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Quantization Rule for Relativistic Klein-Gordon Equation

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Based on the exact quantization rule for the nonrelativistic Schrödinger equation, the exact quantization rule for the relativistic one-dimensional Klein-Gordon equation is suggested. Using the new quantization rule, the exact relativistic energies for exactly solvable potentials, e.g. harmonic oscillator, Morse, and Rosen-Morse II type potentials, are obtained. Consequently the new quantization rule is found to be exact for one-dimensional spinless relativistic quantum systems. Though the physical meanings of the new quantization rule have not been fully understood yet, the present formal derivation scheme may shed light on understanding relativistic quantum systems more deeply.

Key Words : Klein-Gordon equation, Quantization rule, Solvable potentials

Introduction

Since the early days of quantum mechanics, a set of quantization rules has been sought after. These rules supplement the correspondence principle and provide the quantum version of any given classical theory or a classical analog to a quantum system. Therefore, a quantization rule can be regarded as a more understandable description of a quantum system. Meanwhile there have been numerous efforts to find the exact solutions of quantum systems, which attracted much attention in the development of quantum mechanics. Along this line, the exact quantization rule for nonrelativistic system has been developed. It is found that the exact quantization rule method is a powerful tool in finding the eigenvalues of all solvable quantum systems. In the present work, the nonrelativistic quantization rule is extended to the relativistic quantum system without spin.

The exact quantization rule for nonrelativistic one-dimensional quantum system has been found rather recently.[1-5] The one-dimensional nonrelativistic Schrödinger equation for a bound state is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi_n(x)+V(x)\,\Psi_n(x)=E_n\,\Psi_n(x) \tag{1}$$

where m is the mass of the particle and is $\hbar$ the Planck constant divided by 2π. n is the quantum number or a number of node in wave function $\Psi_n(x)$.

Let $p_n(x)\equiv\sqrt{(2m/\hbar^2)[E_n-V(x)]}$ be the classical momentum function for an energy E_n, then the exact quantization rule is

$$\int_{x_{1,n}}^{x_{2,n}} p_n(x)dx=\int_{x_{1,n}}^{x_{2,n}}\sqrt{\frac{2m}{\hbar^2}[E_n-V(x)]}\,dx=n\pi+\gamma(E_n)/\hbar \tag{2}$$

or simply

$$\int_{x_{1,n}}^{x_{2,n}}\sqrt{2m[E_n-V(x)]}\,dx=n\pi\hbar+\gamma(E_n) \tag{3}$$

with

$$\gamma(E_n)=\pi\hbar+\hbar\int_{x_{1,n}}^{x_{2,n}}\frac{\phi_n(x)}{d\phi_n(x)/dx}\left(\frac{dp_n(x)}{dx}\right)dx\,. \tag{4}$$

$x_{1,n}$ and $x_{2,n}$ are two classical turning points ($x_{1,n} < x_{2,n}$), *i.e.* $V(x_{1,n}) = V(x_{2,n}) = E_n$, and $\phi_n(x) = (d\Psi_n(x)/dx)/\Psi_n(x)$ is the log derivative of wave function $\Psi_n(x)$. While in most of the previous works $\hbar = 2m = 1$ is assumed, they are kept in the current work for completeness.

The correction term $\gamma(E_n)$ depends on n, but for some exactly solvable potentials (e.g. all the known translationally shape invariant potentials) it is found to be independent of n,[4] i.e.

$$\gamma(E_n)=\gamma(E_0)=\pi\hbar+\hbar\int_{x_{1,0}}^{x_{2,0}}\frac{W(x)p'_0(x)}{W'(x)}dx\,. \tag{5}$$

$p'_0(x) = dp_0(x)/dx$ is the derivative of the momentum function for the ground state ($n=0$), and $W(x) = -\phi_0(x)$ is the well known superpotential. $W'(x) = dW(x)/dx$ is the derivative of superpotential.

The exact quantization rule (3) and (4) refers to exact wave functions. To calculate the energy using the quantization rule, exact wave functions should be predetermined so that the usage of the exact quantization rule is, in practice, very limited. Nonetheless for exactly solvable potentials, the exact energies obtained from the quantization rule have been reported.[5-10] The quantization rule cannot be algebraically derived from the Schrödinger equation but the Schrödinger equation is utilized in the process of deducing the rule.[2,3] It implies that there must be a certain relationship between the quantization rule and the formality of the Schrödinger equation.

The relativistic Klein-Gordon equation can be represented as a nonrelativistic Schrödinger equation-like equation, i.e. there is a formal similarity between the Klein-Gordon equation and the Schrödinger equation. Exploiting this similarity, one may find an exact quantization rule for the Klein-Gordon equation. In the following sections a new quantization rule for the relativistic Klein-Gordon equation will be suggested and tested against some exactly solvable potentials.

