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Computational Intelligence and Application of Frame Theory in Communication Systems

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Abstract: In this study, we have to discuss the reduction of noise due to peak amplitude and power ratio in multi carrier modulation scheme like Orthogonal Frequency Division Multiplexing (OFDM) process using Frame theory. The frame operator of the frame which is positive, self adjoint, invertible and it commutes with synthesis operator. If X is dual frame in H with frame operator S and analysis operator T, then T is quasi normal operator. If Y is dual frame for X, T is quasi unitary operator. If T and Q are pseudo inverse and sum of its inverse with its ad joint multiplication is frame operator, then X is Bessel's sequence and dual frame.

Keywords: Hilbert Space, Banach Algebra, Frame, Dual Frame, Quasi Normal Operator

Introduction

Frames were formally defined in Hilbert spaces by Duffin and Schaffer (1952) to deal with non harmonic Fourier series. After a couple of years, frames were brought to life Daubechies *et al*. (1986), in the context of Painless nonorthogonal expansions and Peter G. Casazza and their frame theory research centre discussed (Casazza and Christensen, 1997; Casazza, 2000; Casazza and Christensen, 1997; Obeidat *et al*., 2009). Frames are generalizations of orthonormal basis. The linear independence property for a basis which allows each element in the space to be written as a linear combination and this is very restrictive for practical problems. A frames allows each element in the space to be written as a linear combination of the elements in the frames, here linear independence between the frames element is not required. This fact plays important role in signal processing, image processing, coding theory and sampling theory.

Preliminaries and Notations

Let H be Hilbert space and $L(H)$ be a set of all linear bounded operators on H. We can define the following operators:

$$T : I^2 \to H, Ta = \sum_{n=1}^{\infty} a_n f_n \text{ , } for\, a = \{a_n\} \in I^2$$

Is called synthesis operator or pre frame operator and the ad joint operator is given that:

$$T^* : H \to I^2 ,\ T^* f = \{\langle f, f_n \rangle\}_{n=1}^{\infty}$$

Is called the analysis operator. The composition operator T with its adjoint T^* is denoted by:

$$S = T^* T$$

i.e., S: $H \to H$, $Sf = \sum_{n=1}^{\infty} \langle f, f_n \rangle f_n$ for $f \in H$ is called the frame operator.

Every frame at least has a dual .A dual frame which is not the canonical dual frame is called an alternate dual frame.

Before going to definition of Stable and unstable, let us define bounded and unbounded signal or frames. If the signal is bounded, then its magnitude is always be finite i.e., $|f_n| \le m_n$, otherwise unbounded. A system is said to be unstable if the output of the system is unbounded for bounded input. A system is called Stable if the output of system is bounded for every bounded input or BIBO stable.

We begin with frame definitions. Let H be separable Hilbert space with the inner product $\langle \cdot,\cdot \rangle$ linear in the first entry and all index sets are assumed to be countable.

Definition 3.1

Let H be separable Hilbert space and a sequence $\{f_n\}_{n=1}^{\infty} \subset H$ is called an ordinary frames. If there exist constants A, B >0, such that:

$$A \| f \|^2 \le \sum_{n=1}^{\infty} | \langle f, f_n \rangle |^2 \le B \| f \|^2, \ for\,all\ f \in H$$

Definition 3.2

Let H be separable Hilbert space. A sequence $\{f_n\}_{n=1}^{\infty} \subset H$ is called a Bessel Sequence. If here exists constant B >0, such that $\sum_{n=1}^{\infty} | \langle f, f_n \rangle |^2 \le B \| f \|^2, \ for\ f \in H$.

Definition 3.3

Let H be Hilbert space, then:

- A sequence $\{g_n\}_{n=1}^{\infty}$ is called a dual frame for $\{f_n\}_{n=1}^{\infty}$ if $f = \sum_{n=1}^{\infty} \langle f, f_n \rangle g_n$ for $f \in H$
- A sequence $\{g_n\}_{n=1}^{\infty}$ is called a canonical dual frame for $\{S^{-1} f_n\}_{n=1}^{\infty}$ if $f = \sum_{n=1}^{\infty} \langle f, f_n \rangle g_n$ for $f \in H$

Theorem 4.1

Let H be separable Hilbert space and the frame operator S of the dual frame for $\{f_n\}_{n=1}^{\infty}$ is positive, self adjoint, invertible and commutes with synthesis operator, then:

- T is quasi normal operator in Hilbert space H
- If $\{\bar{f}_n\}_{n=1}^{\infty}$ is dual frame for $\{f_n\}_{n=1}^{\infty}$, then T is quasi unitary operator

