Method for including restrained warping in traditional frame analyses

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Restrained warping is important for the torsion deformation and axial stresses of thin-wall open sections. However, this phenomenon is not included in commonly used frame analysis programs. This paper presents a simple method to include the effect of restrained warping of member ends in a frame analysis. In this method, prior to the frame analysis the structural designer increases the section torsion stiffness by a factor. After the analysis the structural designer obtains the extreme bi-moments and axial stresses from the computed torsion moment. The method is demonstrated in three examples. The importance of restrained warping is shown.

Key words: Torsion, frame analysis, restrained warping, design method

1 Introduction

When a beam is loaded in torsion the cross-sections deform in the axial direction. In other words plane cross-sections do not remain plane (Fig. 1). This phenomenon is called warping. Three dimensional frame analysis programs usually apply the torsion theory of De Saint Venant. This theory assumes free warping of the member sections. In reality many joints are such that they prevent the member ends from warping freely. This increases the member torsional stiffness and introduces axial stresses especially in the member ends. For solid and thin-wall closed sections these effects can be often neglected. However, for thin-wall open sections restrained warping is often important.

The torsion theory of Vlasov includes the effect of restrained warping [Vlasov 1959, 1961, Zbirohowski-Koscia 1967]. Compared to the traditional beam theory, the Vlasov theory introduces two extra quantities; the warping constant C_w and the bi-moment B. The warping constant is a cross-section property and a measure for the effort needed to reduce warping. The unit of the warping constant is [length^6]. The bi-moment is a measure for the stresses needed to reduce warping. The unit of the bi-moment is [force length^2]. For an I-section the bi-moment is the moment in the flanges times their distance (Fig. 2). For other cross-section shapes the interpretation is much more difficult.

The Vlasov torsion theory can be implemented in frame analysis programs [Meijers 1998]. The adaptations include one extra degree of freedom (the warping) and an extra force (the bi-moment) for every member end. In the Appendix the stiffness matrix is included for implementation of the Vlasov theory in a frame analysis program. If implemented the structural designer would be able to specify for every member end whether warping is, free, fixed or connected. If free warping would be selected for all member ends the frame analysis would be exactly the same as a traditional De Saint Venant analysis. If fixed warping or connected warping would be selected the member would be stiffer and a bi-moment distribution needs to be displayed. Such a frame program would also allow distributed torsion moment loading, which often occurs in practice. However, none of the commonly used frame analysis programs include the Vlasov torsion theory. The reason might be that Vlasov torsion theory is not well known as yet and that engineers are not educated in the effects of restrained warping. To the authors opinion the Vlasov torsion theory should be standard functionally in three dimensional frame analysis programs.

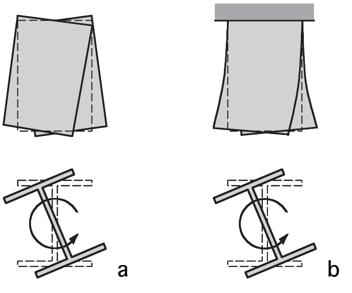


Figure 1: Free warping (a) and restrained warping (b) of an I-section loaded in torsion

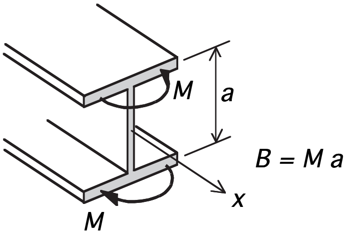


Figure 2: Interpretation of the bi-moment in an I-section

Nonetheless, the effects of restrained warping can already be taken into account in a traditional frame analysis. To this end, a simple method is presented in Chapter 5. The method includes prevented warping at one or both member ends. The method is not valid for connected warping, for example in continuous beams or at a change of cross-section (non-prismatic members). The method also does not allow distributed torsion moment loading. For connected warping and distributed torsion loading the structural designer needs to use a frame program that includes Vlasov torsion theory. In Chapter 2 the Vlasov torsion theory is summarised. In Chapter 3 the analysis method is derived. In Chapter 6 the method is demonstrated for three structures.

It is noted that the torsion theory of De Saint Venant is also referred to as uniform torsion or circulatory torsion. The torsion theory of Vlasov is also referred to as non-uniform torsion or warping torsion.

