

Exactly Solvable Potentials Derived from SWKB Quantization[†]

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The shape invariant potentials are proved to be exactly solvable, *i.e.* the wave functions and energies of a particle moving under the influence of the shape invariant potentials can be algebraically determined without any approximations. It is well known that the SWKB quantization is exact for all shape invariant potentials though the SWKB quantization itself is approximate. This mystery has not been mathematically resolved yet and may not be solved in a concrete fashion even in the future. Therefore, in the present work, to understand (not prove) the mystery an attempt of deriving exactly solvable potentials directly from the SWKB quantization has been made. And it turns out that all the derived potentials are shape invariant. It implicitly explains why the SWKB quantization is exact for all known shape invariant potentials. Though any new potential has not been found in this study, this brute-force derivation of potentials helps one understand the characteristics of shape invariant potentials.

Key Words : SWKB quantization, Solvable potentials, Shape invariant potentials

Introduction

In quantum mechanics it is important to find exactly solvable potentials and understand their characteristics.¹ Of course, up to today not all exactly solvable potentials have been found. An interesting class of solvable potentials is a set of potentials so called “shape invariant potentials (SIP).” All shape invariant potentials are proved to be exactly solvable.^{2,3} Of course there are other exactly solvable potentials that are not shape invariant, for example, Ginocchio potential.⁴ There are two types of SIP. One is the shape invariant potentials by translation (TSIP) and the other is the shape invariant potentials by scaling (CSIP). There are no CSIPs with closed form found yet but many TSIPs with closed form have been found. All TSIPs known, *up to today*, are listed in Table 1.^{3,5}

Another side of understanding quantum mechanics is to investigate the so called “quantization”. As well known, in early days of quantum mechanics “Bohr quantization condition” or “Sommerfeld quantization condition” played a key role in development of quantum physics.⁶ The exact quantization rule (or condition) for nonrelativistic one-dimensional quantum system has been found rather recently.⁷⁻¹²

The one-dimensional nonrelativistic Schrödinger equation for a bound state is

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \Psi_n(x) + V(x) \Psi_n(x) = E_n \Psi_n(x) \quad (1)$$

where m is the mass of the particle and \hbar is the Planck constant divided by 2π . n is the quantum number or a number of node in wave function $\Psi_n(x)$. $V(x)$ is a potential energy function. E_n is the energy for state n . In this work for

convenience the ground state energy is set to be zero, *i.e.* $E_0 = 0$. Without losing generality a proper scaling of coordinate yields a simpler form of Schrödinger equation, *i.e.*

$$-\frac{d^2}{dx^2} \Psi_n(x) + V(x) \Psi_n(x) = E_n \Psi_n(x). \quad (2)$$

Let $p_n(x) \equiv \sqrt{E_n - V(x)}$ be the classical momentum function for an energy E_n , then the exact quantization rule is

$$\int_{x_{1,n}}^{x_{2,n}} p_n(x) dx = \int_{x_{1,n}}^{x_{2,n}} \sqrt{E_n - V(x)} dx = [n + \gamma(E_n)] \pi \quad (3)$$

with

$$\gamma(E_n) = 1 + \frac{1}{\pi} \int_{x_{1,n}}^{x_{2,n}} \frac{\phi_n(x)}{d\phi_n(x)/dx} \left(\frac{dp_n(x)}{dx} \right) dx. \quad (4)$$

$x_{1,n}$ and $x_{2,n}$ are two classical turning points ($x_{1,n} < x_{2,n}$), *i.e.* $V(x_{1,n}) = V(x_{2,n}) = E_n$, and $\phi_n(x) = (d\Psi_n(x)/dx)/\Psi_n(x)$ is the log derivative of wave function $\Psi_n(x)$.

For one-dimensional quantum system, the most well known and widely applied quantization is the WKB (Wentzel, Kramers, Brillouin) quantization.^{13,14} It is an approximate quantization but has been widely used to study many one-dimensional quantum systems. The WKB quantization is

$$\int_{x_{1,n}}^{x_{2,n}} \sqrt{E_n - V(x)} dx = \left(n + \frac{1}{2} \right) \pi. \quad (5)$$

The WKB quantization has been modified to yield the so called supersymmetric WKB (SWKB) quantization that is more accurate than the WKB quantization,^{3,15,16} *i.e.*

$$\int_{x'_{1,n}}^{x'_{2,n}} \sqrt{E_n - W^2(x)} dx = n\pi. \quad (6)$$

where $W(x)$ is the well known superpotential, *i.e.*

[†]This paper is to commemorate Professor Myung Soo Kim's honourable retirement.

$$W(x) = -\phi_0(x) = -\frac{d\Psi_0(x)/dx}{\Psi_0(x)} \quad (7)$$

and $W^2(x'_{1,n}) = W^2(x'_{2,n}) = E_n$.