Proof

We know that (Gavruta, 2012):

$$T : l^2 \to H, Ta = \sum_{i=1}^{\infty} a_n f_n \, , \ for\, a = \{a_n\} \in l^2$$

Is called synthesis operator or pre frame operator and the ad joint operator is given that:

$$T^* : H \to l^2, T^* f = \{\langle f, f_n \rangle\}_{n=1}^{\infty}$$

Is called the analysis operator. The composition operator T with its adjoint T^* it denoted by $S = T^*T$, i.e., $S : H \to H, Sf = \sum_{n=1}^{\infty} \langle f, f_n \rangle f_n$ for $f \in H$ is called the frame operator.

Since T is bounded linear operator:

$$\begin{aligned} TSf &= T\left(\sum_{n=1}^{\infty} \langle f, f_n \rangle f_n \right) \\ &= \left(\sum_{n=1}^{\infty} \langle f, f_n \rangle Tf_n \right) \\ &= \sum_{n=1}^{\infty} \langle f, f_n \rangle \bar{f}_n \\ now &= \sum_{n=1}^{\infty} \langle f, \bar{f}_n \rangle f_n \end{aligned} \tag{1.1}$$

$$\begin{aligned} TSf &= \left(\sum_{n=1}^{\infty} \langle Tf, f_n \rangle f_n \right) \\ &= \sum_{n=1}^{\infty} \langle f, \bar{f}_n \rangle f_n \end{aligned} \tag{1.2}$$

From (1.1) and (1.2), we have $TS = ST$:

$$i.e. TTT^* = T^*TT$$

Therefore T is quasi normal operator.

Since $\{\bar{f}_n\}_{n=1}^{\infty}$ is dual frame for $\{f_n\}_{n=1}^{\infty}$.

From (1.1):

$$TSf = \sum_{n=1}^{\infty} \langle f, f_n \rangle \bar{f}_n \Rightarrow TS = I$$

From (1.2):

$$STf = \sum_{n=1}^{\infty} \langle f, f_n \rangle \bar{f}_n \Rightarrow TS = I$$

So we have:

$$TS = ST = I$$

Therefore T is quasi unitary operator in Hilbert space H.

Theorem 4.2

Let H be separable Hilbert space and let P be an orthogonal projection, $\{f_n\}_{n=1}^{\infty} \subseteq H$ is a dual frame for H if and only if $\{g_n\}_{n=1}^{\infty} \subseteq H$ is a dual frame for $P(H)$, with lower frame bound A' and upper frame bound B and S^q is frame operator.

Proof

Since P is an orthogonal projection in Hilbert space H

$P^2\left(\{f_n\}_{n=1}^{\infty}\right) = P\left(\{f_n\}_{n=1}^{\infty}\right) = \{g_n\}_{n=1}^{\infty}$ is a dual frame.

Hence the proof.

Proposition 4.3

Let H be separable Hilbert space and S^q is quasi normal frame operator which is bounded linear, then $T(TT^*)$ and $(T^*T)T$ have the same non zero Eigen value.

Proof

For:

$$x \in H, 0 \neq Tx \in H \text{ such that} \left(T^*T\right)Tx$$
$$= \lambda x \text{ and } T\left(TT^*\right)x = \mu x$$

Now:

$$\left\langle (\lambda - \mu)x, x \right\rangle = \left\langle \lambda x - \mu x, x \right\rangle$$
$$= \left\langle \lambda x, x \right\rangle - \left\langle \mu x, x \right\rangle$$
$$= \left\langle \left(T^*T\right)Tx, x \right\rangle - \left\langle T\left(TT^*\right)x, x \right\rangle$$
$$= 0$$

Since S^q is quasi normal frame operator.
Therefore $\lambda = \mu$, Hence the proof.

Theorem 4.4

Let H be separable Hilbert space and $T \in H$ is reductive quasi similar to quasi normal operator and S is frame operator of the frame $\{f_n\}_{n=1}^{\infty} \subseteq H$. Then T is quasi normal operator.

Proof

For each $g \in H$, $g - \lim_{m\to\infty}\left(\sum_{n=1}^{\infty} g_{mn}\right)$ and $S_n = T|g_n$ which is reductive quasi similar to quasi normal operator.