2 Vlasov Torsion Theory

In 1940, V.Z. Vlasov developed a torsion theory in which restrained warping is included [Vlasov 1959]. The rotation j of the beam cross-section follows from the differential equation

$$EC_w \frac{d^4\varphi}{dx^4} - GJ\frac{d^2\varphi}{dx^2} = m_x \quad , \tag{1}$$

where, GJ is the torsion stiffness, EC_w is the warping stiffness and m_x is a distributed torsion moment along the beam. The torsion stiffness and warping stiffness are cross-section properties that can be computed with specialised software [ShapeDesigner, ShapeBuilder]. Tables exist for often occurring cross-sections [Timoshenko 1961, Young 1989].
The torsion moment is

$$M_t = GJ\frac{d\varphi}{dx} + \frac{dB}{dx} \quad . \tag{2}$$

The bi-moment is

$$B = -EC_w \frac{d^2\varphi}{dx^2} \quad . \tag{3}$$

The beam-ends can have either an imposed rotation $\varphi=\varphi_0$ or an imposed torsion moment $M_t=M_{t0}$. At the same time they have either an imposed warping $\frac{d\varphi}{dx} = \varphi'_o$. or an imposed bi-moment $B=B_0$. The Vlasov theory reduces to the theory of De Saint Venant if the warping stiffness is zero, the distributed moment is zero and warping is free.

3 Derivation of the Analysis Method

Suppose that warping is prevented at both sides of a beam. Than the boundary conditions are

$$\begin{aligned} \varphi\big|_{x=0} &= \varphi_1, \quad \frac{d\varphi}{dx}\bigg|_{x=0} = 0 \\ \varphi\big|_{x=l} &= \varphi_2, \quad \frac{d\varphi}{dx}\bigg|_{x=l} = 0 \quad . \end{aligned} \tag{4}$$

The solution of differential equation (1) is

$$\varphi = (\varphi_2 - \varphi_1)\frac{\beta\frac{x}{l} + 1 + \left(\beta\frac{x}{l} - 1\right)e^{\beta} - e^{\beta\frac{x}{l}} + e^{\beta(1-\frac{x}{l})}}{\beta + 2 + (\beta - 2)e^{\beta}} + m_x GJ\frac{\beta\frac{x}{l} + 1 - \beta\frac{x^2}{l^2} - (\beta\frac{x}{l} - 1 - \beta\frac{x^2}{l^2})e^{\beta} - e^{\beta\frac{x}{l}} - e^{\beta(1-\frac{x}{l})}}{2\beta l^2(1 - e^{\beta})} \quad .$$

where (5)

$$\beta = l\sqrt{\frac{GJ}{EC_w}} \quad . \tag{6}$$

Using , $M_1 = -M_t\big|_{x=0}$, $B_1 = B\big|_{x=0}$, $M_2 = M_t\big|_{x=l}$ and $B_2 = -B\big|_{x=l}$ the element stiffness matrix is derived. In which, the indices 1, 2 refer to the left and right of the beam, respectively.

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \frac{\beta(1+e^{\beta})}{\beta+2+(\beta-2)e^{\beta}} \begin{bmatrix} \frac{GJ}{l} & -\frac{GJ}{l} \\ -\frac{GJ}{l} & \frac{GJ}{l} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} + m_x \begin{bmatrix} -\frac{1}{2}l \\ -\frac{1}{2}l \end{bmatrix} . \tag{7}$$

The bi-moments at both beam-ends can be expressed in terms of the torsion moment

$$B_1 = -M_t \frac{l}{\beta} \tanh(\tfrac{1}{2}\beta) + m_x \frac{l^2}{\beta^2} \frac{2\beta e^{\beta} + 1 - e^{2\beta}}{1-e^{2\beta}}$$

$$B_2 = -M_t \frac{l}{\beta} \tanh(\tfrac{1}{2}\beta) - m_x \frac{l^2}{\beta^2} \frac{2\beta e^{\beta} + 1 - e^{2\beta}}{1-e^{2\beta}} \quad . \tag{8}$$