The WKB quantization is exact for Harmonic oscillator potential and Morse potential for which the correction term in Eq. (4) turns out to be always 1/2 for any state n .¹⁷ The SWKB quantization is known to be exact for all known TSIPs listed in Table 1, of course including Harmonic

Table 1. All known shape invariant potentials by translation (TSIPs). The energies (E_n) are defined in Eq. (2) with $E_0 = 0$. The constants $A, B, b, \alpha, \omega, l$ are all taken ≥ 0 . Unless otherwise stated, the range of potentials is $-\infty \leq x \leq \infty, 0 \leq r \leq \infty$. Table 4.1 in Ref. 3 is reconstructed here for the purpose of discussion

Harmonic oscillator

$$V(x) = \frac{1}{4}\omega^2\left(x - \frac{2b}{\omega}\right)^2 - \frac{1}{2}\omega$$

$$E_n = n\omega$$

Morse

$$V(x) = A^2 + B^2 \exp(-2\alpha x) - 2B(A + \alpha/2) \exp(-\alpha x)$$

$$E_n = A^2 - (A - n\alpha)^2$$

Coulomb

$$V(r) = -\frac{e^2}{r} + \frac{l(l+1)}{r^2} + \frac{e^4}{4(l+1)^2}$$

$$E_n = \frac{e^4}{4} \left[\frac{1}{(l+1)^2} - \frac{1}{(n+l+1)^2} \right]$$

3-D Harmonic oscillator

$$V(r) = \frac{1}{4}\omega^2 r^2 + \frac{l(l+1)}{r^2} - \left(l + \frac{3}{2}\right)\omega$$

$$E_n = 2n\omega$$

Rosen-Morse I (trigonometric)

$$V(x) = A(A-\alpha) \operatorname{cosec}^2 \alpha x + 2B \cot \alpha x - A^2 + B^2/A^2 \quad (0 \leq \alpha x \leq \pi)$$

$$E_n = (A + n\alpha)^2 - A^2 + B^2/A^2 - B^2/(A + n\alpha)^2$$

Rosen-Morse II (hyperbolic)

$$V(x) = A^2 + B^2/A^2 - A(A + \alpha) \operatorname{sech}^2 \alpha x + 2B \tanh \alpha x \quad (B < A^2)$$

$$E_n = A^2 - (A - n\alpha)^2 + B^2/A^2 - B^2/(A - n\alpha)^2$$

Eckart

$$V(x) = A^2 + B^2/A^2 + (A - \alpha) \operatorname{cosech}^2 \alpha x - 2B \coth \alpha x \quad (B > A^2)$$

$$E_n = A^2 - (A + n\alpha)^2 + B^2/A^2 - B^2/(A + n\alpha)^2$$

Scarf I (trigonometric)

$$V(x) = -A^2 + (A^2 + B^2 - A\alpha) \sec^2 \alpha x - B(2A - \alpha) \tan \alpha x \sec \alpha x$$

$$\left(-\frac{1}{2}\pi \leq \alpha x \leq \frac{1}{2}\pi, A > B\right)$$

$$E_n = (A + n\alpha)^2 - A^2$$

Scarf II (hyperbolic)

$$V(x) = A^2 + [B^2 - A(A + \alpha)] \operatorname{sech}^2 \alpha x + B(2A + \alpha) \operatorname{sech} \alpha x \tanh \alpha x$$

$$E_n = A^2 - (A - n\alpha)^2$$

Generalized Pöschl-Teller

$$V(r) = A^2 + (B^2 + A^2 + A\alpha) \operatorname{cosech}^2 \alpha r - B(2A + \alpha) \coth \alpha r \operatorname{cosech} \alpha r$$

$$(A < B)$$

$$E_n = A^2 - (A - n\alpha)^2$$

oscillator and Morse potentials.¹⁸ Furthermore the correction term in Eq. (4) is known to be constant (independent of n), i.e. for TSIPs,

$$\gamma \equiv \gamma(E_n) = \gamma(E_0) = 1 + \frac{1}{\pi} \int_{x_{1,0}}^{x_{2,0}} \frac{W(x)p'_0(x)}{W'(x)} dx. \quad (8)$$

$p'_0(x) = dp_0(x)/dx$ is the derivative of the momentum function for the ground state ($n=0$), and $W'(x) = dW(x)/dx$ is the derivative of superpotential defined in Eq. (7).

A question of interest is “why does the SWKB quantization (6) yield the exact energies E_n for all TSIPs?” We have examined this puzzling question for a long time but failed in providing an algebraically self consistent answer to the question. To our knowledge, as a matter of fact, no one else has ever succeeded in mathematically proving the fact – “The SWKB quantization is exact for all TSIPs.” Recalling that even the exact quantization rule (3) cannot be algebraically derived solely from the Schrödinger Eq. (2), it may be no wonder why no algebraic proof has been presented.

