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# Indirect extension of the image theory to partial inductance calculations

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**Abstract:** Within the limitations implicit in the magneto-quasi-static approximation, partial inductances are a valuable tool for modeling IC package and interconnect inductances. In the calculation of partial inductances it is not possible to directly apply the image theory due to the absence of charges on the ground plane. In the present paper a correction term is obtained, which allows to calculate partial inductances with the image theory, avoiding in this way the problem of the ground plane segmentation. The formula is valid for conductors parallel to the ground plane. The accuracy of the formula is verified and discussed.

**Keywords:** Partial inductance, image theory

**Classification:** Electromagnetic theory

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## 1 Introduction

Due to the small dimensions of modern IC packages, partial inductances (e.g. in [1]) are sometimes used for modeling interconnects. Their calculation can be accomplished with software of widespread use, such as FASTHENRY [2]. Unfortunately, conducting planes create some difficulties in the calculations, because, in order to accurately model the current below the traces or close to the interconnection points to the ground plane, the segmentation density on the ground plane must be increased. Since a uniform segmentation would require an excessive amount of memory and calculation time, complex segmentation patterns must be used (e.g. in [3]) for high accuracy calculations.

For those situations where the effects of the finiteness of the ground plane are not of interest, the application of the image theory would be very useful. In fact, in this way an image of the trace would efficiently replace the ground plane, avoiding the problems related to its segmentation. It must be observed that the effect of the internal impedance of the ground plane is negligible with respect to the internal impedance of the traces, and the use of a perfectly conducting ground plane would not affect considerably the results.

However, in the magneto-quasi-static approximation the image theory cannot be directly applied to partial inductance calculations, as it will be shown in section 2. Based on the analytical calculation of the point-to-point partial inductance of the ground plane [4], and on some well known formulas [5], an accurate correction term for the image theory for traces parallel to the ground plane will be proposed in section 3. The accuracy of the proposed formulas with respect to calculations with a segmented ground plane will be discussed in section 4.

## 2 Partial inductances, magneto-quasi-statics and image theory

One of the possible definitions (e.g. in [1]) of the partial mutual inductance, $L_{ij}$, between the segments $C_j$ and $C_i$ of a thin conductor along the curve $C$, is in terms of the vector potential $\mathbf{A}$:

$$L_{ij} = \frac{\int_{C_i} \mathbf{A}_j \cdot \mathbf{dl}_i}{I_j} \tag{1}$$

where $I_j$ is the current along the segment $C_j$ of $C$, and $\mathbf{A}_j$ is the magnetic vector potential generated by the current $I_j$. This definition can be generalized to include conductors of finite thickness, as shown in [1]. In the definition, usually the following vector potential is selected:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv' \tag{2}$$

In the magneto-quasi-static approximation the effect of the charge is considered as negligible, and the current distribution is assumed to be solenoidal ($\nabla \cdot \mathbf{J} = 0$). In its original meaning the magneto-quasi-static approximation should be used to calculate self and mutual loop inductances. The ends of the


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exciting port are infinitesimally close to each other, and the opposite charges have no effect in all the space. Due to the relatively low frequencies of interest, the accumulated charge on the conductors is negligible. This is perfectly compatible with the image theory as long as no charge is introduced.

Partial inductances are calculated by injecting and extracting current into a selected port, and calculating the potential difference at the same port for the self inductance, and at a different port for the mutual inductance. To the injection and extraction of current at the port correspond two concentrated oscillating charges in two different spatial positions. In full-wave calculations, the charges induce a charge movement, that is a current, on the ground plane, but in the magneto-quasi-static approximation no charge is allowed to accumulate anywhere, except at the injection and extraction points at the excitation port. The boundary condition on the tangential component of the electric field is not verified on the surface of perfect conductors, because no scalar potential $\Phi$ due to charges can counteract the vector potential.

$$\mathbf{E} = -\nabla\Phi - j\omega\mathbf{A} \tag{3}$$

In partial inductance calculations within the magneto-quasi-static approximations, conductors reduce to a network of resistances and mutually coupled inductances, and the image theory cannot be applied in these conditions.

When the image theory is applied, the ground plane is removed and an image trace carrying a specular current with opposite direction is introduced. In this way, additional charges are also introduced at the ends of the image trace. In the position of the removed ground plane an electric field normal to the plane is present, and no electric field tangential to the plane. In the original configuration, however, an electric field tangential to the plane was present. Therefore, the two configurations are not equivalent.

