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On a Flawed, 16th-Century Derivation of Brahmagupta's Formula for the Area of a Cyclic Quadrilateral

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Abstract. Around 1545, the Indian commentator Gaṇeśa suggested an interesting, but ultimately flawed, derivation of Brahmagupta's formula for the area of a cyclic quadrilateral in terms of its sides. In this paper we show that Gaṇeśa's approach is actually valid and that his proof is easily fixed. We will also investigate to what extent his idea can be generalized to arbitrary (convex) quadrilaterals.

1. Introduction

In the early 6th century, the Indian mathematician Brahmagupta suggested that the area $ABCD$ of a cyclic quadrilateral with vertices A, B, C, D and a, b, c, d the lengths of the sides AB, BC, CD, DA is given by the formula

$$ABCD = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

where $s = (a+b+c+d)/2$. Proofs for Brahmagupta's claim were given by al-Shannī (10th century), Jyeṣṭhadeva (16th century) and others.[1] Although rather different in the details, all of these proofs follow a similar approach. A different type of proof was suggested by Jyeṣṭhadeva's contemporary Gaṇeśa.[2]

Gaṇeśa's "proof" can be found in his commentary on the *Līlāvati* of Bhāskara II. According to Gaṇeśa himself, this commentary was composed in 1545 CE. In this note, we will pursue Gaṇeśa's line of reasoning and show how it can be modified to lead to the desired result.[3] Along the way, we will see that some of his ideas apply to any (convex) quadrilateral, albeit in a less elegant form than for the cyclic case. We start with the big idea.

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[1]For more details on the history of Brahmagupta's formula, see [1].

[2]For biographical information on Gaṇeśa, see [3], p.320.

[3]The original text of the commentary seems to have been included in Āpate's standard edition of the *Līlāvati* (1937). We will mostly rely on [2].

2. Preliminaries

Before we proceed, we will quickly review (mostly without proof) a number of major results regarding quadrilaterals. We need a definition first.

Definition 1. *Let l be a line in the real projective plane. Then an involution on l is a projective transformation on l that is its own inverse.*

Any two distinct points that are images of one another under an involution are said to be conjugate points (under the involution). In addition, any involution has exactly two fixed points. i.e. points that are mapped onto themselves. Any involution is fully determined by the images of any two points on the line, i.e. by two pairs of conjugate points, its two fixed points, or one fixed point and one pair of conjugate points. One example of an involution on a line was first formulated by Girard Desargues in the 1630s.

Theorem 1 (Desargues)**.** *Let $ABCD$ be a quadrilateral in the projective plane and let l be an arbitrary line in the same plane not passing through either A, B, C, or D. Then the three pairs of points of intersection of l with the opposite sides AB and CD, with AD and BC, as well as with the diagonals AC and BD are such that each pair is a pair of conjugate points under the involution determined by the other two pairs of points.*

By the principle of duality, involution is well-defined for a pencil of lines as well. Obviously, the dual version of Theorem 1 provides an example of such an involution. Another example follows from the same theorem by choosing ℓ_∞, the line at infinity, for our line ℓ. In that case, one could say that the directions of the sides and diagonals of $ABCD$ form conjugate pairs under one and the same involution. If we think of the vectors $\overrightarrow{AB}$ and so on as all having their tail at a point O, this can be expressed by saying that the lines these vectors lie on are conjugate lines under one and the same involution on the pencil of lines through O.

Involutions on lines and pencils can also be defined by means of a conic section. We need a definition first.

Definition 2. *Let A_1 and A_2 be two points in the projective plane with a_1 and a_2 their polar lines with respect to a conic $\mathcal{C}$ in the same plane. Then A_1 and A_2 are said to be conjugate points with respect to $\mathcal{C}$ if and only if a_1 lies on A_2 (and therefore a_2 lies on A_1).*

From this definition, it follows immediately that for any point P on a line ℓ not tangent to $\mathcal{C}$, there is exactly one point P' on ℓ that P is conjugate with. Therefore, conjugation with respect to a given conic of the points of a line in the plane of the conic defines an involution. In fact, any involution can be defined as a conjugation of the points of a line with respect to a conic. A conic that is pertinent in the case of the involution on ℓ_∞ defined by a quadrilateral $ABCD$ per Theorem 1 is given by the following theorem.

