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# About Two Characteristic Points Concerning Two Nested Circles and Their Use in Research of Bicentric Polygons

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**Abstract**. This paper is a companion to [8], which primarily deals with two characteristic points defined for two separated circles and their use in research of bicentric polygons with excircle. This paper primarily deals with two characteristic points defined for two nested circles and their use in research of bicentric polygons with incircle. Some useful properties and relations are established and some old and difficult problems are solved using these points.

## 1. The characteristic points of nested circles

We begin with the following definition.

**Definition 1.** Let $C_1$ and $C_2$ be two given circles such that $C_2$ is complete inside $C_1$. Let $R$, $r$, $d$ be lengths (positive numbers) such that $R =$ radius of $C_1$, $r =$ radius of $C_2$, $d = |OI|$, where $O$ is the center of $C_1$ and $I$ is the center of $C_2$. Let $xOy$ be a co-ordinate system with origin $O$, and positive $x$-axis containing $I$. The points $S_i(s_i, 0),\ i = 1, 2$ where

$$s_{1,2} = \frac{R^2 + d^2 - r^2 \mp \sqrt{(R^2 + d^2 - r^2)^2 - 4R^2d^2}}{2d}, \tag{1a}$$

will be called the *characteristic points* of the circles $C_1$ and $C_2$, or of the triple $(R, r, d)$.

It is easy to see that lengths $s_1$ and $s_2$ given by (1a) can be written as

$$s_1 = \frac{R^2 + d^2 - r^2 - t_M t_m}{2d} \quad \text{or} \quad s_1 = \frac{(t_M - t_m)^2}{4d}, \tag{1b}$$

where

$$t_m^2 = (R - d)^2 - r^2, \quad t_M^2 = (R + d)^2 - r^2. \tag{2}$$

See Figure 1a. As can be seen, $t_M$ is the length of the longest tangent that can be drawn from $C_1$ to $C_2$, and $t_m$ is the length of the shortest. These lengths will be often used in the following.

In this connection see also Figure 1b. Later it will be shown that the characteristic point $S_1(s_1, 0)$ is the intersection of the chords $T_1\hat{T}_1$ and $T_2\hat{T}_2$ of $C_1$. From this it will be clear that $s_1 > d$ if $d \neq 0$, but $s_1 = 0$ if $I = O$. Also it will be shown that the point $S_2(s_2, 0)$ is the intersection of the line through the points $T_1$ and $T_2$ with the $x$-axis.

First we prove the following theorem.

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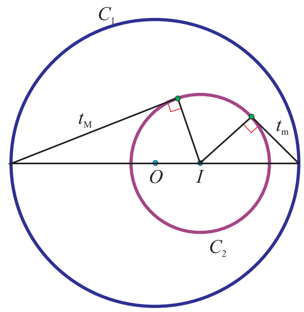


Figure 1a

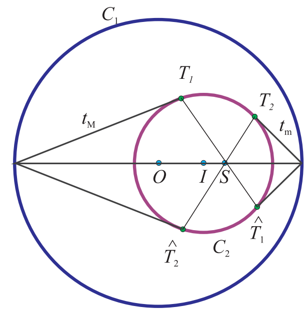


Figure 1b

**Theorem 1.** *Let $C_1$, $C_2$ and $R$, $r$, $d$ be as in Definition 1. Let $PQ$ be any given chord of the circle $C_1$ containing the point $S_1(s_1, 0)$, and $PT_1$ and $QT_2$ the tangents from $P$ and $Q$ to $C_2$ and let*

$$t_1 = |PT_1|, \qquad \hat{t}_1 = |QT_2|. \tag{3}$$

*See Figure 2. Finally, let the coordinates of $P$, $Q$, $T_1$ and $T_2$ with reference to $xOy$ be given by*

$$P(u_1, v_1), Q(u_2, v_2), T_1(x_1, y_1), T_2(x_2, y_2). \tag{4}$$

*Then*

$$t_1 \hat{t}_1 = t_m t_M, \tag{5}$$

*that is,*

$$\left((u_1 - x_1)^2 + (v_1 - y_1)^2\right)\left((u_2 - x_2)^2 + (v_2 - y_2)^2\right) - t_m^2 t_M^2 = 0. \tag{6}$$

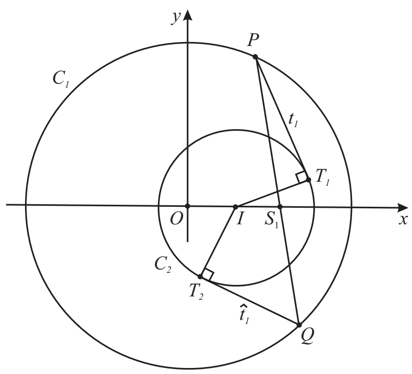


Figure 2

*Proof.* First it is clear that, if $v_1 > 0$, then $v_2 < 0$ and

$$v_1 = \sqrt{R^2 - u_1^2}, \qquad v_2 = -\sqrt{R^2 - u_2^2}. \tag{7}$$

The equation of the straight line through points $P(u_1, v_1)$ and $S_1(s_1, 0)$ is given by

$$y = \frac{v_1}{u_1 - s_1}(x - s_1). \tag{8}$$

It can be easily found that

$$\begin{aligned} u_2 &= \frac{sv_1^2 + \sqrt{s_1^2 v_1^4 - ((u_1 - s_1)^2 + v_1^2)(s_1^2 v_1^2 - R^2(u_1 - s_1)^2)}}{(u_1 - s_1)^2 + v_1^2}, \\ v_2 &= \frac{v_1}{u_1 - s_1}(u_2 - s_1). \end{aligned} \tag{9}$$

One solution of the system

$$(x - d)^2 + y^2 = r^2, \quad (u_1 - d)(x - d) + v_1 y = r^2 \tag{10}$$

is given by

$$\begin{aligned} x_1 &= d + \frac{r^2(u_1 - d) + \sqrt{r^4(u_1 - d)^2 - r^2(r^2 - v_1^2)(v_1^2 + (u_1 - d)^2)}}{(u_1 - d)^2 + v_1^2}, \\ y_1 &= \frac{r^2 - (u_1 - d)(x_1 - d)}{v_1}. \end{aligned} \tag{11}$$

In the same way, it can be found that a solution of the system

$$(x - d)^2 + y^2 = r^2, \quad (u_2 - d)(x - d) + v_2 y = r^2 \tag{12}$$

is given by

$$\begin{aligned} x_2 &= d + \frac{r^2(u_2 - d) + \sqrt{r^4(u_2 - d)^2 - r^2(r^2 - v_2^2)(v_2^2 + (u_2 - d)^2)}}{(u_2 - d)^2 + v_2^2}, \\ y_2 &= \frac{r^2 - (u_2 - d)(x_2 - d)}{v_2}. \end{aligned} \tag{13}$$

Starting from relation (6), using relations (7), (9), (11), (13) and with the help of a computer algebra system, we get, after rationalization and factorization, the following relation

$$\begin{aligned} &- 4d(R - s_1)^2(R + s_1)^2 \left(-dR^2 + d^2 s_1 - r^2 s_1 + R^2 s_1 - ds_1^2\right) \\ &\quad (R - u_1)^3(s_1 - u_1)^4(R + u_1)^3(d^2 + R^2 - 2du_1)^4(R^2 + s_1^2 - 2s_1 u_1)^7 \\ &\qquad\qquad\qquad \left(-dR^2 + d^2 u_1 - r^2 u_1 + R^2 u_1 - du_1^2\right) = 0. \end{aligned}$$

It can be easily seen that above relation is valid for every $u_1$ if the fourth factor $(-dR^2 + d^2 s_1 - r^2 s_1 + R^2 s_1 - ds_1^2)$ is equal to zero, that is, if $s_1$ is given by (1). This proves Theorem 1. □

This theorem will be proved later in an other way which may be interesting in itself. See Theorem 13 below.

**Example 1.** Let $R = 8$, $r = 3$, $d = 2$. Then

$$t_m = 5.196152423\ldots, \quad t_M = 9.539392014\ldots, \quad s_1 = 2.357966268\ldots.$$

If $u_1 = -2.5$, then $v_1 = 7.599342077\ldots$,
$u_2 = 5.847826086\ldots$, $v_2 = -5.459205922\ldots$,
$x_1 = 3.908649086\ldots$, $y_1 = 2.31453262\ldots$,
$x_2 = 0.585482934\ldots$, $y_2 = -2.64558906\ldots$,
$t_1 = 8.306623863\ldots$, $\hat{t}_1 = 5.967302209\ldots$.

$$t_1\hat{t}_1 = t_m t_M = 49.56813493\ldots.$$

**Theorem 2** (Converse of Theorem 1)**.** *Let $R$, $r$, $d$ be as in Theorem 1 and let $PQ$ be any given chord of $C_1$ such that*

$$|PT_1| \cdot |QT_2| = t_m t_M, \tag{14}$$

*where $PT_1$ is tangent of $C_2$ drawn from $P$ and $QT_2$ is tangent of $C_2$ drawn from $Q$. Then the chord $PQ$ contains the point $S_1(s_1, 0)$.*

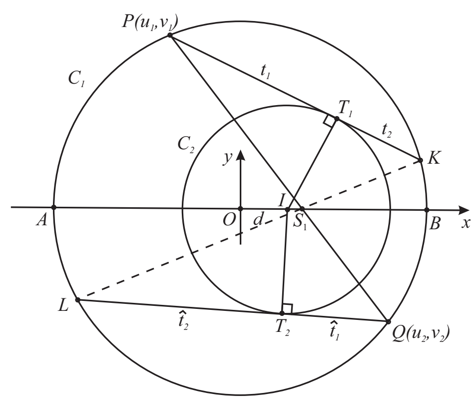


Figure 3

*Proof.* Since $|PT_1|^2 + |T_1I|^2 = |PI|^2$, we have

$$(u_1 - d)^2 + v_1^2 - r^2 = t_1^2$$

from which follows

$$u_1 = \frac{R^2 + d^2 - r^2 - t_1^2}{2d}. \tag{15}$$

In the same way it can be seen that

$$u_2 = \frac{R^2 + d^2 - r^2 - \hat{t}_1^2}{2d}. \tag{16}$$

Since

$$v_1 = \sqrt{R^2 - u_1^2}, \quad v_2 = -\sqrt{R^2 - u_2^2}, \tag{17}$$

the equation of the straight line through $P$ and $Q$ can be written as

$$y - v_1 = \frac{v_1 - v_2}{u_1 - u_2}(x - u_1),$$

where $u_1$ and $u_2$ are given by (15) and (16). Putting $y = 0$ we get the following equation in $x$

$$-v_1 = \frac{v_1 - v_2}{u_1 - u_2}(x - u_1). \tag{18}$$

In this equation we put $\hat{t}_1 = \frac{t_m t_M}{t_1}$ from (5). After rationalization and factorization we get

$$\begin{aligned}(d^2 - r^2 - 2dR + R^2 - t_1^2)(d^2 - r^2 + 2dR + R^2 - t_1^2)&\\ (d^4 - 2d^2r^2 + r^4 - 2d^2R^2 - 2R^2r^2 + R^4 - t_1^4)&\\ (-dR^2 + d^2x - r^2x + R^2x - dx^2) = 0.&\end{aligned} \tag{19}$$

Only the fourth factor gives the point of intersection with $x$-axis and we have

$$x = \frac{R^2 + d^2 - r^2 - \sqrt{(R^2 + d^2 - r^2)^2 - 4R^2d^2}}{2d}. \tag{20}$$