We assume that m_x=0. In the previous stiffness matrices we observe that the torsion stiffness GJ is increased by the factor

$$\frac{\beta(1+e^{\beta})}{\beta+2+(\beta-2)e^{\beta}} \quad . \tag{9}$$

For $\beta > 5$ this can be approximated with 1% accuracy by

$$\frac{\beta}{\beta-2} \quad . \tag{10}$$

For $\beta > 6$ the expression for the bi-moment can be approximated with 1% accuracy by

$$B_1 = B_2 = -\frac{l}{\beta} M_t \quad . \tag{11}$$

Suppose that warping is prevented at one side of a beam (x = 0). Than the boundary conditions are

$$\begin{aligned} &\varphi\big|_{x=0} = \varphi_1 , \quad \frac{d\varphi}{dx}\bigg|_{x=0} = 0 , \\ &\varphi\big|_{x=l} = \varphi_2 , \quad \frac{d^2\varphi}{dx^2}\bigg|_{x=l} = 0 . \end{aligned} \tag{12}$$

The solution of the differential equation is too large to include in this paper. The element stiffness matrix is

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \frac{\beta(1+e^{2\beta})}{\beta+1+(\beta-1)e^{2\beta}} \begin{bmatrix} \frac{GJ}{l} & -\frac{GJ}{l} \\ -\frac{GJ}{l} & \frac{GJ}{l} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} + \frac{m_x}{\beta(\beta+1+(\beta-1)e^{2\beta})} \begin{bmatrix} -\frac{1}{2}(\beta^2-2+4e^{\beta}+(\beta^2-2)e^{2\beta})l \\ -\frac{1}{2}(\beta^2+2\beta+2-4e^{\beta}+(\beta^2-2\beta+2)e^{2\beta})l \end{bmatrix}$$

The bi-moments at both beam-ends are (13)

$$B_1 = -M_t \frac{l}{\beta}\tanh(\beta) + m_x \frac{l^2}{\beta} \frac{1-2e^{\beta}+e^{2\beta}}{1+e^{2\beta}} ,$$

$$B_2 = 0 . \qquad (14)$$

We assume that m_x=0 . In the previous stiffness matrices we observe that the torsion stiffness is increased by the factor

$$\frac{\beta(1+e^{2\beta})}{\beta+1+(\beta-1)e^{2\beta}} . \qquad (15)$$

For $\beta > 3$ this can be approximated with 1% accuracy by

$$\frac{\beta}{\beta-1} . \qquad (16)$$

For $\beta > 3$ the expression for the bi-moment can be approximated with 1% accuracy by

$$B_1 \approx -\frac{l}{\beta} M_t \quad . \qquad (17)$$

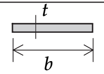
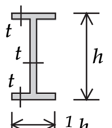
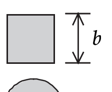
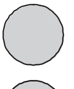
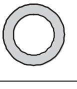
Cross-section		β
Strip $t \ll b$		$\frac{l}{b}\frac{2\sqrt{6}}{\sqrt{1+\nu}}$
I-section		$\frac{l}{h}\frac{t}{h}\frac{8}{\sqrt{1+\nu}}$
Square cross-section		$\frac{l}{b}\frac{23.9}{\sqrt{1+\nu}}$
Round cross-section		∞
Round tube		∞

Table 1: β numbers for several cross-sections

4 Number β

The number β is defined as (6)

$$\beta = l\sqrt{\frac{GJ}{EC_w}} \quad ,$$

which is a member property that determines to what extent torsion according to De Saint Venant can develop when warping is restrained. Table 1 gives an overview of β numbers. For most beams β is larger than 6. The approximations in the previous section are accurate for β > 6. In this case the deviations from the exact values are smaller than 1%. For short thin-wall open sections β can be smaller than 6. As an example, we consider an I-section with a length of 4000 mm. The height is 300 mm, the width is 150 mm and the wall thickness is 10 mm. Poisson's ratio is ν = 0.35. Using Table 1 we find β = 3. For this case the exact formulae needs to be used instead of the approximations. Note that for small values of β a large increase in member torsional stiffness is obtained.
It can be shown that the member torsional stiffness cannot be larger than GI_p/l, where GI_p is the polar moment of inertia. In theory the derived member torsional stiffness can become larger than this maximum for very short members. However, this has no practical consequences because the difference between a very stiff member and an extremely stiff member gives no difference in structural force flow.

