Foundations of Boij-Söderberg Theory for Grassmannians

by

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$\varnothing \oplus \left(\ydiagram{2}^{\oplus 3}\right)$	$\longleftarrow$	$\ydiagram{2,1}^{\oplus 8}$	$\longleftarrow$	$\ydiagram{3,2}^{\oplus 3}$

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ABSTRACT

Foundations of Boij-Söderberg Theory for Grassmannians

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Jake Levinson

Chair: David E Speyer

Boij-Söderberg theory characterizes syzygies of graded modules and sheaves on projective space. This thesis is concerned with extending the theory to the setting of GL_k-equivariant modules and sheaves on Grassmannians $Gr(k, \mathbb{C}^n)$. Algebraically, we study modules over a polynomial ring in kn variables, thought of as the entries of a $k \times n$ matrix. The goal is to characterize equivariant Betti tables of such modules and, dually, cohomology tables of sheaves on $Gr(k, \mathbb{C}^n)$.

We give equivariant analogues of two important features of the ordinary theory: the Herzog-Kühl equations and the pairing between Betti and cohomology tables. As a necessary step and fundamental base case, we consider resolutions and certain more general complexes for the case of square matrices.

Our statements specialize to those of ordinary Boij-Söderberg theory when $k = 1$. Our proof of the equivariant pairing gives a new proof in the graded setting: it relies on finding perfect matchings on certain graphs associated to Betti tables.

CHAPTER 1

Syzygies and Boij-Söderberg theory

1.1 Goals of this thesis

This thesis concerns ***syzygies***: relations among the defining equations of objects arising in algebraic geometry. These relations provide an abundance of useful information that can be difficult to determine from the defining equations alone. For example, a common rule of thumb says that if a space X is described by n equations in m variables, then we would expect X to have dimension $m - n$ (each equation cuts down the dimension of X by 1). Unfortunately, this statement is not always true:

Example 1.1.1. Let $\mathrm{Mat}_{2\times 4}$ be the set of 2×4 matrices (x_{ij}), and let $X \subset \mathrm{Mat}_{2\times 4}$ be the subset of rank-deficient matrices. Then X is defined by setting equal to zero the determinants of each of the 2×2 minors. This gives $\binom{4}{2} = 6$ equations in 8 variables, suggesting that $\dim X = 8 - 6 = 2$. But in fact, $\dim(X) \geqslant 5$, as shown by the matrices of the form

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ tx_{11} & tx_{12} & tx_{13} & tx_{14} \end{bmatrix}.$$

The reason for the discrepancy is that the defining equations are "too similar" – there are algebraic relations (syzygies) between these determinants, such as

$$x_{11} \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} - x_{12} \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix} + x_{13} \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = 0.$$

This situation is common, and makes it difficult to tell even basic facts about many spaces X of interest. Syzygies give one way to solve the problem: the complete description of the syzygies of X tells us, among other things, the dimension of X, its degree, and various measures of its algebraic and geometric complexity, such as its topological Euler characteristic.

In this thesis, the goal is to *classify* which syzygies can occur for various types of objects – notably spaces swept out by lines and planes – thereby outlining the range of possible phenomena. Classifying syzygies was long considered intractable, but the recently-developed *Boij-Söderberg theory* has made surprising inroads using methods from convex geometry. An early paper of Eisenbud and Schreyer [9] gave an essentially-complete classification of syzygies on projective spaces, concerning geometric objects swept out by lines. Along with the concrete benefits – combinatorial criteria to tell which syzygies are possible – the theory uncovered a *duality* between syzygies, which are algebraic, and certain geometric invariants (specifically, cohomology of vector bundles).

The theory has rapidly expanded to other settings than projective spaces. We will consider one that has not been studied previously: the setting of Grassmannians $Gr(k, \mathbb{C}^n)$. Here, the objects in question are swept out by higher-dimensional planes (recovering projective spaces as the case $k = 1$). Grassmannians require more involved combinatorics than projective spaces; as such, one of the goals has been to draw the attention of combinatorialists to the subject. The two main goals, however, are:

- to shed light on syzygies for Grassmannians, improving our understanding of many interesting spaces, such as determinantal loci;
- to understand the "dual" geometric invariants (i.e. cohomology) of vector bundles on $Gr(k, \mathbb{C}^n)$.

The results of this thesis are from two papers, [15, joint with Nic Ford and Steven Sam] and [14, joint with Nic Ford]. We will develop the basic properties of the classification of syzygies in this setting: we describe the base case of the classification completely; we determine the simplest constraints on the general case; and we establish the duality between syzygies and sheaf cohomology. Nonetheless, many steps remain before we will have a complete description: most importantly, in the general case, we have not classified the objects with 'extremal' syzygies.

Remark 1.1.2. In addition to my work on syzygies, I have also done research on Schubert calculus, which has not been included in this thesis. The result of that work are in two papers, [26] and [18, joint with Maria Gillespie].

1.2 What are syzygies?

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring in n variables over $\mathbb{C}$, and let M be an R-module. Let $S = \{m_i\}$ be a generating set for M (assume for simplicity that S is finite), so that every $m \in M$ can be written

$$m = \sum r_i m_i$$

for some choice of scalars $r_i \in R$. (In other words, there is a surjective map of R-modules, $f_0 : R^{|S|} \to M$, sending the i-th generator to m_i.)

If this representation is unique for every element, then M is the simplest kind of module: a *free module* on the generators $\{m_i\}$ (and f_0 is an isomorphism). Most modules, however, are not free, so these representations cannot be unique. Instead, there are ***syzygies***, relations in M of the form

$$0 = \sum r_i m_i$$

with not all $r_i = 0$. The set of all syzygies forms a new module (the kernel of f_0), and we can again seek to describe the module: we find a generating set of syzygies $\{s_i\}$, then ask whether the other syzygies are uniquely representable in terms of the $\{s_i\}$. If they are not, we look for "relations between the relations", and so on. Repeating in this way, we get a sequence of modules and maps

$$\cdots \xrightarrow{f_2} R^{\oplus n_1} \xrightarrow{f_1} R^{\oplus n_0} \xrightarrow{f_0} M \to 0$$

where, at each step, the $(i+1)$-st map surjects onto the kernel of the i-th map:

$$\mathrm{im}(f_{i+1}) = \ker(f_i).$$

This sequence is a ***free resolution*** of M, and encodes much of the algebraic structure of M. If M is graded (with respect to the standard grading on R), then a *graded* free resolution also records the degrees of each of the generators at each step, so that the i-th term has the form $\bigoplus_d R(-d)^{\oplus \beta_{i,d}}$, for certain nonnegative integers $\beta_{i,d}$.

Syzygies were first studied by Hilbert in the 1890s, who proved:

Theorem 1.2.1 (Hilbert's Syzygy Theorem)**.** *Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and let M be a finitely-generated R-module. Then M has a free resolution that terminates in at most n steps. (If M is graded, the resolution can be as well.)*

An important consequence of this theorem is that, for graded modules M over polynomial rings, the Hilbert function

$$\mathrm{hilb}_M(d) := \dim_{\mathbb{C}}(M_d)$$

is a polynomial in d for $d \gg 0$, where $M_d \subset M$ is the space of homogeneous elements of degree d.

In fact, more is true: for graded modules, there is an essentially unique *minimal* free resolution, obtained by choosing minimal generators at each step. As such, the number $\beta_{i,d}$ of degree-d generators in the i-th step of the resolution, is (for each $i, d \in \mathbb{Z}$) an intrinsic invariant of M. These ***Betti numbers*** detect, among other things, the dimension, projective dimension and regularity of M, as well as giving an explicit formula for its Hilbert function.

The subject of this thesis is the structure theory of ***Betti tables*** $\beta(M) := (\beta_{i,d})_{i,d\in\mathbb{Z}}$, which record all the Betti numbers. Which tables of integers arise as Betti tables of modules? This question has been well-studied for graded modules over the ring $\mathbb{C}[x_1, \ldots, x_n]$, and the answers have yielded applications to algebra and to the geometry of sheaves and vector bundles on projective space $\mathbb{P}^{n-1}$ (the geometric counterpart to the graded ring $\mathbb{C}[x_1, \ldots, x_n]$).

Our goal is to study a different class of modules, with an action of the general linear group GL_k, whose geometric counterparts are sheaves and vector bundles on Grassmannians $Gr(k, \mathbb{C}^n)$. These will be modules over a polynomial ring in kn variables, thought of as the entries of a $k \times n$ matrix.

1.3 Boij-Söderberg theory for graded modules

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring using the standard grading by $\mathbb{N}$. Let M a graded, finitely-generated R-module. It is well-known that M has a minimal free resolution of the form

$$M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n \leftarrow 0, \text{ with } F_i = \bigoplus_d R(-d)^{\beta_{i,d}},$$

for certain integers $\beta_{i,d}$. The resolution is unique up to isomorphism, so the numbers $\beta_{i,d}$ are invariants of M:

$$\beta_{i,d}(M) := \#\text{ degree-}d\text{ generators of the }i\text{-th syzygy module of }M.$$

These numbers collectively form the ***Betti table*** of M, thought of as a matrix[1] $\beta(M) := (\beta_{i,d})_{i,d\in\mathbb{Z}}$. This table encodes much of the algebraic structure of M, such as its dimension, projective dimension, regularity, Hilbert function, and whether or not it is Cohen-Macaulay. The Betti table also describes geometrical properties of the associated sheaf on $\mathbb{P}(\mathbb{C}^n)$.

While Betti tables are not new, their structure theory – describing which tables of num-

[1]We typically display the Betti table with entry $\beta_{i,d}$ in column i, row d. This way, left-to-right, each column aligns with a single step of the free resolution.

bers $\beta_{i,d}$ arise as Betti tables of modules – is quite recent. The difficulty is that, although Betti tables form a semigroup (since $M \oplus M'$ has Betti table $\beta(M) + \beta(M')$), the semigroup structure is quite complicated and poorly understood. For example, there are tables of integers β such that the even multiples $2\beta, 4\beta, \ldots$ are realizable as Betti tables, but the odd multiples are not! [10]

The key idea in Boij-Söderberg theory is to ask, instead, which tables can be realized up to a scalar multiple, that is, when is $t \cdot \beta$ a Betti table for some $t \in \mathbb{Q}_{\geqslant 0}$. The goal is to characterize the *cone*

$$BS_n := \mathbb{Q}_{\geqslant 0} \cdot \{\beta(M) : M \text{ graded}\},$$

now called the ***Boij-Söderberg cone*** or ***Betti cone***. [2, 9, 8] In 2006, Boij and Söderberg conjectured an elegant combinatorial description of the cone of Betti tables of Cohen-Macaulay modules [2]: it is a rational polyhedral cone, with a simplicial fan decomposition related to Young's lattice $\mathbb{Y}$. Eisenbud and Schreyer [9] proved the conjectures shortly thereafter and made an additional, surprising discovery: a "duality" with the ***cone of cohomology tables*** of vector bundles $\mathcal{E}$ on projective space. These tables[2] consist of the numbers

$$\gamma(\mathcal{E}) := (\gamma_{i,d}(\mathcal{E})), \text{ where } \gamma_{i,d}(\mathcal{E}) := \dim_{\mathbb{C}} H^i(\mathcal{E}(-d)),$$

giving all the sheaf cohomology of all the twists of $\mathcal{E}$. The duality statement (see below) is that the facets of BS_n are given by inequalities with coefficients from the tables $\gamma(\mathcal{E})$. This discovery, based initially on numerical observation, has since been upgraded to a "categorified" pairing between the underlying modules and vector bundles, resembling an intersection theory [6]. We will write ES_n for the cone of positive rational multiples of tables $\gamma(\mathcal{E})$, also called the ***Eisenbud-Schreyer*** cone.

A good survey of the field, which has developed rapidly, is [13]. The results on Cohen-Macaulay modules were later extended to cover all graded modules [3]; and with graded modules understood, more recent work has focused on modules over multigraded and toric rings [6], and certain homogeneous coordinate rings [17, 24], as well as more detailed homological questions [29, 1, 7]. In these settings, much less is known about the explicit structure of the Boij-Söderberg cone.

In each case, an important feature of the theory is a duality [9, 6] between Betti tables and cohomology tables of sheaves on the associated variety. This duality takes the form of a bilinear pairing of cones, with output in some simpler cone. In the graded case, it produces

[2] We use the same indexing and display conventions for cohomology tables: entry $\gamma_{i,d}$ in column i, row d.

an element of the simplest Boij-Söderberg cone BS_1 (technically its derived analog BS_1^D):

$$BS_n \times ES_n \xrightarrow{\langle -,- \rangle} BS_1^D.$$

In other words, with appropriate conventions, the dot product of a Betti table with a cohomology table is again a Betti table – over the smallest graded ring $\mathbb{C}[x_1]$. The inequalities defining BS_1^D (which are very simple) therefore pull back to nonnegative bilinear pairings between Betti and cohomology tables. Here, the graded case is especially nice: the pulled-back inequalities fully characterize the two cones. That is, the pairing is "perfect" in the sense that

$$\beta \in BS_n \qquad \text{iff} \qquad \langle \beta, \gamma \rangle \in BS_1^D \text{ for all } \gamma \in ES_n.$$

Similarly, if γ is an arbitrary linear combination of cohomology tables (possibly with negative coefficients), then $\gamma \in ES_n$ if and only if γ pairs with all realizable Betti tables β to yield elements of BS_1^D.

Finally, for graded modules, the extremal rays and supporting hyperplanes of the Boij-Söderberg cone are explicitly known. Analogous statements are not known in other settings; likewise, the pairing is not known to be "perfect" (in the graded case, the proof relies on the explicit characterization of rays and facets).

1.3.1 The Boij-Söderberg cone for graded modules

We now describe the cone BS_n in more detail. The statements in this section are due to Eisenbud-Schreyer [9] and Boij-Söderberg [2]; the exposition is partly based on [13].

We focus on the special case where M is assumed to have finite length (equivalently, the induced sheaf $\widetilde{M}$ on $\mathbb{P}(\mathbb{C}^n) = \mathbb{P}^{n-1}$ vanishes). These statements are the most similar to the results of this thesis. The case where M is Cohen-Macaulay is nearly identical, and the general case for graded modules was built out of the Cohen-Macaulay one; in fact, the Betti cone for *all* modules is the same as the cone for all Cohen-Macaulay modules. [3]

Thus, we consider the convex cone

$$BS_n := \mathbb{Q}_{\geqslant 0} \cdot \{\beta(M) : M \text{ a graded module of finite length}\}.$$

The main statement is that BS_n is rational polyhedral, and its extremal rays and facets are explicitly known (see below). The dual cone is

$$ES_n := \mathbb{Q}_{\geqslant 0} \cdot \{\gamma(\mathcal{E}) : \mathcal{E} \text{ a vector bundle on } \mathbb{P}^{n-1}\}.$$

The description of ES_n (which we omit) is quite similar: rational polyhedral, with an explicit list of rays and facets. The main point is that the facets of BS_n come from the extremal rays of ES_n, and vice versa, via the bilinear pairing.

1.3.1.1 The Hilbert function and Herzog-Kühl equations

The ***Herzog-Kühl equations*** are a collection of linear conditions on a Betti table, which detect when the underlying module has finite length (or more generally has a specified codimension). They say, essentially, that the Hilbert polynomial $\mathrm{hilb}_M(t)$ is identically 0. This is the unique polynomial with the property that, for $t \gg 0$, $\mathrm{hilb}_M(t) = \dim_{\mathbb{C}} M_t$. The coefficients,

$$\mathrm{hilb}_M(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}$$

are linear combinations of the Betti numbers. The Herzog-Kühl equations are (up to change of variables) the conditions $a_j = 0$ for all j; the simplest form of the equations is just:

$$\sum_{i,d} (-1)^i \beta_{i,d}\, d^j = 0, \qquad \text{for } j = 0, \ldots, n-1.$$

A module M is finite-length if and only if $\beta(M)$ satisfies these n conditions. Moreover, these turn out to be the only *linear* conditions on Betti tables: the cone BS_n linearly spans the vector space $\mathbb{B} = \bigoplus_{i=0}^{n} \bigoplus_{d \in \mathbb{Z}} \mathbb{Q}$ of "abstract Betti tables".

1.3.1.2 Extremal rays

The extremal rays of BS_n correspond to ***pure Betti tables***. These are the simplest possible tables, having only one nonzero entry in each column (that is, for each i, only one $\beta_{i,d}$ is nonzero). Explicitly, they come from resolutions of the form

$$M \leftarrow R(-d_0)^{\beta_{0,d_0}} \leftarrow \cdots R(-d_n)^{\beta_{n,d_n}} \leftarrow 0,$$

for some increasing sequence of integers $\mathbf{d} := d_0 < \cdots < d_n$. We note that the resolution must have all n steps since M is finite-length.

The key observation is that, because M is finite-length, the $(n+1)$ numbers β_{i,d_i} must satisfy the n Herzog-Kühl equations. It is not hard to see that the equations are linearly independent, so these equations determine the Betti numbers uniquely up to a common scalar factor. In particular, the table cannot be nontrivially decomposed as a sum of other

Betti tables. The explicit solution (coming from Vandermonde determinants) is

$$\beta_{i,d_i} = s \cdot \prod_{j \neq i} \frac{1}{|d_i - d_j|}$$

for some common scalar $s \in \mathbb{Q}_{>0}$. We will write $\beta(\mathbf{d})$ for this table, with $s = 1$.

Thus, *if* $\beta(\mathbf{d})$ is realizable (up to scalar multiple), it automatically generates an extremal ray of BS_n. Constructing such a module M is very nontrivial; see [8] for one construction. We will use some of the same techniques in this thesis (Section 3.2).

Theorem 1.3.1 ([9]). *The Betti tables $\beta(\mathbf{d})$ yield precisely the extremal rays of BS_n.*

1.3.1.3 Facets of BS_n and Rays of ES_n

The facet equations can be summarized as follows: because of the "perfect pairing" of Betti and cohomology tables

$$BS_n \times ES_n \to BS_1^D,$$

the cone BS_n is cut out by the conditions $\beta_{i,d} \geqslant 0$ and all inequalities of the form

$$\varphi\big(\langle -, \gamma \rangle\big) \geqslant 0,$$

with φ a nonnegative linear functional on BS_1^D, and γ an extremal ray of ES_n. The functionals φ are fairly simple: they are certain truncated alternating sums, involving only coefficients of 0 and ± 1. The interesting input is the extremal table γ. It turns out that the extremal rays of ES_n are again tables with as few entries as possible. Specifically, we say a vector bundle $\mathcal{E}$ has ***supernatural cohomology*** (and that $\gamma(\mathcal{E})$ is a supernatural cohomology table) if:

(1) For every $d \in \mathbb{Z}$, at most one $H^i(\mathcal{E}(d))$ is nonzero ($i = 0, \ldots, n-1$), and

(2) The Hilbert polynomial of $\mathcal{E}$, $\mathrm{hilb}_{\mathcal{E}}(d) := \chi(\mathcal{E}(d))$, has distinct integer roots.

Conditions (1) and (2) together state that the table $\gamma(\mathcal{E})$ contains as few nonzero entries as possible: at most one in each row, and none in $n-1$ of the rows (corresponding to the roots d of the Hilbert polynomial).