In the current work we suggest an alternative way of understanding, if not proving, the exactness of SWKB quantization for TSIPs. It can be achieved by finding potentials satisfying both the exact quantization (3) and the SWKB quantization (6) simultaneously and exactly. Of course there are no general ways of determining the integrand from integral form of equation like the quantization expressions given above. But under certain conditions, we find that it is possible to derive a rather general form of potentials from the above integral form of quantization. Our aim is double fold, i.e. i) if all TSIPs are recovered, it will definitely confirm that the SWKB quantization is indeed exact for all TSIPs and ii) if any potential other than the known TSIPs is found, it will be a great achievement because it will give an insight on understanding the physical meaning of the SWKB quantization. Furthermore if the new potential found is shape invariant, it will break the long-believed-notion that all TSIPs have already been found.

In the next section, how to derive an exactly solvable potential from the SWKB quantization is explained by taking a simple example. All the derived potentials are listed in the following section. The findings and limits of the current study are also presented in the same section. Conclusion is provided in the final section.

Derivation of Exactly Solvable Potential from SWKB Quantization

The main purpose of this work is to find exactly solvable potentials satisfying the exact quantization (3) and the SWKB quantization (6) simultaneously. There are certain constraints (or conditions) implicitly embedded in the both quantizations. i) The quantizations should be used for a smoothly varying potential function that has one well. In another words the quantizations are valid for potentials that have only two classical turning points. ii) The correction term $\gamma(E_n)$ should be independent of n in order to determine a potential $V(x)$ simultaneously satisfying the both quantizations (3) and (6).

If the correction term depends on n , there is no way of determining the potential algebraically. See Eq. (8). Consequently the derived potential must meet the conditions i) and ii).

For a potential $V(x)$ whose ground (lowest) bound state energy is zero, *i.e.* $E_0 = 0$, the following relationship from supersymmetry algebra always holds³ (with Eq. (2))

$$V(x) = W^2(x) - W'(x). \quad (9)$$

If we find a superpotential $W(x)$ satisfying both the exact quantization (3) with (8) and the SWKB quantization (6) simultaneously, the potential $V(x)$ will be determined through Eq. (9). Unfortunately there is no general way of finding all $W(x)$. Furthermore for many forms of $W(x)$, the definite integral in (3) or (6) cannot be evaluated. Therefore we have designed an alternative way that is very practical though it might not be complete.

Firstly a transformation of the variable x into a new variable y using $g(y) \equiv dy/dx$ is performed. Then the exact quantization is

$$\int_{y_{1,n}}^{y_{2,n}} \frac{dy}{g(y)} \sqrt{E_n - V(y)} = n\pi + \gamma\pi \quad (10)$$

where $V(y_{1,n}) = V(y_{2,n}) = E_n$. The SWKB quantization is

$$\int_{y'_{1,n}}^{y'_{2,n}} \frac{dy}{g(y)} \sqrt{E_n - W^2(y)} = n\pi \quad (11)$$

where $W^2(y'_{1,n}) = W^2(y'_{2,n}) = E_n$. And the relationship (9) between a potential and its superpotential is now

$$V(y) = W^2(y) - g(y)W'(y). \quad (12)$$

Secondly let us guess the forms (mathematical expressions) for $W(y)$ and $g(y)$. Then the potential $V(y)$ is naturally determined from Eq. (12). With $V(y)$ and $g(y)$ one can evaluate the energies E_n using the exact quantization (10). At the same time with $W(y)$ and $g(y)$ one can also evaluate the energies E_n using the SWKB quantization (11). If the two E_n are equal to each other, it ensures that the potential $V(y)$ is exactly solvable and the SWKB quantization is exact for $V(y)$. In order to make the two E_n simultaneously satisfy (10) and (11), there should exist some conditions (or relationships among parameters) which must be determined. Finally a back transformation of y to x gives a potential $V(x)$ for which the SWKB quantization is exact.

For example, let

$$W(y) = \tau_1 y + \tau_0 \quad (13)$$

and

$$g(y) = a_1 y + a_0. \quad (14)$$

Then, from Eq. (12), the form of the potential $V(y)$ must be

$$V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0 \quad (15)$$

where

$$\sigma_2 = \tau_1^2, \quad \sigma_1 = 2\tau_1\tau_0 - a_1\tau_1, \quad \sigma_0 = \tau_0^2 - \sigma_0\tau_1. \quad (16)$$

From Eq. (16) one obtains the following relationship, *i.e.*

$$\sigma_0 = \frac{\sigma_1^2}{4} + \frac{a_1^2}{4} + \frac{\sigma_1 a_1}{2\sqrt{\sigma_2}} - a_0\sqrt{\sigma_2} \quad (\tau_1 > 0) \quad (17)$$

and

$$\sigma_0 = \frac{\sigma_1^2}{4} + \frac{a_1^2}{4} - \frac{\sigma_1 a_1}{2\sqrt{\sigma_2}} + a_0\sqrt{\sigma_2} \quad (\tau_1 < 0). \quad (18)$$