Now:

$$\left\langle T\left(TT^*\right)g, g \right\rangle = \left\langle \lim_{m\to\infty}\left(\sum_{n=1}^{\infty} T\left(TT^*\right)g_{mn}\right), g \right\rangle$$
$$= \left\langle \lim_{m\to\infty}\left(\sum_{n=1}^{\infty} S_n\left(S_n S_n^*\right)g_{mn}\right), g \right\rangle$$
$$= \left\langle \lim_{m\to\infty}\left(\sum_{n=1}^{\infty} S_n^* S_n S_n g_{mn}\right), g \right\rangle$$
$$\left\langle \lim_{m\to\infty}\left(\sum_{n=1}^{\infty} T^*TTg_{mn}\right), g \right\rangle$$
$$= \left\langle T^*TTg, g \right\rangle$$
$$i.e\left\langle \left(T\left(TT^*\right) - T^*TT\right)g, g \right\rangle = 0$$
$$i.e TTT^* = T^*TT$$

Therefore T is quasi normal operator.

Now:

$$\left\langle TSg, g \right\rangle = \left\langle T\left(\sum_{n=1}^{\infty}\left\langle g, g_n \right\rangle g_n\right), g \right\rangle$$
$$= \left\langle \left(\sum_{n=1}^{\infty}\left\langle g, g_n \right\rangle Tg_n\right), g \right\rangle$$
$$= \sum_{n=1}^{\infty}\left\langle g, g_n \right\rangle \left\langle \left(Tg_n\right), g \right\rangle$$
$$= \sum_{n=1}^{\infty}\left\langle g, g_n \right\rangle \left\langle \sum_{n=1}^{\infty} c_n g_n, g \right\rangle$$

We have:

$$\left\langle TSg, g \right\rangle = K\sum_{n=1}^{\infty}\left|\left\langle g, g_n \right\rangle\right|^2$$

Since $\{g_n\}_{n=1}^{\infty}$ is frame in H:

$$\left\langle TSg, g \right\rangle = K\sum_{n=1}^{\infty}\left|\left\langle g, g_n \right\rangle\right|^2 \leq KB \,\|\, g \,\|^2$$

And:

$$\left\langle TSg, g \right\rangle = K\sum_{n=1}^{\infty}\left|\left\langle g, g_n \right\rangle\right|^2 \geq KB \,\|\, g \,\|^2$$

There exist constants $\bar{B} = KB < \infty$ and $\bar{A} = KA > 0$, we have:

$$\left\langle TSg, g \right\rangle \leq \bar{B} \,\|\, g \,\|^2$$

And:

$$\left\langle TSg, g \right\rangle \geq \bar{A} \,\|\, g \,\|^2$$

Therefore:

$$\bar{A} \,\|\, g \,\|^2 \leq \left\langle TSg, g \right\rangle \leq \bar{B} \,\|\, g \,\|^2$$

where, lower frame bound $\bar{A}$ and upper frame bound $\bar{B}$ with frame operator $S^q = TS$

Lemma 4.5

Let H be Banach algebra and let H be the set of all invertible elements of H, for $x \in H \text{ and } h \in H'; \text{with} \,\|\, h \,\|< 1/2 \,\|\, x^{-1} \,\|^{1-}$, then:

$$x + h \in H$$

And:

$$\left\|\left((x+h)^{-1} + -x^{-1} + x^{-1}hx^{-1}\right)\right\| \leq 2\left\|x^{-1}\right\|^3 \left\|h\right\|^2$$

Theorem 4.6

Let H be separable Hilbert space. If T and Q are pseudo inverses and $(T^{-1} + Q^{-1})^*(T^{-1} + Q^{-1})$ is frame operator, then $\{g_n\}_{n=i}^{\infty}$ is Bessel's sequence and dual frame in Hilbert space H.

Proof

Let T and Q be pseudo inverse in H (Ding, 2003):

$$
\begin{aligned}
&\left\langle \left(T^{-1}+Q^{-1}\right)^{*}\left(T^{-1}+Q^{-1}\right)g,g\right\rangle \\
&=\left\langle \left(T^{-1}+Q^{-1}\right)^{*}\left(T^{-1}+Q^{-1}\right)g,g\right\rangle \\
&=\left\langle \left(T^{-1}+Q^{-1}\right)g,\left(T^{-1}+Q^{-1}\right)g\right\rangle \\
&=\left\langle \left(T^{-1}+Q^{-1}TQ^{-1}\right)g,\left(T^{-1}+Q^{-1}TQ^{-1}\right)g\right\rangle \\
&=\left\langle \begin{matrix}(T^{-1}+Q^{-1}-Q^{-1}+Q^{-1}TQ^{-1})g, \\ (T^{-1}+Q^{-1}-Q^{-1}+Q^{-1}TQ^{-1})g\end{matrix}\right\rangle \\
&=\left\langle \begin{matrix}\left((T+Q)^{-1}+-Q^{-1}+Q^{-1}TQ^{-1}\right)g, \\ \left((T+Q)^{-1}+-Q^{-1}+Q^{-1}TQ^{-1}\right)g\end{matrix}\right\rangle \\
&=\left\|\left((T+Q)^{-1}+-Q^{-1}+Q^{-1}TQ^{-1}\right)g\right\|^{2} \\
&\le\left\|\left((T+Q)^{-1}+-Q^{-1}+Q^{-1}TQ^{-1}\right)\right\|^{2}\|g\|^{2} \\
&\le 2\left\|Q^{-1}\right\|^{3}\|T\|^{2}\|g\|^{2}
\end{aligned}
$$