Let the roots be $\mathbf{r} = r_1 > \cdots > r_{n-1}$, so that the Hilbert polynomial is

$$\mathrm{hilb}_{\mathcal{E}}(d) = s \cdot \prod_i (d - r_i),$$

for some scalar $s \in \mathbb{Q}_{>0}$. The entire cohomology table is determined (up to the scalar s): the roots are specified and distinct, so $\mathrm{hilb}_{\mathcal{E}}(d)$ must change sign when d crosses r_i. It is not hard to show that the row with the nonvanishing entry must then change from H^i to H^{i-1}, and this is the only time the row changes. Thus, putting $r_0 = +\infty, r_n = -\infty$, $H^i(\mathcal{E}(d))$ is nonzero precisely when $r_i > d > r_{i+1}$. (The cases $i = 0, n-1$ are just Serre vanishing and Serre duality.)

Let $\gamma(\mathbf{r})$ be the resulting supernatural cohomology table (with $s = 1$). Finding a vector bundle $\mathcal{E}$ to exhibit $\gamma(\mathbf{r})$ is not so hard: by the Borel-Weil-Bott theorem, the bundle $\mathbb{S}_\mu(\mathcal{Q}^*)$ suffices, where $\mathcal{Q}$ is the tautological quotient bundle in the sequence

$$0 \to \mathcal{O}(-1) \to \mathbb{C}^n \to \mathcal{Q} \to 0,$$

and $\mathbb{S}_\mu$ is the Schur functor for the partition $\mu = (r_1 + 1, r_2 + 2, \ldots, r_{n-1} + n - 1)$.

1.4 Grassmannian Boij-Söderberg theory

The goal of this thesis is to begin extending the theory to the setting of Grassmannians $Gr(k, \mathbb{C}^n)$. All the results are joint with Nic Ford and Steven Sam [15, 14]. On the geometric side, we will be interested in the cohomology of vector bundles and coherent sheaves on $Gr(k, \mathbb{C}^n)$. On the algebraic side, we consider the polynomial ring in kn variables (we always assume $k \leqslant n$), writing

$$R_{k,n} = \mathbb{C}\Big[x_{ij} : \begin{array}{l} 1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant n \end{array}\Big],$$

thinking of the x_{ij} as the entries of a $k \times n$ matrix. The group GL_k acts on $R_{k,n}$, and we are interested in the syzygies of ***equivariant modules*** M, i.e., those with a compatible GL_k action. We always assume that M is finitely-generated.

Aside from the inherent interest of understanding sheaf cohomology and syzygies on Grassmannians, there is hope that this setting might avoid some obstacles faced in other extensions of Boij-Söderberg theory, e.g. to products of projective spaces. For example, in the 'base case' of square matrices $(n = k)$, the 'irrelevant ideal' is the principal ideal generated by the determinant, and the Boij-Söderberg cone has an especially elegant structure (see below, Section 1.5.2).

We define ***equivariant Betti tables*** $\beta(M)$ using the representation theory of GL_k. Let $\mathbb{S}_\lambda(\mathbb{C}^k)$ denote the irreducible GL_k representation of weight λ, where $\mathbb{S}_\lambda$ is the Schur functor. There is a corresponding free module, namely $\mathbb{S}_\lambda(\mathbb{C}^k) \otimes_{\mathbb{C}} R_{k,n}$, and every equivariant free module is a direct sum of these. Then $\beta(M)$ is the collection of numbers

$$\beta_{i,\lambda}(M) := \#\text{ copies of } \mathbb{S}_\lambda(\mathbb{C}^k) \text{ in the generators of the } i\text{-th syzygy module of } M.$$

Thus, by definition, the minimal *equivariant* free resolution of M has the form

$$M \leftarrow F_0 \leftarrow \cdots \leftarrow F_n \leftarrow 0, \text{ with } F_i = \bigoplus_\lambda \mathbb{S}_\lambda(\mathbb{C}^k)^{\beta_{i,\lambda}} \otimes R_{k,n}.$$

Next, for $\mathcal{E}$ a coherent sheaf on $Gr(k, \mathbb{C}^n)$, we will define the **GL-*cohomology table*** $\gamma(\mathcal{E})$, generalizing the usual cohomology table:

$$\gamma_{i,\lambda}(\mathcal{E}) := \dim H^i(\mathcal{E} \otimes \mathbb{S}_\lambda(\mathcal{S})),$$

where $\mathcal{S}$ is the tautological vector bundle on $Gr(k, \mathbb{C}^n)$ of rank k.

Remark 1.4.1. The case $k = 1$ reduces to the ordinary Boij-Söderberg theory, since an action of GL_1 is formally equivalent to a grading; the module $R(-j)$ is just $\mathbb{S}_{(j)}(\mathbb{C}) \otimes R$. Note also that $\mathcal{S} = \mathcal{O}(-1)$ on projective space.

The initial questions of Boij-Söderberg theory concerned finite-length graded modules M, i.e. those annihilated by a power of the homogeneous maximal ideal, and more generally Cohen-Macaulay modules. Similarly, we restrict our focus (for now!) on the following class of modules, which specializes to finite-length modules when $k = 1$:

Condition 1.4.2 (The modules of interest)**.** We consider Cohen-Macaulay modules M such that $\sqrt{\operatorname{ann}(M)} = P_k$, the ideal of maximal minors of the $k \times n$ matrix.

Viewing $\operatorname{Spec}(R_{k,n}) = \operatorname{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ as the affine variety of $k \times n$ matrices, this means M is set-theoretically supported on the locus Z of rank-deficient matrices. That is, the sheaf $\widetilde{M}$ associated to M on $Gr(k, \mathbb{C}^n)$ is zero. For this reason, we refer to P_k as the ***irrelevant ideal*** for this setting. The Cohen-Macaulayness assumption means that

$$\operatorname{pdim}(M) = \operatorname{codim}(Z) = n - k + 1,$$

so its minimal free resolution has length $n - k + 1$. An equivalent characterization is therefore:

Condition 1.4.3. We consider modules M such that

(i) The induced sheaf $\widetilde{M}$ on $Gr(k, n)$ is zero (that is, $\sqrt{\operatorname{ann}(M)} \supseteq P_k$),

(ii) Subject to (i), the minimal free resolution of M is as short as possible.

Definition 1.4.4 (Spaces and cones of Betti tables)**.** We write

$$\mathbb{BT}_{k,n} := \bigoplus_i \bigoplus_\lambda \mathbb{Q} = \text{ the space of abstract Betti tables,}$$
$$\mathbb{CT}_{k,n} := \bigoplus_i \prod_\lambda \mathbb{Q} = \text{ the space of abstract } GL\text{-cohomology tables.}$$

We define the ***equivariant Boij-Söderberg cone*** $BS_{k,n} \subset \mathbb{BT}_{k,n}$ as the positive linear span of Betti tables $\beta(M)$, where M satisfies the assumptions of Condition 1.4.2. We define the ***Eisenbud-Schreyer cone*** $ES_{k,n} \subset \mathbb{CT}_{k,n}$ as the positive linear span of GL-cohomology tables of all coherent sheaves $\mathcal{E}$ on $Gr(k, \mathbb{C}^n)$.

We wish to understand the cones $BS_{k,n}$ and $ES_{k,n}$ generated by equivariant Betti tables of and GL-cohomology tables.

Remark 1.4.5 (Multiplicities and ranks)**.** The irreducible representations $\mathbb{S}_\lambda(\mathbb{C}^k)$ need not be one-dimensional. As such, the corresponding free modules need not have rank 1. We will write a tilde $\widetilde{\beta}$ to denote the *rank* of the type-λ summand in the resolution (rather than its *multiplicity*), and likewise write $\widetilde{\mathbb{BT}}_{k,n}$ and $\widetilde{BS}_{k,n}$ for the spaces of rank Betti tables. Of course, we may switch between them by rescaling each entry, since $\widetilde{\beta_{i,\lambda}} = \beta_{i,\lambda} \cdot \dim(\mathbb{S}_\lambda(\mathbb{C}^k))$.

1.5 Results of this thesis

We will generalize three important results from the existing theory on graded modules: the Herzog-Kühl equations, the pairing between Betti and cohomology tables, and the complete description of the simplest cone $BS_{k,k}$, for equivariant modules over the square matrices.

Below, we introduce the three results. We state the theorems with a minimum of technical detail, giving the full statements in the corresponding chapters of the thesis.

Remark 1.5.1 (Acknowledgments and coauthorship)**.** All the results in this thesis are joint with either Nic Ford [14] or Nic Ford and Steven Sam [15]. I have cited the corresponding theorem statements.

1.5.1 Equivariant Herzog-Kühl equations

In the graded setting, the Herzog-Kühl equations are n linear conditions satisfied by the Betti tables of finite-length modules M. They say, essentially, that the Hilbert polynomial of M vanishes identically, i.e., each of its coefficients is zero.

We give the following equivariant analogue.

Theorem 1.5.2 ([14, Theorem 1.5])**.** *Let M be an equivariant $R_{k,n}$-module with Betti table $\beta(M)$. Assume (twisting up if necessary) that M is generated in positive degree.*

There is a system of $\binom{n}{k}$ linear conditions on $\beta(M)$, indexed by partitions $\mu \geqslant 0$ that fit inside a $k \times (n-k)$ rectangle, called the **equivariant Herzog-Kühl equations***. The following are equivalent:*

(i) $\beta(M)$ *satisfies the equivariant Herzog-Kühl equations,*

(ii) M *is annihilated by a power of the ideal P_k of maximal minors,*

(iii) The sheaf $\widetilde{M}$ on $Gr(k, \mathbb{C}^n)$ vanishes.

In particular, the hypotheses of Condition 1.4.2 are equivalent to the equivariant Herzog-Kühl equations and the additional conditions $\beta_{i,\lambda} = 0$ for all $i > n - k + 1$.

We state and establish these equations in Section 4.1.2, using the combinatorics of standard Young tableaux. Our approach is by equivariant K-theory: namely the condition that $\widetilde{M} = 0$ on $Gr(k, \mathbb{C}^n)$ says that the K-theory class of M is in the kernel of the map

$$K^{GL_k}(\mathrm{Spec}(R_{k,n})) \to K(Gr(k, \mathbb{C}^n))$$

induced by restricting to the locus of full-rank matrices, then quotienting by GL_k.

1.5.2 The Boij-Söderberg cone for square matrices

In the graded setting (i.e. when $k = 1$), the base case $n = 1$ plays an important role as it is the target of the Boij-Söderberg pairing. Since the inequalities characterizing BS_n and ES_n arise from this pairing, it helps that the cone BS_1 is relatively simple to understand. (Note that this case is "purely algebraic" since the corresponding projective space $\mathrm{Proj}(\mathbb{C}[t]) = \mathbb{P}^0$ is just a single point).

For general k, we expect the smallest case $n = k$ to play a similarly important role. It will serve as the base case of the theory, and we will describe $BS_{k,k}$ completely; and it will be the target of the equivariant Boij-Söderberg pairing (Section 1.5.3). Note that this case is again purely algebraic, as the corresponding Grassmannian $Gr(k, \mathbb{C}^k)$ is a point.

Rank tables $\widetilde{\beta}$ turn out to be more significant here, so we will state results in terms of the cone $\widetilde{BS}_{k,k}$.

In the square matrix setting, the modules M satisfying Condition 1.4.2 are Cohen–Macaulay of codimension 1, so their minimal free resolutions are just injective maps $F_1 \hookrightarrow F_0$ of equivariant free modules. Thus, the Betti tables in $\widetilde{BS}_{k,k}$ have only two columns.

We note that, by equivariance, the condition that $\operatorname{ann}(M)$ contains a power of the determinant is the same as requiring M to be torsion. When $k = 1$, the ring $R_{k,k}$ is just $\mathbb{C}[t]$, and its torsion graded modules are trivial to describe. See Section 4 of [6] for a short, complete description of the cone $\widetilde{BS}_{1,1}$. For $k > 1$, however, the cones are algebraically and combinatorially interesting, although simpler than the general case.

The cone $\widetilde{BS}_{k,k}$ is as follows:

Theorem 1.5.3 ([15, Theorem 1.2])**.** *The cone $\widetilde{BS}_{k,k}$ is rational polyhedral. Its supporting hyperplanes are indexed by* **order ideals** *in the extended Young's lattice $\mathbb{Y}_\pm$ of GL_k-representations. Its extremal rays are indexed by* **comparable pairs** *$\lambda \lneq \mu$ from $\mathbb{Y}_\pm$.*

More precisely, the extremal rays come from **pure tables***, written $\widetilde{\beta}\,[\lambda \leftarrow \mu]$, with $\widetilde{\beta_{0,\lambda}} = \widetilde{\beta_{1,\mu}} = 1$ and all other entries zero. Up to scaling, these tables come from free resolutions of the form*

$$M \leftarrow \mathbb{S}_\lambda(\mathbb{C}^k)^{\oplus c_0} \otimes R \leftarrow \mathbb{S}_\mu(\mathbb{C}^k)^{\oplus c_1} \otimes R \leftarrow 0,$$

with all generators in type λ and all syzygies in type μ.

We describe the supporting hyperplanes precisely in Section 3.1.1.

We will need a slightly more general result for the purposes of the equivariant Boij-Söderberg pairing, a derived analog to $\widetilde{BS}_{k,n}$.

Definition 1.5.4. The ***derived Boij-Söderberg cone***, denoted $\widetilde{BS}^D_{k,n}$, is the positive linear span of (rank) Betti tables of bounded minimal complexes $F_\bullet$ of equivariant free modules, such that $F_\bullet$ is exact away from the locus of rank-deficient matrices.

In this definition, we assume only that the homology modules M have $\sqrt{\operatorname{ann}(M)} \supseteq P_k$, not that equality holds. We also do not assume Cohen-Macaulayness. Thus, $\widetilde{BS}^D_{k,k}$ includes, for example, homological shifts of elements of $\widetilde{BS}_{k,k}$, and Betti tables of complexes with more than two terms.

The simplest tables in the derived cone are **homologically shifted pure tables**, written $\widetilde{\beta}[\lambda \xleftarrow{i} \mu]$, for $i \in \mathbb{Z}$ and $\lambda \lneq \mu$. These are the tables with $\widetilde{\beta_{i,\lambda}} = \widetilde{\beta_{i+1,\mu}} = 1$ and all other entries zero. We show:

Theorem 1.5.5 ([14, Theorem 1.8])**.** *The cone $\widetilde{BS}^D_{k,k}$ is rational polyhedral. Its extremal rays are the homological shifts of those of $\widetilde{BS}_{k,k}$ and are spanned by the shifted pure tables $\widetilde{\beta}[\lambda \xleftarrow{i} \mu]$. The supporting hyperplanes are indexed by tuples $(\ldots, S_{-1}, S_1, S_3, \ldots)$ of convex subsets $S_i \subseteq \mathbb{Y}_\pm$, one chosen for every other spot along the complex.*

The key idea in the Theorem 1.5.5 is that these Betti tables are characterized by the existence of certain perfect matchings. This idea is also crucial in our construction of the pairing between Betti and cohomology tables, so we discuss it now.

We introduce a graph-theoretic model of a rank Betti table (in the case of free resolutions, this construction is implicit in [15, Lemma 3.6]).

Definition 1.5.6 (Betti graphs)**.** Let $\widetilde{\beta} \in \widetilde{\mathbb{BT}}_{k,k}$ have nonnegative integer entries. The ***Betti graph*** $G(\widetilde{\beta})$ is defined as follows:

- The vertex set contains $\widetilde{\beta_{i,\lambda}}$ vertices labeled (i, λ), for each (i, λ),
- The edge set contains, for each i, all possible edges $(i, \lambda) \leftarrow (i + 1, \mu)$ with $\lambda \lneq \mu$.

Note that this graph is bipartite: every edge connects an even-indexed and an odd-indexed vertex.

Recall that a **perfect matching** on a graph G is a subset of its edges, such that every vertex of G appears on exactly one chosen edge. A perfect matching on $G(\widetilde{\beta})$ is equivalent to a decomposition of $\widetilde{\beta}$ as a positive integer combination of homologically-shifted pure tables: an edge $(i, \lambda) \leftarrow (i + 1, \mu)$ corresponds to a pure summand $\widetilde{\beta}[\lambda \overset{i}{\longleftarrow} \mu]$. Thus, an equivalent characterization of $\widetilde{BS}^D_{k,k}$ is:

Theorem 1.5.7 ([14, Theorem 1.10])**.** *Let $\widetilde{\beta} \in \mathbb{BT}_{k,k}$ have nonnegative integer entries. Then $\widetilde{\beta} \in \widetilde{BS}^D_{k,k}$ if and only if $G(\widetilde{\beta})$ has a perfect matching.*

Our proof proceeds by exhibiting this perfect matching using homological algebra. The supporting hyperplanes of $\widetilde{BS}^D_{k,k}$ then follow from Hall's Matching Theorem; see Section 3.1.2 for the precise statement.

1.5.3 The pairing between Betti tables and cohomology tables

We now turn to the Boij-Söderberg pairing. This will be a bilinear pairing between abstract Betti tables β and cohomology tables γ, satisfying certain nonnegativity properties when restricted to realizable tables.

Definition 1.5.8. Let $\beta \in \mathbb{BT}_{k,n}$ and $\gamma \in \mathbb{CT}_{k,n}$ be an abstract Betti table and GL-cohomology table. The ***equivariant Boij-Söderberg pairing*** is given by

$$\begin{aligned} \widetilde{\Phi} : \mathbb{BT}_{k,n} \times \mathbb{CT}_{k,n} &\to \widetilde{\mathbb{BT}}_{k,k}, \\ (\beta, \gamma) &\mapsto \widetilde{\Phi}(\beta, \gamma), \end{aligned} \tag{1.5.1}$$

with $\widetilde{\Phi}$ the (derived) *rank* Betti table with entries

$$\widetilde{\varphi_{i,\lambda}}(\beta, \gamma) = \sum_{p-q=i} \beta_{p,\lambda} \cdot \gamma_{q,\lambda}. \tag{1.5.2}$$

In this definition, recall that the homological index of a complex decreases under the boundary map.

Here is how to read the definition of $\widetilde{\Phi}$. (See Example 1.5.11 below.) Form a grid in the first quadrant of the plane, whose (p, q)-entry is the collection of numbers $\beta_{p,\lambda} \cdot \gamma_{q,\lambda}$ for all λ. Only finitely-many of these are nonzero. The line $p - q = i$ is an upwards-sloping diagonal through this grid, and $\widetilde{\varphi_{i,\lambda}}$ is the sum of the λ terms along this diagonal.

Remark 1.5.9. We emphasize that the pairing takes a *multiplicity* Betti table β and a cohomology table γ, and produces a *rank* Betti table $\widetilde{\Phi}$. Intuitively, the entries of γ are dimensions of certain vector spaces (from sheaf cohomology), which, we will see, arise with multiplicities given by β in a certain spectral sequence. In particular, the quantities in (1.5.2) are again dimensions of vector spaces – that is, they give a rank table.

Our final result is the nonnegativity of the pairing ([14, Theorem 1.8]):

Theorem 1.5.10 (Pairing the equivariant cones)**.** *The pairing $\widetilde{\Phi}$ restricts to a map of cones,*

$$BS_{k,n} \times ES_{k,n} \to \widetilde{BS}^D_{k,k}.$$

The same is true with $BS_{k,n}$ replaced by $BS^D_{k,n}$ on the source.

