Arc schemes in logarithmic algebraic geometry

by

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CHAPTER I

Introduction

An arc on an algebraic variety X is a bit of a curve on X. The arcs on X have a parameter space, called the arc scheme of X. The study of the variety X through understanding its arcs began with Nash's 1968 preprint [26], and has attracted special interest since Kontsevich's 1995 lecture at Orsay, in which he introduced a theory of integration on spaces of arcs, called motivic integration, and the successes it generated.

Our study here concerns the development of the theory of arcs in the context of logarithmic algebraic geometry. Log geometry is a subject with many inroads: sometimes it is used as an algebraic analogue of manifolds with boundary, sometimes it is used to study controlled degenerations on varieties, sometimes it brings to a given variety an analogue of the combinatorial structure of a toric variety. Kazuya Kato [19] showed how this geometric theory can be founded by equipping a variety X with a monoid sheaf $\mathcal{M}$, called a log structure. In geometric cases, often the log structure is essentially merely the multiplicative semigroup of monomials in some chosen regular functions on X. The category of log schemes consists of such pairs, a scheme X together with the log structure $\mathcal{M}$, and the log algebro-geometer works in this context. As there are log curves – ordinary curves C with additional log structure at some of their points – so there are log arcs as well.

There is for each log scheme a log arc scheme as well, which parametrises its log arcs. We develop this theory here, building on prior work on log jet schemes [11], [16]. A classical fact about (ordinary) arc schemes is that the arc scheme of an irreducible variety X, over a field of characteristic zero, is irreducible itself. This is a theorem of Kolchin [21]. This means that an arc on the singular locus of X can be deformed into the smooth locus of X; or put the other way, an arc on the singular locus of X is a limit of arcs in the smooth locus of X. For log arcs, such a deformation determines not only a limiting arc but some data about the "trajectory" of the deformation as well. For combinatorially reasonable irreducible log varieties, we show that the log arc scheme is irreducible: all the log arcs are such limits. Specifically, we establish the following result.

Theorem I.1. *(III.44) Let k be a field of characteristic zero. Let $(X, \mathcal{M})$ be a fine log scheme over $(\operatorname{Spec} k, k^*)$, with X irreducible of finite type, and let $J_\infty(X, \mathcal{M})$ be its log arc scheme. Let X_j be the rank j stratum of $(X, \mathcal{M})$ and let r be the minimum rank of $\mathcal{M}$ on X. Then $J_\infty(X, \mathcal{M})$ is irreducible if and only if $\operatorname{codim}_X X_j = j - r$ for all non-empty X_j.*

What else in the study of arc schemes carries over to an analogue in the study of log arc schemes? Certainly it would be attractive to have a theory of log motivic integration as well, meaning a theory of integration on log arc schemes. For such a theory one needs to provide certain ingredients: some kind of measure on log arc schemes, functions to integrate, values for these to take on. From these, to get any work from the theory, the recipe must supply a comparison result between log motivic integrals, as Kontsevich's key change-of-variables formula compares motivic integrals on a variety with motivic integrals on a proper birational model. Steps toward such a theory we offer in Chapter IV.

1.0.1 Logarithmic algebraic geometry

A log structure on an algebraic variety X is "a magic by which a degenerate scheme begins to behave as being non-degenerate," writes Kato [18]. Toric varieties X_Σ, for example, are smooth objects in the category of log schemes – they are "log smooth" – and enjoy good regularity properties under maps $X_\Sigma \to X_{\Sigma'}$ that come from morphisms of their fans $\Sigma \to \Sigma'$ – which are "log scheme morphisms."

In the generality introduced by Kato [19], a log scheme is an algebraic variety X together with a sheaf of monoids $\mathcal{M}$ on X and a morphism $\mathcal{M} \to \mathcal{O}_X$ to the multiplicative monoid of the structure sheaf $\mathcal{O}_X$ which induces an isomorphism on units $\mathcal{M}^* \to \mathcal{O}_X^*$. The sheaf $\mathcal{M}$ is called a log structure on X and the pair $(X, \mathcal{M})$ is called a log scheme. An important classical example is when $\mathcal{M}$ is locally generated as a monoid by local equations for the components of a chosen normal crossing divisor D on X. Sometimes one thinks of D as a "boundary" or "divisor at infinity" for the space $X - D \subseteq X$.

For Kato and those who followed, this was the road to fruitful applications in arithmetic, including the construction of log crystalline cohomology [14] and Mochizuki's proof of Grothendieck's conjecture on anabelian geometry for curves over number fields [24]. Here if $X \to \operatorname{Spec} V$ is a scheme over a discrete valuation ring V with semistable reduction, which just means that the fibre of X over the closed point of $\operatorname{Spec} V$ is a normal crossing divisor, one can treat it with the formalism of log geometry.

For others, as it will largely be for us, log geometry remained a more geometric theory, with X to be a variety over a field k. One then studies the geometry of X when it is paired with an appropriate log structure $\mathcal{M}$, or the geometry of an open subset $U \subset X$ of a complete variety X with log structure along the complement "at infinity" $X - U$ (which now is not required to be a normal crossing divisor in general). For example, the moduli space of stable curves, a compactification of the moduli space of smooth curves, arises this way by considering curves X, not necessarily smooth, with fine saturated log structure at their singular points. Toric varieties as compactifications of algebraic tori may also be viewed in this way.

Spherical varieties, which generalise toric varieties by replacing the torus with an algebraic group G, also naturally have such a description. Recently in [5] the arc spaces of spherical varieties were investigated; see Remark IV.57. It may be quite interesting to study these log geometrically as well.

1.0.2 Motivic integration

Motivic integration is a proven technique for extracting information about an algebraic variety X from its arc scheme $J_\infty(X)$. Kontsevich [22] introduced this theory of integration by analogy with certain p-adic integrals used by Batyrev [1], strengthening and generalising Batyrev's result to work over any algebraically closed field k. The key property that motivic integrals enjoy is the so-called change-of-variables formula for proper birational maps $Y \to X$ of varieties over k, which relates an integral on X to an integral on Y involving the pullback of the integrand on X and the relative canonical class $K_{Y/X}$ of the morphism.

Kontsevich's original application was to show that birational smooth Calabi-Yau varieties have the same Deligne-Hodge polynomial, and hence the same Hodge numbers. This only required constructing the motivic integral on smooth varieties X. Denef and Loeser [8] undertook the task of making precise Kontsevich's proposed theory, at the same time showing how to define motivic integrals for singular varieties, which present significant additional technical difficulties, especially in the construction of the motivic volume on a suitable algebra of subsets of $J_\infty(X)$. Numerous further applications have followed. Among others, we might mention further work of Batyrev on stringy Hodge numbers [3] and Reid's conjecture on the McKay correspondence [2], further work of Denef and Loeser on Igusa zeta functions [9], and Mustaţă's work on some birational invariants by relation to jet schemes [25]. For more on the development and use of the technique, we refer to [23], [7].

1.0.3 Terminology

In this note we are often concerned with terms and concepts (jets and arcs, smoothness and étaleness, and so forth) which appear in both the usual setting of the category of schemes and in the setting of the category of log schemes. Wherever a term is used in the log scheme sense we will indicate this by including "log" in its name (so, log jets and log arcs, log smoothness and log étaleness, and so forth). Sometimes when a term is used in its usual scheme-theoretic sense we will emphasise the distinction by calling it "ordinary" (so, ordinary jets and ordinary arcs, and so forth).

CHAPTER II

Monoids and logarithmic algebraic geometry

We give an exposition, essentially self-contained, of some elements of log geometry. A log scheme, after Kato [19], is a pair $(X, \mathcal{M})$ where $\mathcal{M}$ is a sheaf of monoids with a multiplicative map $\mathcal{M} \to \mathcal{O}_X$ which restricts to an isomorphism $\mathcal{M}^* \to \mathcal{O}_X^*$ on units.

In practice, the log structure $\mathcal{M}$ is often specified by way of a map $P \to \mathcal{O}_X$ from a finitely generated monoid P, which then generates $\mathcal{M}$ in a categorical sense by sheafifying and adding in units. For example, a toric variety is a log scheme in a natural way, with log structure locally generated by cones in the lattice of characters of the torus. This gives a combinatorial interpretation to the data of a log structure. Such a map $P \to \mathcal{O}_X$ is called a chart for the log structure it generates, and in many cases it reflects or controls the behaviour of the map $\mathcal{M} \to \mathcal{O}_X$.

Our first task then will be to recall some of the basic language and behaviour of monoids. Later we extract some information about the geometry of log schemes from them. Especially important to us is the view of a fine log scheme $(X, \mathcal{M})$ as being stratified by locally closed subsets on which $\mathcal{M}$ "does not vary." In the case of a toric variety, the stratification consists of the torus-invariant orbits, understanding of which of course contributes greatly to understanding the geometry of the toric variety. In general the strata are determined from the quotient $\mathcal{M}/\mathcal{O}_X^*$, which is a sheaf of finitely generated monoids and another important object. In good cases, but not quite all cases, this sheaf $\mathcal{M}/\mathcal{O}_X^*$ gives charts for the log structure $\mathcal{M}$ in the sense above at every point. This means that there is a section $\mathcal{M}/\mathcal{O}_X^* \to \mathcal{M}$ of the projection. In these cases such a chart gives an additional measure of control on the log structure.

2.1 Monoids

By a *monoid* we mean always a commutative semigroup with an identity element. Most often we will write the monoid operation multiplicatively, but sometimes additive notation will be apt. In particular we will use the natural numbers $(\mathbb{N}, +)$ under addition, where $\mathbb{N} = \{0, 1, 2, ...\}$, to stand for the free monoid on one generator. When the generator is specified to be some element x of a ring R, we write by abuse of notation $\mathbb{N}x = \{1, x, x^2, ...\}$ to stand for the set of powers of x.

2.1.1 Monoid algebras

To a multiplicative monoid P we associate the monoid algebra $k[P]$, which by definition is the quotient of the polynomial ring on the set P by the relations between monomials in the variables which hold of them in the monoid P (and by identifying the neutral elements $1 \in P$ and $1 \in k$). Equivalently, one may take construct $k[P]$ as the quotient of the polynomial ring on a set of generators of the monoid P by the relations which hold among the monomials in these generators. Thus $k[P]$ is a quotient of a polynomial ring by a *pure binomial ideal*; we also sometimes say that it is a quotient given by *monomial relations*. Conversely such a presentation

$$R = k[\{x_i\}_{i \in I}]/J$$

with a pure binomial ideal J determines a monoid P, generated by the symbols x_i, for which $R = k[P]$.

By abuse of terminology we also call the affine scheme $\operatorname{Spec} k[P]$ a monoid algebra, (partly to avoid conflation with the notion of "monoidal space" which has been used by other authors to refer to something different).

2.1.2 Basic notions and properties

A map $P \to Q$ of monoids induces a k-algebra map $k[P] \to k[Q]$.

A monoid P is *finitely generated* if there is a surjection $\mathbb{N}^r \to P$ for some integer $r \geq 0$. Equivalently, all the elements of P may be written as monomials in some finite generating set of elements of P. Such a description gives $\operatorname{Spec} k[P]$ as a closed subscheme of affine space $\mathbb{A}^r = \operatorname{Spec} k[\mathbb{N}^r]$.

A monoid P has a *group completion* P^{gp}, that is, a group P^{gp} with a map $P \to P^{gp}$ with the universal property that a monoid morphism $P \to G$ to any group G factors through P^{gp}. One may construct P^{gp} as the set of fractions pq^{-1} with $p, q \in P$, where $pq^{-1} = rs^{-1}$ in P^{gp} if there is $t \in P$ such that $tps = trq$.

A monoid P is called *integral* if whenever an equation $pq = pr$ holds in P one has $q = r$. Equivalently, P is integral if and only if the map $P \to P^{gp}$ is an inclusion. This gives also an inclusion $k[P] \to k[P^{gp}]$. If in addition P^{gp} is torsion-free then $k[P^{gp}]$ is a domain, according to a classical result for group rings, so that $k[P]$ is then also a domain. (But P being integral, or integral and torsion-free, is not sufficient for $k[P]$ being a domain in general; see for instance Example II.3 below.)

A monoid both integral and finitely generated is called *fine*.

An integral monoid P is called *saturated* if whenever some power g^n lies in P, where $g \in P^{gp}$ and $n \geq 1$, one has $g \in P$ also.

To a monoid P there is associated a universal integral monoid P^{int}, which may be realised as the image of $P \to P^{gp}$. This in turn has a universal saturation $P^{sat} \subseteq P^{gp}$, consisting of all the elements of P^{gp} for which some positive power lies in P^{int}.

When $X = \operatorname{Spec} k[P]$ is the algebra of a specified monoid P, we write for convenience $X^{gp} = \operatorname{Spec} k[P^{gp}]$ for the monoid algebra of P^{gp}. If P is fine, then X^{gp} is a dense open subset of X, consisting of a finite union of tori, one for each torsion element of P^{gp}, each having dimension $\operatorname{rank} P^{gp}$.

Sometimes we wish to view $P^{gp}/(torsion)$ as a lattice inside a rational vector space $P^{\mathbb{Q}} = P^{gp} \otimes_{\mathbb{Z}} \mathbb{Q}$. If P is finitely generated then the image of $P \to P^{\mathbb{Q}}$ is a (possibly nonsaturated) rational polyhedral cone.

Example II.1. A rational cone in an integer lattice is a fine, saturated, torsion-free monoid. Conversely, if P is a fine saturated torsion-free monoid, then furthermore P^{gp} is torsion-free, for the saturation of any submonoid of P^{gp} includes all the torsion elements of P^{gp}. Then P^{gp} is free and P is a rational cone in P^{gp}. The monoid algebra $\operatorname{Spec} k[P]$ is a (normal) affine toric variety.

Example II.2. The subset $P = \mathbb{N} - \{1\} = \{0, 2, 3, 4, ...\}$ of the natural numbers under addition is a monoid. Writing this multiplicatively as powers of a variable t, its monoid algebra is

$$k[P] = k[t^2, t^3] \simeq k[x, y]/(x^3 - y^2),$$

the co-ordinate ring of a cuspidal plane cubic curve. The inclusion $P \subseteq \mathbb{N}$ gives $P^{gp} \subseteq \mathbb{N}^{gp} = \mathbb{Z}$. In fact $P^{gp} = \mathbb{Z}$, for example because $1 = 3 - 2$ is a difference of elements of P (in other words, because P^{gp} is a subgroup of $\mathbb{Z}$ containing $2, 3$). We see that P is integral and finitely generated, so is fine, but is not saturated. Its saturation is $\mathbb{N}$, and the natural map $k[P] \to k[P^{sat}] = k[t]$ is the normalisation map of the cuspidal curve.

Example II.3. Let Q be generated by two elements x, y subject to the single relation $x^2 = y^2$. Then Q is integral and torsion-free. The spectrum of the monoid algebra $k[Q] = k[x, y]/(x^2 - y^2)$ consists of two lines meeting at the point $x = y = 0$. One has

$$Q^{gp} \simeq (\mathbb{Z}/2\mathbb{Z})\omega \times \mathbb{Z}x$$

by identifying the generator y with $\omega \cdot x$, and

$$Q^{sat} = (\mathbb{Z}/2\mathbb{Z})\omega \times \mathbb{N}x \subseteq Q^{gp}.$$

So $k[Q^{sat}] = k[x, \omega]/(\omega^2 - 1)$. The map $\operatorname{Spec} k[Q^{sat}] \to \operatorname{Spec} k[Q]$ glues the two lines of $\operatorname{Spec} k[Q^{sat}]$ with $\omega = \pm 1$ together at $x = 0$, while $\operatorname{Spec} k[Q^{gp}] \to \operatorname{Spec} k[Q]$ just includes the complement of the node of $\operatorname{Spec} k[Q]$.

2.1.3 Ideals, faces, and quotients

An *ideal* I of P is a submonoid of P closed under multiplication by elements of the monoid, $IP \subseteq I$. An ideal I is called *prime* if whenever $pq \in I$ one of p, q lies in I. Equivalently, an ideal I is prime if the complementary set $P - I$ is a submonoid of P (in fact, a face of P: see below).

For an ideal I of P, there is a quotient monoid $P \to P/I$ which may be realised as the set $P - I$ together with a "zero element," corresponding to the subset I. If a product of elements in $P - I$ lies in I then in P/I the product is zero (i.e., it is the class of I). If I is prime then this does not happen, and P/I may be realised as the *monoid* $P - I$ together with a zero element, (where zero times $P - I$ is zero).

A *face* F of a monoid P is a submonoid of P such that whenever $p, q \in P$ have $pq \in F$, then in fact both p, q lie in F. The complementary set $P - F$ of a face F is a prime ideal of P, and vice versa. The group P^* of invertible elements of P is a face of P, in fact the minimum face of P.

A monoid P with $P^* = 1$ is called *sharp*. A monoid P has a universal map to a sharp monoid, namely to the quotient P/P^*. By this we mean the set of cosets of P^* in P with the induced monoid operation. This is a different kind of quotient of P than those by ideals of P. When F is a face of a fine monoid P, we will write P/F for the sharp quotient

$$F^{-1}P/(F^{-1}P)^*,$$

where $F^{-1}P \subseteq P^{gp}$ is the smallest monoid containing both P and the inverses of the elements of F.

Example II.4. The usual group quotient $\mathbb{Z}/2\mathbb{Z}$ is a monoid of two elements, as is the monoid quotient $\mathbb{N}/\{1, 2, ...\}$, but they are not the same monoid: the latter has a non-trivial idempotent $1 + 1 = 1$. One might write it instead as the set $\{0, \infty\}$, with ∞ being the additive version of what the zero element is for multiplicative monoids.

Put another way, $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{N}/\{1, 2, ...\}$ have operation tables

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \quad \text{and} \quad \begin{array}{c|cc} + & 0 & \infty \\ \hline 0 & 0 & \infty \\ \infty & \infty & \infty \end{array}$$

respectively, and these are not isomorphic.

Example II.5. View the monoid $\mathbb{N}^2$ as the set of integer lattice points

$$\mathbb{N}^2 = \{(a, b) \text{ such that } a, b \geq 0\}$$

of the first quadrant of the plane. Any point $p \in \mathbb{N}^2$ generates an ideal $p + \mathbb{N}^2$, a translated cone in the plane. The ideals of $\mathbb{N}^2$ are unions of such cones (which may then be realised as finite unions of such cones). Equivalently, an ideal of $\mathbb{N}^2$ is the set of points above a descending staircase in the first quadrant.

The monoid $\mathbb{N}^2$ has three prime ideals, being the sets

$$(1,0)+\mathbb{N}^2 = \{(a,b) \text{ such that } a \neq 0\}$$

and

$$(0,1)+\mathbb{N}^2 = \{(a,b) \text{ such that } b \neq 0\}$$

and their union

$$\big((1,0)+\mathbb{N}^2\big) \cup \big((0,1)+\mathbb{N}^2\big) = \{(a,b) \text{ such that } (a,b) \neq 0\}.$$

The complements of these are the three proper faces of $\mathbb{N}^2$, namely the two copies of $\mathbb{N}$ where one co-ordinate or another is zero, and the origin $\{(0,0)\}$.

Example II.6. A field k, considered as a multiplicative monoid, has unit group k^*, and the quotient monoid k/k^* is the (sharp) two-element multiplicative monoid $\{0,1\}$. Here there is an asymmetry between the two points of k/k^* in that the unit group acts freely on the orbit $k^* \subseteq k$ of $1 \in k$ but trivially on the orbit $\{0\} \subseteq k$.

2.2 Log schemes

A log scheme, as introduced by Kato [19], will be a scheme X together with some combinatorial data encoded in a sheaf of monoids $\mathcal{M}$ on X. Any scheme may become a log scheme in many different ways, including trivially, so in complete generality $\mathcal{M}$ does not provide much control. There are various conditions on the pair $(X,\mathcal{M})$ one may ask for to provide some. For example one commonly works in the category of log schemes with fine and saturated log structure, which can avoid many possible pathologies in both $\mathcal{M}$ and X. Fine and saturated log smooth varieties are modelled in a strong sense on toric varieties [20]. Like toric varieties they are normal and Cohen-Macaulay. But the general local model for a log scheme is just of an arbitrary map $X \to \operatorname{Spec} k[P]$ of a scheme X to a monoid algebra, where the map on structure sheaves is induced by a monoid morphism $P \to (\mathcal{O}_X, \cdot)$ that "generates" the log structure on X.

2.2.1 Log structures and charts

A log scheme over k is a pair $(X, \mathcal{M}_X)$, where X is a scheme over k and $\mathcal{M}_X$ is a sheaf of monoids on X with a monoid morphism $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ to the multiplicative monoid $(\mathcal{O}_X, \cdot)$ that induces an isomorphism $\mathcal{M}_X^* = \alpha_X^{-1}\mathcal{O}_X^* \to \mathcal{O}_X^*$ on units. The sheaf $\mathcal{M}_X$ is called a *log structure* on X. The map α_X need not be injective, although in many examples $\mathcal{M}_X$ will be a subsheaf of $\mathcal{O}_X$. The category of log structures on X has an initial object, which is $\mathcal{O}_X^*$ (as a sub-monoid sheaf of $\mathcal{O}_X$), also called the trivial log structure on X, and a final object, which is $\mathcal{O}_X$, since by definition a log structure sheaf $\mathcal{M}$ on X comes with maps $\alpha^{-1} : \mathcal{O}_X^* \to \mathcal{M}$ and $\alpha : \mathcal{M} \to \mathcal{O}_X$.

Any sheaf of monoids P_X on X with a monoid morphism $\alpha : P_X \to \mathcal{O}_X$ (this much data is called a *pre-log structure*) generates an associated log structure P_X^a, which may be realised as the fibred sum (in the category of sheaves of monoids) of the diagram

$$\begin{tikzcd} & P_X \\ \mathcal{O}_X^* & \alpha^{-1}\mathcal{O}_X^* \arrow[l] \arrow[u] \end{tikzcd}$$

In particular if one specifies a monoid P and a map $P \to \mathcal{O}_X(X)$ one obtains a log structure on X generated by P, namely, that associated to the constant sheaf P determines. A monoid P is called a *chart* for the log structure it generates.

A log structure on X is called *coherent* if it has, Zariski-locally, charts by finitely generated monoids. Likewise the log scheme $(X, \mathcal{M}_X)$ is called fine or saturated if it has, Zariski-locally, charts by fine or saturated monoids. A map $P \to \mathcal{M}_X(U)$ over an open set $U \subseteq X$ is a chart if and only if it induces isomorphisms $P_x^a \to \mathcal{M}_{X,x}$ at stalks for $x \in U$. In any case, a map $P \to \mathcal{O}_X(U)$ determines a map $U \to \operatorname{Spec} k[P]$ to the monoid algebra of P. Sometimes this map to $\operatorname{Spec} k[P]$, rather than from the monoid P, is called a chart on U.

Example II.7. A monoid algebra $k[P]$ naturally gives a log scheme, with the log structure given by the chart P. We call this the *standard structure* of $\operatorname{Spec} k[P]$ as a log scheme.

Example II.8. One of the principal motivating examples for logarithmic geometry arises from a scheme X together with a (Cartier) normal crossing divisor D on X. This is realised in Kato's formalism of log structure sheaves as follows. For a point $x \in X$ let $D_1, ..., D_r$ be the prime components of D passing through x and take a chart $\mathbb{N}^r$ near x, with the standard generators of $\mathbb{N}^r$ mapping to local equations for the components D_i. Since the choice of equations is not canonical these charts typically will not glue together by themselves, hence one expands them as monoids to include the units near x. Now various constructions one makes from the log structure, like the sheaf of log differentials, are interpreted as objects with logarithmic (i.e., order one) poles along D. The dual objects, like the sheaf of log tangent vectors (and, we shall see, log jets in general) are thereby interpreted as objects with zeroes of order one along D.

More generally, let U be an open set in X, and take $\mathcal{M}$ to be the subsheaf of $\mathcal{O}_X$ of functions invertible on U. That is, for $V \subseteq X$ open put

$$\mathcal{M}(V) = \{f \in \mathcal{O}_X(V) \text{ such that } f|_U \in \mathcal{O}_X(U \cap V)^*\}.$$

This is a monoid sheaf under multiplication, and is a log structure on X. When

$X - U$ is a normal crossing divisor D, this $\mathcal{M}$ is the same log structure as constructed above. In general, this $\mathcal{M}$ is the pushforward (see Section 2.2.2 below) of the trivial log structure on U along the inclusion $U \to X$.

Example II.9. The affine plane $X = \operatorname{Spec} k[\mathbb{N}^2] = \operatorname{Spec} k[x, y]$ with its standard log structure $\mathcal{M}$, generated as a submonoid sheaf of $\mathcal{O}_X$ by the monomials x and y together with $\mathcal{O}_X^*$, is a special case of the last two examples.

One may go from the chart $P = \mathbb{N}^2$ to the associated log structure $\mathcal{M}$ as follows. On the open subset $U = \operatorname{Spec} k[x, y]_x$ of X the monomial x is a unit, so that in the fibred sum of P and $\mathcal{O}_X(U)^*$ the element $x \in P$ is identified with the unit $x \in \mathcal{O}_X(U)^*$. Consequently on U the monoid $\mathcal{M}$ is generated just by the monomial y together with $\mathcal{O}_X^*$. Likewise on the open set $V = \operatorname{Spec} k[x, y]_y$ the log structure $\mathcal{M}$ is generated by x. On $U \cap V = \operatorname{Spec} k[x, y]_{xy}$ the log structure is trivial, $\mathcal{M}|_{U \cap V} = \mathcal{O}_{U \cap V}^*$.

Example II.10. Consider the affine line $\operatorname{Spec} k[x]$ with log structure $\mathcal{M}$ given by a chart $\mathbb{N} \to k[x]$ taking the generator of $\mathbb{N}$ to the polynomial $x(x - 1)$. That is, the chart is given by the inclusion of algebras $k[x(x - 1)] \to k[x]$. The global sections of $\mathcal{M}$ are not just the monoid sum $k[x]^* \oplus \mathbb{N}$. In fact, thinking of $\mathcal{M}$ as a subsheaf of $\mathcal{O}_X$, both x and $x - 1$ are global sections. For example, away from $x = 0$ the function x is a unit, so is a section of $\mathcal{M}$, while away from $x = 1$ the function

$$x = (x - 1)^{-1} \cdot x(x - 1)$$

is a unit $(x - 1)^{-1}$ times a section $x(x - 1)$, so is a section of $\mathcal{M}$. So x is a global section of $\mathcal{M}$; and similar for $x - 1$.

In this example one has two global charts, by $\mathbb{N}x(x-1)$ and by $\mathbb{N}^2 = \mathbb{N}x \oplus \mathbb{N}(x-1)$, whose (abstract) group completions have different rank. The latter is closer to the global structure of $\mathcal{M}$, in the sense that $\mathcal{M}(X) \simeq \mathbb{N}^2 \oplus k[x]^*$, while the former is closer to the local structure of $\mathcal{M}$ at the points $x = 0, 1$, in the sense that the induced map

$$\mathbb{N}x(x-1) \to \mathcal{M}_p/\mathcal{O}^*_{X,p}$$

is an isomorphism at these two points p.

Remark II.11. A chart $P \to \mathcal{M}_X$ of a fine log scheme $(X, \mathcal{M}_X)$ which induces an isomorphism $P \to \mathcal{M}_{X,x}/\mathcal{O}^*_{X,x}$ is called *good* at x. Good charts do not always exist. The next proposition gives some simple cases where they do.

Proposition II.12. *Let $(X, \mathcal{M})$ be a fine log scheme, x a point of X, and $P = \mathcal{M}_x/\mathcal{O}^*_{X,x}$. Then:*

(1) If P^{gp} is torsion-free there is a chart $P \to \mathcal{M}$ near x.

(2) If X is normal near x then there is a chart $P \to \mathcal{M}$ near x.

(3) If $\operatorname{char} k = 0$, or $\operatorname{char} k = p$ and P has no p-torsion, then there is a chart $P \to h^*\mathcal{M}$ in an étale neighbourhood $h : V \to X$ of x, where $h^*\mathcal{M}$ is the log structure on V generated by $\mathcal{M}$.

Proof. By ([31], II.2.3.6) there is an isomorphism

$$\mathcal{M}(U)/\mathcal{O}_X(U)^* \to \mathcal{M}_x/\mathcal{O}^*_{X,x}$$

in some neighbourhood U of x. So P would be made into a chart by a section of the monoid morphism $\mathcal{M}(U) \to \mathcal{M}(U)/\mathcal{O}_X(U)^*$. To study this map we consider the induced map on group completions,

$$\begin{array}{ccc} \mathcal{M}(U) & \longrightarrow & \mathcal{M}(U)/\mathcal{O}_X(U)^* \\ \downarrow & & \downarrow \\ \mathcal{M}(U)^{gp} & \longrightarrow & (\mathcal{M}(U)/\mathcal{O}_X(U)^*)^{gp}. \end{array}$$

Since $\mathcal{M}(U)$ and $\mathcal{M}(U)/\mathcal{O}_X(U)^*$ are integral monoids, the vertical arrows are injections, and a splitting of the bottom arrow splits the top arrow by restriction.

Now in case (1) it is assumed that $P^{gp} = (\mathcal{M}(U)/\mathcal{O}_X(U)^*)^{gp}$ is free abelian, so a splitting of the bottom arrow exists in this case.

Otherwise, P^{gp} has some torsion part, and the problem becomes to determine that torsion elements of P^{gp} come from torsion elements of $\mathcal{M}(U)^{gp}$. Let $fg^{-1} \in P^{gp}$ be a torsion element, with $f, g \in \mathcal{M}(U)$ and $f^n = ug^n$ for some $u \in \mathcal{O}_X(U)^*$. If X is normal on U, then $u = (\alpha(f)/\alpha(g))^n$ has an n^{th} root $v = \alpha(f)/\alpha(g)$ in $\mathcal{O}_X(U)$. Then $f^n = (vg)^n$ in $\mathcal{M}(U)$, so $f(vg)^{-1}$ is torsion in $\mathcal{M}(U)^{gp}$. Choosing a decomposition of the torsion part of P^{gp} as a sum of cyclic groups and lifting generators fg^{-1} for each now gives (2). Finally, we have at the same time (3), with the morphism $U \to X$ given by extracting the roots $v = u^{1/n}$ near x. $\square$

Example II.13. Let $X = \operatorname{Spec} k[x, y, z, w]/(x^2z - y^2w)$ with its standard structure as a monoid algebra. On the open subset $zw \neq 0$, there are not Zariski-local good charts along the locus $x = 0, y = 0$. The obstruction is that $z/w = (x/y)^2$ is a unit here but x/y is not a regular function.