5 Restrained Warping Method

1) For including the effect of restrained warping we need to know the torsion constant J and the warping constant C_w of the member cross-section. Many frame programs include section libraries with commonly used sections and their properties. If the torsion properties are not available a program for cross-section analysis can be used to compute their values [ShapeDesigner, ShapeBuilder].

2) Subsequently the β number is calculated (6)

$$\beta = l\sqrt{\frac{GJ}{EC_w}} \quad ,$$

where l is the member length, G is the shear modulus and E is Young's modulus. G is defined as

$$G = \frac{E}{2(1+\nu)} \quad , \tag{18}$$

where ν is Poisson's ratio. If β is larger than 6 the subsequent formulas are valid. If β is smaller than 6 these formulas might not be valid and more elaborate formulas might be needed, which are derived in Section 3.

3) Before the frame analysis the torsion stiffness is entered or changed in the frame program. If warping is prevented at one member-end the torsion stiffness J needs to be multiplied by (16)

$$\frac{\beta}{\beta - 1} \ .$$

If warping is prevented at two member-ends the torsion stiffness J needs to be multiplied by (10)

$$\frac{\beta}{\beta - 2} \ .$$

4) After the frame analysis the largest bi-moment is calculated with (11), (17)

$B = -\dfrac{l}{\beta} M_t$ at one beam end (x = 0) if it is restrained there,

$B = +\dfrac{l}{\beta} M_t$ at the other beam end (x = l) if it is restrained there.

5) Finally, the resulting stress state in the cross-section can be computed by a program for cross-section analysis [ShapeDesigner, ShapeBuilder]. This program can also take into account the stresses due to other section moments, shear forces and the normal force.

6 Applications

6.1 ***Cantilever***

A beam is fixed at one end at which rotation and warping are prevented (Fig. 3). At the other end the beam is loaded by a torque T = 2.26 kNm while warping can occur freely. The material and cross-section data of the beam are E = 207000 N/mm², G = 79300 N/mm², J = 269800 mm⁴ and C_w = 1503 10⁷ mm⁶. This example is also analysed in [Chen 1977].

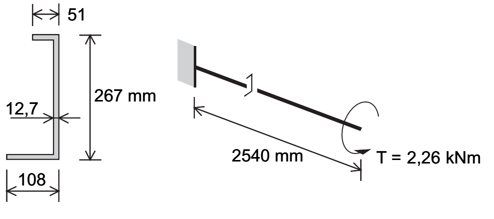


Figure 3: Thin-wall cantilever loaded in torsion

For this beam the β number is

$$\beta = l\sqrt{\frac{GJ}{EC_w}} = 2540\sqrt{\frac{79300 \times 269800}{207000 \times 1503 \times 10^7}} = 6.66 \ .$$

The torsion stiffness included in the frame analysis is

$$\frac{\beta}{\beta-1} J = \frac{6.66}{5.66} 278000 = 327100 \text{ mm}^4 \quad .$$

The computed rotation at the member end is 0.213 rad and the computed torsion moment is $M_t = 226\ 10^4$ Nmm. The rotation is 18% smaller than the same beam with unrestrained warping. Subsequently, the bi-moment is

$$B = -\frac{l}{\beta} M_t = -\frac{2540}{6.66} 226 \times 10^4 = -86190 \times 10^4 \text{ Nmm}^2 \quad .$$

Using the torsion moment and bi-moment a program for cross-section analysis [ShapeDesigner] calculates the shear stress distribution and the normal stress distribution. The largest shear stress is 109 N/mm^2, which occurs at the loaded end along the circumference of the cross-section. The extreme normal stress is -458 N/mm^2, which occurs at the clamped end in the left of the top flange.

The computed axial stress value is 10% larger (in absolute sense) than those found in [Chen 1977]. The analysis in [Chen 1977] is based on exactly the same Vlasov torsion theory but the cross-section analysis is performed analytically using the thin-wall assumption while the cross-section program used in this paper is suitable for both thin- and thick-wall analysis. Therefore, the differences are caused by differences in the cross-section analysis – which is not the subject of this paper. A noticeable detail is that the warping constant computed by the program is 27% smaller than found in the thin-wall analysis. Nonetheless, the stresses differ much less.

