Oscillator frequency spectrum as viewed from resonant energy storage and complex Q factor

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Abstract: The complex power theory is applied to characterize a passive linear circuit as the resonating load of an oscillator. Thanks to Tellegen's theorem relating the port impedance and internal status, the Q factor for use in Leeson's spectrum model is derived in regard to the behavior of the stored energy. The concept of complex Q factor is introduced in terms of the power dissipation slope and the energy difference between inductive and capacitive groups of components. As their relationship results in mathematical complication, it is visualized in Pythagorean charts for ease of phenomenological comprehension.

Keywords: oscillator, Q factor, spectrum, phase noise, stored energy

Classification: Microwave and millimeter wave devices, circuits, and systems

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1 Introduction

Since Leeson introduced a basic model of oscillator spectrum [1], it has been applied to circuit designs and phase noise estimations in a wide variety of oscillators. Even so, the Q factor of the oscillator continues to be an ambiguous term [2]. Unlike passive resonators or filters, the definition of Q factor is not straightforward for circuits in an oscillating state. The conventional definition based on its stored energy ratio to dissipated power does not agree with the phase noise spectrum. This discrepancy was shown by using a transfer function model of a positive feedback circuit [3]. The open-loop group delay was discussed in terms of resonant energy and power [4]. The spectrum-based Q factor applicable to Leeson's formula was derived from the circuit immittance matrix and compared with the energy-based Q factor [5]. However, an itchy question still remains: how the oscillation spectrum is related to the behavior of the energy stored in resonance? This paper clearly gives a persuading elucidation to the problem.

2 Circuit Model

Consider an oscillator consisting of a one-port active device and a one-port passive linear network for simplicity. The network includes resistors, inductors, and capacitors. Some of them work for resonance, some for impedance matching, and some as the output load. We assume a network in any complicated topology of nodes and branches. It may involve multiple tanks and filters of different resonant frequencies.

3 Complex Power

The active device generates RF power. Part of it is dissipated in the passive network and the rest is reflected back to the device. To express this phenomenon, let us define the *complex power*

$$P = \frac{1}{2}vi^* \tag{1}$$

where v and i are the RF voltage and current at the port of the network. The asterisk designates the conjugate of a complex. The real and imaginary parts respectively mean the effective power and reactive power flowing from the device to the network.

According to Tellegen's theorem, the voltage-current product is observed at the port as the sum of that on each branch in the network.

$$vi^* = v_1 i_1^* + v_2 i_2^* + v_3 i_3^* + \cdots + v_k i_k^* + \cdots \tag{2}$$

where v_k and i_k are voltage and current on the k-th branch. Each branch implies a lumped element, but in addition, distributed-constant elements such as transmission lines and stubs may be also included because they can be regarded as ensembles of infinitesimal R, L, and C. Therefore we can sort all the elements into resistive-, inductive-, and capacitive-element groups.

Substituting Eq. (2) into Eq. (1), we get

$$
\begin{aligned}
\mathrm{P} &= \frac{1}{2}\sum^{\mathrm{R}} v_{\mathrm{k}} i_{\mathrm{k}}^* + \frac{1}{2}\sum^{\mathrm{L}} v_{\mathrm{k}} i_{\mathrm{k}}^* + \frac{1}{2}\sum^{\mathrm{C}} v_{\mathrm{k}} i_{\mathrm{k}}^* \\
&= \sum^{\mathrm{R}} \frac{1}{2}\mathrm{R}_{\mathrm{k}} \left|i_{\mathrm{k}}\right|^2 + \sum^{\mathrm{L}} \frac{1}{2}\mathrm{j}\omega\mathrm{L}_{\mathrm{k}} \left|i_{\mathrm{k}}\right|^2 - \sum^{\mathrm{C}} \frac{1}{2}\mathrm{j}\omega\mathrm{C}_{\mathrm{k}} \left|v_{\mathrm{k}}\right|^2 \\
&= \mathrm{P_r} + 2\mathrm{j}\omega\Delta\mathrm{U}, \quad \Delta\mathrm{U} = \mathrm{U_m} - \mathrm{U_e}
\end{aligned}
\tag{3}
$$

where $\mathrm{P_r}$ is the total power dissipated in all the resistive branches. $\mathrm{U_m}$ and $\mathrm{U_e}$ are the total magnetic and electric energy stored in all the reactive branches.

