

# Dedicated $Q$ factor formulas stemming from oscillation frequency stability against source and load deviations

Takashi Ohira<sup>a)</sup>

*Toyohashi University of Technology*

*1-1 Hibarigaoka, Tempaku, Toyohashi 441-8580 Japan*

*a) [ohira@tut.jp](mailto:ohira@tut.jp)*

**Abstract:** This paper derives two kinds of  $Q$  factor formulas for use in estimating frequency stability of sinusoidal oscillators. To cover various circuit topologies, we consider a general black-box model having DC input and RF output ports. Assuming a small deviation in DC supply voltage or RF load impedance, the frequency shift caused by such deviation is formulated by a perturbation technique. As a versatile criterion to indicate the tolerance of circuit itself, we introduce oscillator  $Q$  against each deviation. Since the theory is described in the linear impedance domain, it gives oscillator designers a clear vista on the stability of practical circuits.

**Keywords:** oscillator, stability, pushing, pulling,  $Q$  factor

**Classification:** Microwave and millimeter wave devices, circuits, and systems

## References

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## 1 Introduction

Oscillator circuits generally suffer from deviation of their DC power supply voltage or RF load impedance. To make the circuits tolerant of those disturbances, the  $Q$  factor plays a key role as objective function in the circuit design process to select the best topology or to optimize its element parameters. The  $Q$  factor, however, has several different definitions for active networks [1, 2].

The circuit designers therefore have to carefully choose an appropriate one depending on each particular purpose. This paper derives  $Q$  factor formulas for use in estimating oscillation frequency stability taking into account the DC source pushing and RF load pulling effects.

## 2 Black-box model with source and load

Oscillators are generally regarded to convert given DC energy to RF waves. Any oscillator topology can be represented by a black box shown in Fig. 1, having port #1 for DC input and port #2 for RF output. The box includes everything needed for oscillation such as an active device and passive linear elements.

Regardless of the black box internal topology or element parameters, the circuit's RF behavior observed at port #2 can be fully characterized by its output impedance  $Z(\omega, V)$  along with the load impedance  $Z_L$  to connect. The output impedance depends not only on the observing frequency but also on the voltage  $V$  imposed on port #1 since the circuit includes an active device under DC bias control. Kirchhoff's current law on port #2 leads to well-known oscillation condition

$$Z(\omega, V) + Z_L = 0 \quad (1)$$

This implies that the first term must exhibit a negative resistance while oscillation takes place because the other one i.e. the passive load has always positive real part of its impedance.

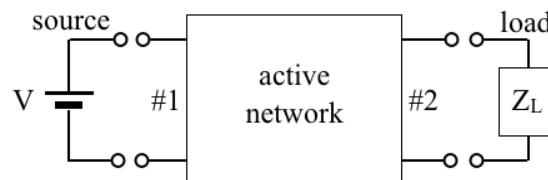


Fig. 1. Oscillator black-box model

## 3 Source pushing

To keep satisfying Eq. (1) even when supplied voltage  $V$  deviates, oscillation frequency  $\omega$  must then change. This is called source pushing on oscillation frequency, and can be formulated by perturbation technique as

$$\omega = \omega_0 + \delta\omega, \quad V = V_0 + \delta V \quad (2)$$

where suffix 0 stands for original quantity and  $\delta$  indicates difference between before and after deviated. Applying Eq. (2) to Eq. (1), then assuming small changes as  $|\delta\omega| \ll \omega_0$  and  $|\delta V| \ll V_0$ , we get two-dimensional Taylor series expansion

$$Z(\omega_0, V_0) + Z_\omega(\omega_0, V_0)\delta\omega + Z_V(\omega_0, V_0)\delta V + \cdots + Z_L = 0 \quad (3)$$

where the fourth term and so on are hidden since they have second or higher order of  $\delta$ . Also note that partial derivative functions are defined as

$$Z_{\omega}(\omega, V) = \frac{\partial}{\partial \omega} Z(\omega, V), \quad Z_V(\omega, V) = \frac{\partial}{\partial V} Z(\omega, V) \quad (4)$$

Extracting the zeroth- and first-order terms from Eq. (3), we get

$$Z(\omega_0, V_0) + Z_L = 0 \quad (5)$$

$$Z_{\omega}(\omega_0, V_0)\delta\omega + Z_V(\omega_0, V_0)\delta V = 0 \quad (6)$$

From Eq. (6) with ease we derive absolute frequency deviation

$$|\delta\omega| = \left| \frac{Z_V(\omega_0, V_0)}{Z_{\omega}(\omega_0, V_0)} \delta V \right| \quad (7)$$