First it can be seen that

$$(R^2 + d^2 - r^2)^2 - 4R^2d^2 = t_m^2 t_M^2. \tag{21}$$

Concerning the first three factors in (19) it is easily seen that these, respectively, can be written as

$$t_m^2 - t_1^2, \quad t_M^2 - t_1^2, \quad t_m^2 t_M^2 - t_1^4.$$

The first and the second expressions occur when $PQ = AB$ (see Figure 3); the third when $t_1 = \hat{t}_1$. Thus, the point of intersection with the $x$-axis in the general case is given by (20), with $x$ replaced by $s_{1,2}$. This proves Theorem 2. □

**Theorem 3.** *Let $T_1$ and $T_2$ be as in Theorem 1* (see Figure 2)*. The point $S_1$ lies on the segment $T_1T_2$.*

*Proof.* It is easy to show that the equation of the line through points $T_1(x_1, y_1)$ and $T_2(x_2, y_2)$ is satisfied by $S_1(s_1, 0)$, that is $-y_1 = \frac{y_1 - y_2}{x_1 - x_2}(s_1 - x_1)$. □

**Theorem 4.** *Let $t_2$ and $\hat{t}_2$ be as in Figure 3. Then $t_2\hat{t}_2 = t_1\hat{t}_1$, that is,*

$$t_2\hat{t}_2 = t_m t_M.$$

*Proof.* The proof is in the same way as the proof of Theorem 1. □

## 2. Characteristic points associated with bicentric polygons

One corollary of this theorem, which will be stated (see [2,3]), refers to bicentric polygons. Before stating it let us mention that a polygon which is both chordal and tangential is simply called a *bicentric* polygon. The following question can be raised: If $C_1$ and $C_2$ are circles such that $C_2$ is completely inside $C_1$, is there an $n$-sided polygon inscribed in $C_1$ and circumscribed around $C_2$? The first who considered this problem for $n = 4$ was the Swiss mathematician Nicolaus Fuss (1755 – 1826). See [2]. He found that for $n = 4$ the following condition must be fulfilled:

$$(R^2 - d^2)^2 - 2r^2(R^2 + d^2) = 0, \tag{22}$$

where $R =$ radius of $C_1$, $r =$ radius of $C_2$, and $d =$ distance between centers of $C_1$ and $C_2$.

Fuss also found conditions for $n = 5, 6, 7, 8$ (see [3]). Subsequently, such conditions are also found for many integers $n > 8$. In honor of Fuss all such conditions are called Fuss' relations.

It seems that many problems concerning bicentric polygons can be proved using properties of the characteristic points in Theorems 1, 2.

In establishing Fuss' relations, Poncelet's celebrated closure theorem [4] plays an important role.

**Theorem** (Poncelet's closure theorem)**.** Let $C$ and $D$ be two nested conics such that there is an $n$-sided polygon inscribed in $D$ and circumscribed around $C$. Then for every point $x \in D$, there is an $n$-sided polygon with $x$ as a vertex, inscribed in $D$ and circumscribed around $C$.

*Remark.* (1) In the following we shall mostly deal with two circles $C_1$ and $C_2$, where $C_2$ is completely inside $C_1$. For brevity in the expression in this dealing we shall say that $C_1$ and $C_2$ are determined by triple $(R, r, d)$ if and only if $(R, r, d) \in \mathbb{R}_+^3$ and

$$R > d + r, \tag{23}$$

where $R =$ radius of $C_1$, $r =$ radius of $C_2$, $d =$ distance between centers of $C_1$ and $C_2$.

In the following it will be shown that relations concerning the characteristic points of these circles are closely connected with bicentric polygons.

Definition A below is a slight modification of Definition 1 in [7].

*Definition A.* Let $S$ be a set given by

$$S = \left\{(R, r, d) : (R, r, d) \in R_+^3 \text{ and } R > r + d\right\}.$$

For a given $(R_0, r_0, d_0) \in S$, we have

$$f_1(R_0, r_0, d_0) = (R_1, r_1, d_1),$$
$$f_2(R_0, r_0, d_0) = (R_2, r_2, d_2),$$

where $R_1$, $r_1$, $d_1$ and $R_2$, $r_2$, $d_2$ are given by

$$R_1^2 = R_0\left(R_0 + r_0 + \sqrt{(R_0 + r_0)^2 - d_0^2}\right), \tag{24a}$$

$$d_1^2 = R_0\left(R_0 + r_0 - \sqrt{(R_0 + r_0)^2 - d_0^2}\right), \tag{24b}$$

$$r_1^2 = (R_0 + r_0)^2 - d_0^2, \tag{24c}$$

$$R_2^2 = R_0\left(R_0 - r_0 + \sqrt{(R_0 - r_0)^2 - d_0^2}\right), \tag{25a}$$

$$d_2^2 = R_0\left(R_0 - r_0 - \sqrt{(R_0 - r_0)^2 - d_0^2}\right), \tag{25b}$$

$$r_2^2 = (R_0 - r_0)^2 - d_0^2, \tag{25c}$$

It can be proved that

$$R_1 > r_1 + d_1, \quad R_2 > r_2 + d_2, \tag{26a}$$

$$R_1 d_1 = R_2 d_2 = R_0 d_0, \tag{26b}$$

$$R_1^2 + d_1^2 - r_1^2 = R_2^2 + d_2^2 - r_2^2 = R_0^2 + d_0^2 - r_0^2. \tag{26c}$$

$$r_1 r_2 = t_M t_m, \tag{27a}$$

where

$$t_M^2 = (R_0 + d_0)^2 - r_0^2, \quad t_m^2 = (R_0 - d_0)^2 - r_0^2. \tag{27b}$$

Also,

$$(R_1 + d_1)^2 - r_1^2 = t_M^2, \quad (R_1 - d_1)^2 - r_1^2 = t_m^2. \tag{28a}$$

$$\frac{R_1^2 - d_1^2}{2r_1} = \frac{R_2^2 - d_2^2}{2r_2} = R_0, \quad \frac{2R_1 d_1 r_1}{R_1^2 - d_1^2} = \frac{2R_2 d_2 r_2}{R_2^2 - d_2^2} = d_0, \tag{28b}$$

$$\begin{aligned} &-\left(R_1^2 + d_1^2 - r_1^2\right) + \left(\frac{R_1^2 - d_1^2}{2r_1}\right)^2 + \left(\frac{2R_1 d_1 r_1}{R_1^2 - d_1^2}\right)^2 \\ &= -\left(R_2^2 + d_2^2 - r_2^2\right) + \left(\frac{R_2^2 - d_2^2}{2r_2}\right)^2 + \left(\frac{2R_2 d_2 r_2}{R_2^2 - d_2^2}\right)^2 = r_0^2. \end{aligned} \tag{28c}$$

More about this and the functions $f_1$ and $f_2$ can be seen in [7, Theorem 1].

**Theorem 5.** *Let $R_0$, $r_0$, $d_0$ and $R_i$, $r_i$, $d_i$, $i = 1, 2$, be as in Definition A. Then*

$$d_i s_i = d_0 s_0, \quad i = 1, 2, \tag{29}$$

*where*

$$s_0 = \frac{(t_M - t_m)^2}{4d_0}, \tag{30a}$$

$$s_i = \frac{(t_M - t_m)^2}{4d_i}. \tag{30b}$$

*Proof.* From (28a) and (30) it follows $4d_0 s_0 = 4d_i s_i,\ i = 1, 2.$ □

**Theorem 6.** *Let $K_1$ and $K_2$ be circles determined by triple $(R_1, d_1, r_1)$, where $R_1$, $r_1$, $d_1$ are given by* (24)*. Then characteristic point of the triple $(cR_1, cd_1, cr_1)$, where $c = \frac{R_0}{R_1}$, is the same as that of the triple $(R_0, d_0, r_0)$, that is,*

$$cs_1 = s_0, \tag{31}$$

*where $s_0$ and $s_1$ are given by* (30)*.*

*Proof.* We can write

$$cs_1 = c \cdot \frac{(t_M - t_m)^2}{4d_1} = \frac{(t_M - t_m)^2}{4\frac{1}{c}d_1} = \frac{(t_M - t_m)^2}{4d_0} = s_0,$$

since, by (26b),

$$\frac{1}{c}d_1 = \frac{R_1}{R_0}d_1 = \frac{R_0 d_0}{R_0} = d_0.$$

□

Relations (25) hold analogously if the triple $(R_1, d_1, r_1)$ is replaced by $(R_2, d_2, r_2)$. In this case we have $c = \frac{R_0}{R_2}$, and $cs_2 = s_0$.

Some important properties concerning bicentric $n$-gons will be now established. Some of these are extension and completion of theorems proved earlier (see [7, 8]).

**Theorem 7.** *Let $n \geq 4$ be an even integer. Let $(R_1, r_1, d_1) \in R_+^3$ be any given solution of Fuss' relation $F_n(R, r, d) = 0$. Let $C_1$ and $C_2$ be circles (in the same plane) such that*
*$R_1$ = radius of $C_1$,*
*$r_1$ = radius of $C_2$,*
*$d_1$ = distance between points $O$ and $I$, where $O$ is the center of $C_1$ and $I$ is the center of $C_2$.*

*Further, let $xOy$ denotes a coordinate system with origin $O$ and positive $x$-axis containing $I$. Finally, let $P(u, v)$ be any given point of $C_1$. Then there is a unique point $Q(\hat{u}, \hat{v})$ of $C_1$ such that*

$$\hat{u} = \frac{-2R_1^2 d_1 + \left(R_1^2 + d_1^2 - r_1^2\right) u}{2d_1 u - \left(R_1^2 + d_1^2 - r_1^2\right)}, \tag{32a}$$

$$\hat{v} = \sqrt{R_1^2 - \hat{u}^2} \text{ or } \quad -\sqrt{R_1^2 - \hat{u}^2}, \tag{32b}$$

*and the line determined by points $PQ$ contains the characteristic point $S_1(s_1, 0)$ of the triple $(R_1, r_1, d_1)$.*

*Proof.* From the equation of the line through $P(u, v)$ and $Q(\hat{u}, \hat{v})$, that is,

$$y - \hat{v} = \frac{v - \hat{v}}{u - \hat{u}}(x - \hat{u}), \tag{33}$$

putting $y = 0$, we obtain

$$\frac{\hat{u} - x}{u - x} = \frac{\hat{v}}{v}.$$

We have to prove that this has solution $x = s_1$ if and only if $\hat{u}$ and $\hat{v}$ are given by (32) and $s_1$ is given by

$$s_1 = \frac{(t_M - t_m)^2}{4d_1} = \frac{\left(\sqrt{(R_1 + d_1)^2 - r_1^2} - \sqrt{(R_1 - d_1)^2 - r_1^2}\right)^2}{4d_1}. \tag{34}$$

See also Figure 4.

Using computer algebra it can be easily shown that the relation

$$\frac{\hat{u}-s_1}{u-s_1}=\frac{\hat{v}}{v}$$

is satisfied if $\hat{u}$ and $\hat{v}$ are given by (32) and $s_1$ is given by (34). □

We remark that instead of the equation (33), the relation (34) can also be established by using the equation of the line through $P(u,v)$ and $S_1(s_1,0)$, that is

$$y=\frac{v}{u-s_1}(x-s_1), \tag{35}$$

and replacing $x$ and $y$ by $\hat{u}$ and $\hat{v}$ given by (32).