First let us determine the energies E_n using the exact quantization (10). The L.H.S. of (10) is

$$\begin{aligned} \text{L.H.S.} &= \int_{y_{1,n}}^{y_{2,n}} \frac{dy}{a_1 y + a_0} \sqrt{E_n - [\sigma_2 y^2 + \sigma_1 y + \sigma_0]} \\ &= \sqrt{\sigma_2} \int_{y_{1,n}}^{y_{2,n}} \frac{dy}{a_1 y + a_0} \sqrt{\left(y - \frac{-\sigma_1 - \sqrt{\sigma_1^2 - 4\sigma_2\sigma_0 + 4\sigma_2 E_n}}{2\sigma_2} \right) \left(y - \frac{-\sigma_1 + \sqrt{\sigma_1^2 - 4\sigma_2\sigma_0 + 4\sigma_2 E_n}}{2\sigma_2} \right)} \\ &= \sqrt{\sigma_2} \int_a^b \frac{dy}{cy + d} \sqrt{(y-a)(b-y)} \end{aligned} \quad (19)$$

where

$$a = y_{1,n} = \frac{-\sigma_1 - \sqrt{\sigma_1^2 - 4\sigma_2\sigma_0 + 4\sigma_2 E_n}}{2\sigma_2}, \quad b = y_{2,n} = \frac{-\sigma_1 + \sqrt{\sigma_1^2 - 4\sigma_2\sigma_0 + 4\sigma_2 E_n}}{2\sigma_2},$$

$c = a_1$, and $d = a_0$. Obviously $a < b$ is satisfied.

The above definite integral (19) can be evaluated by using the formulas,

$$\begin{aligned} &\int_a^b \frac{dy}{cy + d} \sqrt{(y-a)(b-y)} \\ &= \frac{\pi}{c^2} \left[\frac{1}{2}(ac+d) + \frac{1}{2}(bc+d) + \sqrt{(ac+d)(bc+d)} \right] \end{aligned} \quad (20)$$

when $a < b$, $ac + d < 0$, and $bc + d < 0$ or

$$\begin{aligned} &\int_a^b \frac{dy}{cy + d} \sqrt{(y-a)(b-y)} \\ &= \frac{\pi}{c^2} \left[\frac{1}{2}(ac+d) + \frac{1}{2}(bc+d) - \sqrt{(ac+d)(bc+d)} \right] \end{aligned} \quad (21)$$

when $a < b$, $ac + d > 0$, and $bc + d > 0$. Based on the already known integral formulas in Refs. 8 and 19, the above formulas (20) and (21) are derived by us.

When $ac + d < 0$ and $bc + d < 0$, using Eq. (20),

$$\text{L.H.S.} = \frac{\pi\sqrt{\sigma_2}}{a_1^2} \left[\left\{ \frac{1}{2} \left(\frac{\sigma_1}{\sigma_2} \right) a_1 + a_0 \right\} + \sqrt{\left(\frac{\sigma_1}{\sigma_2} \right) a_1 a_0 + \left(\frac{\sigma_0}{\sigma_2} - \frac{E_n}{\sigma_2} \right) a_1^2 + a_0^2} \right].$$

Then the exact quantization (10) is

$$\frac{\pi\sqrt{\sigma_2}}{a_1^2} \left[\left\{ \frac{1}{2} \left(\frac{\sigma_1}{\sigma_2} \right) a_1 + a_0 \right\} + \sqrt{\left(\frac{\sigma_1}{\sigma_2} \right) a_1 a_0 + \left(\frac{\sigma_0}{\sigma_2} - \frac{E_n}{\sigma_2} \right) a_1^2 + a_0^2} \right] = n\pi + \gamma\pi.$$

Solving it for E_n when $\tau_1 > 0$, one obtains energies with the aid of Eq. (17),

$$E_n = \left[-\frac{a_0\sqrt{\sigma_2}}{a_1} + \frac{\sigma_1}{2\sqrt{\sigma_2}} + \frac{1}{2}a_1 \right]^2 - \left[-\frac{a_0\sqrt{\sigma_2}}{a_1} + \frac{\sigma_1}{2\sqrt{\sigma_2}} + (n+\gamma)a_1 \right]^2. \quad (22)$$

When $ac + d > 0$ and $bc + d > 0$, instead of Eq. (20) the formula (21) should be used. But the resulting energy expression is the same as (22).

Now let us determine the energies E_n using the SWKB quantization (11). The L.H.S. of (11) is, when $\tau_1 > 0$,

$$\begin{aligned} \text{L.H.S.} &= \int_{y'_{1,n}}^{y'_{2,n}} \frac{dy}{a_1 y + a_0} \sqrt{E_n - (\tau_1 y + \tau_0)^2} \\ &= \tau_1 \int_{y'_{1,n}}^{y'_{2,n}} \frac{dy}{a_1 y + a_0} \sqrt{\left[y - \left(-\frac{\tau_0}{\tau_1} - \frac{\sqrt{E_n}}{\tau_1} \right) \right] \left[\left(-\frac{\tau_0}{\tau_1} + \frac{\sqrt{E_n}}{\tau_1} \right) - y \right]} \\ &= \tau_1 \int_a^b \frac{dy}{cy + d} \sqrt{(y-a)(b-y)} \end{aligned} \quad (23)$$

where

$$a = y'_{1,n} = -\frac{\tau_0}{\tau_1} - \frac{\sqrt{E_n}}{\tau_1}, \quad b = y'_{2,n} = -\frac{\tau_0}{\tau_1} + \frac{\sqrt{E_n}}{\tau_1}, \quad c = a_1, \quad \text{and} \quad d = a_0.$$

Obviously $a < b$ is satisfied.

