Handling Uncertainty with Qualitative Values

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***Abstract*-Bayesian mathematical model is the oldest method for modelling subjective degree of belief. If we have probabilistic measures with unknown values, then we must choose a different and appropriate model. The belief functions are a bridge between various models handling different forms of uncertainty. The conjunctive rule of Bayes builds a new set of a posteriori probability when two independent and accepted sets of random variable make inference. When two pieces of evidence are accepted with unknown values, the Dempster-Shafer's rule suggests a model for fusion of different degree of belief. In this paper we want to submit the use of MaxEnt principle for modelling the belief. Dealing with non-Bayesian sets, in which the piece of evidence represents the belief instead of the knowledge, the MaxEnt principle gives a tool to reduce the number of subsets representing the frame of discernment. The fusion of a focal set with a set of max entropy causes a Bayesian approximation reducing mass function to a probabilistic distribution.**

***Keywords* – MaxEnt, Probability, Belief, Bayesian Sets.**

I. INTRODUCTION

Let be the probability of A, the Bayes' theory requires the relation

$$P(A)+P(\overline{A})=1$$

in which between lack of disbelief and disbelief there is no distinction. Often in engineering design it is proper to do the fundamental change of replacing the precise value that a probability has with the concept that a probability has a degree of variability in an interval that provides a lower and upper bound. The idea of upper and lower probability, in belief functions, was proposed for handling uncertainty connected with subjectivity. The belief functions are a bridge between various models handling different forms of uncertainty. When there is not enough information on which to evaluate a probability, or when the information is non-specific, ambiguous, or in conflict, then the Bayesian model cannot be used. A method for handling data in presence of uncertainty with qualitative values is the theory of Dempster-Shafer (DS). The DS model includes the Bayesian probability as special case, and introduces the belief function as lower probabilities and the plausibility function as upper probabilities. The numerical measure, in presence of uncertainty, may be assigned to set of elements as well as to a single element. In DS model the probabilities, apportioned to subsets and the mass can move over each element. Let be the frame of discernment the next finite non-empty set

$$\Theta=\{x_1,..x_n\}$$

Θ is the set of all hypothesis. The basic probability is assigned in the range [0,1] to the 2^n subset of Θ consisting of a singleton or conjunction of singleton of n basic elements x_i. The basic probability, a function which assigns the weight to the subset of the frame of discernment, is the mass function $m(.)$. The mass $m(\theta)$ is where we assign the probability that we are unable to assign otherwise. If the belief remain apportioned in single elements, then the DS model corresponds to the Bayesian model of probability. Formally the description of basic probability assignments can be represented with the next equations:

$$\begin{cases} m : P(X)\to[0,1] \;;\; (X : \textit{universal set}) \\ \sum_{A\subseteq\Theta} m(A_i)=1 \\ m(\varnothing)=0 \;;\; (\varnothing : \textit{empty test}) \end{cases}$$

The lower probability $P_*(A_j)$ is defined as

$$P_*(A_j)=\sum_{A_j\subseteq A_i} m(A_j)$$

And the upper probability $P^*(A_j)$ is defined as

$$P^*(A_j)=1-\sum_{A_j\subseteq A_i} m(A_j)$$

The $m(A_i)$ values are the independent basic values of probability inferred on each subset A_i. The belief function of set M is given by

$$Bel(M)=\sum_{A_i\subset M} m(A_i) \qquad Pl(M)=\sum_{A_i\cap M\neq\Theta} m(A_i)$$

The evidential interval that provides a lower and upper bound is

$$EI=[Bl(M),Pl(M)]$$

If m_1 and m_2 are basic probabilities from the independent evidence, and $\{A_{1i}\}, \{A_{2i}\}$ the sets of focal points, then Dempster's model of combination gives the rule of fusion. Given two basic probabilities from the independent evidence, if it is

$$\sum_{A_{1i}\cap A_{2j}\neq\varnothing} m_1(A_i)\;\; m_2(A_j)>0 \quad ; \quad A_k\neq\varnothing$$

then following Dempster's rule

$$m(A)=(m_1\oplus m_2)(A)\overset{def}{=}\frac{\sum_{A_{1i}\cap A_{2j}=A_k} m_1(A_i)m_2(A_j)}{1-\sum_{A_{1i}\cap A_{2j}=\varnothing} m_1(A_i)m_2(A_j)}$$

many combines two or more probabilities.