Indeed this may be a useful index in oscillator design, it is more versatile if we normalize the deviations on both sides by their original values as

$$\frac{|2\delta\omega|}{\omega_0} = \frac{1}{Q} \frac{|\delta V|}{V_0} \quad (8)$$

On the left-hand side  $\delta\omega$  is doubled to measure the peak-to-peak deviation between upper and lower sides around  $\omega_0$ . The coefficient appeared in Eq. (7) is entirely represented by a simple constant in Eq. (8), expressed as  $1/Q$ , since the frequency shift should be inversely proportional to a certain figure of merit for stable oscillation. From Eqs. (7) and (8), we find the constant  $Q$  to be a dimension-free positive scalar quantity expressed as

$$Q = \frac{\omega_0}{2V_0} \left| \frac{Z_{\omega}}{Z_V} \right|_{\omega=\omega_0, V=V_0} \quad (9)$$

This is the first destination of this paper. According to the RF duality theorem, we can also estimate  $Q$  in terms of output admittance  $Y = Z^{-1}$  with

$$Q = \frac{\omega_0}{2V_0} \left| \frac{Y_{\omega}}{Y_V} \right|_{\omega=\omega_0, V=V_0} \quad (10)$$

Since formula Eq. (9) or (10) consists of only the port parameters, circuit designers can calculate  $Q$  without knowing any information on topology or elements inside. What they need is just output impedance of the black box with its derivatives, which can be also observed from outside. It is also worth notifying that hereby derived  $Q$  is not always equal to usual  $Q$  based on other design aspects [1, 2].

#### 4 Load pulling

Another crucial factor that affects oscillation frequency stability is the load impedance deviation. This is actually more crucial than the source pushing because the load is an RF issue. We formulate the problem in a similar way to that described in the previous section. However, it is not so straightforward to extend the above theory to load pulling phenomena.

Let us resume with equation

$$Z(\omega, g) + Z_L = 0 \quad (11)$$

in the same form as Eq. (1) except that  $V$  is no longer a variable. While keeping the DC supply voltage at a constant value, we perturb the frequency  $\omega$  as well as the gain factor  $g$  of the active device used in the circuit. This implies trans-conductance  $g_m$  of an FET for example.

When the load impedance  $Z_L$  changes, gain factor  $g$  should be automatically adjusted as long as the gain-to-loss equilibrium is maintained for steady-state oscillation [1]. Such behavior can be notated as

$$Z_L = Z_{L0} + \delta Z_L, \quad \omega = \omega_0 + \delta\omega, \quad g = g_0 + \delta g \quad (12)$$

Applying Eq. (12) to Eq. (11), then assuming small changes as  $|\delta\omega| \ll \omega_0$  and  $|\delta g| \ll g_0$ , we get two-dimensional Taylor series expansion

$$Z(\omega_0, g_0) + Z_\omega(\omega_0, g_0)\delta\omega + Z_g(\omega_0, g_0)\delta g + \cdots + Z_{L0} + \delta Z_L = 0 \quad (13)$$

where partial derivative functions are defined as

$$Z_\omega(\omega, g) = \frac{\partial}{\partial \omega} Z(\omega, g), \quad Z_g(\omega, g) = \frac{\partial}{\partial g} Z(\omega, g) \quad (14)$$

Although Eq. (13) is similar to Eq. (3), it should be remarked to somewhat differ since  $\delta Z_L$  still remains. Extracting the zeroth- and first-order terms from Eq. (13), we get

$$Z(\omega_0, g_0) + Z_{L0} = 0 \quad (15)$$

$$Z_\omega(\omega_0, g_0)\delta\omega + Z_g(\omega_0, g_0)\delta g + \delta Z_L = 0 \quad (16)$$

This is essentially different from Eq. (6). For a given stimulus  $\delta Z_L$ , there are two unknown variables i.e.  $\delta\omega$  and  $\delta g$  in Eq. (16). This cannot be solely solved. Therefore we create its associating equation as

$$Z_\omega^*(\omega_0, g_0)\delta\omega + Z_g^*(\omega_0, g_0)\delta g + \delta Z_L^* = 0 \quad (17)$$

where asterisk\* in superscript designates complex conjugate. The frequency and gain factor are both real numbers. Regarding Eqs. (16) and (17) as a system of linear equations for unknowns  $\delta\omega$  and  $\delta g$ , we obtain its unique solution in a matrix form

