BAYESIAN ESTIMATION OF GUMBEL TYPE-II DISTRIBUTION

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ABSTRACT

In this paper we consider the Bayesian estimators for the unknown parameters of Gumbel type-II distribution. The Bayesian estimators cannot be obtained in closed forms. Approximate Bayesian estimators are computed using the idea of Lindley's approximation under different loss functions. The approximate Bayes estimates obtained under the assumption of non-informative priors are compared with their maximum likelihood counterparts using Monte Carlo simulation. A real data set is analyzed for illustrative purpose.

Keywords: Bayesian estimator, Maximum likelihood estimator, Lindley's approximation, Monte Carlo simulation, Gumbel type-II distribution

1 INTRODUCTION

The Gumbel type-II distribution was introduced by German mathematician Emil Gumbel (1891-1911) in 1958, and is useful in predicting the chance of meteorological phenomena, such as annual flood flows, earthquakes, and other natural disasters. It has also been found to be satisfactory in describing the life expectancy of components. The random variable X is said to follow a Gumbel type-II distribution with parameters α and β, where the cumulative distribution function (CDF) is given by

$$F\left(x \mid \alpha,\beta\right)=1-\exp\left[-\beta x^{-\alpha}\right], x>0, \alpha,\beta>0, \tag{1}$$

The corresponding probability density function (PDF) of (1) is

$$f\left(x \mid \alpha,\beta\right)=\alpha\beta x^{-(\alpha+1)}\exp\left[-\beta x^{-\alpha}\right], x>0, \alpha,\beta>0. \tag{2}$$

Recently, many authors have contributed to statistical methodology and characterization of Gumbel type-II distribution. For example, Kotz and Nadarajah (2000) discussed some properties of Gumbel distribution. Feroze and Aslam (2012) considered Bayesian analysis of Gumbel type-II distribution under doubly censored samples using different loss functions. Corsini et al. (2002) discussed the maximum likelihood (ML) algorithms and Cramer-Rao (CR) bounds for the location and scale parameters of the Gumbel distribution. Ali Mousa et al. (2002) studied the Bayesian estimation in order to analyze both the parameters of Gumbel distribution based on record values. Similarly, Malinowska and Szynal (2008) obtained Bayesian estimators for two parameters of a Gumbel distribution based on kth lower record values. Nadarajah and Kotz (2004) introduced beta Gumbel (BG) distribution which provides closed-form expressions for the moments, asymptotic distribution of extreme order statistics as well as discussed the maximum likelihood estimation procedure. Miladinovic and Tsokos (2009) studied the sensitivity of Bayesian reliability estimates for modified Gumbel failure models under five different parametric priors using squared error loss function. Al-Baidhani and Sinclair (1987) and Hossain and Howlader (1996) studied different estimation methods in case of complete samples. However, Bayesian estimations under different loss functions are not frequently discussed. Bayesian estimators under different loss functions involve integral expressions, which are not analytically solvable. Therefore, Lindley's approximation technique is suitable for solving such problems.

The main objective of this study is to develop the Bayesian estimators under different loss functions and compare them with maximum likelihood estimators (MLEs) in terms of bias and mean squared error (MSE) of the estimate. The rest of the paper is organized as follows. In Section 2, the MLEs and observed Fisher information matrix for parameters are derived. Bayesian estimation under LINEX (linear exponential) loss function and general entropy loss function are discussed in Section 3. Simulation study is presented in Section 4, and one real data set is analyzed in Section 5. Finally, conclusion is given in Section 6.

2 MAXIMUM LIKELIHOOD ESTIMATION

Let $X = (X_1, X_2, ..., X_n)$ be a random sample of size n from the Gumbel type-II distribution (2). The likelihood function of (α , β) is

$$L(\alpha,\beta) = \prod_{i=1}^{n}\left[\alpha\beta X_i^{-(\alpha+1)}\exp\left(-\beta X_i^{-\alpha}\right)\right].$$

Then the log-likelihood function can be written as

$$\ln L = n\ln\alpha + n\ln\beta - (\alpha+1)\sum_{i=1}^{n}\ln(X_i) - \beta\sum_{i=1}^{n}X_i^{-\alpha},$$

and

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\ln(X_i) + \beta\sum_{i=1}^{n}X_i^{-\alpha}\ln(X_i),$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n}X_i^{-\alpha}.$$

The maximum likelihood estimates of α and β , say $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$ respectively, can be obtained as the solutions of the equations

$$\hat{\alpha}_{ML} = \frac{n}{\sum_{i=1}^{n}\ln(X_i) - \beta\sum_{i=1}^{n}X_i^{-\alpha}\ln(X_i)}, \tag{3}$$

and

$$\hat{\beta}_{ML} = \frac{n}{\sum_{i=1}^{n}X_i^{-\alpha}}. \tag{4}$$

This may be solving using an iteration scheme. We use the Laplace approximation to compute MLEs. Further, the observed Fisher information matrix is obtained by taking the second and mixed partial derivatives of $\ln L$ with respect to α and β . We have

$$I_{(\alpha,\beta)} = \begin{pmatrix} -\frac{\partial^2 \ln L}{\partial \alpha^2} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & -\frac{\partial^2 \ln L}{\partial \beta^2} \end{pmatrix}$$

where

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \beta\sum_{i=1}^{n}X_i^{-\alpha}\left[\ln(X_i)\right]^2,$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{n}{\beta^2},$$
$$\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} = \sum_{i=1}^{n} X_i^{-\alpha} \ln(X_i).$$

