Topological Abel-Jacobi Mapping and Jacobi Inversion

by

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CHAPTER 1

Introduction

1.1 Hodge decomposition and Hodge conjecture

Every smooth projective complex algebraic variety is naturally equipped with the structure of a complex manifold. As such, it has topological invariants such as singular cohomology groups. Central to the subject of complex algebraic geometry is the comparison of the algebraic and topological invariants of a smooth compact complex variety.

On a complex manifold X, differential forms have a natural bigrading. A differential form is of type (p, q) if it is a section of the bundle $(\wedge^p \Omega^{1,0}) \wedge (\wedge^q \Omega^{0,1})$, where $\Omega^{1,0}$ and $\Omega^{0,1}$ are the holomorphic and antiholomorphic cotangent bundle. Thanks to the de Rham theorem, which states that each cohomology class of a differential manifold can be represented by a closed differential form, there is an induced grading on the cohomology groups:

$$H^{p,q}(X) = \{\alpha \in H^{p,q}(X, \mathbb{C}) : \alpha \text{ can be represented by a closed form of type } (p, q)\}.$$

On a projective manifold, we have the following stronger theorem:

Theorem 1.1.1 (Hodge decomposition) *For a projective manifold X, we have*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

This decomposition, which was first proved by Hodge early last century [12], is the first example of the interplay between the topological invariant and algebraic structure of complex algebraic varieties. Hodge theory - the study of this interplay - remains a vital branch of algebraic geometry today.

Fix a smooth projective variety X of complex dimension n. A subvariety is a subset defined locally as the vanishing locus of a collection of polynomials in local coordinates. Each subvariety V of codimension p gives rise to a natural cohomology class in $H^{2p}(X, \mathbb{Z})$ by Poincaré duality. From the viewpoint of de Rham theory, this can be seen as the class of the integral operator $\int_V$ over closed forms. An important observation is that the integral is nonzero only when integrating forms of bidegree $(n-p, n-p)$, hence the cohomology class of V lies in $H^{2p}(X, \mathbb{Z}) \cap H^{p,p}(X)$. The now famous Hodge conjecture states that every rational (p, p) class is of this type:

Conjecture 1.1.1 (Hodge conjecture) *For a smooth projective variety X, every rational cohomology class of type (p, p) (i.e. a class in $H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$) is a linear combination of cohomology classes of algebraic subvarieties of X.*

Despite intense efforts of many powerful mathematicians, the Hodge conjecture remains open. Progress has been made in special cases; notably the $p = 1$, $n = 2$ case was solved by Lefschetz [14] taking the advantage of the fact that every point of the Jacobian of a curve comes from an algebraic cycle, which will be recalled in the following sections.

1.2 Abel-Jacobi mapping for curves

In this section, when we say an algebraic curve, we mean a smooth irreducible complex projective variety of dimension 1. The Abel-Jacobi mapping is the most important tool for studying the Hodge theory of an algebraic curve. The study of this mapping originated in the nineteenth century [1, 13], and a detailed account can be found in [15, 22].

Fix a point p_0 on an algebraic curve C. Since the curve is connected, for any point $p \in C$, we can connect p_0 to p by a path Γ in C. It is a standard result that each class in $H^{1,0}(C)$ can be represented by a *unique holomorphic* 1-form, and integrating this form over Γ gives rise to a complex number. In this way, to each p it associates a linear operator $\int_\Gamma \in H^{1,0}(C)^*$. For two different paths Γ_1 and Γ_2 from p_0 to p, the difference $\Gamma_1 - \Gamma_2$ has boundary 0, so in fact represents a class in $H_1(C, \mathbb{Z})$. Hence the corresponding integral operators differ by the image of the morphism

$$H_1(C, \mathbb{Z}) \to H^{1,0}(C)^*,$$

given by the natural pairing between cohomology and homology. Note that the image of $H_1(C, \mathbb{Z})$ in $H^{1,0}(C)^*$ is a lattice of full rank. Taking the quotient, we can define the Abel-Jacobi mapping.