2.2.2 The category of log schemes

A morphism of log schemes $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a morphism of schemes $f : X \to Y$ with a morphism of sheaves of monoids $\mathcal{M}_Y \to f_*\mathcal{M}_X$ compatible with $\mathcal{O}_Y \to f_*\mathcal{O}_X$; that is, making a commutative diagram

$$\begin{tikzcd} \mathcal{M}_Y \arrow[r] \arrow[d] & f_*\mathcal{M}_X \arrow[d] \\ \mathcal{O}_Y \arrow[r] & f_*\mathcal{O}_X \end{tikzcd}$$

Given a map of schemes $f : X \to Y$ and a (pre-)log structure $\mathcal{M}_Y$ on Y, there is a pullback log structure $\mathcal{M}_X = f^*\mathcal{M}_Y$ on X, which is the log structure associated to the set-theoretic pullback $f^{-1}\mathcal{M}_Y$ with the monoid map $f^{-1}\mathcal{M}_Y \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. Given instead a (pre-)log structure $\mathcal{M}_X$ on X there is a pushforward log structure $\mathcal{M}_Y$ on Y. Writing still f_* for the usual set-theoretic pushforward of sheaves, this is the log structure associated to the fibre product (in the category of sheaves of monoids on Y) of the diagram

$$\begin{tikzcd} & f_*\mathcal{M}_X \arrow[d] \\ \mathcal{O}_Y^* \arrow[r] & f_*\mathcal{O}_X^* \end{tikzcd}$$

In either case, there is an induced map of log schemes $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$. Pullback and pushforward are functorial, and they are adjoint functors in the usual way. Formation of the associated log structure from a chart or other pre-log structure commutes with pullback.

A map of log schemes $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ such that $\mathcal{M}_X \simeq f^*\mathcal{M}_Y$ is called *strict*. A monoid P mapping to $\mathcal{O}_Y$ is a chart on Y if and only if the map $Y \to \operatorname{Spec} k[P]$ is strict. Since strictness is preserved by composition, if $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is strict then P pulls back to a chart on X by $X \to Y \to \operatorname{Spec} k[P]$.

If $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ are log schemes over a base $(S, \mathcal{M}_S)$, they have a fibre product log scheme

$$(X \times_S Y, (\mathcal{M}_X \oplus_{\mathcal{M}_S} \mathcal{M}_Y)^a)$$

with underlying ordinary scheme $X \times_S Y$ and log structure generated by the fibred sum of the diagram

$$\begin{tikzcd} & \mathcal{M}_X \\ \mathcal{M}_Y & \mathcal{M}_S \arrow[l] \arrow[u] \end{tikzcd}$$

of sheaves of monoids on $X \times_S Y$, where we have written just $\mathcal{M}_X$, etc. for the pullbacks of $\mathcal{M}_X$, etc. to $X \times_S Y$.

2.2.3 Stratification of fine log schemes

Let $(X, \mathcal{M})$ be a fine log scheme. At any point x of X there is a stalk $\mathcal{M}_x$ of the log structure sheaf, which is a monoid in the natural way. Consider the group

$$\mathcal{M}_x^{gp}/\mathcal{O}_{X,x}^* = (\mathcal{M}_x/\mathcal{O}_{X,x}^*)^{gp}.$$

This is a finitely-generated abelian group, as one may see by taking a finitely-generated chart of $\mathcal{M}$ near x. Its rank we call the *rank of* $\mathcal{M}$ at x.

The rank of $\mathcal{M}$ is an upper-semicontinuous function on X. Consequently there is a (finite) stratification of X by locally closed subsets X_j on which $\mathcal{M}$ has rank j. In this stratification, if Z is a component of X_j and Z' is a component of X_k such that the closure $\overline{Z}$ meets Z', then Z' is contained in $\overline{Z}$, and $j \leq k$.

If X is irreducible, there is some minimum rank r of $\mathcal{M}$ on X, and X_r is open and dense in X. The complement

$$X - X_r = \bigcup_{j \geq r+1} X_j$$

is a divisor on X, which we might call the *locus* $Z(\mathcal{M})$ of the log structure on X. We have $r > 0$ only if some elements of $\mathcal{M}$ map to the nilradical of $\mathcal{O}_X$.

For elaboration, see ([31], II.2.3).

Example II.14. In the notation of Example II.9, the rank zero stratum of the affine plane X with its standard structure is the complement $U \cap V$ of the co-ordinate axes. On $U - V$ the rank of $\mathcal{M}$ is one, since the quotients $\mathcal{M}_p^{gp}/\mathcal{O}_{X,p}^*$ at each point $p \in U - V$ are copies of $\mathbb{Z}$, generated by the monomial y. Likewise on $V - U$ the rank of $\mathcal{M}$ is one. These punctured lines together form the rank one stratum of $(X, \mathcal{M})$. At the origin o, which is the single point of $X - (U \cup V)$, one has $\mathcal{M}_o^{gp}/\mathcal{O}_{X,o}^* \simeq \mathbb{Z}^2$, generated by the image of the standard chart $\mathbb{N}^2$. So the origin is the rank two stratum of $(X, \mathcal{M})$.

Example II.15. We might instead give the plane $X = \operatorname{Spec} k[x, y]$ a non-standard log structure $\mathcal{M}' \subseteq \mathcal{O}_X$ generated by the monomial xy. That is, the log structure is given by the chart $k[xy] \to k[x, y]$. The difference from the standard structure $\mathcal{M}$ is at the origin, where now the log structure $\mathcal{M}'$ has rank one. So it is the two co-ordinate axes together, including the origin of the plane, which are the rank one locus of $(X, \mathcal{M}')$. In particular this stratum is singular as a subscheme of X.

Example II.16. Consider the affine cone $X = \operatorname{Spec} k[x, y, z, w]/(xw - yz)$ over the quadric surface, with (non-standard) chart given by the projection

$$f : \operatorname{Spec} k[x, y, z, w]/(xw - yz) \to \operatorname{Spec} k[x, y]$$

to the plane. The rank zero stratum of X is the inverse image of the rank zero stratum $xy \neq 0$ of the plane, and the inverse image of the rank one stratum $(x = 0, y \neq 0) \cup (x \neq 0, y = 0)$ of the plane is a line bundle consisting of two components Z, Z' of the rank one stratum X_1 of X. Over the origin $(x, y) = (0, 0)$ the fibre $f^{-1}(0)$ of f is the plane $\operatorname{Spec} k[z, w]$. The lines $zw = 0$, which are the closure of the components Z, Z', are the rank two stratum X_2. The complement of X_2 in $f^{-1}(0)$ is another component Z'' of X_1, for on the locus $zw \neq 0$ in X the monomials x and y are related by units.

2.2.4 Cospecialisation

A point x of a fine log scheme $(X, \mathcal{M})$ which is in the closure of every component of every stratum is called a *central point* for the log scheme. When there is such, the stratum to which x belongs consists of central points. A monoid algebra $\operatorname{Spec} k[P]$ has a central point, namely the point corresponding to its maximal ideal $P - P^*$.

Remark II.17. A point x always has a neighbourhood of which it is a central point: delete the closures $\bar{Z}$ of components of strata of X such that $x \notin \bar{Z}$. This notion can simplify the local picture of the stratification of X, and appears in some basic local results. For example, the restriction map $\mathcal{M}(U)/\mathcal{O}(U)^* \to \mathcal{M}_x/\mathcal{O}^*_{X,x}$ is an isomorphism if x is a central point of U, as we recalled in the proof of Proposition II.12.

Suppose X has a central stratum, with generic point x. Let ξ be the generic point of a stratum of X. Then x lies in the closure of ξ, and there is a cospecialisation map on stalks of the sheaf $\mathcal{M}$

$$\mathcal{M}_x \to \mathcal{M}_\xi$$

which induces a surjection

$$P = \mathcal{M}_x/\mathcal{O}^*_{X,x} \to Q = \mathcal{M}_\xi/\mathcal{O}^*_{X,\xi}$$

modulo units of fine monoids. (See ([31], II.2.3.2).) In particular, there is a face $F \subseteq P$, consisting of the images of elements of $\mathcal{M}_x$ that map to units under

$$\mathcal{M}_x \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,\xi}$$

which maps to the identity class in Q. Then $Q = P/F$. Recall this means that the map $P \to Q$ then factors as

$$P \to F^{-1}P \to Q$$

where $P \to F^{-1}P$ is an inclusion inside P^{gp} and $F^{-1}P \to Q$ is the quotient of $F^{-1}P$ by its unit group.

Suppose there are sections $P, Q \to \mathcal{M}$, that is, P, Q are made into good charts (in the sense of Remark II.11), compatibly with the map $P \to Q$. This amounts to assigning values in $\mathcal{O}^*_{X,\xi}$ for the elements of F. Conversely a chart $P \to \mathcal{M}_x$ sending F into $\mathcal{O}^*_{X,\xi}$ induces a chart $Q \to \mathcal{M}_\xi$.

Example II.18. This discussion generalises the situation in Example II.9. The chart $\mathbb{N}^2 = \mathbb{N}x \oplus \mathbb{N}y$ is good at the origin of $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$, and $\mathbb{N}x$ is good on the open set $y \neq 0$. The maps

$$\mathbb{N}x \oplus \mathbb{N}y \to \mathbb{N}x \oplus \mathbb{Z}y \to \mathbb{N}x$$

on charts take y to the identity $1 \in \mathbb{N}x$. Different source charts $\mathbb{N}x \oplus \mathbb{N}u^{-1}y$ (with their obvious maps to $k[x, y]$) with the same quotient specialise y to different values u.

2.2.5 Closed log subschemes

In the category of ordinary schemes, Z becomes a closed subscheme of X through a map $i : Z \to X$ of schemes for which $i^*\mathcal{O}_X \to \mathcal{O}_Z$ surjects.

In the category of log schemes, one takes the following definition. A log scheme $(Z, \mathcal{M}_Z)$ becomes a closed log subscheme of $(X, \mathcal{M}_X)$ through a map $i : (Z, \mathcal{M}_Z) \to (X, \mathcal{M}_X)$ if $i : Z \to X$ makes Z a closed subscheme of X in the ordinary sense, and the map on log structures $i^*\mathcal{M}_X \to \mathcal{M}_Z$ surjects. Geometrically this second condition means that the locus of the log structure on Z becomes a closed subscheme of the locus of the log structure on X. There is a largest such locus on Z, namely the restriction of the locus on X. This is the case that $i^*\mathcal{M}_X \to \mathcal{M}_Z$ is an isomorphism; that is, that the morphism of log schemes i is strict. Following Kato [19], we will say that the closed log subscheme $i : (Z, \mathcal{M}_Z) \to (X, \mathcal{M}_X)$ is a *closed embedding* if i is strict.

CHAPTER III

Logarithmic jets and arcs

We develop the theory of the log arc scheme $J_\infty(X, \mathcal{M})$, of a log scheme $(X, \mathcal{M})$ over a field $(\operatorname{Spec} k, k^*)$. This builds upon the existing theory of log jet schemes $J_m(X, \mathcal{M})$.

The ordinary jet schemes $J_m(X)$ parametrise the infinitesimally thickened points

$$\operatorname{Spec} k[t]/(t^{m+1}) \to X$$

of X, while the ordinary arc scheme $J_\infty(X)$ parametrises the formal curves at points of X. If an arc

$$\operatorname{Spec} k[\![t]\!] \to X$$

is a bit of a curve on X, then the m-jets are like sketches of bits of curves up to m^{th} order derivatives, possessing specified velocity, acceleration, etc., but not their derivatives of all orders. An interpretation of the log jet schemes $J_m(X, \mathcal{M})$ as analogous parameter spaces in the category of log schemes was given by Karu and Staal [16].

Log jet bundles were studied by Noguchi [30] in the normal crossing case, before Kato's formalism for log structures appeared, and work has continued in this context, e.g. in [10]. The suggestion to found a general theory of log jet schemes using

Kato's log structures was offered by Vojta [33]. Taking up this proposal, the existence of the log jet schemes $J_m(X, \mathcal{M})$, with $(X, \mathcal{M})$ a log scheme over an arbitrary base log scheme, was proved by his student Dutter [11], who constructed their co-ordinate algebras by developing the theory of log Hasse-Schmidt differentials. This was patterned after Vojta's presentation [33] of ordinary jet schemes $J_m(X)$ through ordinary Hasse-Schmidt differentials. These differentials give co-ordinates on, and explicit equations for, jet schemes. Looking at the strata of a fine log scheme gives another important computational approach to log jet schemes.

We also elaborate on the theory of log smooth and log étale maps as they pertain to log jet scehemes. These morphisms were studied first by Kato [19]. Like their counterparts for ordinary schemes, these maps behave well with respect to our infinitesimal objects. The fundamental result on log smooth and log étale maps is Kato's Theorem III.18 characterising when a map of monoids induces a log smooth or log étale map on monoid algebras. As the log geometry category has more smooth objects than the ordinary category of schemes, so does it have more smooth or étale morphisms, whose degeneracies as maps of ordinary schemes will be controlled by the maps on log structures. For example, an equivariant blowup of a toric variety is a log étale morphism. The general characterisation Corollary III.21 of log smooth and log étale maps on log schemes follows from this study of maps of their charts, that is, of monoid algebras.

We end this chapter with an application to log arc schemes. For ordinary arc schemes there is a classical result of Kolchin [21] that the arc scheme of an irreducible variety X over a field of characteristic zero is irreducible. For the log arc scheme of a fine log scheme $(X, \mathcal{M})$, (still in characteristic zero), an irreducibility result cannot be expected without a non-degeneracy condition on the stratification of X, in view of our prior calculations. We introduce such a condition under the name *dimensional regularity,* which is an "expected rank" condition on the strata of $(X, \mathcal{M})$:

Definition III.1. (Definition III.47) Let X_j be the rank j stratum of an irreducible fine log scheme $(X, \mathcal{M})$. We say that $(X, \mathcal{M})$ is *dimensionally regular* if $\operatorname{codim} X_j + j$ is constant, for all j for which X_j is nonempty.

We then show that this is necessary and sufficient:

Theorem III.2. *(Theorem III.44) Let $(X, \mathcal{M})$ be a fine log scheme, with X irreducible. Then $J_\infty(X, \mathcal{M})$ is irreducible if and only if $(X, \mathcal{M})$ is dimensionally regular.*

The proof uses Kolchin's theorem for ordinary arc schemes to deform a log arc on $(X, \mathcal{M})$ into general position in its stratum. Then we can deform it into the next strata level, and repeat until it is in the minimum rank locus of $(X, \mathcal{M})$.

3.1 Log jets and log differentials

We begin with the functorial characterisation of log jet schemes and follow with their concrete realisation in terms of log Hasse-Schmidt differentials.

3.1.1 Ordinary jets and arcs

Let

$$j_m = \operatorname{Spec} k[\![t]\!]/(t^{m+1}),$$

for $m \geq 0$. An ordinary k-valued m-jet on X is a map $j_m \to X$. That is, if X is given over k by some equations $f_1, ..., f_r = 0$ in variables $x_1, ..., x_s$, then an m-jet on X is an assignment of truncated series $x_1(t), ..., x_s(t)$, defined up to order t^m, with co-efficients in k, to the variables such that the equations $f_j(x_1(t), ..., x_s(t)) = 0$ are satisfied up to order t^m. If we think of $x_i(t)$ as being given through its co-efficients

$$x_i(t) = a_{i,0} + a_{i,1}t + a_{i,2}t^2 + ... + a_{i,m}t^m$$

then these equations for the x_i turn on substitution into equations for the co-efficients $a_{i,d}$. The m-jets then have as parameter space the quotient of

$$k[a_{i,d} : 1 \leq i \leq s, 0 \leq d \leq m]$$

given by these equations. This quotient is the Hasse-Schmidt algebra of the affine scheme X [33], and its spectrum is the scheme $J_m(X)$ of m-jets of X.

More generally, an ordinary S-valued m-jet on X is a map $S \times_k j_m \to X$. The functor

$$S \mapsto \mathrm{Hom}(S \times_k j_m, X)$$

is representable, and we write $J_m(X)$ for the (ordinary) *jet scheme* that represents it. There are natural maps $J_m(X) \to J_n(X)$ for $m \geq n$ induced by the truncation $k[\![t]\!]/(t^{m+1}) \to k[\![t]\!]/(t^{n+1})$.

An ordinary k-valued arc on X is a map

$$j_\infty = \mathrm{Spec}\, k[\![t]\!] \to X$$

and an ordinary S-valued arc on X is a map $S \times_k j_\infty \to X$. Concretely, such a map has co-ordinates like as an m-jet does, but now the series $x_i(t)$ involved are not truncated to any order. There is again a scheme of such maps, denoted $J_\infty(X)$. It is the projective limit of the spaces $J_m(X)$ with their truncation maps.

For a detailed construction of these schemes and some of their basic machinery, one might see [12].

3.1.2 Log jets and arcs

Now let us give j_m its trivial log structure, which as a sheaf is just a copy

$$k[\![t]\!]/(t^{m+1})^* = \{a_0 + a_1 t + \cdots a_m t^m \bmod t^{m+1} \text{ such that } a_0 \neq 0\}$$

of the units in $k[\![t]\!]/(t^{m+1})$ at the sole point of j_m. Let S be a scheme, made into a log scheme through its final log structure $\mathcal{O}_S$. Recall from Section 2.2.2 that there is a product log scheme of S and j_m in the category of log schemes over $(\operatorname{Spec} k, k^*)$, whose underlying ordinary scheme is $S \times_k j_m$ and whose log structure is the pushout

$$\mathcal{O}_S \oplus_{k^*} \mathcal{O}_{j_m}^* \to \mathcal{O}_{S\times_k j_m} \simeq \mathcal{O}_S[t]/(t^{m+1})$$

of the respective log structures fibred over k^*. An S-valued log m-jet on $(X, \mathcal{M}_X)$ is then a map of log schemes

$$(S \times_k j_m, \mathcal{O}_S \oplus_{k^*} \mathcal{O}_{j_m}^*) \to (X, \mathcal{M}_X).$$

The functor

$$S \mapsto \operatorname{Hom}_{log}(S \times_k j_m, X)$$

(where we have left the log structures out of the notation on the right side) is representable, and we write $J_m(X, \mathcal{M}_X)$ for the *log jet scheme* that represents it. Again there are truncation maps $J_m(X, \mathcal{M}_X) \to J_n(X, \mathcal{M}_X)$ where $m \geq n$.

Replacing everywhere j_m by $j_\infty = \operatorname{Spec} k[\![t]\!]$, one has the definition of an S-valued log arc. The space of such is denoted $J_\infty(X, \mathcal{M}_X)$, and it is the projective limit of the schemes $J_m(X, \mathcal{M}_X)$ with their truncation maps.

Example III.3. It is worth writing down explicitly the data of a k-valued log jet on a log scheme $(X, \mathcal{M})$. Working locally near the image of $\operatorname{Spec} k[\![t]\!]/(t^{m+1}) \to X$, we may take $X = \operatorname{Spec} A$ with a chart P. Note that although with $S = \operatorname{Spec} k$ we have

$S \times_k j_m = j_m$ as schemes, the specified log structure on $\operatorname{Spec} k \times_k j_m$ comes from the product as log schemes; in fact it is the sum

$$k \oplus_{k^*} k[\![t]\!]/(t^{m+1})^* \stackrel{mult}{\to} \mathcal{O}_{j_m} = k[\![t]\!]/(t^{m+1})$$

of multiplicative monoids fibred over the units k^*, with log structure map given by taking products in $\mathcal{O}_{j_m}$. (In particular, the log structure on $\operatorname{Spec} k \times_k j_m$ is not any of the "obvious" choices, is not integral, and does not inject into $\mathcal{O}_{j_m}$.)

This log structure may be described, by shifting multiplicative constants to the left factor, as the (non-fibred) product

$$k \times (k[\![t]\!]/(t^{m+1}))^*/k^* \simeq k \times (1 + tk[\![t]\!]/(t^{m+1}))$$

of k and the principal units of $k[\![t]\!]/(t^{m+1})$, with the map to $\mathcal{O}_{j_m}$ being just the multiplication map

$$k \times (1 + tk[\![t]\!]/(t^{m+1})) \to k[\![t]\!]/(t^{m+1}).$$

We note that the image of this multiplication map consists only of the units of $k[\![t]\!]/(t^{m+1})$ together with zero.

Altogether, a map of log schemes

$$\operatorname{Spec} k \times_k j_m \to (\operatorname{Spec} A, P^a)$$

is equivalent to a commutative diagram

$$\begin{tikzcd} k \times (1 + tk[\![t]\!]/(t^{m+1})) \arrow[d, "mult"] & P \arrow[l] \arrow[d] \\ k[\![t]\!]/(t^{m+1}) & A. \arrow[l] \end{tikzcd} \tag{3.1}$$

Here the bottom row is the map of the underlying ordinary jet and the top row is the map on log structures. We will have occasion to refer to this diagram a number of times.

Example III.4. There is a similar description to Example III.3 for S-valued log m-jets when $S = \operatorname{Spec} B$ is an affine scheme. Namely, such a log jet on the log scheme $(\operatorname{Spec} A, P^a)$ is equivalent to a commutative diagram

$$\begin{tikzcd} B^* \oplus_{B^*} B[\![t]\!]/(t^{m+1})^* \arrow[d, "mult"] & P \arrow[l] \arrow[d] \\ B[\![t]\!]/(t^{m+1}) & A. \arrow[l] \end{tikzcd} \tag{3.2}$$

The monoid top left is the global sections of the log structure on the product log scheme $\operatorname{Spec} B \times_k j_m$. The appearance of the unit group $B[\![t]\!]/(t^{m+1})^*$ in place of $k[\![t]\!]/(t^{m+1})^*$ is due to the presence of the units on $\operatorname{Spec} B \times_k j_m$ in the log structure. See also [11], Definition 3.6 and following.

Remark III.5. One might also ask after $(S, \mathcal{M}_S)$-valued log m-jets (or arcs), with different log structures $\mathcal{M}_S$ on S than $\mathcal{O}_S$. For given m, this functor on the category of log schemes over $(\operatorname{Spec} k, k^*)$ is representable, say by log schemes $(Y_m, \mathcal{M}_{Y_m})$. Here the underlying scheme Y_m is just the scheme $J_m(X, \mathcal{M}_X)$ described above, considering only the structure $\mathcal{O}_S$ on S, because $S \mapsto (S, \mathcal{O}_S)$ is right adjoint to the forgetful functor from log schemes to schemes (because $(S, \mathcal{O}_S)$ is initial in the category of log schemes underlain by S).

Further, the log structure $\mathcal{M}_{Y_m}$ one obtains on $J_m(X, \mathcal{M}_X)$ from this construction in the category of log schemes is just that pulled back from X via the map $J_m(X, \mathcal{M}_X) \to X$. Indeed, any map $f : (Z, \mathcal{M}_Z) \to (X, \mathcal{M}_X)$ factors through $(Z, f^*\mathcal{M}_X)$, so a log S-jet γ on $(X, \mathcal{M}_X)$ factors through $(S \times_k j_m, \gamma^*\mathcal{M}_X)$, hence an S-point of $J_m(X, \mathcal{M}_X)$ factors through $(S, (\gamma\pi_m)^*\mathcal{M}_X)$. It is for this reason, that the log structure on $J_m(X, \mathcal{M}_X)$ in this sense introduces no new information, that we generally study $J_m(X, \mathcal{M}_X)$ as an ordinary scheme and not a log scheme.

3.1.3 Log Hasse-Schmidt differentials.

The existence of log jet schemes relative to arbitrary morphisms $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ was proven in [11] by construction in terms of log Hasse-Schmidt differentials. This was modelled on the construction in [33] of ordinary jet schemes in terms of ordinary Hasse-Schmidt differentials. For each $m \geq 0$, one constructs locally the m^{th} (log) Hasse-Schmidt algebra, the spectrum of which is the scheme of (log) m-jets.

Locally, for $X = \operatorname{Spec} A$ affine, one has the log Hasse-Schmidt algebra $HS^m_X(\mathcal{M})$, relative to the base log scheme $(\operatorname{Spec} k, k^*)$, as follows. Start with the polynomial ring over A on symbols $d_i f$ and $\partial_j p$, for every $f \in A, p \in \mathcal{M}(X)$, and $1 \leq i, j \leq m$. It is convenient to introduce notation $d_0 f = f$ for every $f \in A$ and $\partial_0 p = 1$ for every $p \in \mathcal{M}(X)$. Then take the quotient by the relations

(1) $d_i(f+g) = d_i f + d_i g$ for $f, g \in A$,

(2) $d_i c = 0$ for $c \in k$ and $i \geq 1$,

(3) $d_i \alpha(p) = \alpha(p)\partial_i p$ for $p \in \mathcal{M}(X)$, and

(4) the ordinary and "logarithmic" Leibniz rules

$$d_k(fg) = \sum_{\substack{0 \leq i,j \leq k \\ i+j=k}} d_i f d_j g$$

for $f, g \in A$ and

$$\partial_k(pq) = \sum_{\substack{0 \leq i,j \leq k \\ i+j=k}} \partial_i p \partial_j q$$

for $p, q \in \mathcal{M}(X)$.

Note that in the logarithmic Leibniz rule the terms $\partial_k p$ and $\partial_k q$ appear on the right side with co-efficients $1 = \partial_0 q = \partial_0 p$. In particular the case $k = 1$ asserts that

$$\partial_1(pq) = \partial_1 p + \partial_1 q.$$

One thinks of d_i as like a divided differential $\frac{1}{i!}d^i$, and ∂_1 as the logarithmic differential $d\log$, where

$$d\log f = \frac{df}{f}.$$

For $i \geq 2$, the differential ∂_i is not repeated logarithmic differentiation, but rather the differential

$$\partial_i p = \frac{d_i \alpha(p)}{\alpha(p)}$$

with a pole along the locus $\alpha(p) = 0$.

One sees, using the ordinary and "logarthmic" quotient rules, that this construction localises properly, so that the spectra of the locally constructed m^{th} log Hasse-Schmidt algebras on some $(X, \mathcal{M})$ glue to a global object, which is the space of log m-jets.

Example III.6. Continuing Example III.3, in terms of the k-valued log jets on $(X, \mathcal{M})$, the connection between the points $HS_X^m(\mathcal{M}) \to k$ of the log Hasse-Schmidt algebra and morphisms

$$P \to k \times 1 + tk[\![t]\!]/(t^{m+1})$$

is that in the latter the first component corresponds to values for the elements $\alpha(p) = d_0\alpha(p)$, and the co-efficients of the series in the second component correspond to values for $\partial_i p$.

Remark III.7. One sees easily from the description by Hasse-Schmidt algebras that the truncation maps $J_m(X, \mathcal{M}) \to J_n(X, \mathcal{M})$ are affine morphisms: for locally on X they just correspond to the maps $HS_X^n(\mathcal{M}) \to HS_X^m(\mathcal{M})$ induced by identifying a symbol $d_i f$ or $\partial_j p$ in $HS_X^n(\mathcal{M})$ with the same in $HS_X^m(\mathcal{M})$. The existence of the log arc scheme $J_\infty(X, \mathcal{M})$, given the existence of the log jet schemes $J_m(X, \mathcal{M})$, is then immediate.

Remark III.8. As above, one has

$$\partial_1\left(\prod p_j^{e_j}\right) = \sum e_j \partial_1 p_j.$$

For $k \geq 2$ one does not have the same identity, but may still write

$$\partial_k\left(\prod p_j^{e_j}\right) = \sum e_j \partial_i p_j + G_k$$

for some universal polynomial G_k in the lower-order differentials $\partial_i p_j$ with $i < k$ and the exponents e_j.

Remark III.9. In general one may not replace $\mathcal{M}(X)$ by an arbitrary chart for $\mathcal{M}$ in the construction of $HS_X^m(\mathcal{M})$. For instance, in Example II.10 when $m = 1$ the chart $\mathbb{N}^2$ gives both log differentials $\partial_1 x$ and $\partial_1(x-1)$, while the chart $\mathbb{N}$ gives only their sum $\partial_1(x(x-1)) = \partial_1 x + \partial_1(x-1)$. However, charts will do for computing the Hasse-Schmidt algebra locally on X.

Remark III.10. So far we have preferred multiplicative notation for our monoids, since in many cases we think of them just as sub-monoid sheaves of $(\mathcal{O}_X, \cdot)$. The log derivative $\partial = \partial_1$ gives an isomorphism from an integral monoid written multiplicatively to the same monoid written additively. Where we interpret the monoid written multiplicatively in terms of monomials, this interprets the monoid written additively as a monoid of first-order log differentials.

3.1.4 Log jets on strata

Every log jet on $(X, \mathcal{M})$ has an underlying ordinary jet, obtained by "forgetting" the map on log structures. That is, there is a natural map

$$J_m(X, \mathcal{M}) \to J_m(X)$$

of schemes over X. It is also the map induced on log jet spaces by the canonical log scheme morphism $(X, \mathcal{M}) \to (X, \mathcal{O}_X^*)$. Typically this map is neither surjective nor injective: not every ordinary jet underlies a log jet, and those that do may become log jets in more than one way.

With some minor hypotheses on a fine log scheme $(X, \mathcal{M})$, there is a simple abstract description of the natural map $J_m(X, \mathcal{M}) \to J_m(X)$ in terms of the stratification of X introduced in Section 2.2.3; namely, on each stratum it is an affine bundle map. To see this, let us study the log jets on $(X, \mathcal{M})$ extending a given ordinary jet γ. For convenience of notation let us suppose that γ is k-valued, i.e. that it gives a jet at a closed point x of X. But really the following discussion makes sense for the generic point ξ of the stratum to which x belongs, replacing k with the residue field of ξ.

Example III.11. Here is the gist of the following in terms of a "good-enough" chart on X. Suppose $(X, \mathcal{M}) = (\operatorname{Spec} A, P^a)$ is affine, and $\gamma : j_m \to X$ is a jet at the point x of X. Considering a log jet underlain by γ, according to (3.1) the map on log structures is a monoid morphism

$$\phi : P \to k \times 1 + t[\![k]\!]/(t^{m+1}).$$

The first component of this map is evaluation at the point x. There is a prime ideal I of P which maps to non-units under ϕ, that is, to elements with first component zero. These then map to zero in $k[\![t]\!]/t^{m+1}$. We see that the underlying ordinary jet γ lies generically in the locus $\alpha(I) \subseteq A$. This is the condition γ must satisfy to underlie a log jet.

