American Journal of Applied Sciences

Original Research Paper

New Hybrid Block Method with Three Off-Step Points for Solving First Order Ordinary Differential Equations

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Article history
Received: 01-12-2015
Revised: 07-12-2015
Accepted: 12-02-2016

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Abstract: A new single-step hybrid block method with three off-step points for the solution of first order ordinary differential equations is proposed. The strategy employed to develop this method is interpolating the power series approximate solution at x_n and off-step points and collocating the derivative of the power at x_{n+1}. The class of linear multistep method derived is then simultaneously applied to first order ordinary differential equations together with the associated initial conditions. The numerical results generated are found to be better when compared with the existing methods in terms of error. Besides its excellent performance in term of accuracy, this method also possesses good properties of numerical method such as zero-stable, consistent and convergent.

Keywords: Hybrid Block, Single-Step, Interpolation, Collocation, Ordinary Differential Equation

Introduction

We are interested in finding the numerical solution of first order initial value problems of Ordinary Differential Equations (ODEs) in the following form:

$$y' = f(x, y), \quad y(x_0) = y_0, \quad a \le x \le b \tag{1}$$

Several scholars such as Awoyemi *et al*. (2007), Badmus and Mishelia (2011), Sunday *et al*. (2013) developed numerical methods which were implemented in predictor-corrector mode for solving (1). The implementation of numerical method in predictor-corrector approach, however, has some setbacks which include lengthy computational time due to more function evaluations needed per step and computational burden which may affect the accuracy of the method in terms of error (James *et al*., 2012). In overcoming the setbacks mentioned above, Sagir (2014) developed a three-step block method without predictor where three points with a single off-step point were considered as interpolation points for solving first order ordinary differential equations. The numerical results generated when the method was applied to first order ordinary differential equations are still not encouraging.

Therefore, in this study a single-step block method with three off-step points for solving (1) in order to improve the accuracy of the existing methods is proposed. This paper is divided into four sections; section 1 gives a brief introduction of our work, section 2 explains the derivation of the method, section 3 establishes the properties of the developed block method which include order, error constants, consistency, zero-stability and convergence and finally section 4 presents the numerical results derived when the method was tested on first order initial value problems of ODEs.

Derivation of the Method

Let the power series of the form

$$y(x) = \sum_{i=0}^{v+m-1} a_i \left(\frac{x - x_n}{h} \right)^i \tag{2}$$

be considered as the approximate solution to (1) where v represents the number of interpolation points and m is the order of ODE. The derivative (2) is given by:

$$y'(x) = \sum_{i=0}^{v+m-1} \frac{i}{h} a_i \left(\frac{x - x_n}{h} \right)^{i-1} \tag{3}$$

Interpolating (1) at $x = x_{n+i}, i = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and collocating (3) at $x = x_{n+1}$ produces equations which can be presented in the following matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \\ 1 & \frac{3}{4} & \frac{9}{16} & \frac{27}{64} & \frac{81}{256} \\ 0 & \frac{1}{h} & \frac{2}{h} & \frac{3}{h} & \frac{4}{h} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ f_{n+1} \end{bmatrix} \quad (4)$$

Gaussian elimination method is then applied to determine the values of the unknown variables a_j, j = 0(1)4, given as below:

$$a_0 = y_n$$

$$a_1 = \left(-\frac{616}{75} y_n + \frac{256}{75} y_{n+\frac{3}{4}} + \frac{384}{25} y_{n+\frac{1}{4}} - \frac{264}{25} y_{n+\frac{1}{2}} - \frac{3h}{25} f_{n+1} \right)$$

$$a_2 = \left(\frac{1684}{75} y_n - \frac{1744}{75} y_{n+\frac{3}{4}} - \frac{16161}{25} y_{n+\frac{1}{4}} + \frac{1636}{25} y_{n+\frac{1}{2}} + \frac{22h}{25} f_{n+1} \right)$$

$$a_3 = \left(-\frac{1856}{75} y_n + \frac{3296}{75} y_{n+\frac{3}{4}} + \frac{2144}{25} y_{n+\frac{1}{4}} - \frac{2624}{25} y_{n+\frac{1}{2}} - \frac{48h}{25} f_{n+1} \right)$$

$$a_4 = \left(\frac{704}{75} y_n - \frac{1664}{75} y_{n+\frac{3}{4}} - \frac{896}{25} y_{n+\frac{1}{4}} + \frac{1216}{25} y_{n+\frac{1}{2}} + \frac{32h}{25} f_{n+1} \right)$$