3 BAYESIAN ESTIMATION

In Bayesian estimation, we consider two types of loss functions. The first one is LINEX loss function, which is asymmetric. The LINEX loss function was introduced by Varian (1975), and several authors, such as Basu and Ebrahimi (1991), Rojo (1987), and Nassar and Eissa (2004), have used this loss function in different estimation problems. This function rises approximately exponentially on one side of zero and approximately linearly on the other side. The LINEX loss function can be expressed as

$$L(\Delta) \propto e^{k\Delta} - k\Delta - 1, k \neq 0. \quad , \tag{5}$$

where $\Delta = (\hat{\theta} - \theta)$, and $\hat{\theta}$ is an estimate of θ. The sign and magnitude of the shape parameter k represents the direction and degree of symmetry, respectively. Moreover, if $k > 0$, the overestimation is more serious compared to the under estimation and vice-versa. For k close to zero, the LINEX loss is approximately squared error loss and therefore almost symmetric. The posterior expectation of the LINEX loss function (5) is

$$E_\theta[L(\hat{\theta} - \theta)] \propto e^{k\hat{\theta}} E_\theta[e^{-k\theta}] - k(\hat{\theta} - E_\theta(\theta)) - 1, \tag{6}$$

where $E_\theta(\cdot)$ denotes the posterior expectation with respect to the posterior density of θ. The Bayes estimator of θ, denoted by $\hat{\theta}_{BL}$ under LINEX loss function, is the value $\hat{\theta}$, which minimizes (6). It is

$$\hat{\theta}_{BL} = -\frac{1}{k} \ln\left\{ E_\theta \left[e^{-k\theta} \right] \right\}, \tag{7}$$

provided that the expectation $E_\theta[e^{-k\theta}]$ exists and is finite. The problem of choosing the value of the parameter k is discussed in Calabria and Pulcini (1996). The second type of loss function is the generalization of the entropy loss, which is discussed by Dey and Liu (1992) and Dey (1987). The general entropy loss is defined as

$$L_{BE}(\hat{\theta}, \theta) \propto \left(\frac{\hat{\theta}}{\theta} \right)^k - k \log\left(\frac{\hat{\theta}}{\theta} \right) - 1, \tag{8}$$

where $\hat{\theta}$ is an estimate of θ. The Bayes estimator relative to the general entropy loss is

$$\hat{\theta}_{BE} = \left[E\left(\theta^{-k} \right) \right]^{-\frac{1}{k}}, \tag{9}$$

provided that $E(\theta^{-k})$ exists and is finite. For k=1, the Bayes estimator (9) coincides with the Bayes estimator under the weighted squared error loss function, and for k= -1, the Bayes estimator (9) coincides with the Bayes estimator under the squared error loss function. Further, the Bayesian estimators under LINEX loss function and general entropy loss function are provided in Appendix.

4 SIMULATION STUDY

To compare the performance of theoretical results, the samples are generated from Gumbel type-II distribution using the inverse transformation technique by considering different values of parameters. Sample size is varied to observe the effect of small and large samples on the estimators. For each sample size, we compute maximum likelihood estimates of α and β and the Bayesian estimates under LINEX loss and entropy loss were computed using Laplace's approximation and Lindley's approximation respectively.

For Bayesian estimators, we consider that α and β each have independent Gamma (a_1, b_1) and Gamma (a_2, b_2) priors. Further, the Bayesian estimators of α and β are also obtained using general uniform priors. We use different values of loss function parameter k=±1 and non-informative priors of both α and β, i.e., $a_1 = b_1 = a_2 = b_2 = 0$ and $a_1 = a_2 = b_1 = b_2 = 2$. The behavior sampling of approximate Bayesian estimators is investigated and compared with the MLEs in terms of their MSEs. The results are presented in Tables 1-4. From the results of simulation study, conclusions are drawn regarding the behavior of the estimators, which are summarized below.

1. As expected, it is observed that the performances of both Bayesian and maximum likelihood estimators become better when sample size increased. Also, it is observed that for large sample sizes, the Bayesian estimates and maximum likelihood estimates become closer in terms of MSEs.
2. When k=1 and $a_1 = a_2 = b_1 = b_2 = 0$, the MSEs of Bayesian estimators under LINEX loss function and general entropy loss function using Gamma priors and general uniform priors are lower than the MSEs of maximum likelihood estimators. Therefore the Bayesian estimators are more stable than maximum likelihood estimators.
3. For k= -1 and $a_1 = a_2 = b_1 = b_2 = 2$, the Bayesian estimators under general entropy loss function and LINEX loss function perform better than MLEs obtained by using Gamma priors and general uniform priors in terms of their MSEs.
4. Figure 1 indicates that MSEs decreased as n increases for all methods of estimation studied. It is to be noted that, when sample size is small, the ML method tends to have larger MSE than the Bayesian method, and one would prefer Bayesian estimators. Clearly, for small sample sizes, the Bayesian estimators should be recommended for Gumbel type-II distribution. From Tables 1-4, we can see that in each scenario, the Bayesian estimators under assumption of general entropy loss function and LINEX loss function outperform the maximum likelihood estimators since MSEs are significantly smaller. It is worth noting that

BLU: Bayesian estimator under LINEX loss function using general uniform prior.
BEU: Bayesian estimator under general entropy loss function using general uniform prior.
BLG: Bayesian estimator under LINEX loss function using Gamma prior.
BEG: Bayesian estimator under general entropy loss function using Gamma prior.

