Journal of Mathematics and Statistics 8 (4): 441-445, 2012
ISSN 1549-3644

doi:10.3844/jmssp.2012.441.445 Published Online 8 (4) 2012 (http://www.thescipub.com/jmss.toc)

4D N = 1 SUPERSYMMETRIC YANG-MILLS THEORIES ON KAHLER-RICCI SOLITON

Bobby Eka Gunara

Indonesia Center for Theoretical and Mathematical Physics,
Theoretical High Energy Physics and Instrumentation Research Group,
Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung,
Jl. Ganesha 10 Bandung 40132, Indonesia

Received 2012-08-31, Revised 2012-09-05; Accepted 2012-09-05

ABSTRACT

In this study we consider four dimensional N = 1 supersymmetric gauge Yang-Mills theory whose complex scalar manifold is Kahler and deforms with respect to a real parameter. The deformation of the geometry is governed by Kahler-Ricci flow equation. This setup implies that some couplings such as shifting quantities, momentum maps and the scalar potential turn out to be evolved with respect to the flow parameter. We also discuss deformation of vacuum structures of the theory in the context of Morse theory.

Keywords: Supersymmetry, Yang-Mills Theory, Kaehler-Ricci Flow

1. INTRODUCTION

The standard model of particle physics based on non-Abelian gauge theory with gauge group SU(3)×SU(2)×U(1) has gained several remarkable success which can be seen from verified experiments in the energy scale below 1000 GeV including the recent discovery of Higgs particle. Despite its success, it leaves many important problems. For example, the first problem is that the standard model neglects the gravity which is described by the general relativity. Secondly, it cannot explain the mass hierarchy. Also, it severe from quadratic divergences. Thus, the standard model has to be extended.

One of the good candidate for the extensions of the standard model is supersymmetry. In order to get a reasonable supersymmetric extension of the standard model, this extention theory must inherit the chiral structure of the standard model. Thus, the only possible extention is N = 1 supersymmetry because extended supersymmetries ($N \geq 2$) cannot accommodate the chiral structure, for a review see, for example, (Louis *et al*., 1998).

Although N = 1 supersymmetry has some phenomenological aspects, our interest is to study closely to the mathematical context. For example, we have previous serial papers studying solitonic solutions of four dimensional N = 1 local supersymmetry (supergravity) on Kahler manifolds satisying Kahler-Ricci flow equation (Cao, 1985). Our results show that in the of both domain walls (Gunara and Zen, 2009a; Gunara *et al*., 2011) and black holes (Gunara, 2012) in general deform with respect to a flow parameter related to Kahler-Ricci equation. Moreover, this flow could change the nature of stability of domain walls and geometry of black holes.

We extend the studies in this study to rigid non-Abelian supersymmetric theories, namely supersymmetric Yang-Mills theories in four dimensions defined on Kahler-Ricci soliton. This follows that some couplings such as shifting quantities, momentum maps and the scalar potential deformed with respect to the flow parameter, see Lemma 1. Moreover, vacuum structures which can be viewed as solutions of field equations of motions indeed evolve with respect to the parameter.

To see the latter, we simply consider a case where at the level of equations of motions all fermions vanishes and the scalars are frozen everywhere such that the gauge fields are trivial. In this case, the ground states can be thought of as supersymmetric critical points of the scalar potential. Taking the assumption that the ground states to be nondegenerate, we find that Morse index of a ground state is affected by Kahler-Ricci flow. In other words, the flow possibly changes the properties of supersymmetric vacua, see Theorem 2. This fact gives us an example of deformed Morse theory.

The application of this study has two major directions. The first case is the dynamics of monopoles or solitonic solutions of N = 1 supersymmetric gauge theories with respect to Kahler-Ricci flow. We would like to see how the flow changes the stability of solutions. For example, this aspect has been observed in the case of domain walls in chiral N = 1 supergravity (Gunara and Zen, 2009b). The second case is the evolutions of real and complex vacuum submanifolds of Kahler manifolds with respect to Kahler-Ricci flow. This aspect is related to the study of evolutions of minimal submanifold under Ricci flow, see for example (Tsatis 2010).