Theorem 2 (Nine-point Conic)**.** *Let $ABCD$ be a quadrilateral in the affine plane with no parallel sides. Let $E = AC \cap BD$, $F = AD \cap BC$, $G = AB \cap DC$,* *while M_{AB} denotes the mid-point of the line segment AB and so on. Then, the nine-point conic of $ABCD$ is the unique conic passing through the nine points E, F, G, M_{AB},...,M_{CD}. In case $ABCD$ is convex or self-intersecting, its associated nine-point conic is a hyperbola. In case $ABCD$ is concave, its nine-point conic is an ellipse.*

It is now easy to verify that the involution on ℓ_∞ defined by a quadrilateral $ABCD$ per Theorem 1 coincides with conjugation of the points of ℓ_∞ with respect to $\mathcal{H}$ of $ABCD$. Alternatively, we could define the involution of the directions of the sides as the conjugation with respect to $\mathcal{H}$ of the lines of the pencil centered at the center of $\mathcal{H}$. Consequently, we have the following result.

Theorem 3. *Let $ABCD$ be a convex quadrilateral in the affine plane with no parallel sides. Then, the fixed lines of the involution of directions of $ABCD$ are real and parallel to the asymptotes of $\mathcal{H}$.*

In other words, the directions of the asymptotes of $\mathcal{H}$ harmonically separate each of the pairs of directions of AB, CD and AC, BD and AD, BC. This immediately leads to the following observation.

Corollary 4. *Let $ABCD$ be a convex quadrilateral with the hyperbola $\mathcal{H}$ for its nine-point conic. Then, for each of the pairs of lines AB, CD and AC, BD and AD, BC, there is a parallelogram that has its sides parallel to the asymptotes of $\mathcal{H}$ and its diagonals parallel to the pair of lines.*

Proof. This follows immediately from the fact that the directions of the sides of a parallelogram separate the directions of the diagonals harmonically... □

We can make the preceding more specific for the case of the diagonals AC and BD, with $E = AC \cap BD$. Let X, X' and Y, Y' be points on AC and BD, respectively, such that XY and $X'Y'$ are conjugate under the aforesaid involution. Furthermore, let x, y, x', y' be the signed lengths of EX, EY, EX', EY'. Then, the ratios $x : y$ and $x' : y'$ can be associated with two points with coordinates $[x : y]$ and $[x' : y']$ on the canonical projective line. By construction, these points are conjugate under an involution on the projective line. Specifically, $[1 : 0]$ (representing AC) is paired with $[0 : 1]$ (representing BD). Furthermore, let e_A, e_C, f_B, and f_D be the lengths of EA, EC, FB, FD, respectively. Then, $[e_A : f_B]$ and $[e_A : f_D]$ are paired with $[e_C : f_D]$ and $[e_C : f_B]$, respectively. As any involution mapping $[x : y]$ to $[x' : y']$ is described by a relation of the form $Axx' + B(xy' + yx') + Cyy' = 0$, it follows that the involution above is described by the relation $e_A e_C yy' = f_B f_D xx'$. We can use this observation to prove the following theorem.

Theorem 5. *Let $ABCD$ be a convex quadrilateral with no parallel sides, with E and e_A, e_C, f_b, f_D defined as above, while e and f are the lengths of AC and BD, respectively. Furthermore, let A^* be on the ray from E through A, B^* on the ray from E through B, C^* on the ray from E through C and D^* on the ray from E through D be such that $A^*E/AC = C^*E/AC = \sqrt{e_A e_C}/e$ and $B^*E/BD = D^*E/BD = \sqrt{f_B f_D}/f$. Then, the two pairs of parallel sides of* *parallelogram $A^*B^*C^*D^*$ are parallel to the asymptotes of the nine-point conic $\mathcal{H}$ of $ABCD$.*

Proof. Since the sides of $A^*B^*C^*D^*$ are in the directions of the fixed lines of the involution defined by the pairs of opposite sides of $ABCD$ it immediately follows that they are parallel to the asymptotes of $\mathcal{H}$. $\square$

This concludes our preliminary section. We are now ready to prove our main result.

