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Unified active Q factor formula for use in noise spectrum estimation from Leeson's and Hajimiri's models

Takanari Minami[a] **and Takashi Ohira**[b]

Toyohashi University of Technology,

1–1 Hibarigaoka, tempaku, Toyohashi, Aichi 441–8580, Japan

a) *minami@comm.ee.tut.ac.jp*

b) *ohira@tut.jp*

Abstract: In this letter, we propose unified active Q factor formula for use in oscillator noise spectrum estimation from Leeson's and Hajimiri's models. This formula is derived from its oscillation condition as unified expression which covers the conventional formulas. Since this formula can reduce to conventional formulas, we confirm that the unified formula is valid regardless of topology and type of active device. In addition, we applied this unified formula to a cascode FET amplifier based oscillator as an example. As the result, we found that it has Q factor twice as high as a single FET oscillator using the same passive network. From these issues, we conclude that active Q factor can be estimated only if we know the oscillation condition and its frequency slope. It will contribute to low-phase-noise oscillator design and optimization.

Keywords: oscillator, Q factor, phase noise, spectrum

Classification: Microwave and millimeter wave devices, circuits, and systems

References

[1] D. B. Leeson: Proc. IEEE **54** [2] (1966) 329.
[2] A. Hajimiri and T. H. Lee: IEEE J. Solid-State Circuits **33** [2] (1998) 179.
[3] B. Razavi: IEEE J. Solid-State Circuits **31** [3] (1996) 331.
[4] T. Ohira: IEEE Trans. Circuits Syst. II **52** [12] (2005) 846.
[5] T. Ohira and K. Araki: IEEE Trans. Circuits Syst. II **54** [9] (2007) 810.

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1 Introduction

Oscillators generally have phase noise which affects the performance of wireless systems. Oscillator designers have to carefully take noise spectrum into account. Noise spectrum behavior can be estimated by Leeson formula, which shows that the higher Q factor is, the lower noise spectrum is [1]. The higher Q factor also reduces $1/f$-noise effect on oscillator noise spectrum as formulated by Hajimiri and Lee [2]. Q factor is first formulated by Razavi in terms of transfer function [3]. Q factor based on Z-, Y-, and S-parameter was formulated [4, 5], which found five formulas for active Q factor. Oscillator designers have to choose the formula that matches the type of active device to be used.

In this letter, we derive a versatile active Q factor formula that can apply to any topology and any type of active device involving the five. We find three issues: 1) the unified active Q factor formula is derived from the oscillation condition, 2) this formula can reduce to the conventional results, and 3) this formula can apply to a multiple active-device topology.

2 General expression

We consider a system model shown as Fig. 1. The system consists of noise source i_{n}, load R_{L}, and an arbitrary oscillator. The oscillator has some arbitrary type of active devices whose gains are described as $\mathbf{g} = [g_1, g_2, \ldots, g_K]$, where K is the number of active devices. Noise source i_n generates current i_{L} through load R_{L}. Since the noise current is much smaller than oscillation current, i_{L} and i_{n} have a linear relation

$$A(\omega, \mathbf{g}) i_{\mathrm{L}} = i_{\mathrm{n}}, \tag{1}$$

where $A(\omega, \mathbf{g})$ is a function of frequency ω and gain $\mathbf{g}$. At frequency $\omega = \omega_0$, current i_{L} is nonzero even if i_{n} vanishes, where ω_0 is the oscillation condition. Therefore, the following condition must be satisfied:

$$A(\omega_0, \mathbf{g}_0) = 0, \quad \mathbf{g}_0 = [g_{10}, g_{20}, \ldots, g_{K0}], \tag{2}$$

where $\mathbf{g}_0$ is minimum necessary active device gain vector for the oscillation to take place. This equation is the oscillation condition. We call $A(\omega, \mathbf{g})$ as *oscillation function* in this letter. Using Eqs. (1) and (2), real power P consumed in R_{L} is given by

$$P = \frac{1}{2} R_{\mathrm{L}} |i_{\mathrm{L}}|^2 = \frac{R_{\mathrm{L}}}{2} \left| \frac{1}{A(\omega_0, \mathbf{g}_0)} \right|^2 |i_{\mathrm{n}}|^2. \tag{3}$$

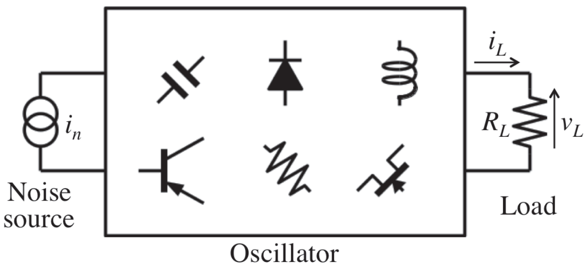

Fig. 1. System model consisting of a noise source, a load, and an arbitrary oscillator.