Quantization Rule for the Klein-Gordon Equation

For a spinless particle of rest mass m, the one-dimensional time-independent Klein-Gordon equation is

$$-\hbar^2 c^2 \frac{d^2}{dx^2}\Psi_n(x)+[mc^2+S(x)]^2\Psi_n(x)=[E_n-V(x)]^2\Psi_n(x) \tag{6}$$

where E_n is the total relativistic energy of the particle for a bound state n and c is the velocity of light. $V(x)$ is called Lorentz vector potential that is the time-component of the (1+1)-vector potential. $S(x)$ is called Lorentz scalar potential that couples to the space-time scalar potential. The vector potential couples to the energy while the scalar potential couples to the mass of the particle. The two couplings are independent and intrinsically different.

To understand relativistic effects in nuclear physics or chemistry it is important to obtain bound state solutions of the Klein-Gordon equation with mixed vector and scalar potentials. For various types of potentials, such as linear, exponential, Coulomb, Rosen-Morse, etc., the exact bound state solutions of the one-dimensional Klein-Gordon equation have been reported.[11-26] It is also reported that the one-dimensional Klein-Gordon equation can be exactly solved for shape invariant potentials.[22-27]

The relativistic Klein-Gordon Eq. (6) can be rewritten as a Schrödinger-like equation,

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi_n(x)+V_{eff}(x)\Psi_n(x)=E_{eff,n}\Psi_n(x) \tag{7}$$

with $V_{eff}(x)=[S^2(x)-V^2(x)+2mc^2S(x)+2E_nV(x)]/2mc^2$ and $E_{eff(f,n)}=(E_n^2-m^2c^2)/2mc^2$.

If one solves Eq. (7), the relativistic energy E_n and the relativistic wave functions $\Psi_n(x)$ can be obtained. The effective potential or potential-like term $V_{eff}(x)$ is apparently energy dependent so that the iterative method should be used to solve Eq. (7). But, in all the previous works, the E_n in $V_{eff}(x)$ is assumed to be constant, i.e. $V_{eff}(x)$ is defined for each E_n. This assumption turns out to be correct for exactly solvable systems.[28,29]

Based on the formal similarity between Eq. (1) and Eq. (7), we suggest a new quantization rule for the relativistic Klein-Gordon Eq. (6) which is formally identical with the exact quantization rule (3) and (4) for the nonrelativistic Schrödinger equation, i.e.

$$\int_{x_{1,n}}^{x_{2,n}}\sqrt{2m[E_{eff,n}-V_{eff}(x)]}\,dx=n\pi\hbar+\gamma_{eff}(E_{eff,n}) \tag{8}$$

with
$$\gamma_{eff}(E_{eff,n})=\pi\hbar+\hbar\int_{x_{1,n}}^{x_{2,n}}\frac{\phi_n(x)p'_{eff,n}(x)}{\phi'_n(x)}dx\,. \tag{9}$$

$x_{1,n}$ and $x_{2,n}$ are two turning points ($x_{1,n}<x_{2,n}$), i.e. $V_{eff}(x_{1,n})=V_{eff}(x_{2,n})=E_{eff,n}$. $p_{eff,n}(x)=\sqrt{2m/\hbar^2[E_{eff,n}-V_{eff}(x)]}$ is an analogy to the classical momentum function $p_n(x)$. $\phi_n(x)$ is the log derivative of the relativistic wave function $\Psi_n(x)$. When $V_{eff}(x)$ is a solvable potential, $\gamma_{eff}(E_{eff,n})$ is again independent of n, i.e.

$$\gamma_{eff}(E_{eff,n})=\gamma_{eff}(E_{eff,0})=\pi\hbar+\hbar\int_{x_{1,0}}^{x_{2,0}}\frac{W(x)p'_{eff,0}(x)}{W'(x)}dx \tag{10}$$

where $W(x)=-\phi_0(x)$ is a superpotential term.

Now we choose some examples to test whether the quantization rule for relativistic one-dimensional system is adequate or not. Of course, the examples are relativistic systems whose exact solutions of the Klein-Gordon equation are already known, and the relativistic energy E_n evaluated using the quantization rule (8) and (9) (or (10)) will be compared with the known solutions.

Applications

Needless to say, the general analytical solutions of the Klein-Gordon equation with arbitrary $S(x)$ and $V(x)$ are not known. Also there is a constraint for the Klein-Gordon equation to have bound state solutions, i.e. when the scalar potential energy ($|S|$) is larger than the vector potential energy ($|V|$), the Klein-Gordon equation always has analytical bound state solutions. We have found that the general form of the scalar and vector potentials for which exact solutions are known is

$$V(x)=V_0f(x) \text{ and } S(x)=S_0f(x)\ \ (S_0>V_0)\,. \tag{11}$$

Here $S(x)$ and $V(x)$ have the function $f(x)$ in common (which is a constraint). The $S_0=V_0$ case is so trivial that it is not of interest. As mentioned before, the $S_0<V_0$ case does not guarantee the existence of bound states. Please see Refs. 12 and 13 (and references therein) for the details of the Klein-Gordon equation.

We will now evaluate the exact relativistic energies of the Klein-Gordon Eq. (6) or (7), using the new quantization rule (8) and (9) for three different choices of $f(x)$.

