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Mathematical proof of Leeson's oscillator noise spectrum model

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Abstract: This letter presents a mathematical formulation that clearly explains the result that was heuristically introduced by Leeson for the oscillator noise spectrum. We consider the voltage and current in a simple equivalent circuit consisting of only linear components. To analyze both the oscillation and noise behaviors simultaneously without resort to frequency-domain transfer functions, we introduce dual coordinates in the time domain. Equivalent device temperature and Q factor are appropriately defined to support Leeson's result. It is successfully clarified, without taking any nonlinear effects into account, that the noise from a white source is converted up into a sharp spectrum around the oscillation frequency.

Keywords: oscillator, phase noise, Q factor, multiple time scale

Classification: Microwave and millimeter wave devices, circuits, and systems

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1 Introduction

In oscillators, noise behaves in a totally different manner from that in other active or passive circuits. It has a relatively sharp spectrum in the vicinity of

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the oscillation frequency. The fractional bandwidth of the noise was formulated in general by Edson in 1960 [1]. For further detail, the noise spectrum density was introduced by Leeson in 1966 [2]. His formula describes the noise distribution, which agrees well with the practically observed spectrum as a function of the frequency offset from the carrier. It is also worth noting that his derivation was heuristic without formal proofs. Later in 1990's, the noise formula was described for LC-tank oscillators by Craninckx and Steyaert [3], and for CMOS oscillators by Razavi [4]. It was then generalized to include the nonlinear and long-delay effects by Hajimiri and Lee [5]. Their studies effectively covered Leeson's model, relying on an open- and closed-loop transfer function technique in the frequency domain. Mathematical proof still remains therefore expected to provide a persuading explanation of noise behavior in the time domain even for a simple oscillator. This letter assumes a linear model of a simple oscillator and affords a clear and rigorous mathematical proof for Leeson's noise spectrum by exploiting a perturbation procedure and a multiple-time-domain technique.

2 Simple Oscillator Formulation

An oscillator circuit generally has an active device, a resonator, and an output port as its minimum required components. Figure 1 shows the simplest equivalent scheme, where the active device is represented by a negative resistor and an inherent series noise source. The resonator is also simply represented by series of an inductor and a capacitor. The output port is terminated with a resistive load. Consider the problem to deduce

$i(t)$: current along the circuit, and
$v(t)$: voltage on the capacitor

as functions of time t for given

$-r$: negative resistance from the active device,
C: series capacitance,
L: series inductance,
R_L: load resistance, and
$n(t)$: noise source voltage.

Let us begin with Kirchhoff's equations

$$i = C\frac{dv}{dt} \quad \text{and} \quad v + L\frac{di}{dt} + (R_L - r)\, i = n.$$

We assume $r = R_L$ for steady-state oscillation. If $r < R_L$, the power dissipated in the load exceeds the power generated in the active device, i.e., the current is not steady but declines. If $r < R_L$, it is not steady either but grows until the active device's gain balances the load's loss, i.e., $r = R_L$.

Based on this steady state assumption, we eliminate $i(t)$ from the above equations, then obtain the single linear second-order ordinary differential equation

$$v + LC\frac{d^2v}{dt^2} = n \tag{1}$$

for $v(t)$. This is an inhomogeneous equation whose general solution cannot be found directly. However, once you find a particular solution for Eq. (1),

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its general solution is expressed as a linear combination of the particular one and the general solution of the associated homogeneous or noise-free equation

$$v + LC\frac{d^2v}{dt^2} = 0.$$

Its general solution is well known as the simple harmonic function

$$v\,(t) = v\,(0)\,e^{j\omega t}, \quad \omega = 1/\sqrt{LC}.$$

As the first step to find a particular solution for Eq. (1), we apply a perturbation technique to this equation by employing multiple time scales [6] as described in the following chapter.