In particular, the defining inequalities of the cone $\widetilde{BS}^D_{k,k}$ (which we give explicitly) pull back to nonnegative bilinear pairings of Betti and cohomology tables, and the Betti graph of $\widetilde{\Phi}(\beta, \gamma)$ has a perfect matching. We emphasize, however, that we do not currently know whether this pairing is "perfect" when $k > 1$: it may be possible to have $\langle \beta, \gamma \rangle \in \widetilde{BS}^D_{k,k}$ for every $\gamma \in ES_{k,n}$, but $\beta \notin BS_{k,n}$. A similar statement applies for γ.

We think of this pairing as a reduction to the base case of square matrices ($k = n$). A geometric consequence is that each equivariant Betti table induces many interesting linear inequalities constraining sheaf cohomology on $Gr(k, \mathbb{C}^n)$.

Our proof proceeds by constructing a perfect matching on $\widetilde{\Phi}(\beta, \gamma)$, not by constructing actual modules over $R_{k,k}$. It would be interesting to see a 'categorified' form of the pairing, in the style of Eisenbud-Erman [6]. Such a pairing would build, from a complex $F_\bullet$ of $R_{k,n}$-modules and a sheaf $\mathcal{E}$, a module (or complex) over $R_{k,k}$. Theorem 1.5.10 would follow from showing that this module is supported along the determinant locus (or that the complex is exact away from the determinant locus).

Example 1.5.11. Let us pair the following tables for $k = 2$, $n = 3$. Both are realizable; the cohomology table is for the sheaf $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(-1)$ on the Grassmannian $Gr(2,3) \cong \mathbb{P}^2$.

$\beta_{p,\lambda}$	0	1	2
$\ydiagram{1}$	4	–	–
$\ydiagram{2}$	–	1	–
$\ydiagram{1,1}$	–	9	–
$\ydiagram{2,1}$	–	3	3
$\ydiagram{3,1}$	–	–	1
$\ydiagram{2,2}$	–	–	1

$\times$

$\gamma_{q,\lambda}$	0	1	2
$\ydiagram{1}$	3	1	–
$\ydiagram{2}$	–	3	–
$\ydiagram{1,1}$	1	–	–
$\ydiagram{2,1}$	–	–	–
$\ydiagram{3,1}$	–	3	–
$\ydiagram{2,2}$	–	–	1

(1.5.3)

We arrange the pairwise products in a first-quadrant grid. The sums along the diagonals $\{p - q = i\}$ result in the rank Betti table $\widetilde{\Phi}$:

$$\begin{array}{c|ccc} & & & \\ & - & - & 1 \cdot \ydiagram{2,2} \\ q & 4 \cdot \ydiagram{1} & 3 \cdot \ydiagram{2} & 3 \cdot \ydiagram{3,1} \\ & 12 \cdot \ydiagram{1} & 9 \cdot \ydiagram{1,1} & - \\ \hline & & p & \end{array} \quad \rightsquigarrow$$

$\widetilde{\varphi_{i,\lambda}}$	-1	0	1
$\ydiagram{1}$	4	12	–
$\ydiagram{2}$	–	3	–
$\ydiagram{1,1}$	–	–	9
$\ydiagram{3,1}$	–	–	3
$\ydiagram{2,2}$	–	1	–

(1.5.4)

Finally, we check that $\widetilde{\Phi} \in \widetilde{BS}^D_{k,k}$. The decomposition of $\widetilde{\Phi}$ into pure tables happens to be unique (this is not true in general):

$$\widetilde{\Phi} = 3\,\widetilde{\beta}[\ydiagram{1} \xleftarrow{-1} \ydiagram{2}] + \widetilde{\beta}[\ydiagram{1} \xleftarrow{-1} \ydiagram{2,2}] + 9\,\widetilde{\beta}[\ydiagram{1} \xleftarrow{0} \ydiagram{1,1}] + 3\,\widetilde{\beta}[\ydiagram{1} \xleftarrow{0} \ydiagram{3,1}].$$

This corresponds to an essentially-unique perfect matching on $G(\widetilde{\Phi})$.

CHAPTER 2

Preliminaries

2.1 Spaces of interest

Let V, W be complex vector spaces with $\dim(V) = k$, $\dim(W) = n$ and $k \leqslant n$. We write

$$X = \mathrm{Hom}(V, W) \cong \mathrm{Mat}_{k\times n}$$

for the space of linear maps $V \to W$ (matrices of size $k \times n$). Thinking of X as an affine variety, its coordinate ring is a polynomial ring in kn variables, namely

$$R = R_{k,n} = \mathrm{Sym}^\bullet(V \otimes W^*) \cong \mathbb{C}\Big[x_{ij} : \begin{smallmatrix} 1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant n \end{smallmatrix}\Big].$$

There is an action of $GL(V) \times GL(W)$ on X. Its orbits are the rank loci, $X_r^\circ := \{T : \mathrm{rank}(T) = r\}$. The following facts about X_r° are standard:

- It is locally closed, smooth and irreducible, has codimension $r(n - k + r)$, and has closure $\overline{X_r^\circ} = \{T : \mathrm{rank}(T) \leqslant r\}$.
- Its closure $\overline{X_r^\circ}$ is defined by the vanishing of the $(r + 1) \times (r + 1)$ minors of the matrix, that is, these minors generate the prime ideal of $\overline{X_r^\circ}$ in $R_{k,n}$ (see, e.g., [32, Section 6.1-6.2]). It is Cohen-Macaulay and its singular locus is $\overline{X_{r-1}^\circ}$.

The most important strata for us are the top one, the open locus U of injective linear maps, i.e. full-rank matrices. Its complement is $Z = \overline{X_{\leqslant k-1}^\circ}$. These are important because we are primarily concerned with the $GL(V)$ action, which is free on U. The quotient of U by $GL(V)$ is the Grassmannian of k-planes in W,

$$Gr(k, W) = \{S \subset W : \dim(S) = k\}.$$

The Grassmannian has a tautological short exact sequence of vector bundles,

$$0 \to \mathcal{S} \to W \to \mathcal{Q} \to 0,$$

whose value on the fiber $[S] \in Gr(k, W)$ is the sequence

$$0 \to S \to W \to W/S \to 0.$$

Many examples of interesting $GL(V)$-equivariant loci in X are given by (closures of) preimages of subvarieties of $Gr(k, W)$ – for example, matrix Schubert varieties. One of the motivations for this work is to study these loci algebraically using $R_{k,n}$ and bundles built out of $\mathcal{S}$ and (to a lesser extent) $\mathcal{Q}$.

2.2 Representation theory of the general linear group

A good introduction to these notions is [16]. The irreducible algebraic representations of $GL(V)$ are indexed by weakly-decreasing integer sequences $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_k)$, where $k = \dim(V)$. We write $\mathbb{S}_\lambda(V)$ for the corresponding representation, call λ its ***weight***, and write $d_\lambda(k)$ for its dimension. We call $\mathbb{S}_\lambda$ a ***Schur functor***. If λ has all nonnegative parts, we write $\lambda \geqslant 0$ and say λ is a ***partition***. In this case, $\mathbb{S}_\lambda(V)$ is functorial for all linear transformations $T : V \to W$. If λ has negative parts, $\mathbb{S}_\lambda$ is only functorial for isomorphisms $T : V \xrightarrow{\sim} W$ (because the induced map involves dividing by $\det(T)$). We often represent partitions by their Young diagrams:

$$\lambda = (3, 1) \longleftrightarrow \lambda = \begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1}\end{array}.$$

Schur functors include symmetric and exterior powers:

$$\lambda = (1^d) = d\left\{\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array}\right. \iff \mathbb{S}_\lambda(V) = \bigwedge\nolimits^{d}(V),$$

$$\lambda = (d) = \overbrace{\begin{array}{|c|c|c|c|}\hline & & & \\ \hline \end{array}}^{d} \iff \mathbb{S}_\lambda(V) = \mathrm{Sym}^d(V),$$

and in general Schur functors roughly describe polynomial combinations of minors of a $k \times k$ matrix, with GL_k acting on the left (cf. Equation (2.3.1)). Occasionally, dual spaces will come up, so we note that there are canonical isomorphisms

$$\mathbb{S}_\lambda(V) \cong \mathbb{S}_{-\lambda^R}(V^*) \cong \mathbb{S}_{-\lambda^R}(V)^*,$$

where $-\lambda^R = (-\lambda_k \geqslant \cdots \geqslant -\lambda_1)$ denotes the reversed, negated weight.

We partially order partitions and integer sequences by containment:

$$\lambda \preceq \mu \text{ if } \lambda_i \leqslant \mu_i \text{ for all } i.$$

We write $\mathbb{Y}$ for the poset of partitions with this ordering, called ***Young's lattice***, and $\mathbb{Y}_\pm$ for the poset of weakly-decreasing integer sequences; we call it the ***extended Young's lattice***.

We'll write $\det(V)$ for the one-dimensional representation $\bigwedge^{\dim(V)}(V) = \mathbb{S}_{1^k}(V)$. We may always twist a representation by powers of the determinant:

$$\det(V)^{\otimes a} \otimes \mathbb{S}_{\lambda_1,\ldots,\lambda_k}(V) = \mathbb{S}_{\lambda_1+a,\ldots,\lambda_k+a}(V)$$

for any integer $a \in \mathbb{Z}$. This operation is invertible and can sometimes be used to reduce to considering the case when λ is a partition.

By semisimplicity, any tensor product of Schur functors is isomorphic to a direct sum of Schur functors with some multiplicities:

$$\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V) \cong \bigoplus_\nu \mathbb{S}_\nu(V)^{\oplus c^\nu_{\lambda,\mu}}.$$

The $c^\nu_{\lambda,\mu}$ are the ***Littlewood–Richardson coefficients***. They are difficult to compute in general; we will only use the following facts (see [16, Section 5, Corollary 2 and Section 8, Corollary 2]:

- If $c^\nu_{\lambda,\mu} \neq 0$ and λ is a partition, then $\mu \preceq \nu$ (and similarly, if μ is a partition, then $\lambda \preceq \nu$) and $|\nu| = |\lambda| + |\mu|$.
- By symmetry of tensor products, we have $c^\nu_{\lambda,\mu} = c^\nu_{\mu,\lambda}$.
- The ***Pieri rule***: when $\lambda = (d)$, we have $c^\nu_{(d),\mu} \leqslant 1$, and it is nonzero if and only if $\mu \preceq \nu$ and the complement of μ in ν is a *horizontal strip*, i.e., does not have more than 1 box in any column.

2.3 Equivariant rings, modules and syzygies

If R is a $\mathbb{C}$-algebra with an action of $GL(V)$, and S is any $GL(V)$-representation, then $S \otimes_{\mathbb{C}} R$ is an ***equivariant free** R**-module***; it has the universal property

$$\mathrm{Hom}_{GL(V),R}(S \otimes_{\mathbb{C}} R, M) \cong \mathrm{Hom}_{GL(V)}(S, M)$$

for all equivariant R-modules M. The basic examples will be the modules $\mathbb{S}_\lambda(V) \otimes R$.

Let $R = R_{k,n} = \mathrm{Sym}^\bullet(\mathrm{Hom}(V, W)^*)$ be the polynomial ring defined above. The structure of R itself as a $GL(V) \times GL(W)$ representation (forgetting the ring structure) is known as the ***Cauchy identity***:

$$R_{k,n} = \mathrm{Sym}^\bullet(\mathrm{Hom}(V, W)^*) \cong \bigoplus_{\lambda \geqslant 0} \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\lambda(W^*). \tag{2.3.1}$$

In the case of square matrices, the ring obtained by inverting the determinant is also simple:

$$R_{k,k}[\tfrac{1}{\Delta}] = \bigoplus_{\lambda \in \mathbb{Y}_\pm} \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\lambda(W^*), \tag{2.3.2}$$

where the sum includes sequences λ that need not be positive. We will write $\mathrm{Isom}(V, W)$ for $\mathrm{Spec}(R_{k,k}[\frac{1}{\Delta}])$, the affine variety of isomorphisms $T : V \xrightarrow{\sim} W$.

Note that the prime ideal of $r \times r$ minors (for $r \leqslant k$), and the maximal ideal $\mathfrak{m} = (x_{ij})$ are $GL(V)$- and $GL(W)$-equivariant. Finally, note that the sheaf on $Gr(k, W)$ induced by the module $\mathbb{S}_\lambda(V) \otimes R$ is the vector bundle $\mathbb{S}_\lambda(\mathcal{S})$.

Let M be a finitely-generated $GL(V)$-equivariant R-module. The module $\mathrm{Tor}_i^R(R/\mathfrak{m}, M)$ naturally has the structure of a finite-dimensional $GL(V)$-representation. We define the ***equivariant Betti number*** $\beta_{i,\lambda}(M)$ as the multiplicity of the Schur functor $\mathbb{S}_\lambda(V)$ in this Tor module, i.e.

$$\mathrm{Tor}_i^R(R/\mathfrak{m}, M) \;\cong\; \bigoplus_\lambda \mathbb{S}_\lambda(V)^{\oplus \beta_{i,\lambda}(M)} \qquad \text{(as } GL(V)\text{-representations)}.$$

By semisimplicity of $GL(V)$-representations, any minimal free resolution of M can be made equivariant, so we may instead define $\beta_{i,\lambda}$ as the multiplicity of the equivariant free module $\mathbb{S}_\lambda(V) \otimes R$ in the i-th step of an equivariant minimal free resolution of M:

$$M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_d \leftarrow 0, \text{ where } F_i = \bigoplus_\lambda \mathbb{S}_\lambda(V)^{\beta_{i,\lambda}(M)} \otimes R.$$

All other notation on Betti tables is as defined in Section 1.4.

2.3.1 Equivariant maps of modules

The maps of any minimal complex have positive degree, but minimal *equivariant* maps must also be compatible with the partial ordering on $\mathbb{Y}_\pm$. More precisely, we have the following:

Lemma 2.3.1 (Weights contract under minimal maps)**.** *Let* $f\colon \mathbb{S}_\nu(V)\otimes R \to \mathbb{S}_\lambda(V)\otimes R$ *be any nonzero map. If* $\nu = \lambda$*, then* f *is an isomorphism. Otherwise,* $\nu \gneq \lambda$ *and* f *is minimal.*

Proof. This follows from the universal property of equivariant free modules,

$$\operatorname{Hom}_{GL(V),R}(\mathbb{S}_\nu(V)\otimes R, \mathbb{S}_\lambda(V)\otimes R) \cong \operatorname{Hom}_{GL(V)}(\mathbb{S}_\nu(V), \mathbb{S}_\lambda(V)\otimes R).$$

We apply the Cauchy identity (2.3.1) for R as a $GL(V)$-representation. We see that

$$\operatorname{Hom}_{GL(V)}(\mathbb{S}_\nu(V), \mathbb{S}_\lambda(V)\otimes R) \cong \bigoplus_{\mu\geqslant 0} \operatorname{Hom}_{GL(V)}\big(\mathbb{S}_\nu(V), \mathbb{S}_\lambda(V)\otimes \mathbb{S}_\mu(V)\big)\otimes \mathbb{S}_\mu(W^*).$$

By the Littlewood–Richardson rule, if $\nu \ngeq \lambda$, every summand is 0. If $\nu = \lambda$, the only nonzero summand comes from $\mu = \varnothing$, and the corresponding map is multiplication by a (nonzero) scalar, so it is an isomorphism. Finally, if $\nu \gneq \lambda$, there is at least one μ for which the corresponding summand is nonzero, and any such μ must satisfy $|\mu| = |\nu| - |\lambda| > 0$, so the corresponding map of R-modules has strictly positive degree (equal to $|\mu|$), hence is a minimal map. $\square$

Remark 2.3.2. Because the ring R involves W^*, not W, the analogous computation for a $GL(W)$-equivariant map shows that the label on a free module $\mathbb{S}_\lambda(W)\otimes R$ *expands* under a minimal map: that is, a nonzero $GL(W)$-equivariant map $\mathbb{S}_\nu(W)\otimes R \to \mathbb{S}_\lambda(W)\otimes R$ exists if and only if $\nu \leq \lambda$ (and is minimal if and only if $\nu \neq \lambda$).

CHAPTER 3

The base case: square matrices

Remark 3.0.1. In this chapter, rank Betti tables play a more significant role than multiplicity tables. As such, we will state results in terms of the cones $\widetilde{BS}_{k,k}$ and $\widetilde{BS}^D_{k,k}$.

We now describe the Boij-Söderberg cone in the base case of square matrices: Theorems 1.5.3, 1.5.5 and 1.5.7. Thus, for the remainder of the chapter, we set $n = k$ (but it is convenient not to identify V and W). We study the coordinate ring $R_{k,k}$, the cone of Betti tables $\widetilde{BS}_{k,k}$, and the derived cone $\widetilde{BS}^D_{k,k}$.

We proceed as follows. We go through the results precisely in Section 3.1, except that we postpone the proofs of two key statements: the existence of certain modules with "pure" equivariant resolutions, and the existence of perfect matchings on Betti graphs of realizable complexes. We construct the desired modules in Section 3.2, which completes our description of $\widetilde{BS}_{k,k}$. We construct the desired perfect matchings (which are necessary for our description of $\widetilde{BS}^D_{k,k}$) in the next chapter, in Section 4.2.1.1, as part of our construction of the equivariant Boij-Söderberg pairing.

3.1 Results on the cones $\widetilde{BS}_{k,k}$ and $\widetilde{BS}^D_{k,k}$

3.1.1 The cone $\widetilde{BS}_{k,k}$

The rank-deficient locus for square matrices $\{T : \det(T) = 0\} \subset \mathrm{Hom}(V, W)$ is codimension 1. Thus, modules satisfying Condition 1.4.2 have free resolutions of length 1,

$$M \leftarrow F^0 \leftarrow F^1 \leftarrow 0,$$

and the rank Betti numbers $\widetilde{\beta_{i,\lambda}}$ are the ranks of the λ summands of F_i, for $i = 0, 1$. By equivariance, the condition that $\mathrm{ann}(M)$ contains a power of the determinant is the same as requiring M to be torsion:

$$\operatorname{rank}(F^0) = \operatorname{rank}(F^1), \text{ that is, } \sum_\lambda \widetilde{\beta_{0,\lambda}} = \sum_\lambda \widetilde{\beta_{1,\lambda}}. \tag{3.1.1}$$

The extremal rays and supporting hyperplanes of $\widetilde{BS}_{k,k}$ are as follows.

Definition 3.1.1 (Pure tables)**.** Fix $\lambda, \mu \in \mathbb{Y}_\pm$ with $\lambda \lneq \mu$. The ***pure table*** $\widetilde{\beta}[\lambda \leftarrow \mu]$ is defined by setting

$$\widetilde{\beta_{0,\lambda}} = \widetilde{\beta_{1,\mu}} = 1$$

and all other entries 0.

Any such table, if realizable, generates an extremal ray of $\widetilde{BS}_{k,k}$. It is nontrivial to show that $\widetilde{\beta}[\lambda \leftarrow \mu]$ is realizable up to scalar multiple for any pair $\lambda \lneq \mu$:

Theorem 3.1.2. *For any pair $\lambda \lneq \mu$, there exists a torsion, $GL(V)$-equivariant R-module M whose minimal free resolution is of the form*

$$M \leftarrow \mathbb{S}_\lambda(V)^{c_0} \otimes R \leftarrow \mathbb{S}_\mu(V)^{c_1} \otimes R \leftarrow 0,$$

for some integers c_0, c_1. Since M is torsion, it follows that $c_0 d_\lambda(k) = c_1 d_\mu(k) = N$ for some N, so the equivariant rank Betti table is $\widetilde{\beta}(M) = N \cdot \widetilde{\beta}[\lambda \leftarrow \mu]$.