The First Example. $f(x)=x/2$. In this case the $V_{eff}(x)$ and $E_{eff,n}$ in Eq. (7) are

$$V_{eff}(x)=\frac{\frac{1}{4}(S_0^2-V_0^2)x^2+(mc^2S_0+E_nV_0)x}{2mc^2} \tag{12}$$

and

$$E_{eff,n}=\frac{E_n^2-m^2c^4}{2mc^2}\,. \tag{13}$$

Let $z=\alpha^{-1}x$ where

$$\alpha=\hbar c\,, \tag{14}$$

then Eq. (7) can be rewritten as (the dummy variable z is renamed back to x)

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi(x)+V_1(x)\Psi(x)=E_{1,n}\Psi(x) \tag{15}$$

with

$$\begin{aligned} V_1(x) &= \frac{\hbar^2}{2m} V_{eff}(x) \\ &= \frac{\hbar^2}{2m}\left[\frac{\alpha^2}{4}(S_0^2 - V_0^2) + \alpha(mc^2 S_0 + E_n V_0)x\right] \\ &= \frac{\hbar^2}{2m}[A^2x^2 + 2ABx] \end{aligned} \tag{16}$$

and

$$\begin{aligned} E_{1,n} &= \frac{\hbar^2}{2m} E_{eff,n} \\ &= \frac{\hbar^2}{2m}(E_n^2 - m^2c^4)\,. \end{aligned} \tag{17}$$

The constants A and B are

$$A^2 = \frac{\alpha^2(S_0^2 - V_0^2)}{4} \text{ and } B = \frac{mc^2 S_0 + E_n V_0}{\sqrt{S_0^2 - V_0^2}}\,. \tag{18}$$

Since it is an exactly solvable system, the quantization rule is, from Eqs. (8) and (10) ,

$$\int_{x_{1,n}}^{x_{2,n}} \sqrt{2m[E_{1,n} - V_1(x)]}\,dx = n\pi\hbar + \gamma_1 \tag{19}$$

with

$$\gamma_1 = \pi\hbar + \hbar\int_{x_{1,n}}^{x_{2,n}} \frac{W(x)p'_{1,0}(x)}{W'(x)}dx \text{ and}$$

$$p_{1,0}(x) = \sqrt{\frac{2m}{\hbar^2}[E_{1,0} - V_1(x)]}\,. \tag{20}$$

Now let us evaluate the left hand side (L.H.S.) and the right hand side (R.H.S.) of Eq. (19) separately and equate L.H.S. to R.H.S. to determine the energy. The L.H.S. is

$$\begin{aligned} \text{L.H.S.} &= \int_{x_{1,n}}^{x_{2,n}} \sqrt{2m[E_{1,n} - V_1(x)]}\,dx \\ &= \int_{x_{1,n}}^{x_{2,n}} \sqrt{2m\left[E_{1,n} - \frac{\hbar^2}{2m}(A^2x^2 + 2ABx)\right]}\,dx \\ &= \hbar A\int_{x_{1,n}}^{x_{2,n}} \sqrt{-x^2 - \frac{2B}{A}x + \frac{2mE_{1,n}}{\hbar^2 A^2}}\,dx \\ &= \frac{\pi\hbar}{2A}\left(B^2 + \frac{2m}{\hbar^2}E_{1,n}\right). \end{aligned} \tag{21}$$

The integral formula I_1 in Appendix is used.

To evaluate the correction term γ_1, the superpotential $W(x)$ should be predetermined and it can be achieved by using the supersymmetry algebra,[30] i.e.

$$V_1(x) = W^2(x) = \frac{\hbar}{\sqrt{2m}}W'(x) + E_{1,0}\,. \tag{22}$$

Recall that the effective potential $V_1(x)$ in Eq. (16) is formally identical with the harmonic oscillator potential.[30] There is no general way of solving Eq. (22). But using the method for finding the superpotential of shape invariant potentials suggested by Cooper and Ginocchio,[31] one may obtain the superpotential $W(x)$ and consequently the ground state effective energy $E_{1,0}$. The results are

$$W(x) = \frac{\hbar}{\sqrt{2m}}(Ax + B) \text{ and } E_{1,0} = \frac{\hbar^2}{2m}(A - B^2)\,. \tag{23}$$

Therefore, the derivative of effective momentum function $p_{1,0}(x)$ is

$$\begin{aligned} p_{1,0}(x) &= \frac{d}{dx}\sqrt{\frac{2m}{\hbar^2}[E_{1,0} - V_1(x)]} \\ &= \frac{d}{dx}\sqrt{\frac{2m}{\hbar^2}\left[\frac{\hbar^2}{2m}(A - B^2) - \frac{\hbar^2}{2m}(A^2x^2 + 2ABx)\right]} \\ &= \frac{-Ax - B}{\sqrt{-x^2 - \frac{2B}{A}x + \frac{A - B^2}{A^2}}} \end{aligned} \tag{24}$$