Table 1. Average estimates and corresponding MSEs (within parenthesis) for α when (k =1 and $a_1=a_2=b_1=b_2= 0$).

n	Estimator↓α→	0.5	1.0	1.5	2.0
20	α-ML	0.5397(0.0123)	1.0765(0.0478)	1.6178(0.1086)	2.1502(0.1889)
	α-BLU	0.5280(0.0113)	1.0698(0.0439)	1.6166(0.1039)	2.1501(0.1824)
	α-BLG	0.5213(0.0112)	1.0508(0.0411)	1.5660(0.0884)	2.0650(0.1474)
	α-BEU	0.5168(0.0094)	1.0554(0.0426)	1.6085(0.1064)	2.1403(0.1887)
	α-BEG	0.5159(0.0092)	1.0370(0.0403)	1.5586(0.0913)	2.0715(0.1594)
30	α-ML	0.5268(0.0068)	1.0470(0.0271)	1.5762(0.0630)	2.0996(0.1129)
	α-BLU	0.5195(0.0063)	1.0429(0.0255)	1.5753(0.0611)	2.1002(0.1100)
	α-BLG	0.5110(0.0062)	1.0305(0.0245)	1.5430(0.0548)	2.0445(0.0954)
	α-BEU	0.5120(0.0056)	1.0330(0.0249)	1.5691(0.0617)	2.1034(0.1109)
	α-BEG	0.5117(0.0054)	1.0210(0.0241)	1.5371(0.0557)	2.0474(0.1002)

50	α-ML	0.5155(0.0037)	1.0282(0.0147)	1.5416(0.0347)	2.0566(0.0573)
	α-BLU	0.5110(0.0034)	1.0258(0.0142)	1.5411(0.0340)	2.0569(0.0564)
	α-BLG	0.5103(0.0032)	1.0168(0.0138)	1.5223(0.0320)	2.0246(0.0517)
	α-BEU	0.5068(0.0033)	1.0199(0.0139)	1.5369(0.0341)	2.0579(0.0570)
	α-BEG	0.5066(0.0032)	1.0127(0.0136)	1.5183(0.0323)	2.0256(0.0531)
80	α-ML	0.5090(0.0021)	1.0174(0.0086)	1.5260(0.0188)	2.0381(0.0345)
	α-BLU	0.5065(0.0020)	1.0159(0.0084)	1.5256(0.0186)	2.0383(0.0341)
	α-BLG	0.5056(0.0020)	1.0114(0.0082)	1.5141(0.0179)	2.0184(0.0322)
	α-BEU	0.5037(0.0020)	1.0122(0.0083)	1.5229(0.0186)	2.0387(0.0340)
	α-BEG	0.5032(0.0020)	1.0077(0.0082)	1.5115(0.0180)	2.0188(0.0327)
100	α-ML	0.5066(0.0016)	1.0134(0.0065)	1.5209(0.0144)	2.0239(0.0252)
	α-BLU	0.5024(0.0016)	1.0123(0.0064)	1.5206(0.0142)	2.0240(0.0249)
	α-BLG	0.5020(0.0016)	1.0087(0.0063)	1.5114(0.0138)	2.0083(0.0240)
	α-BEU	0.5023(0.0015)	1.0092(0.0063)	1.5184(0.0143)	2.0242(0.0250)
	α-BEG	0.5022(0.0015)	1.0057(0.0063)	1.5093(0.0138)	2.0085(0.0243)

Table 2. Average estimates and corresponding MSEs (within parenthesis) for β when ($k=1$ and $a_1=a_2=b_1=b_2=0$).

n	Estimator↓β→	0.5	1.0	1.5	2.0
20	β-ML	0.5098(0.0249)	1.0380(0.0751)	1.6059(0.1902)	2.1958(0.4431)
	β-BLU	0.5081(0.0236)	1.0378(0.0735)	1.6058(0.1803)	2.1758(0.3763)
	β-BLG	0.5073(0.0222)	1.0032(0.0635)	1.5234(0.1413)	2.0398(0.2819)
	β-BEU	0.5068(0.0235)	1.0229(0.0742)	1.5966(0.1900)	2.1950(0.4369)
	β-BEG	0.5059(0.0233)	1.0227(0.0663)	1.5134(0.1349)	2.0598(0.3419)
30	β-ML	0.5075(0.0153)	1.0259(0.0437)	1.5663(0.1052)	2.1192(0.2209)
	β-BLU	0.5043(0.0148)	1.0234(0.0431)	1.5660(0.1030)	2.1113(0.2055)
	β-BLG	0.5040(0.0143)	1.0035(0.0391)	1.5144(0.0876)	2.0235(0.1666)
	β-BEU	0.5023(0.0148)	1.0154(0.0431)	1.5584(0.1041)	2.1217(0.2135)
	β-BEG	0.5020(0.0145)	1.0140(0.0401)	1.5055(0.0919)	2.0316(0.1856)
50	βML	0.5038(0.0090)	1.0146(0.0238)	1.5373(0.0535)	2.0661(0.1065)
	β-BLU	0.5033(0.0088)	1.0191(0.0236)	1.5370(0.0529)	2.0629(0.1033)