2. BRIEF REVIEW: KAHLER-RICCI SOLITON

The devoted to assemble some facts about Kahler-Ricci flow equation which is useful for our analysis in this study. This flow equation was firstly introduced in (Cao, 1985).

A complex Kahler manifold M endowed with metric $g(\tau)$ is said to be Kaehler-Ricci soliton if it satisfies Equation 1:

$$\frac{\partial g_{i\bar{j}}}{\partial\tau} = -2R_{i\bar{j}} \tag{1}$$

where, i, j = 1,…., dim M, τ is a real parameter and $R_{i\bar{j}}$ denote the 2-rank Ricci tensor of M. The simplest solution of (1) is when the initial geometry at $\tau = 0$ is Einstein, namely Equation 2:

$$R_{i\bar{j}}(0) = \Lambda\, g_{i\bar{j}}(0) \tag{2}$$

where, Λ is a real constant and nonzero. Then, we have Equation 3:

$$g_{i\bar{j}}(\tau) = (1-2\Lambda\tau)\; g_{i\bar{j}}(0) \tag{3}$$

whose Kahler potential has the form Equation 4:

$$K(\tau) = (1-2\Lambda\tau)\; K(0) \tag{4}$$

with $K(\tau) \equiv K(z,\bar{z};\tau)$. As we have seen above, for the simplest example, there exists singularity at $\tau = 1/2\Lambda$ where the flow shrinks to zero. This indicates that the singularity could be occurs in general cases, see for example (Topping, 2006; Cao and Zhu, 2006).

Another interesting solution of (1) is when the initial geometry satisfies Equation 5:

$$-2R_{i\bar{j}}(0) = -2\Lambda g_{i\bar{j}}(0) + \nabla_i Y_{\bar{j}}(0) + \bar{\nabla}_{\bar{j}} Y_i(0) \tag{5}$$

where, $Y^i(0)$ is a vector field generating a diffeomorphism which can be expressed in terms of a real function $P(z,\bar{z})$ on M as Equation 6:

$$Y^i = g^{i\bar{j}}\partial_{\bar{j}} P(z,\bar{z}) \tag{6}$$

Such a solution is called gradient Kahler-Ricci soliton (Cao, 1996; 1997). In general, (5) can be split into three cases as follows. For $\Lambda > 0$ the soliton is shrinking, whereas for $\Lambda < 0$ the soliton is expanding. In the case of $\Lambda = 0$ we have a steady gradient Kahler-Ricci soliton.

3. 4D N = 1 SUPERSYMMETRIC YANG-MILLS THEORY ON KAHLER-RICCI SOLITON

We focus on the properties of the deformed N = 1 supersymmetric gauge theory in four dimensions on Kahler-Ricci soliton. The spectrum of the theory consists of vector fields A^Λ_μ and spin-1/2 gauginos λ^Λ with $\mu = 0, \ldots, 3$, $\Lambda = 1,\ldots,n_v$, coupled to complex scalar fields z^i and spin-1/2 fermions x^i with $i = 1,\ldots,n_c$. The complex scalars $(z^i,\bar{z}^{\bar{i}})$ span a Kahler geometry M of dimension n_c. The construction of the N = 1 supersymmetric gauge theory on Kahler-Ricci soliton follows closely (Gunara, 2012; Gunara and Zen, 2009a; 2009c). First, we consider the chiral Lagrangian in (D'Auria and Ferrara, 2001), say $\mathcal{L}(0)$ at $\tau = 0$, where the metric of the scalar manifold is static. Then, replacing all geometric quantities such as the metric $g_{i\bar{j}}(0)$ by the soliton $g_{i\bar{j}}(\tau)$, the bosonic parts of the on-shell N = 1 chiral Lagrangian can be written down as Equation 7:

$$\begin{aligned}\mathcal{L} = &-g_{i\bar{j}}(\tau)D^{\mu}z^{i}D_{\mu}\bar{z}^{\bar{j}} \\ &+R_{\Lambda\Sigma}\mathcal{F}^{\Lambda}_{\mu\nu}\mathcal{F}^{\Sigma\mu\nu}+I_{\Lambda\Sigma}\mathcal{F}^{\Lambda}_{\mu\nu}\tilde{\mathcal{F}}^{\Sigma\mu\nu}-V(\tau)\end{aligned} \tag{7}$$

where the function $V(\tau)$ is the scalar potential of the theory which has the form Equation 8:

$$V(\tau) = g^{i\bar{j}}(\tau)\partial_i W \bar{\partial}_{\bar{j}}\bar{W} + \frac{1}{8}R^{-1|\Lambda\Sigma}P_{\Lambda}(\tau)P_{\Sigma}(\tau) \tag{8}$$