and the correction term γ_1 is

$$\begin{aligned} \gamma_1 &= \pi\hbar + \hbar\int_{x_{1,0}}^{x_{2,0}} \frac{W(x)p'_{1,0}(x)}{W'(x)}dx \\ &= \pi\hbar + \hbar\int_{x_{1,0}}^{x_{2,0}} \frac{\frac{\hbar}{\sqrt{2m}}(Ax + B)}{\frac{\hbar}{\sqrt{2m}}A}\,\frac{-Ax - B}{\sqrt{-x^2 - \frac{2B}{A}x + \frac{A - B^2}{A^2}}}dx \\ &= \pi\hbar + \hbar\int_{x_{1,0}}^{x_{2,0}} \frac{-Ax^2 - 2Bx - \frac{B^2}{A}}{\sqrt{-x^2 - \frac{2B}{A}x + \frac{A - B^2}{A^2}}}dx = \frac{1}{2}\pi\hbar \end{aligned} \tag{25}$$

The integral formulas I_4, I_5, and I_6 Appendix are used. The same value of the correction term is reported in Ref. 32. Therefore, R.H.S. is

$$\text{R.H.S.} = n\pi\hbar + \frac{1}{2}\pi\hbar = \left(n + \frac{1}{2}\right)\pi\hbar\,. \tag{26}$$

Since L.H.S. = R.H.S., Eq. (21) is equal to Eq. (26), i.e.

$$\frac{\pi\hbar}{2A}\left(B^2 + \frac{2m}{\hbar^2}E_{1,n}\right) = \left(n + \frac{1}{2}\right)\pi\hbar\,. \tag{27}$$

After solving Eq. (27) for energy, one obtains

$$E_{1,n} = \frac{\hbar^2}{2m}[(2n + 1)A - B^2]\,. \tag{28}$$

Now let us calculate the relativistic energy E_n. From Eqs. (17) and (28),

$$\frac{\hbar^2}{2m}(E_n^2 - m^2c^4) = \frac{\hbar^2}{2m}[(2n + 1)A - B^2]\,. \tag{29}$$

Inserting the constants A and B (Eq. (18)) and α (Eq. (14)) into Eq. (29), one obtains

$$E_n^2 = m^2c^4 + (2n + 1)\frac{\hbar c}{2}\sqrt{S_0^2 - V_0^2} - \left(\frac{mc^2 S_0 + E_n V_0}{\sqrt{S_0^2 - V_0^2}}\right)^2. \tag{30}$$

After solving the quadratic Eq. (30), the relativistic energy is

$$E_n^{\pm} = \frac{-Q \pm \sqrt{Q^2 + 4PR_n}}{2P} \tag{31}$$

where

$$P = S_0^2$$
$$Q = 2mc^2 S_0 V_0$$
$$R_n = \hbar c (S_0^2 - V_0^2)^{3/2}\left(n + \frac{1}{2}\right) - m^2 c^4 V_0^2 . \tag{32}$$

Here E_n^+ is the relativistic energy for a particle state n and E_n^- for an antiparticle state n. These energies are exact as found in Refs. 13, 22, 23, and 24.

The second example. $f(x) = -e^{-x}$. The $V_{eff}(x)$ and E_{eff} in Eq. (7) are

$$V_{eff}(x) = \frac{(S_0^2 - V_0^2)e^{-2x} - 2(mc^2 S_0 + E_n V_0)e^{-x}}{2mc^2} \quad \text{and}$$

$$E_{eff} = \frac{E_n^2 - m^2 c^4}{2mc^2} \tag{33}$$

which is formally equivalent to the Morse potential.[30]

Let
$$\alpha = \hbar c , \tag{34}$$

then following the same substitution procedure discussed in the previous example, Eq. (7) can be rewritten as

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi(x) + V_1(x)\Psi(x) = E_{1,n}\Psi(x) \tag{35}$$

where

$$V_1(x) = \frac{\hbar^2}{2m}[(S_0^2 - V_0^2)e^{-2\alpha x} - 2(mc^2 S_0 + E_{1,n}V_0)e^{-\alpha x}]$$
$$= \frac{\hbar^2}{2m}[B^2 e^{-2\alpha x} - 2B(A + \alpha/2)e^{-\alpha x}] \tag{36}$$

and

$$E_{1,n} = \frac{\hbar^2}{2m}(E_n^2 - m^2 c^4) \tag{37}$$

$$\text{with } A = \frac{mc^2 S_0 + E_n V_0}{\sqrt{S_0^2 - V_0^2}} - \frac{\alpha}{2} \text{ and } B = \sqrt{S_0^2 - V_0^2} . \tag{38}$$

The L.H.S. of the quantization rule (19) is

$$\text{L.H.S.} = \int_{x_{1,n}}^{x_{2,n}} \sqrt{2m\left[E_{1,n} - \frac{\hbar^2}{2m}(B^2 e^{-2\alpha x} - 2B(A + \alpha/2)e^{-\alpha x})\right]}dx$$
$$= \hbar B \int_{x_{1,n}}^{x_{2,n}} \sqrt{-e^{-2\alpha x} + \frac{2(A + \alpha/2)}{B}e^{-\alpha x} + \frac{2mE_{1,n}}{\hbar^2 B^2}}dx . \tag{39}$$