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Expanding the denominator around ω_0, we get

$$A(\omega_0 + \omega_\mathrm{m}, \mathbf{g}_0) = A(\omega_0, \mathbf{g}_0) + A'(\omega_0, \mathbf{g}_0)\omega_\mathrm{m}, \tag{4}$$
$$A'(\omega, \mathbf{g}) = \frac{\partial}{\partial \omega} A(\omega, \mathbf{g}),$$

where ω_m denote offset frequency on $|\omega_m| \ll \omega_0$. Applying Eqs. (2) and (4), we rewrite Eq. (3) as

$$P_0 = \frac{R_\mathrm{L}}{2} \left| \frac{1}{A'(\omega_0, \mathbf{g}_0)\omega_\mathrm{m}} \right|^2 |i_\mathrm{n}|^2. \tag{5}$$

Looking back to Eq. (3), we find the real power consumed in R_L would be

$$P = \frac{R_\mathrm{L}}{2} \left| \frac{1}{A(\omega_0, \mathbf{0})} \right|^2 |i_\mathrm{n}|^2, \tag{6}$$

if the gain were $\mathbf{g} = \mathbf{0}$. Using Eqs. (5) and (6), we get

$$\frac{P}{P_0} = \left| \frac{A(\omega_0, \mathbf{0})}{A'(\omega_0, \mathbf{g}_0)\omega_\mathrm{m}} \right|^2. \tag{7}$$

Applying this equation to Leeson formula:

$$\frac{P}{P_0} = \left(\frac{\omega_0}{2Q\omega_\mathrm{m}} \right)^2, \tag{8}$$

we get Q factor formula

$$Q = \frac{\omega_0}{2} \left| \frac{A'(\omega_0, \mathbf{g}_0)}{A(\omega_0, \mathbf{0})} \right|. \tag{9}$$

Therefore, we conclude the Q factor can be generally expressed in terms of the oscillation function defined by Eq. (2).

3 Reduction to conventional formulas

According to Ref. [4], modern oscillators mostly employ following active devices: negative resistance (NR); current controlled current source (CCCS); voltage controlled current source (VCCS); voltage controlled voltage source (VCVS); and current controlled voltage source (CCVS). These oscillators are shown in Fig. 2. As an example, we discuss the VCCS oscillator. Its oscillation condition is given by

$$1 + g_\mathrm{m} z_{12}(\omega) = 0, \tag{10}$$

where $z_{12}(\omega)$ is non-diagonal element of matrix $\mathbf{Z}$ for two-port passive linear network shown in Fig. 2 (c), and g_m is the trans-conductance of active device. Using g_m in place of active device gain $\mathbf{g}$, the oscillation function $A(\omega, g_\mathrm{m})$ is defined as

$$A(\omega, g_\mathrm{m}) = 1 + g_\mathrm{m} z_{12}(\omega). \tag{11}$$

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Because Eq. (2) reduces to $A(\omega_0, g_{\rm m0}) = 0$, we get

$$g_{\rm m0} = -\frac{1}{z_{12}(\omega_0)}, \tag{12}$$

$$A'(\omega_0, g_{\rm m0}) = -\frac{z'_{12}(\omega_0)}{z_{12}(\omega_0)}, \tag{13}$$

$$A(\omega_0, 0) = 1, \tag{14}$$

where $g_{\rm m0}$ is the minimum necessary trans-conductance for the oscillation to take place. Substituting Eqs. (13) and (14), Eq. (9) reduces to

$$Q = \frac{\omega_0}{2}\left|\frac{z'_{12}(\omega_0)}{z_{12}(\omega_0)}\right| \qquad \text{for VCCS.} \tag{15}$$