Let $F = P - I$ be the complementary face of the prime ideal I. These are the elements which map to units. Suppose now that the projection $P^{gp} \to P^{gp}/F$ to the quotient generated by F in P^{gp} admits a section. For example, if P is a good chart at x, then $F = \{1\}$ and this is trivial. Then the second component of the morphism

ϕ is uniquely determined by an *arbitrary* group morphism

$$P^{gp}/F \to 1 + t[\![k]\!]/(t^{m+1}).$$

Note that the source group P^{gp}/F has rank equal to the rank of the log structure at x. This computes the fibre of the map $J_m(X, \mathcal{M}) \to J_m(X)$ at x. $\square$

So, we consider the maps of log schemes $j_m \to X$ with given underlying ordinary jet $\gamma : j_m \to X$. The map on log structure sheaves is a monoid morphism

$$\phi : \mathcal{M}_x \to k \times (k[\![t]\!]/(t^{m+1}))^*/k^*$$

with the following properties. First, on composition with the multiplication to $k[\![t]\!]/(t^{m+1})$, the non-units of $\mathcal{M}_x$ map to zero, because the image of the multiplication map consists only of units and zero. So ϕ annihilates the maximal proper ideal $\mathcal{M}_x - \mathcal{M}_x^*$ of $\mathcal{M}_x$ on composition with the map to $k[\![t]\!]/(t^{m+1})$. In particular, the underlying ordinary jet γ lies in the locus $\alpha(\mathcal{M}_x - \mathcal{M}_x^*) = 0$ in X. Second, the map ϕ is $\mathcal{O}_{X,x}^*$-equivariant, where $\mathcal{O}_{X,x}^*$ acts on the second factor of the target through γ.

Let us write $\phi = \alpha \times \beta$ as a product of two monoid morphisms. Note that α is the evaluation map at x. For as α sends $\mathcal{M}_x - \mathcal{M}_x^*$ to zero and agrees with

$$\gamma : \mathcal{M}_x^* = \mathcal{O}_{X,x}^* \to k = \mathcal{O}_{X,x}/\mathfrak{m}_x,$$

it factors through the map $\mathcal{O}_X \to \mathcal{O}_{X,x}/\mathfrak{m}_x$. In particular, this α is compatible with the log structure morphism $\alpha_{X,x} : \mathcal{M}_x \to \mathcal{O}_{X,x}$.

We are interested then in characterising the possible monoid morphisms

$$\beta : \mathcal{M}_x \to (k[\![t]\!]/(t^{m+1}))^*/k^* \simeq 1 + t[\![k]\!]/(t^{m+1})$$

which are $\mathcal{O}_{X,x}^*$-equivariant. Since the target of β is a group it is the same to replace $\mathcal{M}_x$ by $\mathcal{M}_x^{gp}$. Assume either that k has characteristic zero, or that k has characteristic

$p > 0$ and $\mathcal{M}_x^{gp}/\mathcal{O}_{X,x}^*$ has no p-torsion. Then the torsion part T of $\mathcal{M}_x^{gp}/\mathcal{O}_{X,x}^*$ has order relatively prime to the order of the torsion part of $1 + t[\![k]\!]/(t^{m+1})$, so a map

$$\overline{\beta} : \mathcal{M}_x^{gp}/\mathcal{O}_{X,x}^* \to 1 + t[\![k]\!]/(t^{m+1})$$

factors through the quotient by T, hence uniquely determines a map β given a chosen splitting of $\mathcal{M}_x^{gp} \to (\mathcal{M}_x^{gp}/\mathcal{O}_{X,x}^*)/T$. Therefore the possible $\overline{\beta}$ are parametrised by affine space $\mathbb{A}^{mj}$, where $j = \operatorname{rank} \mathcal{M}_x^{gp}/\mathcal{O}_{X,x}^*$ is the rank of $\mathcal{M}$ at x.

Proposition III.12. *([16] 3.2) Let $(X, \mathcal{M})$ be a fine log scheme and, for $m \geq 0$ or $m = \infty$, write $J_m(X, \mathcal{M})_j$ for the space of log m-jets or log arcs over the rank j stratum X_j. Then:*

(1) If $\operatorname{char} k = 0$ *or* $\operatorname{char} k = p$ *and $\mathcal{M}^{gp}/\mathcal{O}_X^*$ has no p-torsion on X_j, the natural map $J_m(X, \mathcal{M})_j \to J_m(X_j)$ is an affine bundle map with fibre $\mathbb{A}^{mj}$.*

(2) The natural map $J_\infty(X, \mathcal{M})_j \to J_\infty(X_j)$ is an affine bundle map with fibre the space of maps $\mathbb{Z}^j \to 1 + tk[\![t]\!]$.

Proof. The above discussion, which is the case $m \geq 0$, applies with the usual notational changes to the case $m = \infty$. In this case $1 + tk[\![t]\!]$ has no torsion, independent of the characteristic of k, so we do not need an additional hypothesis in (2). □

Corollary III.13. *Assume the notation and hypotheses of Proposition III.12(1). Then*

$$\dim J_m(X, \mathcal{M})_j = \dim J_m(X_j) + mj,$$

and, writing $\dim J_m(X, \mathcal{M})_j^{sm}$ *for the log m-jets over the smooth locus of X_j,*

$$\dim J_m(X, \mathcal{M})_j^{sm} = (m+1)(j + \dim X_j) - j.$$

□

Remark III.14. In the case where $\operatorname{char} k = p$ and $\mathcal{M}_x^{gp}/\mathcal{O}_{X,x}^*$ has a p-power-torsion subgroup T_p, there is in addition to the map

$$\tilde{\beta} : (\mathcal{M}_x^{gp}/\mathcal{O}_{X,x}^*)/T \to 1 + t[\![k]\!]/(t^{m+1})$$

some data to a log m-jet given by a map $T_p \to 1 + tk[\![t]\!]/(t^{m+1})$. When m is large enough compared to n the truncation map $J_m(X, \mathcal{M})_j \to J_n(X, \mathcal{M})_j$ has in its image only log n-jets corresponding to the trivial map on T_p, because under the map

$$1 + tk[\![t]\!]/(t^{m+1}) \to 1 + tk[\![t]\!]/(t^{n+1})$$

if $m - n \geq p$ an element not 1 has its torsion order decreased. This gives an alternate explanation of Proposition III.12(2) in the case of positive characteristic.

Example III.15. Here is an illustration of this in terms of differentials in a simple case. Let $(X, \mathcal{M})$ be the affine line $\operatorname{Spec} k[x]$ with its standard structure, generated by x. Both $J_1(X, \mathcal{M})$ and $J_1(X)$ are trivial $\mathbb{A}^1$-bundles over X. The former we may give co-ordinates $d_0 x = x$ and $\partial_1 x$, and the latter, $d_0 x$ and $d_1 x = d_0 x \cdot \partial_1 x$. In other words, the map $J_1(X, \mathcal{M}_X) \to J_1(X)$ here is one chart

$$k[d_0 x, d_1 x] = k[d_0 x, d_0 x \cdot \partial_1 x] \hookrightarrow k[d_0 x, \partial_1 x]$$

of the blowup of an affine plane, at the zero jet in $J_1(X)$. All of the log jets at $x = 0$ map to the zero jet in $J_1(X)$, and otherwise their co-ordinate $\partial_1 x$ is an arbitrary element of k. An ordinary jet at $x = 0$ which is non-zero (that is, does not generically lie in the stratum $x = 0$) underlies no log jet.

3.2 Log smooth and étale maps

The notions of unramified, smooth, and étale maps in the log scheme category may be defined by infinitesimal lifting properties in the same way as in the ordinary scheme-theoretic sense. That is, $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is log unramified, resp. log smooth, resp. log étale if whenever one has a diagram

$$\begin{tikzcd} (T', \mathcal{M}_{T'}) \arrow[r] \arrow[d] & (X, \mathcal{M}_X) \arrow[d] \\ (T, \mathcal{M}_T) \arrow[r] \arrow[ur, dotted] & (Y, \mathcal{M}_Y) \end{tikzcd}$$

where $(T', \mathcal{M}_{T'}) \to (T, \mathcal{M}_T)$ is an infinitesimal thickening, locally on T there is at most one, resp. at least one, resp. exactly one lift $(T, \mathcal{M}_T) \to (X, \mathcal{M}_X)$ making a commutative diagram. Many basic properties of differentials on log schemes will then follow formally in the usual way; see for example [19] or [31].

Remark III.16. What one must supply to make this work precisely is to say what "infinitesimal thickening" means in the log scheme category. Certainly $i : (T', \mathcal{M}_{T'}) \to (T, \mathcal{M}_T)$ should be underlain by an infinitesimal thickening, so that $\mathcal{I} = \ker(\mathcal{O}_T \to \mathcal{O}_{T'})$ is nilpotent, and i should be a strict morphism, so that it is a closed embedding in the log scheme sense.

However, with only this much still much degeneracy is possible in the log structures $\mathcal{M}_T, \mathcal{M}_{T'}$. Following ([31], IV.2.1.1) we also require that the subgroup

$$1 + \mathcal{I} = \ker(\mathcal{O}_T^* \to \mathcal{O}_{T'}^*) \subseteq \mathcal{O}_T^*$$

act freely on $\mathcal{M}_T$. The significance of this technical condition is in the details, but among its implications are: (1) that i is an *exact* morphism, meaning that a map to $\mathcal{M}_T$ is determined by its composite maps to $\mathcal{M}_{T'}$ and $\mathcal{M}_T^{gp}$, (2) that $1+\mathcal{I}$ injects into $\mathcal{M}_T^{gp}$, and is the kernel of the induced map $\mathcal{M}_T^{gp} \to \mathcal{M}_{T'}^{gp}$, and (3) that $1 + \mathcal{I}$ behaves like a kernel of the map $\mathcal{M}_T \to \mathcal{M}_{T'}$ in that if $t_1, t_2 \in \mathcal{M}_T$ have the same image in $\mathcal{M}_{T'}$ then $t_1 = ut_2$ in $\mathcal{M}_T$ for some $u \in 1 + \mathcal{I}$. For these facts see ([31], IV.2.1.2). None of these very mild consequences are automatic of a strict morphism underlain by an ordinary infinitesimal thickening: many pathologies are possible in the category of monoids. But, crucially, they will allow us to have the important Theorem III.18 characterising when maps of monoid algebras are log étale or log smooth.

One observation is that the log smooth or log étale maps which are also smooth or étale in the ordinary sense are the strict ones:

Proposition III.17. *([19] 3.8) Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a morphism of fine log schemes whose underlying map $X \to Y$ is smooth, resp. étale. Then f is log smooth, resp. log étale if and only if f is strict (i.e., $f^*\mathcal{M}_Y \simeq \mathcal{M}_X$).*

Proof. In an infinitesimal lifting situation one has a lift, resp. unique lift, of the underlying map on schemes, and therefore a lift, resp. unique lift as log schemes if and only if there is a map, resp. a unique map, on log structures compatible with the underlying map of schemes. This happens if and only if f is strict. □

Here is the important characterisation of when a map of monoid algebras is log smooth or étale.

Theorem III.18. *([19] 3.4; [31] IV.3.1.9) Let $\phi : Q \to P$ be a morphism of finitely generated monoids, $f : \operatorname{Spec} k[P] \to \operatorname{Spec} k[Q]$ its induced map on monoid algebras, and $\phi^{gp} : Q^{gp} \to P^{gp}$ its induced map on group completions. Then:*

(1) f is log étale if and only if ϕ^{gp} has finite kernel and cokernel, of orders prime to p if $\operatorname{char} k = p > 0$. In particular, if ϕ^{gp} is an isomorphism then f is log étale.

(2) f is log smooth if and only if ϕ^{gp} has finite kernel, and the kernel and the torsion subgroup of the cokernel have orders prime to p if $\operatorname{char} k = p > 0$. In particular, when $\operatorname{char} k = 0$, if ϕ^{gp} injects then f is log smooth.

Proof. We fill out the proof sketch of ([19], 3.4), wherein Kato works only étale-locally. To conclude the argument in the more difficult Zariski-local case, see ([31], IV.3.1.9).

Let an infinitesimal lifting diagram be given. We may assume that $\mathcal{I}^2 = 0$. A map $(T, \mathcal{M}_T) \to (X, \mathcal{M}_X) = (\operatorname{Spec} k[P], P^a)$ is completely determined by the map $P \to \mathcal{M}_T$. Since $(T', \mathcal{M}_{T'}) \to (T, \mathcal{M}_T)$ is exact, such a $P \to \mathcal{M}_T$ is equivalent to the given map $P \to \mathcal{M}_{T'}$ together with a map $P \to \mathcal{M}_T^{gp}$ lifting $Q \to \mathcal{M}_{T'}^{gp}$. Since the targets of these last are groups, it is the same to replace P, Q by their group completions. Hence the infinitesimal lifting problem is equivalent to the lifting problem for the diagram

$$\begin{tikzcd} \mathcal{M}_{T'} & P^{gp} \arrow[l] \\ \mathcal{M}_T \arrow[u] & Q^{gp} \arrow[l] \arrow[u] \end{tikzcd}$$

of monoids.

First, $\ker \phi^{gp}$ maps to $\ker(\mathcal{M}_T \to \mathcal{M}_{T'}) = 1 + \mathcal{I}$ in $\mathcal{M}_T$. But $1 + \mathcal{I} = 1 + \mathcal{I}/\mathcal{I}^2$ does not have torsion elements if $\operatorname{char} k = 0$, or torsion elements of order prime to $\operatorname{char} k = p$, so actually the finite group $\ker \phi^{gp}$ maps to $1 \in \mathcal{M}_T$. We get an induced map from $\operatorname{Im} \phi^{gp} \subseteq P^{gp}$ to $\mathcal{M}_T$.

Second, choose a minimal generating set $p_1, ..., p_r \in P^{gp}$ for the finite abelian group $\operatorname{coker} \phi^{gp}$ and lift their images in $\mathcal{M}_{T'}$ to some elements t_j in $\mathcal{M}_T$. If p_j has finite order e_j, so that $p_j^{e_j} = q_j \in \operatorname{Im} \phi^{gp}$, then $t_j^{e_j} = u_j q_j^{e_j}$ for some unit $u_j \in 1 + \mathcal{I}$. Hence we get our required lift $P^{gp} \to \mathcal{M}_T$ by taking p_j to $v_j q_j$ if $\mathcal{O}_T^*$ has a e_j^{th} root v_j of u_j, (and lifting p_j arbitrarily in the case it does not have finite order). Under the hypothesis on $\operatorname{coker} \phi^{gp}$, this can be arranged by passing to an étale neighbourhood in T. The conclusions follow. $\square$

Example III.19. The monoid algebra $\operatorname{Spec} k[P]$ with its standard structure is log smooth over k if the torsion part of P^{gp} is annihilated on tensoring with k. In particular, in characteristic zero every monoid algebra is log smooth over k.

Example III.20. The normalisation map of the plane cuspidal cubic with its standard structure as a monoid algebra (see Example II.2) is log étale, because the saturation map $P^{sat} \to P$ of an integral monoid induces an isomorphism on their group completions. More generally, every log scheme $(X, \mathcal{M}_X)$ has a universal map from a saturated log scheme $(X^{sat}, \mathcal{M}_{X^{sat}})$, and this is log étale ([31] IV.3.1.12).

Corollary III.21. *([19] 3.5, [31] IV.3.1.14) Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a map of log schemes, with finitely generated charts P, Q on X, Y such that the map $\mathcal{M}_Y \to f_*\mathcal{M}_X$ on log structures is induced by some $\phi : Q \to P$. Then*

(1) f is log étale if ϕ^{gp} has finite kernel and cokernel, of orders primes to p if $\operatorname{char} k = p > 0$*, and the map $X \to Y \times_{\operatorname{Spec} k[Q]} \operatorname{Spec} k[P]$ is étale in the ordinary sense.*

(2) f is log smooth if ϕ^{gp} has finite kernel, and the kernel and the torsion subgroup of the cokernel have orders prime to p if $\operatorname{char} k = p > 0$*, and the map $X \to Y \times_{\operatorname{Spec} k[Q]} \operatorname{Spec} k[P]$ is smooth in the ordinary sense.*

Proof. Follows from Theorem III.18, the formal fact that log smoothness and log étaleness is preserved by base change, and the fact that $X \to Y \times_{\operatorname{Spec} k[Q]} \operatorname{Spec} k[P]$ is strict (with Proposition III.17). $\square$

Remark III.22. In fact, there is a converse to the criterion Corollary III.21: if f is log smooth or log étale, then there exists a chart $Q \to P$ for f satisfying the given conditions. In this case, in (2) one can even take a chart $Q \to P$ such that $X \to Y \times_{\operatorname{Spec} k[Q]} \operatorname{Spec} k[P]$ is étale. For one may enlarge a given chart P by units

$u, u^{-1} \in \mathcal{O}_X^*$, thickening $\operatorname{Spec} k[P]$ by a factor $(\mathbb{A}^1)^*$, until $X \to Y \times_{\operatorname{Spec} k[Q]} \operatorname{Spec} k[P]$ is smooth of relative dimension zero.

Remark III.23. ([20] 11.6, [29] 2.6) For fine saturated log smooth schemes, the log structure is necessarily the pushforward of the trivial structure on its rank zero stratum.

3.2.1 Log arcs and log étale maps

Having introduced log jet and log arc schemes, our special interest in log étale maps arises from the following proposition, which is a formal analogue of the same fact for ordinary étale maps and ordinary jet and arc spaces.

Proposition III.24. *Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a log étale map of log schemes. Then the induced maps $f_m : J_m(X, \mathcal{M}_X) \to J_m(Y, \mathcal{M}_Y)$ for $m \geq 0$ or $m = \infty$ make the diagram*

$$\begin{tikzcd} J_m(X, \mathcal{M}_X) \arrow[r] \arrow[d] & J_m(Y, \mathcal{M}_Y) \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

a fibre square.

Proof. For $m \geq 0$ this follows from the functorial characterisation of log jet spaces and the definition of log étale maps. More precisely, an S-point of $X \times_Y J_m(Y, \mathcal{M}_Y)$ is naturally a diagram of log schemes

$$\begin{tikzcd} S \times_k j_0 \arrow[r] \arrow[d] & X \arrow[d] \\ S \times_k j_m \arrow[r] & Y \end{tikzcd}$$

(where we have omitted the log structures from the notation) because of the natural correspondences

$$\operatorname{Hom}(S, X) = \operatorname{Hom}(S, J_0(X, \mathcal{M}_X)) = \operatorname{Hom}_{log}(S \times_k j_0, X)$$

and

$$\mathrm{Hom}(S, J_m(Y, \mathcal{M}_Y)) = \mathrm{Hom}_{log}(S \times_k j_m, Y).$$

Now $S \times_k j_m \to S \times_k j_0$ is a log thickening, by Proposition III.25 following. So if $X \to Y$ is log étale then such diagrams biject naturally with log scheme lifts $S \times_k j_m \to X$, which are the same things as S-points of $J_m(X, \mathcal{M}_X)$.

Given this, the assertion of the proposition in the case $m = \infty$ follows on taking the projective limit of the maps on log jet spaces. $\square$

Proposition III.25. *For all* $0 \leq n \leq m$ *the truncation maps*

$$\pi_n^m : (S \times_k j_n, \mathcal{O}_S \oplus_{k^*} \mathcal{O}_{j_n}^*) \to (S \times_k j_m, \mathcal{O}_S \oplus_{k^*} \mathcal{O}_{j_m}^*)$$

are log infinitesimal thickenings.

Proof. Obviously the underlying morphism $S \times_k j_n \to S \times_k j_m$ is an infinitesimal thickening in the usual sense. Next, π_n^m is strict if $\mathcal{O}_S \oplus_{k^*} \mathcal{O}_{j_m}^*$ as a monoid sheaf on $S \times_k j_n$ generates the log structure $\mathcal{O}_S \oplus_{k^*} \mathcal{O}_{j_n}^*$. This is true because the sheaf $\mathcal{O}_{j_m}^*$ on j_n generates the trivial structure $\mathcal{O}_{j_n}^*$.

One sees easily that the group $1 + t^{n+1}k[\![t]\!]/(t^{m+1})$ acts freely on the log structure $\mathcal{O}_S \oplus_{k^*} k[\![t]\!]/(t^{m+1})^*$, because it does on $(k[\![t]\!]/(t^{m+1}))/k^* \simeq 1 + tk[\![t]\!]/(t^{m+1})$, satisfying the further condition of Remark III.16. $\square$

Example III.26. Consider one chart $f : \operatorname{Spec} k[x, y] \to \operatorname{Spec} k[x, xy]$ of the blowup of the plane at the origin, with their standard log structures. The map f is log étale, by Theorem III.18. For the map f is the monoid algebra morphism induced by the inclusion

$$\phi : \mathbb{N}x \oplus \mathbb{N}xy \to \mathbb{N}x \oplus \mathbb{N}y$$

of monoids, and ϕ^{gp} is an isomorphism (it is a unimodular linear transformation on $\mathbb{Z}x \oplus \mathbb{Z}y$). Of course, the map of schemes underlying f is far from étale at the origin.

According to Proposition III.24, the induced maps f_m on log m-jet spaces are base changes of f. That is, along the exceptional divisor $E = (x = 0) \subseteq \operatorname{Spec} k[x, y]$ the map f_m is just an $\mathbb{A}^1$-bundle. So f_m gives an *isomorphism* from the log jets at any point of E to the log jets at the origin of $\operatorname{Spec} k[x, xy]$!

We can describe this isomorphism in co-ordinates. At the point $y = 0$ on E, a k-valued log jet γ is a pair of principal series $x(t), y(t) \in 1 + tk[\![t]\!]/(t^{m+1})$, and the image $f_m\gamma$ is the pair $x(t), xy(t) = x(t)y(t)$. Away from $y = 0$, a k-valued log jet γ on E is a principal series $x(t)$ and a series

$$y(t) = d_0y + d_1yt + d_2yt^2 + ... = (d_0y)(1 + \partial_1yt + \partial_2yt^2 + ...)$$

with $d_0y \neq 0$, and the image $f_m\gamma$ is the pair

$$x(t), x(t)y(t)/(d_0y),$$

or $x(t), x(t)y(t) \bmod k^*$, of principal series. In any case the series $x(t)$ is a unit of $k[\![t]\!]/(t^{m+1})$, so that the data of $f_m\gamma$ conversely determine γ, by associating the pair $x(t), xy(t)/x(t)$ to a given jet $x(t), xy(t)$. In additive notation, this is the more familiar statement that, for a group $(G, +)$, the map $(x, y) \to (x, x+y)$ is an automorphism of G^2.

This behaviour is very different from that of the maps f induces on ordinary jets [23], where the jets γ on E break into piecewise $\mathbb{A}^e$-bundles over jets at the origin of $\operatorname{Spec} k[x, xy]$ according to the contact order $e = \operatorname{ord}_t x(t)$ of γ with E. $\square$

Several times so far in calculating with log jet or arc schemes we have replaced a monoid P with its group completion P^{gp}. The following lemma gives a precise sense in which the log jets or arcs of a fine log scheme typically depend only on the group completion of the log structure. According to Proposition III.17, the map f

appearing in it is not a log étale map (it is étale in the ordinary sense but not strict), although the hypothesis on ϕ^{gp} is the same as in Theorem III.18(1).

Lemma III.27. *Let $\phi : \mathcal{M} \to \mathcal{M}'$ be a map of fine log structures on X, with $f : (X, \mathcal{M}') \to (X, \mathcal{M})$ the corresponding map of log schemes. (That is, the underlying map on schemes is the identity on X.) Assume that the induced map*

$$\phi_x^{gp} : \mathcal{M}_x^{gp} \to \mathcal{M}_x'^{gp}$$

at some point x has finite kernel and cokernel, of orders prime to p if $\operatorname{char} k = p$. Then near x, for $m \geq 0$ or $m = \infty$ the maps

$$f_m : J_m(X, \mathcal{M}) \to J_m(X, \mathcal{M}')$$

are isomorphisms.

Proof. The map ϕ_x^{gp} has the same kernel and cokernel as the group completion of the map

$$\overline{\phi}_x : P = \mathcal{M}_x/\mathcal{O}_{X,x}^* \to Q = \mathcal{M}_x'/\mathcal{O}_{X,x}^*,$$

since ϕ_x is an isomorphism on the units $\mathcal{O}_{X,x}^*$ at the stalks of the two log structure sheaves. So by hypotheses we have a map $P \to Q$ such that the induced map $P^{gp} \otimes_{\mathbb{Z}} k \to Q^{gp} \otimes_{\mathbb{Z}} k$ is an isomorphism. In particular the map $P^{gp} \otimes_{\mathbb{Z}} k \to Q^{gp} \otimes_{\mathbb{Z}} k$ is given by an integer matrix invertible over k.

It follows that the log Hasse-Schmidt algebras $HS_X^m(\mathcal{M})$ and $HS_X^m(\mathcal{M}')$ calculated from P and from Q are equal. For as the first-order log differential ∂_1 is a monoid morphism from P or Q to $P^{gp} \otimes_{\mathbb{Z}} k$ or $Q^{gp} \otimes_{\mathbb{Z}} k$, we see that we get the same first log Hasse-Schmidt algebra $HS_X^1(\mathcal{M}) = HS_X^1(\mathcal{M}')$. In view of Remark III.8, we then have $HS_X^m(\mathcal{M}) = HS_X^m(\mathcal{M}')$ for $m \geq 1$ by induction, because the map on log differentials $\partial_m P \to \partial_m Q$ is given, up to a polynomial in lower-order differentials, by the same matrix as the map on lattices $\partial_1 P \to \partial_1 Q$ inside $P^{gp} \otimes_k \mathbb{Z} \to Q^{gp} \otimes_k \mathbb{Z}$. $\square$

3.2.2 Log blowups

Let P be a fine saturated monoid, $\operatorname{Spec} k[P]$ its monoid algebra. (We could also work over $\mathbb{Z}$ instead of a field k.) Let I be an ideal of P. It generates an ideal (I) of $k[P]$, and the blowup

$$\mathrm{Bl}_I(P) = \operatorname{Proj} \oplus_n (I)^n$$

has a covering by affine open sets

$$\operatorname{Spec} k[P\langle q^{-1}I\rangle],$$

for $q \in I$ running over a generating set, where $P\langle S\rangle$ denotes the saturation of the monoid generated by P and a fractional ideal $S \subseteq P^{gp}$ inside P^{gp}. Each chart comes with a dominant map

$$\operatorname{Spec} k[P\langle q^{-1}I\rangle] \to \operatorname{Spec} k[P]$$

induced by $P \hookrightarrow P\langle q^{-1}I\rangle$. This morphism is not strict: the standard log structure on the source is generated by the chart $P\langle q^{-1}I\rangle$ and not just P. Hence the standard log structure on $\mathrm{Bl}_I(P)$ is not that pulled back from P.

Since the ideal I becomes principal in $P\langle q^{-1}I\rangle$, generated by q, we see that these affine charts construct the log blowup in the sense of the universal fine saturated log scheme wherein I becomes principal.

The notion of log blowups was introduced by Kato in his unpublished paper [18]. See also [17] for an exposition on which ours is based.

Example III.28. Let $P = \mathbb{N}^2$ have monoid algebra $\operatorname{Spec} k[x, y]$ with its standard structure. Its blowup at the maximal ideal (x, y) of P is covered by two charts, namely those given by the fractional ideals (x, yx^{-1}) and (y, xy^{-1}). We get the usual toric blowup of the origin with standard structures.

Remark III.29. $\operatorname{Spec} k[P\langle (pq)^{-1}I\rangle]$ is an open subset of $\operatorname{Spec} k[P\langle q^{-1}I\rangle]$ if both $p, q \in I$, since then $P\langle (pq)^{-1}I\rangle$ is generated by P, p^{-1}, q^{-1}. This explains how the affine charts glue, as well as why it's enough to consider only generators of I in our affine cover.

Remark III.30. Suppose that $p \in P - P^*$ becomes a unit in $P\langle q^{-1}I\rangle$, i.e. that $p^{-1} \in P\langle q^{-1}I\rangle$. This means that some power p^{-n} lies in the (possibly unsaturated) monoid generated by $P, q^{-1}I$. So $p^{-n} = p_1 q^{-1} q_1$ for some $p_1 \in P, q_1 \in I$. This means that $q = p^n p_1 q_1$ in P. Since $p \notin P^*$, we see that q is properly divisble as an element of I (by an element of P).

It follows that $\mathrm{Bl}_I(P)$ is covered by *maximal* charts $\operatorname{Spec} k[P\langle q^{-1}I\rangle]$ where q ranges over indivisible elements of I as an ideal (meaning that q is not properly divisible in P by an element of I), and further that

$$P\langle q^{-1}I\rangle^* \cap P = P^*.$$

Example III.31. If P is not saturated, the monoids $P\langle q^{-1}I\rangle$ for q irreducible in I need not be maximal. For example, let $P = \{0, 2, 3, ...\} \subseteq \mathbb{N}$. The log blowup at its maximal ideal $I = \{2, 3, ...\} \subseteq P$ corresponds to the normalisation map $\operatorname{Spec} k[t] \to \operatorname{Spec} k[t^2, t^3] = \operatorname{Spec} k[P]$. Here the chart $P\langle t^{-2}I\rangle$ is the whole blowup, and $P\langle t^{-3}I\rangle$ is the open subset $\operatorname{Spec} k[t, t^{-1}]$.

Example III.32. It is possible to have $P^* = 1$ and still have $P\langle q^{-1}I\rangle$ not sharp. For example, let P be generated by monomials y, xy, x^2y, let $I = P - 1$ be its maximal ideal. Then $q^2 = y \cdot xy^2$, so $1 = q^{-1}y \cdot q^{-1}xy^2$. So both $q^{-1}y = x^{-1}$ and $q^{-1}xy^2 = x$ are units in $P\langle q^{-1}I\rangle$.

Remark III.33. The blowup we have defined stays inside the category of fine saturated log schemes. There is also a notion of unsaturated blowup. Compare Proposition III.36.

Remark III.34. Let us consider how strata behave under a blowup chart

$$Y = \operatorname{Spec} k[P\langle q^{-1}I\rangle] \to X = \operatorname{Spec} k[P].$$

Recall that the strata components of a fine sharp monoid algebra $\operatorname{Spec} k[Q]$ biject with the faces of the monoid Q. Under this bijection a face F of Q corresponds to the stratum component of $\operatorname{Spec} k[Q]$ on which F is invertible. This correspondence is inclusion-preserving in the sense that if one has a containment of faces then the closures of the corresponding strata components have the same containment.

For a face F of P, let the extension F^e of F in $P\langle q^{-1}I\rangle$ denote the smallest face (not necessarily proper) in $P\langle q^{-1}I\rangle$ which contains it. We say that F appears in $P\langle q^{-1}I\rangle$ if F, F^e have the same dimension; equivalently, if F is not contained in the relative interior of a face of $P\langle q^{-1}I\rangle$ of larger dimension. We say that a face F' of $P\langle q^{-1}I\rangle$ is a new face if it is not the extension of a face of P; equivalently, if $F' \cap P$ has smaller dimension than F'.