Let $y = e^{-\alpha x}$, then $y_{1,n} = e^{-\alpha x_{1,n}}$ and $y_{2,n} = e^{-\alpha x_{2,n}}$. Therefore,

$$\text{L.H.S.} = \frac{\hbar B}{\alpha}\int_{y_{2,n}}^{y_{1,n}} \frac{dy}{y}\sqrt{-y^2 + \frac{2(A + \alpha/2)}{B}y + \frac{2mE_{1,n}}{\hbar^2 B^2}}$$
$$= \frac{\pi\hbar}{\alpha}\left[-\sqrt{\frac{-2mE_{1,n}}{\hbar^2}} + A + \alpha/2\right]. \tag{40}$$

The integral formula I_2 in Appendix is used.

To evaluate the R.H.S. of the quantization rule (19), $W(x)$ is predetermined by using the same algebra discussed in the previous example, i.e.

$$W(x) = \frac{\hbar}{\sqrt{2m}}(A - Be^{-\alpha x}) \text{ and } E_{1,0} = -\frac{\hbar^2}{2m}A^2 . \tag{41}$$

Now $p'_{1,0}(x)$ is

$$p'_{1,0}(x) = \frac{-\frac{2m}{\hbar^2}V_1'(x)}{2\sqrt{\frac{2m}{\hbar^2}[E_{1,0} - V_1(x)]}}$$
$$= \frac{\alpha B[Be^{-2\alpha x} - (A + \alpha/2)e^{-\alpha x}]}{\sqrt{-B^2 e^{-2\alpha x} + 2B(A + \alpha/2)e^{-\alpha x} - A^2}} \tag{42}$$

and γ_1 is

$$\gamma_1 = \pi\hbar + \hbar\int_{x_{1,0}}^{x_{2,0}} \frac{W(x)p'_{1,0}(x)}{W'(x)}dx$$
$$= \pi\hbar + \hbar\int_{x_{1,0}}^{x_{2,0}} \frac{(A - Be^{-\alpha x})}{\alpha Be^{-\alpha x}} \frac{\alpha B[Be^{-2\alpha x} - (A + \alpha/2)e^{-\alpha x}]}{\sqrt{-B^2 e^{-2\alpha x} + 2B(A + \alpha/2)e^{-\alpha x} - A^2}}dx$$
$$= \pi\hbar + \frac{\hbar}{\alpha}\int_{y_{2,0}}^{y_{1,0}} dy \frac{-By + (2A + \alpha/2)}{\sqrt{-y^2 + \frac{2}{B}(A + \alpha/2)y - \frac{A^2}{B^2}}}$$
$$+ \frac{\hbar}{\alpha}\int_{y_{2,0}}^{y_{1,0}} \frac{dy}{y} \frac{-\frac{A}{B}(A + \alpha/2)}{\sqrt{-y^2 + \frac{2}{B}(A + \alpha/2)y - \frac{A^2}{B^2}}}$$
$$= \frac{1}{2}\pi\hbar . \tag{43}$$

where $y = e^{-\alpha x}$. The integral formulas I_4, I_5, and I_7 in Appendix are used. The same value of correction term is reported in Ref. 32. Therefore, R.H.S. is

$$\text{R.H.S.} = n\pi\hbar + \frac{1}{2}\pi\hbar = \left(n + \frac{1}{2}\right)\pi\hbar . \tag{44}$$

From the equality of L.H.S. = R.H.S., i.e. Eq. (40) is equal to Eq. (44),

$$\frac{\pi\hbar}{\alpha}\left[-\sqrt{\frac{-2mE_{1,n}}{\hbar^2}} + A + \frac{\alpha}{2}\right] = \left(n + \frac{1}{2}\right)\pi\hbar . \tag{45}$$

Solving Eq. (45) for the energy, one obtains

$$E_{1,n} = \frac{\hbar^2}{2m}[-(A - n\alpha)^2] . \tag{46}$$

From Eqs. (37) and (46), one obtains

$$\frac{\hbar^2}{2m}(E_n^2 - m^2 c^4) = \frac{\hbar^2}{2m}[-(A - n\alpha)^2] \tag{47}$$

and the relativistic energy E_n is found to be

$$E_n^{\pm} = \frac{-Q_n \pm \sqrt{Q_n^2 + 4PR_n}}{2P} \tag{48}$$

where

$$P = S_0^2$$

$$Q = -2\hbar c\left(n+\frac{1}{2}\right)V_0\sqrt{S_0^2-V_0^2}+2mc^2S_0V_0$$

$$R_n = \hbar^2c^2\left(n+\frac{1}{2}\right)^2(S_0^2-V_0^2)-2\hbar c\left(n+\frac{1}{2}\right)mc^2S_0\sqrt{S_0^2-V_0^2}+m^2c^4V_0^2 \quad . \tag{49}$$

These relativistic energies are exact as found in Refs. 11, 20, 23, and 24.