On the other hand, the complex power is also written in terms of the total impedance Z observed at the port of the network as

$$
\mathrm{P} = \frac{1}{2}vi^* = \frac{1}{2}\mathrm{Z}\left|i\right|^2
\tag{4}
$$

From Eqs. (3) and (4), the impedance is related as

$$
\mathrm{Z} = \frac{2\mathrm{P_r}}{\left|i\right|^2} + \frac{4\mathrm{j}\omega\Delta\mathrm{U}}{\left|i\right|^2}
\tag{5}
$$

It is worth notifying that you can estimate the behavior of the internal energy without knowing each branch status. You just have to measure the port impedance Z for given ω, meanwhile you can assume unity for current $|i|$ because the network is linear. This is an important aspect to discuss oscillator Q factor and spectrum in the next chapter.
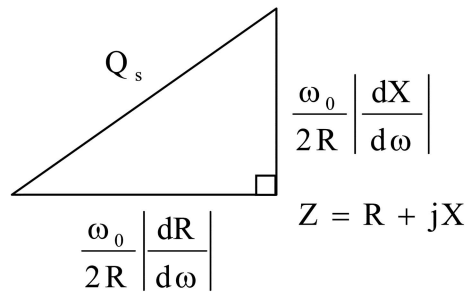
4 Oscillation Spectrum

Start from Leeson's formula

$$
S_{\mathrm{out}} = \left\{ 1 + \left(\frac{\omega_0}{2Q_{\mathrm{s}}\delta\omega}\right)^2 \right\} S_{\mathrm{in}}
\tag{6}
$$

where ω_{o} and $\delta\omega$ are oscillation frequency and offset from it. The output spectrum S_{out} linearly responds to the stimulus S_{in}. The formula says that the oscillator amplifies the signal, whose gain has a sharp frequency selectivity. The key factor to discuss for this formula is Q_{s}; *spectrum-based quality factor.* As reported in [3, 4, 5], it is something different from just the quality factor of a tank. Because Q_{s} dominates the oscillation spectrum, it must be estimated from characteristic of the network overall.
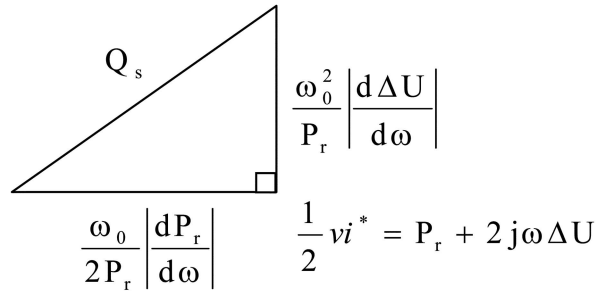
For this purpose, it is convenient for RF engineers to speak in a circuit-parameter domain such as impedance or admittance. Referring to the linear oscillator theory [5], we introduce the *complex Q factor* as the logarithmic derivative of its input impedance

$$
Q_{\mathrm{sc}} = \frac{\omega_0}{2\mathrm{Z}}\frac{\mathrm{dZ}}{\mathrm{d}\omega} = \frac{\omega_0}{2}\frac{\mathrm{d}}{\mathrm{d}\omega}\ln\mathrm{Z}
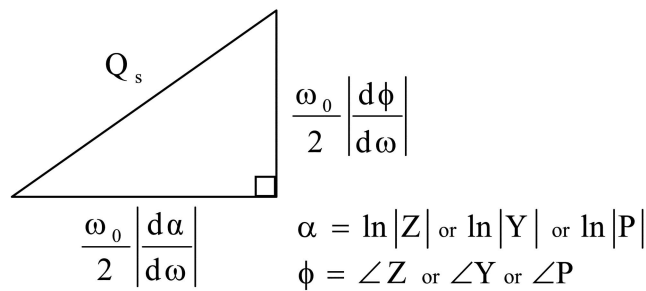\tag{7}
$$

so that $Q_{\mathrm{s}} = |Q_{\mathrm{sc}}|$. Figure 1 (a) shows a Pythagorean chart to quickly get a physical insight into this formula in the impedance domain.

$$\frac{\omega_0}{2\,R}\left|\frac{dX}{d\omega}\right|$$

$$Z = R + jX$$

$$\frac{\omega_0}{2\,R}\left|\frac{dR}{d\omega}\right|$$

(a) impedance domain

$$\frac{\omega_0^2}{P_r}\left|\frac{d\,\Delta U}{d\omega}\right|$$

$$\frac{1}{2}vi^{*} = P_r + 2\,j\omega\,\Delta U$$

$$\frac{\omega_0}{2\,P_r}\left|\frac{dP_r}{d\omega}\right|$$

(b) energy domain

$$\frac{\omega_0}{2}\left|\frac{d\phi}{d\omega}\right|$$

$$\frac{\omega_0}{2}\left|\frac{d\alpha}{d\omega}\right|$$

$$\alpha = \ln\left|Z\right| \text{ or } \ln\left|Y\right| \text{ or } \ln\left|P\right|$$

$$\phi = \angle Z \text{ or } \angle Y \text{ or } \angle P$$

(c) log-magnitude and phase

Fig. 1. Spectrum-based Q factor Pythagorean chart.

