

Generation of Link Mechanism by Shape-Topology Optimization of Trusses Considering Geometrical Nonlinearity*

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Abstract

A two-stage general optimization approach is presented for generating link mechanisms from a highly connected ground structure. The structure is modeled as a pin-jointed truss, which is to be optimized so that a large displacement is generated in the specified direction at the output node. The design variables are the cross-sectional areas of the members and the nodal locations. The equilibrium path of an unstable mechanism is traced by the displacement control method. In the first step, the unnecessary members are removed by solving the optimization problem for minimizing the total structural volume under constraints on the maximum load, the displacement at the specified node, and the stiffnesses at the initial and final states. In the second step, the deviation of the displacement of the output node from the specified direction is minimized. It is shown in the numerical examples that several mechanisms can be naturally found as a result of the two-stage optimization starting from randomly selected initial solutions.

Key words : Optimum Design, Mechanism, Large Deformation, Topology Optimization, Truss, Snapthrough

1. Introduction

There have been many studies for design of link mechanisms. However, most of them are based on trial-and-error, and few mathematical methods have been developed. Kawamoto⁽¹⁾ presented a method for generating a truss mechanism that realizes a specified deformation. Kawamoto *et al.*⁽²⁾ presented an enumeration method of topologies of mechanisms based on graph theory. However, in their method, the numbers of nodes and members that exist in the final design should be specified *a priori*. Kim *et al.*⁽³⁾ generated a mechanism that attains the specified deformation by removing the unnecessary springs from the ground structure with many springs connecting rigid parts. However, the nodal locations and the sizes of the rigid parts are fixed; therefore, the geometry is not fully optimized.

Fujii *et al.*⁽⁴⁾ developed a method based on random start nonlinear programming algorithm. However, the effect of large deformation has not been incorporated in their method. Sekimoto and Noguchi⁽⁵⁾ presented an optimization method for generating a continuum structure of which the output node follows the specified path in the load-displacement space. However, the structure is stable and no link mechanism has been obtained.

Recently, optimization of structures considering geometrical nonlinearity, and/or buckling behavior has been extensively studied^{(6)–(8)}. The authors developed an optimization-based approach to multi-stable compliant bar-joint structure utilizing snapthrough behavior^{(9), (10)}.

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In this paper, we first present the condition for a truss to be an unstable link mechanism using the rank of the equilibrium matrix and the eigenvalues of the stiffness matrix. We next extend the ground structure approach in Ref. (9), (10) to optimize the topology and nodal locations of a link mechanism to trace the specified path of the output node. A two-stage optimization problem is solved for this purpose, where a structure satisfying the mechanism condition is generated in the first stage, and the deviation of the path of the output node from the specified path is minimized in the second stage. The equilibrium path is traced considering geometrical nonlinearity, and the optimization is carried out using nonlinear programming from randomly generated initial solutions.

2. Kinematical indeterminacy of the link mechanism.

Consider a truss consisting of elastic bars connected by pin joints. Let $\mathbf{P} \in \mathbb{R}^n$ denote the vector of external loads, where n is the number of degrees of freedom. The vector consisting of the axial forces of members is denoted by $\mathbf{N} \in \mathbb{R}^m$, where m is the number of members. Then the equilibrium equations are formulated as

$$\mathbf{D}\mathbf{N} = \mathbf{P} \quad (1)$$

where $\mathbf{D} \in \mathbb{R}^{n \times m}$ is called equilibrium matrix. If the rank r of \mathbf{D} is equal to m , then the truss is statically determinate so that \mathbf{N} for the specified \mathbf{P} is found by solving (1). If $r < m$, then the truss is unstable, and there exist $h = m - r$ independent mechanisms, where h is called kinematical indeterminacy. In this paper, we generate a structure with $h = 1$ so that only one mechanism exists in the structure.

We may use one of the following two approaches for modeling and optimization of the link mechanism:

(1) The bars are assumed to be rigid, and the equilibrium path is traced by incorporating the constraints so that the length of each bar remains unchanged. The nodal locations are optimized to minimize the deviation of the path of the output node from the specified path.

(2) The bars are assumed to be elastic, and the equilibrium path is traced using the standard displacement increment method for large-deformation analysis. The optimal topology and nodal locations are found using the ground structure approach with cross-sectional areas of the members and the nodal coordinates as design variables. The maximum absolute value of the load throughout the path is constrained to be 0 so that an unstable link mechanism is generated.