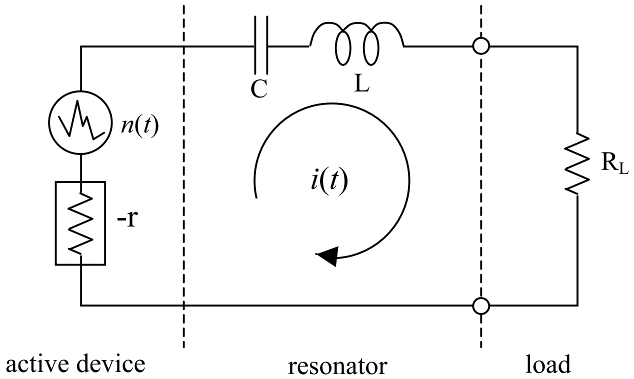

Fig. 1. Simple oscillator equivalent circuit topology

3 Dual Time Scales

The phenomenon of oscillation is regarded as a high-speed rotation of the phase, while the phase noise is its time undulation, which is much slower than its original rotation. We introduce dual time scales $t_0 = t$ and $t_1 = \delta t$ to analyze those two phenomena simultaneously. δ designates a perturbation index implying that the associated term is a small quantity. We map every time-variant quantity upon the plane spanned by two-dimensional coordinate (t_0, t_1). For example, the capacitor voltage is observed as $v(t) = v(t_0, t_1)$. We do the same for derivative quantities. The voltage deviation is expressed as the sum of its two partial differentiations

$$dv\,(t) = dv\,(t_0, t_1) = \frac{\partial v}{\partial t_0}dt_0 + \frac{\partial v}{\partial t_1}dt_1.$$

In addition to those quantities, the chain rule on differentiation leads us to expanding time-derivative operators as

$$\frac{d}{dt} = \frac{dt_0}{dt}\frac{\partial}{\partial t_0} + \frac{dt_1}{dt}\frac{\partial}{\partial t_1} = \frac{\partial}{\partial t_0} + \delta\frac{\partial}{\partial t_1}$$

$$\frac{d^2}{dt^2} = \left(\frac{\partial}{\partial t_0} + \delta\frac{\partial}{\partial t_1}\right)^2 = \frac{\partial^2}{\partial t_0^2} + 2\delta\frac{\partial^2}{\partial t_0 \partial t_1}.$$

Hereafter we omit terms with second- or upper-order power of δ for their negligible smallness.

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4 Separation of Variables

The next step to find a particular solution for Eq. (1) is the separation of variables. We try to factorize the capacitance voltage into the form:

$$v(t) = v(t_0, t_1) = v_0(t_0)v_1(t_1).$$

This course of process sometimes violates the generality of solutions, and may unreasonably confine the possible space for searching for them. However, it suits our case because we only have to find any one of them, as described in Chapter 2.

Applying this factorization and the above-mentioned second-order time-derivative operator to the original Eq. (1), we obtain the partial differential equation

$$v_0(t_0)v_1(t_1) + LC\left(\frac{\partial^2}{\partial t_0^2} + 2\delta\frac{\partial^2}{\partial t_0 \partial t_1}\right)\{v_0(t_0)v_1(t_1)\} = \delta n(t_0, t_1),$$

where an additional perturbation index δ is placed on the right-hand side since the noise voltage is assumed to be considerably smaller than the oscillation amplitude. After sorting the terms on the order of δ, the equation is separated into the unperturbed part

$$v_0(t_0) + LC\frac{\partial^2}{\partial t_0^2}v_0(t_0) = 0,$$

and the perturbed part

$$2LC\frac{\partial}{\partial t_0}v_0(t_0)\frac{\partial}{\partial t_1}v_1(t_1) = n(t_0, t_1).$$

It is straightforward to find that the unperturbed equation yields its general solution

$$v_0(t_0) = v_0(0)e^{j\omega_0 t_0}, \quad \omega_0 = 1/\sqrt{LC}.$$

For the perturbed part, on the contrary, we further continue to manipulate the equation with regard to the noise spectrum as described in the next chapter.

