ELectronics EXpress

Equivalent circuit for Sommerfeld wave

Umberto Paoletti[a)], **Takashi Suga, and Hideki Osaka**

Hitachi, Ltd., Yokohama Research Laboratory, 292 Yoshida-cho, Totsuka-ku, Yokohama 244–0817, Japan

a) *umberto.paoletti.ff@hitachi.com*

Abstract: An equivalent circuit of a single conductor line (Sommerfeld wave) is proposed. The circuit is based on physical quantities, such as the electric current and on the power of the traveling wave, while the line voltage is only indirectly obtained. The characteristic impedance and the per-unit-length parameters are calculated in closed form as functions of the complex wave numbers, which must be determined numerically. The circuit parameters are partly confirmed by literature results, and they differ only where the literature results have no physical meaning (e.g. negative conductance). Furthermore, circuit simulation solutions are used to determine the per-unit-length attenuation, and the results are confirmed by the well known analytical solution. The idea can be extended to other type of single conductor lines, having a more complex cross-sectional profile and inhomogeneous media.

Keywords: Sommerfeld wave, one conductor propagation, equivalent circuit

Classification: Electromagnetic theory

References

[1] A. Sommerfeld, "Über die Fortpflanzung elektrodynamischer Wellen längs eines Drahtes," *Annalen der Physik und Chemie (Neue Folge)*, vol. 67, pp. 233–290, 1899.

[2] J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, 1941.

[3] K. Wang and D. M. Mittleman, "Metal wires for terahertz wave guiding," *Nature*, vol. 432, pp. 376–379, Nov. 2004.

[4] J. A. Deibel, M. Escarra, N. Berndsen, K. Wang, and D. M. Mittleman, "Finite-Element Method Simulations of Guided Wave Phenomena at Terahertz Frequencies," *Proc. IEEE*, vol. 95, no. 8, pp. 1624–1640, Aug. 2007.

[5] K. Wang and D. M. Mittleman, "Dispersion of Surface Plasmon Polaritons on Metal Wires in the Terahertz Frequency Range," *Phys. Rev. Lett.*, vol. 96, no. 15, 157401, 2006.

[6] R. Holland and R. S. John, "EM Pickup and Scattering by a Wire," *IEEE Trans. Electromagn. Compat.*, vol. 42, no. 4, pp. 461–469, Nov. 2000.

[7] T. E. Roberts Jr., "Theory of the Single Wire Transmission Line," *J. Appl. Phys.*, vol. 24, pp. 57–67, 1953.

[8] R. B. Marks and D. F. Williams, "A General Waveguide Circuit Theory," *J. Res. Natl. Inst. Stand. Technol.*, vol. 97, no. 5, pp. 533–562, 1992.

© IEICE 2011

DOI: 10.1587/elex.8.1590
Received August 17, 2011
Accepted September 01, 2011
Published October 10, 2011

IEICE Electronics Express, Vol.8, No.19, 1590–1595

1 Introduction

The first known work on the electromagnetic wave propagation along a single conductor is due to Sommerfeld [1]. A detailed analytical study of the wave propagation on a conductor with cylindrical symmetry can be found also in [2]. As suggested by Sommerfeld himself, the energy confinement of this type of propagation was too loose for practical applications at that time. Recently the interest has been renewed for applications in the THz range (e.g. [3]), where the energy is confined in a region very close to the wire (e.g. [4]). Present studies include possible applications for THz propagation, due to the low-dispersion and low-loss properties of the Sommerfeld wave (e.g. [5]). A slightly different application of interest is EMC (e.g. [6]), where this type of propagation can be used for analyzing the common mode (CM) current on cables or cable shields.

Some attempts to extract an equivalent circuit for the propagation along a single conductor can be found in [6] and [7], but the results are not satisfying. In [6] an equivalent circuit has been calculated based on a definition of voltage as the integral of the radial electric field between the conductor and infinite. The approach followed in [7] is based on the transmitted power and seems to be more general, but the boundary conditions are based on a loss-less conductor with corrugated other coated surface.