With this terminology we have the following description of the strata of the log blowup chart $Y \to X$. The stratum component Z in X corresponding to F is in the image of the blowup chart $Y \to X$ if F appears in $P\langle q^{-1}I\rangle$, and not if not. In the case that it does, the extension F^e is the face of $P\langle q^{-1}I\rangle$ corresponding to smallest stratum of the fibre over Z. If F' is a new face of $P\langle q^{-1}I\rangle$ containing F^e, then the stratum component Z' in Y corresponding to F' is part of the exceptional locus of the map $Y \to X$, and maps to $Z \subseteq X$. (Faces containing F^e which are not new correspond to the transforms of strata components in X which contain Z in their closure.) In particular, in a log blowup strata map generically onto strata, and furthermore the fiber over Z in Y is constant; for this description depends only on the face F corresponding to Z, and not to any point $x \in Z$.

For a fine saturated log scheme $(X, \mathcal{M})$ one constructs the blowup of X along an ideal sheaf $\mathcal{I} \subseteq \mathcal{M}$ as follows. Take local charts $U \to \operatorname{Spec} k[P]$ and blow up the preimage I of $\mathcal{I}$ in P. Then fibre U over this, $U \times_{\operatorname{Spec} k[P]} \mathrm{Bl}_I(P)$. Local universality for blowups of monoid algebras implies that the resulting pieces glue together to a log scheme $\mathrm{Bl}_{\mathcal{I}}(X, \mathcal{M})$, with a standard structure given by the chart morphisms to the pieces $\mathrm{Bl}_I(P)$.

Unlike ordinary blowups for ordinary schemes:

Proposition III.35. *Log blowups are log étale.*

Proof. Log blowups are modelled locally on chart morphisms that induce isomorphisms on groups, which are log étale by Theorem III.18. □

We note a few basic facts about log blowups to place them in some context.

Proposition III.36. *([29] 4.3) Let $(X, \mathcal{M})$ be fine saturated log regular, I a coherent ideal of $\mathcal{M}$. The log blowup $(\tilde{X}, \tilde{M}) \to (X, \mathcal{M})$ is underlain by a map of schemes that factors as $\tilde{X} \to Z \to X$, where $Z \to X$ is the blowup along $I\mathcal{O}_X$ and $\tilde{X} \to Z$ is normalisation.*

Proof. We note that the proof in the reference uses Kato's characterisation of log regularity in terms of the vanishing of certain Tor groups ([20] 6.1.iii). □

Proposition III.37. *([17], 3.8) The log blowup $\mathrm{Bl}_{\mathcal{I}}(X, \mathcal{M}) \to X$ is universal among fine saturated log schemes over X for which the ideal $\mathcal{I}$ of $\mathcal{M}$ is locally principal.* □

This universality means that log blowups of fine saturated log schemes behave like their ordinary counterparts, despite the normalisation step in Proposition III.36. One gets formally many similar results. For example, given two ideals $\mathcal{I}, \mathcal{I}'$ of $\mathcal{M}$, the blowup along $\mathcal{I}$ and then along (the pullback of) $\mathcal{I}'$ and the blowup along $\mathcal{I}'$ and then $\mathcal{I}$ are equal. In fact, both are equal to the blowup along the ideal generated by $\mathcal{I}, \mathcal{I}'$ together in $\mathcal{M}$ ([17], 3.10).

The next proposition characterises, in view of Theorem III.18, the log étale maps on monoid algebras which are log blowups.

Proposition III.38. *([17], 3.12) For a map $Q \to P$ of fine saturated monoids inducing an isomorphism $Q^{gp} \to P^{gp}$, the morphism $\operatorname{Spec} k[P] \to \operatorname{Spec} k[Q]$ is an open subset of a log blowup.*

Proof. Let P be generated in $P^{gp} = Q^{gp}$ by Q and fractions $a_i b^{-1}$ with $a_i, b \in Q$. The required blowup is along the ideal generated by a_i's and b. $\square$

3.3 Log arcs for monoid algebras

The log jet and log arc spaces of a monoid algebra $\operatorname{Spec} k[P]$ are simple to describe, because a map of log schemes $S \times_k j_m \to \operatorname{Spec} k[P]$ is completely determined by the map on log structures $P \to \mathcal{O}_S \oplus_{k^*} \mathcal{O}^*_{j_m}$. The map to the first factor (defined up to scaling by k^*) is the evaluation map of an S-point of $\operatorname{Spec} k[P]$, and the map to the second factor (defined up to scaling by k^*) is typically parametrised by affine space.

Proposition III.39. *Let P be a monoid, not necessarily finitely generated or integral, $X = \operatorname{Spec} k[P]$ its monoid algebra, with the standard log structure $\mathcal{M}$ generated by P, and P^{gp} the group completion of P. Then:*

(1) For $m \geq 0$ or $m = \infty$, the k-rational points of $J_m(X, \mathcal{M})$ are the trivial bundle over X with fiber the space of maps $P^{gp} \to 1 + tk[\![t]\!]/(t^{m+1})$.

(2) Assume $m = \infty$, or else $m \geq 0$ and either $\operatorname{char} k = 0$ or $\operatorname{char} k = p$ and P is p-power-saturated, (that is, if $f \in P^{gp}$ has $f^p \in P^{int}$ then in fact $f \in P^{int}$). If

P^{gp} has finite rank, the fiber of $J_m(X,\mathcal{M}) \to X$ is irreducible, hence $J_m(X,\mathcal{M})$ is irreducible if X is.

Proof. The data of a k-valued log jet or arc on X consists of a diagram

$$\begin{tikzcd} k \times (1 + tk[\![t]\!]/(t^{m+1})) \arrow[d] & P \arrow[l, "\alpha\times\beta"'] \arrow[d] \\ k[\![t]\!]/(t^{m+1}) & k[P] \arrow[l] \end{tikzcd}$$

Such a diagram is completely determined by the top arrow $\alpha \times \beta$. Now $\alpha : P \to k$ is a point of X and β is a map $P \to 1 + tk[\![t]\!]/(t^{m+1})$, equivalently a map

$$\beta : P^{gp} \to 1 + tk[\![t]\!]/(t^{m+1}).$$

So $J_m(X,\mathcal{M})$ is the bundle of such maps β. This gives the claim (1).

For the second, the maps β factor through the quotient of P^{gp} by its torsion in every case except when $\operatorname{char} k = p$, $m \geq 0$, and P^{gp} has p-torsion. But the presence of p-torsion in this case is what is ruled out by the saturation assumption on P. So the maps β are just from a free group, and the claim follows. $\square$

Remark III.40. That the log jet spaces are locally affine bundles over $\operatorname{Spec} k[P]$ when $\operatorname{char} k = 0$ or $\operatorname{char} k = p$ and P^{gp} has no p-torsion, as follows from Proposition III.39(1), is another expression of the fact that $\operatorname{Spec} k[P]$ is log smooth over k with these hypotheses (Example III.19).

Remark III.41. Here is an interpretation of the triviality of the bundle

$$J_\infty(X,\mathcal{M}) \to X = \operatorname{Spec} k[P]$$

over the monoid algebra X. The canonical forgetful map $J_\infty(X,\mathcal{M}) \to J_\infty(X)$ induces a map

$$T(J_\infty(X,\mathcal{M})) \to T(J_\infty(X))$$

on their corresponding total tangent spaces. Concretely, a k-point of $T(J_\infty(X, \mathcal{M}))$ is a collection of principal series

$$p_{log}(s,t) = 1 + \sum_{n \geq 1} (\partial_{0,n}p + \partial_{1,n}p\, s)t^n$$

and values $d_0 p = d_{0,0}p + d_{1,0}p\, s$ satisfying some conditions (the series $p_{log}(0,t)$ define a log arc on X, etc.), and the map to $T(J_\infty(X))$ multiplies these to give series

$$p(s,t) = \sum_{n \geq 0} d_{0,0}p\partial_{0,n}p\, t^n + (d_{0,0}p\partial_{1,n}p + d_{1,0}p\partial_{0,n}p)st^n.$$

Here the series

$$p(0,t) = \sum d_{0,0}p\partial_{0,n}\, t^n$$

give an ordinary arc on X at the point $p = d_{0,0}p$, and the co-efficient series

$$\sum (d_{0,0}p\partial_{1,n}p + d_{1,0}p\partial_{0,n}p)st^n$$

of s gives a tangent direction in $T(J_\infty(X))$ at that arc. What we wish to observe is that if $d_{0,0}p = 0$, that is, if the arc (log or ordinary) lies in a stratum with $p = 0$, then the co-efficient of s does not depend on the numbers $\partial_{1,n}p$, but only the numbers $\partial_{0,n}p$ and the scaling factor $d_{1,0}p$. In other words, it depends only on the log arc $p_{log}(0,t)$ and not on the tangent direction normal to $p = 0$ for $p_{log}(0,t)$. Geometrically this means that the map

$$T(J_\infty(X, \mathcal{M})) \to T(J_\infty(X))$$

factors through piecewise morphisms (on strata)

$$J_\infty(X, \mathcal{M}) \to T(J_\infty(X)).$$

This interprets a log arc $(p_{log}(t))$ alone as specifying an ordinary arc inside a stratum X_j of X together with a deformation of that arc into the rank zero stratum X^{gp} of X (where all $d_{0,0}p \neq 0$).

Remark III.42. Continuing the last Remark, it may be interesting to further study the natural map $J_\infty(X, \mathcal{M}) \to J_\infty(X)$ through its induced maps on (ordinary) jet or arc schemes, particularly the map

$$J_\infty(J_\infty(X, \mathcal{M})) \to J_\infty(J_\infty(X)).$$

This expectation is based in part on the above picture of the map $J_\infty(X, \mathcal{M}) \to J_\infty(X)$ as a kind of blowup, and the usefulness of motivic integration and arc schemes in general for proper birational maps of varieties. For another part, the "wedge scheme" $J_{\infty,\infty}(X) = J_\infty(J_\infty(X))$ of X is less well understood than the arc scheme $J_\infty(X)$ of X, especially in regard of its behaviour under proper birational maps, but when $(X', \mathcal{M}') \to (X, \mathcal{M})$ is log étale the induced map

$$J_\infty(J_\infty(X', \mathcal{M}')) \to J_\infty(J_\infty(X, \mathcal{M}))$$

should be simpler to analyse. $\square$

In the case that the monoid P is finitely generated, but $\operatorname{Spec} k[P]$ has a non-standard log structure generated by only a subset of its elements, in general the fibers of its log jet and log arc spaces will vary.

Proposition III.43. *Let P be a fine monoid, $X = \operatorname{Spec} k[P]$ its monoid algebra over a perfect field k. Let $B = \{x_1, ..., x_r, z_1, ..., z_s\}$ be a generating set for P, and let the (typically non-standard) log structure $\mathcal{M}$ on X be generated by the elements $x_1, ..., x_r$. Assume that X is irreducible and that the open set $U = (z_1 z_2 \cdots z_s \neq 0)$ meets every component of every stratum X_j of X. Then $J_\infty(X, \mathcal{M})$ is irreducible.*

Proof. By hypothesis, $U \cap X_j$ is dense in X_j for each j. By Kolchin's theorem, the ordinary arcs on $X_j \cap U$ are then dense in the ordinary arc space of X_j. In view of Proposition III.12, it is now enough to show that the log arcs over U are irreducible.

But after deleting the locus $z_1 z_2 \cdots z_s = 0$ what remains is isomorphic to X with its standard structure as a monoid algebra, so this follows from the last proposition. $\square$

This situation can occur for example when $\operatorname{Spec} k[P]$ is given a non-standard log structure arising from a face F of P generated by elements $x_1, ..., x_r \in F$.

3.4 Irreducibility over fine log schemes

As an illustration of the theory we have been developing, we consider the question of irreducibility for log arc schemes. For ordinary arc schemes in characteristic zero, a theorem of Kolchin in differential algebra [21] gives the answer: the arc scheme $J_\infty(X)$ of an irreducible variety X is irreducible. The theorem Kolchin proved concerns the prime ideals of differential algebras; for a modern algebro-geometric retelling of this see [13]. But for the special case of the arc scheme of a variety there are simpler proofs. Essentially, to show that every arc on X is a limit of arcs on the smooth locus of X, one begins by deforming a given arc into general position in a proper closed subset, perhaps the singular locus of X, and then uses some strong result in characteristic zero, like resolution of singularities (as in [12]), or Zariski's uniformisation theorem that a valuation ring is an inductive limit of formally smooth algebras (as in [27]), to see that it then deforms all the way into the smooth locus of X.

The irreducibility theorem for arc schemes contrasts the behaviour of jet schemes, where the singular locus of the base scheme X frequently gives rise to “extra” irreducible components in the jet scheme $J_m(X)$. Given a fine log scheme $(X, \mathcal{M})$ there is an additional elementary way to find “extra” components in the log jet scheme $J_m(X, \mathcal{M})$: if the rank of the log structure $\mathcal{M}$ is too large along some stratum of X compared to the codimension of the stratum then the log jet scheme $J_m(X, \mathcal{M})$ is thickened by an affine bundle along that stratum. There is no obstruction to lifting the bundle to higher order, so this carries over to the log arc scheme $J_\infty(X,\mathcal{M})$, which then has the same "extra" component. What we show is that avoiding this situation is sufficient: when the log structure $\mathcal{M}$ has the expected rank everywhere for its stratification, if X is irreducible then so is $J_\infty(X,\mathcal{M})$.

Theorem III.44. *Let k be a field of characteristic zero. Let $(X,\mathcal{M})$ be a fine log scheme over $(\operatorname{Spec} k, k^*)$, with X irreducible of finite type, and let $J_\infty(X,\mathcal{M})$ be its log arc scheme. Let X_j be the rank j stratum of $(X,\mathcal{M})$ and let r be the minimum rank of $\mathcal{M}$ on X. Then $J_\infty(X,\mathcal{M})$ is irreducible if and only if $\operatorname{codim}_X X_j = j-r$ for all non-empty X_j (that is, $(X,\mathcal{M})$ is dimensionally regular in the sense of Definition III.47).*

Our proof does not parallel any of the proofs we mentioned of Kolchin's theorem for arc schemes, but instead makes use of the irreducibility theorem for ordinary arcs on the strata of $(X,\mathcal{M})$.

Below we introduce first our condition of *dimensional regularity* and give it some context in log geometry. It is automatic, for example, when a fine saturated log scheme $(X,\mathcal{M})$ is log smooth, but is much weaker than this; it is just a combinatorial condition on the the stratification of X by the rank of $\mathcal{M}$ (see Proposition III.49). We then prove Theorem III.44. The plan is to show that the closure of the space of log arcs at one level of the stratification includes the log arcs on the smooth locus of the next level down, and then use Kolchin's theorem for ordinary arc schemes to see that the log arcs on the smooth locus of each stratum are dense in the log arc space of the whole stratum.

Remark III.45. We stated Kolchin's theorem and our Theorem III.44 for an irreducible scheme X. Another version of these is the statement that the irreducible components of the arc space $J_\infty(X) = J_\infty(X,\mathcal{O}_X^*)$ or log arc space $J_\infty(X,\mathcal{M})$ biject with the irreducible components of X through the projection maps $\pi : J_\infty(X,\cdot) \to X$, assuming each component of X is separately dimensionally regular. This statement with X allowed reducible implies a fortiori Theorem III.44, but the two are in fact equivalent (and we will pass between them as convenient). The reason for this is that an arc, ordinary or log, on X is contained inside some irreducible component Z of X. That is to say, a map $\operatorname{Spec} k[\![t]\!] \to X$ factors through an inclusion $i : Z \to X$, because $\operatorname{Spec} k[\![t]\!]$ is integral. It follows that the preimage $\pi^{-1}Z$ of Z is simply $J_\infty(Z)$ or $J_\infty(Z, i^*\mathcal{M})$. (This does not occur with jets in place of arcs.)

The same observation shows that an arc, ordinary or log, on X factors through the reduced structure X_{red} of X, so that it makes no difference whether we assume X to be reduced or not. (Again for jets it certainly makes a difference.)

Remark III.46. Arc schemes have been studied in positive characteristic as well (see for example [27]). In Section 3.4.4 we briefly consider how much of our argument for Theorem III.44 carries over to perfect fields of positive characteristic. There the condition of dimensional regularity is not sufficient for the log arc scheme to be irreducible, even when the underlying variety is smooth. We do not attempt to give a necessary and sufficient condition in this case.

Of course, since every ordinary arc scheme is also a log arc scheme (for the trivial log structure), the counterexamples to Kolchin's theorem over fields of positive characteristic apply also to the irreducibility theorem for log arc schemes.

3.4.1 Dimensional regularity

The following condition will appear in our discussion of reducibility and irreducibility for log arc spaces. Recall that the strata X_j of a fine log scheme $(X, \mathcal{M})$ were introduced in Section 2.2.3.

Definition III.47. We call a fine log scheme $(X, \mathcal{M})$ of pure dimension *dimension-*

ally regular if there is a number r, necessarily $r \geq 0$, such that for all non-empty strata X_j of X one has $\operatorname{codim}_X X_j = j - r$. The number r is then the smallest j such that X_j is non-empty.

A toric variety X with its standard structure, for example, is dimensionally regular, with $r = 0$: the strata components of X are its torus orbits, and the rank of each orbit is the orbit's codimension in X. On the other hand, if say the log structure $\mathcal{M}$ on a variety X comes from a reduced divisor D with $n > \dim X$ components passing through a single point x, so that the rank of $\mathcal{M}$ at x is n while the rank of $\mathcal{M}$ at a general point of X is zero, then $(X, \mathcal{M})$ will not be dimensionally regular; see for instance Example III.53, wherein D is three lines in the affine plane X all meeting at a point.

Remark III.48. If X is reducible, there is an induced log structure on any component $i : Z \to X$ by pulling back $\mathcal{M}$ to Z. Now if X is connected and has pure dimension, then $(X, \mathcal{M})$ is dimensionally regular if and only if $(Z, i^*\mathcal{M})$ is dimensionally regular for each component Z of X.

Assume for now that X is integral. The sheaf $\mathcal{I}$ of sections of $\mathcal{M}$ that map to zero in $\mathcal{O}_X$ forms a sheaf of prime ideals of $\mathcal{M}$. The map $\mathcal{M} \to \mathcal{O}_X$ factors through the quotient $\mathcal{M}/\mathcal{I}$, which may be identified with the monoid $\mathcal{M} - \mathcal{I}$ together with a zero element corresponding to the class $\mathcal{I}$. Now X_r is open and dense in X, where r is the minimum rank of $\mathcal{M}$ on X, and $\mathcal{M} - \mathcal{I}$ has rank zero on X_r. So if X is dimensionally regular the number r is just the height of the zero ideal $\mathcal{I}$ (in the sense of the height of prime ideals of a monoid, i.e., the codimension of the complementary face $\mathcal{M} - \mathcal{I}$) at every point.

So when X is integral there would not any loss from our point of view in just asking for $r = 0$, in other words working with $\mathcal{M} - \mathcal{I}$ rather than $\mathcal{M}$, since according to (the proof of) Proposition III.12 the difference is just the multiplication of the log jet spaces $J_m(X, \mathcal{M})$ everywhere by affine factors $\mathbb{A}^{mr}$. On the other hand it is convenient to allow $r > 0$ in general, because of the easy induction it allows. For example:

Proposition III.49. *Let $(X, \mathcal{M})$ be a fine log scheme, let r be the minimum rank of $\mathcal{M}$ on X, and let $Z(\mathcal{M}) = \cup_{j>r} X_j \neq \emptyset$ be the locus in X where the log structure has rank greater than r. Then the following are equivalent:*

(1) $(X, \mathcal{M})$ is dimensionally regular;

(2) $(Z(\mathcal{M}), i^*\mathcal{M})$ is dimensionally regular and X_{r+1} is non-empty;

(3) $(Z(\mathcal{M}), i^*\mathcal{M})$ is dimensionally regular and $Z(\mathcal{M})$ is the closure of X_{r+1};

(4) the set of indices j for which the strata X_j are non-empty is an interval $[a, b]$ in $\mathbb{N}$, and for all $a \leq j \leq \ell \leq b$ the stratum X_ℓ is contained in the closure of X_j.

Proof. The rank of $i^*\mathcal{M}$ on $Z(\mathcal{M})$ is the same as the rank of $\mathcal{M}$ on $Z(\mathcal{M})$ as a subset of X. Since $Z(\mathcal{M})$ is not empty, it has (pure) codimension one, and X is dimensionally regular with minimum rank r if and only if $Z(\mathcal{M})$ is dimensionally regular with minimum rank $r+1$ along all its strata components of codimension one. This gives the equivalence of (1) with (2) and (3). The characterisation (4) follows from the equivalence of (1) and (3) by induction. □

In addition to the conditions here, which are essentially in terms of the combinatorial relationships of the strata, we give another, more geometric characterisation in Proposition III.54.

3.4.2 Dimensional regularity in log geometry

Before proceeding we briefly place our condition of dimensional regularity in context. Essentially the same condition appears in ([16], 2.5), and for essentially the same reason as it appears here, namely, to control the size of the log jet schemes on strata. Thinking of log geometry in general, previously we noted that a toric variety with its standard log structure is dimensionally regular, with $r = 0$, because the strata of a toric variety are its torus orbits and the rank along each orbit is the orbit's codimension.

More than this, after ([20], 2.1) one may define the strata of $(X, \mathcal{M}_X)$ locally scheme-theoretically at point x by taking the ideal $I(\mathcal{M}_X, x)$ generated by the elements of $\alpha(\mathcal{M}_{X,x})$ which vanish at x. Then one says that $(X, \mathcal{M}_X)$ is *log regular* if it is dimensionally regular with constant $r = 0$ and the strata, so defined scheme-theoretically, are regular.

If $(X, \mathcal{M})$ is a fine saturated log smooth scheme then it is log regular. Kato showed that over a perfect field the converse holds.

Proposition III.50. *([20] 8.3) Let $(X, \mathcal{M}_X)$ be a fine and saturated scheme over $(\operatorname{Spec} k, k^*)$, where k is a perfect field. Then X is log smooth if and only if it is log regular.* $\square$

This obviously subsumes the fact that toric varieties are dimensionally regular.

Dimensional regularity, however, is a much weaker condition than log smoothness.

Example III.51. Let X be the affine plane with non-standard log structure given by the chart $k[xy] \to k[x, y]$ (see Example II.15). Then $X_1 = \operatorname{Spec} k[x, y]/(xy)$ is singular. So X is not log smooth. But it is dimensionally regular.

Example III.52. Consider the affine line $X = \operatorname{Spec} k[t]$ with log structure given by the map $k[x,y] \to k[t]$ which takes both generators x, y to t. This is the induced structure on the diagonal $x = y$ of the plane $\operatorname{Spec} k[x,y]$. The rank of the log structure on X is zero away from the origin but jumps to two at $t = 0$, so this log scheme is not dimensionally regular.

Example III.53. Take $\operatorname{Spec} k[x,y]$ with log structure generated by x, y, and $x - y$. So the log structure is supported on the three lines $x = 0, y = 0, x = y$, on which it has rank one, except at the origin, where it has rank three. Similar to the previous example, this can be realised as the induced structure on the plane $z = x - y$ in $\operatorname{Spec} k[x,y,z]$ with its standard structure.

Proposition III.54. *Let $(X, \mathcal{M})$ be an irreducible fine log scheme with a sharp chart P near a point x. For each irreducible component Z of the stratification of X which contains x, choose units $u_1, ..., u_{\dim Z} \in \mathcal{O}_{X,x}$ which are algebraically independent in the residue field $\mathcal{O}_{X,Z}/\mathfrak{m}_Z$ of $\mathcal{O}_{X,Z}$. Let $Q = P \oplus \mathbb{N}^{\dim Z}$ be the chart P expanded to include these units. This Q is also a chart for $(X, \mathcal{M})$ over a neighbourhood U of x over which $u_1, ..., u_{\dim Z}$ are defined.*

Then X is dimensionally regular near x of minimum rank $r = 0$ if and only if the chart morphisms $U \to \operatorname{Spec} k[Q]$ for the various components Z through x are dominant.

Proof. Let j be the rank of $\mathcal{M}$ along Z, and let $x_1, ..., x_j \in \mathfrak{m}_Z$ be the images of generators in P for the log structure on Z.

If $j > \operatorname{codim}_X Z$ then the elements $u_1, ..., u_{\dim Z}, x_1, ..., x_j \in \mathcal{O}_{X,Z}$ number more than the transcendence degree $\dim X$ of the fraction field of $\mathcal{O}_{X,Z}$ over k, so satisfy a polynomial equation $G = 0$ with coefficients in k. The image of $U \to \operatorname{Spec} k[Q]$ is then contained in the proper locus $G = 0$.

On the other hand if $j \leq \operatorname{codim}_X Z$ then there is no such polynomial equation $G = 0$ over k. For supposing there were, let d be the smallest number such that every term of G lies in $\mathfrak{m}_Z^d$, and consider the equation $G = 0$ modulo $\mathfrak{m}_Z^{d+1}$. Writing the monomials in $x_1, ..., x_j$ that appear in G modulo $\mathfrak{m}_Z^{d+1}$ in terms of a basis of $\mathfrak{m}_Z^d / \mathfrak{m}_Z^{d+1}$ over the residue field $\mathcal{O}_{X,Z}/\mathfrak{m}_Z$, we obtain a polynomial with co-efficients in $\mathcal{O}_{X,Z}/\mathfrak{m}_Z$ satisfied by the units $u_1, ..., u_{\dim Z}$, contrary to assumption. □

3.4.3 The irreducibility theorem for log arc schemes

The necessity of dimensional regularity in Theorem III.44 is a straight-forward consequence of Proposition III.12. Recall that for a fine log scheme $(X, \mathcal{M})$ we write $J_m(X, \mathcal{M})_j$ and $J_\infty(X, \mathcal{M})_j$ for the log m-jets and log arcs over the stratum X_j.

Proposition III.55. *Let $(X, \mathcal{M})$ be a fine log scheme with X irreducible. If $(X, \mathcal{M})$ is not dimensionally regular then $J_\infty(X, \mathcal{M})$ is reducible.*

Proof. We may assume that X is reduced. Let r be the minimum rank of $\mathcal{M}$ on X and let $j > r$ be such that X_j has codimension less than $j - r$.

Write $J_m(X, \mathcal{M})_\ell^{sm}$ for the log m-jets of X over the smooth locus of a stratum X_ℓ. According to Corollary III.13, for m large enough we have

$$\dim J_m(X, \mathcal{M})_j^{sm} > \dim J_m(X, \mathcal{M})_r^{sm}.$$

Let $C_m \subseteq J_m(X, \mathcal{M})$ be the image of $J_\infty(X, \mathcal{M})$ under the natural projection. It is the constructible subset of log m-jets which extend to log arcs. If $J_\infty(X, \mathcal{M})$ is irreducible then so is C_m (as a topological space), and further C_m has $J_m(X, \mathcal{M})_r^{sm}$ as a dense subset. But C_m also contains $J_m(X, \mathcal{M})_j^{sm}$, a contradiction. □

We turn to establishing the converse implication. We will make use of the following observation:

Proposition III.56. *Let X_j be a stratum of a fine log scheme $(X, \mathcal{M})$. The irreducible components of $J_\infty(X, \mathcal{M})_j$ biject with the irreducible components of X_j, and the log arcs over the smooth locus X_j^{sm} of X_j are dense in $J_\infty(X, \mathcal{M})_j$.*

Proof. This follows from Kolchin's theorem for the ordinary jets on the stratum X_j, together with the characterisation of the natural map $J_\infty(X, \mathcal{M})_j \to J_\infty(X_j)$ as a bundle map. □

Perhaps tendentiously, this suggests the strategy of describing the components of $J_\infty(X, \mathcal{M})$ by comparing the arcs on one level of the stratification with the arcs on the smooth locus of the next. The simplest situation (apart from the base case $X = X_r$, where the result is just Kolchin's theorem) is in the following lemma. It will be the inductive step of our argument.

Lemma III.57. *Let $(X, \mathcal{M})$ be a fine log scheme with X irreducible and such that $X = X_r \cup X_s$ for some $s > r$, (with X_r, X_s non-empty). Then $J_\infty(X, \mathcal{M})$ is irreducible if and only if $s = r + 1$.*

Proof. The necessity of the condition $s = r+1$ is a special case of Proposition III.55. We consider the converse implication. We may assume that X is reduced.

We may replace X by an open subset which meets X_s and does not meet the singular loci of X_r and X_s, and thereby assume first that the singular locus of X lies inside X_s and second that X_s is smooth. For by Proposition III.56 the arcs over this open subset are dense in $J_\infty(X, \mathcal{M})$. We may further replace the sheaf $\mathcal{M}$ by the complementary face of its zero ideal, and hence assume that $r = 0$ and $s = 1$.

By assumption, either X is smooth or it is singular along X_1. Consider the normalisation $\tilde{X} \to X$, with log structure $\tilde{\mathcal{M}}$ pulled back from X. This is an isomorphism over the smooth locus X_0, and after deleting from X a (positive-codimension) subset of X_1 we may assume that $\tilde{X}$, which was non-singular in codimension one, is in fact smooth, and that the normalisation is unramified, hence étale, over X_1. In particular, the map $J_\infty(\tilde{X}, \tilde{\mathcal{M}}) \to J_\infty(X, \mathcal{M})$ surjects. Thus in any case we are reduced to the situation where X is smooth.

Now let $\mathcal{M}'$ be the log structure on X generated by local equations for the divisor X_1. Near a component Z of X_1 with generic point ξ this has a chart $\mathcal{O}_{X,\xi}/\mathcal{O}^*_{X,\xi} \simeq \mathbb{N}$ and the local valuation maps

$$\mathcal{M}_\xi \to \mathcal{M}_\xi/\mathcal{O}^*_{X,\xi} \to \mathcal{O}_{X,\xi}/\mathcal{O}^*_{X,\xi}$$

determine a morphism $\mathcal{M} \to \mathcal{M}'$ of log structures on X. Lemma III.27 applies to this morphism, and we may replace $\mathcal{M}$ by $\mathcal{M}'$.

In particular we now are reduced to having $(X, \mathcal{M})$ fine, saturated, and log regular. According to Proposition III.50 this means that X is log smooth, which as one may expect is sufficient for the conclusion we seek for abstract reasons. However, a short calculation finishes the argument in any case. For now locally near $Z \subseteq X_1$ there is the chart $X \to \operatorname{Spec} k[x]$, where x is a local equation for Z, a smooth morphism. So there is locally an étale map from X to some affine space $Y = \operatorname{Spec} k[x, z_1, ..., z_s]$ over $\operatorname{Spec} k[x]$, with log structure generated by x. By Proposition III.43, or else by an easy direct computation, the log arc (and log jet) spaces of Y with this log structure are irreducible, and the log arcs over the complement of the hyperplane $x = 0$ are dense. An étale map of schemes which is strict is log étale, so the induced map on log arc spaces is also étale. According to the follow lemma, the claim follows. □

Lemma III.58. *Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a log étale map of log schemes, $f_m : J_m(X, \mathcal{M}_X) \to J_m(Y, \mathcal{M}_Y)$ the induced map on log jet or log arc spaces for $m \in \mathbb{N}$ or $m = \infty$. Then:*

(1) If f is étale (in the ordinary sense) then so is each f_m.