	β-BLG	0.5031(0.0086)	1.0016(0.0223)	1.5075(0.0482)	2.0119(0.0909)
	β-BEU	0.5018(0.0088)	1.0082(0.0235)	1.5319(0.0533)	2.0659(0.1044)
	β-BEG	0.5015(0.0086)	1.0070(0.0226)	1.5012(0.0493)	2.0146(0.0959)
80	β-ML	0.5030(0.0057)	1.0107(0.0153)	1.5204(0.0318)	2.0448(0.0634)
	β-BLU	0.5017(0.0056)	1.0135(0.0153)	1.5203(0.0316)	2.0430(0.0622)
	β-BLG	0.5011(0.0056)	1.0026(0.0147)	1.5022(0.0299)	2.0117(0.0572)
	β-BEU	0.5010(0.0056)	1.0066(0.0152)	1.5169(0.0315)	2.0442(0.0624)
	β-BEG	0.5009(0.0055)	1.0010(0.0148)	1.5006(0.0303)	2.0128(0.0591)
100	β-ML	0.5008(0.0044)	1.0081(0.0119)	1.5120(0.0251)	2.0308(0.0471)
	β-BLU	0.5007(0.0044)	1.0104(0.0118)	1.5119(0.0250)	2.0295(0.0464)
	β-BLG	0.5003(0.0043)	1.0017(0.0115)	1.5005(0.0237)	2.0047(0.0436)
	β-BEU	0.5006(0.0043)	1.0049(0.0118)	1.5102(0.0250)	2.0301(0.0466)
	β-BEG	0.5001(0.0042)	1.0001(0.0116)	1.5000(0.0240)	2.0053(0.0447)

Table 3. Average estimates and corresponding MSEs (within parenthesis) for α when (k = -1 and a_1=a_2= b_1=b_2=2)

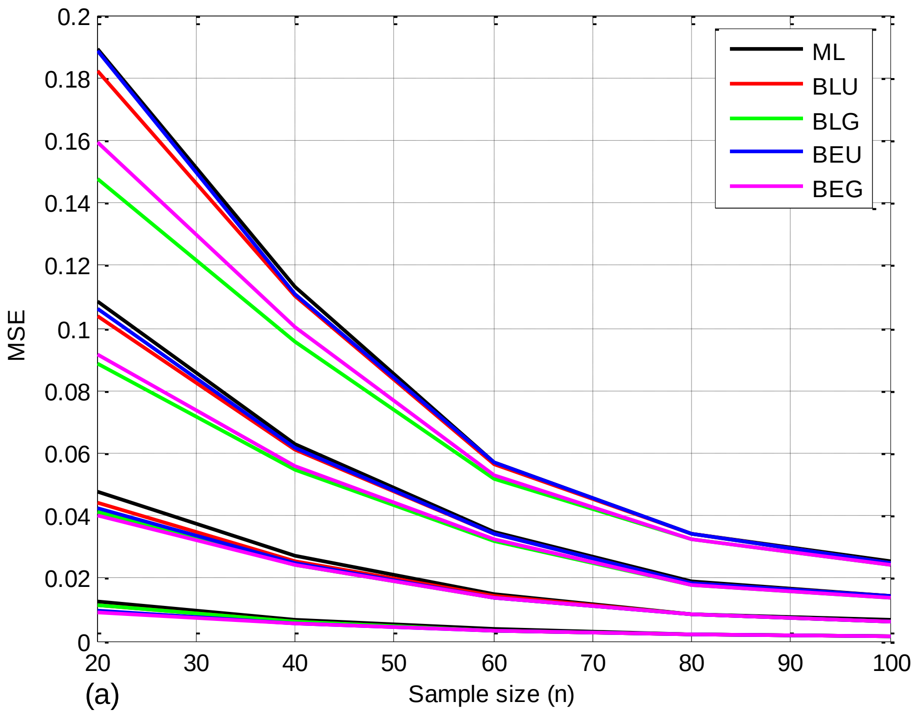
n	Estimator↓α→	0.5	1.0	1.5	2.0
20	α-ML	0.5368(0.0120)	1.0805(0.0514)	1.6120(0.1017)	2.2018(0.1934)
	α-BLU	0.5365(0.0119)	1.0727(0.0512)	1.5910(0.1005)	2.1741(0.1932)
	α-BLG	0.5304(0.0092)	1.0705(0.0478)	1.5597(0.0698)	2.1730(0.1868)
	α-BEU	0.5363(0.0118)	1.0639(0.0467)	1.5495(0.0845)	2.0954(0.1458)
	α-BEG	0.5199(0.0086)	1.0608(0.0392)	1.5212(0.0602)	2.0740(0.1394)
30	α-ML	0.5247(0.0070)	1.0471(0.0279)	1.5667(0.0594)	2.0920(0.0959)
	α-BLU	0.5245(0.0069)	1.0411(0.0278)	1.5522(0.0585)	2.0697(0.0945)
	α-BLG	0.5215(0.0059)	1.0460(0.0261)	1.5411(0.0485)	2.0630(0.0944)
	α-BEU	0.5240(0.0070)	1.0396(0.0261)	1.5264(0.0528)	2.0136(0.0823)
	α-BEG	0.5186(0.0056)	1.0374(0.0239)	1.5158(0.0440)	2.0134(0.0737)
50	α-ML	0.5149(0.0037)	1.0286(0.0146)	1.5415(0.0320)	2.0128(0.0607)
	α-BLU	0.5147(0.0036)	1.0248(0.0143)	1.5327(0.0315)	2.0100(0.0596)
	α-BLG	0.5135(0.0034)	1.0205(0.0141)	1.5290(0.0287)	2.0066(0.0470)
	α-BEU	0.5143(0.0036)	1.0182(0.0140)	1.5179(0.0296)	2.0131(0.0547)
	α-BEG	0.5140(0.0033)	1.0139(0.0134)	1.5143(0.0269)	2.0130(0.0473)