When $ac + d < 0$ and $bc + d < 0$, using Eq. (20),

$$\text{L.H.S.} = \frac{\tau_1 \pi}{a_1^2} \left[\frac{1}{2} \left(-\frac{2\tau_0}{\tau_1} \right) a_1 + a_0 + \sqrt{\left(-\frac{2\tau_0}{\tau_1} \right) a_1 a_0 + \left[\left(\frac{\tau_0}{\tau_1} \right)^2 - \frac{E_n}{\tau_1^2} \right] a_1^2 + a_0^2} \right].$$

Then the SWKB quantization (11) is

$$\frac{\tau_1 \pi}{a_1^2} \left[\frac{1}{2} \left(-\frac{2\tau_0}{\tau_1} \right) a_1 + a_0 + \sqrt{\left(-\frac{2\tau_0}{\tau_1} \right) a_1 a_0 + \left[\left(\frac{\tau_0}{\tau_1} \right)^2 - \frac{E_n}{\tau_1^2} \right] a_1^2 + a_0^2} \right] = n\pi.$$

Solving it for E_n , one obtains energies with the aid of Eq. (17),

$$E_n = \left[-\frac{a_0\sqrt{\sigma_2}}{a_1} + \frac{\sigma_1}{2\sqrt{\sigma_2}} + \frac{1}{2}a_1 \right]^2 - \left[-\frac{a_0\sqrt{\sigma_2}}{a_1} + \frac{\sigma_1}{2\sqrt{\sigma_2}} + \frac{1}{2}a_1 + na_1 \right]^2 \quad (24)$$

When $ac + d > 0$ and $bc + d > 0$, the resulting energy expression is again the same as Eq. (24).

Comparing the SWKB E_n (24) with the exact E_n (22), one immediately finds that the two expressions are identical when $\gamma = 1/2$. It verifies that the potential $V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0$ (15) (with Eqs. (13), (14) and (16)) satisfies the exact quantization (10) and the SWKB quantization (11) simultaneously. Consequently we have successfully found an exactly solvable potential from the SWKB quantization.

Integration of $g(y)$ given in Eq. (14) yields

$$y = C \exp(a_1 x) - (a_0/a_1) \quad (25)$$

where C is an arbitrary constant. Therefore the potential $V(x)$ is

$$\begin{aligned} V(x) &= \sigma_2 [C \exp(a_1 x) - a_0/a_1]^2 + \sigma_1 [C \exp(a_1 x) - a_0/a_1] + \sigma_0 \\ &= \sigma_2 C^2 \exp(2a_1 x) + (\sigma_1 - 2\sigma_2 a_0/a_1) C \exp(a_1 x) \\ &\quad + \sigma_0 - \sigma_1 a_0/a_1 + \sigma_2 (a_0/a_1)^2. \end{aligned} \quad (26)$$

Redefining parameters as

$$A = -\frac{a_0\sqrt{\sigma_2}}{a_1} + \frac{\sigma_1}{2\sqrt{\sigma_2}} + \frac{1}{2}a_1, \quad B = -\tau_1 C, \quad \text{and} \quad \alpha = -a_1,$$

one obtains with the aid of Eq. (16)

$$V(x) = B^2 \exp(-2\alpha x) - 2B(A + \alpha/2) \exp(-\alpha x) + A^2, \quad (27)$$

$$E_n = A^2 - (A - n\alpha)^2. \quad (28)$$

Therefore, in this example, one has successfully derived a potential (27) for which the SWKB quantization is exact. It turns out that the potential (27) is none other than Morse potential (See Table 1). Of course, the same procedure can be utilized for other choices of $W(y)$ and $g(y)$.

All Exactly Solvable Potentials from SWKB Quantization

In the previous section an example showing how to derive an exactly solvable potential from the SWKB quantization is presented. With the choice (or guess) of $W(y) = \tau_1 y + \tau_0$ and $g(y) = a_1 y + a_0$, we have obtained a potential of $V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0$. There may be other forms of $W(y)$ and $g(y)$ that may yield the same $V(y)$. For example, the same $V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0$ form can be obtained from the choice of $W(y) = \tau_1 y + \tau_0$ and $g(y) = a_2 y^2 + \sigma_1 y + a_0$. This choice must produce a different potential. Therefore one has to make all possible choices of $W(y)$ and $g(y)$ to derive all the potentials satisfying the quantizations.

The systematic choice was made as follows. Recall that $V(y)$ and $W^2(y)$ must have one well with two turning points (condition i) in the previous section).