(2) If X and Y are irreducible, and f is étale and surjective, then $J_m(X, \mathcal{M}_X)$ is irreducible if and only if $J_m(Y, \mathcal{M}_Y)$ is.

Proof. (1) follows from Proposition III.24, which says that f_m is a base change of f. In (2), the base changes f_m are likewise étale and surjective. It's immediate that $J_m(Y, \mathcal{M}_Y)$ is irreducible if $J_m(X, \mathcal{M}_X)$ is. Conversely if $J_m(X, \mathcal{M}_X)$ is reducible then it has a component Z lying over some proper closed subset $X_1 \subseteq X$. The image of an open subset U of Z is then open in $J_m(Y, \mathcal{M}_Y)$ and lying over the proper closed subset $Y_1 = f(\bar{X}_1) \subseteq Y$. But if $J_m(Y, \mathcal{M}_Y)$ is irreducible it has no open subsets contained in the fibre over Y_1. So $J_m(Y, \mathcal{M}_Y)$ is reducible. □

We are ready to argue for Theorem III.44. We re-state the Theorem now.

Proposition III.59. *Let $(X, \mathcal{M})$ be a fine log scheme with X irreducible. Then $J_\infty(X, \mathcal{M})$ is irreducible if and only if $(X, \mathcal{M})$ is dimensionally regular.*

Proof. After Proposition III.55 it remains to show that if $(X, \mathcal{M})$ is dimensionally regular then $J_\infty(X, \mathcal{M})$ is irreducible.

Let r be the minimum rank of $\mathcal{M}$ on X. By Lemma III.57 applied to the components of

$$X_r \cup X_{r+1} = X - \cup_{j>r+1} X_j,$$

the log arcs $J_\infty(X, \mathcal{M})_r$ are dense in $J_\infty(X, \mathcal{M})_{r+1}$. Since $\cup_{j>r+1} X_j$ is contained in the closure of X_{r+1}, by Proposition III.49, by induction $J_\infty(X, \mathcal{M})_{r+1}$ is dense in $\cup_{j \geq r+2} J_\infty(X, \mathcal{M})_j$. Therefore the irreducible $J_\infty(X, \mathcal{M})_r$ is dense in $J_\infty(X, \mathcal{M})$. So $J_\infty(X, \mathcal{M})$ is irreducible. □

3.4.4 Remarks on positive characteristic.

Let us consider briefly the case where k has $\operatorname{char} k = p > 0$. For ordinary arc schemes, given an irreducible variety X with non-smooth locus $Y \subseteq X$, the scheme $J_\infty(X) - J_\infty(Y)$ is irreducible ([27], 3.15). In other words, if $J_\infty(X)$ is reducible, it is because there are arcs lying generically in the non-smooth locus of X which are not limits of arcs on the smooth locus of X. This can occur over any field k of positive characteristic.

Considering fine log schemes, we easily see that the conditions of Theorem III.44 are no longer sufficient even when the underlying variety is smooth in the ordinary sense:

Example III.60. Let $X = \operatorname{Spec} k[x]$ with log structure $\mathcal{M}$ generated by x^p. Away from $x = 0$ the k-valued log arcs are the ordinary arcs $x(t) = d_0x + d_1xt + ...$ with $d_0x \neq 0$, which then have

$$x(t)^p = (d_0x)^p + (d_1x)^p t^p + ... = (d_0x)^p(1 + (\partial_1 x)^p t^p + ...).$$

The limits of these at $x = 0$ are log arcs, underlain by the zero ordinary arc, in which only powers of t^p appear in the principal series $x^p(t)$. But a log arc at the origin corresponds to an arbitrary principal series $x^p(t) = 1 + \partial_1(x^p)t + ...$ So the log arcs over the locus $x \neq 0$ are not dense in $J_\infty(X, \mathcal{M})$.

Remark III.61. Here is one way to look at this example. There is the map

$$g : (\operatorname{Spec} k[x], x) \to (\operatorname{Spec} k[x], x^p)$$

to X from the affine line with its standard structure. This is modelled on the map $p\mathbb{N} \to \mathbb{N}$ on charts, whose corresponding map on monoid algebras is the inseparable map $\operatorname{Spec} k[x] \to \operatorname{Spec} k[x^p]$. Now the induced map $p\mathbb{Z} \to \mathbb{Z}$ on group completions does not become an isomorphism on tensoring with k. So although g is an isomorphism away from the origin $x = 0$, it does not induce a surjection of log arc spaces at the origin.

Aside from this obstruction, our proof of Theorem III.44 fails in general in positive characteristic at the normalisation step in Lemma III.57. Again the issue is of having log structure on a locus that gives rise to an inseparable map.

Of course one may give a sufficient criterion just by assuming the conclusions these steps were meant to obtain. So let $(X, \mathcal{M})$ be a fine log scheme over a perfect field k with $\operatorname{char} k = p$. Assume

(1) $(X, \mathcal{M})$ is dimensionally regular of minimum rank r,

(2) for each $j \geq r$, the arc scheme $J_\infty(X_j)$ is irreducible,

(3) for each $j \geq r$, for each generic point ξ of the scheme $\overline{X_j}$ in the codimension one locus $X_{j+1} = \overline{X_j} - \overline{X_{j+2}}$, the local ring $\mathcal{O}_{\overline{X_j},\xi}$ is regular and the valuation map

$$\phi : \mathcal{M}_\xi/\mathcal{O}^*_{X,\xi} - I \to \mathcal{O}_{\overline{X_j},\xi}/\mathcal{O}^*_{\overline{X_j},\xi} \simeq \mathbb{N}$$

(where I is the zero ideal of $\mathcal{M}_\xi/\mathcal{O}^*_{X,\xi}$) is such that $\phi^{gp} \otimes k$ is an isomorphism; that is, $\mathcal{M}^{gp}_\xi/\mathcal{O}^*_{X,\xi} - I$ has rank one as an abelian group and no p-torsion part, and the image of ϕ is not contained in the submonoid $p\mathbb{N}$ of multiples of p.

Then $J_\infty(X, \mathcal{M})$ is irreducible if X is. That a log smooth scheme, whose log arc scheme is necessarily irreducible, need not satisfy (3) illustrates that this criterion is not necessary.

Example III.62. Nor is condition (2) here necessary in general. Consider the affine space $X = \operatorname{Spec} k[x, y, z]$ over a field of characteristic p with log structure $\mathcal{M}$ given by the inclusion

$$k[x^p - yz^p] \to k[x, y, z].$$

That is, X has log structure along the locus X_1 given by the equation $x^p - yz^p = 0$. It is observed in ([28], Rmq. 1) that $J_\infty(X_1)$ is reducible, because in X_1 along the nonsmooth locus $z = 0$ there are arcs with $d_1 y \neq 0$, but on the open set $z \neq 0$ every arc has $d_1 y = 0$.

Now let $x(t), y(t), z(t)$ be any arc in $J_\infty(X_1)$. That is, $x(t)^p - y(t)z(t)^p = 0$. The log arcs $J_\infty(X, \mathcal{M})_1$ are pairs consisting of these ordinary arcs together with arbitrary principal series $f(t) \in 1 + tk[\![t]\!]$. Given such data, we may choose deformations $x_s(t), y_s(t), z_s(t)$, giving an ordinary arc in X_0 for any $s \neq 0$ and specialising to the given arc in X_1 at $s = 0$, such that the principal series defined by

$$f_s(t) = \frac{x_s(t)^p - y_s(t)z_s(t)^p}{x_s(0)^p - y_s(0)z_s(0)^p}$$

for $s \neq 0$ is a deformation of the series $f_0(t) = f(t)$. Then when s is specialised to zero the deformation arc in $J_\infty(X_0)$ has limit the chosen log arc in $J_\infty(X, \mathcal{M})_1$. Since $J_\infty(X_0)$ is irreducible it follows therefore that $J_\infty(X, \mathcal{M})$ is irreducible.

CHAPTER IV

Integration on log arc schemes

We develop a motivic integral on log arc schemes $J_\infty(X, \mathcal{M})$. Our approach will be to construct integrals and make calculations in a way that is modelled on analogous calculations for motivic integration on ordinary arc schemes $J_\infty(X)$.

We work over a field k of characteristic zero. Recall that if $(X, \mathcal{M})$ is a fine log scheme with a good chart P at a stratum component Z there is locally an isomorphism

$$J_\infty(X, \mathcal{M})_Z \simeq J_\infty(Z) \times \mathrm{Hom}(P, 1 + tk[\![t]\!])$$

describing the log arcs on Z. Since log motivic integration should restrict to ordinary motivic integration in trivialising cases, what will happen with $J_\infty(Z)$ here should just be induced by the ordinary theory. From this point of view, it is the second factor $\mathrm{Hom}(P, 1+tk[\![t]\!])$ that we need to study. This is the first task we will take up. The analogy here begins with the basic case of ordinary motivic integration on the power series ring $k[\![t]\!]$ (which geometrically is the space of arcs on the affine line), or p-adic integration on the complete ring $\mathbb{Z}_p$ of p-adic integers – and now to log motivic integration on the group $\Lambda = 1 + tk[\![t]\!]$. Once we have a measure μ on $\mathrm{Hom}(P, \Lambda)$ we may define a log motivic integral on the stratum component Z by

$$\int_{J_\infty(X,\mathcal{M})_Z} \phi_Z \, d\mu = \int_{J_\infty(Z)\times \mathrm{Hom}(P,\Lambda)} \phi_Z \, d\mu,$$

integrating a suitable function ϕ_Z defined on $J_\infty(X, \mathcal{M})_Z$ against the product measure on $J_\infty(Z) \times \mathrm{Hom}(P, \Lambda)$. Summing such pieces over the strata of $(X, \mathcal{M})$ gives our log motivic integral on $(X, \mathcal{M})$.

There is also a geometric interpretation of the group $\mathrm{Hom}(P, 1 + tk[\![t]\!])$ in terms of the chart morphism $X \to \operatorname{Spec} k[P]$. As we shall see, in characteristic zero there is an isomorphism

$$(1 + tk[\![t]\!], \cdot) \simeq (tk[\![t]\!], +)$$

of Λ and the additive group of the maximal ideal of $k[\![t]\!]$. Now morphisms $P \to tk[\![t]\!]$ may be viewed as ordinary arcs on the log tangent space $T(P)$ to $\operatorname{Spec} k[P]$.

There is one more thing which a theory of integration must provide to deserve the name, which is a way to compare various integrals, for example under certain morphisms $(Y, \mathcal{N}) \to (X, \mathcal{M})$, which amount to the calculation of integrals by parametrisations. For ordinary motivic integration the fundamental result is the change-of-variables formula for a proper birational morphism $Y \to X$, which encodes how the measure on the arc spaces $J_\infty(Y), J_\infty(X)$ changes under such a map. Such a comparison formula is the basis for applying the integration method in practice.

In the log case it is less obvious what the "admissible" morphisms $(Y, \mathcal{N}) \to (X, \mathcal{M})$ should be that are to have a transformation formula. For one thing, most proper birational maps $Y \to X$ do not behave very well with respect to the strata of log schemes; even the blowup of the affine plane at a rank one point (which has the non-standard structure on $\mathbb{A}^2$ of Example II.15) introduces over the point's stratum the projection map $\operatorname{Spec} k[x, y]/(xy) \to \operatorname{Spec} k[x]$ with an exceptional line $x = 0$. This projection induces a map on arc schemes which, from the point of view of motivic integration, is rather pathological. For example, the arcs at the points of the exceptional locus $x = 0$ with $y \neq 0$ all map to the zero arc on $\operatorname{Spec} k[x]$, which is a set of measure zero.

Log blowups, which are the log geometric version of equivariant blowups of toric varieties, do not have such pathology. On the contrary, they are all log étale, even, and induce very simple behaviour on maps of log arc schemes. Despite this they have some good utility. For example, W. Niziol [29] showed that fine saturated log smooth schemes have resolution of singularities (of the underlying ordinary scheme) by log blowups, generalising the fact that toric varieties have resolution of singularities by equivariant blowups. See also [17] for a "flattening" theorem for morphisms of fine saturated log smooth schemes using log blowups.

For the log blowup $\pi : (Y, \mathcal{N}) \to (X, \mathcal{M})$ along an ideal $\mathcal{I} \subseteq \mathcal{M}$, we have the transformation rule:

Theorem IV.1. *(Theorem IV.25) Let ξ be the generic point of a stratum $Z = X_\xi$ of X, and E the locus of Y mapping to X_ξ. Then*

$$\int_{J_\infty(X,\mathcal{M})_\xi} \phi \, d\mu = \frac{[Z]}{[E]} \sum_{\eta \in E} \lambda^{d(\eta)} \int_{J_\infty(Y,\mathcal{N})_\eta} \pi^* \phi \, d\mu,$$

where the sum $\eta \in E$ ranges over the generic points of strata of Y contained in E, and $d(\eta) = \operatorname{rank} \xi - \operatorname{rank} \eta$.

The quantity λ which appears in this statement is the measure of $\Lambda = \operatorname{Hom}(\mathbb{N}, \Lambda)$; see Section 4.2.1 for more on this. Note that, in contrast to the transformation rule for proper birational maps for ordinary motivic integrals, no Jacobian factor appears in this formula. The reason is that a log blowup $\pi : Y \to X$ induces isomorphisms of the log arc spaces at $y \in Y$ and $x = \pi(y) \in X$ that preserve measure up to factors of the normalisation λ.

Another task is to try to compare some ordinary integrals with log integrals, or, more generally, to compare log integrals under maps $(X, \mathcal{M}') \to (X, \mathcal{M})$ that only change the log structure. We shall see that log motivic integrals on varieties which are both smooth and log smooth essentially *are* ordinary integrals. So applying Niziol's resolution algorithm will give a comparison theorem for integrals on fine saturated log smooth schemes, in terms of the combinatorics of the resolution. We study another approach based on characterising suitable ordinary integrals in terms of data less extensive than that of a full resolution. For toric varieties X, this turns out to mean passing to the Nash modification $X' \to X$ [6].

4.1 Ordinary motivic integration

Let us briefly recall how to construct motivic integrals on ordinary (not log) arc schemes. For a quite readable introduction to this circle of ideas at more length, we might suggest [4].

Recall that the jet scheme $J_m(X)$ of X, for $m \geq 0$, parametrises maps

$$\operatorname{Spec} k[\![t]\!]/(t^{m+1}) \to X.$$

There are canonical maps $J_m(X) \to J_n(X)$ for $m \geq n$ corresponding to truncation of series, and the arc scheme $J_\infty(X)$ of X, which parametrises maps

$$\operatorname{Spec} k[\![t]\!] \to X,$$

is the projective limit of the jet schemes $J_m(X)$ with respect to the truncation maps.

When X is smooth of dimension d, one defines a "measure" μ, called the motivic volume, on the jet schemes of X by setting

$$\mu(Z) = [Z]\,\mathbb{L}^{-md}$$

for $Z \subseteq J_m(X)$ locally closed. Here $[Z]$ is the class of Z in the Grothendieck ring of varieties

$$K_0(\mathrm{var}/k)$$

and $\mathbb{L} = [\mathbb{A}^1]$ is the class of the affine line. Thus μ takes values in the localisation

$$\mathcal{V} = K_0(\mathrm{var}/k)[\mathbb{L}^{-1}]$$

of the Grothendieck ring where $\mathbb{L}$ is inverted. The normalisation factor $\mathbb{L}^{-md}$ makes the definition of μ compatible with the truncation maps, which are just affine bundle maps when X is smooth. Hence we get a motivic volume function $\mu = \mu_X$ on the arc scheme $J_\infty(X)$, at least for subsets of $J_\infty(X)$ which are fibres of a projection to some $J_m(X)$. When X is not smooth, Denef and Loeser [8] showed how to construct a suitable μ on $J_\infty(X)$. This construction presents significant further technical difficulties which we will not discuss here.

Now consider a function $\phi : J_\infty(X) \to K_0(\mathrm{var}/k)$ which takes on only countably many values. A typical example may be to consider a function

$$\phi_Y(\gamma) = \mathbb{L}^{-\mathrm{ord}_Y(\gamma)}$$

defined using the contact order $\mathrm{ord}_Y(\gamma)$ of an arc γ with a fixed closed subscheme $Y \subseteq X$. In any case one defines the integral of ϕ by the formula

$$\int_{J_\infty(X)} \phi \, d\mu = \sum_{v \in K_0(\mathrm{var}/k)} v \cdot \mu(\{\gamma \text{ such that } \phi(\gamma) = v\})$$

if the level sets appearing on the right side of this equation are measurable. In the example $\phi = \phi_Y$, the expression on the right side is a series in $\mathbb{L}^{-1}$ with coefficients in $K_0(\mathrm{var}/k)$. This makes sense in the completion $\overline{\mathcal{V}}$ of $\mathcal{V}$ wherein a sequence $(v_j) \subseteq \mathcal{V}$ converges to zero if $\lim \dim v_j = -\infty$. It is in this ring $\overline{\mathcal{V}}$ where at last the motivic integral itself takes values.

The fundamental change-of-variables formula for a proper birational map $Y \to X$ is the formula

$$\int_{J_\infty(X)} \phi \, d\mu = \int_{J_\infty(Y)} \phi \cdot \phi_{K_{Y/X}} \, d\mu,$$

which captures how the induced map $J_\infty(Y) \to J_\infty(X)$ behaves with respect to the measures on the two arc schemes. In fact, the factor

$$\phi_{K_{Y/X}} = \mathbb{L}^{-\operatorname{ord}_{K_{Y/X}}}$$

is the contact order function of the divisor $K_{Y/X}$ of the Jacobian of the morphism $Y \to X$, (hence the formula's name). See [8], [23] for more on this. This formula allows one to compare integrals on X with integrals on, say, a resolution of singluarities of X, or a log resolution of a pair (X, Z), where calculations may be much more tractable.

All this we would like to have an analogue in the log scheme category.

4.2 The group $\Lambda = 1 + tk[\![t]\!]$

Let k be a field of characteristic zero. We consider the group $\Lambda = \Lambda(k) = 1+tk[\![t]\!]$ of principal series in one variable over k under multiplication. Abstractly, Λ is isomorphic to the direct product $\prod_i (k, +)$ of countably many copies of the additive group of k. A realisation of this is given by an isomorphism to the additive group of the ideal $tk[\![t]\!]$ through the logarithmic series. That is, the series

$$\log(1+s) = s - \frac{1}{2}s^2 + \frac{1}{3}s^3 \mp ...$$

defines a map

$$\log : 1 + tk[\![t]\!] \to tk[\![t]\!]$$

with inverse

$$\exp : tk[\![t]\!] \to 1 + tk[\![t]\!]$$

given by

$$\exp(s) = 1 + s + \frac{1}{2}s^2 + \frac{1}{6}s^3 + ...$$

These series converge t-adically on the given domains. For a series $x(t) = 1 + \sum_{j\geq 1} a_j t^j$, we might refer to the co-efficients b_i in $\log x(t) = \sum_{i\geq 1} b_i t^i$ as its "logarithmic co-ordinates," and refer to them as the co-ordinates $(b_1, b_2, ...)$ of $x(t)$ in the direct product of countably many copies of $(k, +)$. One has

$$x(t) = \exp(\sum b_i t^i) = \exp(b_1 t) \cdot \exp(b_2 t^2) \cdot ...$$

with the product on the right side converging t-adically. Since

$$\exp(b_i t^i) = 1 + b_i t^i + \frac{b_i^2}{2} t^{2i} ...$$

starts in degree t^i (after the initial term 1), we see that the first m logarithmic co-ordinates $b_1, ..., b_m$ determine and are determined by the first m co-efficients $a_1, ..., a_m$ of $x(t)$. In other words, we have the same isomorphism modulo t^{m+1}.

Remark IV.2. If we write a monoid P multiplicatively with elements p as the monoid $(P, \cdot)$, let us write it additively with elements $\partial p = d \log p$ (and identity $0 = \partial(1)$) as the monoid $(\partial P, +)$. The logarithm induces an isomorphism

$$\mathrm{Hom}(P, \Lambda) \simeq \mathrm{Hom}(\partial P, tk[\![t]\!]),$$

where the left side involves monoids under multiplication and the right side involves monoids under addition.

Let $k[P]$ be the monoid algebra of P and let $T(P)$ be the log tangent space to $\mathrm{Spec}\, k[P]$ at its origin. The origin of $\mathrm{Spec}\, k[P]$ is the point corresponding to the maximal ideal $P - P^*$, and the log tangent space there is the affine space with co-ordinates $\partial p = \partial_1 p$ for $p \in P - P^*$ modulo relations $\sum a_i \partial p_i = \sum b_j \partial p_j$, where $\prod p_i^{a_i} = \prod p_j^{b_j}$ in P written multiplicatively. In other words, the log tangent space $T(P)$ is the k-algebra on the set ∂P with the same (additive) relations. Now an element of $\mathrm{Hom}(\partial P, tk[\![t]\!])$ is nothing other than an ordinary arc on $T(P)$ whose closed point lies at the origin $0 \in T(P)$ (as all the series in $tk[\![t]\!]$ have constant term zero).

Let us write $J_\infty(T(P))_0$ for this set of arcs, the fiber over the origin of $T(P)$ of the projection $J_\infty(T(P)) \to T(P)$. What we have said altogether is that the log arc space $J_\infty(X, \mathcal{M})$ decomposes over a stratum component Z with a good chart P as a product

$$J_\infty(X, \mathcal{M})_Z \simeq J_\infty(Z) \times J_\infty(T(P))_0.$$

As both factors in this product have motivic volume functions on them, this suggests that we can integrate functions piecewise on strata by integrating with respect to the product measure.

Remark IV.3. There is an additional algebraic interpretation of the logarithm map through Witt vectors, as follows. Write a series $x(t) \in 1 + tk[\![t]\!]$ as

$$x(t) = \prod_{i \geq 1} 1 - \alpha_i t^i.$$

The product here converges t-adically. The numbers α_i are determined by the series $x(t)$ and may be computed successively by considering the equality modulo t^{i+1}. There is an isomorphism

$$(W(k), +) \to (1 + tk[\![t]\!], \cdot)$$

with the additive group of the Witt ring over k by identifying $x(t)$ with the sequence $\alpha = (\alpha_j)$. Now the Witt polynomials $w_n(\alpha)$ which give the operations on the ring $W(k)$ have the generating series

$$-t\, d \log x(t) = \sum_{n \geq 1} w_n(\alpha) t^n.$$

See [32] for a survey on Witt vectors, and ([32], 3.4) for this last formula in particular.

4.2.1 Measure on Λ

As integration over ordinary arc schemes gives a motivic volume to the ring $k[\![t]\!]$, being the set of ordinary arcs on the affine line, and its ideals, with values in the localised Grothendieck ring

$$\mathcal{V} = K_0(\mathrm{var}/k)[\mathbb{L}^{-1}],$$

so do we wish to give a motivic volume on the group $\Lambda = 1 + tk[\![t]\!]$. One option is simply to pull back the usual measure through the logarithm $\Lambda \to tk[\![t]\!]$. Geometrically this just corresponds to working with the log tangent space interpretation for log arcs of Remark IV.2. For concreteness, we will use this definition.

Definition IV.4. Our measure μ on Λ is the pullback of the motivic volume on $tk[\![t]\!]$ through the isomorphism $\log : \Lambda \to tk[\![t]\!]$, renormalised to have total volume $\mu(\Lambda) = \lambda$ an indeterminate λ. In other words, for $A \subseteq \Lambda$, we set $\mu(A) = \lambda\, \mu(\log A)$, if the set $\log A = \{\log x : x \in A\}$ is measureable.

So, our measure takes values

$$\mu : \Lambda \to \mathcal{V}[\lambda]$$

in a polynomial ring over $\mathcal{V}$. Geometrically, $\Lambda = \mathrm{Hom}(\mathbb{N}, 1 + tk[\![t]\!])$ is the space of log arcs at the origin of the line $\mathrm{Spec}\, k[\mathbb{N}]$ with its standard log structure, and this means that this point on $\mathrm{Spec}\, k[\mathbb{N}]$ has volume λ.

As another justification for this definition, let us describe an alternate, more naïve approach to constructing a measure on Λ, which we shall see is just another description of the same μ. Start with a tautological normalisation $\mu(\Lambda) = \lambda$, and beyond this ask for μ to be homogeneous with respect to the group structure of Λ. For us this means in particular

(1) for $u \in \Lambda$ and $A \subseteq \Lambda$ a measureable set, $\mu(uA) = \mu(A)$; and

(2) if a finite subgroup $G \simeq (k,+)^m$ of Λ acts on Λ, then a coset of G has measure $\mathbb{L}^{-m}\mu(\Lambda) = \mathbb{L}^{-m}\lambda$.

Of course, μ should be finitely additive on disjoint measureable sets of Λ as well.

Let us compute the motivic volumes of some important subsets of Λ as a consequence of these requirements.

Example IV.5. For $e \geq 1$ the sets

$$\Lambda_{\geq e} = \{1 + a_e t^e + a_{e+1}t^{e+1} + ... : a_j \in k\}$$

have measure

$$\mu(\Lambda_{\geq e}) = \mathbb{L}^{-e+1}\lambda.$$

This is because $\Lambda_{\geq e}$ is a coset of the action of

$$G = \{\exp(a_1 t)\exp(a_2 t^2)\cdots\exp(a_{e-1}t^{e-1}) : a_1, ..., a_{e-1} \in k\}.$$

In other words, the set $\Lambda_{\geq e}$ just corresponds to the subgroup $\{(0, ..., 0, b_e, b_{e+1}, ...)\}$ of series having logarithmic co-ordinates zero before index e, and this is a coset of $G = \{(a_1, ..., a_{e-1}, 0, ...)\} \subseteq tk[\![t]\!]$. More generally, the same is true if we replace $\Lambda_{\geq e}$ by any set of principal series whose terms in degrees $1, ..., e-1$ are specified, and left arbitrary after, since this is a translate $u\Lambda_{\geq e}$ by an element $u \in \Lambda$ whose first $e-1$ co-efficients are the specified values.

In terms of the logarithm isomorphism, this means that we have assigned the ideals of $tk[\![t]\!]$ and their cosets the “correct” values. It follows that the usual basic measureable sets constructed from these sets also are given measures compatible with the usual motivic volume on $tk[\![t]\!]$.

Example IV.6. The set

$$\Lambda_e = \{1 + a_e t^e + ... : a_e \neq 0\}$$

is the set difference $\Lambda_{\geq e} - \Lambda_{\geq e+1}$, so has measure

$$\mu(\Lambda_e) = \mathbb{L}^{-e}(\mathbb{L} - 1)\lambda.$$

Multiplying by $u \in \Lambda$, we see that a translate $u\Lambda_e$, being a set of series whose co-efficients in degree up to $e - 1$ are specified and whose co-efficient in degree e avoids any given value, likewise has measure $\mathbb{L}^{-e}(\mathbb{L} - 1)\lambda$. One way to intrepret this calculation is to say that principle (2) above is also compatible with action by factors $(k^*, \cdot)$ of the multiplicative group of k, each one scaling measure by a factor $\mathbb{L} - 1$.

Example IV.7. The intersection of two translates $u\Lambda_e \cap v\Lambda_e$ whose co-efficients in degree e avoid specified, distinct values $a, b \in k$ respectively has measure

$$\mu(u\Lambda_e \cap v\Lambda_e) = \mathbb{L}^{-e}(\mathbb{L} - 2)\lambda.$$

For the union $u\Lambda_e \cup v\Lambda_e$ is just a set $w\Lambda_{\geq e}$, so that $u\Lambda_e \cap v\Lambda_e$ has measure

$$\mu(u\Lambda_e) + \mu(v\Lambda_e) - \mu(u\Lambda_e \cup v\Lambda_e) = (2\mathbb{L}^{-e}(\mathbb{L} - 1) - \mathbb{L}^{-e}\mathbb{L})\lambda = \mathbb{L}^{-e}(\mathbb{L} - 2)\lambda.$$

We record some of the equivalent guises of our measure μ on Λ.

Proposition IV.8. *Let P be a fine monoid, and r be the rank of P^{gp}. Then:*

(1) There is a canonical measure on $\mathrm{Hom}(P, \Lambda) = \mathrm{Hom}(P^{gp}, \Lambda)$, *which may be given by choosing a basis of P^{gp} and pulling back the product measure on Λ^r through the induced isomorphism* $\mathrm{Hom}(P^{gp}, \Lambda) \simeq \mathrm{Hom}(\mathbb{Z}^r, \Lambda) = \Lambda^r$. *In other words, the measure induced on* $\mathrm{Hom}(P^{gp}, \Lambda)$ *by choosing a basis of P^{gp} does not depend on the choice of basis.*

(2) Pullback of sets through $\mathrm{Hom}(P^{gp}, \Lambda) \xrightarrow{\sim} J_\infty(T(P))_0$ *multiplies measure by* λ^r.

(3) Pullback of sets through $\mathrm{Hom}(P, k[\![t]\!]^*) \to \mathrm{Hom}(P, k[\![t]\!]^*/k^*) \simeq \mathrm{Hom}(P, \Lambda)$ *multiplies measure by* $(\mathbb{L}-1)^r\lambda^{-r}$.

(4) The n^{th} *power automorphisms* $x(t) \mapsto x(t)^n$ *on* Λ *preserve measure.*

Proof. Since μ is translation invariant on Λ, an automorphism $\Lambda^r \to \Lambda^r$ given by the action of an invertible integer matrix preserves measure. For example, for maps given by elementary matrices like $(x, y) \mapsto (x, xy)$ we have seen this before. This gives (1).

Claim (2) is the assertion that the exponential $tk[\![t]\!] \to \Lambda$ multiplies measure by λ. This can be checked modulo t^{m+1}. Likewise (3) is the statement that our measure on Λ agrees with the ordinary measure induced by inclusion $1 + tk[\![t]\!] \subseteq k[\![t]\!]^* \subseteq k[\![t]\!]$ up to a factor of λ.

Finally, one way to see (4) is that after taking logarithms the automorphism on $tk[\![t]\!]$ is multiplication by n, which preserves measure. $\square$

4.2.2 A valuation function on Λ

A theory of integration needs functions to integrate. The basic example in ordinary motivic integration is to attach a valuation $\phi : J_\infty(X) \to \mathbb{N} \cup \{\infty\}$ to the arcs on a variety X and try to integrate $\mathbb{L}^{-\phi}$. For example, $\phi(\gamma)$ could be the contact order $\mathrm{ord}_Y(\gamma)$ of an arc γ with a fixed closed subscheme Y of X. If Y is defined by the ideal $\mathcal{I}$, this means that $\mathrm{ord}_Y(\gamma)$ is the t-adic valuation in $k[\![t]\!]$ of $\gamma^*\mathcal{I}$. In other words, the contact order is given by the vanishing of co-efficients in the equations defining Y in X on evaluation at the arc γ.