5 Carrier Noise Spectrum

An active device generally generates random noise over a broad band. At frequencies as high as the oscillation is considered, the noise is observed as white or a uniform spectrum. For convenience, we deal with the continuously distributed spectrum as an integration of narrow-band line components. The principle of superposition makes it possible to estimate the noise effects on the oscillation, spectral component by component, thanks to the linearity of equations. Thus, only one component at the frequency $\delta\omega_1$ offset from the oscillation or carrier ω_0

$$n(t) = n(t_0, t_1) = n_0 e^{j(\omega_0 + \delta\omega_1)t} = n_0 e^{j\omega_0 t_0 + j\omega_1 t_1}$$

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is sufficient for consideration. Substituting this noise expression and the unperturbed voltage into the perturbation equation, we obtain

$$\frac{2j}{\omega_0}v_0(0)e^{j\omega_0 t_0}\frac{\partial}{\partial t_1}v_1(t_1) = n(0,0)e^{j\omega_0 t_0 + j\omega_1 t_1}.$$

For this equation, a particular solution

$$v_1(t_1) = -\frac{\omega_0 n(0,0)}{2\omega_1 v_0(0)}e^{j\omega_1 t_1}$$

is found with ease. Having hereby both $v_0(t_0)$ and $v_1(t_1)$, we multiply them to deduce

$$v(t) = v_0(t_0)v_1(t_1) = -\frac{\omega_0}{2\omega_1}n_0 e^{j\omega_0 t_0}e^{j\omega_1 t_1}.$$

From this voltage, the current is subsequently deduced as

$$i = C\frac{dv}{dt} = -\frac{\omega_0}{2\omega_1}C\frac{d}{dt_0}\left(n_0 e^{j\omega_0 t_0}e^{j\omega_1 t_1}\right) = -j\frac{\omega_0^2}{2\omega_1}Cn_0 e^{j\omega_0 t_0}e^{j\omega_1 t_1}.$$

As we are interested in a practically observable quantity, i.e., the power spectrum at the output port or on the load R_L, we calculate it as

$$P_n = \frac{1}{2}R_L|i|^2 = \frac{\omega_0^4}{8\omega_1^2}C^2 R_L|n_0|^2 = \frac{R_L}{8\omega_1^2 L^2}|n_0|^2.$$

6 Equivalent Device Temperature and Q Factor

The final approach in this work is performed by introducing the equivalent thermal resistance for the noise source and Q factor of the oscillator. We replace the noise source by an equivalent noise resistor with its temperature T via the relation $|n_o|^2 = 2k_B TBR_n$, where k_B is Boltzman's constant and B is the bandwidth to observe the noise. By this replacement, the noise power output is rewritten as

$$P_n = \frac{R_L R_n}{4\omega_1^2 L^2}k_B TB.$$

This equation means that a white noise source perturbs the oscillation spectrum into a $1/f^2$ function of the offset frequency. The noise power is usually normalized by the bandwidth and carrier output power as

$$\frac{P_n}{BP_0} = \frac{R_L R_N}{4\omega_1^2 L^2}\frac{k_B T}{P_0}.$$

Finally, we introduce the Q factor

$$Q = \frac{2\pi f_0 L}{\sqrt{R_n R_L}}$$

so as to yield our result

$$\left.\frac{N}{C}\right|_{SSB} = \frac{1}{2}\left\{1 + \left(\frac{f_0}{2f_1 Q}\right)^2\right\}\frac{k_B T}{P_0}$$

where a unity was added to involve the thermal background noise of the system. This affords mathematically rigorous proof of the expression heuristically described by Leeson.

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7 Conclusion

A clear explanation has been given for the oscillator noise spectrum formula that Leeson heuristically introduced. A simple equivalent circuit consisting of linear components, i.e., a negative resistor, a white noise source, an LC resonator, and a load, is sufficient for the formulation. As a more informative and acceptable way than frequency-domain transfer functions, the dual time coordinates are convenient for analyzing the oscillation and noise behaviors. Appropriate definitions of equivalent device temperature and Q factor consequently support Leeson's result. Even though the circuit model is a linear system without modulation effects, it successfully deduces that the noise from a white source is converted up into a sharp spectrum around the carrier frequency.

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