In case of other oscillators: using NR, CCCS, VCVS and CCVS, the oscillation conditions are also given by

$$z(\omega) - r = 0 \qquad \text{for NR,} \tag{16}$$

$$z_{11}(\omega) + \beta z_{12}(\omega) = 0 \qquad \text{for CCCS,} \tag{17}$$

$$y_{11}(\omega) - \mu y_{12}(\omega) = 0 \qquad \text{for VCVS,} \tag{18}$$

$$1 - r_{\rm m} y_{12}(\omega) = 0 \qquad \text{for CCVS.} \tag{19}$$

where $z_{11}(\omega)$ and $z_{12}(\omega)$ are elements of matrix $\mathbf{Z}$ for two-port passive linear network shown in Fig. 2 (b), β is the current gain, $y_{11}(\omega)$ and $y_{12}(\omega)$ are elements of matrix $\mathbf{Y}$ for two-port passive linear network shown in Fig. 2 (d) or (e), μ is the voltage gain, and $r_{\rm m}$ is the trans-resistance. Similarly, the oscillation function can be defined as

$$A(\omega, r) = z(\omega) - r \qquad \text{for NR,} \tag{20}$$

$$A(\omega, \beta) = z_{11}(\omega) + \beta z_{12}(\omega) \qquad \text{for CCCS,} \tag{21}$$

$$A(\omega, \mu) = y_{11}(\omega) - \mu y_{12}(\omega) \qquad \text{for VCVS,} \tag{22}$$

$$A(\omega, r_{\rm m}) = 1 - r_{\rm m} y_{12}(\omega) \qquad \text{for CCVS.} \tag{23}$$

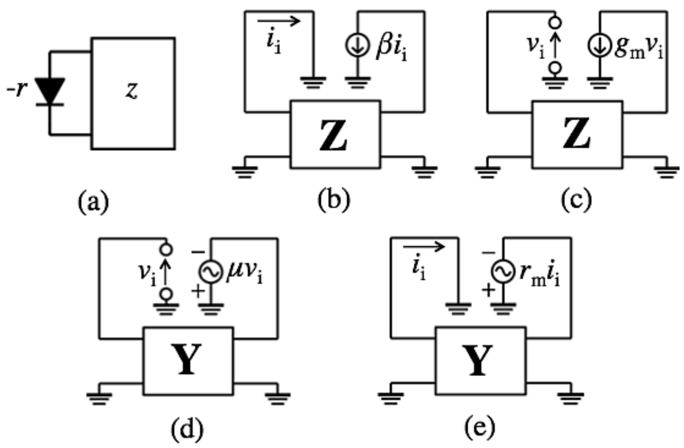

Fig. 2. Modern five oscillators: (a) NR, (b) CCCS, (c) VCCS, (d) VCVS, and (e) CCVS oscillators.

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Therefore, these active Q factor formulas are derived as

$$Q = \frac{\omega_0}{2}\left|\frac{z'(\omega_0)}{z(\omega_0)}\right| \quad \text{for NR,} \tag{24}$$

$$Q = \frac{\omega_0}{2}\left|\frac{z'_{11}(\omega_0)}{z_{11}(\omega_0)} - \frac{z'_{12}(\omega_0)}{z_{12}(\omega_0)}\right| \quad \text{for CCCS,} \tag{25}$$

$$Q = \frac{\omega_0}{2}\left|\frac{y'_{11}(\omega_0)}{y_{11}(\omega_0)} - \frac{y'_{12}(\omega_0)}{y_{12}(\omega_0)}\right| \quad \text{for VCVS,} \tag{26}$$

$$Q = \frac{\omega_0}{2}\left|\frac{y'_{12}(\omega_0)}{y_{12}(\omega_0)}\right| \quad \text{for CCVS.} \tag{27}$$

The derived formulas all accord with what appeared in Ref. [4], which means that Eq. (9) covers the formulas for typical oscillators.

4 Application to cascode amplifier based oscillators

As a simple example for an active Q factor analysis application, we consider a transmission line feedback FET oscillator shown in Fig. 3 (a). To analyze the oscillator, we add imaginary ports #1 and #2. In accordance with the previous section, its $z_{12}(\omega)$ is given by

$$z_{12}(\omega) = \frac{-jZ_0}{\sin\omega\tau - jg_\mathrm{d}Z_0\cos\omega\tau}. \tag{28}$$

Therefore, the oscillation condition is derived as

$$\sin\omega_0\tau - jZ_0(g_\mathrm{d}\cos\omega_0\tau + g_\mathrm{m0}) = 0. \tag{29}$$

Solving the condition, we get

$$\omega_0\tau = (2m-1)\pi, \tag{30}$$

$$g_\mathrm{m0} = \frac{1}{g_\mathrm{d}}. \tag{31}$$

On the other hand, the oscillation function can be defined as

$$A(\omega, g_\mathrm{m}) = \sin\omega\tau - jZ_0(g_\mathrm{d}\cos\omega\tau + g_\mathrm{m}). \tag{32}$$