The third example. $f(x)$=tanh$x$. The $V_{eff}(x)$ and $E_{eff,n}$ in Eq. (7) are

$$V_{eff}(x) = \frac{-(S_0^2-V_0^2)\mathrm{sech}^2x+2(mc^2S_0+E_nV_0)\tanh x}{2mc^2} \tag{50}$$

and

$$E_{eff} = \frac{E_n^2-m^2c^4-S_0^2+V_0^2}{2mc^2} \tag{51}$$

which is formally equivalent to the Rosen-Morse II potential.[30]

Similar variable change in the previous example gives the Schrödinger-like equation of interest, i.e.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi(x)+V_1(x)\Psi(x) = E_{1,n}\Psi(x) \tag{52}$$

where

$$\begin{aligned} V_1(x) &= \frac{\hbar^2}{2m}[-(S_0^2-V_0^2)\mathrm{sech}^2\alpha x+2(mc^2S_0+E_nV_0)\tanh\alpha x] \\ &= \frac{\hbar^2}{2m}[-(A^2+\alpha A)\mathrm{sech}^2\alpha x+2B\tanh\alpha x] \end{aligned} \tag{53}$$

and
$$E_{1,n} = \frac{\hbar^2}{2m}(E_n^2-m^2c^4-S_0^2+V_0^2) \tag{54}$$

with, $A = \dfrac{-\alpha+\sqrt{\alpha^2+4(S_0^2-V_0^2)}}{2}$, $B = mc^2S_0+E_nV_0$, and $\alpha = \hbar c$. (55)

The superpotential and the ground state energy are

$$W(x) = \frac{\hbar}{\sqrt{2m}}\left(A\tanh\alpha x+\frac{B}{A}\right) \text{ and } E_{1,0} = \frac{\hbar^2}{2m}\left(-A^2-\frac{B^2}{A^2}\right). \tag{56}$$

Then, the L.H.S. of the quantization rule (19) is

$$\begin{aligned} \text{L.H.S.} &= \int_{x_{1,n}}^{x_{2,n}}\sqrt{2m\left[E_{1,n}+\frac{\hbar^2}{2m}[(A^2+\alpha A)\mathrm{sech}^2\alpha x-2B\tanh\alpha x]\right]}dx \\ &= \frac{\hbar\sqrt{A^2+\alpha A}}{\alpha}\int_{y_{1,n}}^{y_{2,n}}\frac{dy}{1-y^2}\sqrt{-y^2-\frac{2B}{A^2+\alpha A}y+\frac{2m}{\hbar^2}\frac{E_{1,n}}{A^2+\alpha A}+1} \\ &= \frac{\pi\hbar}{2\alpha}\left[2\sqrt{A^2+\alpha A}-\sqrt{2B-\frac{2m}{\hbar^2}E_{1,n}}-\sqrt{-2B-\frac{2m}{\hbar^2}E_{1,n}}\right] \end{aligned} \tag{57}$$

where y = tanhαx. The integral formula I_3 in Appendix is used.

Now $p'_{1,0}(x)$ is

$$p'_{1,0}(x) = \frac{\alpha[-(A^2+\alpha A)y-B](1-y^2)}{\sqrt{\frac{2m}{\hbar^2}E_{1,0}+(A^2+\alpha A)(1-y^2)-2By}}. \tag{58}$$

The R.H.S. of the quantization rule (19) is

$$\begin{aligned} \text{R.H.S.} &= n\pi\hbar+\pi\hbar+\hbar\int_{x_{1,0}}^{x_{2,0}}\frac{\left(A\tanh\alpha x+\frac{B}{A}\right)p'_{1,0}(x)}{A\alpha\mathrm{sech}^2\alpha x}dx \\ &= (n+1)\pi\hbar+\hbar\int_{y_{1,0}}^{y_{2,0}}\frac{dy}{\alpha(1-y^2)}\frac{\left(Ay+\frac{B}{A}\right)p'_{1,0}(y)}{A\alpha(1-y^2)} \\ &= (n+1)\pi\hbar+\frac{\hbar}{\alpha}\int_{y_{1,0}}^{y_{2,0}}\frac{dy}{1-y^2}\frac{-\sqrt{A^2+\alpha A}y^2-\frac{\alpha B+2AB}{A\sqrt{A^2+\alpha A}}y-\frac{B^2}{A^2\sqrt{A^2+\alpha A}}}{\sqrt{-y^2-\frac{2B}{A^2+\alpha A}y-\frac{A^2+B^2/A^2}{A^2+\alpha A}+1}} \\ &= (n+1)\pi\hbar-\frac{\pi\hbar}{\alpha}[A+\alpha-\sqrt{A(A+\alpha)}]\,. \end{aligned} \tag{59}$$

where y = tanhαx. The integral formulas I_8, I_9, and I_{10} in Appendix are used.