In the first method, the numbers of nodes and bars should be appropriately assigned so that the link mechanism of order 1 is to be generated. Furthermore, many constraints are required to restrict the deformation of bars. Therefore, we use the second approach, which is effective in both of optimization and analysis.

Let L_i and A_i denote the length and cross-sectional area of the i th member. Young's modulus is denoted by E . The diagonal matrix that has $A_i E / L_i$ in the i th diagonal element is denoted by $\mathbf{C} \in \mathbb{R}^{m \times m}$. Then the stiffness matrix $\mathbf{K} \in \mathbb{R}^{n \times n}$ is given as

$$\mathbf{K} = \mathbf{D}\mathbf{C}\mathbf{D}^T \quad (2)$$

Since \mathbf{C} is full-rank, \mathbf{K} and \mathbf{D} have the same rank r . Suppose that all the rigid-body motions are appropriately constrained. There exists only one mode Φ of mechanism satisfying $\mathbf{K}\Phi = \mathbf{0}$

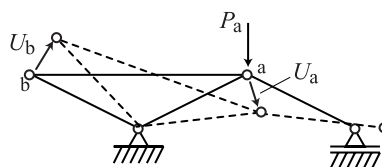


Fig. 1 A link mechanism.

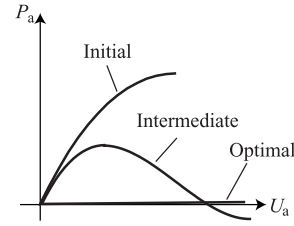


Fig. 2 Relation between input force P_a and input displacement U_a of initial, intermediate and optimal solutions.

in a structure with one degree of kinematical indeterminacy. Let λ_i denote the i th lowest eigenvalue of \mathbf{K} . Then $\lambda_1 = 0$ and $\lambda_i > 0$ ($i = 2, \dots, n$) are satisfied. In this case, the structure is stabilized by constraining one degree-of-freedom corresponding to a nonzero component of Φ , and the equilibrium path is traced by using a finite element analysis program utilizing the displacement increment method, or the arc-length method.

3. Optimization problem for generating a link mechanism.

Link mechanisms are generated by removing unnecessary members through optimization from a highly connected ground structure. In the following, a plane structure is assumed for simplicity.

The final state of a mechanism is defined as shown in Fig. 1 so that the displacement U_b in the specified direction of node 'b' reaches the specified value \bar{U}_b as a result of input displacement U_a (> 0) at node 'a' as

$$U_b = \bar{U}_b \quad (3)$$

The solid and dotted lines in Fig. 1 show the initial and final states, respectively.

Since the truss is stable with many members at the initial design of optimization, its maximum load before reaching the final state is not zero; i.e., the truss does not satisfy the condition of link mechanism. Therefore, we assign the condition so that the maximum value P_a^{\max} of the load P_a throughout the loading process in the specified direction at the input node 'a' is equal to 0. Hence, an unstable link mechanism is generated. Fig. 2 illustrates the load-displacement relation for the initial, intermediate and optimal solutions. Since the maximum load cannot be negative, as illustrated in Fig. 2, the constraint on the maximum load is relaxed as

$$P_a^{\max} \leq 0 \quad (4)$$

It is verified in the numerical examples in Section 4 that the inequality constraint (4) is satisfied in equality at the optimal solution. However, due to a small tolerance ε for the constraint violation, (4) is actually given as

$$P_a^{\max} \leq \varepsilon \quad (5)$$

The design variables are the vectors of cross-sectional areas $\mathbf{A} \in \mathbb{R}^m$ and nodal coordinates $\mathbf{X} \in \mathbb{R}^n$. The total structural volume $V(\mathbf{A}, \mathbf{X})$ is minimized to obtain a mechanism with less numbers of nodes and members. However, a structure with very small stiffness and slender members is obtained if only (4) is given as constraint. Let U_{bx}^{f0} and U_{by}^{f0} denote the displacements in x - and y -directions of node 'b' against unit loads in x - and y -directions at node 'b', respectively, at the final deformed state, where node 'a' is constrained in the direction of input displacement. The following constraints are given for ensuring the stiffness of the structure^{(9),(10)}:

$$U_{bx}^{f0} \leq \bar{U}_{bx}^{f0}, \quad (6)$$

$$U_{by}^{f0} \leq \bar{U}_{by}^{f0} \quad (7)$$

where \bar{U}_{bx}^{f0} and \bar{U}_{by}^{f0} are the specified values, and the tangent stiffness at the deformed state is used for evaluating U_{bx}^{f0} and U_{by}^{f0} .