[1] The simplest choice for $V(y)$ is obviously a form of quadratic function, i.e. $V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0$ ($\sigma_2 > 0$) for $(-\infty \leq y \leq \infty)$. The simplest choice for $W^2(y)$ is also a quadratic function so that $W(y)$ is a linear function, i.e. $W(y) = \tau_1 y + \tau_0$. Then Eq. (12) gives the following equation

$$\sigma_2 y^2 + \sigma_1 y + \sigma_0 = (\tau_1 y + \tau_0)^2 - \tau_1 g(y). \quad (29)$$

So that there are three possible forms for $g(y)$ to satisfy Eq. (29), i.e. $g(y) = a_0$, $g(y) = a_1 y + a_0$ or $g(y) = a_2 y^2 + a_1 y + a_0$.

In summary there are three possible choices for a potential whose form is $V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0$.

(I) The first choice is $V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0$, $W(y) = \tau_1 y + \tau_0$, and $g(y) = a_0$. For this choice, following the procedure discussed in the previous section, one finds that the potential $V(x)$ turns out to be Harmonic oscillator potential.

(II) The second choice is $V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0$, $W(y) = \tau_1 y + \tau_0$, and $g(y) = a_1 y + a_0$. This is the case discussed in the previous section as an example. As already shown in Eq. (27), the potential $V(x)$ is Morse potential.

For the third choice of $V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0$, $W(y) = \tau_1 y + \tau_0$, and $g(y) = a_2 y^2 + a_1 y + a_0$, depending on parameters a_2 , a_1 and a_0 , four possible forms of $V(x)$ are derived, i.e.

(III) when $a_1^2 - 4a_2 a_0 = 0$, $V(x)$ is Coulomb potential,

(IV) when $a_1^2 - 4a_2 a_0 < 0$, $V(x)$ is Rosen-Morse I potential,

(V) when $a_1^2 - 4a_2 a_0 > 0$ and $(2a_2 y + a_1)^2 < a_1^2 - 4a_2 a_0$, $V(x)$

is Rosen-Morse II potential, and

(VI) when $a_1^2 - 4a_2a_0 > 0$ and $(2a_2y + a_1)^2 > a_1^2 - 4a_2a_0$, $V(x)$ is Eckart potential.

[2] Besides a quadratic function there could be another form of potential that has one well with two turning points, i.e. $V(y) = \sigma_2 y^2 + \sigma_0 + \sigma_{-2} y^{-2}$ ($\sigma_2 > 0$, $\sigma_{-2} > 0$, $\sigma_0 < 0$) for $(0 \leq y \leq \infty)$. Then the simple choice for $W(y)$ is $W(y) = \tau_1 y + \tau_{-1} y^{-1}$ ($\tau_1 \neq 0$, $\tau_{-1} \neq 0$). The Eq. (12) yields $g(y) = a_0$ or $g(y) = a_2 y^2 + a_0$ ($a_2 \neq 0$). In summary there are two possible choices when $V(y) = \sigma_2 y^2 + \sigma_0 + \sigma_{-2} y^{-2}$ is assumed.

(VII) The first choice is $V(y) = \sigma_2 y^2 + \sigma_0 + \sigma_{-2} y^{-2}$, $W(y) = \tau_1 y + \tau_{-1} y^{-1}$, and $g(y) = a_0$. For this choice, one finds that the potential $V(x)$ turns out to be 3-D Harmonic oscillator potential.

For the second choice of $V(y) = \sigma_2 y^2 + \sigma_0 + \sigma_{-2} y^{-2}$, $W(y) = \tau_1 y + \tau_{-1} y^{-1}$, and $g(y) = a_2 y^2 + a_0$, depending on parameters a_2 and a_0 , three possible forms of $V(x)$ are derived, i.e.

(VIII) when $a_2 a_0 > 0$, $V(x)$ is Scarf I potential,

(IX) when $a_2 a_0 < 0$ and $(2a_2 y)^2 < -4a_2 a_0$, $V(x)$ is Scarf II potential, and

(X) when $a_2 a_0 < 0$ and $(2a_2 y)^2 > -4a_2 a_0$, $V(x)$ is Generalized Pöschl-Teller potential.

The derived 10 (I through X) potentials for which the SWKB quantization is exact are summarized in Table 2. Comparing with Table 1, one finds that the derived 10 potentials are exactly the same (neither one less nor one more) as the known ten TSIPs in Table 1. It indeed verifies the fact that the SWKB quantization is exact for all known TSIPs.

[3] An immediate question arising is if there are any other exactly solvable potential forms that might be derived from the SWKB quantization. Of course there could be a lot more. In the choices of (I) through (VI) the potential was $V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0$ and the superpotential was $W(y) = \tau_1 y + \tau_0$. When $V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0$ is chosen, another possible

form of $W(y)$ is $W(y) = \tau_1 y + \tau_{-1} y^{-1}$. Then there are three possible forms for $g(y)$, i.e. $g(y) = -\tau_{-1}$, $g(y) = a_2 y^2 - \tau_{-1}$, and $g(y) = a_2 y^2$.