80	α-ML	0.5078(0.0021)	1.0185(0.0084)	1.5231(0.0182)	2.0064(0.0346)
	α-BLU	0.5075(0.0021)	1.0160(0.0084)	1.5175(0.0181)	2.0053(0.0342)
	α-BLG	0.5071(0.0020)	1.0155(0.0082)	1.5162(0.0171)	2.0047(0.0301)
	α-BEU	0.5073(0.0021)	1.0120(0.0082)	1.5086(0.0174)	2.0023(0.0325)
	α-BEG	0.5061(0.0019)	1.0115(0.0080)	1.5073(0.0165)	2.0018(0.0297)
100	α-ML	0.5034(0.0016)	1.0138(0.0066)	1.5100(0.0151)	2.0006(0.0265)
	α-BLU	0.5030(0.0016)	1.0118(0.0065)	1.5100(0.0150)	2.0001(0.0263)
	α-BLG	0.5028(0.0016)	1.0110(0.0065)	1.5048(0.0144)	2.0002(0.0239)
	α-BEU	0.5031(0.0016)	1.0086(0.0064)	1.5000(0.0143)	2.0001(0.0253)
	α-BEG	0.5025(0.0015)	1.0008(0.0063)	1.5007(0.0140)	2.0000(0.0235)

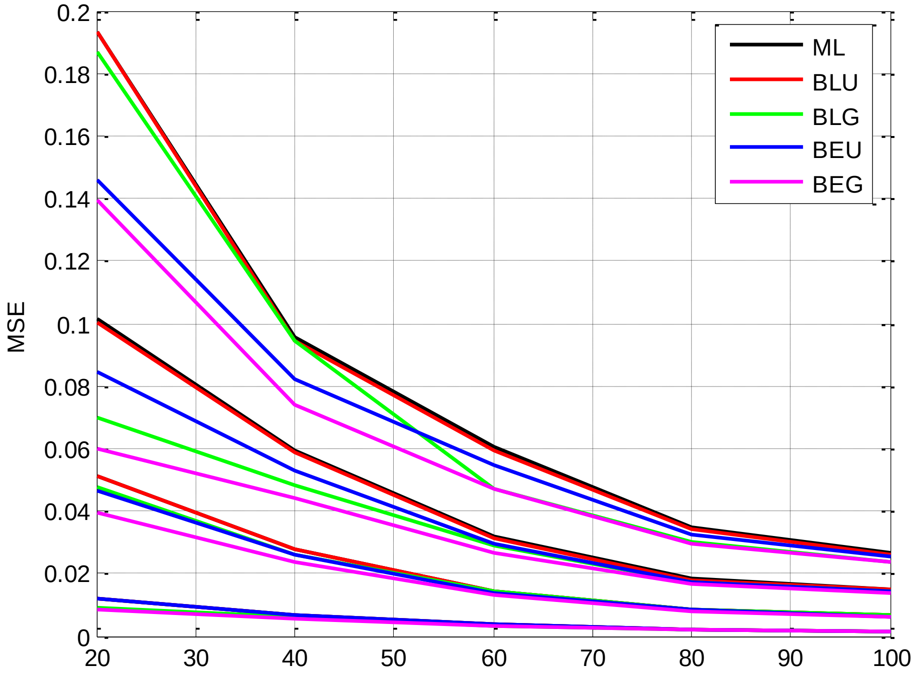
Table 4. Average estimates and corresponding MSEs (within parenthesis) for β when (k = -1 and a_1=a_2=b_1=b_2=2).

n	Estimator↓β→	0.5	1.0	1.5	2.0
20	β-ML	0.5061(0.0246)	1.0362(0.0737)	1.6004(0.1911)	2.1960(0.1783)
	β-BLU	0.5049(0.0239)	1.0164(0.0691)	1.5706(0.1892)	2.1705(0.1763)
	β-BLG	0.5038(0.0237)	1.0152(0.0590)	1.5512(0.1513)	2.1508(0.1699)
	β-BEU	0.5060(0.0238)	1.0265(0.0617)	1.5404(0.1429)	2.1650(0.1669)
	β-BEG	0.5015(0.0213)	1.0247(0.0501)	1.5357(0.1329)	2.1578(0.1626)
30	β-ML	0.5047(0.0156)	1.0254(0.0443)	1.5669(0.1027)	2.1120(0.1309)
	β-BLU	0.5040(0.0154)	1.0124(0.0422)	1.5457(0.0992)	2.1112(0.1205)
	β-BLG	0.5028(0.0151)	1.0122(0.0391)	1.5285(0.0684)	2.0935(0.1166)
	β-BEU	0.5043(0.0149)	1.0230(0.0394)	1.5284(0.0849)	2.1326(0.1305)
	β-BEG	0.5040(0.0141)	1.0197(0.0353)	1.5210(0.0603)	2.1215(0.1156)
50	β-ML	0.5008(0.0091)	1.0135(0.0250)	1.5398(0.0545)	2.0733(0.1125)
	β-BLU	0.5005(0.0090)	1.0059(0.0243)	1.5269(0.0530)	2.0600(0.1124)
	β-BLG	0.5003(0.0089)	1.0026(0.0233)	1.5218(0.0448)	2.0519(0.1030)
	β-BEU	0.5006(0.0088)	1.0129(0.0234)	1.5084(0.0487)	2.0420(0.0968)
	β-BEG	0.5005(0.0085)	1.0111(0.0220)	1.5072(0.0413)	2.0349(0.0850)