Definition 1.2.1 *For an algebraic curve C, the Jacobian of the curve is defined by*

$$J(C) := \frac{H^{1,0}(C)^*}{H_1(C, \mathbb{Z})}.$$

Fixing a point p_0, the Abel-Jacobi mapping is given by

$$\begin{aligned} AJ : C &\to J(C), \\ p &\to \int_\Gamma = \int_{p_0}^{p}. \end{aligned}$$

Here the Jacobian $J(C)$ has a natural structure of a complex torus induced from the complex structure of the vector space $H^{1,0}(C)^*$. Furthermore, the Abel-Jacobi mapping is holomorphic [11, Section 2.2]. A more detailed argument shows that $J(C)$ is in fact an abelian variety [11, Section 2.2], i.e. a projective variety with an abelian group structure, and the Abel-Jacobi mapping is a morphism between projective varieties.

A divisor D on a curve C is a formal linear combination

$$D = \sum_i n_i p_i.$$

of points on C. When $\sum_i n_i = 0$, we can find a 1-chain Γ on C such that

$$\partial\Gamma = D.$$

In this case, the divisor is homologically trivial, i.e. the associated degree-zero cohomology class vanishes. More generally, the Abel-Jacobi image of a homologically trivial divisor D is defined to be the class of $\int_\Gamma$ in $J(C)$. Using the group structure on the Jacobian $J(C)$, we can take the sum $\sum_i n_i \cdot AJ(p_i)$ of the Abel-Jacobi images of all the points, and this gives an equivalent definition. We then have the Jacobi inversion theorem [13]:

Theorem 1.2.1 (Jacobi inversion) *For an algebraic curve C, each point on the Jacobian $J(C)$ is the Abel-Jacobi image of some divisor on C of degree* 0*.*

1.3 Lefschetz' theorem on (1,1) classes

Among the cases in which Hodge conjecture is proved, the most important one is the case when $p = 1$, usually known as the Lefschetz theorem on $(1, 1)$ classes.

Theorem 1.3.1 (Lefschetz theorem on $(1, 1)$ classes) *Each class in $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ is a linear combination of cohomology classes of subvarieties in X.*

Lefschetz originally proved this theorem for varieties of dimension two ([14], for details see [24, Chapter VII]), using the following beautiful geometric idea. The first step is to choose a Lefschetz pencil on the variety X, that is a one-dimensional family of hyperplane sections of X $\pi : \mathscr{C} \to \mathbb{P}^1$, all smooth but for finitely many exceptions which have exactly one ordinary double point. The existence of such a pencil is shown for example in [23, Section II.2.1]. Now as each smooth hypersurface $C_b := \pi^{-1}(b)$ of X is a smooth algebraic curve, we can consider its Jacobian $J(C_b)$. The Jacobians turn out to form an algebraic family $\mathscr{J} \to \mathbb{P}^1_{sm}$, called the Jacobian fibration, parametrized by the locus $\mathbb{P}^1_{sm}$ in $\mathbb{P}^1$ corresponding to smooth curves. Any given $(1,1)$ class on X induces a section of $\mathscr{J} \to \mathbb{P}^1_{sm}$, with certain restricted boundary behavior when approaching the singular fibers. Such a section is called a normal function, and in some sense, the Hodge conjecture is equivalent to saying that normal functions can be constructed out of algebraic cycles. In this special case, the magic is the Jacobi inversion theorem. Using this theorem with some extra work we deduce the algebraicity of the $(1,1)$ class.

1.4 Griffiths' Abel-Jacobi mapping

One theme in Hodge theory is the higher dimensional generalization of the above notions. For a smooth projective variety X, Griffiths' Abel-Jacobi mapping is a group homomorphism from the group $A^k_{hom}(X)$ of homologically trivial algebraic cycles of codimension k to the intermediate Jacobian $J^{2k-1}(X)$. The Hodge filtration $F^\bullet$ on the singular cohomology $H^k(X,\mathbb{C})$ is the decreasing filtration defined by

$$F^pH^k(X,\mathbb{C}) := \bigoplus_{m\geq p} H^{m,k-m}(X).$$

The intermediate Jacobian $J^{2k-1}(X)$ is given by

$$J^{2k-1}(X) := \frac{H^{2k-1}(X,\mathbb{C})}{F^kH^{2k-1}(X) \oplus H^{2k-1}(X,\mathbb{Z})} \cong \frac{F^kH^{2k-1}(X,\mathbb{C})^*}{H_{2k-1}(X,\mathbb{Z})}.$$

Here $H_{2k-1}(X,\mathbb{Z})$ is identified with a lattice in $F^kH^{2k-1}(X,\mathbb{C})^*$ via the pairing between homology and cohomology. Because $F^kH^{2k-1}(X,\mathbb{C})^*$ is a complex vector space and $H_{2k-1}(X,\mathbb{Z})$ is a lattice of the same dimension, the intermediate Jacobians are naturally compact complex tori. However, as opposed to the curve case, these complex tori cannot in general be given the structure of projective varieties.