In the present paper a new approach for the extraction of the equivalent circuit parameters for the Sommerfeld line is proposed, based on the concept of traveling waves in [8]. From the current on the wire and the power of the traveling wave, the characteristic impedance and the per-unit-length circuit parameters are calculated analytically and compared with literature results.

The present work is important from both the theoretical and practical points of view. From the theoretical point of view, the new idea of using the traveling wave concept to extract the equivalent circuit for the Sommerfeld line paves the way to a more convenient approach to handle all the one conductor lines, with or without a coating insulator. From the practical point of view, the analytically calculated equations for the Sommerfeld line are made available to all the above mentioned applications. Furthermore, the resulting equations offer a convenient starting point for further research with the aim of their possible generalization to other single conductor lines.

2 Electromagnetic field distribution

The fundamental TM mode (sometimes called E-mode) for a cylindrical conductor of radius r_c, finite conductivity σ_c and magnetic permeability μ_c, has only three field components, which can be expressed (e.g. [1] and [2]) in terms of the Bessel function of first kind and order n, J_n, and the Hankel functions of the first kind and order n, $H_n = H_n^{(1)}$. When the total current along a conductor on the z-axis at the frequency $\omega = 2\pi f$ is expressed as $I = I_0\, e^{j(\omega t - hz)}$, the electromagnetic field inside the conductor (region 1) becomes:

$$E_z^{(1)} = \frac{I_0}{2\pi r_c} \frac{u_c}{\sigma_c} \frac{J_0(u_c r)}{J_1(u_c r_c)}\, e^{j(\omega t - hz)} \qquad (1)$$

© IEICE 2011
DOI: 10.1587/elex.8.1590
Received August 17, 2011
Accepted September 01, 2011
Published October 10, 2011

IEICE Electronics Express, Vol.8, No.19, 1590–1595

$$H_{\phi}^{(1)} = \frac{I_0}{2\pi r_c} \frac{J_1(u_c r)}{J_1(u_c r_c)} e^{j(\omega t - hz)}$$
$$E_r^{(1)} = \frac{\omega\,\mu_c\,h}{k_c^2} H_{\phi}^{(1)} \approx \frac{jh}{\sigma_c} H_{\phi}^{(1)}$$
$$u_c^2 = k_c^2 - h^2,$$

and in the dielectric medium outside the conductor (region 2) it becomes:

$$E_z^{(2)} = \frac{I_0}{2\pi r_c} \frac{-ju}{\omega\,\epsilon} \frac{H_0(ur)}{H_1(ur_c)} e^{j(\omega t - hz)} \tag{2}$$
$$H_{\phi}^{(2)} = \frac{I_0}{2\pi r_c} \frac{H_1(ur)}{H_1(ur_c)} e^{j(\omega t - hz)}$$
$$E_r^{(2)} = \frac{\omega\,\mu_0\,h}{k^2} H_{\phi}^{(2)}$$
$$u^2 = k^2 - h^2,$$

where $k_c \approx \sqrt{-j\omega\,\sigma_c\,\mu_c} = (1-j)/\delta_c$ is the wave number inside the conductor, δ_c is the skin depth, μ_0 is the vacuum magnetic permeability, ϵ is the dielectric constant in the external medium, u_c and u are the transverse complex phase constant inside and outside the conductor, respectively. The complex phase constant h can be determined by imposing the continuity of the tangential field components on the surface of the conductor:

$$\frac{u_c\,\mu_c}{k_c^2} \frac{J_0(u_c r_c)}{J_1(u_c r_c)} = \frac{u\,\mu_0}{k^2} \frac{H_0(ur_c)}{H_1(ur_c)}. \tag{3}$$

Additionally, it is convenient to define a surface impedance Z_s, as the ratio of the longitudinal electric field at the surface of the conductor and the total current I_0:

$$Z_s = \frac{E_z|_{r=r_c}}{I_0} = \frac{-ju}{2\pi r_c\,\omega\,\epsilon} \frac{H_0(ur_c)}{H_1(ur_c)} = \frac{u_c}{2\pi r_c\,\sigma_c} \frac{J_0(u_c r_c)}{J_1(u_c r_c)}. \tag{4}$$