Here, $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(\tau)$ is Kahler metric, whereas $R_{\Lambda\Sigma}$ and $I_{\Lambda\Sigma}$ are respectively the real and imaginer parts of holomorphic gauge kinetic functions $F_{\Lambda\Sigma}$. The covariant derivative $D_{\mu}z^{i}$ is given by $D_{\mu}z^{i} = \partial_{\mu}z^{i} + k^{i}_{\Lambda}A^{\Lambda}_{\mu}$ where k^{i}_{Λ} is a holomorphic Killing vector generating isometries of M satisfying Equation 9:

$$\left[k^{i}_{\Lambda}, k^{i}_{\Sigma}\right] = f^{\Gamma}_{\ \Lambda\Sigma}\, k^{i}_{\Gamma} \tag{9}$$

The field strength $\mathcal{F}^{\Lambda}_{\mu\nu}$ is defined as Equation 10:

$$\mathcal{F}^{\Lambda}_{\mu\nu} = \partial_{\mu}A^{\Lambda}_{\nu} - \partial_{\nu}A^{\Lambda}_{\mu} + f^{\ \Lambda}_{\Sigma\Gamma}A^{\Sigma}_{\mu}A^{\Gamma}_{\nu} \tag{10}$$

while its dual field is $\tilde{\mathcal{F}}^{\Lambda\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}^{\Lambda}_{\rho\sigma}$.

The holomorphic superpotential $W \equiv W(z)$ is arbitrary. The real momentum maps $P_{\Lambda}(\tau) \equiv P_{\Lambda}(z, \bar{z}; \tau)$ has the form Equation 11:

$$P_{\Lambda}(\tau) = -i\, k^{i}_{\Lambda}\partial_{i}K(\tau) \tag{11}$$

The Lagrangian (7) is invariant under supersymmetric transformation (up to four-fermion terms) Equation 12:

$$\begin{aligned}&\delta\chi^{i} = i\, D_{\mu}z^{i}\gamma^{\mu}\varepsilon + N^{i}(\tau)\varepsilon, \\ &\delta\lambda^{\Lambda} = \mathcal{F}^{\Lambda -}_{\mu\nu}\gamma^{\mu\nu}\varepsilon + N^{\Lambda}(\tau)\varepsilon, \\ &\delta z^{i} = \bar{\chi}^{i}\varepsilon,\ \delta A^{\Lambda}_{\mu} = \frac{i}{2}\bar{\lambda}^{\Lambda}\gamma_{\mu}\varepsilon + h.c\end{aligned} \tag{12}$$

where the shifting quantities $N^{i}(\tau)$and $N^{\Lambda}(\tau)$ are given by Equation 13:

$$\begin{aligned}&N^{i}(\tau) = g^{i\bar{j}}\bar{\partial}_{\bar{j}}K(\tau) \\ &N^{\Lambda}(\tau) = i\, R^{-1|\Lambda\Sigma}P_{\Sigma}(\tau)\end{aligned} \tag{13}$$

Now, we can write down the dynamic equations of the shifting quantities (13) and the scalar potential (8).

Lemma 1 Equation 14:

$$\begin{aligned}&\frac{\partial N^{i}}{\partial\tau}(\tau) = g^{i\bar{j}}\bar{\partial}_{\bar{j}}\frac{\partial K}{\partial\tau}(\tau), \\ &\frac{\partial N^{\Lambda}}{\partial\tau}(\tau) = i\, R^{-1|\Lambda\Sigma}\frac{\partial P_{\Sigma}}{\partial\tau}(\tau), \\ &\frac{\partial V}{\partial\tau}(\tau) = 2R^{i\bar{j}}(\tau)\partial_{i}W\bar{\partial}_{\bar{j}}\bar{W} + \frac{1}{4}R^{-1|\Lambda\Sigma}P_{\Lambda}(\tau)\frac{\partial P_{\Sigma}}{\partial\tau}(\tau)\end{aligned} \tag{14}$$

Proof

One can use (1), (8) and (13) in a straightforward way.