For an algebraic cycle $\Sigma \in A^k_{hom}(X)$, by definition we can choose a differential chain Γ, such that $\partial\Gamma = \Sigma$. Now for every element in $F^kH^{2k-1}(X,\mathbb{C})$, it is a consequence of Hodge theory that we can pick a closed form representing the cohomology class such that the holomorphic degrees of the forms are at least k. Now the integral operator $\int_\Gamma$, paired with the above representatives, can be viewed as a linear operator on $F^kH^{2k-1}(X,\mathbb{C})$. Griffiths' Abel-Jacobi image of Σ is defined to be the projection of this operator to $J^{2k-1}(X)$:

$$\begin{array}{ccccc} A^k_{hom}(X) & \to & F^kH^{2k-1}(X,\mathbb{C})^* & \to & J^{2k-1}(X) \\ \Sigma & \to & \displaystyle\int_\Gamma & \to & AJ(\Sigma) \end{array}$$

This doesn't depend on the choice of the differential chain and the closed forms, we refer to [23, Section I.12.1] for the details of this construction.

The study of these objects leads to many important results, e.g. the proof of the irrationality of cubic threefolds by Clemens and Griffiths [6]. However, unlike the curve case, in higher dimensions the Abel-Jacobi mapping is rarely surjective [23, Corollary I.12.19]. This non-surjectivity is one obstruction to the geometric interpretation of normal functions, and the first part of my thesis overcomes this obstruction in a canonical way. The obstruction to characterizing the 'algebraicity' of this canonical geometric representative remains.

1.5 Topological Abel-Jacobi mapping and Jacobian inversion

In Chapter 2, we construct an extension of Griffiths' Abel-Jacobi mapping beyond just algebraic cycles. Let X be a smooth projective variety of odd dimension $2n-1$, and fix an ample line bundle on X. We only look at the primitive part

$$J^{2n-1}(X)_{prim} = \frac{F^n H^{2n-1}(X,\mathbb{C})^*_{prim}}{H_{2n-1}(X,\mathbb{Z})_{prim}/torsion}$$

of the intermediate Jacobian in the middle dimension. Recall $H_{2n-1}(X,\mathbb{Z})_{prim}$ is the kernel of the Lefschetz operator, i.e. cupping with the Chern class of the given ample line bundle. By Grothendieck's inductive approach to the Hodge problem ([16, Theorem 12.12]), this is the most interesting part. For a smooth hypersurface $Y \overset{i}{\hookrightarrow} X$, there is a natural pushforward morphism between the singular cohomology groups $H^{2n-2}(Y,\mathbb{Z}) \to H^{2n}(X,\mathbb{Z})$. Define the vanishing cohomology of Y, $H^{2n-2}(Y,\mathbb{Z})_{van}$, as the kernel of this morphism.

Our topological Abel-Jacobi mapping is a group homomorphism $AJ : H^{2n-2}(Y,\mathbb{Z})_{van} \to J^{2n-1}(X)_{prim}$; it has two important properties:
(i) When the class in $H^{2n-2}(Y,\mathbb{Z})_{van}$ can be represented by an algebraic cycle, the topological Abel-Jacobi image of this class agrees with Griffiths' Abel-Jacobi image of the algebraic cycle in $J^{2n-1}(X)_{prim}$ defined in Section 1.4;
(ii) When Y deforms in a linear series, and the class in $H^{2n-2}(Y,\mathbb{Z})_{van}$ moves to cohomology classes of nearby fibers by the Gauss-Manin connection, the topological Abel-Jacobi image varies differentially in $J^{2n-1}(X)_{prim}$.

The construction of the topological Abel-Jacobi mapping is given in Section 2.1. The strategy is to define a closed current for each vanishing cohomology class, using harmonic representatives and several canonically defined differential operators from Kähler geometry. We show that this construction is canonical, and works for all projective manifolds of odd dimensions. The proof of the two properties is also given in Section 2.1, using harmonic decomposition of currents on Kähler manifolds.