Substituting the values of a_s' into (2) to give a continuous implicit scheme of the form:

$$\begin{aligned} &y(x) \\ &= y_n + \frac{(x-x_n)^4}{h^4} \left(\frac{704}{75} y_n - \frac{1664}{75} y_{n+\frac{3}{4}} - \frac{896}{75} y_{n+\frac{1}{4}} + \frac{1216}{25} y_{n+\frac{1}{2}} + \frac{32h}{25} f_{n+1} \right) \\ &+ \frac{(x-x_n)^2}{h^2} \left(\frac{1684}{75} y_n - \frac{1744}{75} y_{n+\frac{3}{4}} - \frac{1616}{25} y_{n+\frac{1}{4}} + \frac{1636}{25} y_{n+\frac{1}{2}} + \frac{22h}{25} f_{n+1} \right) \\ &- \frac{(x-x_n)^3}{h^3} \left(\frac{1856}{75} y_n - \frac{3296}{75} y_{n+\frac{3}{4}} - \frac{2144}{25} y_{n+\frac{1}{4}} + \frac{2624}{25} y_{n+\frac{1}{2}} + \frac{48h}{25} f_{n+1} \right) \\ &- \frac{(x-x_n)}{h} \left(\frac{616}{75} y_n - \frac{256}{75} y_{n+\frac{3}{4}} - \frac{384}{25} y_{n+\frac{1}{4}} + \frac{264}{25} y_{n+\frac{1}{2}} + \frac{3h}{25} f_{n+1} \right) \end{aligned} \quad (5)$$

Evaluating Equation 5 at the non-interpolating point $x = x_{n+1}$ yields:

$$y_{n+1} + \frac{3}{25} y_n - \frac{48}{25} y_{n+\frac{3}{4}} - \frac{16}{25} y_{n+\frac{1}{4}} + \frac{36}{25} y_{n+\frac{1}{2}} = \frac{3h}{25} f_{n+1} \quad (6)$$

Equation 5 is then differentiated and evaluated at the off-grid points $\left(x = x_{n+i, i=\frac{1}{4},\frac{1}{2},\frac{3}{4}} \right)$ to produce:

$$\begin{aligned} f_{n+\frac{1}{4}} &= \frac{-1}{25h} \left(26 y_n + 34 y_{n+\frac{3}{4}} + 78 y_{n+\frac{1}{4}} - 138 y_{n+\frac{1}{2}} \right) + \frac{1}{25} f_{n+1.} \\ f_{n+\frac{1}{2}} &= \frac{1}{75h} \left(28 y_n + 152 y_{n+\frac{2}{3}} + -216 y_{n+\frac{1}{4}} + 36 y_{n+\frac{1}{2}} \right) - \frac{1}{25} f_{n+1.} \\ f_{n+\frac{3}{4}} &= \frac{-1}{75h} \left(34 y_n - 394 y_{n+\frac{3}{4}} - 198 y_{n+\frac{1}{4}} + 558 y_{n+\frac{1}{2}} \right) + \frac{3}{25} f_{n+1} \end{aligned} \quad (7)$$