3. Constructing a pair of inscribed parallelograms

Let $ABCD$ be a convex quadrilateral with no parallel sides. Let the asymptotes of the nine-point conic $\mathcal{H}$ of $ABCD$ be the axes of an oblique coordinate system and denote the center of $\mathcal{H}$ by O. Furthermore, let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$, $\bar{f}$ denote the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$, $\overrightarrow{AC}$, $\overrightarrow{BD}$ (with a, b, c, d, e, f their lengths). Finally, let $\bar{p}$ and $\bar{q}$ denote the vectors $\overrightarrow{A^*B^*}$ and $\overrightarrow{B^*C^*}$ (with p and q their lengths). The following result now applies (See Figure 1).

Theorem 6. *For a quadrilateral $ABCD$ in the affine plane with no parallel sides, let A' be the unique point on AB such that $2\overrightarrow{AA'}$ is the sum of $\bar{a}$ and the oblique projection of $\bar{c}$ onto $\bar{a}$ in the direction of $\bar{p}$ and let B', C', D' be defined analogously. Similarly, let B'' be the unique point on AB such that $2\overrightarrow{AA'}$ is the sum of $\bar{a}$ and the oblique projection of $\bar{c}$ onto $\bar{a}$ in the direction of $\bar{q}$ with A'', C'', D'' analogously. Then, $A'B'C'D'$ is a parallelogram inscribed in $ABCD$ such that $\overrightarrow{A'B'}$ is the oblique projection of $\bar{e}$ onto $\bar{p}$ in the direction of $\bar{q}$, while $\overrightarrow{B'C'}$ is the oblique projection of $\bar{f}$ onto $\bar{q}$ in the direction of $\bar{p}$. Similarly, $A''B''C''D''$ is a parallelogram inscribed in $ABCD$ such that $\overrightarrow{A''B''}$ is the oblique projection of $\bar{f}$ onto $\bar{p}$ in the direction of $\bar{q}$, while $\overrightarrow{B'C'}$ is the oblique projection of $\bar{e}$ onto $\bar{q}$ in the direction of $\bar{p}$. Finally, the oriented areas of $A'B'C'D'$ and $A''B''C''D''$ are each equal to the oriented area of $ABCD$.*

Proof. Let $\bar{a} = a_1\bar{p} + a_2\bar{q}$, $\bar{c} = c_1\bar{p} + c_2\bar{q}$ and so on. By Corollary 4, $a_1c_2 = -a_2c_1$. Therefore, the projection of $\bar{c}$ onto $\bar{a}$ in the direction of $\bar{p}$ is given by $-c_1\bar{p} + c_2\bar{q}$. In other words, $\overrightarrow{AA'} = \frac{1}{2}((a_1 + c_1)\bar{p} + (a_2 - c_2)\bar{q})$ while $\overrightarrow{BA'} = \frac{1}{2}((-a_1 + c_1)\bar{p} + -(a_2 + c_2)\bar{q})$ with analogous expressions for $\overrightarrow{BB'}$ and $\overrightarrow{CB'}$ and so on. Now, note that $\bar{a} + \bar{b} + \bar{c} + \bar{d} = \bar{0}$ by construction. Therefore, $\overrightarrow{A'B'} = \overrightarrow{A'B} + \overrightarrow{BB'}$ equals $\frac{1}{2}((a_1 + b_1 - c_1 - d_1)\bar{p} + (a_2 + b_2 + c_2 + d_2)\bar{q}) = e_1\bar{p}$. Similarly, $\overrightarrow{B'C'} = f_2\bar{q}$, while $\overrightarrow{C'D'} = -\overrightarrow{A'B'}$ and $\overrightarrow{D'A'} = -\overrightarrow{B'C'}$. We conclude that $A'B'C'D'$ is a parallelogram which by construction is inscribed in $ABCD$ with its sides parallel to the asymptotes of $\mathcal{H}$. Finally, let $\bar{a} \times \bar{b}$ denote the oriented area of the parallelogram spanned by $\bar{a}$ and $\bar{b}$ and so on. Then, the oriented area of $ABCD$ equals $\frac{1}{2}(\bar{e} \times \bar{f}) = \frac{1}{2}(e_1\bar{p} + e_2\bar{q}) \times (f_1\bar{p} + f_2\bar{q}) = \frac{1}{2}(e_1f_2 - e_2f_1)(\bar{p} \times \bar{q})$. As $e_1f_2 = -e_2f_1$, the latter expression equals $e_1f_2(\bar{p} \times \bar{q}) = (e_1\bar{p}) \times (f_2\bar{q})$