Property (ii) allows us to study the topological Abel-Jacobi mapping for a family of hypersurfaces. In particular, when we take an ample linear series on X, the mapping is defined for the local system of vanishing cohomology of hypersurfaces over the smooth locus of the linear series. This local system can be viewed as an infinitely sheeted covering space of the smooth locus, hence is equipped with the structure of a complex manifold. Our first main theorem, proved in Section 3.2, is the following topological Jacobi inversion theorem:

Theorem 1.5.1 *The topological Abel-Jacobi mapping is a canonically-defined smooth extension of the Griffiths' Abel-Jacobi mapping on algebraic cycles. It is independent of the choice of Kähler metric on X. When the hypersurface linear series is sufficiently ample, there exists a connected component of the local system of vanishing cohomology (viewed as a complex manifold), such that the topological Abel-Jacobi mapping restricted to this component is a surjection to* $J^{2n-1}(X)_{prim}$.

The proof makes use of a nice result of Schnell [20] and a detailed analysis of the boundary behavior of the topological Abel-Jacobi mapping. Note that when $n = 1$ this reduces to the classical Jacobi inversion theorem for curves.

In Chapter 4, we study another type of topological Abel-Jacobi mapping. This mapping only works for sufficiently ample hypersurfaces in a given even dimensional variety, but has the benefit of being holomorphic. The idea originates from an approach to Griffiths' Abel-Jacobi mapping for Calabi-Yau threefolds, given by Clemens and Voisin [5], which will be recalled at the beginning of Chapter 4.

1.6 Extended locus of Hodge classes

We have seen the important role of Lefschetz pencils in the study of integral $(1, 1)$ classes. However, recent advances indicate that in the study of Hodge loci in higher dimensions, we need to consider the complete family of hypersurfaces in an ample linear series, rather than just one-dimensional families. In particular, the notion of singularities of normal functions over the complete family was introduced [9], and it is shown that the existence of such singularities is equivalent to the Hodge conjecture [9, 2].

In this section we want to recall Schnell's construction of the extended locus of Hodge classes [19], which is closely related to the circle of ideas mentioned above, and also to our work in Chapter 4.

Fix a smooth projective variety X of odd dimension $2n-1$ and an ample line bundle $\mathscr{L}$ over X. Consider $\mathbb{P} := \mathbb{P}(H^0(X, \mathscr{L}))$ as the parameter space of hypersurfaces in $\mathscr{L}$ and $\mathbb{P}_{sm}$ as the locus of smooth hypersurfaces. For each smooth hypersurface Y, we can define the vanishing cohomology

$$H^{2n-2}(Y, \mathbb{Z})_{van} := ker(H^{2n-2}(Y, \mathbb{Z}) \to H^{2n}(X, \mathbb{Z})).$$

Over $\mathbb{P}_{sm}$, this forms a local system $\mathcal{H}_{Y,2n-2,\mathbb{Z}}$, and we use $\mathcal{U}$ to denote the topological space associated to this local system.

Now following Saito [17, 18] and Schnell [21], there exists a mixed Hodge module $\mathcal{M}$, which is an extension of the local system $H^{2n-2}(Y, \mathbb{Z})_{van}$ over $\mathbb{P}_{sm}$. Consider the holomorphic morphism

$$\iota : \mathcal{U}(K) \to T(F_{-n}\mathcal{M}),$$

given by the cup product, where $\mathcal{U}(K)$ is the union of connected components of $\mathcal{U}$ over which the self intersection is bounded by K, and $T(F_{-n}\mathcal{M})$ is the analytic space defined as the spectrum of the symmetric algebra of the coherent sheaf $F_{-n}\mathcal{M}$. Concretely, this is given by

$$H^{2n-2}(Y, \mathbb{Z})_{van} \to F^n H^{2n-2}(Y)^*_{van},$$

$$\alpha \to < \alpha, \cdot > .$$

It is proved by Schnell that the closure of $\iota(\mathcal{U}(K))$ is an analytic subset of $T(F_{-n}\mathcal{M})$. Denote the normalization of the closure of the image by $\bar{\Psi}$. Now we factor ι through $\bar{\Psi}$, and it is shown that the morphism $\iota : \mathcal{U}(K) \to \bar{\Psi}$ is finite. By a standard construction in complex analysis, we can build an extension of $\mathcal{U}(K)$ as a finite (ramified) cover of $\bar{\Psi}$. We get the following important theorem in [19]