We postpone the proof of Theorem 3.1.2 until Section 3.2. **Note**: this theorem is a joint result with Nic Ford and Steven Sam [15, Theorem 4.1].

The following inequalities on $\widetilde{BS}_{k,k}$ are, by contrast, easy to establish.

Definition 3.1.3 (Antichain inequalities)**.** Let $S \subseteq \mathbb{Y}_\pm$ be a downwards-closed set. Let

$$\Gamma = \{\lambda : \lambda \lneq \mu \text{ for some } \mu \in S\}.$$

For any rank Betti table $(\widetilde{\beta_{i,\lambda}})$, the ***antichain inequality (for** S**)*** is then:

$$\sum_{\lambda \in \Gamma} \widetilde{\beta_{0,\lambda}} \geqslant \sum_{\lambda \in S} \widetilde{\beta_{1,\lambda}}. \tag{3.1.2}$$

(The terminology of 'antichains' is due to [15], where the inequality (3.1.2) is stated in terms of the maximal elements of S, which form an antichain in $\mathbb{Y}_\pm$.)

Proposition 3.1.4. *A realizable Betti table satisfies the antichain inequalities.*

Proof. By minimality of the underlying map of modules, the summands corresponding to S in F_1 must map (injectively) into the summands corresponding to Γ in F_0. (See Lemma 2.3.1.) $\square$

Finally, we recall the graph-theoretic model of $\widetilde{\beta}$ introduced in Section 1.5.2. It is especially simple in this case:

Definition 3.1.5. The ***Betti graph*** $G(\widetilde{\beta})$ is the directed bipartite graph with left vertices L and right vertices R, defined as follows:

- The set L (resp. R) contains $\widetilde{\beta_{0,\lambda}}$ (resp. $\widetilde{\beta_{1,\lambda}}$) vertices labeled λ, for each λ,
- The edge set contains all possible edges $\lambda \leftarrow \mu$, from R to L, for $\lambda \lneq \mu$.

A perfect matching on $G(\widetilde{\beta})$ expresses $\widetilde{\beta}$ as a positive integer sum of pure tables: an edge $\lambda \leftarrow \mu$ corresponds to a summand

$$\widetilde{\beta} = \cdots + \widetilde{\beta}\,[\lambda \leftarrow \mu] + \cdots .$$

It is easy to see that the cone spanned by the pure tables is contained in the cone defined by the antichain inequalities. The fact that these cones agree follows from Hall's Matching Theorem for bipartite graphs:

Theorem 3.1.6 (Hall's Matching Theorem, 1935 [20])**.** *Let G be a bipartite graph with left vertices L and right vertices R, with $|L| = |R|$. Then G has a perfect matching if and only if the following holds for all subsets $S \subseteq R$ (equivalently, for all subsets $S \subseteq L$): let $\Gamma(S)$ be the set of vertices adjacent to S. Then $|\Gamma(S)| \geqslant |S|$.*

In the antichain inequality (3.1.2), S corresponds to a set of vertex labels on the right-hand-side of the Betti graph $G(\widetilde{\beta})$. The set Γ consists of the labels of vertices adjacent to S. The numbers of these vertices are the right- and left-hand-sides of the inequality. (Given the structure of $G(\widetilde{\beta})$, it suffices to consider downwards-closed sets S.) Thus, *assuming* Theorem 3.1.2, we have proven the following characterization of $\widetilde{BS}_{k,k}$:

Theorem 3.1.7 ([15, Theorem 3.8])**.** *The cone $\widetilde{BS}_{k,k}$ is defined by the rank equation* (3.1.1)*, the conditions $\widetilde{\beta_{i,\lambda}} \geqslant 0$, and the antichain inequalities* (3.1.2)*. Its extremal rays are the pure tables $\widetilde{\beta}\,[\lambda \leftarrow \mu]$, for all choices of $\lambda \lneq \mu$ in $\mathbb{Y}_{\pm}$.*

Moreover, if $\widetilde{\beta} \in \widetilde{\mathbb{B}}_{k,k}$ has nonnegative integer entries, then $\widetilde{\beta} \in \widetilde{BS}_{k,k}$ if and only if the Betti graph $G(\widetilde{\beta})$ has a perfect matching.

3.1.2 The derived cone

We now generalize Theorem 3.1.7 to describe the derived cone $\widetilde{BS}^D_{k,k}$. We are interested in bounded free equivariant complexes of $R_{k,k}$-modules,

$$\cdots \leftarrow F_i \leftarrow F_{i+1} \leftarrow F_{i+2} \leftarrow \cdots,$$

all of whose homology modules are torsion.

The supporting hyperplanes of $\widetilde{BS}^D_{k,k}$ are quite complicated and we do not establish them directly, as in Proposition 3.1.4. We instead generalize the descriptions in terms of extremal rays and perfect matchings, which remain fairly simple. We then deduce the inequalities from Hall's Theorem.

The extremal rays of $\widetilde{BS}^D_{k,k}$ will be homological shifts of those of $\widetilde{BS}_{k,k}$:

Definition 3.1.8 (Homologically-shifted pure tables)**.** Fix $i \in \mathbb{Z}$ and $\lambda, \mu \in \mathbb{Y}_\pm$ with $\lambda \lneq \mu$. We define the ***homologically-shifted pure table*** $\widetilde{\beta}[\lambda \xleftarrow{i} \mu]$ by setting

$$\widetilde{\beta_{i,\lambda}} = \widetilde{\beta_{i+1,\mu}} = 1$$

and all other entries 0.

The supporting hyperplanes will be defined by the following inequalities. Recall that a ***convex subset*** S of a poset P is the intersection of an upwards-closed set with a downwards-closed set.

Definition 3.1.9 (Convexity inequalities)**.** For each odd i, let $S_i \subseteq \mathbb{Y}_\pm$ be any convex set. For each even i, define

$$\Gamma_i = \{\lambda : \mu \lneq \lambda \text{ for some } \mu \in S_{i-1}\} \cup \{\lambda : \lambda \lneq \mu \text{ for some } \mu \in S_{i+1}\}.$$

For any rank Betti table $(\widetilde{\beta_{i,\lambda}})$, the ***convexity inequality (for the*** S_i***'s)*** is then:

$$\sum_{i \text{ even}} \sum_{\lambda \in \Gamma_i} \widetilde{\beta_{i,\lambda}} \geqslant \sum_{i \text{ odd}} \sum_{\lambda \in S_i} \widetilde{\beta_{i,\lambda}}. \tag{3.1.3}$$

(We may, if we wish, switch 'even' and 'odd' in this definition. We will see that either collection of inequalities yields the same cone.)

We recall the general definition of the Betti graph:

Definition 3.1.10 (Betti graphs for complexes)**.** Let $\widetilde{\beta} \in \widetilde{\mathbb{B}}_{k,k}$ have nonnegative integer entries. The ***Betti graph*** $G(\widetilde{\beta})$ is defined as follows:

- The vertex set contains $\widetilde{\beta_{i,\lambda}}$ vertices labeled (i, λ), for each (i, λ),
- The edge set contains, for each i, all possible edges $(i, \lambda) \leftarrow (i + 1, \mu)$ with $\lambda \lneq \mu$.

Note that this graph is bipartite: every edge connects an even- and an odd-indexed vertex.

Proposition 3.1.11. *Let $F_\bullet$ be a bounded minimal complex of equivariant free modules over $R_{k,k}$. Then the Betti graph $G(\widetilde{\beta}(F_\bullet))$ has a perfect matching.*

We postpone the proof – which exhibits such a matching using homological algebra – until Section 4.2.1.1.

Note that, in the convexity inequalities of Definition 3.1.9, each segment S_i corresponds to a set of vertex labels in $G(\widetilde{\beta})$. The set Γ_i then contains the labels of vertices adjacent to S_{i-1} and S_{i+1}. The numbers of these vertices give the right- and left-hand-sides of the inequality (3.1.3), so the inequality follows from Hall's Matching Theorem and 3.1.11.

We can now characterize the derived Boij-Söderberg cone $\widetilde{BS}^D_{k,k}$, assuming Theorem 3.1.2 and Proposition 3.1.11.

Theorem 3.1.12 (The derived Boij-Söderberg cone, for square matrices)**.** *Let $\widetilde{\beta}$ be an abstract rank Betti table. Without loss of generality, assume the entries of $\widetilde{\beta}$ are nonnegative integers. The following are equivalent:*

(i) $\widetilde{\beta} \in \widetilde{BS}^D_{k,k}$;

(ii) $\widetilde{\beta}$ *satisfies all the convexity inequalities, together with the rank condition*

$$\sum_{i,\lambda} (-1)^i \widetilde{\beta_{i,\lambda}} = 0;$$

(iii) $\widetilde{\beta}$ *is a positive integral linear combination of homologically shifted pure tables;*

(iv) The Betti graph $G(\widetilde{\beta})$ *has a perfect matching.*

Proof. We have (iv) ⇒ (iii) as each edge of a perfect matching indicates a pure table summand for $\widetilde{\beta}$. Next, (iii) ⇒ (ii) since the desired (in)equalities hold for each homologically-shifted pure table individually. Hall's Matching Theorem gives the statement (ii) ⇔ (iv) and shows that we may exchange 'even' and 'odd' in the definition of the convexity inequalities. Homologically-shifted pure tables are realizable by Theorem 3.1.2, hence (iii) ⇒ (i). Finally, any realizable Betti graph has a perfect matching by Proposition 3.1.11, so that (i) ⇒ (iv). □

Remark 3.1.13 (Decomposing Betti tables)**.** There are efficient algorithms for computing perfect matchings of graphs; see e.g. [27, §1.2]. A standard proof of Hall's Theorem implicitly uses the following algorithm, which is inefficient but conceptually clear. Let $\widetilde{B} \in \widetilde{BS}^D_{k,k}$ be a Betti table.

Case 1: Suppose every convexity inequality holds strictly. Choose any pure table $\widetilde{\beta}[\lambda \xleftarrow{i} \mu]$ whose entries occur with nonzero values in $\widetilde{\beta}$. Then

$$\widetilde{\beta}_{\text{rest}} = \widetilde{\beta} - \widetilde{\beta}[\lambda \xleftarrow{i} \mu] \in \widetilde{BS}^D_{k,k}.$$

Continue the algorithm on $\widetilde{\beta}_{\text{rest}}$.

Case 2: Suppose, instead, some convexity inequality is actually an equality. Write

$$\widetilde{\beta} = \widetilde{\beta}_{\text{convex}} + \widetilde{\beta}_{\text{rest}},$$

where $\widetilde{\beta}_{\text{convex}}$ is the table with the same entries as $\widetilde{\beta}$ in the positions (i, λ) involved in the convexity equality, and zeroes elsewhere. It follows that both summands $\widetilde{\beta}_{\text{convex}}, \widetilde{\beta}_{\text{rest}} \in \widetilde{BS}_{k,k}$; continue the algorithm separately for each.

We contrast the algorithm above with the usual algorithm [9, §1] for decomposing graded Betti tables. For graded tables, the decomposition is "greedy" and deterministic. It relies on a partial ordering on pure graded Betti tables, which induces a decomposition of the Boij–Söderberg cone as a simplicial fan.

Unfortunately, the natural choices of partial ordering on the equivariant pure tables $\widetilde{\beta}[\lambda \xleftarrow{i} \mu]$ do not yield valid greedy decomposition algorithms (even for the non-derived cone $\widetilde{BS}_{k,k}$). For example, suppose the Betti graph G consists of a single long path. Then G has a unique perfect matching, but whether an edge is used depends on its distance along the path, not on the partitions labelling its vertices. Hence, an algorithm that (for instance) greedily selects the lexicographically-smallest pure table will fail. Similarly, if G is a cycle, then G has two perfect matchings, so a deterministic algorithm must have a way of selecting one.

Also, unlike in the graded case, we do not know a good simplicial decomposition of $\widetilde{BS}_{k,k}$; it would be interesting to find such a structure.

3.2 Constructing Pure Resolutions

The main result of this section is Theorem 3.1.2. We recall the statement:

Theorem 3.2.1. *Let $\lambda \lneq \mu$ be weakly-decreasing integer sequences. There exists a torsion, $GL(V)$-equivariant R-module M whose minimal free resolution is of the form*

$$M \leftarrow \mathbb{S}_\lambda(V)^{c_0} \otimes R \leftarrow \mathbb{S}_\mu(V)^{c_1} \otimes R \leftarrow 0, \tag{3.2.1}$$

for some integers c_0, c_1.

(By equivariance, M is annihilated by a power of the determinant if and only if M is torsion.) Since M is torsion, the ranks must agree, $c_0 d_\lambda(k) = c_1 d_\mu(k) = N$ for some N, so the equivariant rank Betti table $\widetilde{\beta}(M)$ is, up to scaling by N, the pure table

$$\widetilde{\beta}\,[\lambda \leftarrow \mu] = \begin{array}{c|cc} \widetilde{\beta_{i,\lambda}} & 0 & 1 \\ \hline \lambda & 1 & - \\ \mu & - & 1 \end{array}$$

An obvious choice is to take $c_0 = d_\mu(k)$ and $c_1 = d_\lambda(k)$, which suggests constructing a 'small' resolution of the form

$$M \leftarrow \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(W) \otimes R \leftarrow \mathbb{S}_\mu(V) \otimes \mathbb{S}_\lambda(W) \otimes R \leftarrow 0, \tag{3.2.2}$$

equivariant for both $GL(V)$ and $GL(W)$. (Since R contains representations involving W^*, the label on the W side must expand, not contract, under the map. See Remark 2.3.2.)

Conjecture 3.2.2. *A small pure resolution* (3.2.2) *exists, for any pair $\lambda \lneq \mu$.*

In general, we do not know how to construct a small resolution as in (3.2.2); see Proposition 3.2.5 below for one case. Instead, we construct a larger multiple of the resolution (3.2.1), by reducing to the case of linear resolutions (where the map has degree 1). This suffices for the purpose of describing the Boij–Söderberg cone. Then, for linear resolutions, we will construct the small resolution (3.2.2).

Lemma 3.2.3 (Reduction to the linear case)**.** *If a small pure resolution* (3.2.2) *exists in the case where $|\mu| = |\lambda| + 1$, then some pure resolution exists for any pair $\lambda \lneq \mu$.*

Proof. Let $|\mu| - |\lambda| = r$. Choose a chain of weights

$$\mu = \alpha^{(r)} \gneq \alpha^{(r-1)} \gneq \cdots \gneq \alpha^{(0)} = \lambda, \text{ with } |\alpha^{(i)}| = |\lambda| + i \text{ for all } i.$$

By hypothesis, for $i = 1, \ldots, r$, there exists a sequence of bi-equivariant, linear injections

$$f_i \colon \mathbb{S}_{\alpha^{(i)}}(V) \otimes \mathbb{S}_{\alpha^{(i-1)}}(W) \otimes R \hookrightarrow \mathbb{S}_{\alpha^{(i-1)}}(V) \otimes \mathbb{S}_{\alpha^{(i)}}(W) \otimes R.$$

Let g be the composite map

$$\begin{tikzcd}
F_1 = \mathbb{S}_{\alpha^{(r)}}(V) \otimes \mathbb{S}_{\alpha^{(r-1)}}(W) \otimes \cdots \otimes \mathbb{S}_{\alpha^{(1)}}(W) \otimes \mathbb{S}_{\alpha^{(0)}}(W) \otimes R \arrow[d, "f_r \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}"'] \\
\mathbb{S}_{\alpha^{(r)}}(W) \otimes \mathbb{S}_{\alpha^{(r-1)}}(V) \otimes \cdots \otimes \mathbb{S}_{\alpha^{(1)}}(W) \otimes \mathbb{S}_{\alpha^{(0)}}(W) \otimes R \arrow[d, "\mathrm{id} \otimes f_{r-1} \otimes \cdots \otimes \mathrm{id}"'] \\
\vdots \arrow[d, "\mathrm{id} \otimes \cdots \otimes f_2 \otimes \mathrm{id}"'] \\
\mathbb{S}_{\alpha^{(r)}}(W) \otimes \mathbb{S}_{\alpha^{(r-1)}}(W) \otimes \cdots \otimes \mathbb{S}_{\alpha^{(1)}}(V) \otimes \mathbb{S}_{\alpha^{(0)}}(W) \otimes R \arrow[d, "\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes f_1"'] \\
F_0 = \mathbb{S}_{\alpha^{(r)}}(W) \otimes \mathbb{S}_{\alpha^{(r-1)}}(W) \otimes \cdots \otimes \mathbb{S}_{\alpha^{(1)}}(W) \otimes \mathbb{S}_{\alpha^{(0)}}(V) \otimes R.
\end{tikzcd}$$

Clearly g is again injective. Since $\operatorname{rank}(F_1) = \operatorname{rank}(F_0)$, we are done. □

Remark 3.2.4. By the Pieri rule, when $|\mu/\lambda| = 1$, there are no choices to make: the bi-equivariant map (3.2.2) of free modules exists and is unique up to scaling. This follows from a computation similar to that of Lemma 2.3.1. So either the map is injective or it is not! The linear case is thus both sufficient and morally necessary for the general case of the theorem to hold.

In many cases, it is easy to construct a small bi-equivariant resolution (3.2.2). The following case is simple:

Proposition 3.2.5. *Suppose there exists d so that $\lambda_i \leqslant d \leqslant \mu_j$ for all i, j. (Equivalently, assume $\lambda_1 \leqslant \mu_k$.) Then a small pure resolution* (3.2.2) *exists.*

Proof. We construct the map geometrically. After twisting down by d, we may suppose instead $\lambda \leqslant 0 \leqslant \mu$. Write $\lambda = -\epsilon^R$ for some partition $\epsilon \geqslant 0$. Over $X = \operatorname{Hom}(V, W)$, there are canonical, bi-equivariant maps of vector bundles

$$\mathcal{T} \colon V \times X \to W \times X,$$
$$\mathcal{T}^* \colon W^* \times X \to V^* \times X,$$

both isomorphisms away from the determinant locus. (The corresponding maps of free modules are given by the square matrices (x_{ij}) and (x_{ji}).) Recall that the Schur functor $\mathbb{S}_\lambda$

is functorial for linear transformations when $\lambda \geqslant 0$, so there are induced maps

$$\mathbb{S}_\mu(\mathcal{T})\colon \mathbb{S}_\mu(V) \times X \to \mathbb{S}_\mu(W) \times X,$$
$$\mathbb{S}_\epsilon(\mathcal{T}^*)\colon \mathbb{S}_\epsilon(W^*) \times X \to \mathbb{S}_\epsilon(V^*) \times X.$$

Let $g = \mathbb{S}_\mu(\mathcal{T}) \otimes \mathbb{S}_\epsilon(\mathcal{T}^*)$. Note that g is generically an isomorphism of vector bundles, so the corresponding map of R-modules is injective:

$$g\colon \mathbb{S}_\mu(V) \otimes \mathbb{S}_\epsilon(W^*) \otimes R \to \mathbb{S}_\epsilon(V^*) \otimes \mathbb{S}_\mu(W) \otimes R.$$

Finally, we apply the canonical isomorphism $\mathbb{S}_\epsilon(E^*) \cong \mathbb{S}_{-\epsilon^R}(E)$ (for any vector space E) to the free R-modules above to get the desired map. □

We now consider a pair of weights differing by 1. The argument we will use works in more generality (see Remark 3.2.12), but we restrict to this case for notational simplicity. By twisting by the determinant, we assume λ and μ are partitions. We know that the map (3.2.2) exists and is unique up to scalar multiple.