Combining Equation 6 and 7 give the following equations presented in a matrix form as below:

$$R \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \end{bmatrix} = T \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + U \begin{bmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{bmatrix} \quad (8)$$

where:

$$R = \begin{bmatrix} \frac{-16}{25} & \frac{36}{25} & \frac{-48}{25} & 1 \\ \frac{-78}{(25h)} & \frac{138}{(25h)} & \frac{-34}{(25h)} & 0 \\ \frac{-72}{(25h)} & \frac{12}{(25h)} & \frac{152}{(75h)} & 0 \\ \frac{66}{(25h)} & \frac{-186}{(25h)} & \frac{394}{(75h)} & 0 \end{bmatrix}, T = \begin{bmatrix} 0 & 0 & 0 & \frac{-3}{(25h)} \\ 0 & 0 & 0 & \frac{26}{(25h)} \\ 0 & 0 & 0 & \frac{-28}{(75h)} \\ 0 & 0 & 0 & \frac{34}{(75h)} \end{bmatrix}$$

and:

$$U = \begin{bmatrix} 0 & 0 & 0 & \frac{3h}{25} \\ 1 & 0 & 0 & \frac{-1}{25} \\ 0 & 1 & 0 & \frac{1}{25} \\ 0 & 0 & 1 & \frac{-3}{25} \end{bmatrix}$$

Multiplying (8) with

$$\begin{bmatrix} \frac{-16}{25} & \frac{36}{25} & \frac{-48}{25} & 1 \\ \frac{-78}{(25h)} & \frac{138}{(25h)} & \frac{-34}{(25h)} & 0 \\ \frac{-72}{(25h)} & \frac{12}{(25h)} & \frac{152}{(75h)} & 0 \\ \frac{66}{(25h)} & \frac{-186}{(25h)} & \frac{394}{(75h)} & 0 \end{bmatrix}^{-1}$$

We will get a block method of the form:

$$V\begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \end{bmatrix} = W\begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + Y\begin{bmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{bmatrix} \quad (9)$$

For:

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, W = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Y = \begin{bmatrix} \frac{55h}{96} & \frac{-59h}{96} & \frac{37}{96} & \frac{-3h}{32} \\ \frac{2h}{3} & \frac{-5h}{12} & \frac{h}{3} & \frac{-h}{12} \\ \frac{21h}{32} & \frac{-9h}{32} & \frac{15h}{32} & \frac{-3h}{32} \\ \frac{2h}{3} & \frac{-h}{3} & \frac{2h}{3} & 0 \end{bmatrix}$$

which can also be represented as:

$$\begin{aligned} y_{n+\frac{1}{4}} &= y_n - \frac{3h}{32} f_{n+1} + \frac{37h}{96} f_{n+\frac{3}{4}} + \frac{55h}{96} f_{n+\frac{1}{4}} - \frac{59h}{96} f_{n+\frac{1}{2}} \\ y_{n+\frac{1}{2}} &= y_n - \frac{h}{12} f_{n+1} + \frac{h}{3} f_{n+\frac{3}{4}} + \frac{2h}{3} f_{n+\frac{1}{4}} - \frac{5h}{12} f_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} &= y_n - \frac{3h}{32} f_{n+1} + \frac{15h}{32} f_{n+\frac{3}{4}} + \frac{21h}{32} f_{n+\frac{1}{4}} - \frac{9h}{32} f_{n+\frac{1}{2}}. \\ y_{n+1} &= y_n + \frac{2h}{3} f_{n+\frac{3}{4}} + \frac{2h}{3} f_{n+\frac{1}{4}} - \frac{h}{3} f_{n+\frac{1}{2}} \end{aligned} \quad (10)$$

Properties of the Method

Based on the definition given by Lambert (1973) and Henrici (1962), the order of the developed method is $[4, 4, 4, 4]^T$ with vector error constant:

$$(\overline{c}_5) = \left[3.4044e^{(-4)}, 3.1467e^{(-4)}, 3.2959e^{(-4)}, 3.0382e^{(-4)}\right]^T$$

The first characteristic polynomial of (8) is given by:

$$\rho(\tau) = \left| \tau \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right|$$

$$= \tau^3(\tau - 1)$$

which satisfies the zero-stability condition stated by Lambert (1973) that is $|\tau_j| \leq 1$ and when $|\tau_j| = 1$, the multiplicity must not exceed 1.

Since the order of method (8) is greater than 1, it is therefore consistent. Furthermore, the developed method is also convergent because it is zero stable and consistent.