6.2 ***Box-Girder***

A box-girder bridge has a length l = 60 m. The cross-section dimensions and properties are shown in Figure 4.[1] The concrete Young's modulus is $E = 0.30\ 10^{11}$ N/m^2 and Poisson's ratio ν = 0.15. At both ends the bridge is supported while warping is free (Fig. 5). In the middle the bridge is loaded by a torque $T = 269\ 10^5$ Nm. This loading occurs when the bridge is supported at mid span by two temporary columns of which one fails due to an accident. Therefore, the torque T is due to the support reaction of the remaining temporary column.

[1] This example is adapted from lecture notes by dr.ir. C. van der Veen on reinforced and prestressed concrete design for the Dutch Concrete Association.

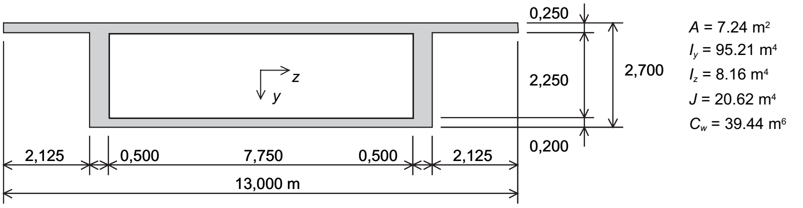


Figure 4: Bridge cross-section

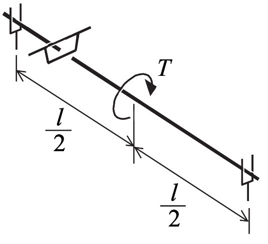


Figure 5: Structural model of the bridge including fork supports and torsion loading

Due to symmetry the warping in the middle section is prevented. The structural model consists of two beam elements. The β number is

$$\beta = l\sqrt{\frac{GJ}{EC_w}} = l\sqrt{\frac{J}{2(1+\nu)C_w}} = 30\sqrt{\frac{20.62}{2 \times 1.15 \times 39.44}} = 14.30\,.$$

Both elements have warping restrained at one side. Therefore, the torsion constant for the frame analysis is

$$\frac{\beta}{\beta - 1} J = \frac{14.30}{13.30} 20.62 = 22.17 \text{ m}^4\,.$$

The frame program computes a rotation of 129 10^{-5} rad in the middle of the bridge. This is 14% smaller than the unrestrained behaviour. The program computes the torsion moment M_t = -1345 10^4 Nm for the left beam and = 1345 10^4 Nm for the right-hand beam. Therefore, the largest bi-moment in the right-hand end of the left beam is

$$B = \frac{l}{\beta} M_t = \frac{60}{14.30}(-1345\ 10^4) = -5643\ 10^4 \text{ Nm}^2\,.$$

The largest bi-moment in the left end of the right-hand beam is

$$B = -\frac{l}{\beta} M_t = -\frac{60}{14.30} 1345\ 10^4 = -5643\ 10^4 \text{ Nm}^2\,.$$

Using the torsion moment and bi-moment a program for cross-section analysis calculates the shear stress distribution and the normal stress distribution. The largest shear stress is -1.74 N/mm², which occurs everywhere in the bottom flange left of the symmetry plane. The largest normal stress is 7.28 N/mm², which occurs in the symmetry plane in the left side of the bottom flange (Fig. 6). It shows that also for closed thin-wall sections the stresses due to restrained warping can be important. In the bridge also stresses due to the flexural moment, shear and prestressing occur but these have not been considered in this example.

Figure 6: Distribution of the normal stresses due to restrained warping in the middle cross-section

6.3 ***Traffic sign***

The structure of a traffic signpost consists of two inclined columns and a horizontal beam (Fig. 7). All members are steel sections IPE400. The top joint and the foundation joints are such that warping cannot occur. At the free end of the horizontal beam warping can occur.