Theorem 1.6.1 *There is a normal analytic space $\bar{\mathcal{U}}(K)$ containing $\mathcal{U}(K)$ as a dense open subset, and a finite holomorphic mapping*

$$\bar{\iota} : \bar{\mathcal{U}}(K) \to T(F_{-n}\mathcal{M}),$$

whose restriction to $\mathcal{U}(K)$ coincides with ι. Moreover, $\bar{\iota}$ and $\bar{\mathcal{U}}(K)$ are unique up to isomorphism.

According to the uniqueness, when $K_1 < K_2$, $\bar{\mathcal{U}}(K_1)$ is naturally contained in $\bar{\mathcal{U}}(K_2)$ as a subspace. We can take the direct limit and form an extension of $\mathcal{U}$. We will denote this complex analytic space as $\bar{\mathcal{U}}$. In this case, Schnell proved that $\bar{\mathcal{U}}$ is holomorphically convex. The inverse image $\iota^{-1}(0)$ in $\bar{\mathcal{U}}$ is the extended locus of Hodge classes.

CHAPTER 2

Topological Abel-Jacobi mapping

2.1 Construction

Assume X is a smooth irreducible projective variety of dimension $2n-1$, and Y is a smooth ample hypersurface in X. By definition, $H^{2n-1}(X,\mathbb{R})_{prim}$ is the kernel of the restriction mapping between the cohomology groups

$$i^* : H^{2n-1}(X,\mathbb{R}) \to H^{2n-1}(Y,\mathbb{R}),$$

and $H^{2n-1}(X,\mathbb{Z})_{prim}$, $H^{2n-1}(X,\mathbb{C})_{prim}$ can be defined in a similar way. These groups naturally carry Hodge structures, as substructures of the corresponding cohomology groups of X.

We will introduce the notion of primitive intermediate Jacobian. Using Poincaré duality, we have the following diagram,

$$\begin{array}{ccc}
H^{2n-1}(X,\mathbb{Z})_{prim}/torsion & = & H_{2n-1}(X,\mathbb{Z})_{prim}/torsion \\
\downarrow & & \downarrow \\
H^{2n-1}(X,\mathbb{R})_{prim} & = & H^{2n-1}(X,\mathbb{R})^*_{prim},
\end{array}$$

where the second vertical mapping is given by integrating forms over chains. In order to have a complex structure, we identify $H^{2n-1}(X,\mathbb{R})_{prim}$ with the complex vector space $F^nH^{2n-1}(X,\mathbb{C})_{prim}$, when $F^\bullet$ denotes the Hodge filtration.

Definition 2.1.1 *The primitive intermediate Jacobian of X is defined to be the compact complex torus*

$$J^{2n-1}(X)_{prim} = \frac{F^n H^{2n-1}(X,\mathbb{C})^*_{prim}}{H_{2n-1}(X,\mathbb{Z})_{prim}/torsion}.$$

Recall that the vanishing cohomology group $H^{2n-2}(Y,\mathbb{Z})_{van}$ is defined to be the kernel of the Gysin morphism

$$\begin{tikzcd} i_* : H^{2n-2}(Y,\mathbb{Z}) \arrow[r] \arrow[d, "P.D."'] & H^{2n}(X,\mathbb{Z}) \\ H_{2n-2}(Y,\mathbb{Z}) \arrow[r, "i_*"] & H_{2n-2}(X,\mathbb{Z}), \arrow[u, "P.D.^{-1}"'] \end{tikzcd}$$

where $P.D.$ is the Poincaré duality morphism.

For a vanishing cohomology class $\alpha \in H^{2n-2}(Y,\mathbb{Z})_{van}$, we can choose a topological cycle Σ on Y, such that Σ represents the Poincaré dual of α. In other words, the current $\int_\Sigma$ associated to Σ, which we will denote also as Σ, represents α. By definition, we see that the homology class of Σ in X is zero, so we can choose a topological chain Γ in X whose boundary is Σ. As currents, we have

$$d\Gamma = \Sigma.$$

Now on Y, we can consider the harmonic form h_α representing α. This defines a current

$$\Sigma_{\alpha,Y} = \int_Y h_\alpha \wedge (-).$$

We will use Σ_α to denote the push forward of $\Sigma_{\alpha,Y}$ into X. Σ_α is an exact current on X. So for a fixed metric on X, using differential operators on currents, we have a current on X given by $d^*G\Sigma_\alpha$.