4. DEFORMATION OF VACUUM STRUCTURES

We discuss vacuum structures of the theory which can be viewed as the solution of field equations of motions derived from the Lagrangian (7). The equations of motions can be obtained by varying (7) with respect to z^{i} and A^{Λ}_{μ}. The fermions vanish at the level of equations of motions. Then, we have Equation 15:

$$\begin{aligned}&D^{\mu}D_{\mu}z^{i} = -g^{i\bar{j}}(\tau)\left(\bar{\partial}_{\bar{j}}R_{\Lambda\Sigma}\mathcal{F}^{\Lambda}_{\mu\nu}\mathcal{F}^{\Sigma\mu\nu} + \bar{\partial}_{\bar{j}}I_{\Lambda\Sigma}\mathcal{F}^{\Lambda}_{\mu\nu}\tilde{\mathcal{F}}^{\Sigma\mu\nu}\right) \\ &\qquad\qquad -g^{i\bar{j}}(\tau)\bar{\partial}_{\bar{j}}V(\tau) \\ &D_{\mu}\left(R_{\Lambda\Sigma}\mathcal{F}^{\Sigma\mu\nu} + I_{\Lambda\Sigma}\tilde{\mathcal{F}}^{\Sigma\mu\nu}\right) = -g_{i\bar{j}}(\tau)k^{i}_{\Lambda}D^{\nu}\bar{z}^{\bar{j}} - g_{i\bar{j}}(\tau)k^{\bar{j}}_{\Lambda}D^{\nu}z^{i}\end{aligned} \tag{15}$$

Now, let us consider the solution of (14) as follows. When the scalars z^{i} become fixed, namely $z^{i} = z^{i}_{0}$ which follows $A^{\Lambda}_{\mu} = 0$. Moreover, (14) becomes simply Equation 16:

$$\bar{\partial}_{\bar{j}}V(\tau) = 0 \tag{16}$$

then we have Equation 17:

$$\partial_{i}W = 0,\ P_{\Lambda}(\tau) = 0 \tag{17}$$

From supersymmetric variation (12), the condition (17) describes N = 1 supersymmetric vacua. In general, the vacuum geometry $N_0 \subseteq M$ is real and deforms with respect to τ. In this study we particularly consider N_0 to be discrete. In order to characterize the ground states, we have to consider the second-order derivative of the scalar potential (8) with respect to $(z, \bar{z})$ called Hessian matrix evaluated at $p_0(\tau)=\left(z_0(\tau),\bar{z}_0(\tau)\right)$, whose nonzero component has the form Equation 18:

$$\begin{aligned}\partial_i\bar{\partial}_{\bar{j}}V(p_0;\tau) &= g^{k\bar{l}}(\tau)\\ \partial_i\partial_k W\bar{\partial}_{\bar{j}}\bar{\partial}_{\bar{l}}\bar{W} &+ \frac{1}{8}R^{-1|\Lambda\Sigma}k_{i\Lambda}(\tau)k_{\bar{j}\Sigma}(\tau)\end{aligned} \quad (18)$$

where we have used (11). Note that all quantities in (18) have been evaluated at p_0. In the rest of the study we simply consider the case of Kahler-Einstein metric satisfying (3). So, equation (17) becomes Equation 19:

$$\begin{aligned}\partial_i\bar{\partial}_{\bar{j}}V(p_0;\tau) &= \sigma^{-1}(\tau)g^{k\bar{l}}(0)\partial_i\partial_k W\bar{\partial}_{\bar{j}}\bar{\partial}_{\bar{l}}\bar{W}\\ &+\frac{1}{8}\sigma^2(\tau)R^{-1|\Lambda\Sigma}k_{i\Lambda}(0)k_{\bar{j}\Sigma}(0)\end{aligned} \quad (19)$$

Where Equation 20:

$$\sigma(\tau)\equiv 1-2\Lambda\tau \quad (20)$$

and $g_{k\bar{l}}(0)$ is a Kahler-Einstein metric. In (19) it is easy to see that when the flow becomes ill-defined at $\tau = 1/2\Lambda$, the theory also turns to be singular. Then (19) leads to the following statements.