3 Equivalent circuit

The fundamental idea is to calculate the complex characteristic impedance based on the complex power and the total current on the conductor I_0. According to [8] this is possible for any uniform waveguide, and therefore also for a single conductor line. The complex power associated with the traveling wave, P, can be calculated by means of the integral of the longitudinal component of the complex Poynting vector $\mathbf{S}$ on the cross-section S:

$$P = \frac{1}{2} \int_S S_z\,dS = \pi \int_0^{\infty} E_r\,H_{\phi}^*\,r\,dr = \frac{1}{2} |I_0|^2 Z_0 = \frac{1}{2} \frac{|V_0|^2}{Z_0}, \tag{5}$$

where Z_0 is the still undefined characteristic impedance of the Sommerfeld wave, and V_0 is the still undefined line voltage. As observed in [8], the magnitude of the characteristic impedance depends on the definition (either 'power-current' or 'power-voltage'), and on some normalizations, but its phase is unequivocally associated with the phase of the power, except for the sign. By combining last equation with Eqs. (1), (2) and (4) it is possible to

© IEICE 2011
DOI: 10.1587/elex.8.1590
Received August 17, 2011
Accepted September 01, 2011
Published October 10, 2011

IEICE Electronics Express, Vol.8, No.19, 1590–1595

define the characteristic impedance in terms of the power and the current (power-current definition), which after some analytical calculations can be expressed as follows:

$$Z_0 = \frac{2P}{|I_0|^2} = \frac{2h}{u_c^2 - u_c^{*2}}\, Im[Z_s] + \frac{2jh}{u^2 - u^{*2}}\, Re[Z_s]\,. \tag{6}$$

Last equation apparently differs from the expression calculated in [7], because the author of that paper considered a lossless conductor with a surface impedance different from zero, for example due to a dielectric coating or a corrugated surface. It must be noticed that in that case also the power transmitted inside the dielectric material close to the interface should be considered. More study is necessary in order to verify whether Eq. (6) is valid in that case as well. Last equation is also different from the expression in [6], as expected, since a different definition of impedance has been used.

From the characteristic impedance and the propagation constant $\gamma = jh$ it is possible to calculate the per-unit-length parameters with Eq. (7).

$$\begin{aligned} Z &= Z_0\,\gamma = R + j\omega L \\ Y &= \frac{\gamma}{Z_0} = G + j\omega C \end{aligned} \tag{7}$$

According to [8], a second equivalent procedure to calculate the per-unit-length equivalent circuit parameters is that of integrating the squared components of the electromagnetic fields. By combining Eqs. (33)–(36) in [8] with Eqs. (1), (2) and (6), and assuming that $\epsilon'' = 0$ in the external medium, $\epsilon''_c = \sigma_c/\omega$ inside the conductor and $\mu'' = 0$ everywhere, the per-unit-length circuit parameters of the Sommerfeld wave can be calculated by integrating over the cross-sectional surface $S = S_1 \cup S_2$, where S_1 and S_2 are the cross sections inside and outside the conductor, respectively.

