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Another Archimedean Circle in an Arbelos

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Abstract. The incircle of a triangle associated with an arbelos is Archimedean.

We make an addition to recent contributions to the Archimedean circles associated to an arbelos. See [1, 3, 4, 5] and the catalogue [2].

Consider an arbelos bounded by semicircles AB, AC, CB of radii $a+b$, a, b, and centers O, D, E respectively. Construct the semicircles with diameters AE, DB (and centers K, L respectively), and the common tangent of these semicircles touching AE at M, DB at N, and intersecting the semicircle AB at F and G (see Figure 1). If the tangents to the semicircle AB at F and G intersect at H, we prove that the incircle of triangle FGH is an archimedean circle of the arbelos, namely, its radius is $\frac{ab}{a+b}$.

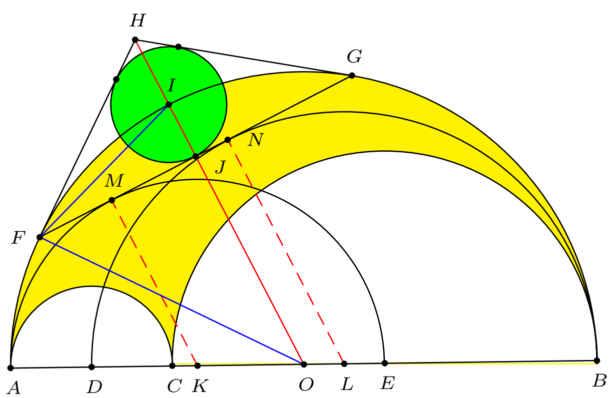

Figure 1

Let OH intersect the semicircle AB and the chord FG at I and J respectively. Since

$$\begin{aligned}\angle IFH = 90^\circ - \angle OFI &= 90^\circ - \frac{1}{2}(180^\circ - \angle IOF) \\ &= \frac{1}{2}\angle IOF = \frac{1}{2}\angle JOF = \frac{1}{2}\angle JFH.\end{aligned}$$

FI bisects angle GFH. Since I also lies on the bisector of angle FHG, it is the incenter of triangle FGH. The radius of the incircle of triangle FGH is $IJ = IO - OJ = (a+b) - OJ$.

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To find the length of OJ, note that it is parallel to both KM and LN of the trapezoid $KMNL$. Since $OL = \frac{a}{2}$ and $KO = \frac{b}{2}$, $OL : KO : KL = a : b : a + b$, and

$$\begin{aligned} OJ = &\ \frac{OL}{KL} \cdot KM + \frac{KO}{KL} \cdot LN \\ = &\ \frac{a}{a+b}\left(a + \frac{b}{2}\right) + \frac{b}{a+b}\left(\frac{a}{2} + b\right) \\ = &\ \frac{a^2 + ab + b^2}{a+b}. \end{aligned}$$

It follows that

$$IJ = (a+b) - \frac{a^2 + ab + b^2}{a+b} = \frac{ab}{a+b}.$$

This is the radius of an Archimedean circle in the arbelos.

References

[1] T. O. Dao, Two pairs of Archimedean circles in the arbelos, *Forum Geom.*, 14 (2014) 201–202.
[2] F. M. van Lamoen, Online catalogue of Archimedean circles, `http://home.kpn.nl/lamoen/wiskunde/Arbelos/Catalogue.htm`
[3] F. M. van Lamoen, A special point in the Arbelos leading to a pair of Archi-medean circles, *Forum Geom.*, 14 (2014) 253–254.
[4] Q. H. Tran, Two more pairs of Archimedean circles in the arbelos, *Forum Geom.*, 14 (2014) 249–251.
[5] P. Yiu, Three constructions of Archimedean circles in an arbelos, *Forum Geom.*, 14 (2014) 255–260.

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