For a series $x(t) = 1 + a_e t^e + \ldots$ with $a_e \neq 0$, we set

$$\mathrm{val}(x) = e.$$

In other words, $\mathrm{val}(x)$ is the t-adic valuation of $\log x(t)$. We also write

$$|x| = \mathbb{L}^{-\mathrm{val}(x)}.$$

The sets Λ_e are the level sets $\mathrm{val}(x) = e$ and the sets $\Lambda_{\geq e}$ are the sets $\mathrm{val}(x) \geq e$. We note that $\mathrm{val}(xy) = \min(\mathrm{val}(x), \mathrm{val}(y))$ if $\mathrm{val}(x) \neq \mathrm{val}(y)$, or if $\mathrm{val}(x) = \mathrm{val}(y) = e$ and $\partial_e(xy) = \partial_e x + \partial_e y \neq 0$. On the other hand if $\mathrm{val}(x) = \mathrm{val}(y) = e$ and $\partial_e x + \partial_e y = 0$ then $\mathrm{val}(xy) > \mathrm{val}(x), \mathrm{val}(y)$.

The function $\mathrm{val}(x)$ measures agreement of x with the unit series 1. If we can interpret the map $P \to \Lambda$ as giving the data of a deformation for some arcs, then this means measuring the agreement with a given deformation. A translate $u \cdot \mathrm{val}(x) = \mathrm{val}(u^{-1}x)$ measures agreement with a series $u \in \Lambda$ instead. If we integrate each of val and $u \cdot \mathrm{val}$ against a translation-invariant measure μ the result is the same, since one integral is a change-of-coordinates of the other. In the context of integrating over log arc spaces $J_\infty(X, M)$ with a good chart $P \simeq \mathcal{M}_x/\mathcal{O}^*_{X,x}$ at a point x, this means that the integral over morphisms $\mathrm{Hom}(P, \Lambda)$ at a point will be the same for any section $P \to \mathcal{M}_x$. That is:

Proposition IV.9. *Let Z be a stratum of $(X, \mathcal{M})$ which has a good chart P. Then the integral*

$$\int_{J_\infty(X,\mathcal{M})_Z} \phi \, d\mu = \int_{J_\infty(Z)\times \mathrm{Hom}(P,\Lambda)} \phi \, d\mu$$

of a function ϕ on P does not depend on the choice of the chart morphism $P \to \mathcal{M}_{X,Z}$. $\square$

This justifies our use of good charts to define and compute integrals on fine log schemes. We note that although the valuation of a series $p(t)$ under a map $P \to \Lambda$ for $p \in P$ depends on the chart morphism, the valuation of a log arc $\mathcal{M}_x \to \Lambda$ (underlain by a given ordinary arc) does not.

Remark IV.10. Continuing from Remark IV.2, when $P \to Q$ is a cospecialisation map, there is a corresponding closed embedding $T(Q) \to T(P)$ of log tangent spaces induced by the same map $\partial P \to \partial Q$. If $P \to Q$ is the quotient by a face F, then $T(Q)$ is included in $T(P)$ as the locus $\partial F = 0$. In this embedding, $T(Q)$ passes through the origin of $T(P)$. So the map $\partial P \to \partial Q$ induces on the one hand an inclusion $J_\infty(T(Q))_0 \to J_\infty(T(P))_0$ and on the other a restriction of closed subschemes of $T(P)$ that pass through the origin to $T(Q)$. Both maps are determined by the equations $\partial F = 0$. What results is that an integrand on one stratum of X naturally cospecialises to an integrand on any stratum containing it.

Remark IV.11. Continuing from Remark IV.3, we see that for $x(t) = \prod_{i \geq 1} 1 - \alpha_i t^i$ the quantity $\mathrm{val}(x)$ is also the minimum index n for which $w_n(\alpha) \neq 0$. More generally, the ring $k^{\mathbb{N}}$ has a map

$$k^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$$

to the power set of $\mathbb{N}$, taking a vector (β_n) to its set of non-zero indices,

$$\beta \mapsto \{n \text{ such that } \beta_n \neq 0\}.$$

This even is a monoid morphism

$$(k^{\mathbb{N}}, \cdot) \to (\mathcal{P}(\mathbb{N}), \cap)$$

from the multiplicative monoid of $k^{\mathbb{N}}$ (which models a convolution operation on series) to the power set of $\mathbb{N}$ with the intersection operation. The map val is then composition with the map

$$\mathcal{P}(\mathbb{N}) \to \mathbb{N}$$

sending a non-empty subset S to its minimum element. The composite $k^{\mathbb{N}} \to \mathbb{N}$ is not a monoid morphism, though. (Rather, the "complementary" structure $(\mathcal{P}(\mathbb{N}), \cup)$

has a monoid morphism to $(\mathbb{N} \cup \{\infty\}, \min)$, with the convention that the empty set has minimum ∞.)

Obviously this produces some other functions one could imagine integrating, by taking other maps from $\mathcal{P}(\mathbb{N})$. The geometric significance of these, if any, is not clear.

4.2.3 Integration on Λ

We perform a few simple calculations to illustrate integration over groups

$$\operatorname{Hom}(P, \Lambda) \simeq \Lambda^{\operatorname{rank} P^{gp}}$$

as we have developed so far, and to provide some examples for future reference. Here our integrands arise from the functions $|p| = \mathbb{L}^{-\operatorname{val} p(t)}$ for $p \in P$ mapping to series $p(t) \in \Lambda$.

Example IV.12. Consider the affine line $X = \operatorname{Spec} k[x]$ with its standard log structure (that is, $(X, \mathcal{M}) = (\operatorname{Spec} k[\mathbb{N}], \mathbb{N}^a)$). We compute $\int_\Lambda \mathbb{L}^{-\operatorname{val}(x)}\, d\mu$, which is the contribution of the origin of X to the integral of the divisor $x = 0$ over $J_\infty(X, \mathcal{M})$. The sets Λ_e for $e \geq 1$ partition Λ, and $\mathbb{L}^{-\operatorname{val}(x)} = \mathbb{L}^{-e}$ on Λ_e. So,

$$\int_\Lambda \mathbb{L}^{-\operatorname{val}(x)}\, d\mu = \lambda \sum_{e \geq 1} \mathbb{L}^{-e}\mathbb{L}^{-e}(\mathbb{L}-1) = (\mathbb{L}-1)\lambda \sum_{e \geq 1} \mathbb{L}^{-2e} = \frac{(\mathbb{L}-1)\mathbb{L}^{-2}}{1-\mathbb{L}^{-2}}\lambda = \frac{1}{\mathbb{L}+1}\lambda.$$

We get the same result as for integrating the same divisor $x = 0$ on the ordinary arc scheme $J_\infty(X)$, essentially because $\operatorname{val}(x) = \operatorname{ord}_t(\log x)$, except that the appearance of λ remembers that the origin of X as a log scheme has rank one.

Example IV.13. Consider the affine plane with log structure along one line, say $X = \operatorname{Spec} k[x, y]$ with log structure generated by x. We consider

$$\int_{\Lambda \times tk[\![t]\!]} \mathbb{L}^{-\operatorname{val}(x)} d\mu,$$

which is the contribution of one point on the line $x = 0$ to the integral of the divisor $x = 0$ over $J_\infty(X, \mathcal{M})$. Since $tk[\![t]\!]$ has volume 1, and the integrand is constant on this factor, we get the same result as above, $\dfrac{1}{\mathbb{L}+1}\lambda$.

Example IV.14. Consider the affine plane $X = \operatorname{Spec} k[x, y]$ with its standard log structure. We consider

$$\int_{\Lambda^2} \mathbb{L}^{-\operatorname{val}(xy)}\, d\mu,$$

the contribution of the origin of X to the integral of the divisor $xy = 0$ over $J_\infty(X, \mathcal{M})$.

There are easy ways to compute this integral and less-easy ways. One easy way is with a suitable change of co-ordinates on the multiplicative group Λ^2, say from (x, y) to $(u, w) = (x, xy)$. Then

$$\int_{\Lambda^2} \mathbb{L}^{-\operatorname{val}(xy)}\, d\mu = \int_{\Lambda^2} \mathbb{L}^{-\operatorname{val}(w)}\, d\mu = \frac{1}{\mathbb{L}+1}\lambda^2,$$

because like in the previous example the integrand $\mathbb{L}^{-\operatorname{val}(w)}$ is constant on one factor of the domain Λ^2. Note that the "blowup formula" $(x, y) \to (x, xy)$ on an algebra $k[x, y]$ is merely an invertible linear transformation on the group $\Lambda x \times \Lambda y$.

Another "easy" way, given prior knowledge of some basic ordinary motivic integrals, is to use the compatibility of the integral with the logarithm map on Λ to write this as an integral on $\mathbb{A}^2 = T(\mathbb{N}^2) = \operatorname{Spec} k[\partial x, \partial y]$. We have

$$\int_{\Lambda^2} |xy|\, d\mu = \lambda^2 \int_{J_\infty(\mathbb{A}^2)_0} |\partial x + \partial y|\, d\mu = \lambda^2 \int_{J_\infty(\mathbb{A}^2)_0} \mathbb{L}^{-\operatorname{ord}(\partial x + \partial y)}\, d\mu.$$

The integral in the latter expression we recognise as the contribution of a point on a line to the integral of that line on $\mathbb{A}^2$, and one can apply a well-known formula for integrating normal crossing divisors ([4], 2.6) to compute this ordinary motivic integral. Note that the co-ordinates (u, w) on Λ^2 correspond to co-ordinates $\partial u = \partial x$ and $\partial w = \partial x + \partial y$ on $\mathbb{A}^2$.

A rather more involved way is to decompose Λ^2 into cells where $\mathrm{val}(x), \mathrm{val}(y)$ are specified quantities, try to compute the integral on each of these, and sum these contributions up. The difficulty, compared to the previous examples, is that $\mathrm{val}(xy)$ is not determined by just $\mathrm{val}(x)$ and $\mathrm{val}(y)$. That is, $\mathrm{val}(xy)$ is not constant on some of these cells, so those will have to be decomposed further. For practice, perhaps, and to make the distinction from integrating a divisor $xy = 0$ on the ordinary arc scheme $J_\infty(X)$, we record this calculation following.

We compute the contribution from the following subsets of Λ^2:

(1) $\{(x, y) \in \Lambda^2 : \mathrm{val}(x) \neq \mathrm{val}(y)\}$. Suppose first that $\mathrm{val}(x) = e < \mathrm{val}(y)$. So, $x \in \Lambda_e$ and $y \in \Lambda_{\geq e+1}$, and $\mathrm{val}(xy) = e$. We get a contribution of

$$\lambda^2 \sum_{e \geq 1} \mathbb{L}^{-e}\mathbb{L}^{-e}(\mathbb{L}-1)\mathbb{L}^{-e} = \lambda^2(\mathbb{L}-1)\sum_{e\geq 1}\mathbb{L}^{-3e} = \frac{\mathbb{L}-1}{\mathbb{L}^3-1}\lambda^2.$$

We get the same calculation in the case that $\mathrm{val}(y) = e < \mathrm{val}(x)$, for another contribution of $\frac{\mathbb{L}-1}{\mathbb{L}^3-1}\lambda^2$. So we get $\frac{2(\mathbb{L}-1)}{\mathbb{L}^3-1}\lambda^2$ altogether from integrating over this subset of Λ^2.

(2) $\{(x, y) \in \Lambda^2 : \mathrm{val}(x) = \mathrm{val}(y) = \mathrm{val}(xy)\}$. Let $\mathrm{val}(x) = \mathrm{val}(y) = e \geq 1$. Given $x(t)$, this means that the series $y(t)$ avoids two values for its co-efficient of t^e, namely $\partial_e y \neq 0, -\partial_e x$. So for given e we integrate $|xy| = \mathbb{L}^{-e}$ on a set of measure $\mathbb{L}^{-e}(\mathbb{L}-1)\mathbb{L}^{-e}(\mathbb{L}-2)\lambda^2$. We get a contribution

$$\begin{aligned}\lambda^2 \sum_{e\geq 1}\mathbb{L}^{-e}\mathbb{L}^{-e}(\mathbb{L}-1)\mathbb{L}^{-e}(\mathbb{L}-2) &= (\mathbb{L}-1)(\mathbb{L}-2)\lambda^2\sum_{e\geq 1}\mathbb{L}^{-3e} \\ &= \frac{(\mathbb{L}-1)(\mathbb{L}-2)}{\mathbb{L}^3-1}\lambda^2\end{aligned}$$

to the integral from these terms.

(3) $\{(x, y) \in \Lambda^2 : \mathrm{val}(x) = \mathrm{val}(y) < \mathrm{val}(xy)\}$. Let $\mathrm{val}(x) = \mathrm{val}(y) = e \geq 1$ and suppose, given $x(t)$, that $y(t)$ agrees with $x^{-1}(t)$ for exactly $j \geq 1$ places

starting at the co-efficient of t^e; so, $y(t)$ agrees with $x^{-1}(t)$ up to and including the term t^{e+j-1}, and $\partial_{e+j} y \neq -\partial_{e+j} x$. Given e and j, the set of such x, y has measure $\mathbb{L}^{-e}(\mathbb{L}-1)\mathbb{L}^{-e-j}(\mathbb{L}-1)\lambda^2$. With this notation, we have $\mathrm{val}(xy) = e+j$. Altogether we get a contribution

$$\begin{aligned}
\lambda^2 \sum_{e\geq 1}\sum_{j\geq 1} \mathbb{L}^{-e-j}\mathbb{L}^{-2e-j}(\mathbb{L}-1)^2 &= (\mathbb{L}-1)^2\lambda^2 \sum_{e\geq 1}\mathbb{L}^{-3e}\sum_{j\geq 1}\mathbb{L}^{-2j} \\
&= \frac{(\mathbb{L}-1)^2}{(\mathbb{L}^2-1)(\mathbb{L}^3-1)}\lambda^2
\end{aligned}$$

from this case.

Taken together, we have computed

$$\begin{aligned}
\int_{\Lambda^2} \mathbb{L}^{-\mathrm{val}(xy)}\, d\mu &= \left(\frac{2(\mathbb{L}-1)+(\mathbb{L}-2)(\mathbb{L}-1)}{(\mathbb{L}^3-1)} + \frac{(\mathbb{L}-1)^2}{(\mathbb{L}^2-1)(\mathbb{L}^3-1)}\right)\lambda^2 \\
&= \frac{(\mathbb{L}^2+\mathbb{L}+1)(\mathbb{L}-1)}{(\mathbb{L}+1)(\mathbb{L}^3-1)}\lambda^2 \\
&= \frac{1}{\mathbb{L}+1}\lambda^2,
\end{aligned}$$

as we found before.

Remark IV.15. We get the same result with xy replaced by any $x^a y^b$, with $(a,b) \in \mathbb{Z}^2 - \{(0,0)\}$. For any non-zero (a,b) where a, b have no common factor extends to an integral basis of $\mathbb{Z}^2$, while if they have a common factor d, the map $z \mapsto z^d$ is a measure-preserving automorphism of Λ. Hence a change of co-ordinates computes

$$\int_{\Lambda^2} |x^a y^b|\, d\mu = \int_{\Lambda^2} \mathbb{L}^{-\mathrm{val}(x^a y^b)}\, d\mu = \frac{1}{\mathbb{L}+1}\lambda^2.$$

Note that when $a, b \neq 0$ this is not the same as integrating

$$|x^a||y^b| = \mathbb{L}^{-\mathrm{val}(x^a)-\mathrm{val}(y^b)},$$

and neither of these is the same as integrating

$$|x|^a|y|^b = \mathbb{L}^{-a\,\mathrm{val}(x)-b\,\mathrm{val}(y)}.$$

That is, the value $|\cdot| = \mathbb{L}^{-\operatorname{val}(\cdot)}$ is not multiplicative on Λ, because $\operatorname{val} : \Lambda \to \mathbb{N}$ is not a homomorphism. In terms of integration on the log tangent space $T(\mathbb{N}^2)$, these three integrands correspond respectively to

$$|a\partial x + b\partial y|, |a\partial x||b\partial y|, \text{ and } |\partial x + \partial y|^{a+b},$$

which are distinct in general. This is because $\partial : P \to \partial P$ *is* a morphism:

$$\operatorname{val}(xy) = \operatorname{val}(\partial xy) = \operatorname{val}(\partial x + \partial y),$$

because of the compatibility of the valuation map with the logarithm $\Lambda \to tk[\![t]\!]$.

Remark IV.16. After Example IV.14 we should consider the integral

$$\int_{J_\infty(X,\mathcal{M})_0} |x||y||xy| \, d\mu$$

on $X = \operatorname{Spec} k[x, y]$. Now the "easy way" does not avail: there is no basis u, v of Λ^2 with respect to which the integrand is constant on the sets of fixed $\operatorname{val} u, \operatorname{val} v$. The reason is that after taking logarithms we have the integral on $T(\mathbb{N}^2) = \operatorname{Spec} k[\partial x, \partial y]$

$$\int_{J_\infty(\mathbb{A}^2)_0} |\partial x||\partial y||\partial x + \partial y)| \, d\mu.$$

Let us write x, y again for co-ordinates on $T(\mathbb{N}^2)$ in place of $\partial x, \partial y$ to have this ordinary motivic integral in more standard notation. The integrand here does not define a normal crossing divisor: $\operatorname{ord} x(t)$ and $\operatorname{ord} y(t)$ do not determine $\operatorname{ord}(x+y)(t)$.

Computationally this may be handled as follows, which is nothing but the "less-easy" way of Example IV.14 repeated in different terms. If $\operatorname{ord} x(t) < \operatorname{ord} y(t)$, we may write $y(t) = x(t)z(t)$ for some series $z(t) \in tk[\![t]\!]$. Now $\operatorname{ord}(x + y)(t) = \operatorname{ord} x(t)$, and integration with these variables x, z should handle this case. The complication is only that the multiplication map

$$k[\![t]\!]^2 \to k[\![t]\!]^2$$

sending $(x,z) \mapsto (x,xz) = (x,y)$ does not preserve measure, so the integral with respect to x,z should be adjusted to account for this. The case $\operatorname{ord} y(t) < \operatorname{ord} x(t)$ is obviously the same, letting say $x(t) = y(t)w(t)$. In the remaining case $\operatorname{ord} x(t) = \operatorname{ord} y(t)$ either substitution will do. Say if $y(t) = x(t)z(t)$ with $z(t) \in k[\![t]\!]^*$ then $\operatorname{ord}(x+y)(t)$ is $\operatorname{ord} x(t) = \operatorname{ord} y(t)$ if $z(0) \neq -1$ or $\operatorname{ord} x(t) + \operatorname{ord}(z(t) - z(0))$ if $z(0) = -1$. Geometrically what we have described are two charts $y = xz$, $x = yw$ of the blowup of $\mathbb{A}^2$ at the origin. The cases $\operatorname{ord} x \neq \operatorname{ord} y$ correspond to the geometric points $x, z = 0$ and $y, w = 0$ of the exceptional divisor E. The case $\operatorname{ord} x = \operatorname{ord} y$ corresponds to the rest of E, with the special case $z(0) = -1$ being where E meets the proper transform of the line $x + y = 0$. In other words, this co-ordinate substitution has replaced our original divisor with one with normal crossings, at the computational cost of introducing the Jacobian of the multiplication $(x,z) \mapsto (x,xz)$. This is the meaning and strength of Kontsevich's change-of-variables formula in this example.

But what of the log motivic integral? Returning to the old notation, the changes of variables $\partial y = \partial x \cdot \partial z$ and $\partial x = \partial y \cdot \partial w$ give the blowup of $T(\mathbb{N}^2)$. The importation into the log scheme category is not the blowup $y = xz, x = yw$ of $\operatorname{Spec} k[x,y]$ at the origin. That does not induce a blowup on log tangent spaces: it is only a co-ordinate change on Λ^2, with trivial Jacobian. Instead we would want a map $P \to Q$ of monoids such that the map $\operatorname{Spec} k[Q] \to \operatorname{Spec} k[P]$ gives $T(Q) \to T(P)$ as part of a blowup of affine space. But this is not possible: the map on log tangent spaces is just the map $P \to Q$ written additively as $\partial P \to \partial Q$, so is a linear transformation.

Example IV.17. Consider the plane $\operatorname{Spec} k[x,y]$ with non-standard log structure given by the chart $k[xy] \to k[x,y]$, as in Example II.15. Integration of a monomial in x,y at the origin is integration over two disjoint (up to measure zero sets) copies of $\Lambda \times tk[\![t]\!]$, one for each component of the arc space of the rank one stratum $xy = 0$

at the origin. So if we integrate $|xy|$, say, then at the origin we get a contribution $\dfrac{2\lambda}{\mathbb{L}+1}$.

4.3 Integration

Here is our log motivic integral:

Definition IV.18. Using the piecewise splitting

$$J_\infty(X,\mathcal{M})_\xi \simeq J_\infty(X_\xi) \times \mathrm{Hom}(P^{gp},\Lambda)$$

for a fine log scheme $(X,\mathcal{M})$ with ξ the generic point of a stratum component and P a good chart at ξ, we consider integrals

$$\int_{J_\infty(X,\mathcal{M})} \phi\, d\mu = \sum_\xi \int_{J_\infty(X_\xi)\times\mathrm{Hom}(P,\Lambda)} \phi|_\xi\, d\mu$$

with respect to the product measure on $J_\infty(X,\mathcal{M})_\xi$, for chosen integrands $\phi|_\xi$ on each stratum.

Example IV.19. For a fine log scheme $(X,\mathcal{M})$ with smooth strata X_j of rank j, the motivic volume of $J_\infty(X,\mathcal{M})$ is

$$\mu(X,\mathcal{M}) = \int_{J_\infty(X,\mathcal{M})} 1\, d\mu = \sum_j [X_j]\lambda^j.$$

If we evaluate at $\lambda = 1$ we get the ordinary motivic volume $\mu(X) = [X] = \sum_j [X_j]$. Otherwise in general the grade by λ remembers the rank of the log structure on the strata of X.

Remark IV.20. This motivic volume respects taking disjoint unions, and also respects taking products of fine log schemes, because the strata of $(X,\mathcal{M}) \times (Y,\mathcal{N})$ are the products $X_j \times Y_\ell$ along which $\mathcal{M}\oplus\mathcal{N}$ has rank $j+\ell$. More generally, one has the formula

$$\int_{J_\infty(X,\mathcal{M})} \phi_X\, d\mu \int_{J_\infty(Y,\mathcal{N})} \phi_Y\, d\mu = \int_{J_\infty(X\times Y,\mathcal{M}\oplus\mathcal{N})} \phi_X\phi_Y\, d\mu$$

for the same reason.

Example IV.21. For $X = \operatorname{Spec} k[x,y]$ with its standard log structure $\mathcal{M}$, we compute a few integrals using our calculations in Section 4.2.3. First, this log scheme has motivic volume

$$\mu(X,\mathcal{M}) = \int_{J_\infty(X,\mathcal{M})} 1\, d\mu = \lambda^2 + 2(\mathbb{L}-1)\lambda + 1.$$

One way to make sense of this is that the affine line $Y = \operatorname{Spec} k[\mathbb{N}]$ has volume $\lambda + (\mathbb{L}-1)$, and $\operatorname{Spec} k[\mathbb{N}^2] = \operatorname{Spec} k[\mathbb{N}] \times \operatorname{Spec} k[\mathbb{N}]$ as log schemes.

Let us try another, say,

$$\int_{J_\infty(X,\mathcal{M})} |x|\, d\mu.$$

We compute the contributions on strata and sum. On the rank zero stratum $(\mathbb{A}^1 - 0)^2$ we are integrating the value of a unit x over $(k[\![t]\!]^*)^2$, i.e. as an ordinary motivic integral. We have $|x| = 1$, so we get a contribution $(\mathbb{L}-1)^2$.

On the stratum component with generic point (x), according to Example IV.13 we get a contribution $\frac{1}{\mathbb{L}+1}\lambda$ from each point. This stratum component is a torus $\mathbb{A}^1 - 0$, so from it we get $\frac{\mathbb{L}-1}{\mathbb{L}+1}\lambda$. From the other rank one component, with generic point (y), we get $(\mathbb{L}-1)\lambda$: we are integrating the value of a unit over $\Lambda \times k[\![t]\!]^*$.

Finally, the rank two component, the origin, contributes $\frac{1}{\mathbb{L}+1}\lambda^2$, like as in Example IV.12. Altogether we have found

$$\int_{J_\infty(X,\mathcal{M})} |x|\, d\mu = \frac{1}{\mathbb{L}+1}\lambda^2 + (\mathbb{L}-1)(1 + \frac{1}{\mathbb{L}+1})\lambda + (\mathbb{L}-1)^2.$$

More, we get the same result on integrating $|x^a|$ or, symmetrically, $|y^b|$ instead for any $a, b \neq 0$, (but not the same on integrating $|x|^a \geq 2$).

In the same way, except recalling Example IV.14 to compute the contribution of the origin, we find

$$\int_{J_\infty(X,\mathcal{M})} |xy|\,d\mu = \frac{1}{\mathbb{L}+1}\lambda^2 + \frac{2(\mathbb{L}-1)}{\mathbb{L}+1}\lambda + (\mathbb{L}-1)^2.$$

Note that on the rank one strata here the monomial xy is a unit times a (non-unit) element of the log structure, so that the calculation of Example IV.13 still applies after a co-ordinate change on $\Lambda \times k[\![t]\!]^*$. Put another way, the monoid $\mathbb{N}xy$ is a chart on these strata, and we are computing with it.

Like before,

$$\int_{J_\infty(X,\mathcal{M})} |x^a y^b|\,d\mu = \frac{1}{\mathbb{L}+1}\lambda^2 + \frac{2(\mathbb{L}-1)}{\mathbb{L}+1}\lambda + (\mathbb{L}-1)^2$$

if $a, b > 0$.

4.3.1 Integration for strict étale maps

A strict étale morphism $f : (Y,\mathcal{N}) \to (X,\mathcal{M})$ is log étale, preserves the rank of $\mathcal{M}$ under pullback, and induces isomorphisms $J_m(Y,\mathcal{M})_y \to J_m(X,\mathcal{M})_{f(y)}$, for $m \geq 0$ or $m = \infty$, of log jet or log arc spaces at points. Therefore:

Proposition IV.22. *Let $f : (Y,\mathcal{N}) \to (X,\mathcal{M})$ be a strict étale morphism. Let $A \subseteq J_\infty(X,\mathcal{M})$ be measureable and let f^*A be its inverse image in $J_\infty(Y,\mathcal{N})$. Then*

$$\int_{f^*A} f^*(\phi)\,d\mu = \deg f \int_A \phi\,d\mu$$

for any integrand ϕ on A. □

We note two useful consequences of this fact.

Remark IV.23. For a fine log scheme $(X,\mathcal{M})$ in this Chapter, typically we suppose we have a good chart P at any chosen point x. Recall that this means that the composite

$$P \to \mathcal{M}_x \to \mathcal{M}_x/\mathcal{O}^*_{X,x}$$

is an isomorphism; in other words, the chart P is given by a section

$$\mathcal{M}_x/\mathcal{O}^*_{X,x} \to \mathcal{M}_x.$$

This assumption costs little, for according to Proposition II.12 a fine log scheme in characteristic zero has good charts étale locally. That is, in general there is a strict étale morphism $f : U \to X$ near x such that the chart P on $(X, \mathcal{M})$ pulls back to a good chart on U near a point y with $f(y) = x$. Now Proposition IV.22 lets us compute integrals on X by passing to the étale neighbourhoods U.

Remark IV.24. Recall from Corollary III.21 and following that if $(X, \mathcal{M})$ is log smooth then it has, locally on X, a strict étale morphism $X \to \operatorname{Spec} k[P]$ to a monoid algebra. Consequently Proposition IV.22 essentially reduces the calculation of log integrals on log smooth varieties to their calculation on monoid algebras.

4.3.2 Integration for log blowups of monoid algebras

Our main result for log motivic integrals is the following transformation rule for integrals under a log blowup.

Theorem IV.25. *Let $\pi : P \to Q$ be a log blowup, and ϕ an integrand on $X = \operatorname{Spec} k[P]$. Let ξ be the generic point of a stratum $Z = X_\xi$ of X, and E the locus of $Y = \operatorname{Spec} k[Q]$ mapping to X_ξ. Then*

$$\int_{J_\infty(X,\mathcal{M})_\xi} \phi \, d\mu = \frac{[Z]}{[E]} \sum_{\eta \in E} \lambda^{d(\eta)} \int_{J_\infty(Y,\mathcal{N})_\eta} \pi^* \phi \, d\mu,$$

where the sum $\eta \in E$ ranges over the generic points of strata of Y contained in E and $d(\eta) = \operatorname{rank} \xi - \operatorname{rank} \eta$.

Proof. A log blowup induces isomorphisms $J_\infty(Y, \mathcal{N})_y \to J_\infty(X, \mathcal{M})_x$ on log arc spaces at points y mapping to x. Pullback of sets along this map multiplies measures by $\lambda^{-d(\eta)}$. So, integration along the fibre E' of the blowup over ξ contributes

$$[E'] \int_{J_\infty(X,\mathcal{M})_x} \phi \, d\mu = \frac{[E]}{[Z]} \int_{J_\infty(X,\mathcal{M})_x} \phi \, d\mu.$$

Summing over strata, the claim follows. $\square$

See also Remark III.34 describing the strata appearing in a log blowup of monoid algebras.

Example IV.26. Consider the blowup of $X = \operatorname{Spec} k[x, y]$ with its standard log structure at the ideal $I = (x, y)$. From Example IV.13, we have

$$\int_{J_\infty(X,\mathcal{M})_0} |y| \, d\mu = \frac{1}{\mathbb{L}+1}\lambda^2.$$

One chart of this blowup is the map $Y = \operatorname{Spec} k[x, w] \to \operatorname{Spec} k[x, y]$ with $w = x^{-1}y$. Here $\operatorname{Spec} k[x, w]$ has its standard structure, generated by x, w at its origin and by x at the other points of the exceptional divisor $x = 0$. The integrand $|y|$ pulls back to $|xw|$. Now Example IV.14 says that

$$\int_{J_\infty(Y,\mathcal{N})_0} |xw| \, d\mu = \frac{1}{\mathbb{L}+1}\lambda^2,$$

and likewise for the point of the exceptional divisor $E \subseteq \mathrm{Bl}_I(\mathbb{N}^2)$ not in this chart. Finally, Example IV.13 shows, after the co-ordinate substitution $x, y = xw$ on $\Lambda \times k[\![t]\!]^*$, that

$$\int_{J_\infty(Y,\mathcal{N})_{(0,c)}} |xw| \, d\mu = \frac{1}{\mathbb{L}+1}\lambda,$$

for a point $(x, w) = (0, c)$ on the exceptional divisor with $c \neq 0$.