Remark 2.1.1 *Instead of the harmonic representative, we can take any d-closed and ∂-closed differential form representing α, and the argument in this section still works. In this paper we will stick to the harmonic forms since it simplifies several proofs in later sections.*

Note that Σ and $\Sigma_{\alpha,Y}$ represent the same cohomology class on Y, so their difference is a coboundary. We will write

$$dB = \Sigma_{\alpha,Y} - \Sigma,$$

where B is supported on Y. Consider the following current

$$A_\alpha = -d^* G\Sigma_\alpha + B + \Gamma.$$

The following proposition shows that A_α defines a mapping from $H^{2n-2}(Y,\mathbb{Z})_{van}$ to $J^{2n-1}(X)_{prim}$.

Proposition 2.1.1 *A_α is a closed current on X. It determines a point $AJ(\alpha) \in J^{2n-1}(X)_{prim}$, which is independent of the choice of Σ and Γ, and also the choice of metrics on X and Y. Moreover, if α is represented by an algebraic cycle, then this point in $J^{2n-1}(X)_{prim}$ coincides with the Abel-Jacobi image of this algebraic cycle.*

Proof: We will first show that A_α is closed. Note that Σ_α represents the cohomology class $i_*(\alpha) = 0$, so it is an exact current, and

$$\Sigma_\alpha = dd^* G\Sigma_\alpha.$$

So we can compute

$$\begin{aligned} dA_\alpha &= -dd^* G\Sigma_\alpha + dB + d\Gamma \\ &= -\Sigma_\alpha + \Sigma_\alpha - \Sigma + \Sigma \\ &= 0. \end{aligned}$$

Since A_α is closed, it takes zero value when paired with exact forms, we see A_α is a well defined element in $F^n H^{2n-1}(X,\mathbb{C})^*_{prim}$. The corresponding point $AJ(\alpha) \in J^{2n-1}(X)_{prim}$ is determined by the projection of this element.

In order to show that $AJ(\alpha)$ doesn't depend on the choice of Σ and Γ, assume a cycle Σ' on Y also represents α, and Γ' is a chain on X whose boundary is Σ'. We can define B'

similarly, and consider the current

$$A'_\alpha = -d^* G\Sigma_\alpha + B' + \Gamma'.$$

Note that Σ and Σ' are cycles on Y representing the same cohomology class, so they differ by a boundary on Y, write

$$\Sigma - \Sigma' = dT.$$

Now

$$\begin{aligned} A_\alpha - A'_\alpha &= B + \Gamma - B' - \Gamma' \\ &= (\Gamma - \Gamma' - T) + (B - B' + T). \end{aligned}$$

The first term is a topological cycle on X, a multiple of which can be written as the sum of an element of $H_{2n-1}(X - Y)$ and an element of the image of $H_{2n-1}(Y)$, so its image in $F^n H^{2n-1}(X, \mathbb{C})^*_{prim}$ is contained in the image of $H_{2n-1}(X-Y, \mathbb{Q}) \cap H_{2n-1}(X, \mathbb{Z})_{prim}$. The second term is a closed current supported on Y. Since for any differential form on X representing a primitive cohomology class, its restriction to Y is exact and pairs to zero with closed forms, we see the Abel-Jacobi image of $A_\alpha - A'_\alpha$ is 0.

Now we want to show that the Abel-Jacobi mapping doesn't depend on the Kähler metric of X. Using the fact that Σ_α is $d^c - closed$, we see that

$$d^* G\Sigma_\alpha = d^c d^{c*} d^* G^2 \Sigma_\alpha.$$

So for two Kähler metrics on X, the difference of the two expressions of $d^* G\Sigma_\alpha$ is $d - closed$ and $d^c - exact$, hence $d - exact$. We see that the choice of Kähler metrics on X doesn't change the Abel-Jacobi mapping.