The Dempster's rule is easy to use and gives a quick mathematical model for handling uncertainty including Bayesian theory. The reliability of result depends on the interpretation of the basic probability assignment. When the conflict K, between the sources of independent basic probability, becomes important then the DS rules present some limitations.

$$K = \sum_{A_{1i} \cap A_{2j} = \varnothing} m_1(A_i) m_2(A_j)$$

The DS rules present some weakness, more than once reported by Zadeh and Dubois&Prade, because if the conflict K is important, then the result of fusion is unacceptable. The rules are mainly based on the extension of the domain of the probability functions. In the applications there exists many cases where DS rules assign low belief to elements of sets with larger cardinality. Many algorithms have been suggested, and many alternative rules have been proposed to overcome the difficult of the computational complexity of reasoning and to escape the limitations of Dempster's rule. A greater number of suggested algorithms can be represented changing fusion's rules

$$\begin{cases} m_{ij}(A) = \mathrm{d}\left(m_1(A_i) m_1(A_j)\right) \\ \mathrm{d} \in \{+, -, g\,/, Max, Min, ...\} \end{cases}$$

choosing solutions in relation with the application and with the needs of capturing epistemic uncertainty. The alternative rules, proposed by Dubois&Prade Yager and Smets, are well-known. Many rules are justified or criticized but all certainly show that there exists a great number of possible rules of combinations. The calculus of upper $P^*(A_i)$ and lower $P_*(A_i)$ probabilities has the same dual interpretation as standard Bayesian calculus. If we don't accept that the DS rules assign certainty to element of sets with lower cardinality, than it means that we don't accept the rules based on the extension of the domain of the probabilistic functions.

The basic probability assignments $m = 2^{\Theta} \to [0,1]$ assign a numerical value to focal elements $m(A_j)$. If we reduce focal elements of the frame of discernment $\Theta = \{x_1, x_2\}$ in singletons basic elements, then the fusion takes the same structure of Bayesian rule.

In a fusion, if n_A is the power of set Θ_A and n_B is the power of set Θ_B, then the power of the set resulting from a fusion is

$$\Theta_{A \cap B} = \Theta_A \oplus \Theta_B \to n_{A \cap B} = Min(n_A, n_B).$$

A lot of limitations of fusion rule are imputable to the evaluation of independence of the two distributions $m_1(A_{1i})$ and $m_2(A_{2j})$.

The problem of the independence is a critical factor in combining evidence. Given the events θ = $\{A,B,C\}$, the elements $2^{\Theta} = 2^3$ of the frame of discernment of all hypothesis are:

$$\{A, B, C, A \cup B, A \cup C, B \cup C, A \cup B \cup C, \varnothing\}$$

The masses of probability of a distribution can be:

$$m(.) = (m_1 \oplus m_2)(A) = \{m(A), m(B), m(C), m(A \cup B), m(A \cup C), m(B \cup C), m(A \cup B \cup C)\}$$

The masses $\{m(A), m(B), m(C)\}$ are uncoupled values of probability whereas the masses

$$\{m(A \cup B), m(A \cup C), m(B \cup C), m(A \cup B \cup C)\}$$

are coupled values of distribution.

In the fusion system, the reduction of uncertainty and complexity is the central problem because the resulting data are the input design parameter for many applications, especially the real time.

We need that probability is assigned to singleton basic elements with uncoupled distribution. The goal of this paper is the definition of a rule for decoupling distributions on basic of MaxEnt Principle.

II. FUSION WITH MAXENT DISTRIBUTION

The basic probability assignment (bpa) is the primitive of evidence theory. The bpa does not always refer to probability in classical sense, in many applications it is useful to interpret bpa as classic probability. The number of focal elements influences the complexity of combining pieces of evidence. A way for lowering the limitations of fusion rules is the reduction of the number of focal elements decoupling probabilities.

Special importance is given to the Bayesian max entropy distributions of probability with the same probability allotted only in all singleton elements. Given the set θ = $\{A,B,C\}$ with 3 basic elements, the Basic Max Entropy (BME) distribution on the elements 2^3 is:

$$m^{BME}(.) = \{m(A) = 1/3,\ m(B) = 1/3,\ m(C) = 1/3, m(A \cup B) = 0, m(A \cup C) = 0, m(B \cup C) = 0, m(A \cup B \cup C) = 0\}$$

The Basic-Max-Entropy distributions capture the epistemic max uncertainty of the Bayesian probabilistic distribution.

1. Remarkable Features of Fusions

A. Let's consider the following two distributions:

$$m_1(.) = m_1(A) = 0.1 \quad m_1(B) = 0.4 \quad m_1(C) = 0.2 \quad m_1(A \cup B) = 0.3$$
$$m_2(.) = m_2(A) = 0.5 \quad m_2(B) = 0.2 \quad m_2(C) = 0.3 \quad \_\_$$

The distribution $m_1(.)$ has a coupled probability $m_1(A \cup B)$, while the distribution $m_2(.)$ is Bayesian.