(XI) The first choice is $V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0$, $W(y) = \tau_1 y + \tau_{-1} y^{-1}$, and $g(y) = -\tau_{-1}$. For this choice, $V(x)$ turns out to be 3D-Harmonic oscillator potential with one less number of parameters. The number of parameters is 5 (σ_2 , σ_1 , σ_0 , τ_1 , τ_{-1}) while there are 6 parameters (σ_2 , σ_1 , σ_0 , τ_1 , τ_{-1} , a_0) in the case (VII) that produces the same 3D-Harmonic oscillator potential. Therefore it has one less number of parameters.

The second choice is $V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0$, $W(y) = \tau_1 y + \tau_{-1} y^{-1}$, and $g(y) = a_2 y^2 - \tau_{-1}$. This choice yields three forms of $g(y)$ depending on the sign of $a_2 \tau_{-1}$.

(XII) When $a_2 \tau_{-1} < 0$, $V(x)$ is Scarf I potential with one less number of parameters.

(XIII) When $a_2 \tau_{-1} > 0$ and $(2a_2 y)^2 > 4a_2 \tau_{-1}$, $V(x)$ is Generalized Pöschl-Teller potential with one less number of parameters.

(XIV) When $a_2 \tau_{-1} > 0$ and $(2a_2 y)^2 < 4a_2 \tau_{-1}$, $V(x)$ is Scarf II potential with one less number of parameters.

(XV) The third choice is $V(y) = \sigma_2 y^2 + \sigma_0 + \sigma_{-2} y^{-2}$, $W(y) = \tau_1 y + \tau_{-1} y^{-1}$, and $g(y) = a_2 y^2$. For this choice $V(x)$ is 3-D Harmonic oscillator with one less number of parameters.

The choices of (XI) through (XV) yielded 5 more potentials but they all belong to known TSIPs. No new potentials are found.

[4] Besides $V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0$ ($\sigma_2 > 0$) and $V(y) = \sigma_2 y^2 + \sigma_0 + \sigma_{-2} y^{-2}$, there might be another forms of $V(y)$ that has one well with two turning points. We have considered two more possible forms as follows.

(XVI) When $V(y) = \sigma_m y^m + \sigma_0 + \sigma_{-m} y^{-m}$, $W(y) = \tau_m y^{m/2} + \tau_{-m} y^{-m/2}$, and $g(y) = a_m y^{m/2+1}$ ($m = 3, 4, 5, \dots$), $V(x)$ is 3-D Harmonic oscillator potential. It is simply the generalization of the case (XV).

(XVII) When $V(y) = \sigma_1 |y| + \sigma_0$, $W(y) = \tau_1 |y|^{1/2}$, and $g(y) = a_0 |y|^{1/2}$, $V(x)$ is Harmonic oscillator potential with one less number of parameters.

(XVIII) When $V(y) = \sigma_1 |y| + \sigma_0$, $W(y) = \tau_1 |y|^{1/2}$, and $g(y) = a_2 |y|^{3/2} + a_0 |y|^{1/2}$ with $a_2 a_0 > 0$, is Rosen-Morse I potential with one less number of parameters.

(XIX) When $V(y) = \sigma_1 |y| + \sigma_0$, $W(y) = \tau_1 |y|^{1/2}$, and $g(y) = a_2 |y|^{3/2} + a_0 |y|^{1/2}$ with $a_2 a_0 < 0$, $V(x)$ is Rosen-Morse II potential with one less number of parameters.

Again the potentials derived are none other than known TSIPs. In summary the choices of (XI) through (XIX) did not yield any new potentials.

[5] Finally more possible forms satisfying the two constraints (i) and (ii) (presented in the previous section) and Eq. (12) are chosen and examined. They are

(XX) $V(y) = \sigma_2 y^2 + \sigma_0 + \sigma_{-2} y^{-2}$, $W(y) = \tau_0 y + \tau_{-1} y^{-1}$, $g(y) = a_4 y^4 + a_3 y^3 + a_2 y^2 - 2\tau_0 y - \tau_{-1}$;

(XXI) $V(y) = \sigma_2 y^2 + \sigma_0 + \sigma_{-2} y^{-2}$, $W(y) = \tau_1 y$, $g(y) = a_0 + a_{-2} y^{-2}$ (or $g(y) = a_2 y^2 + a_0 + a_{-2} y^{-2}$);

(XXII) $V(y) = \sigma_2 y^2 + \sigma_0 + \sigma_{-2} y^{-2}$, $W(y) = \tau_{-1} y^{-1}$, $g(y) = a_4 y^4 + a_2 y^2 + a_0$;

(XXIII) $V(y) = \sigma_2 y^2 + \sigma_0 + \sigma_{-2} y^{-2}$, $W(y) = \tau_1 y + \tau_0$, $g(y) = 2\mu_0 y + a_0 + a_{-2} y^{-2}$ (or $g(y) = 2\mu_0 y + a_2 y^2 + a_0 + a_{-2} y^{-2}$);