Theorem 2

Let the scalar potential (8) be Morse function and $\tau \neq 1/2\Lambda$, so that the determinant of (19) is nonzero. Suppose that $p_0(\tau) = q_0$ is an isolated ground state (nondegenerate) of Morse index λ for $\tau < 1/2\Lambda$ with $\Lambda > 0$. Then, there exists real local coordinates $X_r(\tau)$ around q_0 with $r = 1,\ldots,2n_c$ such that Equation 21:

$$\begin{aligned}V(\tau) = V(p_0;\tau) &- X_1^2(\tau) - \ldots - X_\lambda^2(\tau) + X_{\lambda+1}^2(\tau)\\ &+\ldots + X_{2n_c}^2(\tau)\end{aligned} \quad (21)$$

Taking the assumption the real and the imaginary parts of the following quantities Equation 22:

$$V_{i\bar{j}}(p_0(\tau);\tau)\equiv\sigma(\tau)\partial_i\bar{\partial}_{\bar{j}}V(p_0(\tau);\tau) \quad (22)$$

are positive for all τ and i, j. Let $p_0(\tau)=\hat{q}_0$ be another isolated ground state for $\tau > 1/2\Lambda$. Then, Kahler-Ricci soliton changes the index λ to $2n_c$-λ such that near $\hat{q}_0$ we have $\hat{X}_r(\tau)\neq X_r(\tau)$ and Equation 23:

$$\begin{aligned}V(\tau) = V(p_0;\tau) &+ \hat{X}_1^2(\tau) + \ldots + \hat{X}_\lambda^2(\tau) - \hat{X}_{\lambda+1}^2(\tau)\\ &-\ldots - \hat{X}_{2n_c}^2(\tau)\end{aligned} \quad (23)$$

for $\tau > 1/2\Lambda$.

Proof

First of all, we define the Hessian matrix of the scalar potential (8) of the theory as Equation 24:

$$H_V \equiv \sigma^{-1}(\tau)\begin{pmatrix} V_{i\bar{j}} & V_{ij}\\ V_{\bar{j}\bar{i}} & V_{\bar{i}j}\end{pmatrix}(p_0) \quad (24)$$

where, p_0 is an isolated critical point near which the scalar potential (8) can be expanded as Equation 25:

$$V(\tau) = V(p_0;\tau) + \sum_{p,q=1}^{2n}\frac{\partial^2 V}{\partial x^p\partial x^q}(p_0;\tau)\delta x^p\delta x^q \quad (25)$$

where we have defined real coordinates $z^i\equiv x^i + i\,x^{i+n_c}$ such that $\delta x^i\equiv x^i - x_0^i$.

Since we only consider nondegenerate case, the matrix (24) does not have zero eigenvalues and is non singular because $\tau\neq 1/2\Lambda$. Let us now rewrite (25) as Equation 26:

$$V(\tau) = V(p_0;\tau) + \frac{\varepsilon(\sigma)}{|\sigma(\tau)|}\sum_{p,q=1}^{2n}V_{pq}(p_0;\tau)\delta x^p\delta x^q \quad (26)$$

Where Equation 27:

$$\varepsilon(\sigma)\equiv\begin{cases}1 & \text{for } \tau < 1/2\Lambda\\ -1 & \text{for } \tau > 1/2\Lambda\end{cases} \quad (27)$$

and then, one can define new coordinates Equation 28:

$$Y_r(\tau)\equiv\left|\tilde{V}_{rr}(p_0;\tau)\right|^{1/2}\left(\delta x^r + \sum_{p=r+1}^{2n}\delta x^p\frac{\tilde{V}_{pq}(p_0;\tau)}{\tilde{V}_{rr}(p_0;\tau)}\right) \quad (28)$$

for $1 \le r \le 2n_c$. Thus, we can rewrite (25) in the simplest bilinear form (21) and (22) with identification Equation 29:

$$Y_r(\tau)\equiv\begin{cases}X_r(\tau) & \text{for } \tau < 1/2\Lambda\\ \hat{X}_r(\tau) & \text{for } \tau > 1/2\Lambda\end{cases} \quad (29)$$