In practical terms, with reference to Fig. 1, this means that the effective partial inductance of a conductor over a ground plane, $L'_{tr}$, is not equal to the difference between the self inductance of the trace without ground plane, $L_{tr}$, and the mutual inductance with the image trace, $M_{tr}$:

$$L'_{tr} \neq L_{tr} - M_{tr} \tag{4}$$

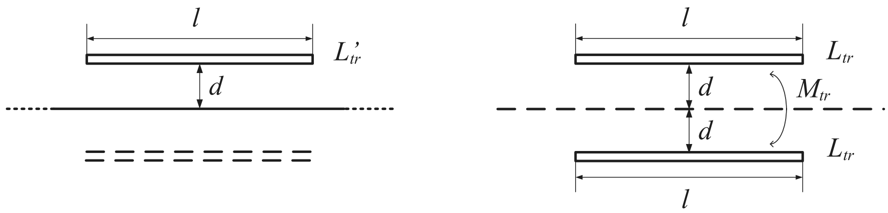


**Fig. 1.** Conductor above a ground plane and its image.

## 3 Correction term for the image theory

As it has been underlined in the previous section, the image theory cannot be applied to partial inductance calculation in the magneto-quasi-static approx-


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imation. At the same time, the image theory can still be applied to current loops. Therefore, the inductance of the loop formed by a trace parallel to the ground, the ground plane and two additional vertical thin conductors connecting the trace to the ground plane, will be half of the inductance formed by the same trace, its image, and two vertical thin conductors connecting the traces. In terms of the partial self and mutual inductances, this can be expressed as follows:

$$\begin{aligned} L_{loop} &= L'_{tr} + 2L_{cyl}(a,d) + L_g - 2M_{fil}(l,d) - 2M_{trg} \\ &= \frac{1}{2}\Big(2L_{tr} + 2L_{cyl}(a,2d) - 2M_{fil}(l,2d) - 2M_{tr}\Big) \end{aligned} \tag{5}$$

where $L'_{tr}$ is the partial self inductance of the trace in presence of the ground plane (effective partial inductance), $L_g$ is the point-to-point ground inductance of the ground plane [4], $M_{trg}$ is the mutual inductance between the trace and the ground plane, $L_{tr}$ is the partial self inductance of the trace when no ground plane is present, and $M_{tr}$ is the mutual inductance between the trace and its image. The partial inductances related to the vertical connections, $L_{cyl}$ and $M_{fil}$, have been expressed as function of their radius $a$, length $d$ or $2d$, and distance $l$.

The point-to-point ground inductance, $L_g$, has been defined as the partial inductance between an injection and an extraction point on the ground plane at distance $l$ from each other, and can be calculated analytically for a infinite perfectly conducting ground plane [4]:

$$L_g = \frac{\mu\, l}{2\pi} \tag{6}$$

It can be observed that this definition of point-to-point ground inductance is different from the definition of ground inductance found for example in [6].

The mutual inductance between the trace and the ground plane, $M_{trg}$, is defined similarly to the point-to-point ground inductance [4], and can be expressed in terms of a two-dimensional integral, which can be efficiently calculated numerically:

$$M_{trg} = \frac{\mu}{8\pi^2}\int_S \left(\frac{1}{\sqrt{{r_{i0}}^2 + d^2}} - \frac{1}{\sqrt{{r_{e0}}^2 + d^2}}\right)\ln\frac{r_{e0}}{r_{i0}}\, ds \tag{7}$$

where $r_{i0}$ and $r_{e0}$ are the distances from the projections on the ground plane $S$ of the injection and extraction points, respectively, and $d$ is the distance between trace and ground plane.

The cross section of the vertical conductors is not fixed, and for simplicity a cylindrical conductor of radius $a$ much smaller than its length $d$ ($a \ll d$) will be chosen, with a closed form expression [5] for the self inductance, $L_{cyl}$:

$$L_{cyl} = \frac{\mu\, d}{2\pi}\left[\frac{a}{d} - \sqrt{1+\frac{a^2}{d^2}} + \ln\left(\frac{d}{a} + \sqrt{1+\frac{d^2}{a^2}}\right)\right] \tag{8}$$

The mutual inductance between the vertical connectors can be approximated by the mutual inductance, $M_{fil}$, between two filamentary conductors


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of equal length $d$, and at distance $l$ from each other, for which also a closed form expression exists [5]:

$$M_{fil} = \frac{\mu d}{2\pi}\left[\frac{l}{d} - \sqrt{1+\frac{l^2}{d^2}} + \ln\left(\frac{d}{l} + \sqrt{1+\frac{d^2}{l^2}}\right)\right] \tag{9}$$