Hence, the optimization problem for generating a truss that satisfies the mechanism condition is formulated as

$$\text{minimize } V(\mathbf{A}, \mathbf{X}) \quad (8a)$$

$$\text{subject to } P_a^{\max}(\mathbf{A}, \mathbf{X}) \leq 0, \quad (8b)$$

$$U_{bx}^{f0}(\mathbf{A}, \mathbf{X}) \leq \bar{U}_{bx}^{f0}, \quad (8c)$$

$$U_{by}^{f0}(\mathbf{A}, \mathbf{X}) \leq \bar{U}_{by}^{f0}, \quad (8d)$$

$$\mathbf{A}^L \leq \mathbf{A}, \quad (8e)$$

$$\mathbf{X}^L \leq \mathbf{X} \leq \mathbf{X}^U \quad (8f)$$

where \mathbf{X}^U and \mathbf{X}^L are the upper and lower bounds for \mathbf{X} . $\mathbf{A}^L = (A_1^L, \dots, A_m^L)^T$ consists of a sufficiently small lower bound for A_i , and the member with $A_i = A_i^L$ is removed after optimization.

The optimization algorithm is summarized as

Step 1 Assign initial values for \mathbf{A} and \mathbf{X} .

Step 2 Apply forced displacement in the specified direction at the input node 'a' and carry out path-tracing analysis to find the deformation at the final state satisfying (3).

Step 3 Evaluate the objective and constraint functions and their sensitivity coefficients.

Step 4 Update the design variables \mathbf{A} and \mathbf{X} in accordance with the optimization algorithm.

Step 5 Go to Step 2 if the stopping criteria are not satisfied.

It can be decided that the final state cannot be obtained if the direction of displacement of node 'b' is reverse to the specified one at Step 2. Furthermore, the optimization problem stated above has strong nonlinearity that leads to many local optimal solutions. Therefore, in the following examples, the initial values of \mathbf{A} and \mathbf{X} are randomly generated, and the analysis is carried out only when the displacement U_b at the first step of the path-tracing analysis is in the specified direction.

The optimal solution obtained by solving this first problem is unstable and satisfies the condition of single-order mechanism. However, the absolute value of the displacement \bar{U}_b perpendicular to the specified direction of node 'b' may be very large, because no constraint is given for that displacement component. Therefore, another optimization problem, stated as follows, is solved to minimize the maximum absolute value \bar{U}_b^{\max} of \bar{U}_b considering only \mathbf{X} as design variables:

$$\text{minimize } \bar{U}_b^{\max}(\mathbf{X}) \quad (9a)$$

$$\text{subject to } \mathbf{X}^L \leq \mathbf{X} \leq \mathbf{X}^U \quad (9b)$$

Hence, a link mechanism that exhibits specified deformation can be generated by solving a two-stage optimization problem, and all members are replaced with a rigid bar with sufficiently large cross-sectional areas.

A mechanism with minimum number of members without unnecessary members can be obtained by solving the first optimization problem of minimizing the total structural volume, if the optimization process converges to the global minimum. However, since both of the

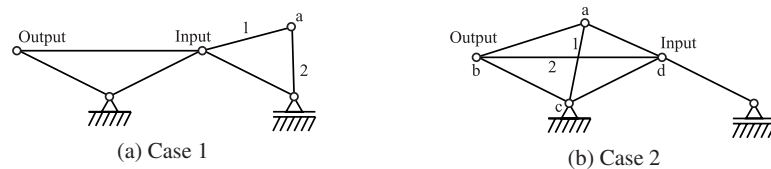


Fig. 3 Illustration of unnecessary members.