Applying Eq. (7) to Eq. (5), and then imposing the condition that imaginary part of the port impedance must vanish, we finally reach

$$Q_{\mathrm{sc}} = \frac{\omega_0}{2P_r}\frac{dP_r}{d\omega} + j\frac{\omega_0^2}{P_r}\frac{d\Delta U}{d\omega} \qquad (8)$$

at $\omega = \omega_o$. Another Pythagorean chart is shown in Fig. 1 (b) to depict the same $Q_s = |Q_{\mathrm{sc}}|$ as in (a) but in the energy domain.

Though Eqs. (7) and (8) look different styles, they can be translated into polar coordinates resulting uniquely in

$$Q_{\mathrm{sc}} = \frac{\omega_0}{2}\frac{d\alpha}{d\omega} + j\frac{\omega_0}{2}\frac{d\phi}{d\omega} \qquad (9)$$

where α and ϕ are log-magnitude and phase in either immittance or energy domain. Figure 1 (c) gives a common vista on Eq. (9). The formulas described here are summarized in Table I.

Table I. Q Factor Formulas at a Glance

	immittance domain	energy domain
circuit status	$Z = R + jX$ $Y = G + jB$	$P = \dfrac{1}{2} vi^* = P_r + 2j\omega\Delta U$ $\Delta U = U_m - U_e$
resonance condition	$X = 0$ $B = 0$	$\Delta U = 0$ $U_m = U_e$
complex Q factor	$Q_{sc} = \dfrac{\omega_0}{2Z}\dfrac{dZ}{d\omega} = \dfrac{\omega_0}{2Y}\dfrac{dY}{d\omega}$ $= \dfrac{\omega_0}{2R}\dfrac{dR}{d\omega} + j\dfrac{1}{2R}\dfrac{dX}{d\omega}$ $= \dfrac{\omega_0}{2G}\dfrac{dG}{d\omega} + j\dfrac{\omega_0}{2G}\dfrac{dB}{d\omega}$	$Q_{sc} = \dfrac{\omega_0}{2P}\dfrac{dP}{d\omega}$ $= \dfrac{\omega_0}{2P_r}\dfrac{dP_r}{d\omega} + j\dfrac{\omega_0^2}{P_r}\dfrac{d\Delta U}{d\omega}$
polar expression	$Q_{sc} = \dfrac{\omega_0}{2}\dfrac{d\alpha}{d\omega} + j\dfrac{\omega_0}{2}\dfrac{d\phi}{d\omega}$	
	$\alpha = \ln\lvert Z\rvert \ \ \text{or} \ \ \ln\lvert Y\rvert$ $\phi = \angle Z \ \text{or} \ \angle Y$	$\alpha = \ln\lvert P\rvert$ $\phi = \angle P$
modulus Q	$Q_s = \lvert Q_{sc}\rvert = \sqrt{\Re e\{Q_{sc}\}^2 + \Im m\{Q_{sc}\}^2}$	

Compare Eq. (9) with a predecessor's work. The Q factor described in Eq. (31) of reference [4] is somewhat similar to our Eq. (9), but actually incomplete because they took only the phase slope $d\phi/d\omega$ into account for spectrum analysis. For even simpler circuits like a single-tank resonator, the conventional Q factor, i.e., the ratio of stored energy to power dissipation per unit cycle eventually coincides with our spectrum-based Q factor.

5   Conclusion

The Q factor plays a crucial role in oscillator spectrum. It is necessary for proper estimation to see the port impedance of the entire network, not only of a constituent resonator. The phase slope or group delay is not sufficient but a partial information to evaluate the Q factor. The complex power analysis clearly relates the Q factor and energy stored in the circuit. Pythagorean visualization helps systematic understanding of the relationship. The modulus of the complex Q factor contributes to the rigorous spectrum characterization.

Acknowledgments