Therefore, substituting Eqs. (30), (31) and (32) into Eq. (9) gives

$$Q = \frac{(2m-1)\pi}{2}\frac{1}{Z_0 g_\mathrm{d}}. \tag{33}$$

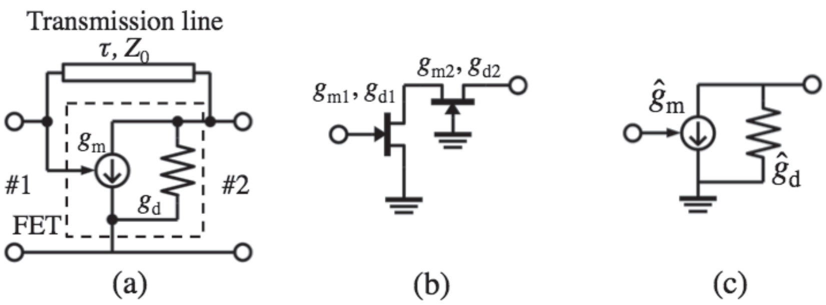

Fig. 3. (a) A transmission line feedback FET oscillator, (b) a cascode FET amplifier, and (c) its equivalent circuit.

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The Q factor increases with decreasing g_{d} or Z_0.

To get smaller g_{d}, we consider a cascode FET amplifier shown in Fig. 3 (b). Its equivalent circuit is described as shown in Fig. 3 (c), where $\hat{g}_{\mathrm{m}}$ is equivalent trans-conductance and $\hat{g}_{\mathrm{d}}$ is equivalent drain-conductance

$$\hat{g}_{\mathrm{m}}(g_{\mathrm{m1}}, g_{\mathrm{m2}}) = \frac{g_{\mathrm{d2}} + g_{\mathrm{m2}}}{g_{\mathrm{d1}} + g_{\mathrm{d2}} + g_{\mathrm{m2}}} g_{\mathrm{m1}}, \tag{34}$$

$$\hat{g}_{\mathrm{d}}(g_{\mathrm{m2}}) = \frac{g_{\mathrm{d2}}}{g_{\mathrm{d1}} + g_{\mathrm{d2}} + g_{\mathrm{m2}}} g_{\mathrm{d1}}. \tag{35}$$

Equivalent drain-conductance $\hat{g}_{\mathrm{d}}$ is smaller than FET's drain-conductance g_{d1} or g_{d2}. The oscillation function can be given by rewriting Eq. (32) as

$$A(\omega, g_{\mathrm{m1}}, g_{\mathrm{m2}}) = \sin\omega\tau - jZ_0\{\hat{g}_{\mathrm{d}}(g_{\mathrm{m2}})\cos\omega\tau + \hat{g}_{\mathrm{m}}(g_{\mathrm{m1}}, g_{\mathrm{m2}})\}. \tag{36}$$

Therefore, we get the active Q factor as

$$Q = \frac{\omega}{2}\left|\frac{A'(\omega_0, g_{\mathrm{m10}}, g_{\mathrm{m20}})}{A(\omega_0, 0, 0)}\right| = \frac{(2m-1)\pi}{2}\frac{1}{Z_0}\frac{g_{\mathrm{d2}} + g_{\mathrm{d1}}}{g_{\mathrm{d2}} g_{\mathrm{d1}}}, \tag{37}$$

where g_{m10} and g_{m20} are the minimum necessary trans-conductances for the oscillation to take place. In case of $g_{\mathrm{d}} = g_{\mathrm{d2}} = g_{\mathrm{d1}}$, we get

$$Q = (2m-1)\pi\frac{1}{Z_0 g_{\mathrm{d}}}. \tag{38}$$

Compering Eqs. (33) and (38), a cascode FET amplifier based oscillator has Q factor twice as high as FET amplifier based one in a transmission line feedback topology. It means that a cascode amplifier improves phase noise by 6-dB.

5 Conclusion

We have derived the versatile active Q factor formula that can apply to any topology and any type of active device involving the five topologies in the conventional formulas. The formula is derived from the oscillation condition and can reduce to the conventional formulas. We have applied the unified formula to a cascode amplifier based oscillator. In the case of the cascode amplifier consisting of the same FETs, it gives active Q factor twice as high as an FET oscillator in a transmission line feedback oscillator. From these issues, we conclude that the active Q factor can be estimated without knowing the specific topology or impedance information but only by knowing the oscillation condition and its frequency slope. Therefore, oscillation condition analysis enables phase noise prediction. It will contribute to low-phase-noise oscillator design and optimization.

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