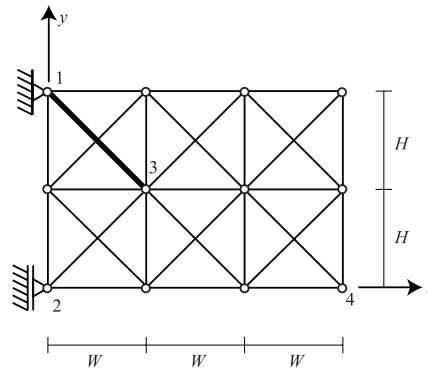


Fig. 4 A plane truss model.

analysis and optimization problems are highly nonlinear, the optimization process often converges to a local minimum. Furthermore, the purpose of this paper is to generate mechanisms of various shapes and topologies. Therefore, the second problem is solved for all local optimal solutions of the first problem. However, it is still highly likely that unnecessary members exist after solving the two-stage problem, because the number of members are fixed in the second problem. Hence, the following conditions are applied after two-stage optimization to remove the obviously unnecessary members that does not contribute to the input-output relation:

(1) If no axial force exists in a member after constraining the output node in the specified direction and applying the input force, then that member is to be removed; i.e., if there exist only one or two members connected to a node except the supports and the input node, then the members connected to the node are unnecessary. For example, members 1 and 2 connecting to node 'a' in Fig. 3(a) should be removed.

(2) If there exists a locally statically indeterminate part, and stability is not lost after removing a member, then the member can be removed. For example, in Fig. 3(b), the part defined by nodes 'a', 'b', 'c' and 'd' is statically indeterminate, and one of the members 1 and 2 can be removed.

The performance of the structure after removing the unnecessary members is verified by carrying out path-tracing analysis.

4. Numerical examples.

Link mechanisms are generated by solving a two-stage optimization problem from the ground structure as shown in Fig. 4, where $W = H = 200$ mm. The truss has pin-support at node 1, and only the displacement in x -direction is fixed at node 2. The ground structure is made of 3×2 square units, and the intersecting members are not connected at their centers.

The mechanisms are generated so that node 4 moves 200 mm in y -direction as a result of anti-clockwise rotation of node 3 around the pin support 1. For this, a sufficiently large lower bound A_i^L is assigned for the cross-sectional area of the member connecting support 1 and node 3, indicated by thick line in Fig. 4. Since the mechanism obtained after optimization is unstable, its property does not depend on the values of the cross-sectional areas. Hence, a forced displacement U_a is assigned in y -direction at node 3, and the final state is defined so that the displacement in y -direction of node 4 reaches 200 mm. The units of force and length are N and mm, respectively, and the upper-bound displacements in the first problem are given as $\bar{U}_{bx}^{f0} = \bar{U}_{by}^{f0} = 50$ mm.

The cross-sectional areas of all the members are considered as independent design variables. Small lower bound 0.01 mm^2 is given for all members except that connecting support 1 and node 3. The x -coordinate of node 3, the x, y -coordinates of node 4, and the coordinates of the supports in the fixed directions are fixed. For other nodes, the feasible regions defined in (8f) are bounded by ± 60 mm from the locations in Fig. 4 to assure sufficiently large search

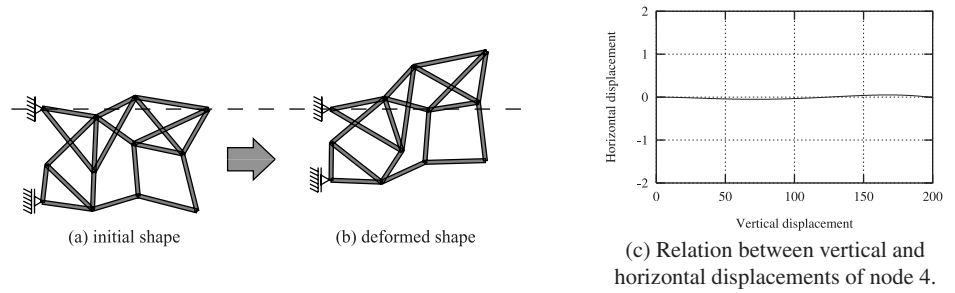


Fig. 5 Optimal solution 1.

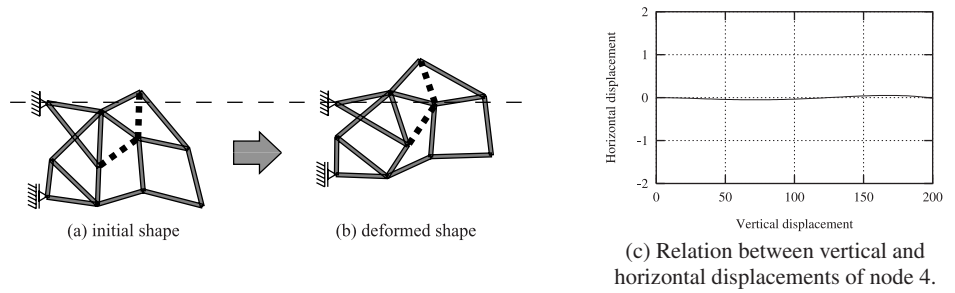


Fig. 6 Optimal solution 1 after removing unnecessary members.