The remainder of this chapter is devoted to the proof of the following:

Theorem 3.2.6. *When $|\mu| - |\lambda| = 1$, the bi-equivariant map* (3.2.2) *is injective.*

3.2.1 Borel–Weil–Bott and Eisenbud–Fløystad–Weyman

We first recall the statement of the Borel–Weil–Bott Theorem for the projective space $\mathbb{P}(W^*)$. We have the short exact sequence of vector bundles

$$0 \to \mathcal{S} \to W \to \mathcal{O}(1) \to 0.$$

Note that we use the sequence with W, not W^*, even though we work over $\mathbb{P}(W^*)$. We fix this notation for the remainder of the chapter.

Given a permutation σ, define its ***length*** to be $\ell(\sigma) = \#\{i < j \mid \sigma(i) > \sigma(j)\}$.

Theorem 3.2.7 (Borel–Weil–Bott, [32, Corollary 4.1.9])**.** *Let $\beta = (\beta_1, \ldots, \beta_{k-1})$ be a weakly decreasing integer sequence and let $d \in \mathbb{Z}$. The cohomology of $\mathbb{S}_\beta(\mathcal{S})(d)$ is determined as follows. Write*

$$(a_1, \ldots, a_k) = (d, \beta_1, \ldots, \beta_{k-1}) - (0, 1, \ldots, k-1).$$

1. *If $a_i = a_j$ for some $i \neq j$, every cohomology group of $\mathbb{S}_\beta(\mathcal{S})(d)$ vanishes.*

2. *Otherwise, a unique permutation σ sorts the a_i into decreasing order, $a_{\sigma(1)} > a_{\sigma(2)} > \cdots > a_{\sigma(k)}$. Put $\lambda = (a_{\sigma(1)}, \ldots, a_{\sigma(k)}) + (0, 1, \ldots, k-1)$. Then*

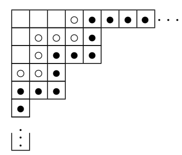
$$H^{\ell(\sigma)}\big(\mathbb{S}_\beta(\mathcal{S})(d)\big) = \mathbb{S}_\lambda(W),$$

and $H^i(\mathbb{S}_\beta(\mathcal{S})(d)) = 0$ for $i \neq \ell(\sigma)$.

We will also use the following result of Eisenbud-Fløystad-Weyman on the existence of certain equivariant graded free resolutions.

First, for a partition λ, we say (i, j) is an ***outer border square*** if $(i, j) \notin \lambda$ and $(i-1, j-1) \in \lambda$ (or $i = 1$ or $j = 1$). Similarly, we say it is an ***inner border square*** if $(i, j) \in \lambda$ and $(i+1, j+1) \notin \lambda$.

For example, with $\lambda = (4, 4, 2, 2)$, the inner $\boxed{\circ}$ and outer $\boxed{\bullet}$ border squares are:



Let α be a partition with k parts, and let $\alpha' \supsetneq \alpha$ be obtained by adding at least one outer border square in row 1, and all possible outer border squares in rows $2, \ldots, k$. Let $\alpha^{(0)} = \alpha$, and for $i = 1, \ldots, k$, let $\alpha^{(i)}$ be obtained by adding the chosen outer border squares only in rows $1, \ldots, i$.

Theorem 3.2.8 ([8, Theorem 3.2])**.** *Let E be a k-dimensional complex vector space and $R = \mathrm{Sym}(E)$ its symmetric algebra. There is a finite-length, $GL(E)$-equivariant R-module M whose equivariant minimal free resolution is, with $\alpha^{(i)}$ defined as above,*

$$F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_k \leftarrow 0, \qquad F_i = \mathbb{S}_{\alpha^{(i)}}(E) \otimes R.$$

Since the construction is equivariant, it works in families:

Theorem 3.2.9. *Let X be a complex variety and $\mathcal{E}$ a rank k vector bundle over X. Let $\mathcal{E}^* \to X$ be the dual bundle. There is a sheaf $\mathcal{M}$ of $\mathcal{O}_{\mathcal{E}^*}$-modules with a locally-free resolution*

$$\mathcal{F}_0 \leftarrow \mathcal{F}_1 \leftarrow \cdots \leftarrow \mathcal{F}_n \leftarrow 0, \qquad \mathcal{F}_i = \mathbb{S}_{\alpha^{(i)}}(\mathcal{E}) \otimes \mathcal{O}_{\mathcal{E}^*}.$$

This follows by applying the EFW construction to the sheaf of algebras $\mathcal{O}_{\mathcal{E}^*} = \mathrm{Sym}(\mathcal{E})$. The resolved sheaf $\mathcal{M}$ is locally given by M above. Note that $\mathcal{M}$ is coherent as an $\mathcal{O}_X$-module, though we will not need this.

Remark 3.2.10. The construction we presented is also a direct corollary of a special case of Kostant's version of the Borel–Weil–Bott theorem, for example see [11, §6] for some discussion and references. We expect that other cases of Kostant's theorem are relevant for constructing complexes in the non-square matrix case.

3.2.2 Informal summary of the argument

We have fixed $\lambda \lnsim \mu$, a pair of partitions differing by a box. There is a unique Eisenbud–Fløystad–Weyman (EFW) complex with, in one step, a linear differential of the form

$$\mathbb{S}_\lambda(E) \otimes \mathrm{Sym}(E) \leftarrow \mathbb{S}_\mu(E) \otimes \mathrm{Sym}(E).$$

The rest of the complex is uniquely determined by this pair of shapes, and is functorial in E. We 'sheafify' the complex as in Theorem 3.2.9, with $X = \mathbb{P}(W^*)$ and $\mathcal{E} = V \otimes \mathcal{O}(-1)$. So, we get a complex of modules over $\mathrm{Sym}(V \otimes \mathcal{O}(-1))$, with terms of the form

$$\mathcal{O}(-d_i) \otimes \mathbb{S}_\alpha(V) \otimes \mathrm{Sym}(V \otimes \mathcal{O}(-1)).$$

The bundle $\mathcal{E}$ fits in the short exact sequence

$$0 \to V \otimes \mathcal{O}(-1) \to V \otimes W^* \to V \otimes \mathcal{S}^* \to 0$$

obtained by tensoring the tautological sequence by the constant bundle V. We base change along the flat extension $\mathrm{Sym}(V \otimes \mathcal{O}(-1)) \hookrightarrow \mathrm{Sym}(V \otimes W^*)$, which is locally an inclusion of polynomial rings. We twist so that the 0-th term in the complex has degree $d = \mu_1$, then tensor through by $\mathbb{S}_\beta(\mathcal{S})$, where β is chosen so that all the terms of the resulting complex *except* the desired pair have no cohomology. We obtain the desired map as an induced map of sheaf cohomology, and show that it is injective using the hypercohomology spectral sequence.

Example 3.2.11. Let $k = 4$ and let $\lambda = (6, 1, 1, 0)$, $\mu = (6, 2, 1, 0) = \ytableaux{\ ,\ ,\ ,\ ,\ ,\ \\ \ ,\star \\ \ }$ (the added box is starred). Working on $\mathbb{P}(W^*)$, the corresponding locally free resolution of sheaves (with the twisting degrees indicated) is, after twisting and base-changing,

$$\mathscr{F}_\bullet \;:=\; \ydiagram{1,1,1}(6) \leftarrow \ydiagram{6,1,1}(1) \xleftarrow{\star} \ydiagram{6,2,1} \leftarrow \ydiagram{6,2,2}(-1) \leftarrow \ydiagram{6,2,2,2}(-3),$$

where $\alpha(d)$ stands for the sheaf $\mathcal{O}(d) \otimes \mathbb{S}_\alpha(V) \otimes \mathrm{Sym}(V \otimes W^*)$ on $\mathbb{P}(W^*)$. The desired linear differential is marked with a $\star$. We put $\beta = (7, 1, 0)$ and tensor through by $\mathbb{S}_\beta(\mathcal{S})$

(note that $\mathcal{S}$ has rank 3). Let $\mathcal{M}$ be the resolved sheaf.

We run the hypercohomology spectral sequence for $\mathscr{F}_\bullet$. Since $\mathscr{F}_\bullet$ is quasi-isomorphic to $\mathcal{M}$ in homological degree 0, its hypercohomology is just $\mathbf{H}^q(\mathscr{F}_\bullet) \cong H^q(\mathcal{M})$. Note that this is zero if $q < 0$.

Observe that $\mathbb{S}_\beta(\mathcal{S})(d)$ has no cohomology when $d \in \{6, -1, -3\}$, but

$$H^1\big(\mathbb{S}_\beta(\mathcal{S})(1)\big) = \mathbb{S}_{621}(W), \qquad H^1\big(\mathbb{S}_\beta(\mathcal{S})\big) = \mathbb{S}_{611}(W).$$

So, the E_1 page of the spectral sequence is just:

$$\begin{array}{ccc}
- & - & - \\
\\
- & \mathbb{S}_{611}(V) \otimes \mathbb{S}_{621}(W) \otimes R \xleftarrow{\;f\;} & \mathbb{S}_{621}(V) \otimes \mathbb{S}_{611}(W) \otimes R, \\
\\
- & - & -
\end{array}$$

The dotted line is the main diagonal. All other E_1 terms are zero, so the sequence converges on E_2, giving $\mathbf{H}^0(\mathscr{F}) \cong \operatorname{coker}(f) \cong H^0(\mathcal{M})$ and $\mathbf{H}^{-1}(\mathscr{F}) \cong \ker(f) = 0$. This gives the desired sequence of R-modules:

$$0 \leftarrow H^0(\mathcal{M}) \leftarrow \mathbb{S}_{611}(V) \otimes \mathbb{S}_{621}(W) \otimes R \leftarrow \mathbb{S}_{621}(V) \otimes \mathbb{S}_{611}(W) \otimes R \leftarrow 0.$$

Remark 3.2.12. There are two straightforward ways to generalize the construction that we have sketched above. First, in the map marked $\star$ above, there is no reason to assume that the two partitions differ by a single box, and the same construction allows them to differ by multiple boxes as long as they are in the same row. In this case, the Pieri rule still implies that the map (3.2.2) is unique up to scalar.

Second, in the above example we chose β so that $\mathbb{S}_\beta(\mathcal{S})(d)$ has no cohomology for all d besides the twists appearing in the target and domain of a single differential (in this case, the one marked $\star$). Alternatively, we could choose β so that $\mathbb{S}_\beta(\mathcal{S})(d)$ has no cohomology for all but two of the terms in the complex (not necessarily consecutive terms). The end result is also a map of the form (3.2.2) where λ and μ differ by a connected border strip. In general the map (3.2.2) is not unique up to scalar, however.

3.2.3 Combinatorial setup

We have two partitions $\lambda \subsetneq \mu$ differing by a box. We write

$$\mu = (\mu_1, \ldots, \mu_k), \text{ and we assume } \mu_r > \mu_{r+1},$$
$$\lambda = \mu \text{ except for } \lambda_r = \mu_r - 1.$$

We define shapes $\alpha^{(i)}$, $i = 0, \ldots, k$, as follows. Consider the squares formed by:

- the inner border squares of μ inside rows $1, \ldots, r-1$,
- the rightmost square in row r of μ,
- the outer border squares of μ outside rows $r+1, \ldots, k$.

(See Figure 3.2.1.) Then $\alpha = \alpha^{(0)}$ is obtained by deleting all these squares, and $\alpha^{(i)}$ is obtained by including those squares in rows $1, \ldots, i$. Clearly, $\alpha^{(r)} = \mu$ and $\alpha^{(r-1)} = \lambda$.

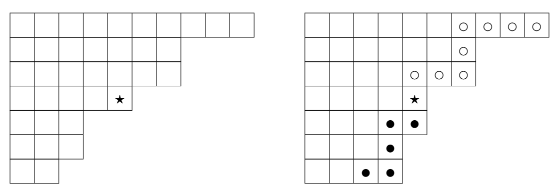


Figure 3.2.1: **Left:** The partition μ; the starred box in the 4th row is removed to form λ. **Right:** The outer strip is formed by connecting the *inner border squares* ($\circ$) in rows 1 to $r-1$ to the *outer border squares* ($\bullet$) outside rows $r+1, \ldots, k$. The empty squares form $\alpha^{(0)}$; then $\alpha^{(i)}$ is obtained by adding all marked squares ($\circ, *, \bullet$) up to row i. Note that $\alpha^{(3)} = \lambda$ and $\alpha^{(4)} = \mu$.

For each i, we let $d_i := \mu_1 - |\alpha^{(i)}/\alpha^{(0)}|$. In terms of μ, this is:

$$d_i := \begin{cases} \mu_{i+1} - i & \text{if } i = 0, \ldots, r-1, \\ \mu_r - r & \text{if } i = r, \\ \mu_i - i + 1 & \text{if } i = r+1, \ldots, k. \end{cases}$$

Finally, we define $\beta = (\mu_1+1, \ldots, \mu_{r-1}+1, \mu_{r+1}, \ldots, \mu_k)$. Note that this definition satisfies

$$\beta - (1, \ldots, k-1) = (d_0, \ldots, d_{r-2}, d_{r+1}, \ldots, d_k).$$

Let $\mathcal{S} \subset W \times \mathbb{P}(W^*)$ be the tautological rank-$(k-1)$ subbundle on $\mathbb{P}(W^*)$. We consider, for each d_i, the bundle $\mathbb{S}_\beta(\mathcal{S})(d_i)$. We check:

Lemma 3.2.13. *If $i \notin \{r-1, r\}$, then $\mathbb{S}_\beta(\mathcal{S})(d_i)$ has no cohomology. For $d = d_r$, the only nonvanishing cohomology of $\mathbb{S}_\beta(\mathcal{S})(d)$ is $H^{r-1} = \mathbb{S}_\mu(W)$. For $d = d_{r-1}$, the only nonvanishing cohomology is $H^{r-1} = \mathbb{S}_\lambda(W)$.*

Proof. We apply Borel–Weil–Bott: we have to sort

$$(d, \beta_1, \dots, \beta_k) - (0, 1, \dots, k-1) = (d, d_0, \dots, d_{r-2}, d_{r+1}, \dots, d_k).$$

If $i \notin \{r-1, r\}$, then d_i occurs twice, so there is no cohomology. For $i = r-1$ or r, sorting takes $r-1$ swaps, so in both cases H^{r-1} is nonvanishing. To see that the cohomology group is $\mathbb{S}_\mu(W)$ for d_{r-1} and $\mathbb{S}_\lambda(W)$ for d_r, we must check that

$$\begin{aligned} \mu &= (d_0, \dots, d_{r-2}, d_{r-1}, d_{r+1}, \dots, d_k) + (0, 1, \dots, k-1), \\ \lambda &= (d_0, \dots, d_{r-2}, d_r, d_{r+1}, \dots, d_k) + (0, 1, \dots, k-1) \end{aligned}$$

These are clear from the computation above. □

3.2.4 The proof of Theorem 3.2.6

Let $\alpha^{(i)}$ and d_i be defined as above. Consider the projective space $\mathbb{P}(W^*)$, with tautological line bundle $\mathcal{O}(-1) \subset W^*$ and rank-$(k-1)$ bundle $\mathcal{S} \subset W$. Set $\mathcal{E} := V \otimes \mathcal{O}(-1)$. By Theorem 3.2.9, we have an exact complex

$$F_0 \leftarrow \cdots \leftarrow F_i \leftarrow F_{i+1} \leftarrow \cdots, \text{ where } F_i = \mathbb{S}_{\alpha^{(i)}}(\mathcal{E}) \otimes \mathrm{Sym}(\mathcal{E}).$$

Note that

$$\mathbb{S}_\lambda(\xi) = \mathbb{S}_\lambda(V) \otimes \mathcal{O}(-|\lambda|).$$

For legibility, we write $\mathcal{O}(-\lambda)$ for $\mathcal{O}(-|\lambda|)$. Thus, we have a locally free resolution

$$\mathbb{S}_{\alpha^{(0)}}(V) \otimes \mathcal{O}(-\alpha^{(0)}) \otimes \mathrm{Sym}(\mathcal{E}) \leftarrow \cdots \leftarrow \mathbb{S}_{\alpha^{(i)}}(V) \otimes \mathcal{O}(-\alpha^{(i)}) \otimes \mathrm{Sym}(\mathcal{E}) \leftarrow \cdots$$

of sheaves of $\mathrm{Sym}(\mathcal{E})$-modules.

Next, let $\mathscr{R} = \mathcal{O}_{\mathbb{P}(W^*)} \otimes \mathrm{Sym}(V \otimes W^*)$. Observe that $\mathrm{Sym}(\xi) \hookrightarrow \mathscr{R}$ is a flat ring extension (locally it is an inclusion of polynomial rings). Now base change to $\mathscr{R}$, which preserves exactness. Finally, we tensor by $\mathbb{S}_\beta(\mathcal{S}) \otimes \mathcal{O}(\alpha^{(0)} + \mu_1)$. Our final complex has terms

$$\mathbb{S}_{\alpha^{(i)}}(V) \otimes \mathbb{S}_\beta(\mathcal{S})(d_i) \otimes \mathscr{R}.$$

Call the complex $\mathscr{F}_\bullet$ and let $\mathcal{M}$ be the sheaf it resolves.

We run the hypercohomology spectral sequence for $\mathscr{F}_\bullet$. Since $\mathscr{F}_\bullet$ is quasi-isomorphic to $\mathcal{M}$ in degree zero, its hypercohomology is just $\mathbf{H}^q(\mathscr{F}_\bullet) \cong H^q(\mathcal{M})$. The E_1 page of the spectral sequence has terms

$$H^q(\mathbb{S}_{\alpha^{(p)}}(V) \otimes \mathbb{S}_\beta(\mathcal{S})(d_p) \otimes \mathscr{R}) = \mathbb{S}_{\alpha^{(p)}}(V) \otimes H^q\big(\mathbb{S}_\beta(\mathcal{S})(d_p)\big) \otimes \mathscr{R}.$$

(We emphasize that $\mathbb{S}_{\alpha^{(p)}}(V)$ and $\mathscr{R}$ are trivial bundles.) By Lemma 3.2.13, the middle tensor factor is zero unless $p = r-1, r$, in which case the nonvanishing term is H^{r-1}, with

$$H^{r-1}\big(\mathbb{S}_\beta(\mathcal{S})(d_{r-1})\big) = \mathbb{S}_\mu(W), \qquad H^{r-1}\big(\mathbb{S}_\beta(\mathcal{S})(d_r)\big) = \mathbb{S}_\lambda(W).$$

In particular, the E_1 page contains only the map

$$\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(W) \otimes R \xleftarrow{f} \mathbb{S}_\mu(V) \otimes \mathbb{S}_\lambda(W) \otimes R,$$

with the left term located on the main diagonal. Since there are no other E_1 terms, the sequence converges on E_2 and so $\mathbf{H}^0(\mathscr{F}) \cong \operatorname{coker}(f) \cong H^0(\mathcal{M})$ and $\mathbf{H}^{-1}(\mathscr{F}) \cong \ker(f) = 0$. In particular, the map above is a resolution of $H^0(\mathcal{M})$ by free R-modules. $\square$

CHAPTER 4

The general case: rectangular matrices

4.1 The equivariant Herzog-Kühl equations

In this section we derive the equivariant analogue of the Herzog-Kühl equations. This will be a system of linear conditions on the entries of an equivariant Betti table. It will detect when the resolved module M is supported only along the locus of rank-deficient matrices.