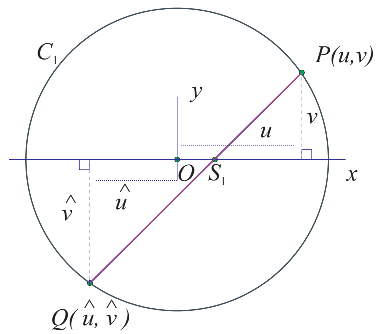


Figure 4: $(-\hat{u}+s_1):(u-s_1)=-\hat{v}:v$.

**Corollary 8.** *The relation* (32a) *is equivalent to*

$$u=\frac{-2R_1^2d_1+\left(R_1^2+d_1^2-r_1^2\right)\hat{u}}{2d_1\hat{u}-\left(R_1^2+d_1^2-r_1^2\right)}. \tag{36}$$

The proof is straightforward.

**Theorem 9.** *Let $u$ and $\hat{u}$ be as in Theorem 7. Then*

$$t\hat{t}=t_Mt_m, \tag{37}$$

*where*

$$t^2=R_1^2+d_1^2-r_1^2-2d_1u, \qquad \hat{t}^2=R_1^2+d_1^2-r_1^2-2d_1\hat{u}, \tag{38}$$

$$t_M^2=(R_1+d_1)^2-r_1^2, \qquad t_m^2=(R_1-d_1)^2-r_1^2. \tag{39}$$

*Proof.* Replacing $u$ in the relation (32a) by

$$\frac{R_1^2+d_1^2-r_1^2-t^2}{2d_1}$$

obtained from $t^2$ given by (38) we easily get the relation

$$\left(R_1^2+d_1^2-r_1^2-2d_1\hat{u}\right)t^2=t_M^2t_m^2$$

or

$$\hat{t}^2t^2=t_M^2t_m^2.$$

This theorem can be also proved in the following way.

If in the relation $(t\hat{t})^2 = \left(R_1^2 + d_1^2 - r_1^2 - 2d_1u\right)\left(R_1^2 + d_1^2 - r_1^2 - 2d_1\hat{u}\right)$ we replace $(t\hat{t})^2$ by $(t_Mt_m)^2 = \left(R_1^2 + d_1^2 - r_1^2\right)^2 - 4d_1^2R_1^2$, then we get

$$-4d_1^2R_1^2 = -2d_1\hat{u}\left(R_1^2 + d_1^2 - r_1^2\right) - 2d_1u\left(R_1^2 + d_1^2 - r_1^2\right) + 4d_1^2u\hat{u},$$

which is equivalent to the relation (32a). □

*Remark.* (2) As can be seen, proving Theorem 9 we in fact prove Theorem 1 in an another way which may be interesting in itself.

**Theorem 10.** *Let $P(u,v)$, $Q(\hat{u},\hat{v})$ and $S_1(s_1,0)$ be as in Theorem 7. Then*

$$|PS|\,|QS| = R_1^2 - s_1^2. \tag{40}$$

*Proof.* First let us remark that $R_1 > s_1$ since

$$s_1 = \frac{(t_M - t_m)^2}{4d_1} = \frac{t_M^2 - 2t_Mt_m + t_m^2}{4d_1} = \frac{R_1^2 + d_1^2 - r_1^2 - t_Mt_m}{2d_1},$$

$$\begin{aligned} &2R_1d_1 > R_1^2 + d_1^2 - r_1^2 - 2t_Mt_m \\ &\Longrightarrow 0 > (R_1 - d_1)^2 - r_1^2 - t_Mt_m \text{ or } 0 > t_m^2 - t_mt_M. \end{aligned} \tag{41}$$

Now,

$$\begin{aligned} |PS|^2 \cdot |QS|^2 &= \left((u - s_1)^2 + v^2\right)\left((\hat{u} - s_1)^2 + \hat{v}^2\right) \\ &= \left(R_1^2 - 2us_1 + s_1^2\right)\left(R_1^2 - 2\hat{u}s_1 + s_1^2\right). \end{aligned}$$

This is equal to $(R_1^2 - s_1^2)^2$ if and only if

$$-2R_1^2\hat{u}s_1 - 2R_1^2us_1 + 4u\hat{u}s_1^2 - 2us_1^3 - 2\hat{u}s_1^3 = 4R_1^2s_1^2. \tag{42}$$

This can be rewritten as

$$\hat{u}\left(R_1^2s + s_1^3 - 2s_1^2u\right) = 2R_1^2s_1^2 - \left(R_1^2s_1 + s_1^3\right)u.$$

From this,

$$\hat{u} = \frac{2R_1^2s_1 - (R_1^2 + s_1^2)u}{-2s_1u + R_1^2 + s_1^2}. \tag{43}$$

Replacing $s_1$ by $\dfrac{R_1^2 + d_1^2 - r_1^2 - t_Mt_m}{2d_1}$ (see (1b)) it is easy to find that the above relation can be written as

$$\hat{u} = \frac{2R_1^2d_1 - \left(R_1^2 + d_1^2 - r_1^2\right)u}{-2d_1u + \left(R_1^2 + d_1^2 - r_1^2\right)}.$$

Thus, the relation (40) is valid if $\hat{u}$ is given by (32a). □

**Theorem 11.** *Let $P(u,v)$, $Q(\hat{u},\hat{v})$ and $S_1(s_1,0)$ be as in Theorem 7. Then*

$$\frac{|PQ|}{t + \hat{t}} = \frac{2R_1}{\sqrt{(R_1 + d_1)^2 - r_1^2} + \sqrt{(R_1 - d_1)^2 - r_1^2}}, \tag{44}$$

*where $t$ and $\hat{t}$ are given by* (38).

*Proof.* We have to prove that

$$\frac{(u-\hat{u})^2+(v-\hat{v})^2}{(t+\hat{t})^2}=\left(\frac{2R_1}{t_M+t_m}\right)^2, \tag{45}$$

where $t_M$ and $t_m$ are given by (39). The proof goes in the same way as the proofs of the previous two theorems. Of course, in this theorem there is some more calculations since there are some terms which need to be rationalized. If obtained relation after rationalization is denoted by $f(u,\hat{u})$ then $f(u,\hat{u})=0$ for $\hat{u}$ given by (32a). This proves Theorem 11. □

*Remark.* (3) In [6, Theorem1] it is proved that for $n=4$ it holds

$$\frac{|PQ|}{t+\hat{t}}=\sqrt{\frac{2R_1^2}{R_1^2+d_1^2}}. \tag{46}$$

In the following theorem some results in [5, pp. 52–53] will be used. It was proved that for the lengths of tangents to a bicentric polygons,

$$(t_2)_{1,2}=\frac{(R^2-d^2)t_1\pm r\sqrt{(t_M^2-t_1^2)(t_1^2-t_m^2)}}{r^2+t_1^2}. \tag{47}$$

If $t_1$ is given tangent length, then one of $(t_2)_{1,2}$ is consecutive and other is proceeding.

**Theorem 12.** *Let $A_1\ldots A_n$ be any given bicentric $n$-gon whose circumcircle is $C_1$ and incircle $C_2$ as it is described in Theorem 7. Let $xOy$ be a coordinate system as in Figure 3 and let the vertices $A_1,\ldots,A_n$ be given by $A_i(u_i,v_i),\ i=1,\ldots,n$. Finally, let $t_1,\ldots,t_n$ be tangent lengths from the vertices $A_i(u_i,v_i)$ of the $n$-gon $A_1\ldots A_n$, that is,*

$$t_i^2=R_1^2+d_1^2-r_1^2-2d_1u_i,\quad i=1,\ldots,n. \tag{48}$$

*If $t_1^2=R_1^2+d_1^2-r_1^2-2d_1u_1$ is given, then the consequent of $u_1$ is $(u_2)_1$ or $(u_2)_2$ given by*

$$\begin{aligned}(u_2)_1=\frac{1}{\left(d_1^2+R_1^2-2d_1u_1\right)^2}&\left(-d_1^4u_1+2r_1^2R_1^2u_1-R_1^4u_1+2d_1^2\left(r_1^2-3R_1^2\right)u_1\right.\\&-2\sqrt{r_1^2\left(R_1^2-d_1^2\right)^2\left(d_1^2-r_1^2+R_1^2-2d_1u_1\right)\left(R_1^2-u_1^2\right)}\\&\left.+2d_1^3\left(R_1^2+u_1^2\right)+2d_1R_1^2\left(R_1^2+u_1^2-2r_1^2\right)\right)\end{aligned} \tag{49a}$$

$$\begin{aligned}(u_2)_2=\frac{1}{\left(d_1^2+R_1^2-2d_1u_1\right)^2}&\left(-d_1^4u_1+2r_1^2R_1^2u_1-R_1^4u_1+2d_1^2\left(r_1^2-3R_1^2\right)u_1\right.\\&+2\sqrt{r_1^2\left(R_1^2-d_1^2\right)^2\left(d_1^2-r_1^2+R_1^2-2d_1u_1\right)\left(R_1^2-u_1^2\right)}\\&\left.+2d_1^3\left(R_1^2+u_1^2\right)+2d_1R_1^2\left(R_1^2+u_1^2-2r_1^2\right)\right).\end{aligned} \tag{49b}$$

*Proof.* From relation (47) (rewriting $t_2$ instead of $(t_2)_2$) it follows that

$$\left(t_2^2 + \frac{(R_1^2 - d_1^2)^2 t_1^2}{(R_1^2 + d_1^2 - r_1^2 - 2d_1u_1)^2} - \frac{4d_1^2 r_1^2 (R_1^2 - u_1^2)}{(R_1^2 + d_1^2 - r_1^2 - 2d_1u_1)^2}\right)^2 = \left(\frac{2(R_1^2 - d_1^2)t_1t_2}{R_1^2 + d_1^2 - 2d_1u_1}\right)^2, \tag{50}$$

where

$$t_1 = \sqrt{R_1^2 + d_1^2 - r_1^2 - 2d_1u_1}, \quad t_2 = \sqrt{R_1^2 + d_1^2 - r_1^2 - 2d_1u_2}. \tag{51}$$

Replacing $t_1$ and $t_2$ in the relation (50) by the right sides of the above two relations, we get

$$au_2^2 + bu_2 + c = 0, \tag{52a}$$

where

$$\begin{aligned} a &= -4d_1^2r_1^2R_1^2 + 4r_1^4R_1^2 + 4d_1^2R_1^4 - 4r_1^2R_1^4 - 4d_1^3R_1^2u_1 \\ &\quad + 8d_1r_1^2R_1^2u_1 - 4d_1R_1^4u_1 + d_1^4u_1^2 + 2d_1^2R_1^2u_1^2, \\ b &= 2\left(d_1^2 - 2r_1^2 + R_1^2 - 2d_1u_1\right)\left(-2d_1R_1^2 + d_1^2u_1 + R_1^2u_1\right), \\ c &= d_1^2 + R_1^2 - 2d_1u_1. \end{aligned} \tag{52b}$$

Using computer algebra it can be easily found that the solution of the equation given by (52) are given by (49). □

Here is an example.

**Example 2.** Let $n = 6$ and $(R_1, r_1, d_1)$ be a solution of Fuss' relation $F_6(R, r, d) = 0$ such that

$$R_1 = 8.340410321\ldots, \ r_1 = 6.812488532\ldots, \ d_1 = 1.198981793\ldots.$$

(For brevity in the following the points (sign) ... after calculated values will be omitted.)