Some comments are in order. The extension of Theorem 2 for degenerate vacua is in a straightforward way. It is worth mentioning that Theorem 2 is the evidence of deformed Morse theory related to deformation of (vacuum) submanifolds of Kahler geometry. Since the flow (1) could change the index of a ground state, so in general it could indeed affect the geometrical nature of the submanifolds. The latter aspects will be considered elsewhere.

5. CONCLUSION

So far, we have constructed four dimensional N = 1 supersymmetric Yang-Mills theory on Kahler-Ricci soliton. As we have seen, this setup implies that some couplings, namely the shifting quantities, the momentum maps and the scalar potential evolved with respect to the flow parameter, that is equation (13) in Lemma 1.

Moreover, we also have showed that the nondegenerate vacua of the theory is evolved with respect to the flow parameter. It is also possible that their Morse index changes caused by Kahler-Ricci flow, see Theorema 2.

6. ACKNOWLEDGEMENT

The study of this study is supported by Hibah Kompetensi DIKTI 2012 No. 781a/I1.C01/PL/2012.

7. REFERENCES

Cao, H.D. and X.P. Zhu, 2006. Hamilton-Perelman’s proof of the poincare conjecture and the geometrization conjecture. Asian J. Math., 10: 165-492.

Cao, H.D., 1985. Deformation of Kahler matrics to Kahler-Einstein metrics on compact Kahler manifolds. Inventiones Math., 81: 359-372. DOI: 10.1007/BF01389058

Cao, H.D., 1996. Existence of Gradient Kaehler-Ricci Solitons. In: Elliptic and Parabolic Methods in Geometry, Chow, B. (Ed.), A K Peters, Wellesley, MA, ISBN-10: 1568810644, pp: 1-16.

Cao, H.D., 1997. Limits of solutions to the Kahler-Ricci flow. J. Differential Geometry, 45: 257-272.

D’Auria, R. and S. Ferrara, 2001. On fermion masses, gradient flows and potential in supersymmetric theories. J. High Energy Phys., 5: 34-34. DOI: 10.1088/1126-6708/2001/05/034

Gunara, B.E. and F.P. Zen, 2009a. Kahler-Ricci flow, Morse theory and vacuum structure deformation of N = 1 supersymmetry in four dimensions. Adv. Theoretical Math. Phys., 13: 217-257.

Gunara, B.E. and F.P. Zen, 2009b. Deformation of curved BPS domain walls and supersymmetric flows on 2d Kahler-Ricci soliton. Commun. Math. Phys., 287: 849-866. DOI: 10.1007/s00220-009-0744-1

Gunara, B.E. and F.P. Zen, 2009c. Flat Bogomolnyi-Prasad-Sommerfeld domain walls on two-dimensional Kahler-Ricci soliton. J. Math. Phys., 50: 063514-063522. DOI: 10.1063/1.3155786

Gunara, B.E., 2012. Spherical symmetric dyonic black holes and vacuum geometries in 4 D N=1 supergravity on Kahler-ricci soliton. Reports Math. Phys., 69: 281-309. DOI: 10.1016/S0034-4877(12)60032-9

Gunara, B.E., F.P. Zen and Arianto, 2011. N = 1 supergravity BPS domain walls on Kahler–Ricci soliton. Reports Math. Phys., 67: 395-413. DOI: 10.1016/S0034-4877(11)60021-9

Louis, J., I. Brunner and S. Huber, 1998. The supersymmetric standard model. High Energy Phys.

Topping, P., 2006. Lectures on Ricci flow. Cambridge University Press.

Tsatis, E., 2010. Mean curvature flow on Ricci solitons. J. Phys. A: Math. Theor., 43: 045202-045202. DOI: 10.1088/1751-8113/43/4/045202