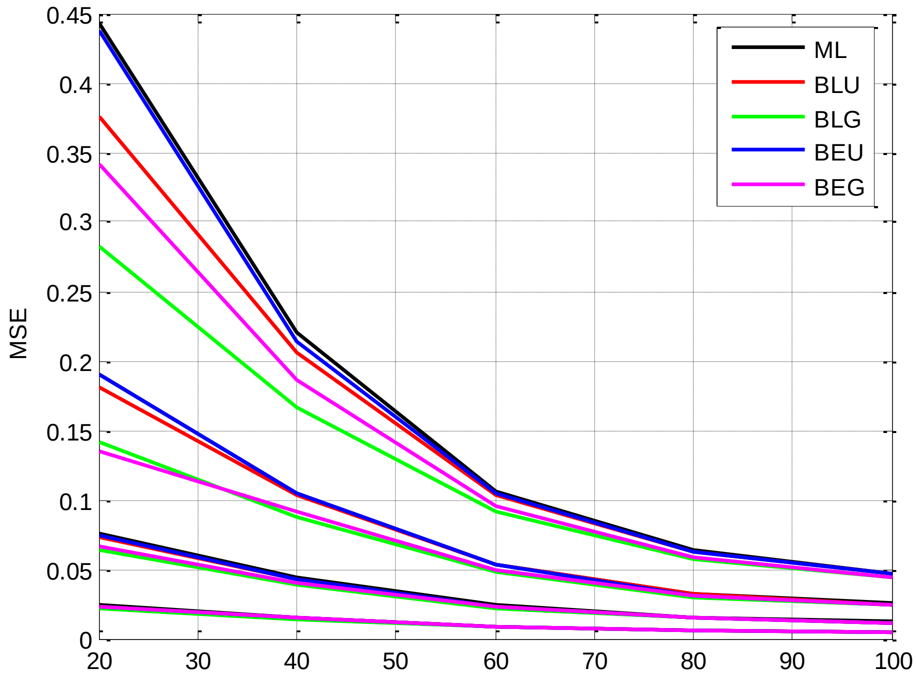
80	Β-ML	0.5003(0.0058)	1.0075(0.0147)	1.5217(0.0316)	2.0399(0.0602)
	β-BLU	0.5002(0.0058)	1.0028(0.0145)	1.5136(0.0310)	2.0309(0.0600)
	β-BLG	0.5001(0.0058)	1.0020(0.0143)	1.5119(0.0284)	2.0107(0.0479)
	β-BEU	0.5003(0.0057)	1.0050(0.0141)	1.5049(0.0296)	2.0028(0.0551)
	β-BEG	0.5001(0.0056)	1.0032(0.0136)	1.5030(0.0271)	2.0020(0.0460)
100	β-ML	0.5001(0.0045)	1.0017(0.0118)	1.5156(0.0245)	2.0021(0.0458)
	β-BLU	0.5000(0.0044)	1.0010(0.0116)	1.5009(0.0241)	2.0015(0.0457)
	β-BLG	0.5000(0.0044)	1.0010(0.0115)	1.5005(0.0225)	2.0013(0.0388)
	β-BEU	0.5000(0.0043)	1.0003(0.0130)	1.5002(0.0233)	2.0010(0.0429)
	β-BEG	0.5000(0.0043)	1.0001(0.0111)	1.5001(0.0218)	2.0003(0.0377)



(a)


MSE
Sample size (n)
ML
BLU
BLG
BEU
BEG
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
20
30
40
50
60
70
80
90
100

(b)


MSE
Sample size (n)
ML
BLU
BLG
BEU
BEG
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
20
30
40
50
60
70
80
90
100

(c)

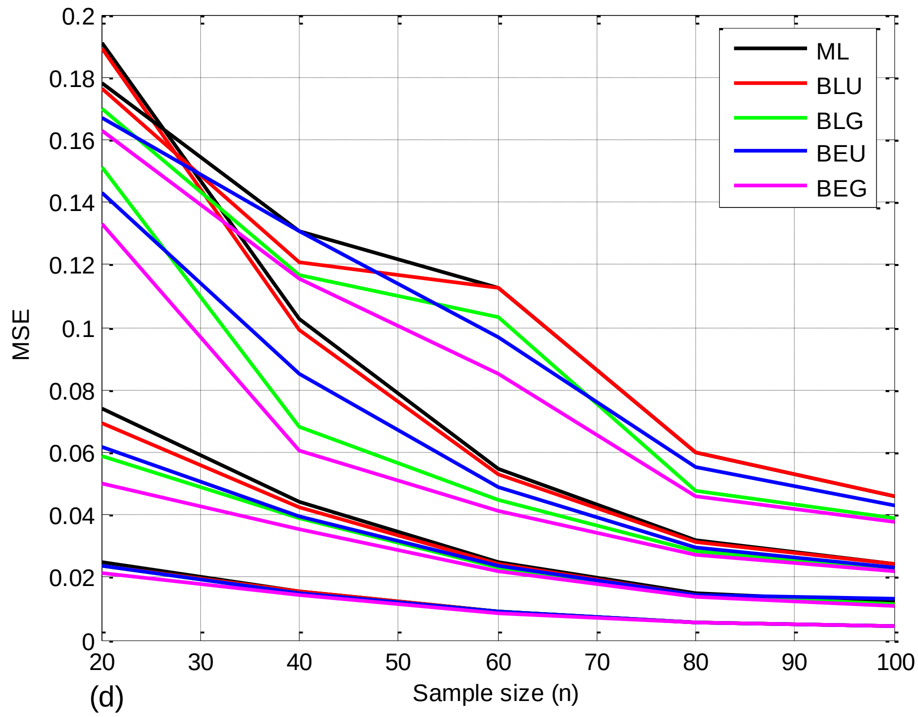


Figure 1. Plot of sample size and MSEs of α and β using different methods of estimation such as k = ±1, $a_1=b_1=a_2=b_2=0$ and $a_1=a_2=b_1=b_2=2$. In abscissa (a) MSEs of α (when k =1), (b) MSEs of α (when k =-1), (c) MSEs of β (when k =1), (d) MSEs of β (when k =-1).