4.1.1 K-theory rings

For background on equivariant K-theory, we refer to the original paper by Thomason [31]; a more recent discussion is [28]. We recall that, if X is a smooth variety, its K-theory $K(X)$ is the free abelian group on the isomorphism classes of coherent sheaves[1] $\mathcal{E}$ on X, modulo the relation $[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$ for all short exact sequences

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0.$$

If X has an action of an algebraic group G, its equivariant K-theory $K^G(X)$ is defined the same way, only the sheaves and sequences are required to be G-equivariant. If $X = \mathrm{Spec}(R)$ is affine, we will say 'R-module' instead of 'sheaf on X'.

We will study the $GL(V)$-equivariant K-theory of the affine space $\mathrm{Hom}(V, W)$. We consider the decomposition of $\mathrm{Hom}(V, W)$ into the closed locus X_{k-1} of rank-deficient matrices and the open complement U of full-rank matrices. Excision in equivariant K-theory ([31, Theorem 2.7]) gives the right-exact sequence of abelian groups

$$K^{GL(V)}(X_{k-1}) \xrightarrow{i_*} K^{GL(V)}(\mathrm{Hom}(V, W)) \xrightarrow{j^*} K^{GL(V)}(U) \to 0.$$

[1]Technically we are defining $K_0(X)$, while $K^0(X)$ is defined using only vector bundles. Since all our spaces will be smooth, the natural map $K^0(X) \to K_0(X)$ is an isomorphism, so we will ignore the difference.

The pullback j^*, induced by the open inclusion $j : U \hookrightarrow X$, is a map of rings. The pushforward i_*, induced by the closed embedding $i : X_{k-1} \hookrightarrow X$, is only a map of abelian groups. Its image is the ideal I generated by the classes of modules supported along X_{k-1}.

We do not attempt to describe the first term, though we will describe (below) its image under i_*. The second term is, by ([31, Theorem 4.1] or [28, Example 2 and Corollary 12]),

$$K^{GL(V)}(\mathrm{Hom}(V, W)) \cong \mathbb{Z}[t_1^{\pm}, \ldots, t_k^{\pm}]^{S_k},$$

the ring of symmetric Laurent polynomials in k variables (essentially the representation ring of $GL(V)$). Here, the class of the equivariant R-module $\mathbb{S}_\lambda(V) \otimes_{\mathbb{C}} R$ is identified with the Schur polynomial $s_\lambda(t_1, \ldots, t_k)$. If M is a finitely-generated R-module, its equivariant minimal free resolution expresses the K-class $[M]$ as a finite alternating sum of Schur polynomials. In other words, the equivariant Betti table determines the K-class:

$$[M] = \sum_{i,\lambda} (-1)^i \beta_{i,\lambda}(M) s_\lambda(t).$$

An equivalent approach is to write

$$M \cong \bigoplus_\lambda \mathbb{S}_\lambda(V)^{c_\lambda(M)} \text{ as a } GL(V)\text{-representation},$$

and define the equivariant Hilbert series of M,

$$\begin{aligned} H_M(t) &= \sum_\lambda c_\lambda(M) s_\lambda(t) \\ &= \frac{f(t)}{\prod_{i=1}^k (1 - t_i)^n} \end{aligned}$$

for some symmetric function $f(t)$. Then $f(t)$ is the K-theory class of M. (If we forget the $GL(V)$ action and remember only the grading of M, we recover the usual Hilbert series.)

To see that these definitions agree, note that the second definition is additive in short exact sequences, hence is well-defined on K-classes. Replacing M by its equivariant minimal free resolution, it suffices to consider indecomposable free modules $M = \mathbb{S}_\lambda(V) \otimes_{\mathbb{C}} R$. This tensor product multiplies the entire series by $s_\lambda(t)$ and does the same to $f(t)$, so it suffices to consider $M = R$. For this case, we have

$$R = \mathrm{Sym}(V \otimes \mathbb{C}^n) \cong \mathrm{Sym}(\overbrace{V \oplus \cdots \oplus V}^{n \text{ times}}) = \mathrm{Sym}(V)^{\otimes n},$$

and the equivariant Hilbert series of $\mathrm{Sym}(V)$ is

$$\sum_d h_d(t) = \prod_{i=1}^{k} \frac{1}{(1-t_i)},$$

where $h_d(t)$ is the homogeneous symmetric polynomial (corresponding to $\mathrm{Sym}^d(V)$). Thus $H_R(t)$ is the n-th power of the above.

Remark 4.1.1. It will be convenient to restrict to modules M generated in positive degree, i.e. $\beta_{i,\lambda}(M) \neq 0$ implies $\lambda \geqslant 0$. In this case, the class of M is a polynomial, not a Laurent polynomial. We write $K_+^{GL(V)}(\mathrm{Hom}(V,W))$ for this subring.

Finally, we have for the third term (cf. [31, Proposition 6.2])

$$K^{GL(V)}(U) \cong K(U/GL(V)) = K(Gr(k,W)),$$

because the action of $GL(V)$ is free on U. The structure of this ring is well-known from K-theoretic Schubert calculus (e.g. [23] or [4]). We will only need to know the following: it is a free abelian group with an additive basis consisting of $\binom{n}{k}$ generators, indexed by partitions μ fitting inside a $k \times (n-k)$ rectangle. These correspond to the classes $[\mathcal{O}_\mu]$ of structure sheaves of Schubert varieties. It is easy to check that $K_+^{GL(V)}(\mathrm{Hom}(V,W)) \to K_0(Gr(k,W))$ is also surjective (because, e.g., matrix Schubert varieties are generated in positive degree).

4.1.2 Modules on the rank-deficient locus and the equivariant Herzog-Kühl equations

From the surjection $K_+^{GL(V)}(\mathrm{Hom}(V,W)) \to K_0(Gr(k,W))$, we see that the ideal

$$I' := I \cap K_+^{GL(V)}(\mathrm{Hom}(V,W)),$$

as a linear subspace, has co-rank $\binom{n}{k}$. We wish to find exactly this many linear equations cutting out the ideal, indexed appropriately by partitions. That is, given a K-class written in the Schur basis,

$$f = \sum_{\lambda \geqslant 0} a_\lambda s_\lambda \in K_+^{GL(V)}(\mathrm{Hom}(V,W)),$$

we wish to have coefficients $b_{\lambda\mu}$ for each $\mu \subseteq \boxplus$, such that

$$f \in I' \text{ if and only if } \sum_{\lambda \geqslant 0} a_\lambda b_{\lambda\mu} = 0 \text{ for all } \mu \subseteq \boxplus .$$

We will then apply these equations in the case where f is the class of a module M, and

$$a_\lambda = \sum_i (-1)^i \beta_{i,\lambda}(M)$$

comes from the equivariant Betti table of M. Our approach is to prove the following:

Theorem 4.1.2. *We have* $I' = \mathrm{span}_{\mathbb{C}}\{s_\lambda(1-t) : \lambda \nsubseteq \boxplus\}$.

We will prove Theorem 4.1.2 in the next section. Here is how it leads to the desired equations. Let $b_{\lambda\mu}$ be the change-of-basis coefficients defined by sending $t_i \mapsto 1 - t_i$. So, by definition,

$$s_\lambda(1-t) = \sum_\mu b_{\lambda\mu} s_\mu(t).$$

Note that we have, equivalently,

$$s_\lambda(t) = \sum_\mu b_{\lambda\mu} s_\mu(1-t).$$

Thus

$$f = \sum_\lambda a_\lambda s_\lambda(t) = \sum_{\lambda,\mu} a_\lambda b_{\lambda\mu} s_\mu(1-t).$$

The polynomials $s_\mu(1-t)$ for all $\mu \geqslant 0$ form an additive basis for the $K_+^{GL(V)}(\mathrm{Hom}(V,W))$. Thus, $f \in I'$ if and only if the coefficient of $s_\mu(1-t)$ is 0 for all $\mu \subseteq \boxplus$. That is,

$$0 = \sum_\lambda a_\lambda b_{\lambda\mu} \text{ for all } \mu \subseteq \boxplus .$$

The following description of $b_{\lambda\mu}$ is due to Stanley. Recall that, if $\mu \subseteq \lambda$ are partitions, the ***skew shape*** λ/μ is the Young diagram of λ with the squares of μ deleted. A ***standard Young tableau*** is a filling of a (possibly skew) shape by the numbers $1, 2, \ldots, t$ (with t boxes in all), such that the rows increase from left to right, and the columns increase from top to bottom. We write f^σ for the number of standard Young tableaux of shape σ.

Proposition 4.1.3. *[30] If $\mu \nsubseteq \lambda$ then $b_{\lambda\mu} = 0$. If $\mu \subseteq \lambda$, then*

$$b_{\lambda\mu} = (-1)^{|\mu|} \frac{f^{\lambda/\mu} f^\mu}{f^\lambda} \binom{|\lambda|}{|\mu|} \frac{d_\lambda(k)}{d_\mu(k)}.$$

An equivalent formulation is

$$b_{\lambda\mu} = (-1)^{|\mu|} \frac{f^{\lambda/\mu}}{|\lambda/\mu|!} \prod_{(i,j)\in\lambda/\mu} (k + j - i).$$

Corollary 4.1.4 (Equivariant Herzog-Kühl equations)**.** *Let M be an equivariant R-module with equivariant Betti table $\beta_{i,\lambda}$. Assume M is generated in positive degree.*

The set-theoretic support of M is contained in the rank-deficient locus if and only if:

$$\text{For each } \mu \subseteq \boxplus: \quad \sum_{i,\lambda \supseteq \mu} (-1)^i \underbrace{\beta_{i,\lambda} d_\lambda(k)}_{(=\widetilde{\beta_{i,\lambda}})} \frac{f^{\lambda/\mu} f^\mu}{f^\lambda} \binom{|\lambda|}{|\mu|} = 0. \tag{4.1.1}$$

Note that $\beta_{i,\lambda}$ is the **multiplicity** *of the λ-isotypic component of the resolution of M (in cohomological degree i), whereas $\beta_{i,\lambda} d_\lambda(k) = \widetilde{\beta_{i,\lambda}}$ is the* **rank** *of this isotypic component.*

Proof. ($\Rightarrow$): The only thing to note is that, for simplicity, we have rescaled the μ-indexed equation by $(-1)^{|\mu|} d_\mu(k)$.

($\Leftarrow$): Let $\mathcal{O}(1)$ be the ample line bundle corresponding to the Plücker embedding of $Gr(k, W)$. (Any ample line bundle will do; we are really just using that $Gr(k, W)$ is projective.) Let $\widetilde{M}$ be the sheaf induced by M on $Gr(k, W)$. Since $\mathcal{O}(1)$ is ample, $\widetilde{M} \otimes \mathcal{O}(d)$ is globally generated and has no higher cohomology when $d \gg 0$.

If the equations are satisfied, then $\widetilde{M}$ corresponds to the trivial K-theory class on $Gr(k, W)$, as does $\widetilde{M} \otimes \mathcal{O}(d)$ for any d. In particular $\chi(\mathcal{M} \otimes \mathcal{O}(d)) = 0$ for all d. But then $H^0(\widetilde{M} \otimes \mathcal{O}(d)) = 0$, so by global generation $\widetilde{M} = 0$. This implies the support restriction. $\square$

The coefficient in Equation (4.1.1) has the following interpretation. Consider a uniformly-random filling T of the shape λ by the numbers $1, \ldots, |\lambda|$. Say that T ***splits along*** $\mu \sqcup \lambda/\mu$ if the numbers $1, \ldots, |\mu|$ lie in the subshape μ. Then:

$$\frac{f^{\lambda/\mu} f^\mu}{f^\lambda} \binom{|\lambda|}{|\mu|} = \frac{\mathrm{Prob}(T \text{ splits along } \mu \sqcup \lambda/\mu \mid T \text{ is standard})}{\mathrm{Prob}(T \text{ splits along } \mu \sqcup \lambda/\mu)}. \tag{4.1.2}$$

4.1.3 Proof of Theorem 4.1.2

First, we recall the following fact about K-theory of Grassmannians. It is essentially a consequence of the tautological exact sequence of vector bundles on $Gr(k, W)$,

$$0 \to \mathcal{S} \to W \to \mathcal{Q} \to 0.$$

Proposition 4.1.5 ([12], page 21)**.** *The following identity holds of formal power series over $K(Gr(k, W))$:*

$$\left(\sum_p [\textstyle\bigwedge^p \mathcal{S}] u^p\right) \cdot \left(\sum_q [\textstyle\bigwedge^q \mathcal{Q}] u^q\right) = (1+u)^n.$$

We rearrange the equality as

$$\Big(\sum_q \big[\textstyle\bigwedge^q \mathcal{Q}\big]u^q\Big) = (1+u)^n \cdot \frac{1}{\Big(\sum_p \big[\bigwedge^p \mathcal{S}\big]u^p\Big)}$$
$$= (1+u)^n \cdot \sum_p \big(-1)^p [\mathrm{Sym}^p(\mathcal{S})\big]u^p.$$

The key observation is that the left-hand side is a polynomial in u of degree $n-k$, since $\mathcal{Q}$ has rank $n-k$. Thus, the coefficient f_ℓ of u^ℓ of the right-hand side vanishes for $\ell \geqslant n-k$. In other words, viewing f_ℓ as a symmetric polynomial, we have $f_\ell \in I'$ for $\ell > n-k$.

We compute the coefficient f_ℓ. Recall that $[\mathrm{Sym}^p(\mathcal{S})] = h_p$, the p-th homogeneous symmetric polynomial. We have

$$\begin{aligned}
\sum_\ell f_\ell u^\ell &= (1+u)^n \sum_p (-1)^p h_p u^p \\
&= \sum_{q=0}^{n} \sum_{p=0}^{\infty} u^{p+q}(-1)^p h_p \binom{n}{q} \\
&= \sum_{\ell=0}^{\infty} u^\ell \sum_{p=\ell-n}^{\ell} (-1)^p h_p \binom{n}{\ell-p},
\end{aligned}$$

so our desired coefficients are

$$f_\ell = \sum_{p=\ell-n}^{\ell} (-1)^p h_p \binom{n}{\ell-p},$$

where in the last two lines we use the convention $h_p = 0$ for $p < 0$. We next show:

Lemma 4.1.6. *We have $I' \supseteq (h_{n-k+1}(1-t), \ldots, h_n(1-t))$.*

Proof. Equivalently, we change basis $t \mapsto 1-t$, calling the (new) ideal J, and we show

$$J \supseteq (h_{n-k+1}, \ldots, h_n).$$

We consider the elements $f_{n-k+i}(1-t) \in J$ for $i = 1, \ldots, k$.

$$f_{n-k+i}(1-t) = \sum_{p=-k+i}^{n-k+i} (-1)^p h_p(1-t) \binom{n}{n-k+i-p}.$$

Since $i \leqslant k$ we have

$$= \sum_{p=0}^{n-k+i} (-1)^p h_p(1-t) \binom{n}{n-k+i-p}.$$

We apply the second formula from Proposition 4.1.3. Note that all terms are single-row partitions, $\lambda = (p)$ and $\mu = (s)$, with $s \leqslant p$, so $f^{\lambda/\mu} = 1$ and the change of basis is:

$$b_{\lambda\mu} = (-1)^s \frac{f^{\lambda/\mu}}{|\lambda/\mu|!} \cdot (k+s)\cdots(k+p-1) = (-1)^s \binom{k+p-1}{k+s-1}.$$

Hence,

$$\begin{aligned} f_{n-k+i}(1-t) &= \sum_{p=0}^{n-k+i} \sum_{s=0}^{p} (-1)^{p+s} \binom{n}{n-k+i-p} \binom{k+p-1}{k+s-1} h_s. \\ &= \sum_{s=0}^{n-k+i} (-1)^s h_s \sum_{p=s}^{n-k+i} (-1)^p \binom{n}{n-k+i-p} \binom{k+p-1}{k+s-1}. \end{aligned}$$

We reindex, sending $p \mapsto n-k+i-p$, and reverse the order of the inner sum:

$$= (-1)^{n-k+i} \sum_{s=0}^{n-k+i} (-1)^s h_s \sum_{p=0}^{n-k+i-s} (-1)^p \binom{n}{p} \binom{n+i-p-1}{k+s-1}.$$

The terms h_s for $s \leqslant n-k$. First, we show that all the lower terms h_s, with $s \leqslant n-k$, vanish. For these terms, we view the large binomial coefficient as a polynomial function of p. It has degree $k+s-1$, with zeroes at $p = (n-k+i-s)+1, \ldots, n+i-1$, so we may freely include these terms in the inner sum. It is convenient to extend the inner sum only as far as $p = n$, obtaining

$$\sum_{p=0}^{n} (-1)^p \binom{n}{p} \binom{n-i-p-1}{k+s-1}.$$

Recall from the theory of finite differences that

$$\sum_{p=0}^{d} (-1)^p \binom{d}{p} g(p) = 0$$

whenever g is a polynomial of degree $< d$. Since the above sum has degree $k+s-1 \leqslant n-1$, it vanishes. Thus, dropping the lower terms, we are left with

$$f_{n-k+i}(1-t) = (-1)^i \sum_{s=1}^{i} (-1)^s h_{n-k+s} \sum_{p=0}^{i-s} (-1)^p \binom{n}{p} \binom{n+i-p-1}{n+s-1}. \tag{4.1.3}$$

Showing $h_{n-k+i} \in J$ for $i = 1, \ldots, k$. From equation (4.1.3), we see directly that the coefficient of h_{n-k+i} in $f_{n-k+i}(1-t)$ is 1. This is the leading coefficient, so the claim follows by induction on i. $\square$

Corollary 4.1.7. *We have*

$$J = (h_i : i > n-k) = \operatorname{span}_{\mathbb{C}}\{s_\lambda : \lambda \nsubseteq \boxplus\}.$$

Proof. The equality of ideals

$$(h_{n-k+1}, \ldots, h_n) = (h_i : i > n-k)$$

follows from Newton's identities and induction. The equality

$$(h_i : i > n-k) = \operatorname{span}_{\mathbb{C}}\{s_\lambda : \lambda \nsubseteq \boxplus\}$$

follows from the Pieri rule (for $\subseteq$) and the Jacobi-Trudi formula (for $\supseteq$). See [16] for these identities. This shows J *contains* this linear span. But then quotienting by J leaves at most $\binom{n}{k}$ classes. This is already the rank of $K(Gr(k,W))$, so we must have equality. $\square$

Changing bases $t \mapsto 1-t$ a final time completes the proof of Theorem 4.1.2.