The values $t_M$ and $t_m$ are given by

$$\begin{aligned} t_M &= \sqrt{(R_1 + d_1)^2 - r_1^2} = 6.7677574552, \\ t_m &= \sqrt{(R_1 - d_1)^2 - r_1^2} = 2.14242886. \end{aligned}$$

Let $t_1$ be a length such that $t_M \geq t_1 \geq t_m$, say $t_1 = 4$. Then, as can be easily concluded, there is a bicentric hexagon $A_1 \ldots A_6$ such that its first tangent is $t_1 = 4$. The other tangent lengths of this hexagon can be calculated using formula (47) and find that

$$\begin{aligned} &t_2 = 2.3947586766, \qquad t_3 = 2.2572852505, \qquad t_4 = 3.5765564793, \\ &t_5 = 5.973973936, \qquad t_6 = 6.3378015311. \end{aligned}$$

These tangent lengths can also be calculated using $u_1$ given by

$$u_1 = \frac{R_1^2 + d_1^2 - r_1^2 - t_1^2}{2d_1} = 3.58220619 \tag{53}$$

and formulas given by (49). For example, taking $u_1$ given by (53) and using relation (49a) we get

$$(u_2)_1 = 7.862976314, \quad (u_2)_2 = 6.49623254.$$

It can be verified that

$$(u_2)_1 = \frac{R_1^2 + d_1^2 - r_1^2 - t_2^2}{2d_1}, \quad (u_2)_2 = \frac{R_1^2 + d_1^2 - r_1^2 - t_6^2}{2d_1}.$$

Thus,

$$(t_2)^2 = \frac{R_1^2 + d_1^2 - r_1^2 - 2d_1(u_2)_1}{2d_1}, \quad (t_6)^2 = \frac{R_1^2 + d_1^2 - r_1^2 - 2d_1(u_2)_2}{2d_1}.$$

In the same way we can proceed and get $t_3$, $t_4$, $t_5$.

Here, let us remark that the relations (49) may be very useful in some investigations concerning bicentric polygons.

**Theorem 13.** *Let $A_1 \dots A_n$ be $n$-gon as in Theorem 12 with vertices $A_i = A_i(u_i, v_i)$, $i = 1, \dots, n$. Let $t_1, \dots, t_n$ be the tangent lengths of the $n$-gon from the vertices $A_i = A_i(u_i, v_i)$, $i = 1, \dots, n$, that is*

$$t_i^2 = R_1^2 + d_1^2 - r_1^2 - 2d_1 u_i, \quad i = 1, \dots, n. \tag{54}$$

*Let $n \geq 4$ be an even integer. Then*

$$u_{i+\frac{n}{2}} = \frac{2R_1^2 d_1 - \left(R_1^2 + d_1^2 - r_1^2\right) u_i}{-2d_1 u_i + \left(R_1^2 + d_1^2 - r_1^2\right)}, \quad i = 1, \dots, \frac{n}{2}. \tag{55}$$

*In other words, the chords $A_i A_{i+\frac{n}{2}}$, $i = 1, \dots, \frac{n}{2}$, of the circle $C_1$ contain the points $S_1(s_1, 0)$ such that the points $A_i(u_i, v_i)$ and $A_{i+\frac{n}{2}}(u_{i+\frac{n}{2}}, v_{i+\frac{n}{2}})$, $i = 1, \dots, \frac{n}{2}$, have the properties as the points $P(u, v)$ and $Q(\hat{u}, \hat{v})$ in the previous theorems, that is*

$$t_i t_{i+\frac{n}{2}} = t_M t_m, \quad i = 1, \dots, \frac{n}{2}, \tag{56}$$

*where $t_M$ and $t_m$ are given by* (39).

*Proof.* First let us remark that the notation used in Theorems 7 and 9 will be used. So, if $u$ and $\hat{u}$ are as in Theorem 7 and $t$ is given by $t^2 = R_1^2 + d_1^2 - r_1^2 - 2d_1 u$, then $\hat{t}^2 = R_1^2 + d_1^2 - r_1^2 - 2d_1 \hat{u}$.

In the first way we prove that $\hat{t}_1, \hat{t}_2$ are consequent if $t_1$ and $t_2$ are consequent. In other words, we prove that

$$\hat{t}_1 + \hat{t}_2 = \sqrt{(\hat{u}_1 - \hat{u}_2)^2 + (\hat{v}_1 - \hat{v}_2)^2}, \tag{57}$$

where $t_1$ and $t_2$ are consecutive tangent lengths of the $n$-gon $A_1 \dots A_n$, that is, the relation (47) is valid and can be written as

$$t_2 = \frac{t_1(R^2 - d^2) - k}{r^2 + t_1^2}, \tag{58}$$

where

$$k=\sqrt{t_1^2\left(R^2-d^2\right)^2+\left(r^2+t_1^2\right)\left(4R^2d^2-r^2t_1^2-\left(R^2+d^2-r^2\right)^2\right)}.$$

Also let us remark that

$$\hat{v}_i^2=R_1^2-\hat{u}_i^2,\quad i=1,2.$$

It can be easily shown (even by hand, without using computer algebra) that the relation (57) implies the following relation

$$\left(8\hat{t}_1\hat{t}_2\hat{v}_1\hat{v}_2\right)^2=\left(\left(2R_1^2-2\hat{u}_1\hat{u}_2-\hat{t}_1^2-\hat{t}_2^2\right)^2-4(\hat{t}_1\hat{t}_2)^2-4(\hat{v}_1\hat{v}_2)^2\right)^2.$$

Replacing $\hat{u}_i,\ i=1,2$, with

$$\frac{-2R_1^2d_1+\left(R_1^2+d_1^2-r_1^2\right)u_i}{2d_iu_i-\left(R_1^2+d_1^2-r_1^2\right)},\quad i=1,2,$$

respectively, we get

$$\begin{aligned}&(d_1-r_1-R_1)^2(d_1+r_1-R_1)^2(d_1-r_1+R_1)^2(d_1+r_1+R_1)^2\\&\big(d_1^4u_1^2+2d_1^4u_1u_2+d_1^4u_2^2-4d_1^3R_1^2u_1-4d_1^3R_1^2u_2-4d_1^3u_1^2u_2\\&\quad-4d_1^3u_1u_2^2-4d_1^2r_1^2R_1^2-4d_1^2r_1^2u_1u_2+4d_1^2R_1^4+2d_1^2R_1^2u_1^2\\&\quad+12d_1^2R_1^2u_1u_2+2d_1^2R_1^2u_2^2+4d_1^2u_1^2u_2^2+8d_1r_1^2R_1^2u_1\\&\quad+8d_1r_1^2R_1^2u_2-4d_1R_1^4u_1-4d_1R_1^4u_2-4d_1R_1^2u_1^2u_2-4d_1R_1^2u_1u_2^2\\&\quad+4r_1^4R_1^2-4r_1^2R_1^4-4r_1^2R_1^2u_1u_2+R_1^4u_1^2+2R_1^4u_1u_2+R_1^4u_2^2\big)=0.\end{aligned}\tag{59a}$$

Now, if in the fifth (last) factor of the above relation we put

$$\frac{\left(R_1^2+d_1^2-r_1^2\right)-t_i^2}{2d_1},\quad i=1,2,$$

instead of $u_i,\ i=1,2$, respectively, then we get

$$\begin{aligned}&\left(d_1^4-2d_1^2r_1^2-2d_1^2R_1^2+2d_1^2t_1t_2+r_1^4-2r_1^2R_1^2+r_1^2t_1^2+r_1^2t_2^2+R_1^4-2R_1^2t_1t_2+t_1^2t_2^2\right)\\&\left(d_1^4-2d_1^2r_1^2-2d_1^2R_1^2-2d_1^2t_1t_2+r_1^4-2r_1^2R_1^2+r_1^2t_1^2+r_1^2t_2^2+R_1^4+2R_1^2t_1t_2+t_1^2t_2^2\right)=0.\end{aligned}\tag{59b}$$

Finally, if $t_2$ in the above relation be replaced by the right side of the relation (58), then the second factor of the above relation vanishes.

This proves the validity of (57).

Now, using this result, the proof of Theorem 13 follows from Poncelet's closure theorem. Namely, by this theorem there is a bicentric $n$-gon whose first tangent has length $\hat{t}_1$ and beginning point $\hat{A}_1(\hat{u}_1,\hat{v}_1)$. This $n$-gon is obtained such that we proceed in the same way with $t_2$, $t_3$, then with $t_3,t_4,\ldots$, finally with $t_n,t_1$. In this way we get closure:

$$\{\hat{A}_1,\ldots,\hat{A}_n\}=\{A_1\ldots,A_n\},$$

where

$$\hat{A}_i=A_{i+\frac{n}{2}}\text{ and }\hat{t}_i=t_{i+\frac{n}{2}},\quad i=1,\ldots,\frac{n}{2}.$$

Thus,

$$A_1A_2\cdots A_{\frac{n}{2}}\hat{A}_1\hat{A}_2\cdots\hat{A}_{\frac{n}{2}} = A_1A_2\cdots A_{n-1}A_n.$$

This proves Theorem 13. □

*Remark.* (4) As it is seen, the proof of Theorem 13 is rather involved and we have solved one of the old and difficult problems concerning bicentric polygons.

Here is an example. With the hexagon $A_1\dots A_6$ from Example 2, we have

$$t_1t_4 = t_2t_5 = t_3t_6 = t_Mt_m.$$

**Theorem 14.** *Let the triple* $(R_1, r_1, d_1)$ *and the bicentric* $n$*-gon* $A_1\dots A_n$ *be as in Theorem 7. Let* $t_i,\ i = 1,\dots,n$*, be the tangent lengths of the* $n$*-gon* $A_1\dots A_n$ *and let* $T_i,\ i = 1,\dots,n$ *be the touching points of the segments* $A_iA_{i+1},\ i = 1,\dots,n$*, and the circle* $C_2$*, respectively. In other words*

$$t_i = |A_iT_i|\,,\quad i = 1,\dots,n. \tag{60}$$

*Let*

$$(k^iR_1, k^ir_1, k^id_1),\quad i = 1,2,\dots \tag{61}$$

*be a set of triples such that*

$$k = \frac{r_1}{R_1}. \tag{62}$$

*Then for each* $i = 1,2,\dots$ *there is a bicentric* $n$*-gon from the class* $C(k^iR_1, k^ir_1, k^id_1)$ *such that its tangent lengths are* $k^it_1,\dots,k^it_n$*.*

*Proof.* The triples given by (61) also satisfy Fuss' relation $F_n(R,r,d) = 0$ as the triple $(R_1, r_1, d_1)$. □

**Corollary 15.** *Let* $S_i(s_i, 0)$ *denote the characteristic point of the triple* $\left(k^iR_1, k^ir_1, k^id_1\right)$*. Then*

$$s_i = k^{i-1}s_1, \tag{63}$$

*where*

$$s_1 = \frac{(t_M - t_m)^2}{4d_1}, \tag{64}$$

$$t_M^2 = (R_1 + d_1)^2 - r_1^2,\quad t_m^2 = (R_1 - d_1)^2 - r_1^2. \tag{65}$$

*Proof.* This follows from

$$s_i = \frac{\left(k^{i-1}t_M - k^{i-1}t_m\right)^2}{4k^{i-1}d_i} = \frac{k^{i-1}(t_M - t_m)^2}{4d_1}.$$

□

**Example 3.** Let $n = 6$ and let the triple $(R_1, r_1, d_1)$, where

$$R_1 = 8.340410321,\ r_1 = 6.812488532,\ d_1 = 1.198981793$$

be a solution of Fuss' relation $F_6(R,r,d) = 0$.