$$\begin{aligned} C &= \frac{1}{|Z_0\,I_0|^2}\left[\int_S \epsilon'\,|E_r|^2\,dS\right] \approx \frac{|h|^2}{|Z_0|^2}\,\frac{1}{\omega}\,\frac{2j}{u^2 - u^{*2}}\,Re[Z_s]\,, \\ L &= \frac{1}{|I_0|^2}\left[\int_S \mu\,|H_\phi|^2\,dS - \int_S \epsilon'\,|E_z|^2\,dS\right] \\ &\approx -\mu\,\sigma_c\,\frac{2j}{u_c^2 - u_c^{*2}}\,Im[Z_s] + \frac{2j}{u^2 - u^{*2}}\left(\mu\,\omega\,\epsilon\,Re[Z_s] - \frac{1}{\omega}\,Re\big[u^{*2}\,Z_s\big]\right), \\ G &= \frac{\sigma_c}{|Z_0\,I_0|^2}\left[\int_{S_1} |E_r|^2\,dS\right] = -\frac{|h|^2}{|Z_0|^2}\,\frac{2j}{u_c^2 - u_c^{*2}}\,Im[Z_s]\,, \\ R &= \frac{\sigma_c}{|I_0|^2}\left[\int_{S_1} |E_z|^2\,dS\right] = -\frac{2j}{u_c^2 - u_c^{*2}}\,Im\big[u_c^{*2}\,Z_s\big]\,. \end{aligned} \tag{8}$$

4 Comparison with literature results

As it can be observed in Figs 1 and 2 for a copper wire of 1 mm radius, the capacitance and inductance per unit length calculated with Eqs. (7) and (8) are very similar to those in [6]. On the other side, the series resistance R_{pul} according to [6] is negative at low frequencies and very large at high frequencies, and it is somehow compensated by a large and negative shunt

© IEICE 2011
DOI: 10.1587/elex.8.1590
Received August 17, 2011
Accepted September 01, 2011
Published October 10, 2011

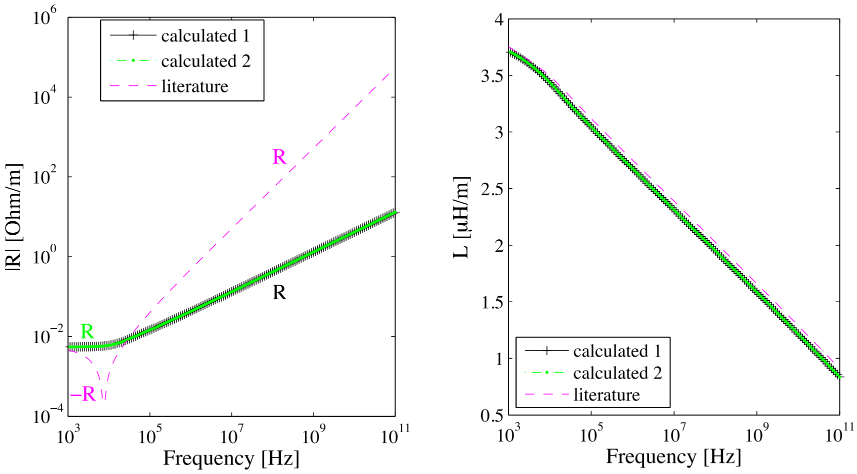

Fig. 1. Series parameters with Eq. (7) (calculated 1), Eq. (8) (calculated 2), and [6] (literature).

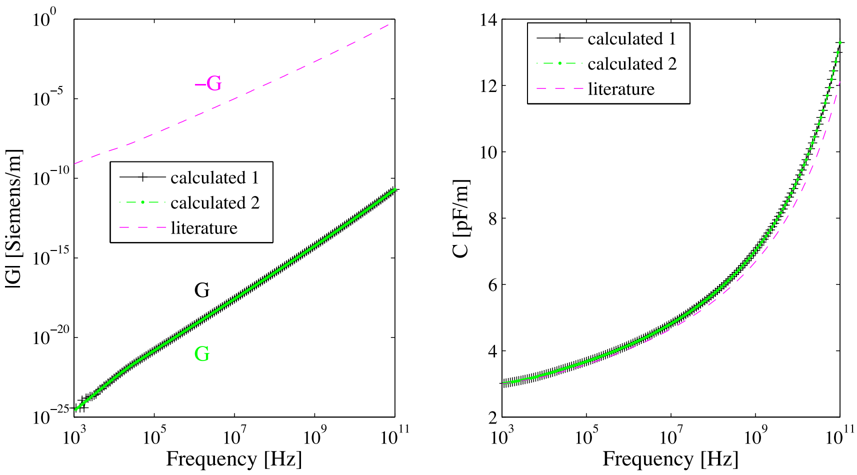

Fig. 2. Shunt parameters with Eq. (7) (calculated 1), Eq. (8) (calculated 2), and [6] (literature).