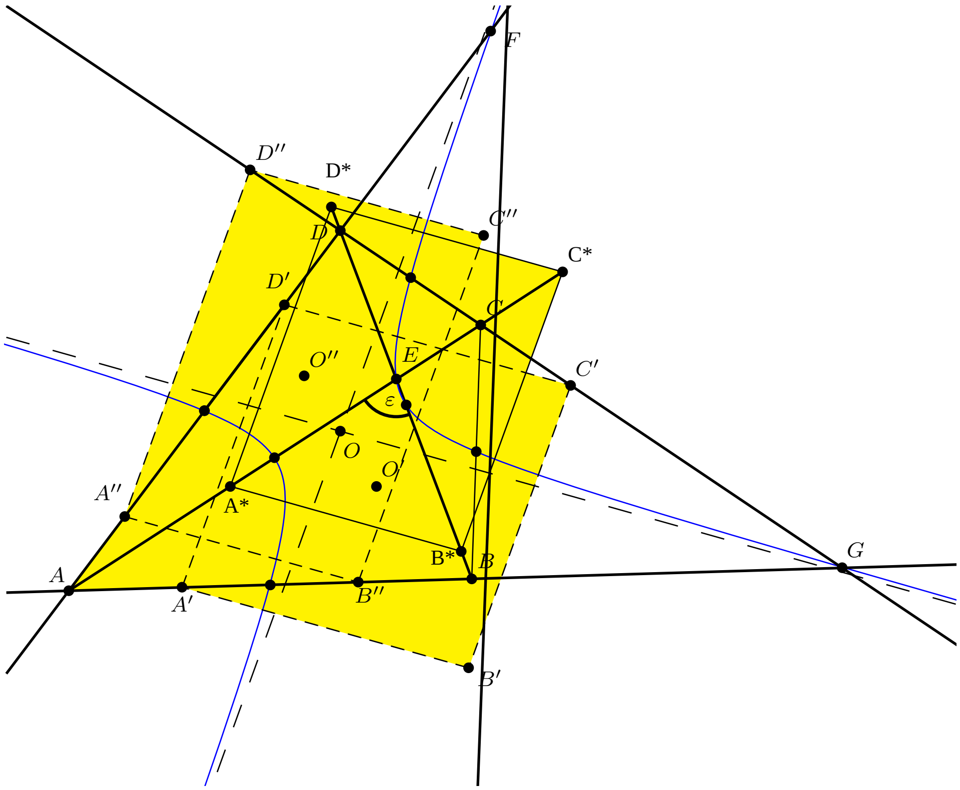

Figure 1. The two parallelograms inscribed in $ABCD$ with equal area to $ABCD$

or $\overrightarrow{A'B'} \times \overrightarrow{B'C'}$. This proves that the oriented area of $A'B'C'D'$ equals that of $ABCD$. The properties of $A''B''C''D''$ now follow similarly. $\square$

If $ABCD$ does have parallel sides, i.e. if $ABCD$ is a trapezoid, the proof above does not apply. In this case, however, a slightly modified version can be fairly easily found and $A'B'C'D'$ and $A''B''C''D''$ end up being parallelograms with a pair of opposite sides on the parallel sides of $ABCD$.