By taking the limit for the radius of the vertical connectors which tends to zero ($a \to 0$), it is now possible to calculate the inductance of the trace in presence of the ground plane, $L'_{tr}$ in Eq. (5), in terms of the difference of the self and mutual inductances of the trace and its image, $(L_{tr} - M_{tr})$, which is the theoretical value obtained by a direct application of the image theory:

$$L'_{tr} \approx L_{tr} - M_{tr} + 2M_{trg} + \frac{\mu\, d \ln 2}{\pi} - \frac{\mu\, d}{\pi}\left[\sqrt{1+\frac{l^2}{d^2}} - \sqrt{1+\frac{l^2}{4d^2}} + \ln\left(\frac{2d+\sqrt{l^2+4d^2}}{d+\sqrt{l^2+d^2}}\right)\right] \tag{10}$$

where the inductances $L_{tr}$ and $M_{tr}$ can be calculated numerically, or with the help of approximated formulas, such as those found in [5].

It can be observed, that no assumption regarding the length or the shape of the trace has been made. In fact, the formula is valid also for short traces, or even for curved traces, or for traces with any number of corners, as long as the traces are parallel to the ground plane. Some limitations on the thickness and width of the trace exist, but not for PCB traces or package pins of practical use. The flexibility of the formula is due to the fact that it relies on numerical calculations for the self inductance of the trace in absence of the ground plane, $L_{tr}$, and for the mutual inductance with the image trace, $M_{tr}$. In this way all the dependencies on the cross section, as well as the skin and proximity effects of the trace, are taken into account.

## 4 Comparison with numerical results

In order to verify the accuracy of the proposed formula, the partial self inductances of traces of different length over a large ground plane have been compared with the difference between the self and mutual inductances of a trace and its image, and with Eq. (10). The simulations have been conducted with FASTHENRY. The cross section of the trace was $0.32 \times 0.1$ mm, which are typical values for a QFP package pin. The distance of the center of the trace from the ground plane was 0.5 mm, which is within the range of typical values for the PCB thickness. The length of the trace was varied between 1 and 10 mm. The ground plane had dimensions $130 \times 130$ mm, a thickness of 35 $\mu$m, and has been segmented with $260 \times 260$ equal segments. The conductivity of all the conductors was $5.8 \times 10^7$ S/m (copper). The simulations have been conducted with a well developed skin effect, but similar results have been obtained with current flowing homogeneously inside the trace. The results are shown in Fig. 2.

It can be observed that the results obtained by directly applying the image theory differ noteworthy from the values of the effective inductance


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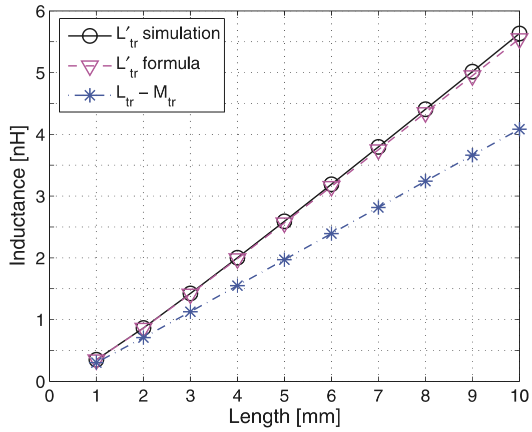


**Fig. 2.** Comparison of the effective trace inductance.

in presence of the ground plane, whereas those obtained with the proposed formula agree very well with them. When expressed in percentage, the results with the image theory differ up to 27% for the considered lengths, whereas the difference for those obtained with Eq. (10) is less than 1.5%. This difference increases with the length of the trace, but it is difficult to determine the reason, because it is in the range of the accuracy of the simulation results with the ground plane, due to the rough segmentation used, and to the finiteness of the ground plane. This indicates that the accuracy of the proposed formula could be even higher than what it can appear in Fig. 2.

## 5 Conclusions

The inapplicability of the image theory to partial inductance calculations has been explained theoretically and verified with an example. The proposed correction term allows to take into account with high accuracy the effect of a wide enough ground plane on a trace parallel to it, without having to model the ground plane itself. This greatly simplifies the preparation of the model, reduces the calculation time and increases the accuracy.

The proposed formula can be used in conjunction with simulation results, or with results obtained from semi-analytical approximations. Furthermore, when it makes use of numerical results obtained with an image trace, both skin and proximity effects can be taken into account.

Finally, it should be reminded, that even though in this example a straight trace was used, similar results can be obtained for curved traces, or traces having corners, as long as they are parallel to the ground plane. Extensions to conductors skew with respect to the ground plane are also possible, but they will not be shown here.

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