Equating Eq. (57) to Eq. (59), one obtains

$$\frac{\pi\hbar}{2\alpha}\left[2\sqrt{A^2+\alpha A}-\sqrt{2B-\frac{2m}{\hbar^2}E_{1,n}}-\sqrt{-2B-\frac{2m}{\hbar^2}E_{1,n}}\right] = (n+1)\pi\hbar-\frac{\pi\hbar}{\alpha}[A+\alpha-\sqrt{A(A+\alpha)}]\,. \tag{60}$$

Solving Eq. (60) for the energy,

$$E_{1,n} = \frac{\hbar^2}{2m}\left[-(A-n\alpha)^2-\frac{B^2}{(A-n\alpha)^2}\right]. \tag{61}$$

From Eqs. (54) and (61), the relativistic energy E_n is

$$E_n^{\pm} = \frac{-Q\pm\sqrt{Q^2-4P_nR_n}}{2P_n} \tag{62}$$

where

$$\begin{aligned} P_n &= \frac{1}{4}(-\hbar c+\sqrt{\hbar^2c^2+4(S_0^2-V_0^2)}-2\hbar cn)^2+V_0 \\ Q &= 2mc^2S_0V_0 \\ R_n &= \frac{1}{16}(-\hbar c+\sqrt{\hbar^2c^2+4(S_0^2-V_0^2)}-2\hbar cn)^4+m^2c^4S_0^2 \\ &\quad -\frac{1}{4}(-\hbar c+\sqrt{\hbar^2c^2+4(S_0^2-V_0^2)}-2\hbar cn)^2(m^2c^4+S_0^2-V_0^2)\,. \end{aligned} \tag{63}$$

These energies are exact as found in Refs. 14, 15, and 23.

For the three exactly solvable examples presented above, the quantization rule (8) and (10) for the Klein-Gordon equation is found to be exact. As a matter of fact the quantization rule (8) and (9) should be exact for any other potential in general.

Conclusion

A quantization rule for relativistic system cannot be algebraically derived from the Klein-Gordon equation. However, by exploiting the formal similarity between the nonrelativistic Schrödinger equation and the relativistic Klein-Gordon equation, a quantization rule for relativistic system is successfully obtained. The exact quantization rule for one-dimensional spinless relativistic quantum systems, in compact form, is

$$\int_{x_{1,n}}^{x_{2,n}} p_n(x)dx = (n+1)\pi + \int_{x_{1,n}}^{x_{2,n}} \frac{\phi_n(x)p'_n(x)}{\phi'_n(x)}dx \tag{64}$$

where $p_n(x) = \frac{1}{\hbar c}\sqrt{E_n^2 - 2E_nV(x) + V^2(x) - S^2(x) - 2mc^2S(x) - m^2c^4}$ and $\phi_n(x) = (d\Psi_n(x)/dx)/\Psi_n(x)$. $x_{1,n}$ and $x_{2,n}$ are two turning points ($x_{1,n} < x_{2,n}$). $S(x)$ is a scalar potential and $V(x)$ is a vector potential. For exactly solvable systems, it is found that

$$\int_{x_{1,n}}^{x_{2,n}} \frac{\phi_n(x)p'_n(x)}{\phi'_n(x)}dx = \int_{x_{1,0}}^{x_{2,0}} \frac{\phi_0(x)p'_0(x)}{\phi'_0(x)}dx. \tag{65}$$

The relativistic quantization rule seems to have no practical usages but its form may suggest the deeper understanding on relativistic quantum systems. In the current work, since the nonrelativistic Schrödinger equation is bosonic in nature, the relativistic bosonic particle Klein-Gordon equation is selected instead of the relativistic Dirac equation for fermions. It will be interesting to see if any quantization can be easily formulated for the relativistic Dirac equation.

Though the current formal derivation seems to be trivial, this type of formal analysis may help one understand other quantization conditions. For example, the well-known supersymmetric WKB quantization is found to be exact for all translationally shape invariant potentials. This mystery has never been algebraically proved even though there were many attempts. We are performing a formal analysis on supersymmetric WKB quantization and the results will be reported in the near future.

Appendix

The following integral formulas are used in the current calculation. The formulas I_1 through I_7 are found in the literatures[6,9,33] and the last three formulas I_8, I_9, and I_{10} are algebraically derived by us.

$$I_1 = \int_a^b dy\sqrt{(y-a)(b-y)} = \frac{\pi}{8}(b-a)^2 \qquad (a<b)$$

$$I_2 = \int_a^b \frac{dy}{y}\sqrt{(y-a)(b-y)} = \frac{\pi}{2}(a+b) - \pi\sqrt{ab} \qquad (0<a<b)$$

$$I_3 = \int_a^b \frac{dy}{1-y^2}\sqrt{(y-a)(b-y)} = \frac{\pi}{2}[2-\sqrt{(1-a)(1-b)}-\sqrt{(1+a)(1+b)}] \qquad (-1<a<b<1)$$

$$I_4 = \int_a^b \frac{dy}{\sqrt{(y-a)(b-y)}} = \pi \qquad (a<b)$$

$$I_5 = \int_a^b \frac{ydy}{\sqrt{(y-a)(b-y)}} = \frac{\pi}{2}(a+b) \qquad (a<b)$$

$$I_6 = \int_a^b \frac{y^2dy}{\sqrt{(y-a)(b-y)}} = \frac{3\pi}{8}(a+b)^2 - \frac{\pi}{2}ab \qquad (a<b)$$