The set resulting from the fusion is:

$$m_3(.) = [m_1(.) \oplus m_2(.)] = \begin{bmatrix} 0.1 & 0.4 & 0.2 & 0.3 \\ 0.5 & 0.2 & 0.3 & \_\_ \end{bmatrix} = \begin{bmatrix} A & B & C \\ 0.500 & 0.350 & 0.150 \end{bmatrix}$$

In the fusion of the non-Bayesian set $m_1(.)$ with the Bayesian set $m_2(.)$, the mass $m_1(A \cup B)$ is allotted in the basic elements of the set $m_3(.)$. The feature of the set $m_3(.)$ is the number of masses equal to the number of basic elements of the Bayesian set $m_2(.)$.

B. From the aggregation of a Bayesian set of max entropy $m^{MaxEnt}(.)$ with a generic set $m_1(.)$ we get a fusion in which the set of max entropy does not add new information to the set $m_1(.)$:

$$m_1(.) \oplus m^{MaxEnt}(.) = m_1(.)$$

$$m_1(.) = \left[m_1(.) \oplus m^{MaxEnt}(.)\right] = \begin{bmatrix} 0.500 & 0.350 & 0.150 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} =$$

$$\begin{bmatrix} 0{,}500 & 0.350 & 0.150 \end{bmatrix}$$

The result remarks the well-known Bayesian fusion.

2. Maxent Principle Inference

If we have a distribution with coupled values of probability, for obtaining the probability allotted in a set of only basic elements, we must have a fusion with a proper Bayesian set. If we don't have a proper Bayesian set, then we can employ the MaxEnt Principle. In absence of information MaxEnt suggests us to select the distribution of max entropy. Given a set $\Theta\{x_1, x_n\}$ of basic elements, on the basis of the suggestion of MaxEnt principle, it is possible the definition of the next Belief-MaxEnt theorem: *The fusion of the MaxEnt set $m^{MaxEnt}(.)$ with all generic sets of probability assignments $m(.)$ gives a Bayesian set $m^{BME}(.)$ with probability mass allotted only in the basic elements.*

$$m^{BME}(.) = \left[m(.) \oplus m^{MaxEnt}(.)\right] = \begin{bmatrix} A & B & C & A \cup B \cup C \\ 0.1 & 0.3 & 0.2 & 0.4 \\ 1/3 & 1/3 & 1/3 & - \end{bmatrix} =$$

$$\begin{bmatrix} A & B & C & - \\ 0{,}381 & 0{,}333 & 0{,}286 & - \end{bmatrix}$$

The fusion carries out a *Bayesian approximation* reducing mass functions to a probabilistic distribution. The set $m^{MaxEnt}(.)$ works as a distiller extracting, from generic belief assignments, a set with mass of probability decoupled and allotted only in the basic elements. The decoupled Bayesian set is:

$$m^{BME}(.) = \{m(A) = 0.381; m(B) = 0.333; m(C) = 0.286\}$$

The fusion of the $m^{BME}(.)$ with $m^{MaxEnt}(.)$ gives

$$\left[m^{BME}(.) \oplus m^{MaxEnt}(.)\right] = \begin{bmatrix} A & B & C & - \\ 0.381 & 0.333 & 0.286 & - \\ 1/3 & 1/3 & 1/3 & - \end{bmatrix} =$$

$$\begin{bmatrix} A & B & C & - \\ 0{,}381 & 0{,}333 & 0{,}286 & - \end{bmatrix}$$

$$m^{BME}(.) \oplus m^{MaxEnt}(.) = m^{BME}(.)$$

The result shows the special action of set $m^{MaxEnt}(.)$ of max entropy in the fusion: The fusion of a $m^{MaxEnt}(.)$ distribution with a Bayesian set does not change the Bayesian set.

III. BELIEF-MAXENT THEOREM AS RULE OF COMBINATION

The MaxEnt Principle provides an alternative combination rule to Dempster's rule. In addition MaxEnt principle, lowering of the number of focal elements removes some limitations. We can utilize the MaxEnt Principle as distiller for decoupling probability.

Now we can see some results provided by application of Belief-MaxEnt theorem.