4.2 The pairing between Betti tables and cohomology tables

In this section, we establish the numerical pairing between Betti tables and cohomology tables. We recall that the pairing is defined as follows (Definition 1.5.8):

$$\begin{aligned} \widetilde{\Phi} : \mathbb{BT}_{k,n} \times \mathbb{CT}_{k,n} &\to \widetilde{\mathbb{BT}}_{k,k}, \\ (\beta, \gamma) &\mapsto \widetilde{\Phi}(\beta, \gamma), \end{aligned} \tag{4.2.1}$$

with $\widetilde{\Phi}$ the (derived) rank Betti table with entries

$$\widetilde{\varphi_{i,\lambda}}(\beta,\gamma) = \sum_{p-q=i} \beta_{p,\lambda} \cdot \gamma_{q,\lambda}. \tag{4.2.2}$$

Recall also that the convention is that homological degree (p and i) decreases under the boundary map of the complex.

The remainder of this section is devoted to the proof of the following:

Theorem 4.2.1 (Pairing the equivariant Boij-Söderberg cones)**.** *The pairing* $\widetilde{\Phi}$ *restricts to a pairing of cones,*

$$BS^D_{k,n} \times ES_{k,n} \to \widetilde{BS}^D_{k,k}.$$

In light of our description (Theorem 3.1.12) of the derived cone $\widetilde{BS}^D_{k,k}$, the goal will be to exhibit a perfect matching on the Betti graph of $\widetilde{\Phi}(\beta,\gamma)$. Along the way, we will also complete the proof of Theorem 3.1.12 itself, showing that $\widetilde{BS}^D_{k,k}$ is characterized by the existence of such matchings (Corollary 4.2.10).

Remark 4.2.2. The pairing is based on the hypercohomology spectral sequence for a complex of sheaves $\mathscr{F}^\bullet$. The proof, however, relies on an explicit realization of this spectral sequence via a double complex (taking an injective resolution of $\mathscr{F}^\bullet$, then taking sections). Any injective resolution that is functorial in the underlying maps of sheaves will do; we use the Cech complex.

Proof. Let $\beta = \beta(F^\bullet)$ be the Betti table of a minimal free equivariant complex $F^\bullet$ of finitely-generated R-modules, with $R = R_{k,n}$ the coordinate ring of the $k \times n$ matrices. Assume $F^\bullet$ is exact away from the locus of rank-deficient matrices, so descending $F^\bullet$ to $Gr(k,\mathbb{C}^n)$ gives an exact sequence of vector bundles $\mathscr{F}^\bullet$:

$$F^\bullet = \bigoplus_\lambda \mathbb{S}_\lambda(V)^{\beta_{\bullet,\lambda}} \otimes R \qquad \xrightarrow{\text{descends to}} \qquad \mathscr{F}^\bullet = \bigoplus_\lambda \mathbb{S}_\lambda(\mathcal{S})^{\beta_{\bullet,\lambda}}, \tag{4.2.3}$$

with $\mathcal{S}$ the tautological subbundle on $Gr(k,\mathbb{C}^n)$. Let $\gamma = \gamma(\mathcal{E})$ be the GL-cohomology table of a coherent sheaf $\mathcal{E}$ on $Gr(k,n)$. Observe that $\mathcal{E} \otimes \mathscr{F}^\bullet$ is again exact.

We study the hypercohomology spectral sequence for $\mathcal{E} \otimes \mathscr{F}^\bullet$. Let $E^{\bullet,\bullet}$ be the result of taking the Cech resolution, then taking global sections: a double complex of vector spaces, with $\operatorname{Tot}(E^{\bullet,\bullet})$ exact (since, by exactness of $\mathcal{E} \otimes \mathscr{F}^\bullet$, the spectral sequence abuts to zero).

By functoriality of the Cech complex, each term splits as a direct sum $E^{\bullet,\bullet} = \bigoplus_\lambda E^{\bullet,\bullet,\lambda}$ according to the λ summands in (4.2.3), while the differentials satisfy

$$d_v(E^{\bullet,\bullet,\lambda}) \subset E^{\bullet,\bullet+1,\lambda} \qquad \text{and} \qquad d_h(E^{\bullet,\bullet,\lambda}) \subseteq \bigoplus_{\mu \lesssim \lambda} E^{\bullet-1,\bullet,\mu}.$$

(The statement about d_h also uses minimality of the original complex $F^\bullet$.) We run the spectral sequence for $E^{\bullet,\bullet}$ beginning with the vertical maps, giving on the E_1 page,

$$E_1^{p,q} = \bigoplus_\lambda H^q(\mathcal{E} \otimes \mathbb{S}_\lambda(\mathcal{S}))^{\beta_{p,\lambda}}.$$

Observe that the λ summand has dimension $\beta_{p,\lambda}(F^\bullet)\gamma_{q,\lambda}(\mathcal{E})$. The (i,λ) coefficient produced in the Boij-Söderberg pairing, $\widetilde{\varphi_{i,\lambda}}(F^\bullet, \mathcal{E})$, is the sum of this quantity along the diagonal $\{p - q = i\}$. That is, $\widetilde{\Phi}$ is akin to a Betti table for $\mathrm{Tot}(E_1)$:

$$\widetilde{\varphi_{i,\lambda}} = \dim_{\mathbb{C}} \mathrm{Tot}(E_1)_{i,\lambda}.$$

We emphasize, however, that there is no actual GL_k-action on $\mathrm{Tot}(E_1)$, nor an $R_{k,k}$-module structure. Instead, we will show by homological techniques that, for a wide class of double complexes including $E^{\bullet,\bullet}$, there is a perfect matching on a graph associated to $\mathrm{Tot}(E_1)$; in our setting, this will give the desired perfect matching on the Betti graph of $\widetilde{\Phi}$. $\square$

The key properties of the double complex $E^{\bullet,\bullet}$ constructed above are that

(1) Each term $E^{p,q}$ has a direct sum decomposition labeled by a poset P;

(2) The vertical maps d_v are label-preserving;

(3) The horizontal maps are strictly-label-decreasing.

By (1) and (2), the E_1 page (the homology of d_v) again has a direct sum decomposition labeled by P, $E_1^{p,q} = \bigoplus_{\lambda \in P} E_1^{p,q,\lambda}$. We define the following graph:

Definition 4.2.3. The $\mathbf{E_1}$ ***graph*** $G = G(E^{\bullet,\bullet})$ is the following directed graph:

- The vertex set has $\dim(E_1^{p,q,\lambda})$ vertices labeled (p, q, λ), for each $p, q \in \mathbb{Z}$ and $\lambda \in P$;
- The edge set includes all possible edges $(p, q, \lambda) \to (p', q', \lambda')$ whenever $\lambda' \lneq \lambda$ and $(p', q') = (p - r, q - r + 1)$ for some $r > 0$.

The edges of G respect the strictly-decreasing-P-labels condition, and are shaped like the higher-order differentials of the associated spectral sequence.

We show:

Theorem 4.2.4. *Let $E^{\bullet,\bullet}$ be a double complex of vector spaces satisfying (1)-(3). If $\mathrm{Tot}(E^{\bullet,\bullet})$ is exact, its E_1 graph has a perfect matching.*

We think of this theorem as a combinatorial analog of the fact that the associated spectral sequence (beginning with the homology of d_v) converges to zero. We explore this idea further in Section 4.2.1.

Finishing the proof of Theorem 4.2.1. With $E^{\bullet,\bullet}$ as above, we identify the vertices of the E_1 graph and the Betti graph of $\widetilde{\Phi}(\beta, \gamma)$; for any such identification, the edges of the E_1 graph become a subset of the Betti graph's edges. (We may recover the missing edges by allowing $r \leqslant 0$ in Definition 4.2.3.) Hence, the perfect matching produced by Theorem 4.2.4 is valid for the Betti graph, completing the proof of Theorem 4.2.1. $\square$

4.2.1 Perfect matchings in linear and homological algebra

Our approach uses linear maps to produce perfect matchings. The starting point is the following construction:

Definition 4.2.5. Let $T : V \to W$ be a map of vector spaces, having specified bases $\mathcal{V}, \mathcal{W}$. The ***coefficient graph*** G of T is the directed bipartite graph with vertex set $\mathcal{V} \sqcup \mathcal{W}$ and edges

$$E = \{v \to w : T(v) \text{ has a nonzero } w\text{-coefficient}\}.$$

Note that the adjacency matrix of G is T with all nonzero coefficients replaced by 1's.

Proposition 4.2.6. *For finite-dimensional vector spaces, the coefficient graph of an isomorphism admits a perfect matching.*

We will say the corresponding bijection $\mathcal{V} \leftrightarrow \mathcal{W}$ is ***compatible with*** T, a combinatorial analog of the fact that T is an isomorphism. The proof of existence is simple, but essentially nonconstructive in practice. Here are two ways to do it:

(i) All at once: since $\det(T) \neq 0$, some monomial term of $\det(T)$ is nonzero. This exhibits the perfect matching.

(ii) By induction, using the Laplace expansion: expand $\det(T)$ along a row or column; some term $a_{ij} \cdot$ (complementary minor) is nonzero, and so on.

Method (ii) actually satisfies a slightly stronger condition: the resulting matching is compatible with both T and T^{-1} (since, up to scaling by $\det(T)$, the complementary minors are the entries of the inverse matrix.)

Similarly, if T is merely assumed to be injective or surjective, we may produce a maximal matching in this way (choose some nonvanishing maximal minor).

We generalize Proposition 4.2.6 to the setting of homological algebra in three ways: to infinite-dimensional vector spaces, to long exact sequences, and to double complexes (motivated by spectral sequences).

Proposition 4.2.7. *For vector spaces of arbitrary dimension, the coefficient graph of an isomorphism admits a perfect matching.*

We will not need Proposition 4.2.7 (which uses the axiom of choice) for our proof of Theorem 4.2.4, so we prove it in the appendix. See Remark 4.2.14 for additional discussion on the role of the axiom of choice.

4.2.1.1 Long exact sequences and the proof of Theorem 3.1.12

We generalize to the case of long exact sequences. Let

$$\cdots \leftarrow V_i \xleftarrow{\delta} V_{i+1} \leftarrow \cdots$$

be a long exact sequence, with $\mathcal{V}_i$ a fixed basis for V_i. (The vector spaces may be finite- or infinite-dimensional.)

Definition 4.2.8. The ***coefficient graph*** G for $(V_\bullet, \delta)$ (with respect to $\mathcal{V}_\bullet$) is the directed graph with vertex set $\bigsqcup_i \mathcal{V}_i$ and an edge $v \to v'$ whenever $\delta(v)$ has a nonzero v'-coefficient.

Proposition 4.2.9. *The coefficient graph of a long exact sequence has a perfect matching.*

Proof. Choose subsets $\mathcal{F}_i \subset \mathcal{V}_i$ descending to bases of $\mathrm{im}(\delta) \subset V_{i-1}$, using Zorn's Lemma in the infinite case. Let $\mathcal{G}_i = \mathcal{V}_i - \mathcal{F}_i$, and let $F_i = \mathrm{span}(\mathcal{F}_i)$ and $G_i = \mathrm{span}(\mathcal{G}_i)$. The composition $\tilde{\delta} : F_{i+1} \hookrightarrow V_{i+1} \xrightarrow{\delta} V_i \twoheadrightarrow G_i$ is an isomorphism and has the same coefficients as δ, restricted to $\mathcal{F}_{i+1}$ and $\mathcal{G}_i$. This can be checked using the snake lemma on

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & F_{i+1} \arrow[r, equal] \arrow[d] & F_{i+1} \arrow[r] \arrow[d, "\tilde{\delta}"] & 0 \\
0 \arrow[r] & F_i \arrow[r] & V_i \arrow[r] & G_i \arrow[r] & 0,
\end{tikzcd}$$

using exactness and our choices of F_i. Thus Proposition 4.2.6 (or 4.2.7 in the infinite case) yields a matching of $\mathcal{F}_{i+1}$ with $\mathcal{G}_i$. □

At this point, we complete the proof of Theorem 3.1.12, characterizing the derived Boij-Söderberg cone $\widetilde{BS}^D_{k,k}$ of the square matrices.

Corollary 4.2.10. *If $\widetilde{\beta} \in \widetilde{BS}^D_{k,k}$, then the Betti graph $G(\widetilde{\beta})$ has a perfect matching.*

Proof. Let $\widetilde{\beta}$ be the Betti table of a minimal free equivariant complex $(F^\bullet, \delta)$ of finitely-generated R-modules, with $R = R_{k,k}$ the coordinate ring of the $k \times k$ matrices, and $F^\bullet \otimes R[\frac{1}{\det}]$ exact.

Choose, for each F^i, a $\mathbb{C}$-basis of each copy of $\mathbb{S}_\lambda(V)$ occuring in F^i. Label each basis element x by the corresponding partition λ. It follows from minimality that $\delta(x_\lambda)$ is an R-linear combination of basis elements labeled by partitions $\lambda' \lneq \lambda$.

Since the homology modules are torsion, $F_\bullet \otimes \mathrm{Frac}(R)$ is an exact sequence of $\mathrm{Frac}(R)$-vector spaces, with bases given by the x_λ's chosen above. By the previous proposition, its coefficient graph has a perfect matching. This graph has the same vertices as the Betti graph $G(\widetilde{\beta})$, and its edges are a subset of $G(\widetilde{\beta})$'s edges. $\square$

Remark 4.2.11. Rather than tensoring with $\mathrm{Frac}(R)$, we may instead specialize to any convenient invertible $k \times k$ matrix $T \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^k)$, such as the identity matrix. This approach is useful for computations, since the resulting exact sequence consists of finite-dimensional $\mathbb{C}$-vector spaces.

4.2.1.2 Double complexes and the proof of Theorem 4.2.1

Finally, we generalize to the setting of double complexes and spectral sequences. Let $(E^{\bullet,\bullet}, d_v, d_h)$ be a double complex of vector spaces, with differentials pointing up and to the left:

$$\begin{tikzcd}
 & \\
\arrow[u, dashed, "q \text{ axis}"] \arrow[r, dashed, "p \text{ axis}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
E^{p-1,q+1} & E^{p,q+1} \arrow[l, "d_h"'] \\
E^{p-1,q} \arrow[u, "d_v"] & E^{p,q} \arrow[l, "d_h"] \arrow[u, "d_v"']
\end{tikzcd}$$

We assume the squares anticommute, so the total differential is

$$d_{tot} = d_h + d_v, \qquad \text{and} \qquad d_h d_v + d_v d_h = 0.$$

We will always assume the total complex $\mathrm{Tot}(E^{\bullet,\bullet})$ has a finite number of columns. Note that we do *not* assume a basis has been specified for each $E^{\bullet,\bullet}$. We recall the complexes $E^{\bullet,\bullet}$ of interest:

(1) Each term $E^{p,q}$ has a direct sum decomposition

$$E^{p,q} = \bigoplus_{\lambda \in P} E^{p,q,\lambda},$$

with labels λ from a poset P.

(2) The vertical differential d_v is *graded* with respect to this labeling, and

(3) The horizontal differential d_h is *downwards-filtered.*

The conditions (2) and (3) mean that

$$d_v(E^{p,q,\lambda}) \subseteq E^{p,q+1,\lambda}, \text{ and } d_h(E^{p,q,\lambda}) \subseteq \bigoplus_{\lambda' \lneq \lambda} E^{p-1,q,\lambda'},$$

so the vertical differential preserves the label and the horizontal differential strictly decreases it.

We are interested in the homology of the vertical map d_v. Since d_v is P-graded, so is its homology $E_1^{p,q,\lambda} = H(d_v)^{p,q,\lambda}$. We recall that the $\mathbf{E_1}$ ***graph*** $G(E^{\bullet,\bullet})$ is defined as follows:

- The vertex set has $\dim(E_1^{p,q,\lambda})$ vertices labeled (p, q, λ) for each $p, q \in \mathbb{Z}$ and $\lambda \in P$;
- The edge set includes all possible edges $(p, q, \lambda) \to (p', q', \lambda')$ whenever $\lambda' \lneq \lambda$ and $(p', q') = (p - r, q - r + 1)$ for some $r > 0$.

The edges of G respect the strictly-decreasing-P-labels condition, and moreover are shaped like higher-order differentials of the associated spectral sequence, i.e. they point downwards-and-leftwards. We wish to show:

Theorem 4.2.12. *If* $\mathrm{Tot}(E^{\bullet,\bullet})$ *is exact, the* E_1 *graph of* $E^{\bullet,\bullet}$ *has a perfect matching.*

Remark 4.2.13. Consider summing the E_1 page along diagonals. Call the resulting complex $\mathrm{Tot}(E_1)$. If it were exact, the matching would exist by Proposition 4.2.9, and in fact would only use the edges corresponding to $r = 1$. Since $\mathrm{Tot}(E_1)$ is not exact in general, the proof works by modifying its maps to make it exact.

Explicitly, we will exhibit a quasi-isomorphism from $\mathrm{Tot}(E^{\bullet,\bullet})$ to a complex with the same terms as $\mathrm{Tot}(E_1)$, but different maps – whose nonzero coefficients are only in the spots permitted by the E_1 graph. Since $\mathrm{Tot}(E^{\bullet,\bullet})$ is exact, so is the new complex, so we will be done by Proposition 4.2.9.

Remark 4.2.14 (The role of the axiom of choice)**.** In our intended usage (Theorem 4.2.1), the terms $E_1^{p,q,\lambda}$ are all finite-dimensional, so we do not need the axiom of choice to produce the perfect matching from Proposition 4.2.9. The reduction to E_1 does require choices (on the E_0 page). For the Cech complex used in the proof of Theorem 4.2.1, the axiom of countable choice suffices. In general the proof relies on a sufficiently strong choice axiom, even though our combinatorial conclusions do not depend on the choices made.

Proof. First, we split all the vertical maps: for each p, q, λ, we define subspaces $B, H, B^* \subseteq E$ (suppressing the indices) as follows. We put $B = \operatorname{im}(d_v)$; we choose H to be linearly disjoint from B and such that $B + H = \ker(d_v)$; then we choose B^* linearly disjoint from $B + H$, such that $B + H + B^* = E$.

In particular, d_v maps the subspace B^* isomorphically to the subsequent subspace B, and the space H descends isomorphically to $H(d_v)$, the E_1 term. The picture of a single column of the double complex looks like the following:

$$\begin{tikzcd}
 & \vdots & \\
B & H & B^* \arrow[ull, "\sim"'] \\
B & H & B^* \arrow[ull, "\sim"'] \\
 & H & B^* \arrow[ull, "\sim"']
\end{tikzcd} \qquad \text{(note that } d_v(B) = d_v(H) = 0\text{)}$$

For the horizontal map, we have $d_h(B) \subset B$ and $d_h(H) \subset B + H$, and the poset labels λ strictly decrease.

Our goal will be to choose bases carefully, so as to match the H basis elements to one another, in successive diagonals, while decreasing the poset labels.

We first choose an arbitrary basis of each H and B^* space. We descend the basis of B^* to a basis of the subsequent B using d_v. Note that every basis element has a position (p, q) and a label λ. We will write x_λ if we wish to emphasize that a given basis vector x has label λ.

We now change basis on the entire diagonal $E^i := \bigoplus_{p-q=i} E^{p,q}$. We leave the H and B^* bases untouched, but replace all the B basis vectors, as follows. Let $b_\lambda \in B^{p,q,\lambda}$ and let $b^*_\lambda = d_v^{-1}(b_\lambda) \in (B^*)^{p,q-1,\lambda}$ be its 'twin'. We define

$$\tilde{b}_\lambda := d_{tot}(b^*_\lambda) = b_\lambda + d_h(b^*_\lambda).$$

We replace b_λ by $\tilde{b}_\lambda$, formally labeling the new basis vector by (p, q, λ). We write $\widetilde{B}^{p,q,\lambda}$ for the span of these $\tilde{b}$'s, so in particular, $\widetilde{B}^{p,q,\lambda} := d_{tot}\big((B^*)^{p,q-1,\lambda}\big)$.