5 DATA ANALYSIS

In this section we consider the real data set obtained from Nichols and Padgett (2006), which represents the breaking stress of carbon fibres (in Gba). The data consist of 100 observations and are presented in Table 5.

Table 5. Breaking stress of carbon fibres (in Gba).

3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59,3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65

Table 6. Point estimates and standard deviations (SD) of α and β.

Estimator	α	SD	β	SD
ML	1.7687	0.1119	3.0880	0.3271
BLU	1.7678	0.1117	3.0634	0.3271
BLG	1.7569	0.1117	3.0238	0.3261
BEU	1.7670	0.1118	3.0820	0.3264
BEG	1.7561	0.1116	3.0412	0.3270

The point estimates of α and β and their standard deviations (SD) are summarized in Table 6. It is observed that the Bayesian estimates under general entropy loss function and LINEX loss function are close to the ML estimates. When we compare the ML estimators with Bayesian estimators using Lindley's approximation in terms of their standard deviations, the approximate Bayesian estimators perform better than the MLEs.

6 CONCLUSION

In this paper we consider classical and Bayesian estimators under the assumption of LINEX loss and general entropy loss functions. Neither Bayesian nor maximum likelihood estimators can be obtained in closed forms. Lindley's approximation is used to obtain the Bayesian estimates, and it is concluded that the approximation works very well even for small sample sizes though the computation of Lindley's technique based on the maximum likelihood estimators. We compare the performance of different methods by Monte Carlo simulations. Simulations showed that the Bayesian estimators under general entropy loss function and LINEX loss function perform better than the maximum likelihood estimators. However, it is observed that for large sample sizes the Bayesian and maximum likelihood estimates become closer in terms of their MSEs.

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9 APPENDIX

For Bayesian estimation, we need prior distribution of α and β. Assuming that α and β each have independent Gamma (a_1, b_1) and Gamma (a_2, b_2) priors respectively for $a_1, b_1, a_2, b_2 > 0$, i.e.,

$$\pi_1(\alpha) \propto \alpha^{a_1-1} e^{-b_1\alpha} \text{ and } \pi_2(\beta) \propto \beta^{a_2-1} e^{-b_2\beta}.$$

Based on the priors, the joint posterior density of α and β can be written as

$$f(\alpha, \beta \mid x) = \frac{L(data \mid \alpha, \beta)\pi_1(\alpha)\pi_2(\beta)}{\int_0^\infty \int_0^\infty L(data \mid \alpha, \beta)\pi_1(\alpha)\pi_2(\beta) d\alpha d\beta} \tag{10}$$

Therefore, the Bayesian estimator of any function of α and β, say g(α ; β) , under the LINEX loss function is

$$\hat{g}(\alpha, \beta) = E_{\alpha,\beta|data}[g(\alpha, \beta)] = \frac{\int_0^\infty \int_0^\infty g(\alpha, \beta) L(data \mid \alpha, \beta)\pi_1(\alpha)\pi_2(\beta) d\alpha d\beta}{\int_0^\infty \int_0^\infty L(data \mid \alpha, \beta)\pi_1(\alpha)\pi_2(\beta) d\alpha d\beta} \tag{11}$$

It is not possible for (11) to have a closed form. Therefore, we adopt Lindley's approximation (1980) procedure to approximate the ratio of the two integrals such as (11), which can be evaluated as

$$\hat{g} = g(\hat{\alpha}, \hat{\beta}) + \frac{1}{2}\left[\sum_{i=1}^{2}\sum_{j=1}^{2} l_{ij}s_{ij} + l_{30}A_{12} + l_{03}A_{21} + l_{21}B_{12} + l_{12}B_{21}\right] + p_1C_{12} + p_2C_{21}, \tag{12}$$

where

$$l_{ij} = \frac{\partial^{i+j} l(\alpha_1, \alpha_2)}{\partial\alpha_1^i \partial\alpha_2^i}, i, j = 0,1,2,3, i+j=3,$$

$$p_1 = \frac{\partial \ln \pi(\alpha, \beta)}{\partial\alpha}, p_2 = \frac{\partial \ln \pi(\alpha, \beta)}{\partial\beta}, l_{12} = \frac{\partial^2 g(\alpha, \beta)}{\partial\alpha\partial\beta}, l_{21} = \frac{\partial^2 g(\alpha, \beta)}{\partial\beta\partial\alpha},$$

$$l_{11} = \frac{\partial^2 g(\alpha, \beta)}{\partial\alpha^2}, l_{22} = \frac{\partial^2 g(\alpha, \beta)}{\partial\beta^2}, l_1 = \frac{\partial g(\alpha, \beta)}{\partial\alpha}, l_2 = \frac{\partial g(\alpha, \beta)}{\partial\beta},$$

$$A_{ij} = (l_i s_{ii} + l_j s_{ij})s_{ii}, B_{ij} = 3l_i s_{ii} s_{ij} + l_j(s_{ii}s_{jj} + 2s_{ij}^2), C_{ij} = l_i s_{ii} + l_j s_{ji}, i, j = 1,2,$$

where $l(\cdot)$ is the log-likelihood function of the observed data, s_{ij} is the $(i, j)th$ element of the inverse of Fisher's information matrix. Therefore, the approximate Bayesian estimators of α and β under LINEX loss function are

