g(j)]) \leq L_3 x(n_0) - f(\gamma) \sum_{j=n_0}^{n-1} q(j).$$

If  $n \in \mathbb{N}$  is large enough, in view of (2.27), we have  $L_3 x(n) < 0$ . Then, for all large  $n_0 \in \mathbb{N}$ ,

$$L_3 x(n) < -f(\gamma) \sum_{j=n_0}^{n-1} q(j),$$

or

$$\Delta L_2 x(n) \leq -f^{1/\alpha_3}(\gamma) \left( a_3(n) \sum_{j=n_0}^{n-1} q(j) \right)^{1/\alpha_3}.$$

The rest of the proof is similar to the proof of Theorem 6 and hence it is omitted.

The proof of sufficiency is similar to the proof of necessity part of Theorem 7 and hence omitted. This completes the proof. ■

Finally, we can apply our results obtained here to neutral difference equations of the form

$$L_4(x(n) + p(n)x[\tau(n)]) + f(x[g(n)]) = 0, \quad (2.29)$$

where  $p(n) : \mathbb{N} \rightarrow \mathbb{R}^+ \cup \{0\}$ ,  $p(n) \not\equiv 1$ ,  $\tau(n) : \mathbb{N} \rightarrow \mathbb{R}$ ,  $\Delta\tau(n) > 0$  and  $\lim_{n \rightarrow \infty} \tau(n) = \infty$ . Here, we let  $\{x(n)\}$  be a solution of (1.1),  $x(n) > 0$  for  $n \geq n_0 \in \mathbb{N}$  and set

$$y(n) = x(n) + p(n)x[\tau(n)], \quad n \geq n_0.$$

Then, (2.29) takes the form

$$L_4y(n) + q(n)f(x[g(n)]) = 0. \quad (2.30)$$

It is easy to see that if  $0 \leq p(n) \leq 1$  and  $\tau(n) < n$  for  $n \geq n_0$ ,

$$x(n) \geq [1 - p(n)]y(n) \text{ for } n \geq n_1 \geq n_0. \quad (2.31)$$

Also, if  $p(n) \geq 1$  and  $\tau(n) > n$  for  $n \geq n_0$ ,

$$x(n) \geq \frac{1}{p[\tau^{-1}(n)]} \left( 1 - \frac{1}{p[\tau^{-1} \circ \tau^{-1}(n)]} \right) y[\tau^{-1}(n)], \quad n \geq n_1, \quad (2.32)$$

where  $\tau^{-1}$  is the inverse function of  $\tau(n)$ . Using (2.31) or (2.32) in equation (2.30), we arrive to inequalities of type (1.1). The rest of the details are left to the reader.

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