$$I_7 = \int_a^b \frac{dy}{(cy+d)\sqrt{(y-a)(b-y)}} = \frac{\pi}{\sqrt{(d+ca)(d+cb)}} \qquad (a<b, c\neq 0)$$

$$I_8 = \int_a^b \frac{dy}{(1-y^2)\sqrt{(y-a)(b-y)}} = \frac{\pi}{2}\left[\frac{1}{\sqrt{(1+a)(1+b)}} + \frac{1}{\sqrt{(1-a)(1-b)}}\right] \qquad (-1<a<b<1)$$

$$I_9 = \int_a^b \frac{ydy}{(1-y^2)\sqrt{(y-a)(b-y)}} = \frac{\pi}{2}\left[-\frac{1}{\sqrt{(1+a)(1+b)}} + \frac{1}{\sqrt{(1-a)(1-b)}}\right] \qquad (-1<a<b<1)$$

$$I_{10} = \int_a^b \frac{y^2dy}{(1-y^2)\sqrt{(y-a)(b-y)}} = \frac{\pi}{2}\left[-2 + \frac{1}{\sqrt{(1+a)(1+b)}} + \frac{1}{\sqrt{(1-a)(1-b)}}\right] \qquad (-1<a<b<1)$$

References

1. Ou, F. C.; Cao, Z. Q.; Shen, Q. S. *J. Chem. Phys.* **2004**, *121*, 8175.
2. Ma, Z. Q.; Xu, B. W. *Int. J. Modern Phys. E* **2005**, *14*, 599.
3. Ma, Z. Q.; Xu, B. W. *Europhys. Lett.* **2005**, *69*, 685.
4. Grandati, Y.; Bérard, A. *Phys. Lett. A* **2011**, *375*, 390.
5. Qiang, W. C.; Dong, S. H. *Europhys. Lett.* **2010**, *89*, 10003.
6. Dong, S. H.; Morales, D.; García-Ravelo, J. *Int. J. Modern Phys. E* **2007**, *16*, 189.
7. Qiang, W. C.; Dong, S. H. *Phys. Lett. A* **2007**, *363*, 169.
8. Dong, S. H.; Gonzalez-Cisneros, A. *Ann. Phys.* **2008**, *323*, 1136.
9. Gu, X. Y.; Dong, S. H. *Phys. Lett. A* **2008**, *372*, 1972.
10. Dong, S. H. *Int. J. Quantum Chem.* **2009**, *109*, 701.
11. Chen, G. *Phys. Lett. A* **2005**, *339*, 300.
12. de Castro, A. S. *Phys. Lett. A* **2005**, *338*, 81.
13. de Souza Dutra, A.; Chen, G. *Phys. Lett. A* **2006**, *349*, 297.
14. Yi, L.; Diao, Y.; Liu, J.; Jia, C. *Phys. Lett. A* **2004**, *333*, 212.
15. Garcia, M. G.; de Castro, A. S. *Ann. Phys.* **2009**, *324*, 2372.
16. Barakat, T. *Ann. Phys.* **2009**, *324*, 725.
17. McQuarrie, B. R; Vrscay, E. R. *Phys. Rev. A* **1993**, *47*, 868.
18. Barton, G. *J. Phys. A*: *Math. Theor.* **2007**, *40*, 1011.
19. Hall, R. L. *Phys. Lett. A* **2007**, *372*, 12.
20. Alhaidari, A. D.; Bahlouli, H.; Al-Hasan, A. *Phys. Lett. A* **2006**, *349*, 87.
21. Zhao, X. Q.; Jia, C. S.; Yang, Q. B. *Phys. Lett. A* **2005**, *337*, 189.
22. Jana, T. K.; Roy, P. *Phys. Lett. A* **2009**, *373*, 1239.
23. Jana, T.; Roy, P. *Phys. Lett. A* **2007**, *361*, 55.
24. Chen. G.; Chen, Z.; Xuan, P. *Phys. Lett. A* **2006**, *352*, 317.
25. Diao, Y.; Yi, L.; Jia, C. *Phys. Lett. A* **2004**, *332*, 157.
26. Qiang, W. C.; Zhou, R. S.; Gao, Y. *Phys. Lett. A* **2007**, *371*, 201.
27. Znojil, M. *J. Phys. A*: *Math. Gen.* **2004**, *37*, 9557.
28. Sun, H. *Phys. Lett. A* **2009**, *374*, 116.
29. Sun, H. *Bull. Korean Chem. Soc.* **2010**, *31*, 3573.
30. Cooper, F.; Khare, A.; Sukhatme, U. *Phys. Rep.* **1995**, *251*, 267.
31. Cooper, F.; Ginocchio, J. N. *Phys. Rev. D* **1987**, *36*, 2458.
32. Yin, C.; Cao, Z.; Shen, Q. *Ann. Phys.* **2010**, *325*, 528.
33. Hruška, M.; Keung, W. Y.; Sukhatme, U. *Phys. Rev. A* **1997**, *55*, 3345.