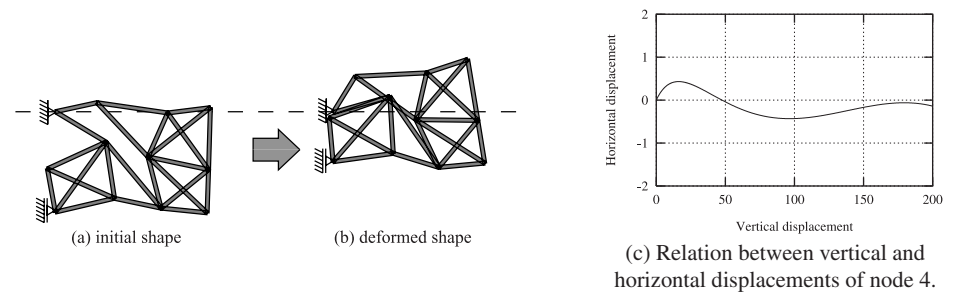


Fig. 7 Optimal solution 2.

region and to prevent no further existence of crossing members.

The rotated engineering strain, which is computed from the member lengths at the current and reference (initial) state, is used for the strain-displacement relation. Young's modulus is 2.0 N/mm^2 , but the value does not have any effect on the final design, because the final mechanism is unstable with rigid bars. The displacement increment method is used for path-tracing analysis, where the displacement increment is 1.0 mm , and the y -directional displacement of node 3 is used as parameter. No iteration is carried out at each step, and the unbalanced load is canceled at the following step. The accuracy of the analysis program is verified by the commercial software package ADAMS 2005. Optimization is carried out by IDESIGN Ver. 3.5⁽¹¹⁾, where sequential quadratic programming is used. The tolerance of the constraints is $\varepsilon = 1.0 \times 10^{-4}$. Finite difference approach is used for sensitivity analysis, because computational efficiency is not important in this study.

In order to generate the mechanisms with many types, optimization is carried out from many randomly generated initial solutions. Since the ratios rather than the magnitudes are important for the cross-sectional areas, the initial values are generated as follows using a parameter A_0 and a uniform random number $R_i \in [0, 1)$ as

$$A_i = A_0 + 2A_0(R_i - 0.5) \quad (10)$$

where $A_0 = 10$ in the following examples. For the x -coordinate x_i of node i that is taken as a variable, the initial value is uniformly distributed around the coordinate x_i^0 in Fig. 4 as

$$x_i = x_i^0 + 120.0(R_i - 0.5) \quad (11)$$

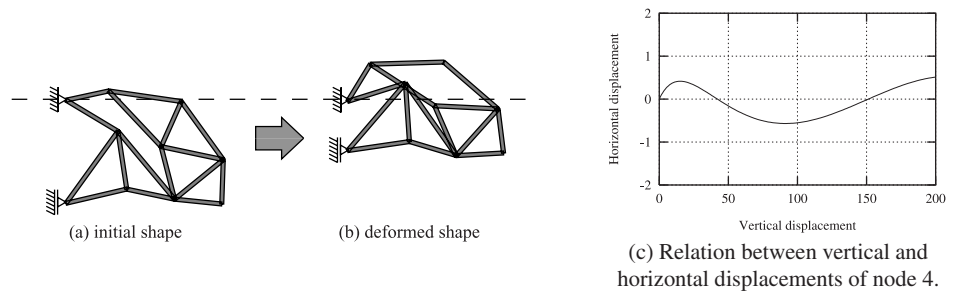


Fig. 8 Optimal solution 2 after removing unnecessary members.