Table 2. Exactly solvable potentials $V(x)$ derived from the SWKB quantization (6). $V(y)$ and $W(y)$ are, respectively, a potential and a superpotential in y with $g(y) \equiv dy/dx$

Choices of $W(y)$ and $g(y)$	Potential $V(x)$
[1] $V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0$ ($\sigma_2 > 0$) ($-\infty \leq y \leq \infty$)	
$W(y) = \tau_1 y + \tau_0$ ($\tau_1 \neq 0$)	
$g(y) = a_0$ ($a_0 \neq 0$)	Harmonic oscillator
$g(y) = a_1 y + a_0$ ($a_1 \neq 0$)	Morse
$g(y) = a_2 y^2 + a_1 y + a_0$ ($a_2 \neq 0$)	
$a_1^2 - 4a_2 a_0 = 0$	Coulomb
$a_1^2 - 4a_2 a_0 < 0$	Rosen-Morse I
$a_1^2 - 4a_2 a_0 > 0$ & $(2a_2 y + a_1)^2 < a_1^2 - 4a_2 a_0$	Rosen-Morse II
$a_1^2 - 4a_2 a_0 > 0$ & $(2a_2 y + a_1)^2 > a_1^2 - 4a_2 a_0$	Eckart
[2] $V(y) = \sigma_2 y^2 + \sigma_0 + \sigma_{-2} y^{-2}$ ($\sigma_2 > 0$, $\sigma_{-2} > 0$, $\sigma_0 < 0$) ($0 \leq y \leq \infty$)	
$W(y) = \tau_1 y + \tau_{-1} y^{-1}$ ($\tau_1 \neq 0$, $\tau_{-1} \neq 0$)	
$g(y) = a_0$ ($a_0 \neq 0$)	3-D Harmonic oscillator
$g(y) = a_2 y^2 + a_0$ ($a_2 \neq 0$)	
$a_2 a_0 > 0$	Scarf I
$a_2 a_0 < 0$ & $(2a_2 y)^2 < -4a_2 a_0$	Scarf II
$a_2 a_0 < 0$ & $(2a_2 y)^2 > -4a_2 a_0$	Generalized Pöschl-Teller

(XXIV) $V(y) = \sigma_2 y^2 + \sigma_0 + \sigma_{-2} y^{-2}$, $W(y) = \tau_0 + \tau_{-1} y^{-1}$, $g(y) = -2\mu_0 y + a_4 y^4 + a_0$ (or $g(y) = -2\mu_0 y + a_4 y^4 + a_2 y^2 + a_0$).

The choices of (XX) through (XXIV) did not yield any potential function satisfying both the exact and SWKB quantizations. It is probably due to the form of $g(y) \equiv dy/dx$ which cannot be integrated.

Does the above derivation ([1] through [5]) exhaust all possible forms of $V(y)$ or $W(y)$? It is believed that the answer to the question is “yes”. Cooper *et al.*⁵ who first derived TSIPs listed in Table 1 actually guessed the forms of potentials within the constraints of shape invariance in a similar way utilized in this work. Since their pioneering work, no one has found any other TSIP. Therefore it is sure that the SWKB quantization is exact for all known TSIPs and there are not any other potentials exactly satisfying the SWKB quantization. What we have shown here is that the validity of the SWKB quantization is a sufficient condition for TSIPs but may not be a necessary condition.

Conclusion

From the SWKB quantization we have derived exactly solvable potentials. Another words we have succeeded in deriving potentials for which the SWKB quantization is exact. The derived potentials turn out to be identical with all known shape invariance potentials by translation. No other potentials are found. It explains (though not proves) why the SWKB quantization is exact for all known shape invariant potentials by translation.

Have we derived all exactly solvable potentials for which the SWKB quantization is exact? The answer to the question is perhaps “yes”. Of course there are other exactly solvable potentials, for example Ginocchio potential, for which the SWKB quantization is not exact at all. Ginocchio potential is not shape invariant and the correction term $\gamma(E_n)$ in Eq. (8) is not a constant (n -dependent). It implies that the n -dependency of $\gamma(E_n)$ has something to do with the validity of the SWKB quantization. Since the correction term has a quantum nature that does not appear in classical systems, it casts

a new understanding on quantum mechanics – quantum systems are known to be quantized but the pattern (or the way of appearance) of quantization is critical in deciding the exact solubility of the quantum systems.

Finally we would like to add a short but significant observation on shape invariant potentials. The potentials in Table 1 have been known to form a class called TSIP. But as shown in Table 2, the TSIPs can be further classified into two subclasses. The one subclass (Harmonic oscillator, Morse, Coulomb, Rosen-Morse I, Rosen-Morse II, Eckart) has a transformed form of $V(y) = \sigma_2 y^2 + \sigma_1 y + \sigma_0$ and the other subclass (3-D Harmonic oscillator, Scarf I, Scarf II, Generalized Pöschl-Teller) has a form of $V(y) = \sigma_2 y^2 + \sigma_0 + \sigma_{-2} y^{-2}$.

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