In other words, every point on E contributes $\frac{1}{\mathbb{L}+1}\lambda^2$, if we re-weigh the rank one points of E with an extra factor of λ. Adding these up, and then dividing by the class $[E] = \mathbb{L} + 1$ of the exceptional divisor, we get $\frac{1}{\mathbb{L}+1}\lambda^2$ again. This is the claim of the transformation formula in this case.

Remark IV.27. Let $Y = \operatorname{Spec} k[Q] \to X = \operatorname{Spec} k[P]$ be log étale. If $\phi^{gp} : P^{gp} \to Q^{gp}$ is an isomorphism, then $Y \to X$ factors as an open immersion into a log blowup of X, according to Proposition III.38. If $\ker \phi^{gp} \neq 0$, then the map $Y \to X$ is not dominant. The case $\operatorname{coker} \phi^{gp} \neq 0$ corresponds to ramified (in the ordinary sense) covers.

4.4 Ordinary arcs on monoid algebras

We study ordinary integration on monoid algebras $\operatorname{Spec} k[P]$. The main result, also introducing some of our notation, can be summarised as follows.

Theorem IV.28. *There is a canonical decomposition of the space of arcs*

$$J_\infty^*(P) \subseteq J_\infty(\operatorname{Spec} k[P])$$

of a monoid algebra $\operatorname{Spec} k[P]$ *which lie generically in no stratum (of positive rank) of* $\operatorname{Spec} k[P]$ *into cells*

$$J_\infty^v(P), \text{ for } v \in \operatorname{Hom}(P, \mathbb{N}).$$

Furthermore, there exists a function ϕ_P *on* $\operatorname{Hom}(P, \mathbb{N})$*, conewise linear on some subdivision* Σ_P *of the cone* $\operatorname{Hom}(P, \mathbb{N})$*, such that the cells' measure is given by*

$$\mu(J_\infty^v(P)) = [\operatorname{Spec} k[P^{gp}]]\, \mathbb{L}^{-\phi_P(v)}$$

for all $v \in \operatorname{Hom}(P, \mathbb{N})$*. In fact,* $\phi_P(v)$ *is given by the minimum*

$$\phi_P(v) = \min\{v(p_1) + v(p_2) + ...v(p_d) : p_1, p_2, ..., p_d \in P \text{ is a rational basis for } P^{\mathbb{Q}}\}$$

of $v(p_1 + p_2 ... + p_d)$ *over bases* $p_1, p_2, ..., p_d \in P$ *of* $P^{\mathbb{Q}} = P^{gp} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark IV.29. The dual monoid $\operatorname{Hom}(P, \mathbb{N}) = \operatorname{Hom}(P^{sat}, \mathbb{N})$ is a (saturated) polyhedral cone in $\operatorname{Hom}(P^{gp}, \mathbb{Z}) = \operatorname{Hom}(P^{gp}/(torsion), \mathbb{Z})$.

Remark IV.30. The decomposition $J^v_\infty(P)$ for arcs on a toric variety was given by Ishii [15]. The motivic classes $\mu(J^v_\infty(P))$ were calculated in [6] in terms of the Newton polyhedra of "logarithmic Jacobian ideals" of P. Our approach in Section 4.4.2 below is essentially the same.

Corollary IV.31. *Let $F : J^*_\infty(P) \to \mathcal{V}$ be an integrand on $\operatorname{Spec} k[P]$ for which $F(\gamma)$ depends only on the cell $J^v_\infty(P)$ to which γ belongs. We then write the integrand also as $F : \operatorname{Hom}(P, \mathbb{N}) \to \mathcal{V}$. Then*

$$\int_{J_\infty(P)} F(\gamma)\, d\mu = \sum_{v \in \operatorname{Hom}(P,\mathbb{N})} F(v)\mu(J^v_\infty(P)) = [\operatorname{Spec} k[P^{gp}]] \sum_{v \in \operatorname{Hom}(P,\mathbb{N})} F(v)\, \mathbb{L}^{-\phi_P(v)}.$$

Proof. Note that the complement of $J^*_\infty(P)$ in $J_\infty(P)$ has measure zero, since it consists of arcs generically contained in proper strata of $\operatorname{Spec} k[P]$. This explains the domain of F and of the integral. The formula then follows from the above Theorem IV.28 and the definition of the motivic integral. $\square$

This means that one may compute the integral of any such F as the given formal sum, given knowledge of the function ϕ_P, which encodes the relevant information about the motivic volume on $\operatorname{Spec} k[P]$.

In this subsection we describe the decomposition of the arc scheme of $\operatorname{Spec} k[P]$. In those following we compute the motivic volumes $\mu(J^v_\infty(P))$, and discuss some simple integrals on monoid algebras.

Let P be a fine monoid. The set of arcs

$$J_\infty(P) = \operatorname{Hom}(k[P], k[\![t]\!])$$

on $\operatorname{Spec} k[P]$ bijects naturally with the set of monoid morphisms

$$\operatorname{Hom}(P, (k[\![t]\!], \cdot))$$

from P to the multiplicative monoid of power series, by the universal property of monoid algebras. In particular, it has a natural monoid structure itself, given by multiplication of series. Concretely, this is the observation that if two maps $\gamma_1 = (x_1(t)), \gamma_2 = (x_2(t))$ satisfy some monomial relations then so does the product $\gamma_1\gamma_2$, (and conversely, if the series $x_i(t)$ are non-zero). Viewing the monoid $(k[\![t]\!], \cdot)$ as the zero element together with the product

$$(k[\![t]\!]^\times, \cdot) \simeq \mathbb{N} \times k[\![t]\!]^* \simeq \mathbb{N} \times k^* \times \Lambda$$

through the decomposition

$$x(t) = t^e(a_e + a_{e+1}t + ...) = t^e \cdot a_e \cdot (1 + \frac{a_{e+1}}{a_e}t + ...)$$

where $a_e \neq 0$, we obtain the following description of the ordinary arcs of $X = \operatorname{Spec} k[P]$. They are first stratified by the prime ideal $I \subseteq P$ which maps to zero in $k[\![t]\!]$. The ideal $I \subseteq k[P]$ defines the stratum of X in which the arc *generically* lies. We are interested in the set $J^*_\infty(P)$ of arcs with $I = P - P^*$, that generically lie in no proper stratum of $\operatorname{Spec} k[P]$. (One can study the arcs which generically lie in proper strata by passing to the quotients P/F, where $F = P - I$ is the complementary face of I, and considering $J^*_\infty(P/F)$.)

In terms of the above decomposition of $k[\![t]\!]$, this is the set

$$J^*_\infty(P) \simeq \operatorname{Hom}(P, \mathbb{N}) \times X^{gp} \times \operatorname{Hom}(P, \Lambda),$$

where $X^{gp} = \operatorname{Spec} k[P^{gp}]$ is the rank zero stratum of X. (So X^{gp} is a disjoint union of tori of dimension $\operatorname{rank} P^{gp}$.) We grade $J^*_\infty(P)$ by $v \in \operatorname{Hom}(P, \mathbb{N})$, writing

$$J^v_\infty(P) = v \times X^{gp} \times \operatorname{Hom}(P, \Lambda)$$

for such v. These are the arcs where $x(t)$ has order $v(x)$. That is, these arcs are given by equations

$$x(t) = x_0 t^{v(x)}(1 + \partial_1 x\, t + ...),$$

for $x \in P$, with the $x_0 \neq 0$ the co-ordinates of a point in X^{gp}. They have a multiplication

$$J^v_\infty(P) \times J^{v'}_\infty(P) \to J^{v+v'}_\infty(P)$$

as subsets of $J_\infty(X)$. Obviously the fibre of this map is just the set $X^{gp} \times \mathrm{Hom}(P, \Lambda)$ of ordinary arcs on X^{gp} over any point of the target. This grading and multiplication descend to

$$J^v_m(P) \times J^{v'}_m(P) \to J^{v+v'}_m(P),$$

where

$$J^v_m(P) = \pi_m J^v_\infty(P) \subseteq J_m(\mathrm{Spec}\, k[P]),$$

but the fibres of the product maps on jets are harder to determine, even when one of v, v' is zero and the multiplication is just the action by the jets of X^{gp}. The $J^v_m(P)$ have additional structure as well. For example, let $\delta = \min(v+v') \leq \min(v)+\min(v')$, with minima taken over $P - P^* = P - 1$. Then there is a well-defined, surjective multiplication map

$$J^v_{m-\delta}(P) \times J^{v'}_{m-\delta}(P) \to J^{v+v'}_m(P),$$

which in general still has a nontrivial fibre.

This illustrates the difficulty in computing the motivic volume of the cells $J^v_\infty(P)$, for it is the images $J^v_m(P)$ under the projection $\pi_m : J_\infty(P) \to J_m(P)$ that determine the classes $\mu(J^v_\infty(P))$. Indeed,

$$\mu(J^v_\infty(P)) = \lim_{m\to\infty} [\pi_m J^v_\infty(P)]\, \mathbb{L}^{-md} = \lim_{m\to\infty} [J^v_m(P)]\, \mathbb{L}^{-md},$$

where $d = \mathrm{rank}(P^{gp})$, according to [8].

4.4.1 Support functions

As an illustration, we consider the case where v is zero on a non-trivial face F of P.

In general, a map $v \in \mathrm{Hom}(P, \mathbb{N})$ determines a face F of P mapping to the identity $0 \in \mathbb{N}$ (which is the minimal face of $\mathbb{N}$). Here $F \subseteq P$ is the dual to the face of $\mathrm{Hom}(P, \mathbb{N})$ in whose relative interior v lies, and the *closed* point (not the generic point) of the arcs $J_\infty^v(P)$ lies in the stratum of X determined by F. Let us call v a support function for the face F of P. For fixed F, the set of such is naturally $\mathrm{Hom}(P/F, \mathbb{N})$.

The simplest case is when a face F has codimension one, that is, when the quotient P/F has rank one. Since the target $\mathbb{N}$ is saturated, the support functions of facets F are then just maps $\mathbb{N} \to \mathbb{N}$, although if P is not saturated then the image $v(P)$ may not be a saturated submonoid of $\mathbb{N}$.

Proposition IV.32. *Let v be a support function of a facet $F \subseteq P$. Then*

$$\mu(J_\infty^v(P)) = [X^{gp}]\,\mathbb{L}^{-\min_{x \in P-F} v(x)}.$$

Proof. An arc γ in $J_\infty^v(P)$ is determined by a point of X^{gp} together with an element of $\mathrm{Hom}(P, \Lambda)$. This in turn is determined by a monoid morphism in $\mathrm{Hom}(F, \Lambda)$ and the series $x(t) \in k[\![t]\!]^\times$ for a chosen $x \in P - F$. Any single x will do in characteristic zero, since it completes a basis of F^{gp} to a $\mathbb{Q}$-basis of P^{gp} and we can take n^{th} roots in Λ. But the series $x(t)$ are determined by $v(x)$, their leading co-efficients x_0, and the image

$$\overline{\gamma} \in \mathrm{Hom}(\mathbb{N}, \Lambda) = \mathrm{Hom}(P/F, \Lambda),$$

by taking a splitting $P/F \to P$. That is, there is a series, call it $z(t) \in \Lambda$, corresponding to the image $\overline{\gamma}(1) \in \Lambda$, such that for any $x \in P - F$ we have $x(t) = x_0 t^{v(x)} z(t)^{v(x)}$.

Modulo t^{m+1}, then, for m large enough (depending only on v), we have the following description of the image $\pi_m J^v_\infty(P)$ in $J_m(X)$. A truncated arc corresponds to a point of X^{gp}, an arbitrary map in $\mathrm{Hom}(F^{gp}, \Lambda/t^{m+1}) \simeq (\Lambda/t^{m+1})^{\mathrm{rank}\, P^{gp}-1}$, and series $\overline{x(t)} = x_0 t^{v(x)} \overline{z(t)}^{v(x)}$. This last part modulo t^{m+1} is determined by the truncation of $z(t)$ up to the power $t^{m-v(x)}$, hence all these series at once are determined by the truncation of $z(t)$ up to the power $t^m - \min_{x\in P-F} v(x)$. Obviously different such truncated series $z(t)$ give different m-jets. Hence we have computed the class

$$[\pi_m J^v_\infty(P)] = [X^{gp}]\, \mathbb{A}^{m(\mathrm{rank}\, P^{gp}-1)+m-\min_{x\in P-F} v(x)}.$$

The motivic volume of $J^v_\infty(P)$ is the limit of the normalised classes

$$\lim\, [\pi_m J^v_\infty(P)]\, \mathbb{L}^{-m \dim X},$$

where $\dim X = \mathrm{rank}\, P^{gp}$. So $\mu(J^v_\infty(P)) = [X^{gp}]\, \mathbb{L}^{-\min_{x\in P-F} v(x)}$, as claimed. $\square$

More generally, for any face F of P, the same argument, with the single series $z(t)$ replaced by a rational basis for $(P/F)^{gp}$, shows the following.

Proposition IV.33. *If v is a support function of a face F of P, not necessarily a facet, then*

$$\mu(J^v_\infty(P)) = \frac{[\mathrm{Spec}\, k[P^{gp}]]}{[\mathrm{Spec}\, k[(P/F)^{gp}]]}\, \mu(J^v_\infty(P/F)).$$

$\square$

Example IV.34. Let P be a non-singular simplicial cone of dimension d. Its dual cone $\mathrm{Hom}(P, \mathbb{N})$ is also non-singular and simplicial, say with a generating set $v_1, ..., v_d$, which is a basis of $\mathrm{Hom}(P^{gp}, \mathbb{Z})$, the dual basis to $x_1, ..., x_d \in P$. The function v_j is a support functions for the facet F_j of P generated by the x_i with $i \neq j$, and the minimum value of v_j on $P - F_j$ is one (for example, attained at the point $x_j \in P$).

In fact, for $v = \sum a_j v_j \in \mathrm{Hom}(P, \mathbb{N})$, we have

$$\mu(J_\infty^v(P)) = \mathbb{T}^d \prod \mathbb{L}^{-a_j} = \mathbb{T}^d \mathbb{L}^{-\sum a_j},$$

where $\mathbb{T} = \mathbb{L} - 1$ is the class of the torus $\mathbb{A}^1 - \{0\}$, although we have not yet proven this.

Example IV.35. For $X = k[t^2, t^3]$ and $v(t^n) = n$, the minimum of v on $P - F = \{t^2, t^3, ...\}$ is two. So $\mu(J_\infty^v(P)) = \mathbb{T}\mathbb{L}^{-2}$. More, the motivic volume of the cuspidal singularity is the sum of the classes $\mu(J_\infty^{ev}(P)) = \mathbb{T}\mathbb{L}^{-2e}$ for $e \geq 1$, which is $\mathbb{T}/(\mathbb{L}^2 - 1) = 1/(\mathbb{L} + 1)$.

Remark IV.36. It is not the case that every $v \in \mathrm{Hom}(P, \mathbb{N})$ is a sum of support functions (for example, consider $\mathrm{Hom}(P, \mathbb{N})$ a singular simplicial cone). Nor is $\mu(J_\infty^v(P))/[X^{gp}]$ additive on all of $\mathrm{Hom}(P, \mathbb{N})$ in general.

4.4.2 Min sequences

We finish the calculation of the motivic volumes $\mu(J_\infty^v(P))$. Recall that our claim is that there is a function ϕ_P on $\mathrm{Hom}(P, \mathbb{N})$, continuous and conewise linear on some subdivision Σ_P of $\mathrm{Hom}(P, \mathbb{N})$, such that

$$\mu(J_\infty^v(P)) = [X^{gp}]\, \mathbb{L}^{-\phi_P(v)}$$

for all $v \in \mathrm{Hom}(P, \mathbb{N})$. This ϕ_P will be given in terms of combinatorial or convex geometric data of P. It suffices to show that

$$[\pi_m J_\infty^v(P)] = [J_m^v(P)] = [X^{gp}]\, \mathbb{L}^{md - \phi_P(v)}$$

for m large enough, depending on v, and some constant $\phi_P(v)$ depending only on v (and not m).

The determination of the jets $J_m^v(P)$, and hence their motivic classes $[J_m^v(P)] \in K_0(\mathrm{var}/k)$, is more complicated than that of the arcs $J_\infty^v(P)$, because m-jets correspond to maps

$$P \to (k[\![t]\!]/(t^{m+1}), \cdot)$$

rather than maps $P \to (k[\![t]\!], \cdot)$, and $(k[\![t]\!]/(t^{m+1}), \cdot)$ has a more complicated structure than $(k[\![t]\!], \cdot)$. Namely, it is graded by $\mathbb{N}/(m+1) = \{0, 1, 2, ..., m, \infty\}$, with the grade $n \leq m$ piece isomorphic to $k^* \times \Lambda/(t^{m-n+1})$, where we write

$$\Lambda/(t^{m-n+1}) = (1 + tk[\![t]\!]/(t^{m-n+1}), \cdot).$$

This isomorphism is given by writing a truncated series $x(t)$ mod t^{m+1} as

$$t^n a_0(1 + a_1 t + ... + a_{m-n}t^{m-n}).$$

The grade ∞ piece is the zero element, which formally agrees with this if we take the notational convention that $t^\infty = 0$. As a result, when describing the possible maps

$$P \to (k[\![t]\!]/(t^{m+1}), \cdot)$$

we will need to keep track of which graded piece of the target various elements of P map to.

Let us call a set of elements $p_1, ..., p_d \in P$ a $\mathbb{Q}$-basis of P if the images of $p_1, ..., p_d$ are a basis for the rational vector space $P^{\mathbb{Q}} = P^{gp} \otimes_{\mathbb{Z}} \mathbb{Q}$, and a $\mathbb{Z}$-basis of P if further they generate the lattice $P^{gp}/(torsion) \subseteq P^{\mathbb{Q}}$.

Proposition IV.37. *1. There is a surjection*

$$\mathrm{Hom}(P, \mathbb{N}/(m+1)) \times X^{gp} \times \mathrm{Hom}(P, \Lambda/(t^{m+1})) \to J_m^*(P)$$

of monoids, given by multiplication, with the identification

$$\mathbb{N}/(m+1) = \{0, 1, ..., m, \infty\} = \{1, t, ..., t^m, 0\}.$$

2. *Let* $\operatorname{char} k = 0$*. Let* $p_1, ..., p_d$ *be a* $\mathbb{Q}$*-basis of* P*, and let* $v \in \operatorname{Hom}(P, \mathbb{N})$*. Then the above map induces a surjection*

$$X^{gp} \times \prod_{1 \le j \le d} \Lambda/(t^{m-v(p_j)+1}) \to J_m^v(P).$$

Proof. The first claim is a restatement of the previous discussion. For the second, a map $P \to \Lambda/(t^{m+1})$ is determined by the images of the elements p_j, which are series p_j in the grade $v(p_j)$ piece of $\Lambda/(t^{m+1})$. That is, the choice of $\mathbb{Q}$-basis followed by truncation of series gives a morphism

$$\operatorname{Hom}(P, \Lambda/(t^{m+1})) \to \prod_j \Lambda/(t^{m+1}) \to \prod_j \Lambda/(t^{m-v(p_j)+1})$$

which gives a factorisation of the surjection

$$v \times X^{gp} \times \operatorname{Hom}(P, \Lambda/(t^{m+1})) \to J_m^v(P).$$

□

In particular, it follows that if our function ϕ_P exists, then

$$\phi_P(v) \le \sum_j v(p_j)$$

for any $\mathbb{Q}$-basis $p_1, ..., p_d$ of P. Our claim is that in fact $\phi_P(v)$ is the minimum over $\mathbb{Q}$-bases of P of such sums. To establish this we will construct, given $v \in \operatorname{Hom}(P, \mathbb{N})$, a $\mathbb{Q}$-basis $s_1, ..., s_d$ of P with minimum sum and show that $J_m^v(P)$ is a torsor under the induced action by $X^{gp} \times \prod_j \Lambda/(t^{m-v(s_j)+1})$.

In fact, a suitable basis can be chosen greedily. For $v \in \operatorname{Hom}(P, \mathbb{N})$ we construct a *min sequence* $s = (s_1, ..., s_d) \in P^d$ for v on P as follows. First we write $s_0 = 1$, the identity of P. Obviously $v(s_0) = 0$. Inductively, for $1 \le j \le d$ in succession we take s_j to minimise v on $P - L_j$, where L_j is the linear subspace of $P^{\mathbb{Q}}$ generated by the elements $s_0, s_1, ..., s_{j-1}$. Sometimes s_j is not uniquely determined, because v

may be simultaneously minimised at more than one point of $P - L_j$. In any case, the elements of a min sequence s are in linearly general position in $P^{\mathbb{Q}}$, so form a $\mathbb{Q}$-basis of P.

If $s = (s_1, ..., s_d)$ is a min sequence for some $v \in \mathrm{Hom}(P, \mathbb{N})$, we call it a min sequence of P. If further $v(s_j) > 0$ for $j \geq 1$ we call s a non-degenerate min sequence of P. If v is a support function of a non-trivial face F of P then a min sequence for v is equivalent to a choice of a $\mathbb{Q}$-basis of F followed by a min sequence for P/F.

Remark IV.38. We make some basic observations about min sequences.

1. A non-degenerate min sequence consists of irreducible elements of P. For suppose s_j is reducible with j minimal, $s_j = pq$ for some $p, q \in P - P^*$. If v witnesses that s is non-degenerate, then as $v(s_j) = v(p) + v(q)$ we have $1 \leq v(p), v(q) < v(s_j)$. Since p, q were not chosen in the construction in place of s_j, they must both be in L_j. But then so is s_j, a contradiction.

2. In particular, a fine monoid P has finitely many non-degenerate min sequences.

3. A non-degenerate min sequence need not lie in a face of the convex hull of the irreducible elements of P in $P^{\mathbb{Q}}$, but may contain interior elements of this polytope.

4. A support function v still has a non-degenerate min sequence. Start with a $\mathbb{Q}$-basis of F which has minimum volume in F^{gp} and extend to a min sequence s of P using the ordering induced by v. Now a perturbation v' of v into the interior of $\mathrm{Hom}(P, \mathbb{N}) \otimes_{\mathbb{Z}} \mathbb{Q}$ has min sequence s', where the beginning segment of s' is some ordering of the chosen $\mathbb{Q}$-basis of F, and thereafter s, s' agree. Then s' is non-degenerate, and is also a min sequence for v. This observation

amounts to noting that the conewise linear function ϕ_P on $\mathrm{Hom}(P, \mathbb{N})$ we will give is continuous at the boundary of $\mathrm{Hom}(P, \mathbb{N})$.

Remark IV.39. If $s = (s_1, ..., s_d)$ is a min sequence for v, then $v(s_1), ..., v(s_d)$ is the smallest sequence in dominance order of values of v on d linearly independent elements of P. That is, if $p_1, ..., p_d$ is another $\mathbb{Q}$-basis of P, then

$$v(s_1) + ... + v(s_k) \leq v(p_1) + ... + v(p_k)$$

for all $1 \leq k \leq d$. For suppose $p_1, ..., p_d$ is a counterexample with $k \geq 2$ minimum. Thus

$$v(s_1) + ... + v(s_{k-1}) \leq v(p_1) + ... + v(p_{k-1}),$$

but

$$v(s_1) + ... + v(s_{k-1}) + v(s_k) > v(p_1) + ... + v(p_{k-1}) + v(p_k).$$

In particular, $v(p_k) < v(s_k)$. Since p_k was not chosen in the construction of s, it must lie in the linear span of $s_1, ..., s_{k-1}$. Note now that if $q_1, ..., q_k$ is any permutation of the elements $p_1, ..., p_k$, then the ordered basis $q_1, ..., q_k, p_{k+1}, ..., p_d$ also gives a minimal counterexample. In particular, *all* of $p_1, ..., p_k$ lie in the linear span of $s_1, ..., s_{k-1}$. This contradicts that $p_1, ..., p_k$ are linearly independent in $P^{\mathbb{Q}}$. One might compare ([6], 5.1).

To emphasise the convex geometry of P, let us consider the polytope $\Delta(s) \subseteq P^{\mathbb{Q}}$ which is the convex hull of the elements of a min sequence s, or $\Delta_0(s)$ the convex hull of $\Delta(s)$ and $s_0 = 1$. We also consider the point $p_s = \sum_{p \in s} p$, the far corner of a parallelotope of which $\Delta_0(s)$ is one half. Note that if s is non-degenerate then $\Delta_0(s)$ has no points of P in its interior, (as such a point p would be in linearly general position with respect to any $d-1$ elements of s and would have $v(p) < v(s_j)$ for some j, contradicting the construction). In particular, if P is saturated then $\Delta_0(s)$

has no points of P^{gp} in its interior, so s actually gives a $\mathbb{Z}$-basis of P^{gp}, (because every element of P has a translate which is a lattice point in a fundamental domain for $P^{gp}/(s_1, ..., s_d)$.)

We are ready to prove our claim:

Theorem IV.40. *Let*

$$\phi_P(v) = v(p_s) = \sum_j v(s_j),$$

with $s = (s_1, ..., s_d)$ *a min sequence for* v. *Then* $J^v_m(P)$ *is a torsor for* $X^{gp} \times \prod_j \Lambda/(t^{m-v(s_j)+1})$. *In particular,* $\mu(J^v_\infty(P)) = [X^{gp}]\,\mathbb{L}^{-\phi_P(v)}$.

Proof. We need to show that, for $\gamma \in J^v_m(P)$, the multiplication map

$$\gamma \times X^{gp} \times \prod_{1 \le j \le d} \Lambda/(t^{m-v(s_j)+1}) \to J^v_m(P)$$

after Proposition IV.37 injects. If there is a non-trivial relation among the series $s_1(t), ..., s_j(t)$, with j taken minimal, then there is a relation of their principal truncated series modulo $t^{m-v(s_j)+1}$, because $v(s_1), ..., v(s_j)$ is a nondecreasing sequence. But the submonoid P_j generated by $s_1, ..., s_j$ has rank j, with $\mathbb{Q}$-basis $s_1, ..., s_j$, and the group $\mathrm{Hom}(P_j, \Lambda/(t^{m-v(p_j)+1}))$ is free, a contradiction. $\square$

Remark IV.41. It follows that a min sequence s for v does minimise the sum $\sum_j s_j(v)$ over $\mathbb{Q}$-bases of P, and that this minimum value is $\phi_P(v)$. In particular, we have $\phi_P(v) = \min_s v(p_s)$, with minimum taken over all the min sequences of P, or just over all the non-degenerate min sequences of P. Such a minimum, over a finite set of linear functions, is continuous and conewise linear on the subdivision Σ_P of $\mathrm{Hom}(P, \mathbb{N})$ which is the normal fan in $\mathrm{Hom}(P, \mathbb{N})$ of the polytope Δ_P in $P^{\mathbb{Q}}$ generated by the "evaluation points" p_s, for various min sequences s. This Σ_P is the Nash modification of P [6].

This construction of ϕ_P also shows that if v, v' have min sequences s, s' which are permutations of each other then $\phi_P(v)+\phi_P(v') = \phi_P(v+v')$. That is, ϕ_P is linear on cones corresponding to *unordered* min sequences. To see this, we only need re-order the factors $\Lambda/(t^{m-v(p_{s'_j})+1})$ to agree with the $\Lambda/(t^{m-v(p_{s_j})+1})$ and observe that then the multiplication map

$$J^v_m(P) \times J^{v'}_m(P) \to J^{v+v'}_m(P)$$

has constant fibre X^{gp}, and hence

$$\mu(J^v_\infty(P))\mu(J^{v'}_\infty(P)) = [X^{gp}]\mu(J^{v+v'}_\infty(P)).$$

Consequently, in general the subdivision Σ_P of $\mathrm{Hom}(P, \mathbb{N})$ is coarser than the subdivision into cones of constant (ordered) min sequences, because the evaluation points p_s are unchanged by permutation of the elements of s.

Example IV.42. Let $P = \langle y, xy, x^2y\rangle$ be generated by monomials y, xy, x^2y. Then the dual cone $\mathrm{Hom}(P, \mathbb{N})$ has rays generated by $e, -e + 2f$, where e, f is the dual basis to x, y. The non-degenerate min sequences of P are (y, xy) and (x^2y, xy), which have evaluation points xy^2 and x^3y^2. These are equal on the ray $x = 0$ of $\mathrm{Hom}(P, \mathbb{N})$. This ray consists of those $v \in \mathrm{Hom}(P, \mathbb{N})$ for which $v(y) = v(xy) = v(x^2y)$, so that v takes its minimum on $P - \{1\}$ at all these points simultaneously. Two of the choices of ordered min sequences for v give min sequences for different perturbations of v away from the ray $x = 0$.

We get $\phi_P(ae + bf) = \min(a + 2b, 3a + 2b)$, conewise linear on the subdivision by the ray $x = 0$. In particular, this gives the correct weights to the points on the boundary of $\mathrm{Hom}(P, \mathbb{N})$, which correspond to the support functions of proper faces of P. Note that the primitive point f on the ray $x = 0$ has $\phi_P(f) = 2$, and not $\phi_P = 1$, like the primitive points on the other rays. (If P is saturated then the primitive codimension one support functions always have $\phi_P = 1$, according to Proposition IV.32).

Example IV.43. Similarly to the last example, taking $Q = \langle y, xy, x^2y, x^3y\rangle$ we have $\phi_Q(ae+bf) = \min(5a+2b, a+2b)$, conewise linear after subdividing at the ray $x = 0$. The primitive point f of this ray still has $\phi_Q(f) = 2$.

Example IV.44. Let us consider the case of the quadric cone,

$$P = \langle x, y, z, w : xw = yz\rangle.$$

For example, we can realise P by identifying variables x, y, z, w as vectors

$$(1, 0, 0), (0, 1, 0), (1, 0, 1), (0, 1, 1)$$

in $\mathbb{N}^3$. The non-degenerate min sequences (which now have length 3) are two of x, y, z, w (but not x, w or y, z) followed by $xw = yz$. Hence ϕ_P will be conewise linear on a star subdivision of $\mathrm{Hom}(P, \mathbb{N})$. In the example, the cone $\mathrm{Hom}(P, \mathbb{N}) \subseteq (\mathbb{N}^3)^*$ has rays generated by $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, and $(1, 1, -1)$. The "middle" point $v = (1, 1, 0)$, which generates the star subdivision Σ_P of $\mathrm{Hom}(P, \mathbb{N})$, takes its minimum on $P - \{1\}$ simultaneously on the four points x, y, z, w.

In terms of the multiplication of jets, it has an isomorphism $\gamma \times J_{m-1}^{(0,0,0)}(P) \to J_m^{(1,1,0)}(P)$ for any $\gamma \in J_m^v(P)$, for example the jet $x(t) = y(t) = z(t) = w(t) = t$. This map gives an alternate explanation of the exceptional value $\phi_P(v) = 3$ (compared to $\phi_P = 1$ at the primitive points of the rays of $\mathrm{Hom}(P, \mathbb{N})$).