Now, using these values we get

$$t_M = 6.67757441,\quad t_m = 2.142428529,\quad k = 0.816804971,$$

$$s_1 = 4.288544723, \quad s_2 = 3.502904648, \quad s_3 = 2.86118993, \text{ and so on.}$$

Let $R_{i+1}$, $r_{i+1}$, $d_{i+1}$ be given by

$$R_{i+1} = k^i R_1, \quad r_{i+1} = k^i r_1, \quad d_{i+1} = k^i d_1, i = 1, 2, \dots .$$

Thus

$$R_2 = 6.81248861, \quad r_2 = 5.564474498, \quad d_2 = 0.979334288,$$
$$R_3 = 5.564474498, \quad r_3 = 4.545090431, \quad d_3 = 0.799925115, \text{ and so on.}$$

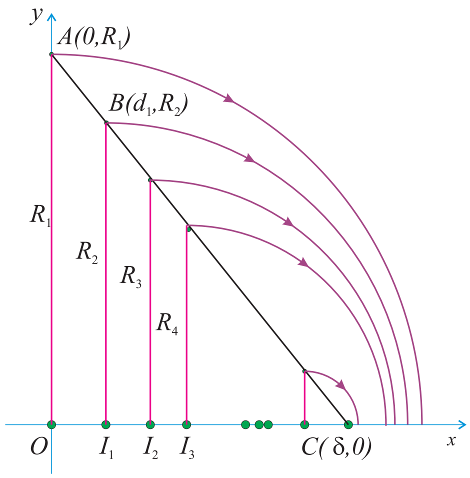


Figure 5

The following properties may be interesting. In Figure 5,

$$\delta = d_1 + d_2 + d_3 + \cdots = \frac{d_1}{1-k} = 6.544838032.$$

First, let us remark that the line determined by points $A(0, R_1)$ and $B(d_1, R_2)$, where $R_2 = r_1$, contains the point $C(\delta, 0)$ and the points whose coordinates are $(d_2, R_3)$, $(d_3, R_4)$ and so on. Also, let us remark that there are points $S_i(s_i, 0),\ i = 1, 2, \dots, n$, on the positive $x$-axis such that

$$s_1 = |OS_1| < r_1, \quad s_2 = |I_1 S_2| < r_2, \quad s_3 = |I_2 S_3| < r_3, \text{ and so on.}$$

*Remark.* (5) If instead of $k = \frac{r_1}{R_1}$, we take $K = \frac{R_1}{r_1}$ then we have analogous situation. Only in this case each of the values $K^i R_1$, $K^i r_1$, $K^i d_1 \to \infty$ when $i \to \infty$.

**Theorem 16.** *Let* $(R_0, r_0, d_0) \in \mathbb{R}_+^3$ *be a solution of Fuss' relation* $F_n(R, r, d) = 0$ *and let* $(R_1, r_1, d_1) \in \mathbb{R}_+^3$ *be given by* (24)*, that is,*

$$(R_1, r_1, d_1) = \left(\sqrt{R_0(R_0 + r_0 + r_1)}, r_1, \sqrt{R_0(R_0 + r_0 - r_1)}\right)$$

*where*

$$r_1 = \sqrt{(R_0 + r_0)^2 - d_0^2}.$$

*Let* $(R_1, r_1, d_1)$ *be a solution of Fuss' relation* $F_{2n}(R, r, d) = 0$ *and let* $C_1$, $C_2$, $K_1$, $K_2$ *be circles in the same plane such that* $O$ *is the center of* $C_1$ *and* $K_1$ (see Figure 6). *The center of* $C_2$ *is denoted by* $I_0$ *and center of* $K_2$ *is denoted by* $I_1$ *and*

$$\begin{aligned} R_0 &= \text{ radius of } C_1, \quad r_0 = \text{ radius of } C_2, \\ d_0 &= \text{ distance between centers of } C_1 \text{ and } C_2. \\ R_1 &= \text{ radius of } K_1, \quad r_1 = \text{ radius of } K_2, \\ d_1 &= \text{ distance between centers of } K_1 \text{ and } K_2. \end{aligned}$$

*Let* $xOy$ *be a coordinate system with origin* $O$ *and positive* $x$*-axis containing the centers* $I_0$ *and* $I_1$*. Then there are bicentric* $n$*-gon* $A_1 \cdots A_n$ *inscribed in* $C_1$ *and circumscribed around* $C_2$ *and bicentric* $2n$*-gon inscribed in* $K_1$ *and circumscribed around* $K_2$ *such that the following is valid:*

> *If* $t_1 \dots, t_n$ *are tangent lengths of the* $n$*-gon* $A_1 \cdots A_n$ *and* $u_1, \dots, u_{2n}$ *are tangent lengths of the* $2n$*-gon* $B_1 \cdots B_{2n}$ *then*
$$u_{2i-1} = t_i, \quad i = 1, \dots, n. \tag{66}$$

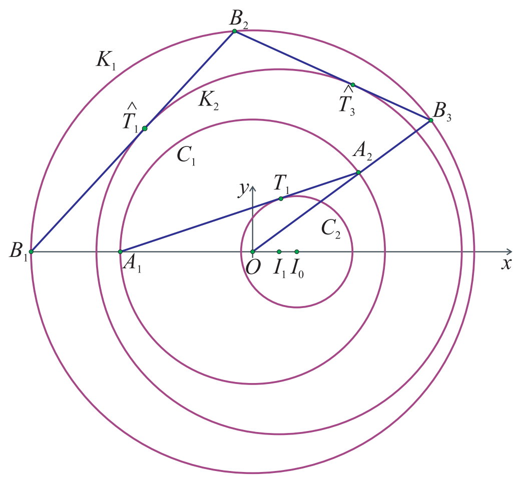


Figure 6: $A_1$ and $A_2$ are two consequent vertices of an $n$-gon $A_1 \dots A_n$ inscribed in $C_1$ and circumscribed around $C_2$.

*Proof.* The point $A_1$ is given by $A_1(u_1, 0)$, where $u_1 = -R_0$, and the point $A_2$ (as a consequent of $A_1$) is given by $A_2(u_2, v_2)$, where $u_2$ (by Theorem 12) is given by

$$u_2 = \frac{R_0}{(R_0 + d_0)^4} \left( d_0^4 - 2R_0^2 r_0^2 + R_0^4 - 2r_0^2 d_0^2 + 6R_0^2 d_0^2 + 4R_0 d_0^3 + 4R_0^3 d_0 - 4R_0 d_0 r_0^2 \right). \tag{67}$$

Of course, $v_2^2 = R_0^2 - u_2^2$.

The point $B_1$ and $B_3$ are elements of $K_1$ given by $B_1(\hat{u}_1, 0)$ and $B_3(\hat{u}_3, \hat{v}_3)$, where

$$\hat{u}_1 = cu_1 = -R_1, \quad \hat{u}_3 = cu_2, \text{ where } c = \frac{R_1}{R_0}. \tag{68}$$

First we prove that

$$|A_1T_1| = \left|B_1\hat{T}_1\right|, \quad |A_2T_1| = \left|B_3\hat{T}_3\right|, \tag{69}$$

where relations

$$R_0^2 + d_0^2 - r_0^2 = R_1^2 + d_1^2 - r_1^2, \quad R_0d_0 = R_1d_1$$

given by (28b) and (28c) will be used. The proof is as follows:

$$|A_1T_1|^2 = R_0^2 + d_0^2 - r_0^2 - 2d_0u_1 = R_0^2 + d_0^2 - r_0^2 + 2d_0R_0, \tag{70a}$$

$$\left|B_1\hat{T}_1\right|^2 = R_1^2 + d_1^2 - r_1^2 - 2d_1\hat{u}_1 = R_1^2 + d_1^2 - r_1^2 + 2d_1R_1, \tag{70b}$$

since

$$R_1^2 + d_1^2 - r_1^2 = R_0^2 + d_0^2 - r_0^2, \quad 2d_1cu_1 = 2d_1R_1\frac{R_1}{R_0}u_1 = 2\frac{d_0R_0}{R_0}u_1 = 2d_0u_1.$$

In the same way it can be shown that the second relation given by (69) is also valid. That the tangent length is given by $t^2 = R^2 + d^2 - r^2 - 2du$ can be seen in the proof of Theorem 2.

Now we prove that there is a point $B_2 \in K_1$ between $B_1$ and $B_3$ such that $B_2$ is a consequent of $B_1$ and $B_3$ is a consequent of $B_2$. The proof is as follows.

By Theorem 12 the consequent of $B_1$ is given by $B_2(\hat{u}_2, \hat{v}_2)$, where

$$\hat{u}_2 = \frac{R_1}{(R_1+d_1)^4}\left(d_1^4 - 2R_1^2r_1^2 + R_1^4 - 2r_1^2d_1^2 + 6R_1^2d_1^2 + 4R_1d_1^3 + 4R_1^3d_1 - 4R_1d_1r_1^2\right). \tag{71}$$

From this, using computer algebra, it is easy to show that $B_3$ is consequent of $B_2$.

Now, if we take $A_3 \in C_1$ which is consequent of $A_2$, then for $A_2$ and $A_3$ analogously holds as for $A_1$ and $A_2$. So, in this way we can proceed and get closure, that is, a bicentric $2n$-gon inscribed in $K_1$ and circumscribed around $K_2$ whose tangent lengths are such that holds (66). □

For example, from (70) and Figure 6 it can be seen that

$$t_1 = |A_1T_1| = \left|B_1\hat{T}_1\right| = u_1,$$
$$t_2 = |A_2T_1| = \left|B_3\hat{T}_3\right| = u_3,$$

analogously for $A_3$ and $B_5$, and so on.

*Remark.* (6) If we take $A_1$ on the $x$-axis we get (with less calculation) a bicentric $2n$-gon inscribed in $K_1$ and circumscribed around $K_2$ symmetric about the $x$-axis. By Poncelet's closure theorem, it follows that for every point $X \in K_1$ we get a bicentric $2n$-gon inscribed in $K_1$ and circumscribed around $K_2$.

Now we state the following corollaries of Theorem 16.

**Corollary 17.** $u_iu_{i+n} = t_Mt_m$ *for* $i = 1, \ldots, n$.

See Theorem 13.

**Corollary 18.** *$A_1A_2 \parallel B_1B_3$ and $c\,|A_1A_2| = |B_1B_3|$ for $c$ given by* (68) (see Figure 6).

*Proof.* Let $f$ denote the homothety whose center is $O$ and coefficient $c$ is given by (68). This homothety maps $A_1A_2$ onto $B_1B_3$. □

**Corollary 19.** *Let $B_1 \dots B_{2n}$ be a bicentric $2n$-gon as described in Theorem 16. Then $B_1B_3 \dots B_{2n-1}$ and $B_2B_4 \dots B_{2n}$ are bicentric $n$-gons inscribed in $K_1$ and circumscribed around a circle $\hat{K}_2$ with center $I_2$ and radius $cr_0$ such that $|OI_2| = c\,|OI_0| = cd_0$.*

*Proof.* First it is clear that $F_n(R_0, r_0, d_0) = 0 \Longrightarrow F_n(cR_0, cr_0, cd_0) = 0$, that is, $F_n(R_0, r_0, d_0) = 0 \Longrightarrow F_n(R_1, cr_0, cd_0) = 0$ since $cR_0 = R_1$.