$$\begin{aligned} \begin{bmatrix} \delta\omega \\ \delta g \end{bmatrix} &= - \begin{bmatrix} Z_\omega(\omega_0, g_0) & Z_g(\omega_0, g_0) \\ Z_\omega^*(\omega_0, g_0) & Z_g^*(\omega_0, g_0) \end{bmatrix}^{-1} \begin{bmatrix} \delta Z_L \\ \delta Z_L^* \end{bmatrix} \\ &= -\frac{1}{J} \begin{bmatrix} Z_g^*(\omega_0, g_0) & -Z_g(\omega_0, g_0) \\ -Z_\omega^*(\omega_0, g_0) & Z_\omega(\omega_0, g_0) \end{bmatrix} \begin{bmatrix} \delta Z_L \\ \delta Z_L^* \end{bmatrix} \end{aligned} \quad (18)$$

where  $J$  stands for Jacobian determinant

$$J = \begin{vmatrix} Z_\omega(\omega_0, g_0) & Z_g(\omega_0, g_0) \\ Z_\omega^*(\omega_0, g_0) & Z_g^*(\omega_0, g_0) \end{vmatrix} = 2j\Delta \quad (19)$$

$$\Delta = \text{Im}\{Z_\omega(\omega_0, g_0)Z_g^*(\omega_0, g_0)\} \quad (20)$$

Employing these notations, we can write the frequency deviation as

$$\begin{aligned}\delta\omega &= -\frac{1}{J}\{Z_g^*(\omega_0, g_0)\delta Z_L - Z_g(\omega_0, g_0)\delta Z_L^*\} \\ &= \frac{1}{\Delta}\text{Im}\{Z_g(\omega_0, g_0)\delta Z_L^*\}\end{aligned}\quad (21)$$

To get a physical insight of it, we introduce polar expressions of complex numbers

$$Z_g(\omega_0, g_0) = |Z_g(\omega_0, g_0)| e^{j\varphi}, \quad \delta Z_L = |\delta Z_L| e^{j\theta} \quad (22)$$

which lead Eq. (21) to

$$\delta\omega = \frac{1}{\Delta}|Z_g(\omega_0, g_0)\delta Z_L| \sin(\varphi - \theta) \quad (23)$$

From this equation, we find the frequency shift depends on the phase of load deviation. Revolving  $\theta$  from zero to  $2\pi$  as shown on a dashed circumference in Fig. 2, the frequency shift reaches its maximum

$$|\delta\omega|_{\max} = \left| \frac{\delta Z_L}{\Delta} Z_g(\omega_0, g_0) \right| \quad (24)$$

when  $\varphi = \theta + \pi/2$ . For general purpose in oscillator design, it is more versatile if we normalize the deviations on both sides by their original values as

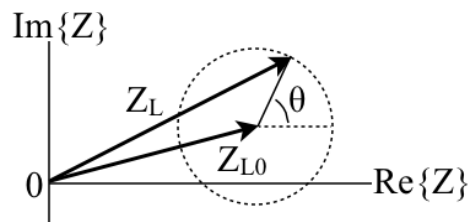
$$\frac{2|\delta\omega|_{\max}}{\omega_0} = \frac{1}{Q} \left| \frac{\delta Z_L}{Z_{L0}} \right| \quad (25)$$

For the same reason as described in Eq. (8),  $\delta\omega$  is doubled again. The coefficient appeared in Eq. (24) is entirely represented by a simple constant in Eq. (25), expressed as  $1/Q$ , since the frequency shift should be inversely proportional to a certain figure of merit for stable oscillation. From Eqs. (24) and (25), we find the constant  $Q$  to be a dimension-free positive scalar quantity expressed as

$$Q = \frac{\omega_0}{2} \left| \frac{\Delta}{Z_{L0} Z_g(\omega_0, g_0)} \right| \quad (26)$$

Thanks to the relation of Eq. (15), we can rewrite it as

$$\begin{aligned}Q &= \frac{\omega_0}{2} \left| \frac{\Delta}{Z(\omega_0, g_0) Z_g(\omega_0, g_0)} \right| \\ &= \frac{\omega_0}{2} \left| \frac{\text{Im}\{Z_\omega Z_g^*\}}{Z Z_g} \right|_{\omega=\omega_0, g=g_0}\end{aligned}\quad (27)$$



**Fig. 2.** Deviated load impedance locus on a complex plane

We get the final expression without directly including the load impedance, so that circuit designers can apply the formula conveniently to any active network. According to the duality theorem, one can also estimate  $Q$  in terms of output admittance  $Y = Z^{-1}$  with

$$Q = \frac{\omega_0}{2} \left| \frac{\text{Im}\{Y_\omega Y_g^*\}}{YY_g} \right|_{\omega=\omega_0, g=g_0} \quad (28)$$

## 5 Conclusion

The  $Q$  factor that dominates frequency stability characteristics of oscillators was successfully bridged from circuit port parameter for the first time. Once engineers get this  $Q$ , they can put it into their computer optimization program to find the most stable oscillator against source and load deviations. This is much more elegant and insightful than seeking the solution by repeating nonlinear time-domain simulations.