The two parallelograms $A'B'C'D'$ and $A''B''C''D''$ are connected in various ways. Most notably, the centers O' and O'' of the two parallelograms are reflections of one another in the center O of the nine-point conic $\mathcal{H}$ of $ABCD$. Also, the line through the midpoints of A' and A'' and of C' and C'' is parallel to $\bar{e}$ and passes through O. Likewise, the line through the midpoints of B' and B'' and of D' and D'' is parallel to $\bar{f}$ and passes through O as well. Finally, the points of intersection $A'D' \cap D''C''$ and $C'B' \cap B''A''$ both lie on BD, while the points of intersection $D''A'' \cap A'B'$ and $B''C'' \cap C'D'$ both lie on AC. For the purposes of this paper, however, there is no need to investigate these properties any further.

4. Finding angles and sides

Our next task is to find expressions for the angles between the sides of $A'B'C'D'$ and $A''B''C''D''$ as well as for their lengths.

Theorem 7. *For $ABCD$ and $A^*B^*C^*D^*$ as defined above, let ϵ denote the signed angle from $\bar{e}$ to $\bar{f}$. Furthermore, let ζ be the signed angle from $\bar{p}$ to $\bar{q}$. Then*

$$\frac{\sin(\epsilon)}{\tan(\zeta)} = \frac{(f_B f_D - e_A e_C)}{2\sqrt{e_A e_C f_B f_D}}.$$

Proof. We have $pq\sin(\zeta) = \bar{p} \times \bar{q}$, which equals

$$\left(\frac{\sqrt{e_A e_C}}{e}\bar{e} - \frac{\sqrt{f_B f_D}}{f}\bar{f}\right) \times \left(\frac{\sqrt{e_A e_C}}{e}\bar{e} + \frac{\sqrt{f_B f_D}}{f}\bar{f}\right) = \frac{2\sqrt{e_A e_C f_B f_D}}{ef}\sin(\epsilon).$$

Similarly $pq\cos(\zeta) = \frac{e_A e_C - f_B f_D}{ef}$. The desired formula now immediately follows from these two equalities. □

As for the lengths of the sides of $A'B'C'D'$ and $A''B''C''D''$, let the lengths of the sides $A'B'$ and $B'C'$ be denoted by p' and q', while the lengths of the sides $A''B''$, $B''C''$ are denoted by p'' and q''. We now have the following relations.

Theorem 8. *Let $ABCD$ be a convex quadrilateral, with p, p', p'' and q, q', q'' defined as above. Then*

$$p' = \frac{ep}{2\sqrt{e_A e_C}},\ q' = \frac{fq}{2\sqrt{f_B f_D}} \text{ and } p'' = \frac{fp}{2\sqrt{f_B f_D}},\ q'' = \frac{eq}{2\sqrt{e_A e_C}}.$$

Proof. Both $\overrightarrow{A^*B^*}$ and $\overrightarrow{A'B'}$ are oblique projections onto $\bar{q}$ in the direction of $\bar{q}$ of parallel vectors. Therefore, the first relation follows from similarity. The other three relations are derived similarly. □

Corollary 9. *Let $ABCD$ be a convex quadrilateral, with p, p', p'' and q, q', q'' defined as above. Then*

$$p' = \frac{e}{2\sqrt{e_A e_C}}\sqrt{e_A e_C + f_B f_D - 2\sqrt{e_A e_C f_B f_D}\cos(\epsilon)},$$

$$q' = \frac{f}{2\sqrt{f_B f_D}}\sqrt{e_A e_C + f_B f_D + 2\sqrt{e_A e_C f_B f_D}\cos(\epsilon)},$$

$$p'' = \frac{f}{2\sqrt{f_B f_D}}\sqrt{e_A e_C + f_B f_D - 2\sqrt{e_A e_C f_B f_D}\cos(\epsilon)},$$

$$q'' = \frac{e}{2\sqrt{e_A e_C}}\sqrt{e_A e_C + f_B f_D + 2\sqrt{e_A e_C f_B f_D}\cos(\epsilon)},$$

where, as before, ϵ is the signed angle between $\bar{e}$ and $\bar{f}$.