It is clear that $\widetilde{B}, H, B^*$ collectively gives a new basis for the entire diagonal, unitriangular in the old basis. Notice also that the old basis element $b_\lambda \in B^{p,q,\lambda}$ becomes, in general, a linear combination of $\widetilde{B}, H, B^*$ elements in all positions down-and-left of p, q, with leading term $\tilde{b}_\lambda$:

$$b_\lambda = \tilde{b}_\lambda + \sum_{i>0} x^{p-i,q-i}, \text{ with } x^{p-i,q-i} \in \bigoplus_{\lambda' \lneq \lambda} E^{p-i,q-i,\lambda'}.$$

The lower terms have strictly smaller labels $\lambda' \lneq \lambda$. (In fact, slightly more is true: if a label λ' occurs in the i-th term, the poset P contains a chain of length $\geqslant i$ from λ' to λ.)

We now inspect the coefficients of $(\mathrm{Tot}(E^{\bullet,\bullet}), d_{tot})$ in the new basis. We have

$$\begin{aligned} d_{tot}(b^*_\lambda) &= \tilde{b}_\lambda, \\ d_{tot}(\tilde{b}_\lambda) &= 0 \ \big(= d^2_{tot}(\tilde{b}^*_\lambda)\big), \end{aligned}$$

so the B^* elements map one-by-one onto the $\widetilde{B}$ elements, with the same λ labels; the latter elements then map to 0.

For a basis element $h_\lambda \in H$, however, the coefficients change but remain 'P-filtered'. If $d_{tot}(h_\lambda)$ included (in the old basis) some nonzero term $t \cdot b_\mu$, then in the new basis we have

$$d_{tot}(h_\lambda) = d_h(h_\lambda) = \cdots + t \cdot \big(\tilde{b}_\mu - d_h(b^*_\mu)\big) + \cdots .$$

Since t is nonzero, we have $\mu \lneq \lambda$; and the additional terms coming from $d_h(b^*_\mu)$ all have labels $\mu' \lneq \mu$. Thus all labels occurring in $d_{tot}(h_\lambda)$ in the new basis are, again, strictly smaller than λ. We note that $d_{tot}(h_\lambda)$ is a linear combination of $\widetilde{B}, H, B^*$ elements down-and-left of h_λ along the subsequent diagonal:

$$\begin{tikzcd}
 & & & \widetilde{B}, H & H \arrow[l] \arrow[dll] \arrow[dddllll] \\
 & & \widetilde{B}, H, B^* & & \\
 & \iddots & & & \\
\widetilde{B}, H, B^* & & & &
\end{tikzcd}$$

Finally, we observe that the spaces $\widetilde{B} + B^*$ collectively span a subcomplex of $\mathrm{Tot}(E)$, so we have a short exact sequence of complexes

$$0 \to \mathrm{Tot}(\widetilde{B} + B^*) \to \mathrm{Tot}(E) \to \mathrm{Tot}(H) \to 0.$$

By construction, $\mathrm{Tot}(\widetilde{B} + B^*)$ is exact, so $\mathrm{Tot}(E) \to \mathrm{Tot}(H)$ is a quasi-isomorphism. Note that $\mathrm{Tot}(H)$ and $\mathrm{Tot}(E_1)$ have "the same" terms, but different maps, as desired. Since $\mathrm{Tot}(E)$ is exact, so is $\mathrm{Tot}(H)$. The desired matching then exists by Proposition 4.2.9. $\square$

Remark 4.2.15. Our initial attempts to establish the Boij-Söderberg pairing (Theorems 4.2.1 and 4.2.4) used the higher differentials on the $E_1, E_2, \ldots$ pages, rather than the E_0 page as above – aiming to systematize "chasing cohomology of the underlying sheaves". The following example shows that such an approach fails on general double complexes.

Example 4.2.16 (A cautionary example)**.** Consider the following double complex. Each partition denotes a single basis vector with that label.

$$\begin{tikzcd}
 & & \ydiagram{2} & \ydiagram{3} \arrow[l] \\
 & \ydiagram{2,1} \oplus \ydiagram{1} & \ydiagram{3,1} \oplus \ydiagram{2} \arrow[l, "f"'] \arrow[u] & \\
\ydiagram{1,1} & \ydiagram{2,1} \arrow[l] \arrow[u] & &
\end{tikzcd}$$

The vertical map d_v preserves labels and the horizontal map d_h decreases labels. The unlabeled arrows correspond to coefficients of 1, and the map f is given by

$$f(\ydiagram{2}) = \ydiagram{1}, \qquad f\left(\ydiagram{3,1}\right) = \ydiagram{2,1} - \ydiagram{1}.$$

Note that the rows are exact, so the total complex is exact as well, and the spectral sequence abuts to zero. The only nonzero higher differentials are on the E_1 and E_3 pages. These pages, and (for contrast) the complex H constructed in Theorem 4.2.4, are as follows.

$$E_1 : \begin{tikzcd}
 & & 0 & \ydiagram{3} \\
 & \ydiagram{1} & \ydiagram{3,1} \arrow[l] & \\
\ydiagram{1,1} & 0 & &
\end{tikzcd}
\qquad
E_3 : \begin{tikzcd}
 & & 0 & \ydiagram{3} \arrow[llldd, dashed] \\
 & 0 & 0 & \\
\ydiagram{1,1} & 0 & &
\end{tikzcd}
\qquad
H : \begin{tikzcd}
 & & 0 & \ydiagram{3} \arrow[lld] \\
 & \ydiagram{1} & \ydiagram{3,1} \arrow[l] \arrow[lld] & \\
\ydiagram{1,1} & 0 & &
\end{tikzcd}$$

All the arrows are coefficients of ± 1. In particular, *no* combination of the E_1 and E_3 differentials gives a valid matching (the E_3 arrow violates the P-filtered condition). In contrast, H finds the (unique) valid matching $\left\{ \ydiagram{1} \leftarrow \ydiagram{3}\, ,\ \ydiagram{1,1} \leftarrow \ydiagram{3,1} \right\}$.

APPENDIX A

Local cohomology and sheaf cohomology for Grassmannians

We have mostly approached sheaf cohomology from the perspective of geometry. There is, however, an (almost) equivalent approach to it using ***local cohomology***, which is more algebraic – built directly using modules. For graded modules and sheaves on $\mathbb{P}^n$, this relationship is well-known, but the analog in the equivariant setting is not. We discuss it here briefly for the sake of exposition. We assume familiarity with the main properties of local cohomology as presented in [22, 21].

We recall that, if R is a noetherian ring, $I \subset R$ an ideal and M an R-module, the ***(zeroth) local cohomology of*** M ***along*** I is defined as the R-module

$$H_I^0(M) := \{x \in M : I^d \cdot x = 0 \text{ for } d \gg 0\}.$$

This is a left-exact functor, and its right derived functors $H_I^i(M)$ are the ***local cohomology modules of*** M ***along*** I.

A.1 Local and sheaf cohomology for graded modules

If R is graded polynomial ring and M is a graded R-module, there is a map

$$f : M \to \bigoplus_{d \in \mathbb{Z}} H^0(\widetilde{M}(d)),$$

where $\widetilde{M}$ is the induced sheaf on $\mathbb{P}^n$. This map is neither injective nor surjective in general: some elements of M_d may be supported only at the origin in $\mathbb{A}^{n+1}$, so they give the zero section of $\widetilde{M}(d)$. On the other hand, $\widetilde{M}(d)$ may have extra sections obtained by gluing. The following facts are standard:

- We have $\ker(f) = H^0_{\mathfrak{m}}(M)$ and $\operatorname{coker}(f) = H^1_{\mathfrak{m}}(M)$, giving an exact sequence

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to \bigoplus_{d\in\mathbb{Z}} H^0(\widetilde{M}(d)) \to H^1_{\mathfrak{m}}(M) \to 0.$$

- Moreover, for $i \geqslant 1$ there are canonical isomorphisms

$$\bigoplus_{d\in\mathbb{Z}} H^i(\widetilde{M}(d)) \xrightarrow{\sim} H^{i+1}_{\mathfrak{m}}(M).$$

Thus, the sheaf cohomology numbers $h^1, h^2, \ldots$ are the dimensions of the graded pieces of the $i = 2, 3, \ldots$ local-cohomology of M (as a graded module), while the h^0 term is

$$h^0(\widetilde{M}(d)) = \dim_{\mathbb{C}} M_d + \dim_{\mathbb{C}} H^1_{\mathfrak{m}}(M)_d - \dim_{\mathbb{C}} H^0_{\mathfrak{m}}(M)_d.$$

A.2 Local and sheaf cohomology for equivariant modules

We now consider GL_k-equivariant modules M over the polynomial ring $R_{k,n}$. We write $X = \mathrm{Mat}_{k\times n}$, $U \subset X$ for the open subset of full-rank matrices and $Z = X - U$ for the rank-deficient matrices, defined by the prime ideal $\mathfrak{p} \subset R_{k,n}$ of maximal minors. Let $\pi : U \to Gr(k, \mathbb{C}^n)$ be the projection.

The local cohomology modules $H^i_{\mathfrak{p}}(M)$ are naturally GL_k-equivariant. We will compare them to the sheaf cohomology of the induced sheaf $\widetilde{M}$ on $Gr(k, \mathbb{C}^n)$. By abuse of notation, we write $M|_U$ for the sheaf obtained by restricting M to U (and we do not distinguish between M and the sheaf induced by M on X). In particular, $\widetilde{M}$ is the sheaf of invariants $\widetilde{M} = (M|_U)^{GL_k}$.

Proposition A.2.1. *There is a GL_k-equivariant exact sequence of $R_{k,n}$-modules,*

$$0 \to H^0_{\mathfrak{p}}(M) \to M \to \bigoplus_{\lambda} \mathbb{S}_\lambda(\mathbb{C}^k) \otimes H^0\big(\mathbb{S}_\lambda(\mathcal{S}^*) \otimes \widetilde{M}\big) \to H^1_{\mathfrak{p}}(M) \to 0,$$

and for $i \geqslant 1$ there are canonical isomorphisms:

$$\bigoplus_{\lambda} \mathbb{S}_\lambda(\mathbb{C}^k) \otimes H^i\big(\mathbb{S}_\lambda(\mathcal{S}^*) \otimes \widetilde{M}\big) \xrightarrow{\sim} H^{i+1}_{\mathfrak{p}}(M).$$

Corollary A.2.2. *For $i \geqslant 1$, the GL-cohomology table entry $\gamma_{1,-\lambda^R}(\widetilde{M})$ is the* multiplicity *of the λ summand of $H^{i+1}_{\mathfrak{p}}(M)$. For $i = 0$, we have*

$$\gamma_{0,-\lambda^R}(\widetilde{M}) = \mathrm{mult}(M_\lambda) + \mathrm{mult}(H^1_{\mathfrak{p}}(M)_\lambda) - \mathrm{mult}(H^0_{\mathfrak{p}}(M)_\lambda).$$

Remark A.2.3. The direct sum is really the sheaf cohomology $H^i(U, M|_U)$, which has the structure of $R_{k,n}$-module (in fact, $H^0(U, \mathcal{O}_U) = R_{k,n}$). The main idea for relating its equivariant structure to the cohomology of $\widetilde{M}$ is that, essentially by geometric invariant theory, $M|_U \cong \mathcal{O}_U \otimes (M|_U)^{GL_k} = \mathcal{O}_U \otimes \widetilde{M}$. This is because U is a GL_k principal bundle over $Gr(k, \mathbb{C}^n)$, namely the open subset $\mathcal{I}som(\mathbb{C}^k, \mathcal{S})$ of the vector bundle $\mathcal{H}om(\mathbb{C}^k, \mathcal{S})$ of square matrices, defined locally by the nonvanishing of the determinant.

Proof. The decomposition $X = U \sqcup Z$ induces a long exact sequence

$$\cdots \to H^i_{\mathfrak{p}}(M) \to H^i(X, M) \to H^i(U, M|_U) \to H^{i+1}_{\mathfrak{p}}(M) \to \cdots.$$

Since X is affine, we have $H^i(X, M) = 0$ for all $i > 0$ and $H^0(X, M) = M$. This gives

$$0 \to H^0_{\mathfrak{p}}(M) \to M \to H^0(U, M|_U) \to H^1_{\mathfrak{p}}(M) \to 0$$

and isomorphisms $H^i(U, M|_U) \xrightarrow{\sim} H^{i+1}_{\mathfrak{p}}(M)$ for $i \geqslant 1$. All these maps are GL_k-equivariant. It remains to show that $H^i(U, M|_U)$ has the desired direct sum decomposition.

Consider the projection map $U \to Gr(k, \mathbb{C}^n)$. This is a principal GL_k-bundle, so by geometric invariant theory and the fact that GL_k is reductive we have a canonical isomorphism of sheaves,

$$\mathcal{O}_U \otimes_{\mathcal{O}_{Gr}} \widetilde{M} \xrightarrow{\sim} M|_U. \tag{A.2.1}$$

(Note: it is easy to check this isomorphism directly when M is an equivariant free module, using functoriality of $\mathbb{S}_\lambda$ to reduce to the case $M = \mathbb{C}^k \otimes R_{k,n}$. The case of general M follows from the Five Lemma. So GIT is not really necessary.)

So, we will compute the sheaf cohomology H^i of the left-hand side of Equation (A.2.1). Since $U \to Gr = Gr(k, \mathbb{C}^n)$ is an affine morphism, cohomology commutes with the projection, so we can compute H^i over Gr instead. By Remark A.2.3 and the Cauchy identity (Equation (2.3.2)), $\mathcal{O}_U$ is the sheaf of $\mathcal{O}_{Gr}$-algebras

$$\mathcal{O}_U = \mathcal{S}ym(\mathbb{C}^k \otimes \mathcal{S}^*)[\tfrac{1}{\Delta}] = \bigoplus_\lambda \mathbb{S}_\lambda(\mathbb{C}^k) \otimes \mathbb{S}_\lambda(\mathcal{S}^*),$$

where $\Delta \in \det(\mathbb{C}^k) \otimes \det(\mathcal{S}^*) \subset \mathcal{O}_U$ is the local equation of the determinant, unique up to scaling. The $\mathbb{C}^k$ tensor factors pass through the cohomology computation, giving:

$$\begin{aligned} H^i(U, M) &= H^i(Gr, \mathcal{O}_U \otimes \widetilde{M}) \\ &= \bigoplus_{\lambda} \mathbb{S}_\lambda(\mathbb{C}^k) \otimes H^i\big(\mathbb{S}_\lambda(\mathcal{S}^*) \otimes \widetilde{M}\big). \end{aligned} \qquad \square$$

APPENDIX B

Perfect matchings of infinite-dimensional vector spaces

We give the proof of Proposition 4.2.7; we thank David Lampert [25] for contributing to this proof. Throughout, let V, W be vector spaces of arbitrary dimension, with specified bases $\mathcal{V}, \mathcal{W}$. Let $T : V \to W$ be an isomorphism.

Lemma B.0.4. *There is a pair $(v, w) \in \mathcal{V} \times \mathcal{W}$ such that the following holds:*

(i) $T(v)$ has a nonzero w-coefficient and $T^{-1}(w)$ has a nonzero v-coefficient,

(ii) Let $V^c = \operatorname{span}(\mathcal{V} - \{v\})$ and $W^c = \operatorname{span}(\mathcal{W} - \{w\})$. Then the composition

$$\widetilde{T} : V^c \hookrightarrow V \xrightarrow{T} W \twoheadrightarrow W^c$$

is an isomorphism. (The projection onto W^c is the identity on W^c and sends $w \mapsto 0$.)

Proof. Fix $v \in \mathcal{V}$ and write $T(v) = \sum a_i w_i$, and assume every a_i in the sum is nonzero. Then $v = \sum a_i T^{-1}(w_i)$, so some w_i contributes a nonzero v-coefficient. Choose one such w. The pair (v, w) clearly satisfies (i). For condition (ii), consider the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & V^c \arrow[r, equal] \arrow[d, "T"] & V^c \arrow[d, "\tilde{T}"] \\ (w) \arrow[r] & W \arrow[r] & W^c, \end{tikzcd}$$

The rows are exact, so by the snake lemma we have

$$0 \to \ker(\widetilde{T}) \to (w) \to W/T(V^c) \to \operatorname{coker}(\widetilde{T}) \to 0.$$

Note that $W/T(V^c)$ is one-dimensional, spanned nontrivially by $T(v)$. So, the map $(w) \to W/T(V^c)$ is either an isomorphism or zero. But it cannot be zero, since $w \notin T(V^c)$ (that is, $T^{-1}(w) \notin V^c$). It follows that $\ker(\widetilde{T}) = \operatorname{coker}(\widetilde{T}) = 0$. $\square$

Proposition B.0.5. *The coefficient graph of T has a perfect matching.*

Proof. Fix a well-ordered set Ω with cardinality $|\Omega| > |\mathcal{V}| \big(= |\mathcal{W}|\big)$. Write I_ω for the interval $\{\omega' \in \Omega : \omega' < \omega\} \subset \Omega$ for any $\omega \in \Omega$.

Let $f : \Omega \to \mathcal{V} \times \mathcal{W}$ be a partial function, writing $f(\omega) = (v_\omega, w_\omega)$ for $\omega \in \mathrm{dom}(f) \subseteq \Omega$. We say f is *good* if it is a partial matching and the following conditions hold:

(i) For each $(v, w) \in \mathrm{im}(f)$, $T(v)$ has a nonzero w-coefficient and $T^{-1}(w)$ has a nonzero v-coefficient.

(ii) Let $V^c = \mathrm{span}(\mathcal{V} - \{v_\omega : \omega \in \mathrm{dom}(f)\})$; define W^c similarly. Then the composition

$$T' : V^c \hookrightarrow V \xrightarrow{T} W \twoheadrightarrow W^c$$

is an isomorphism. (The projection onto W^c is the identity on W^c and sends $w_\omega \mapsto 0$ for all $\omega \in \mathrm{dom}(f)$.)

Note that the empty function is good. We construct by transfinite induction a good f, such that $\mathrm{im}(f)$ is a perfect matching between $\mathcal{V}$ and $\mathcal{W}$.

Assume f is defined on I_ω for some $\omega \in \Omega$ and is good. By (ii), either $\mathcal{V}^c = \mathcal{W}^c = \varnothing$ or both are nonempty. If both are nonempty, apply the previous lemma to the map T' (which has the same coefficients as T, restricted to $\mathcal{V}^c, \mathcal{W}^c$). Let $(v, w) \in \mathcal{V}^c \times \mathcal{W}^c$ be the resulting pair, and define $f(\omega) = (v, w)$. Then f is good on $I_\omega \sqcup \{\omega\}$. So, by transfinite induction, we may extend the definition of f until $\mathcal{V}^c = \varnothing = \mathcal{W}^c$, at which point $\mathrm{im}(f)$ is the desired perfect matching. This must occur eventually, since $|\Omega| > |\mathcal{V}|$. $\square$

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