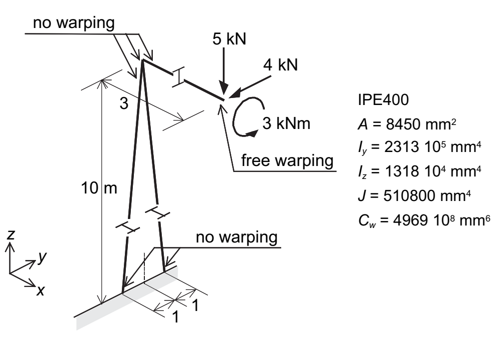


Figure 7: Structural model of a traffic sign

The length of the inclined columns is 10.05 m. Young's modulus is $E = 2.1\ 10^5$ N/mm² and Poisson's ratio is $\nu = 0.35$. All values are assumed to be design values, therefore, safety factors are already included. The β numbers of the columns and the beam are, respectively

$$\beta_c = l\sqrt{\frac{GJ}{EC_w}} = l\sqrt{\frac{J}{2(1+\nu)C_w}} = 10050\sqrt{\frac{510800}{2\times1.35\times4969\times10^8}} = 6.20 \ ,$$

$$\beta_b = 3000\sqrt{\frac{510800}{2\times1.35\times4969\times10^8}} = 1.85 \ .$$

The torsion constants for the frame analysis are respectively

$$\frac{\beta_c}{\beta_c - 2} J = \frac{6.20}{5.20} 510800 = 609000 \text{ mm}^4,$$

$$\frac{\beta_b (1 + e^{2\beta_b})}{\beta_b + 1 + (\beta_b - 1) e^{2\beta_b}} J = \frac{1.85(1 + e^{3.70})}{2.85 + 0.85 e^{3.70}} 510800 = 1052000 \text{ mm}^4.$$

The frame program computes a rotation of the free beam end of 0.107 rad in the direction of the moment loading. This is 51% smaller than computed with free warping of all member ends. The horizontal displacement of the free beam end is 107 mm, which is 1% smaller than with free warping. The torsion moment in the beam is 3.00 kNm. The torsion moment in the columns is 0.15 kNm. Consequently, the bi-moments are

$$B_c = -\frac{l}{\beta} M_t = -\frac{10.05}{6.20} 0.15 = -0.24 \text{ kNm}^2 \quad \text{in the fixed beam end,}$$

$$B_b = -\frac{l}{\beta} \tanh(\beta) M_t = -\frac{3.00}{1.85} 0.95 \times 3.00 = -4.62 \text{ kNm}^2 \quad \text{in the column ends.}$$

With a cross-section analysis program the normal stress is computed due to the bi-moments. The largest normal stress due to B = -4.62 kNm^2 is σ_{xx} = 164 N/mm^2. It occurs at the fixed end of the horizontal beam in the top and bottom flanges.
It is noted that for I-beams a formula can be derived for restrained warping stresses [Young 1989].

$$\sigma_{xx,\max} = \frac{B}{\frac{1}{6}(h - t) t b^2} \tag{19}$$

where h is the height of the cross-section, b is he width of the cross-section and t is the thickness of the flanges. The thickness of the web is not relevant.

7 Remarks

Though large stresses can occur due to restrained warping, this does not necessarily mean that traditional frame analysis is unsafe for the ultimate limit state. The reason is that traditional frame analysis provides a force flow that is everywhere in perfect equilibrium. The lower bound theorem of plasticity theory states that any equilibrium system that does not violate the ultimate stress conditions provides a safe solution for the limit load of the structure. In other words, the material will yield and the peak restrained warping stresses will not occur. Therefore, traditional frame analysis is safe provided that the materials used are sufficiently ductile to allow redistributions.
The advantages of using Vlasov torsion theory is mainly related to the serviceability limit state.

It can be used to show that the deformations will be smaller than found with the traditional theory. It can also be used for crack control in structural concrete box-girders.

8 Conclusions

A simple method is presented to include the effect of restrained warping of member-ends in traditional frame analyses. The method consists of just 5 steps and the formulas can be easily memorised.

Restrained warping can substantially reduce the structural deformation. It can also introduce large axial stresses in the restrained sections.

Restrained warping is not only important for thin-wall open sections. Also in thin-wall closed sections the normal stresses due to the restrained sometimes can be considerable.

Despite the large stresses that can occur when warping is restrained, a traditional frame analysis is a safe method of computing the ultimate load provided that the materials applied are sufficiently ductile.