By lemma 4.5:

$$
\begin{aligned}
&\le B\|g\|^{2} \\
&i.e\left\langle \left(T^{-1}+Q^{-1}\right)^{*}(T^{-1}+Q^{-1})g,g\right\rangle \le B\|g\|^{2} \\
&i.e\left(T^{-1}+Q^{-1}\right)^{*}\left(T^{-1}+Q^{-1}\right)g=\sum_{n=1}^{\infty}\left\langle g,g_n\right\rangle g_n \\
&i.e\ g=\sum_{n=1}^{\infty}\left\langle g,\bar{g}_n\right\rangle g_n
\end{aligned}
$$

Therefore $\{\bar{g}_n\}_{n=1}^{\infty}$ is dual frame of the frame $\{g_n\}_{n=1}^{\infty}$.

Proposition 4.7

Let H be Hilbert space and T_1 and T_2 be self ad joint operators in H, then:

- $S = T_2T_1 \geq 0$ is self adjoin operator
- If T_1 and T_2 are shift invariant operators in H
- If the sequence $(a_n)\to a\in H$, then the frame operator S is stable

Proof

T_1 and $T_2$are self ad joint operators in Hilbert space:

$$T_1 : l^2 \to H, T_1\, a=\sum_{n=1}^{\infty}a_n f_n\ ,\ for\, all\, a=\{a_n\}\in l^2$$

$$T_2 : H \to l^2, T_2\, f=\left\{\left\langle f,f_n\right\rangle\right\}_{n=1}^{\infty},for\, all\, f\in H$$

For every $f_n\in l^2$, there is $g_n^o \in H$ such that $T_1\left(f_n\right)=g_n^o$ and $g_n=T_2\left(g_n^o\right)$.

Since T_1 and T_2 are self ad joint operators in H.

We get $S = T_2T_1$ which is non negative and self ad joint in H.

To prove S is shift invariant.

Since T_1 and T_2 are shift invariant operators in H:

$$
\begin{aligned}
&g_{n-k}^{o}=T_1\left(f_{n-k}\right) and\ g_{n-k}=T_2\left(g_{n-k}^{o}\right) \\
&g_{n-k}=T_2\left(T_1\left(f_{n-k}\right)\right) \\
&g_{n-k}=T_2T_1\left(g_{n-k}^{o}\right) \\
&g_{n-k}=S\left(g_{n-k}^{o}\right)
\end{aligned}
$$

Therefore S is shift invariant.

Since the sequence $(a_n)\to a$ in the Hilbert space which is bounded (BIBO).

Therefore the system is stable.

Theorem 4.8

Let $\{f_n\}_{n=1}^{\infty}\subseteq M\ and \{g_n\}_{n=1}^{\infty}\subseteq N$ be subspaces of Hilbert space H which is orthonormal basis and if there exists analysis operator T^*, Frames with junk $\{R_\sigma\}\in H$ such that $\|R_\sigma\|^2 = \|\tau\|^2 + \|\sigma\|^2$, for all $\tau\in M$, $\sigma\in N$, then $T^*R_\sigma = \tau$.

This is reconstruction of original information (Arefijamaal and Zekaee, 2013).

Conclusion

We conclude that the main problem of communication systems is noise, which is eliminated by frame theory operator in the modes of linear, Shift invariant and orthogonal. The orthonormal Frames in Hilbert space used for reduce noise to received original data. Frames play an important role not only the theoretic but also many applications in Engineering and Technology.

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Author's Contribution

K. Rajupillai: Manuscript described and writing work.

S. Palaniammal: Manuscript correction and guiding.

K. Bommuraju: Discussion for technical communication.

Ethics

Today noise is biggest problem in communication system. The analysis of orthonormal operator in quantization error or noise is similar to the analysis of quantization error or noise due to A/D process. Reconstruction of original information which is eliminated noise.

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