$$\hat{\alpha}_{BLG} = -\frac{1}{k}\ln\left[e^{-k\hat{\alpha}} + \frac{1}{2}\left\{\left(k^2 e^{-k\hat{\alpha}}\right)s_{11} - \left(\frac{2n}{\hat{\alpha}^3} + \hat{\beta}\sum_{i=1}^{n} X_i^{-\hat{\alpha}}(\ln X_i)^3\right)\left(ke^{-k\hat{\alpha}}\right)s_{11}^2\right.\right.$$
$$\left. - \frac{2n}{\hat{\beta}^3}\left(ke^{-k\hat{\alpha}}\right)s_{22}s_{11} + 3s_{11}s_{12}\left(ke^{-k\hat{\alpha}}\right)\sum_{i=1}^{n} X_i^{-\hat{\alpha}}(\ln X_i)^2\right\} \tag{13}$$
$$\left. -\left(ke^{-k\hat{\alpha}}\right)s_{11}\left(\frac{a_1 - 1}{\hat{\alpha}} - b_1\right) - \left(\frac{a_2 - 1}{\hat{\beta}} - b_2\right)\left(ke^{-k\hat{\alpha}}\right)s_{12}\right],$$

$$\hat{\beta}_{BLG} = -\frac{1}{k}\ln\left[e^{-k\hat{\beta}} + \frac{1}{2}\left\{\left(k^2 e^{-k\hat{\beta}}\right)s_{22} - \left(\frac{2n}{\hat{\alpha}^3} + \hat{\beta}\sum_{i=1}^{n} X_i^{-\hat{\alpha}}(\ln X_i)^3\right)\left(ke^{-k\hat{\beta}}\right)\right.\right.$$
$$\left. s_{12}s_{11} - \frac{2n}{\hat{\beta}^3}\left(ke^{-k\hat{\beta}}\right)s_{22}^2 + \left(s_{11}s_{22} + 2s_{12}^2\right)\left(ke^{-k\hat{\beta}}\right)\sum_{i=1}^{n} X_i^{-\hat{\alpha}}(\ln X_i)^2\right\} \tag{14}$$
$$\left. -\left(\frac{a_1 - 1}{\hat{\alpha}} - b_1\right)\left(ke^{-k\hat{\beta}}\right)s_{21} - \left(\frac{a_2 - 1}{\hat{\beta}} - b_2\right)\left(ke^{-k\hat{\beta}}\right)s_{22}\right].$$

The Bayesian estimators of α and β under general entropy loss function are

$$\hat{\alpha}_{BEG} = \left[\hat{\alpha}^{-k} + \frac{1}{2}\left\{k(k+1)\hat{\alpha}^{-(k+2)}s_{11} - \left(\frac{2n}{\hat{\alpha}^3} + \hat{\beta}\sum_{i=1}^{n} X_i^{-\hat{\alpha}}(\ln X_i)^3\right)k\hat{\alpha}^{-(k+1)}s_{11}^2\right.\right.$$
$$\left. - \frac{2n}{\hat{\beta}^3}k\hat{\alpha}^{-(k+1)}s_{22}s_{11} + 3s_{11}s_{12}k\hat{\alpha}^{-(k+1)}\sum_{i=1}^{n} X_i^{-\hat{\alpha}}(\ln X_i)^2\right\} \tag{15}$$
$$\left. -k\hat{\alpha}^{-(k+1)}\left(\frac{a_1 - 1}{\hat{\alpha}} - b_1\right)s_{11} - k\hat{\alpha}^{-(k+1)}\left(\frac{a_2 - 1}{\hat{\beta}} - b_2\right)s_{12}\right]^{-\frac{1}{k}},$$

$$\hat{\beta}_{BEG} = \left[\hat{\beta}^{-k} + \frac{1}{2}\left\{k(k+1)\hat{\beta}^{-(k+2)}s_{22} - \left(\frac{2n}{\hat{\alpha}^3} + \hat{\beta}\sum_{i=1}^{n} X_i^{-\hat{\alpha}}(\ln X_i)^3\right)k\hat{\beta}^{-(k+1)}\right.\right.$$
$$\left. s_{12}s_{11} - \frac{2n}{\hat{\beta}^3}k\hat{\beta}^{-(k+1)}s_{22}^2 + \left(s_{11}s_{22} + 2s_{12}^2\right)k\hat{\beta}^{-(k+1)}\sum_{i=1}^{n} X_i^{-\hat{\alpha}}(\ln X_i)^2\right\} \tag{16}$$
$$\left. -k\hat{\beta}^{-(k+1)}\left(\frac{a_1 - 1}{\hat{\alpha}} - b_1\right)s_{21} - k\hat{\beta}^{-(k+1)}\left(\frac{a_2 - 1}{\hat{\beta}} - b_2\right)s_{22}\right]^{-\frac{1}{k}},$$

where $s_{11} = \dfrac{v}{uv - w^2}$, $s_{22} = \dfrac{u}{uv - w^2}$, $s_{12} = s_{21} = \dfrac{w}{uv - w^2}$, $u = \dfrac{n}{\hat{\alpha}^2} + \hat{\beta}\sum_{i=1}^{r} X_i^{-\hat{\alpha}} (\ln X_i)^2$, $w = \sum_{i=1}^{r} X_i^{-\hat{\alpha}} (\ln X_i)$, $v = \dfrac{n}{\hat{\beta}^2}$, and $\hat{\alpha}$ and $\hat{\beta}$ in equations (12)-(16) are the maximum likelihood estimators of α and β from (3) and (4). Similarly, the approximate Bayesian estimators of α and β under LINEX loss function and general entropy loss function using general uniform priors i.e. $\pi_3(\alpha) \propto \alpha^{-a_1}$ and $\pi_4(\beta) \propto \beta^{-b_1}$ can be obtained.

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