Numerical Examples

The following problems available in the previous literatures were solved in order to compare the performance of the new method with existing ones

Problem 1: $y' = xy, y(0) = 1, h = 0.1$

Exact solution: $y = e^{\frac{x^2}{2}}$

The numerical results of our method and the method developed by Odekunle *et al.* (2012) for solving Problem 1 are tabulated in Table 1 below.

Problem 2: $y' = 0.2y$, $y(0) = 10000$, $h = 0.1$

Exact solution: $y = 1000e^{\frac{x^2}{2}}$

The above problem is spatial case of differential equation of growth model which describes the growth rate of bacteria in a colony every hour by assuming that the bacteria grows continuously without any restriction. This problem was solved by Sagir (2014) using three-step hybrid block method of order four. The numerical results comparing our method with Sagir (2014) for solving Problem 2 are depicted in Table 2 below.

Table 1. Comparison of the new method with Odekunle *et al.* (2012) for solving Problem 1

x	Exact solution	Computed solution	Error in new method	Error in Odekunle *et al.* (2012)
0.1	1.005012520859401000	1.005012521195772800	$3.363718\ e^{-10}$	$2398\ e^{-7}$
0.2	1.020201340026755800	1.020201341302355800	$1.275600\ e^{-9}$	$1.6913\ e^{-7}$
0.3	1.046027859908716900	1.046027862194445400	$2.285728\ e^{-9}$	$8.7243\ e^{-7}$
0.4	1.083287067674958600	1.083287069062809600	$1.387851\ e^{-9}$	$3.0098\ e^{-6}$
0.5	1.133148453066826300	1.133148447836481300	$5.230345\ e^{-9}$	$1.7466\ e^{-6}$
0.6	1.197217363121810200	1.197217339115218500	$2.400659\ e^{-8}$	$4.1710\ e^{-6}$
0.7	1.277621313204886600	1.277621247849590700	$6.535530\ e^{-8}$	$9.6465\ e^{-6}$
0.8	1.377127764335957000	1.377127618495140200	$1.458408\ e^{-7}$	$6.7989\ e^{-6}$
0.9	1.499302500056766800	1.499302208459504200	$2.915973\ e^{-7}$	$1.2913\ e^{-5}$
1.0	1.648721270700128000	1.648720727033248700	$5.436669\ e^{-7}$	$2.6575\ e^{-5}$

DOI: 10.3844/ajassp.2016.209.212

Table 2. Comparison of the new method with Sagir (2014) for solving Problem 2

x	Exact solution	Computed solution	Error in new method	Error in Sagir (2014)
0.1	1020.201340	1020.201340	00000000e+00	00000000e+00
0.2	1040.810774	1040.810774	00000000e+00	00000000e+00
0.3	1061.836547	1061.836547	00000000e+00	00000000e+00
0.4	1083.287068	1083.287068	00000000e+00	00000000e+00
0.5	1105.170918	1105.170919	1.000000000 e^{-6}	00000000e+00
0.6	1127.496852	1127.496852	00000000e+00	00000000e+00
0.7	1150.273799	1150.273710	1.000000000 e^{-6}	1.000000000 e^{-6}
0.8	1173.510871	1173.510872	1.000000000 e^{-6}	4.800000000 e^{-5}
0.9	1197.217363	1197.217364	1.000000000 e^{-6}	5.100000000 e^{-5}
1.0	1221.402758	1221.402759	1.000000000 e^{-6}	6.600000000 e^{-5}

Discussion of the Results

In Table 1, the results produced from the new method for solving Problem 1 have better accuracy when compared with the results in Odekunle *et al*. (2012). It can also be seen in Table 2 that the new method is more efficient in terms of error than Sagir (2014).

Conclusion

A new one-step hybrid block method of order four for solving first ODEs has been developed in this study. The numerical solutions are produced with less computational efforts when compared with non-block method. The new method is consistent, zero-stable and convergent. In term of accuracy, this method claims superiority over the existing methods. Hence, the new developed method should be opted to solve first order initial value problems of ODEs directly.

Acknowledgement

Thanks to the Universiti Utara Malaysia for the financial assistance rendered in publishing this paper.

Author's Contribution

All authors equally contributed in this work.

Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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