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Appendix

Stiffness matrix for the Vlasov Torsion Theory

This matrix can be used for implementation of restrained warping in structural analysis software.

$$x = 0 \;\; \to \;\; \varphi = \varphi_1 \,, \;\; \frac{d\varphi}{dx} = \theta_1 \,, \;\; -M_t = M_1 \,, \;\; B = B_1$$
$$x = l \;\; \to \;\; \varphi = \varphi_2 \,, \;\; \frac{d\varphi}{dx} = \theta_2 \,, \;\; M_t = M_2 \,, -B = B_2$$

$$\begin{bmatrix} M_1 \\ M_2 \\ B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} l \\ \frac{1}{2} l \\ \frac{1}{2} \frac{l^2}{\beta^2 \lambda} \\ -\frac{1}{2} \frac{l^2}{\beta^2 \lambda} \end{bmatrix} m_x = \begin{bmatrix} \frac{GJ}{l}\eta & -\frac{GJ}{l}\eta & GJ\lambda & GJ\lambda \\ -\frac{GJ}{l}\eta & \frac{GJ}{l}\eta & -GJ\lambda & -GJ\lambda \\ GJ\lambda & -GJ\lambda & \frac{EC_w}{l}\xi & \frac{EC_w}{l}\mu \\ GJ\lambda & -GJ\lambda & \frac{EC_w}{l}\mu & \frac{EC_w}{l}\xi \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\eta = \frac{\beta(e^\beta + 1)}{\beta + 2 + (\beta - 2)e^\beta}$$
$$\lambda = \frac{e^\beta - 1}{\beta + 2 + (\beta - 2)e^\beta}$$
$$\mu = \frac{\beta}{e^\beta - 1} \, \frac{e^{2\beta} - 2\beta e^\beta - 1}{\beta + 2 + (\beta - 2)e^\beta}$$
$$\xi = \frac{\beta}{e^\beta - 1} \, \frac{\beta + 1 + (\beta - 1)e^{2\beta}}{\beta + 2 + (\beta - 2)e^\beta}$$

$$\beta = l\sqrt{\frac{GJ}{EC_w}}$$

For large values of β the stiffness matrix can be approximated as

$$\begin{bmatrix} M_1 \\ M_2 \\ B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} l \\ \frac{1}{2} l \\ \frac{1}{2} l\sqrt{\frac{EC_w}{GJ}} \\ -\frac{1}{2} l\sqrt{\frac{EC_w}{GJ}} \end{bmatrix} m_x = \begin{bmatrix} \frac{GJ}{l} & -\frac{GJ}{l} & \frac{\sqrt{GJEC_w}}{l} & \frac{\sqrt{GJEC_w}}{l} \\ -\frac{GJ}{l} & \frac{GJ}{l} & -\frac{\sqrt{GJEC_w}}{l} & -\frac{\sqrt{GJEC_w}}{l} \\ \frac{\sqrt{GJEC_w}}{l} & -\frac{\sqrt{GJEC_w}}{l} & \sqrt{GJEC_w} & \frac{EC_w}{l} \\ \frac{\sqrt{GJEC_w}}{l} & -\frac{\sqrt{GJEC_w}}{l} & \frac{EC_w}{l} & \sqrt{GJEC_w} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

If β to zero from above the stiffness matrix reduces to

$$\begin{bmatrix} M_1 \\ M_2 \\ B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}l \\ \frac{1}{2}l \\ \frac{1}{12}l^2 \\ -\frac{1}{12}l^2 \end{bmatrix} m_x = EC_w \begin{bmatrix} \frac{12}{l^3} & -\frac{12}{l^3} & \frac{6}{l^2} & \frac{6}{l^2} \\ -\frac{12}{l^3} & \frac{12}{l^3} & -\frac{6}{l^2} & -\frac{6}{l^2} \\ \frac{6}{l^2} & -\frac{6}{l^2} & \frac{4}{l} & \frac{2}{l} \\ \frac{6}{l^2} & -\frac{6}{l^2} & \frac{2}{l} & \frac{4}{l} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \theta_1 \\ \theta_2 \end{bmatrix}$$