Also let us remark that from the Corollary 18 can be concluded that there are two bicentric $n$-gons $A_1 \dots A_n$ and $D_1 \dots D_n$ inscribed in $C_1$ and circumscribed around $C_2$ such that the first has sides parallel with the corresponding sides of the $n$-gon $B_1B_3 \dots B_{2n-1}$ and the second has sides parallel with the corresponding sides of the $n$-gon $B_2B_4 \dots B_{2n}$. □

**Corollary 20.** *Let $u_1, \dots, u_{2n}$ be tangent lengths of the $2n$-gon $B_1 \dots B_{2n}$. Then, $cu_i,\ i = 1, 3, 5, \dots, 2n-1$, are the tangent lengths of the $n$- gon $B_1B_3 \dots B_{2n-1}$, and*
*$cu_i,\ i = 2, 4, 6, \dots, 2n$, are the tangent lengths of the $n$- gon $B_2B_4 \dots B_{2n}$ .*

*Proof.* It follows from the above corollaries. □

Here is an example where $n = 3$. See Figure 7.

**Example 4.** The incircle of the triangles $B_1B_3B_5$ and $B_2B_4B_6$ is denoted by $\hat{K}_2$. There are two triangles $A_1A_2A_3$ and $D_1D_2D_3$ inscribed in $C_1$ and circumscribed around $C_2$. The first is similar to the triangle $B_1B_3B_5$ and the second is similar to the triangle $B_2B_4B_6$. If $u_1 \dots, u_6$ are the tangent lengths of the hexagon $B_1 \dots B_6$, then

$u_1, u_3, u_5$ are the tangent lengths of the triangle $A_1A_2A_3$,

$u_2, u_4, u_6$ are the tangent lengths of the triangle $D_1D_2D_3$,

where, for example, $u_1 = |A_1T_1|$, $u_3 = |A_2T_1|$, $u_5 = |A_3T_3|$.

By Theorem 16, this holds analogously for each bicentric $n$-gon $A_1 \dots A_n$ and the corresponding bicentric $2n$-gon $B_1 \dots B_{2n}$.

**Theorem 21.** *Let the triple $(R_0, r_0, d_0)$ be as in Theorem 16 and let the triple $(R_2, r_2, d_2)$ be given by* (25)*, that is,*

$$(R_2, r_2, d_2) = \left(\sqrt{R_0(R_0 - r_0 + r_2)}, r_2, \sqrt{R_0(R_0 - r_0 - r_2)}\right), \tag{72a}$$

*where*

$$r_2 = \sqrt{(R_0 - r_0)^2 - d_0^2}. \tag{72b}$$

*Then $F_{2n}(R_2, r_2, d_2) = 0$.*

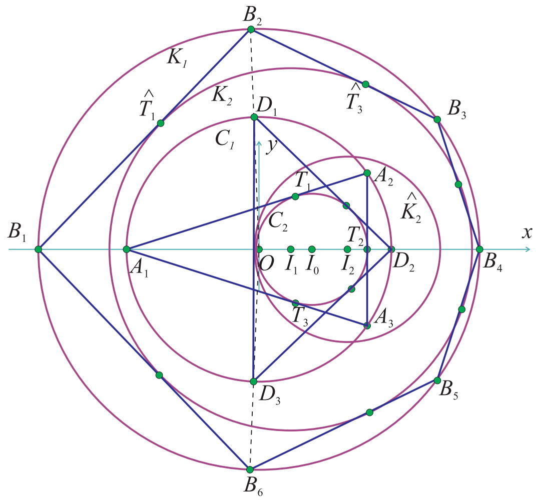


Figure 7: $R_0 = 5,\ r_0 = 2.1,\ d_0 = 2$ refers to circle $C_1$ and $C_2$. The chords $B_1B_3$, $B_3B_5$, $B_5B_1$ of the circle $K_1$ are tangential segments of the circle $\hat{K}_2$.

The proof is analogous to that of Theorem 16, and we have analogous corollaries.

## 3. The $n$-closure and related considerations

Let $S$ denote the set of all ordered triples $(R, r, d)$, where $(R, r, d) \in \mathbb{R}^3_+$ and $R > r + d$. Let $f_1$ and $f_2$ be functions defined on the set $S$ as in Definition A. We have

$$f_1(R_0, r_0, d_0) = (R_1, r_1, d_1), \tag{73}$$

$$f_2(R_0, r_0, d_0) = (R_2, r_2, d_2), \tag{74}$$

where $(R_1, r_1, d_1)$ and $(R_2, r_2, d_2)$ are given by (24) and (25) respectively.

Let $f$ be any given composition of the function $f_1$ and $f_2$. For example, $f = f_1^2 f_2 f_1 f_2^3 f_1$. Then it is appropriate to write this composition as

$$(R_{11212221}, r_{11212221}, d_{11212221}),$$

since

$$\begin{aligned} f_1^2 f_2 f_1 f_2^3 f_1(R_0, r_0, d_0) &= f_1^2 f_2 f_1 f_2^3(R_1, r_1, d_1) \\ &= f_1^2 f_2 f_1 f_2^2(R_{12}, r_{12}, d_{12}), \text{ and so on.} \end{aligned}$$

Concerning such indices let us remark that the situation is in some way connected with fact that there are $2^k$ integers with $k$ digits from the set $\{1, 2\}$. So, if $k = 3$, we have indices

$$111, 112, 121, 122, 211, 212, 221, 222.$$

See also Figure 8, where instead of $(R_i, r_i, d_i)$, $i = 0, 1, 2, \ldots$, are (for brevity) written only corresponding indices.

Before stating some examples, we define some terms which will be used.

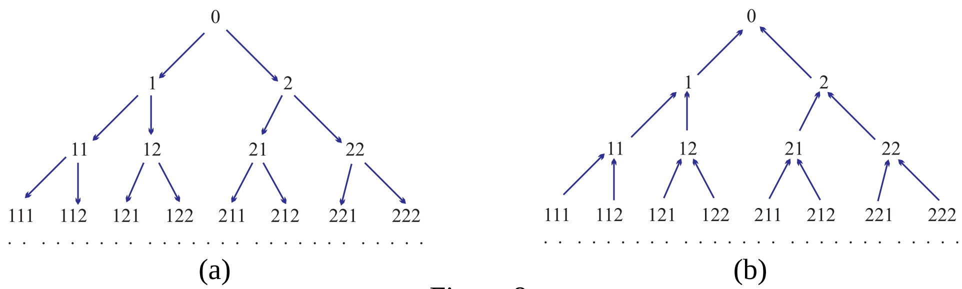


Figure 8:
The Figure 8a geometrically represents functions $f_1$ and $f_2$ and their compositions, and the Figure 8b geometrically represents function $g$ given by (80) and its compositions.

**Definition 2.** Let $(R_0, r_0, d_0)$ be a triple such that $F_n(R_0, r_0, d_0) = 0$. We say that this triple has $n$-closure.

Now, let $C_1$ and $C_2$ be circles such that $C_2$ is completely inside $C_1$, and let $R_0$ = radius of $C_1$, $r_0$ = radius of $C_2$, $d_0$ = distance between centers of $C_1$ and $C_2$. Let $A_1 \ldots A_n$ be an $n$-gon inscribed in $C_1$ and circumscribed around $C_2$. We say that this $n$-gon has $k$-circumscription if

$$\sum_{i=1}^{n} \arctan \frac{t_i}{r_0} = k\pi,$$

where $t_1, \ldots, t_n$ are the tangent lengths of the $n$-gon $A_1 \ldots A_n$. The number $k$ in this case is called the *rotation number* for $n$.

Let $(R_0, r_0, d_0)$ be as in Definition 2. Then $f_1(R_0, r_0, d_0)$ has $2n$-closure. Also, the triple $f_2(R_0, r_0, d_0)$ has $2n$-closure for every $n > 3$. But for $n = 3$, we get a bicentric hexagon which is a double triangle.

Here are some examples referred to Theorems 16 and 21 and composition of the functions $f_1$ and $f_2$.

Let $n = 3$ and let $(R_0, r_0, d_0) = (5, 2.1, 2)$. Then

$$\begin{aligned} f_1(5, 2.1, 2) &= (R_1, r_1, d_1), \\ f_1^2(5, 2.1, 2) &= (R_{11}, r_{11}, d_{11}), \end{aligned}$$

where

$$\begin{aligned} R_1 &= 8.340410221, & r_1 &= 6.812488532, & d_1 &= 1.198981793, \\ R_{11} &= 15.886048415, & r_{11} &= 15.105389214, & d_{11} &= 0.629483163. \end{aligned}$$

Since $t_M = \sqrt{(R_0 + d_0)^2 - r_0^2} = 6.67757441$, $t_m = \sqrt{(R_0 - d_0)^2 - r_0^2} = 2.142428529$, we should take $t_1$ such that $t_m < t_1 < t_M$. Let's say $t_1 = 4$. Now, let $A_1A_2A_3$ be a triangle from the class $C(R_0, r_0, d_0)$, and let $B_1, \ldots, B_6$ and $C_1 \ldots C_{12}$ be bicentric hexagon and bicentric 12-gon, the first from the class $C(R_1, r_1, d_1)$ and the second from the class $C(R_{11}, r_{11}, d_{11})$, such that their first tangent length is also $t_1 = 4$. Then the following are valid.

The tangent lengths of the triangle $A_1A_2A_3$ are

$$t_1 = 4,\ t_2 = 2.257285251,\ t_3 = 5.973973936. \tag{75a}$$

The tangent lengths of the bicentric hexagon $B_1 \dots B_6$ are

$$\begin{aligned} u_1 &= 4, & u_2 &= 2.394578677, & u_3 &= 2.257285251,\\ u_4 &= 3.576556479, & u_5 &= 5.973973936, & u_6 &= 6.337801531, \end{aligned} \tag{75b}$$

where $u_1 = t_1$, $u_3 = t_2$, $u_5 = t_3$.

The tangent lengths of the bicentric 12-gon $C_1 \dots C_{12}$ are

$$\begin{aligned} v_1 &= 4, & v_2 &= 3.010399453, & v_3 &= 2.394578677,\\ v_4 &= 2.148970243, & v_5 &= 2.257285251, & v_6 &= 2.727553891,\\ v_7 &= 3.576556479, & v_8 &= 4.752268309, & v_9 &= 5.973973936,\\ v_{10} &= 6.657247101, & v_{11} &= 6.337801531, & v_{12} &= 5.245075438, \end{aligned} \tag{75c}$$

where $v_1 = u_1$, $v_3 = u_2$, $v_5 = u_3$, $v_7 = u_4$, $v_9 = u_5$, $v_{11} = u_6$.

Here is a partition of the tangent lengths of the bicentric hexagon

$$\{\{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}\}\,. \tag{76}$$

This partition has the property that there are two triangles from the class $C(R_0, r_0, d_0)$ such that the first has tangent lengths $u_1, u_3, u_5$ and the second has tangent lengths $u_2, u_4, u_6$.

Analogously for the tangent lengths $v_1, \dots, v_{12}$ of the bicentric 12-gon $C_1 \dots C_{12}$; in this case we have the following partition

$$\{\{v_1, v_5, v_9\}, \{v_3, v_7, v_{11}\}, \{v_2, v_6, v_{10}\}, \{v_4, v_8, v_{12}\}\}\,. \tag{77}$$

This partition has the property that there are four triangles from the class $C(R_0, r_0, d_0)$ such that their tangent lengths are

$$v_1, v_5, v_9, \tag{78a}$$
$$v_3, v_7, v_{11}, \tag{78b}$$
$$v_2, v_6, v_{10}, \tag{78c}$$
$$v_4, v_8, v_{12}, \tag{78d}$$

respectively.