Proof. This is a straightforward application of the Law of Cosines and Theorem 8. □

5. The case of the cyclic quadrilateral

For the general quadrilateral, the expressions above probably cannot be simplified. For the cyclic case, however, we have the following result.

Theorem 10. *Let $ABCD$ be a (convex) cyclic quadrilateral with no parallel sides, with p', q', p'', q'' defined as above. Then $A'B'C'D'$ and $A''B''C''D''$ are rectangles and*

$$p' = \sqrt{\frac{e}{f}(s-b)(s-d)}, \qquad q' = \sqrt{\frac{f}{e}(s-a)(s-c)}$$

and

$$p'' = \sqrt{\frac{f}{e}(s-b)(s-d)}, \qquad q'' = \sqrt{\frac{e}{f}(s-a)(s-c)},$$

where $s = \frac{1}{2}(a+b+c+d)$.

Proof. If $ABCD$ can be inscribed in a circle, then obviously $e_A e_C = f_B f_D$. Therefore, $1/\tan\zeta = 0$, by Corollary 7. In other words, the sides of $A'B'C'D'$ and $A''B''C''D''$ are at right angles. It also follows from Corollary 9 that $p' = \frac{e}{2}\sqrt{1-\cos\epsilon}$, while $q' = \frac{e}{2}\sqrt{1+\cos\epsilon}$. Next, note that for every quadrilateral $ABCD$, $2ef\cos\epsilon = b^2+d^2-a^2-c^2$ (Bretschneider's Formula, see [1]), while for any (convex) cyclic quadrilateral $ef = ac+bd$ (Ptolemy's Theorem). Elimination of ef and $\cos\epsilon$ and some straightforward algebraic manipulation now gives the desired result. □

Corollary 11. *Let $ABCD$ be a cyclic quadrilateral with no parallel sides. Then, its area $ABCD$ is given by the formula*

$$ABCD = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

Proof. The statement immediately follows by combining Theorem 6 and Theorem 10. □

Again, the proof above does not apply to the only type of cyclic quadrilateral with parallel sides, i.e. the isosceles trapezoid. It is easily verified, however, that the statement of the theorem is true for this case as well. This concludes our derivation of the area formula for the cyclic quadrilateral as inspired by Gaṇeśa's flawed attempt to derive the same formula.

6. Conclusion

At this point, one might ask how all of this relates to Gaṇeśa's derivation. In light of the proofs of the results contained in this paper, it might seem highly unlikely that any 16th-century mathematical practitioner (regardless of the mathematical culture in which he was operating) would have been able to come up with a line of reasoning like ours. The answer is that Gaṇeśa did not either. It is true that essentially he gave the statement of Theorem 10 and used the argument of Corollary 11, implicitly assuming Theorem 6. But then, he only did so for the case of the cyclic quadrilaterals. Even for this more simple situation, however, Gaṇeśa's reasoning is hardly satisfactory. Thus, the construction of the points of the two parallelograms $A'B'C'D'$ and $A''B''C''D''$ is a lot easier, as the asymptotes of the nine-point conic for a cyclic quadrilateral $ABCD$ are parallel to the angle bisectors of AEB. Therefore A' simply is the point on AB such that AA' has length $(a+c)/2$ and so on. This is exactly how Gaṇeśa constructs one of the two inscribed parallelograms $A'B'C'D'$ and $A''B''C''D''$, to then compute the area of the cyclic quadrilateral from the area of the inscribed parallelogram (which only requires tools and properties that were reasonably well-known to the mathematical culture in which Gaṇeśa operated). Of course, he still would have had to prove that his inscribed parallelogram is a rectangle and that the area of this rectangle equals that of $ABCD$. As it is, there is no proof of either in his work. At best, we could say that Gaṇeśa had the right intuition, but perhaps not the tools to fully back up his claims.

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