Given the two belief assignments

$$\begin{cases} m_1(.) = \{m_1(A) = 0.1, \ \ m_1(B) = 0.4, \ \ m_1(C) = 0.2, \ \ m_1(A \cup B) = 0.3\} \\ m_2(.) = \{m_2(A) = 0.5, \ \ m_2(B) = 0.1, \ \ m_2(C) = 0.3, \ \ m_1(A \cup B) = 0.1\} \end{cases}$$

If one applies the Belief-MaxEnt theorem for decoupling the two distributions, $m_1(.)$ and $m_2(.)$, and successively gets the final set with a Bayesian fusion, the other one gets the next results.

Decoupling set $m_1(.)$ we have:

$$m_I^{BME}(.) = \left[m(.) \oplus m_2^{MaxEnt}(.)\right] = \begin{bmatrix} A & B & C & A \cup B \\ 0.1 & 0.4 & 0.2 & 0.3 \\ 1/3 & 1/3 & 1/3 & - \end{bmatrix} =$$

$$\begin{bmatrix} A & B & C & - \\ 0{,}204 & 0{,}559 & 0{,}236 & - \end{bmatrix}$$

Decoupling set $m_2(.)$ we have:

$$m_{II}^{BME}(.) = \left[m_2(.) \oplus m^{MaxEnt}(.)\right] = \begin{bmatrix} A & B & C & - \\ 0{,}545 & 0{,}182 & 0{,}373 & - \end{bmatrix}$$

Appling Bayesian fusion to the two sets, we have the final aggregation:

$$\left[m_I^{BME}(.) \oplus m_{II}^{BME}(.)\right] = \begin{bmatrix} A & B & C & - \\ 0{,}414 & 0{,}379 & 0{,}207 & - \end{bmatrix}$$

The final result of fusion is a set with mass of belief allotted only in the basic elements

$$m(.) = \{m(A) = 0.414, m(B) = 0{,}379, m(C) = 0{,}207\}.$$

This way of application can be used as new method for decoupling and fusing two distributions of probabilities.

If one applies the DS rule of fusion, the other one gets:

$$m(.) = \left[m_1(.) \oplus m_2(.)\right] = \begin{bmatrix} A & B & C & A \cup B \\ 0.1 & 0.4 & 0.2 & 0.3 \\ 0.5 & 0.1 & 0.3 & 0.1 \end{bmatrix} =$$

$$\begin{bmatrix} A & B & C & A \cup B \\ 0{,}512 & 0{,}268 & 0.146 & 0.073 \end{bmatrix}$$

The new aggregation contains the coupled mass $m(A \cup B) = 0.073$ having coupled the elements A and B. Now if we use *Belief-MaxEnt* theorem as distiller of the set $m(.)$, we have the decoupled set:

$$\left[m^{BME}(.)=m(.)\oplus m_2^{MaxEnt}(.)\right]=\begin{bmatrix} A & B & C & A\cup B \\ 0.512 & 0.268 & 0.146 & 0.073 \\ 1/3 & 1/3 & 1/3 & - \end{bmatrix}=$$

$$\begin{bmatrix} A & B & C & - \\ 0{,}546 & 0{,}318 & 0.136 & - \end{bmatrix}$$

The result of aggregation is a set with mass of belief allotted only in the basic elements

$$m^{BME}(.)=\{m(A)=0.546, m(B)=0{,}318, m(C)=0{,}136\}.$$

in which the mass $m(A\cup B)=0.073$ is reassigned to the basic elements. The second method is useful for lowering uncertainty, decoupling probability, in sets obtained using DS rules.

IV. CONCLUSIONS

The use of *Belief-MaxEnt Theorem* is a new method for aggregation and modelling more distributions of masses of probability.

Dealing with non-Bayesian sets, in which the pieces of evidence represent the belief instead of the knowledge, the *Belief-MaxEnt* theorem carries out a *Bayesian approximation* and gives a tool for reducing the number if subsets represent the frame of discernment, adding great simplification to the process of aggregation.

The use of *Belief-MaxEnt* theorem in the fusion of basic probabilities is synthesized in the next way of applications:

- First way of application: *Belief-MaxEnt* theorem as new method for fusion and decoupling of two, or more, distributions of probability. The results show that the new method adds a great simplification to the process of aggregation.
- Second way of application**:** *Belief-MaxEnt* theorem can be used as distiller for reduction of uncertainty and complexity. If we combine MaxEnt set with a combination of basic probabilities, we get a Bayesian set with less complexity and decoupled probability.
- Third way of application: *Belief-MaxEnt* theorem can be used as distiller of the sets resulting from the fusion via Dempster's rule for reducing the uncertainty and complexity and allotting the probability in a Bayesian set.

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