In the same way we can proceed and find that this holds analogously for the tangent lengths of the corresponding bicentric 24-gon from the class $C(f_1^3(R_0, r_0, d_0))$, that is, from $C(R_{111}, r_{111}, d_{111})$. More generally, for any integer $m \geq 1$, there is a partition of the tangent lengths of the corresponding bicentric $3 \cdot 2^m$-gon from the class $C(f_1^m(R_0, r_0, d_0))$ such that this holds analogously as for $m = 1, 2, 3$.

Also from Theorem 16 and Theorem 21 analogous results can be concluded if instead of $n = 3$ we take $n > 3$ and any given composition of the function $f_1$ and $f_2$ given by Definition A.

In connection with Theorem 16 and Theorem 21 we state the following conjecture which is a modification of Conjecture 2 given in [5, page 56].

**Conjecture 1.** *Let $(R_0, r_0, d_0)$ be as in Theorem 16 and let $P_1 \ldots P_n$ and $Q_1 \ldots Q_n$ be $n$-gons from the class $C(R_0, r_0, d_0)$ such that the sum of the tangent lengths of the $n$-gon $P_1 \ldots P_n$ is minimal and the sum of the tangent lengths of the $n$-gon $Q_1 \ldots Q_n$ is maximal. Then both of those two $n$-gons are axial symmetric in the $x$-axis. Let the sum of the tangent lengths of the $n$-gon $P_1 \ldots P_n$ be denoted by $a$ and the sum of the tangent lengths of the $n$-gon $Q_1 \ldots Q_n$ be denoted by $b$. Then the following is valid:*

> *For every $n$-gon $A_1 \ldots A_n$ from the class $C(R_0, r_0, d_0)$ there is an $n$-gon $B_1 \ldots B_n$ from the same class such that*
>
> $$(t_1 + \cdots + t_n)(u_1 + \cdots + u_n) = ab, \tag{79}$$
>
> *where $t_1, \ldots, t_n$ are the tangent lengths of the $n$-gon $A_1 \ldots A_n$ and $u_1, \ldots, u_n$ are the tangent lengths f the $n$-gon $B_1 \ldots B_n$.*

*Let such two $n$-gons be called conjugate $n$-gons. Thus for every $n$-gon from the class $C(R_0, r_0, d_0)$ there is an $n$-gon from the same class conjugate to it.*

Here are some examples where $n = 3$ and $(R_0, r_0, d_0) = (5, 2.1, 2)$.

First, it can be easily found that for axial symmetric triangles from the class $C(5, 2.1, 2)$ we have $ab = 150.5559966$. Now using the tangent lengths $u_1, \ldots, u_6$ given by (75b) and partition given by (76) it can be verified that triangle whose tangent lengths are $u_1, u_3, u_5$ is conjugate to triangle whose tangent lengths are $u_2, u_4, u_6$, that is, $(u_1 + u_3 + u_5)(u_2 + u_4 + u_6) = 150.5559966$. Also, using the tangent lengths given by (75c) (see also (77)) it can be verified that triangle whose tangent lengths are $v_1$, $v_5$, $v_9$ is conjugate to triangle whose tangent lengths are $v_3$, $v_7$, $v_{11}$, and triangle whose tangent lengths are $v_2$, $v_6$, $v_{10}$ is conjugate to triangle whose tangent lengths are $v_4$, $v_8$, $v_{12}$. In other words,

$$(v_1 + v_5 + v_9)(v_3 + v_7 + v_{11}) = (v_2 + v_6 + v_{10})(v_4 + v_8 + v_{12}) = 150.5559966.$$

In order that the rule of obtaining conjugate bicentric polygons be more noticeable here will be also in short about bicentric 24-gon $D_1 \ldots D_{24}$ from the class $C(R_{111}, r_{111}, d_{111})$ obtained starting from the triple $(5, 2.1, 2)$. Let $w_1, \ldots, w_{24}$ denote tangent lengths of this 24-gon and let $w_1$ be 4 as in the previous examples. Then

$$\begin{aligned}
& (w_1 + w_9 + w_{17})(w_5 + w_{13} + w_{21}) \\
= \; & (w_3 + w_{11} + w_{19})(w_7 + w_{15} + w_{23}) \\
= \; & (w_2 + w_{10} + w_{18})(w_6 + w_{14} + w_{22}) \\
= \; & (w_4 + w_{12} + w_{20})(w_8 + w_{16} + w_{24}) \\
= \; & 150.5559966.
\end{aligned}$$

Thus in this case there are $4$ pairs of conjugate triangles from the class $C(5, 2.1, 2)$ which refer to the 24-gon $D_1 \ldots D_{24}$.

More generally, for a given $m > 1$, there are $2^{m-1}$ pairs of conjugate triangles from the class $C(5, 2.1, 2)$ which refer to bicentric polygons with $3 \cdot 2^m$ vertices. Analogously holds if instead $n = 3$ we take $n > 3$. Of course, this holds on the supposition that Conjecture 1 is true. We hope that the Conjecture will be validated in the near future.

Figure 7 shows how conjugate bicentric polygons can be constructed. For example, $A_1A_2A_3$ and $D_1D_2D_3$ are conjugate triangles from the class $C(5, 2.1, 2)$.

Analogously can be concluded if instead of $n = 3$ we can take $n > 3$.

It is clear from Theorem 16 and Theorem 21 that the functions $f_1$ and $f_2$ play key roles in this work. These functions are given in [7], where some of their important properties are established. In the present article we have established some other of their important properties given by Theorem 16 and Theorem 21. In this connection let us mention that in [7] we have also defined a function $g$ such that the following is valid: If

$$f_1(R_0, r_0, d_0) = (R_1, r_1, d_1), \quad f_2(R_0, r_0, d_0) = (R_2, r_2, d_2), \tag{80a}$$

then

$$g(R_1, r_1, d_1) = (R_0, r_0, d_0), \quad g(R_2, r_2, d_2) = (R_0, r_0, d_0). \tag{80b}$$

This function is given by

$$g(R,r,d) = \left(\frac{R^2-d^2}{2r}, \sqrt{-(R^2+d^2-r^2)+\left(\frac{R^2-d^2}{2r}\right)^2+\left(\frac{2Rrd}{R^2-d^2}\right)^2}, \frac{2Rrd}{R^2-d^2}\right). \tag{81}$$

We have subsequently found that $\sqrt{-(R^2+d^2-r^2)+\left(\frac{R^2-d^2}{2r}\right)^2+\left(\frac{2Rrd}{R^2-d^2}\right)^2}$ can be written rationally as $\frac{d^4-2d^2r^2-2d^2R^2-2r^2R^2+R^4}{2r(d^2-R^2)}$.

See Figure 8b, for example. Starting from the triple $(R_{112}, r_{112}, d_{112})$ we get

$$g^3(R_{112}, r_{112}, d_{112}) = (R_0, r_0, d_0).$$

Thus, using sequences like these in Theorem 16 we can get some other relations useful in research of bicentric polygons.

Also let us emphasize here that using the function $g$ the following theorem can be easily proved.

**Theorem 22.** *The converses of Theorems 16 and 21 are also valid, that is, if the triples* $(R_i, r_i, d_i),\ i = 1, 2$, *are such that* $F_{2n}(R_i, r_i, d_i) = 0,\ i = 1, 2$, *then there is a triple* $(R_0, r_0, d_0)$ *such that* $F_n(R_0, r_0, d_0) = 0$ *and* $f_i(R_0, r_0, d_0) = (R_i, r_i, d_i),\ i = 1, 2$.

*Proof.* If the triple $(R, r, d)$ in the relation (81) is one of the triples $(R_i, r_i, d_i),\ i = 1, 2$, then we have a relation which can be written as $g(R_i, r_i, d_i) = (R_0, r_0, d_0),\ i = 1, 2$, that is, $F_{2n}(R_i, r_i, d_i) = 0 \to F_n(g(R_i, r_i, d_i)) = 0,\ i = 1, 2$.

Also let us remark that the system

$$Rd = R_0d_0, \quad R^2 + d^2 - r^2 = R_0^2 + d_0^2 - r_0^2, \quad R^2 - d^2 = 2R_0r$$

in $R$, $r$, $d$ has two solutions given by (24) and (25), and that the solution of the above system in $R_0$, $r_0$, $d_0$ is given by (81), that is, $g(R, r, d) = (R_0, r_0, d_0)$.

□

**Corollary 23.** *Using relation* (27a) *and* (28b) *the triple* $c(R_0, r_0, d_0)$*, where* $c = \frac{R_1}{R_0}$*, can be written as*

$$\left(R_1,\ \frac{2R_1 t_M t_m}{R_1^2 - d_1^2},\ \frac{2R_1 r_1 d_1}{R_1^2 - d_1^2}\right),$$

*where* $cr_0$ *and* $cd_0$ *are also expressed only using* $R_1$*,* $r_1$*,* $d_1$.

Of course, the triple $c(R_0, r_0, d_0)$ can be also expressed as

$$\frac{2R_1 r_1}{R_1^2 - d_1^2}\left(\frac{R_1^2 - d_1^2}{2r_1}, \sqrt{-\left(R_1^2 + d_1^2 - r_1^2\right) + \left(\frac{R_1^2 - d_1^2}{2r_1}\right)^2 + \left(\frac{2R_1 r_1 d_1}{R_1^2 - d_1^2}\right)^2}, \frac{2R_1 r_1 d_1}{R_1^2 - d_1^2}\right).$$

## 4. Another type of characteristic points for two nested circles

About interesting geometrical properties of the triples $(R_0, r_0, d_0)$ and $c(R_0, r_0, d_0)$ see Corollaries 17–20.

In the following we briefly consider one more characteristic point defined for two nested circles. Definition 1 will be extended as follows. Instead of $R$, $r$, $d$, we use $R_0$, $r_0$, $d_0$, and let the points $S_1(s_1, 0)$ and $S_2(s_2, 0)$ be given by

$$s_{1,2} = \frac{R_0^2 + d_0^2 - r_0^2 \mp \sqrt{(R_0^2 + d_0^2 - r_0^2)^2 - 4R_0^2 d_0^2}}{2d_0} \tag{82a}$$

or

$$s_{1,2} = \frac{R_0^2 + d_0^2 - r_0^2 \mp t_M t_m}{2d_0}, \tag{82b}$$

since

$$(R_0^2 + d_0^2 - r_0^2)^2 - 4R_0^2 d_0^2 = \left((R_0 + d_0)^2 - r_0^2\right)\left((R_0 - d_0)^2 - r_0^2\right) = t_M^2 t_m^2.$$

Then both of the points $S_1(s_1, 0)$ and $S_2(s_2, 0)$ can be called characteristic points determined by the triple $(R_0, r_0, d_0)$.

It is easy to show that

$$s_{1,2} = \frac{(t_M \mp t_m)^2}{4d_0}. \tag{82c}$$

The point $S_1(s_1, 0)$ is the intersection of the $x$-axis and the line through the points $T_1$ and $\hat{T}_1$ drawn in Figure 1b given by

$$T_1\left(d_0 - \frac{r_0^2}{R_0 + d_0}, \frac{r_0 t_M}{R_0 + d_0}\right), \quad \hat{T}_1\left(d_0 + \frac{r_0^2}{R_0 - d_0}, -\frac{r_0 t_m}{R_0 - d_0}\right). \tag{83}$$

The point $S_2(s_2, 0)$ is the intersection of the $x$-axis and the line through the points $T_1$ given by (83) and

$$T_2\left(d_0 + \frac{r_0^2}{R_0 - d_0}, \frac{r_0 t_m}{R_0 - d_0}\right).$$

Now we consider the relation (43) as an equation in $s$ given by

$$\hat{u} = \frac{2R_0^2 s - \left(R_0^2 + s^2\right) u}{-2su + R_0^2 + s^2}, \tag{84a}$$

where we here use notation $R_0$, $r_0$ $d_0$ instead of notation $R_1$, $r_1$, $d_1$. As will be shown, this relation plays a key role in using characteristic points. First it can be easily shown that this relation is equivalent to

$$\hat{u} = \frac{-2R_0^2 d_0 + \left(R_0^2 + d_0^2 - r_0^2\right) u}{2d_0 u - \left(R_0^2 + d_0^2 - r_0^2\right)}. \tag{84b}$$

Namely, if $s$ in the relation (84a) is replaced by right side of the any of the relations

$$s_1 = \frac{(t_M - t_m)^2}{4d_0}, \quad s_2 = \frac{(t_M + t_m)^2}{4d_0},$$

(see (82)) we get relation (84b).