In terms of the variables x, y, z, w, here is how this sort of calculation plays out. Take for example $u = (2, 1, -1)$. Then $u(x) = 2, u(y) = 1, u(z) = 1, u(w) = 0$. We parametrise the possible m-jets by X^{gp} times three principal series, with m coefficients for w, $m - 1$ for y, and $m - 1$ for z. The series $x(t)$ is then determined by $x = yzw^{-1}$. We get $3m-2$ parameters, giving $\phi_P(u) = 2$ after the normalisation by the factor $\mathbb{L}^{-3m}$. For $u' = (1,1,1) = (0,0,1) + v$, say, we have $u'(x) = 1, u'(y) = 1, u'(z) = 2, u'(w) = 2$. We get $3m-4$ parameters by choosing series for x, y, z, say. Now we cannot write $w = yzx^{-1}$, but it is the case that the principal series part of w, which we might denote $w/(w_0t^2)$, is determined by $w/t^2 = (y/t)(z/t^2)(x/t)^{-1}$, with the same notational convention, modulo t^{m-2}. So we find $\phi_P(u') = 4$.

We note that here the function ϕ_P is not conewise linear on an arbitrary resolution of $\mathrm{Hom}(P, \mathbb{N})$ (that is, an arbitrary subdivision into nonsingular simplicial cones). For example, $\mathrm{Hom}(P, \mathbb{N})$ has a "small" resolution by adding in either of the two-dimension cones generated by $(1,0,0), (0,1,0)$ or $(0,0,1), (1,1,-1)$. Instead ϕ_P is only conewise linear after subdividing at their intersection, generated by $v = (1,1,0)$.

Remark IV.45. The polytopes $\Delta(s)$, and the subdivision of $\mathrm{Hom}(P, \mathbb{N})$, depend on P and not just P^{sat}. The subdivisions Σ_P and $\Sigma_{P^{sat}}$ need not be comparable, essentially because the irreducible elements of P and P^{sat} need not bear much relation. For example, let $P^{sat} = \langle y, xy, x^2y\rangle$, with P consisting of the elements $x^ay^b \in P$ with $a+b \geq n$ for some fixed n, say $n = 3$. Here $\mathrm{Hom}(P, \mathbb{N})$ is the saturated cone generated by vectors $(-1,2)$ and $(1,0)$. Now P^{sat} and P both have two non-degenerate min sequences, namely (y, xy) and (x^2y, xy) for P^{sat}, and (y^n, xy^{n-1}) and $(x^ay^b, x^{a-1}y^{b+1})$, with $a + b = n$ and a maximal, for P. We see that $\Sigma_{P^{sat}}$ is obtained by subdividing $\mathrm{Hom}(P, \mathbb{N})$ by the ray through $(0,1)$, while Σ_P is obtained by subdividing by the ray through $(1,1)$ instead.

Remark IV.46. The subdivsion Σ_P of $\mathrm{Hom}(P, \mathbb{N})$ need not be nonsingular, or even simplicial, even if P is saturated. The same holds for the refinement of this subdivision into cones corresponding to the different ordered min sequences of P.

For example, consider $P = \langle x, y, z, x^2y^3z^{-1}\rangle \subseteq \mathbb{Z}^3$. Then P is the union of two simplicial cones $\langle x, y, z\rangle$ and $\langle x, y, x^2y^3z^{-1}\rangle$, so is saturated, with exactly the four irreducible elements $x, y, z, x^2y^3z^{-1}$. Let us consider the ordered or unordered min sequence (x, y, z). Let (a, b, c) be co-ordinates on $\mathrm{Hom}(P, \mathbb{N})^{gp}$ corresponding to x, y, z. The cone corresponding to the ordered min sequence (x, y, z) is given by inequalities

$$0 \leq a \leq b \leq c \leq 2a + 3b - c$$

or, equivalently, by inequalities

$$0 \leq a \leq b \leq c, 0 \leq 2a + 3b - 2c.$$

The cone for the unordered min sequence, which is the largest cone on which

$$\phi_P(a, b, c) = a + b + c,$$

is given by inequalities

$$0 \leq a, b, c \leq 2a + 3b - c.$$

In this case one sees easily that the relations $a, b \leq 2a + 3b - c$ are implied by $c \leq 2a + 3b - c$, so that the cone is also given by the inequalities

$$0 \leq a, b, c, 2a + 3b - 2c.$$

Now neither of these cones is simplicial. The point is that the plane $2a+3b-2c = 0$ in each case meets two faces of the simplicial cone given by the other inequalities $a, b, c \geq 0$ in the relative interior of two of its facets, so that the part $0 \leq 2a+3b-2c$ of it has four facets. In particular these cones are not simplicial, as claimed.

Remark IV.47. Even in dimension 2, the subdivision Σ_P need not be a resolution. For example, consider P the saturated cone generated by y, x^5y^{-1}. There are 2 unordered non-degenerate min sequences on P, but one subdivision is not enough to resolve $\mathrm{Hom}(P, \mathbb{N}) = \langle(1, 0), (1, 5)\rangle$.

4.4.3 Ordinary integration on monoid algebras.

We consider integrands $F(\gamma)$ on $J^*_\infty(P)$, with values in $\mathcal{V} = K_0(\mathrm{var}/k)[\mathbb{L}^{-1}]$, which are invariant under the action of the arcs $J_\infty(X^{gp})$. That is, F is constant on the orbits of this action, and hence the value $F(\gamma)$ depends only on the $v \in \mathrm{Hom}(P, \mathbb{N})$ for which $v \in J^v_\infty(P)$. Sometimes we will speak instead of integrating functions $F(v)$ on $\mathrm{Hom}(P, \mathbb{N})$.

Recall from Theorem IV.40 that $\mu(J^v_\infty(P)) = [X^{gp}]\,\mathbb{L}^{-\phi_P(v)}$, with ϕ_P conewise linear on a subdivision Σ_P of $\mathrm{Hom}(P, \mathbb{N})$. We then have the formula

$$\int_{J^*_\infty(P)} F(\gamma)\,d\mu = \sum_{v\in \mathrm{Hom}(P,\mathbb{N})} F(v)\,\mu(J^v_\infty(P)) = [X^{gp}] \sum_{v\in \mathrm{Hom}(P,\mathbb{N})} F(v)\,\mathbb{L}^{-\phi_P(v)},$$

whenever the sum on the right converges in $\overline{\mathcal{V}}$.

A special case of interest is when $F(v) = \mathbb{L}^{-p(v)}$ with $p \in P$, or sometimes $p \in P^{gp}$, a linear function on $\mathrm{Hom}(P, \mathbb{N})$. The sum converges if p lies in the interior of P, so that $F(v) > 0$ for any $v \neq 0$, and for some other choices of p as well, due to the presence of the factor $\mathbb{L}^{-\phi_P(v)}$. The study of these sums generalises to the consideration of functions $F(v)$ conewise linear on some subdivision Σ of $\mathrm{Hom}(P, \mathbb{N})$. Now if $\sigma \in \Sigma$ is a maximal cone of Σ in general the element $p \in P$ or P^{gp} for which $F(v) = \mathbb{L}^{p(v)}$ on σ need not lie in σ^*, but instead $(\sigma^*)^{gp} = P^{gp}$.

Altogether, these integrals appear as sums of the form $\sum_{v\in \mathrm{Hom}(P,\mathbb{N})} \mathbb{L}^{-\phi(v)}$ for ϕ conewise linear on some subdivision of $\mathrm{Hom}(P, \mathbb{N})$ on which both F, ϕ_P are conewise linear. Separating this sum by cones σ of this subdivision, we are left to consider sums of form

$$\sum_{v\in\sigma} \mathbb{L}^{-p(v)}$$

for $p \in P^{gp} = (\sigma^*)^{gp}$ linear on σ. Consequently, we think of the theory of motivic integration for equivariant integrands on $X = \mathrm{Spec}\, k[P]$ as equivalent to the theory of certain formal sums on $\mathrm{Hom}(P, \mathbb{N})$.

The following observation relates integrals for unsaturated monoids to integrals for their saturation. Note that $\phi_P(v) \geq \phi_{P^{sat}}(v)$ for any $v \in \mathrm{Hom}(P, \mathbb{N}) = \mathrm{Hom}(P^{sat}, \mathbb{N})$, since the construction of ϕ_P by min sequences takes minima over smaller sets.

Proposition IV.48. *Let $A \subseteq \mathrm{Hom}(P, \mathbb{N})$ and an integrand $F(v)$ on A be given. Then*

$$\int_A F(v)\, \mathbb{L}^{-\phi_P + \phi_{P^{sat}}}\, d\mu_{P^{sat}} = \int_A F(v)\, d\mu_P.$$

Proof. Both integrals equal

$$\sum_{v \in A} F(v)\, \mathbb{L}^{-\phi_P}.$$

□

Thus it is essentially sufficient to develop motivic integration on saturated monoids in order to understand integration on general fine monoids P.

Example IV.49. Let $\sigma = \mathrm{Hom}(P, \mathbb{N}) \simeq \mathbb{N}^d$ be nonsingular, with primitive elements $v_1, ..., v_d$ on its rays, and let $x_1, ..., x_d \in \sigma^* = P$ be the dual basis. Writing $v = \sum a_j v_j$ and $p = \sum b_j x_j$, we have a sum

$$\sum_{(a_j) \in \mathbb{N}^d} \mathbb{L}^{-p(\sum a_j v_j)} = \sum_{(a_j) \in \mathbb{N}^d} \mathbb{L}^{-\sum a_j b_j} = \prod_j \sum_{a_j \in \mathbb{N}} \mathbb{L}^{a_j b_j}.$$

This sum converges if each $b_j > 0$, that is, if p lies in the interior of σ^*, to

$$\prod_j \frac{\mathbb{L}^{b_j}}{1 - \mathbb{L}^{b_j}}.$$

Example IV.50. Let σ be a (possibly singular) simplicial cone, with primitive elements $v_1, ..., v_d$ on its rays. Let $\tau \subseteq \sigma$ be the cone generated by the v_j. Then σ is a finite union of translates of τ, by the points $t_1, ..., t_n$ of σ in the fundamental parallelogram spanned by the v_j. Indeed, σ^{gp}/τ^{gp} is finite of this order, and $t_1, ..., t_n$ is a system of coset representatives. Let $p \in (\sigma^*)^{gp}$ have $p(v_j) = b_j, p(t_i) = c_i$. Then

$$\sum_{v\in\sigma} \mathbb{L}^{-p(v)} = \left(\sum_{1\le i\le n} \mathbb{L}^{-c_i}\right) \prod_j \frac{\mathbb{L}^{b_j}}{1-\mathbb{L}^{b_j}},$$

as we may see by re-writing the sum over $v \in \sigma$ as a sum over cosets $t_i + \tau$ for $1 \le i \le n$, and using the previous example to compute the sum for τ.

Example IV.51. Other explicit calculations are possible. For example, let σ be generated by vectors $(-1,2),(0,1)$, and $(1,0)$. This is the cone $\mathrm{Hom}(P,\mathbb{N})$ of Example IV.42. The function $\phi = \phi_P$ giving the weights for the measure on σ takes values $\phi(-1,2) = \phi(1,0) = 1, \phi(0,1) = 2$ and is conewise linear on the subdivision of σ by the ray through $(0,1)$. Let us compute $S = \sum_{v\in\sigma} \mathbb{L}^{-\phi(v)}$. Decompose σ into its interior $(0,1)+\sigma$ and the two rays $\mathbb{N}(-1,2)$ and $\mathbb{N}(1,0)$, and the origin $(0,0)$. Summing $\mathbb{L}^{-\phi(v)}$ over these four parts gives contributions $\mathbb{L}^{-2}S, \mathbb{L}/(1-\mathbb{L}), \mathbb{L}/(1-\mathbb{L})$, and 1. (For the first of these we have used that the point $(0,1)$ lies in both the maximal cones on which ϕ is linear.) That is,

$$S = \mathbb{L}^{-2}S + \frac{2\mathbb{L}}{1-\mathbb{L}} + 1 = \mathbb{L}^{-2}S + \frac{1+\mathbb{L}}{1-\mathbb{L}},$$

so that

$$S = \frac{1+\mathbb{L}}{(1-\mathbb{L})(1-\mathbb{L}^{-2})} = \frac{\mathbb{L}^2}{(\mathbb{L}-1)^2} = \mathbb{T}^{-2}\mathbb{L}^2.$$

Multiplying by $\mathbb{T}^2$ gives the motivic volume $\mathbb{L}^2$ of the toric variety X_σ. Actually, every point of X_σ has measure 1. For example, the torus-invariant point of X_σ has measure

$$\mathbb{T}^2 \sum_{v\in\sigma^{int}} \mathbb{L}^{-\phi(v)} = \mathbb{T}^2 \sum_{v\in(0,1)+\sigma} \mathbb{L}^{-\phi(v)} = \mathbb{T}^{-2}\mathbb{L}^{-\phi(0,1)}S = 1.$$

The calculation of the previous example does not apply to this one directly because ϕ is not linear on all of σ (it has $\phi(0,2) = 4$). The sense of this calculation is that in general if v is an irreducible element of σ then $v + \sigma$ is a finite union of translates of faces of σ, allowing one to compute the sums by induction on dimension.

Example IV.52. If a simplicial subdivision Σ of σ is given, a sum $\sum_{v\in\sigma} F(v)$ over the points of σ can be computed in terms of sums over the cones of Σ. If F is conewise linear on Σ then these are all of elementary type, as in Example IV.49.

Looking in the opposite direction, one can try to put σ inside a larger simplicial cone τ on which $\sum_{v\in\tau} F(v)$ still converges, then subdivide the region between σ and τ as a fan T, and compute $\sum_\sigma F(v) = \sum_\tau F(v) - \sum_T F(v)$. Here is an example of this type. Write co-ordinates on $\mathbb{N}^3$ as triples (a, b, c), let $F(v) = \mathbb{L}^{-\phi(v)}$ for a linear functional ϕ on $\mathbb{N}^3$ strictly positive away from the origin, and let σ be the cone $a, b, c, b + c - a \geq 0$. Its complement in $\mathbb{N}^3$ is the simplicial cone τ' given by $b, c \geq 0, b + c \leq a$. Then

$$\sum_{v\in\sigma} F(v) = \sum_{v\in\mathbb{N}^3} F(v) - \sum_{v\in\tau'} F(v) + \sum_{v\in\sigma\cap\tau'} F(v),$$

and the three sums on the right side of this equation are of elementary type. We will see that this type of construction, with σ inside a nonsingular cone τ can interpret the sum $\sum_{v\in\sigma F(v)}$ as a certain log motivic integral on X_σ.

4.5 Integration on monoid algebras

We describe here a simple type of log integrand which leads to a comparison with the formal sums over cones that we considered previously in studying ordinary motivic integrals on monoid algebras. Subsequently we will describe how the two may be related.

Remark IV.53. Recall from Proposition III.39 that the log arc scheme on $(X, \mathcal{M})$ is a trivial affine bundle (of infinite dimension). More, we see easily from the proof that the isomorphism of the log arcs $J_\infty(X,\mathcal{M})_x$ at point $x \in X_j$ in the rank j stratum X_j with a point $y \in X_\ell$ respects the measure μ after scaling by a factor $\lambda^{\ell-j}$. In view of this, generally it is more convenient to study log integrals at a single point of X, which (when P is sharp) we may suppose is the central point 0 of X.

Let $x_1, ..., x_d \in P^{gp}$ be a basis for $P^{\mathbb{Q}}$ and consider an integral

$$\int_{J_\infty(X,\mathcal{M})_0} |x_1|^{a_1} \cdot |x_2|^{a_2} \cdots |x_d|^{a_d} \, d\mu$$

for some $a_1, ..., a_d \geq 0$. Recall that this equals the integral

$$\int_{T(P)} |(\partial x_1)^{a_1} (\partial x_2)^{a_2} \cdots (\partial x_d)^{a_d}| \, d\mu$$

on the log tangent space $T(P) \simeq \mathbb{A}^d$ to X at the origin. The integrand is equivariant on $T(P)$ with the log structure induced by $P \simeq \partial P$, which is the standard structure on $\mathbb{A}^d \simeq \operatorname{Spec} k[\mathbb{N}^d]$, so corresponds to a lattice sum on $\operatorname{Hom}(\mathbb{N}^d, \mathbb{N}) \simeq \mathbb{N}^d$.

Here is another way to view this integral. Consider the monoid $\mathbb{N}^d$ with generators labelled $\operatorname{val} x_1, \operatorname{val} x_2, ... \operatorname{val} x_d$. We think of $\operatorname{val} x_j \geq 1$ as giving the order of a principal series $x_j(t)$ in some log arc, in the sense of Section 4.2.2. Now let ϕ be the linear function on $\mathbb{N}^d$ with $\phi(\operatorname{val} x_j) = a_j + 1$. Then

$$\int_{J_\infty(X,\mathcal{M})_0} |x_1|^{a_1} |x_2|^{a_2} \cdots |x_d|^{a_d} \, d\mu = \mathbb{T}^d \lambda^d \sum_{(e_1,...,e_d) \in \mathbb{N}^d_{>0}} \mathbb{L}^{-\phi(e_1,...,e_d)}. \tag{4.1}$$

This follows simply on decomposing the group

$$\operatorname{Hom}(P, \Lambda) = \Lambda x_1 \times \Lambda x_2 \times \cdots \times \Lambda x_d \simeq \Lambda^d$$

into sets

$$\Lambda_{e_1} \times \Lambda_{e_2} \times \cdots \times \Lambda_{e_d}$$

where $\Lambda_e \subseteq \Lambda$ is the set of series $\{1 + c_e t^e + ... : c_e \neq 0\}$ with valuation e. According to Example IV.6, we have $\mu(\Lambda_e) = \mathbb{L}^{-e} \mathbb{T} \lambda$, so that

$$\mu(\Lambda_{e_1} \times \Lambda_{e_2} \times \cdots \times \Lambda_{e_d}) = \mathbb{T}^d \lambda^d \mathbb{L}^{-(e_1+e_2+...+e_d)}.$$

Since the integrand $|x_1|^{a_1}|x_2|^{a_2}\cdots|x_d|^{a_d}$ takes value $\mathbb{L}^{-(a_1e_1+a_2e_2+\ldots+a_de_d)}$ on this set, Equation 4.1 follows immediately from the definition of ϕ.

4.5.1 Ordinary integrals as log integrals

We describe a kind of transformation rule for ordinary integrals on monoid algebras $X = \operatorname{Spec} k[P]$. Ultimately this gives a comparison to log integrals on a certain affine bundle $X' \to X$ over X.

Let P be a fine monoid, $\operatorname{Hom}(P,\mathbb{N})$ its dual. Let $v_1, \ldots, v_n$ be the irreducible elements of $\operatorname{Hom}(P,\mathbb{N})$, and consider the map $(\mathbb{N}^n)^* \to \operatorname{Hom}(P,\mathbb{N})$ taking standard generators $e_1^*, \ldots, e_n^*$ on $(\mathbb{N}^n)^*$ to these points. This induces a surjection $(\mathbb{Z}^n)^* \to \operatorname{Hom}(P,\mathbb{N})^{gp} = \operatorname{Hom}(P,\mathbb{Z})$ on group completions. Dual to this is an inclusion $P^{gp} \to \mathbb{Z}^n$. Write $\mathcal{K}^*, \mathcal{K}$ for the kernel and cokernel of these respective maps. We have dual exact sequences

$$0 \to \mathcal{K}^* \to (\mathbb{Z}^n)^* \to \operatorname{Hom}(P,\mathbb{N})^{gp} \to 0, \tag{4.2}$$

$$0 \leftarrow \mathcal{K} \leftarrow \mathbb{Z}^n \leftarrow P^{gp} \leftarrow 0. \tag{4.3}$$

For $w \in (\mathbb{N}^n)^*$, consider the coset $w + \mathcal{K}^* \subseteq (\mathbb{Z}^n)^*$. Write $\Delta(w)$ for its intersection $(w+\mathcal{K}^*) \cap (\mathbb{N}^n)^*$ with the first orthant. Obviously $\Delta(w)$ and $\Delta(u)$ are congruent if w, u map to the same point in $\operatorname{Hom}(P,\mathbb{N})$, and the number of such points $u \in (\mathbb{N}^n)^*$ is the number $\#\Delta(w)$ of lattice points in $\Delta(w)$. Considering a general $\Delta(w)$ as a subset of $\mathcal{K}^*$, it determines an ample line bundle on a toric variety $X_{\mathcal{K}}$, whose fan is the normal fan in $\mathcal{K}$ of such $\Delta(w)$.

Remark IV.54. There is a related construction, say with $\operatorname{Hom}(P,\mathbb{N})$ simplicial, taking generators of $(\mathbb{N}^d)^*$ to the primitive points $v_1, \ldots, v_d$ on the rays of $\operatorname{Hom}(P,\mathbb{N})$. In general the corresponding sequences (4.2), (4.3) are only exact after tensoring with

$\mathbb{Q}$. This is because this map $(\mathbb{N}^d)^* \to \mathrm{Hom}(P,\mathbb{N})$ has non-trivial, finite cokernel if $\mathrm{Hom}(P,\mathbb{N})$ is singular. After Example IV.50, though, from the point of view of integration having such a cokernel only amounts to multiplication by a constant. So in such a case one might work with this map instead. The toric variety $X_{\mathcal{K}}$ is then the Gale dual of $X = \operatorname{Spec} k[P]$.

Let an integral $\sum_{v \in \mathrm{Hom}(P,\mathbb{N})} F(v)$ on $\mathrm{Hom}(P,\mathbb{N})$ be given. Pulling back along the map $(\mathbb{N}^n)^* \to \mathrm{Hom}(P,\mathbb{N})$ gives on the one hand

$$\sum_{v \in \mathrm{Hom}(P,\mathbb{N})} F(v) = \sum_{w \in (\mathbb{N}^n)^*} \frac{1}{\#\Delta(w)} F(w),$$

or, put the other way,

$$\sum_{v \in \mathrm{Hom}(P,\mathbb{N})} \#\Delta(v) F(v) = \sum_{w \in (\mathbb{N}^n)^*} F(w),$$

where by $\Delta(v)$ we mean any of the polytopes $\Delta(w)$ where $w \in (\mathbb{N}^n)^*$ maps to v.

Example IV.55. Let $Q = \mathrm{Hom}(P,\mathbb{N})$ be generated by elements v_1, v_2, v_3, v_4 with $v_1 + v_4 = v_2 + v_3$, and consider an integral $\sum_{v \in Q} F(v)$ for some F. We have the surjection $\pi : \mathbb{N}^4 \to Q$ taking standard generators of $\mathbb{N}^4$ to these elements. Then

$$\sum_{v \in Q} F(v) = \sum_{u \in \mathbb{N}^4} \frac{1}{\#\Delta(u)} F(\pi u),$$

where $\Delta(u) = (u + \ker \pi) \cap \mathbb{N}^4$ counts the number of points of $\pi^{-1}\pi u$. In co-ordinates $u = (a,b,c,d)$ on $\mathbb{N}^4$, we have

$$\#\Delta(u) = 1 + \min(b,c) + \min(a,d).$$

Suppose for instance that $F(\pi u) = \mathbb{L}^{-\phi(a,b,c,d)}$ with $\phi(a,b,c,d) = a + b + c + d$, say. (This is the smallest linear function ϕ on $\mathbb{N}^4$ for which the formal sum above converges.) Splitting the domain $\mathbb{N}^4$ into cones according to which of its co-ordinates are larger, we get contributions like

$$S = \sum_{a \geq d, b \geq c} \frac{1}{1+c+d} \mathbb{L}^{-(a+b+c+d)},$$

and so forth. We can evaluate this sum explicitly by standard manipulations with formal series. For fixed c, d we have in this sum the term

$$S(c,d) = \frac{1}{1+c+d} \mathbb{L}^{-c-d} \left(\sum_{a \geq d} \mathbb{L}^{-a} \right) \left(\sum_{b \geq c} \mathbb{L}^{-b} \right) = \frac{1}{1+c+d} \frac{\mathbb{L}^2}{\mathbb{T}^2} \mathbb{L}^{-2(c+d)}.$$

To sum these pieces over $c, d \geq 0$, consider the formal series identity

$$\frac{d}{dz} z \sum_{c,d \geq 0} \frac{1}{1+c+d} z^{c+d} = \sum_{c,d \geq 0} z^{c+d} = \frac{1}{(1-x)^2}.$$

Re-arranging this gives

$$\sum_{c,d \geq 0} \frac{1}{1+c+d} z^{c+d} = \frac{1}{z(1-z)},$$

so that on letting $z = \mathbb{L}^{-2}$ we find

$$S = \sum_{c,d \geq 0} S(c,d) = \frac{\mathbb{L}^2}{\mathbb{T}^2} \frac{\mathbb{L}^4}{\mathbb{L}^2(1-\mathbb{L}^2)} = \frac{\mathbb{L}^4}{\mathbb{T}^3[\mathbb{P}^1]}.$$

The rationality of this expression (in $\mathbb{L}$) is not an accident: the sum comes from an integral on Q which must have a rational form. (For example it is computable by taking a toric resolution of singularities; in other words, by taking a simplicial subdivision of $\mathrm{Hom}(P, \mathbb{N})$.) This rationality is not automatic from the form of the sum $\sum_{w \in (\mathbb{N}^n)^*} \frac{1}{\#\Delta(w)} \mathbb{L}^{-\phi(w)}$, but says something about the polynomials $\Delta(w)$ and the integrand $\mathbb{L}^{-\phi(w)}$. It might be interesting to know what this rationality means for the numbers $\#\Delta(w)$.

Remark IV.56. Here is one way to interpret this construction. Choose a nonsingular simplicial cone σ^* in $\mathrm{Hom}(P, \mathbb{N})^{gp}$ containing $\mathrm{Hom}(P, \mathbb{N})$, equivalently a nonsingular simplicial cone σ in P^{gp} lying inside P^{sat}. Summation on σ^* may be interpreted as giving a log integral on X by viewing $\sigma^* \simeq \oplus_j \mathbb{N}\,\mathrm{val}(x_j)$, where the x_j are the generators of the dual cone σ in P^{gp}. But, given an integral on X, we only have an integrand defined on the cone $\mathrm{Hom}(P,\mathbb{N}) \subseteq \sigma^*$. This cone, in these co-ordinates, is defined by some inequalities in the symbols $\mathrm{val}\, x_j$. Unfortunately such a subset does not have a good (better) interpretation in terms of the structure of $\mathrm{Hom}(P,\Lambda)$. To try to correct this, we take an injective map $P \to Q$ for some monoid Q – in this case, $Q = \mathbb{N}^n$ – for which the cone $\mathrm{Hom}(P,\mathbb{N})$ pulls back to a "better" subset of $\mathrm{Hom}(Q,\mathbb{N})$ – in this case, the whole monoid of valuations $\mathbb{N}^n$ of a given basis of Q^{gp}.

If $F(v) = \mathbb{L}^{-\phi(v)}$ with ϕ linear on $\mathrm{Hom}(P,\mathbb{N})$, we can view these as integrals on $(\mathbb{N}^n)^* \subseteq (\mathbb{Z}^n)^*$. In other words, they are integrals on affine space $\mathbb{A}^n$: let $x_1, ..., x_n$ be the dual basis in $\mathbb{N}^n$, and consider the integrand $\prod |x_j|^{\phi(x_j)-1}$ on $\mathrm{Spec}\, k[\mathbb{N}^n]$. These sums have the shape of the log integral on $\mathbb{A}^n$ we saw in Section 4.5, with a scalar factor $\dfrac{1}{\#\Delta(w)}$ included in the one case. Consequently we can interpret these sums as log integrals on $\mathbb{A}^n$, up to scaling factors.

Now the map $P \to \mathbb{N}^n$ gives a log smooth map $\mathbb{A}^n \to X = \mathrm{Spec}\, k[P]$, because the corresponding map on group completions has no kernel. (Recall Theorem III.18 characterising log smooth or étale morphisms on monoid algebras.) Further, the map $\mathbb{A}^n \to X$ factors as a log étale map to affine space $X' = X \times \mathbb{A}^{n-\dim X}$ over X. We can give an explicit such factorisation: start with a $\mathbb{Q}$-basis of P and add in standard generators of $\mathbb{N}^n$ to get a $\mathbb{Q}$-basis of $\mathbb{N}^n$. Let $P' \subseteq \mathbb{N}^n$ be generated by this $\mathbb{Q}$-basis. The inclusion $P \to P'$ induces an affine bundle map $X' = \mathrm{Spec}\, k[P'] \to X$, and the inclusion $P' \to \mathbb{N}^n$ induces an open chart $\mathbb{A}^n \to X'$ of a log blowup, by Proposition III.38. The log integrand on $\mathbb{A}^n$ is given in terms of the variables

$$x_1, ..., x_n \in \mathbb{Z}^n = (\mathbb{N}^n)^{gp} = (P')^{gp},$$

so it makes sense as an integrand on X' as well. Theorem IV.25 applies in this case, transforming our log integral into a log integral on X'.

Remark IV.57. The map $\mathbb{A}^n \to X$ we constructed above appears in a similar role in the interesting recent preprint [5]. There the authors compute an arithmetic invariant which they define of the toric variety X, now over a finite field k, the "intersection complex function," as what in our language we might call an l-adic integral on X. Specifically, they view this function as a formal series

$$\mathrm{IC}_X = \sum_{v \in \mathrm{Hom}(P,\mathbb{N})} m_v \, v \in \mathbb{Q}_l[\![\mathrm{Hom}(P,\mathbb{N})]\!]$$

over $\mathbb{Q}_l$ with a variable for each element of the cocharacter cone $\mathrm{Hom}(P,\mathbb{N})$ of X. They show that this series is the pushforward of the sum

$$\sum_{w \in \mathbb{N}^n} w \in \mathbb{Z}[\![\mathbb{N}^n]\!]$$

on $\mathbb{A}^n$, and hence that the co-efficient m_v is nothing other than what we previously called $\#\Delta(v)$. Thus these co-efficients track the number of decompositions of v into combinations of irreducible elements of $\mathrm{Hom}(P,\mathbb{N})$.

Their larger interest consists in working with certain spherical varieties X, which are analogues of toric varieties with the torus T^d replaced by an algebraic group G, say. In other words, there is a dense open embedding $G \to X$ such that the multiplication $G \times G \to G$ induces a multiplication $X \times X \to X$. Thus the variety X becomes an algebraic monoid. For example, one might have X the space of $n \times n$ matrices with $G = \mathrm{GL}_n$ embedded as the matrices with non-zero determinant. In general X is not commutative, because usually G is not. The authors prove some results in this setting analogous to the toric case. For example, they show that the arc space of X generically has a "finite-dimensional model," which in the toric case corresponds to the fact that every arc is, up to the action of the arcs on the torus

T^d, represented by an actual toric morphism $\mathbb{A}^1 \to X$. It might be very interesting to try to analyse this situation from the log geometric viewpoint.

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