conductance G_{pul}, which can be a source of instability in time domain simulations, as suggested by the authors themselves. On the contrary, Eqs. (7) and (8) bring to much more reasonable results for the resistance and the conductance. At frequencies below 10 Hz Eqs. (7) and (8) bring to negative values of the inductance, but it is not clear whether it corresponds to the physics of the phenomenon or it is an incorrect numerical solution of Eq. (3). In any case the practical effect is not significant due the low frequencies.

5 Comparison between simulation and analytical results

In order to verify the correctness of the equivalent circuit some Spice simulations with a wire of 3 different materials (steel, brass and copper) and radii (0.1 mm, 1 mm and 10 mm) have been conducted. The traditional pseudo-scattering parameters have been calculated with 50 Ω reference impedance at each port. From the pseudo-scattering parameters, the open-circuit impedance matrix has been calculated, and from this one the traveling wave

© IEICE 2011
DOI: 10.1587/elex.8.1590
Received August 17, 2011
Accepted September 01, 2011
Published October 10, 2011

scattering parameters with complex reference impedance have been calculated. The transformation formulas can be found in [8].

The complex and frequency dependent impedance of Eq. (6) has been used as reference impedance. Since this impedance represents a matched termination, no reflected wave is expected, and the power transmitted from port 1 to port 2 can be expressed as $|S_{21}|^2$ for unitary input power. For a conductor of 1 m length this power ratio should correspond to the attenuation per unit length that can be calculated from the attenuation factor in Eq. (1) ($\alpha = Re[\gamma] = -Im[h]$). Figure 3 shows a good agreement between theoretical and simulation results for the attenuation, confirming the correctness of the equivalent circuit.

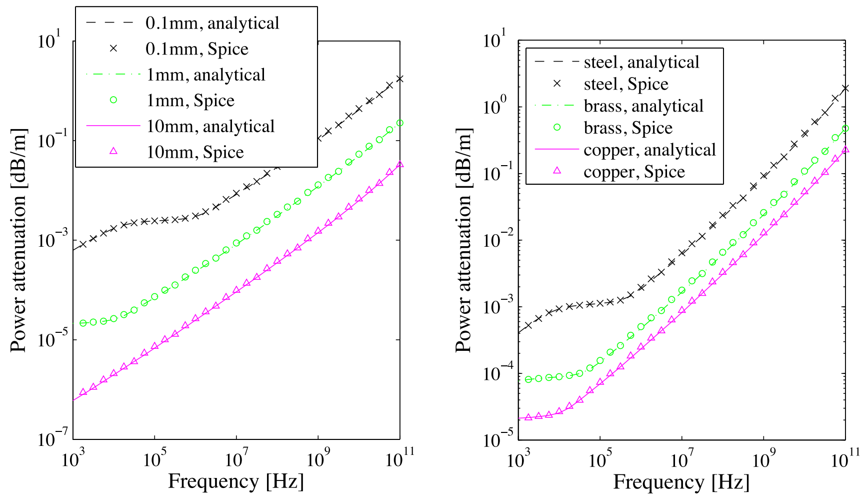

Fig. 3. Attenuation obtained from Spice circuit simulations ($|S_{21}|^2$) and directly from Eq. (1) for different radii (copper) and conductors (r_c =1 mm).

6 Conclusion

The proposed approach to obtain the equivalent circuit for the propagation of electromagnetic waves on a single conductor offers a convenient way to handle such structures based on the physical quantities of current and power transported by the traveling wave. In the case of the Sommerfeld wave on a conductor of any radius and finite conductivity the equivalent circuit parameters can be analytically obtained from the surface impedance.

© IEICE 2011
DOI: 10.1587/elex.8.1590
Received August 17, 2011
Accepted September 01, 2011
Published October 10, 2011