Thus, the equation in $s$ given by (84a) has the solutions $s_1$ and $s_2$. These solutions can be also written as

$$s_{1,2} = \frac{u\hat{u} + R_0^2}{u + \hat{u}} \mp \sqrt{\left(\frac{u\hat{u} + R_0^2}{u + \hat{u}}\right)^2 - R_0^2}$$

and it is easily seen that $s_1 s_2 = R_0^2$.

Also, if $u$ and $\hat{u}$ in (84b) are interchanged, then

$$u = \frac{2R_0 s_i - \left(R_0^2 + s_i^2\right) \hat{u}}{-2s_i\hat{u} + R_0^2 + s_i^2} = \frac{-2R_0^2 d_0 + \left(R_0^2 + d_0^2 - r_0^2\right) \hat{u}}{2d_0 \hat{u} - \left(R_0^2 + d_0^2 - r_0^2\right)}, \quad i = 1, 2. \tag{85}$$

The above relations in $u$, $\hat{u}$, $s_1$, $s_2$ are very important since they open the way to the use of both of the characteristic points $S_1$ and $S_2$. The relation (84a) is connected with both of the characteristic points $S_1$ and $S_2$.

**Theorem 24.** *Let $C_1$ and $C_2$ be two nested circles such that*
*$R_0$ = radius of $C_1$, $r_0$ = radius of $C_2$,*
*$d_0 =$ distance between centers of $C_1$ and $C_2$.*
*Let $xOy$ be a coordinate system with origin $O$ at the center of $C_1$ and positive $x$-axis containing the center of $C_2$. Let $P(u, v)$ be any given point of $C_1$ and let $\hat{P}(\hat{u}, \hat{v})$ be a point of $C_1$ such that the chord $P\hat{P}$ of $C_1$ contains the characteristic point $S_1(s_1, 0)$, that is,*

$$\hat{u} = \frac{-2R_0^2 d_0 + \left(R_0^2 + d_0^2 - r_0^2\right) u}{2d_0 u - \left(R_0^2 + d_0^2 - r_0^2\right)}, \quad \hat{v}^2 = R_0^2 - \hat{u}^2.$$

*Then the point $Q(\hat{u}, -\hat{v})$ of $C_1$ has the property that the chord $PQ$ of $C_1$ contains the characteristic point $S_2(s_2, 0)$* (see Figure 9).

*Proof.* The condition that the line through the points $P$ and $Q$ contains characteristic point $S_2$ can be written as

$$-v = \frac{v + \hat{v}}{u - \hat{u}}(s_2 - u)$$

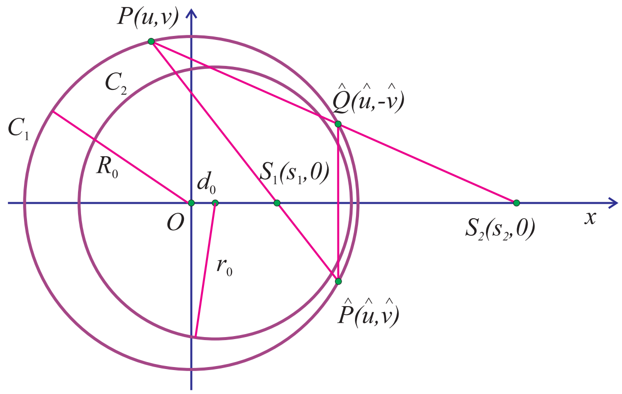


Figure 9. Geometrical interpretation of the points $P$, $\hat{P}$, $S_1$ and the points $P$, $Q$ $S_2$.

or

$$-\frac{v}{\hat{v}} = \frac{u - s_2}{\hat{u} - s_2}. \tag{86a}$$

The condition that the line through the points $P$ and $\hat{P}$ contains point $S_1$ is given by

$$\frac{v}{\hat{v}} = \frac{u - s_1}{\hat{u} - s_1}. \tag{86b}$$

Thus, the condition that the line through $P$ and $\hat{P}$ contains the characteristic point $S_1$ and the line through $P$ and $Q$ contains the characteristic point $S_2$ is given by

$$\left(\frac{v}{\hat{v}}\right)^2 = \left(\frac{u - s}{\hat{u} - s}\right)^2, \tag{87}$$

where $s = s_1$ in the first case and $s = s_2$ in the second case. We make use of the relations

$$v^2 = R_0^2 - u^2, \quad \hat{v}^2 = R_0^2 - \hat{u}^2, \quad \hat{u} = \frac{-2R_0^2 d_0 + \left(R_0^2 + d_0^2 - r_0^2\right) u}{2d_0 u - \left(R_0^2 + d_0^2 - r_0^2\right)}$$

which hold for any $u$ such that the point $P(u, v)$ belongs to the circle $C_1$. Using computer algebra we get the following equation in $s$:

$$\begin{aligned}
&d_0^4 R_0^2 su - d_0^4 su^3 - d_0^3 R_0^4 s - d_0^3 R_0^4 u - d_0^3 R_0^2 s^2 u + d_0^3 R_0^2 u^3 + d_0^3 s^2 u^3 + d_0^3 su^4 \\
&- 2d_0^2 r_0^2 R_0^2 su + 2d_0^2 r_0^2 su^3 + d_0^2 R_0^6 + d_0^2 R_0^4 s^2 + 2d_0^2 R_0^4 su - 2d_0^2 R_0^2 su^3 - d_0^2 R_0^2 u^4 \\
&- d_0^2 s^2 u^4 + d_0 r_0^2 R_0^4 s + d_0 r_0^2 R_0^4 u + d_0 r_0^2 R_0^2 s^2 u - d_0 r_0^2 R_0^2 u^3 - d_0 r_0^2 s^2 u^3 - d_0 r_0^2 su^4 \\
&- d_0 R_0^6 s - d_0 R_0^6 u - d_0 R_0^4 s^2 u + d_0 R_0^4 u^3 + d_0 R_0^2 s^2 u^3 + d_0 R_0^2 su^4 + r_0^4 R_0^2 su \\
&- r_0^4 su^3 - 2r_0^2 R_0^4 su + 2r_0^2 R_0^2 su^3 + R_0^6 su - R_0^4 su^3 = 0
\end{aligned}$$

whose roots are

$$s_1 = \frac{R_0^2 + d_0^2 - r_0^2 - \sqrt{(R_0^2 + d_0^2 - r_0^2)^2 - 4R_0^2 d_0^2}}{2d_0},$$
$$s_2 = \frac{R_0^2 + d_0^2 - r_0^2 + \sqrt{(R_0^2 + d_0^2 - r_0^2)^2 - 4R_0^2 d_0^2}}{2d_0}.$$

□

**Corollary 25.** *The equation in $s$ given by* (84) *is the same as the equation given by* (87)*. Each of them has only the solutions $s_1$ and $s_2$.*

**Corollary 26.** *Let $n \geq 4$ be an even integer with Fuss' relation $F_n(R_0, r_0, d_0) = 0$. There are two bicentric $n$-gons $A_1 \cdots A_n$ and $B_1 \cdots B_n$ with the following properties.*

($\mathbf{i}_1$) $A_1 = P(u, v)$, $B_{1+\frac{n}{2}} = Q(\hat{u}, -\hat{v})$.

($\mathbf{i}_2$) *For each $A_i(u_i, v_i),\ i = 1, \ldots, \frac{n}{2}$, there is $B_{i+\frac{n}{2}}\left(u_{i+\frac{n}{2}}, -v_{i+\frac{n}{2}}\right)$ such that the line through the points $A_i$ and $B_{i+\frac{n}{2}}$ contains point $S_2$.*

($\mathbf{i}_3$) *For each $A_i(u_i, v_i),\ i = 1, \ldots, \frac{n}{2}$, there is $A_{i+\frac{n}{2}}\left(u_{i+\frac{n}{2}}, v_{i+\frac{n}{2}}\right)$ such that the chord $A_i A_{i+\frac{n}{2}}$ of $C_1$ contains point $S_1$.*

($\mathbf{i}_4$) *The point $A_i$ and $B_i,\ i = 1, \ldots, \frac{n}{2}$, are symmetric about the $x$-axis.*

($\mathbf{i}_5$) *For each $i = 1, \ldots, \frac{n}{2}$,*

$$|A_i S_2| \left|A_{i+\frac{n}{2}} S_2\right| = s_2^2 - R_)^2,$$
$$|B_i S_2| \left|B_{i+\frac{n}{2}} S_2\right| = s_2^2 - R_)^2.$$

*Proof.* ($\mathbf{i}_2$): The proof easily follows from the equation of the line through the points $A_i$ and $B_{i+\frac{n}{2}}$.

($\mathbf{i}_4$): From the Figure 9 can be easily seen that the chord $Q\hat{P}$ of $C_1$, that is, the chord $B_{1+\frac{n}{2}} A_{1+\frac{n}{2}}$, is perpendicular to the $x$-axes.

($\mathbf{i}_5$): The proof is in the same way as the proof that holds the relation (40). □

Here is an example. Using Example 4, where $n = 6$, can be easily found that the vertices of the hexagon $A_1 \cdots A_6$ (determined by given tangent lengths) are

$A_1(3.58220619, 7.531948163)$, $A_2(7.862976314, 2.781375164)$,
$A_3(8.129674451, -1.863018422)$, $A_4(4.920109639, -6.734609525)$,
$A_5(-4.628245672, -6.938428231)$, $A_6(-6.49623254, 5.230813236)$.

In this case is $\hat{A}_1 = A_4$, $\hat{A}_2 = A_5$, $\hat{A}_3 = A_6$, .

The vertices of the hexagon $B_1 \cdots B_6$ are such that if $A_i(u_i, v_i),\ i = 1, \ldots, 6$, then $B_i(u_i, -v_i),\ i = 1, \ldots, 6$.

In Example 4 it is shown that $t_M = 6.7677574552$, $t_m = 2.14242886$. Thus $s_1 = 4.288544701$, $s_2 = 16.22052397$. It is easy to verify the assertions ($\mathbf{i}_1$) – ($\mathbf{i}_5$). Also, the relations like those given by (84) and (85) can be verified.

*Concluding Remark.* The main result of the present paper refers to the given definition of characteristic points for two nested circles and their properties useful in research of bicentric polygons. Some old and difficult problems are solved. It seems that there are many problems concerning bicentric polygons for which the characteristic points can be very useful. In this connection we remark that the characteristic points can be also useful in research of bicentric $2n$-gons from the class $C_{2n}(R_i, r_i, d_i)$ which is obtained from the class $C_n(R_0, r_0, d_0)$ using function $f_1$ and $f_2$ given in Definition A. Some results from this area are given in Theorem 5, 6, 14. Also some results concerning functions $f_1$ and $f_2$ given in [7] are extended.

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