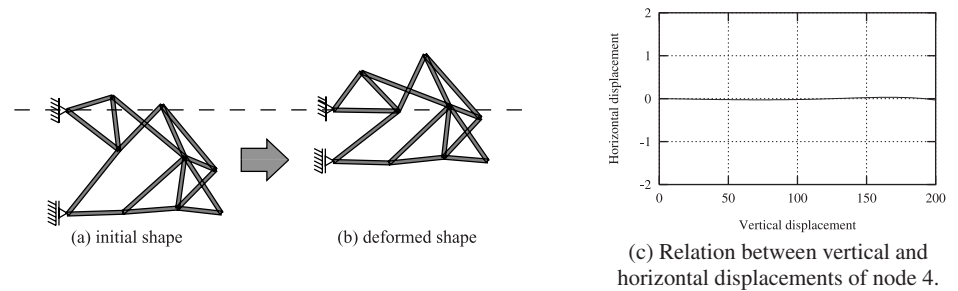


Fig. 9 Optimal solution 3 after removing unnecessary members.

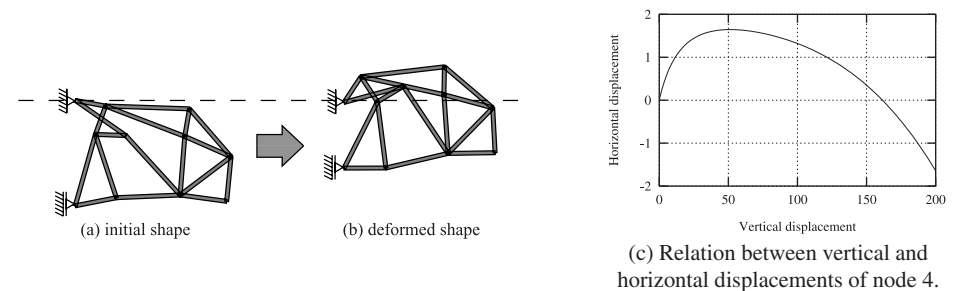


Fig. 10 Optimal solution 4 after removing unnecessary members.

The y -coordinates are given similarly.

Fig. 5(a) shows a solution of the two-stage optimization problem, where the maximum absolute value of the displacement in x -direction of node 2 is minimized in the second stage. Fig. 5(b) shows the final deformed shape, where the dashed line in x -direction is an auxiliary line to show clearly the pin support does not move in the deformation process. The relation between the y - and x -directional displacements of node 4 is plotted in Fig. 5(c). Note that the vertical and horizontal axes correspond to horizontal and vertical displacements, respectively, and their scales are different. As is seen, node 4 moves almost straight in y -direction.

The configuration after removing the unnecessary members is shown in Fig. 6. Similarly to Fig. 5, the figures (a) and (b) correspond to the shapes before and after deformation, respectively, and (c) is the plot between x - and y -directional displacements of node 4. Although only removal of members is allowed in the ground structure approach in the first stage, the structure is further simplified if addition of members is allowed as indicated in dotted lines. The numbers of members and degrees-of-freedom of displacement are 18 and 19, respectively; i.e., the degree of kinematic indeterminacy is 1, if the equilibrium matrix is full-rank.

Another optimal solution, denoted by solution 2, is obtained from different initial random seeds. The configurations with and without unnecessary members are shown in Figs. 7 and 8, respectively. Other two optimal solutions, denoted by solutions 3 and 4, after removing the unnecessary members are shown in Figs. 9 and 10. Although the horizontal displacement of node 4 is a little large in solutions 2 and 4, that for solution 3 is very small. The number of members and the number of degrees-of-freedom of solutions 2, 3, 4 are 16, 16, 18, and 17, 17,

19, respectively. Therefore, the degree of kinematic indeterminacy is 1 for these solutions, if the equilibrium matrix is full-rank.

5. Conclusions

The following conclusions are drawn from this study.

(1) Link mechanisms can be generated by a two-stage optimization process considering geometrical nonlinearity. In the first stage, unnecessary members are removed from a highly connected ground structure by considering cross-sectional areas and nodal locations as design variables. In the second stage, the direction of displacement of the output node is optimized by adjusting the nodal locations while the topology and cross-sectional areas are fixed.

(2) The equilibrium path of a mechanism with one degree of kinematic indeterminacy can be traced by displacement increment approach. Therefore, a link mechanism can be generated using a conventional software package of finite element analysis.

(3) A mechanism that transfers rotation to translational displacement can be successfully found using the proposed method.

(4) The stiffness constraints have been assigned only at the final state. In the future study, the constraints should be given for the stiffness throughout the loading history.

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