Now for a primitive cohomology class $\theta \in F^n H^{2n-1}(X, \mathbb{C})_{prim}$, we choose the harmonic form h_θ as the representative of θ. Since $h_\theta|_Y$ is $d - exact$ and $\partial - closed$, by $\partial\bar{\partial} - lemma$, we have

$$h_\theta|_Y = dd^c\gamma,$$

for some differential form γ on Y. Then

$$\begin{aligned} A_\alpha(h_\theta) &= \int_\Gamma h_\theta + \int_X h_\theta \wedge B \\ &= \int_\Gamma h_\theta + \int_Y dd^c\gamma \wedge B \\ &= \int_\Gamma h_\theta + \int_Y (\Sigma_{\alpha,Y} - \Sigma)(d^c\gamma) \\ &= \int_\Gamma h_\theta + \int_Y h_\alpha \wedge d^c\gamma - \int_\Sigma d^c\gamma \\ &= \int_\Gamma h_\theta - \int_\Sigma d^c\gamma. \end{aligned}$$

Here the second term in the fourth line vanishes by the Stokes theorem because h_α is $d^c-closed$. In particular, we see that this is independent of the choice of the Kähler metric on Y.

Moreover, if Σ is an algebraic cycle, the restriction of the Weil operator C to Σ is the identity on forms of top degree on Σ, hence $\int_\Sigma d^c\gamma = \int_\Sigma dc\gamma = 0$ by Stokes theorem. In this case, the only contribution comes from the current Γ, which coincides with the Abel-Jacobi image of an algebraic cycle. □

Definition 2.1.2 *The group homomorphism from $H^{2n-2}(Y,\mathbb{Z})_{van}$ to $J^{2n-1}(X)_{prim}$, as defined above, is the topological Abel-Jacobi mapping.*

2.2 First order derivatives

Now we fix an ample line bundle L over X, and use $\mathbb{P}_{sm} \subset \mathscr{L} = \mathbb{P}H^0(X,L)$ to denote the locus where for $t \in \mathbb{P}_{sm}$, the corresponding hypersurfaces Y_t is smooth. Over $\mathbb{P}_{sm}$, we have the local system $\mathcal{H}^{2n-2}(\mathbb{Z})_{van}$, which is fiberwise given by $H^{2n-2}(Y_t,\mathbb{Z})_{van}$. Equipped with the Gauss-Manin connection, $\mathcal{H}^{2n-2}(\mathbb{Z})_{van}$ can be viewed as a covering space of $\mathbb{P}_{sm}$, hence is equipped with a natural complex structure. We will use $\mathcal{U}$ to denote this complex manifold.

For a holomorphically embedded one-dimensional complex disc $T \subset \mathbb{P}_{sm}$, consider the corresponding family of hypersurfaces $\mathscr{Y} \to T$. Given a flat section α of $\mathcal{H}^{2n-2}(\mathbb{Z})_{van}$, for each point $t \in T$, we have $\alpha_t \in H^{2n-2}(Y_t, \mathbb{Z})_{van}$, for which the topological Abel-Jacobi mapping is defined. In order to compute the derivatives of the topological Abel-Jacobi mapping, we fix a primitive cohomology class $\theta \in F^n H^{2n-1}(X, \mathbb{C})_{prim}$, and consider the complex function $AJ(\alpha_t)(\theta) : T \to \mathbb{C}$. Choose a transversely holomorphic trivialization $\sigma : X \times T \to X \times T$, such that σ also trivializes the family $\mathscr{Y} \to T$. For a tangent vector field v over T, we have a canonical global lifting $\widetilde{v}$ of v to $X \times T$, and let $\bar{v} = \sigma_*(\widetilde{v})$. Note that by the choice of the trivialization, $\bar{v}$ and v are of the same type. Use $p : X \times T \to X$ to denote the projection morphism.

Let h_{α_t} be the harmonic forms on Y_t representing α_t, we can take a lifting form H_α on $\mathscr{Y}$ such that i) $H_\alpha|_{Y_t} = h_{\alpha_t}$, ii) $i_u H_\alpha = 0$, for any horizontal vector field u on $\mathscr{Y}$. Also let h_θ be the harmonic form on X representing θ. It is easy to see that $p^* h_\theta|_{\mathscr{Y}}$ is $d - exact$, write $p^* h_\theta|_{\mathscr{Y}} = d\Omega$ for some differential form Ω on $\mathscr{Y}$. Assume $\Omega|_{Y_t} = \omega_t$, then we have $d\omega_t = h_\theta|_{Y_t}$.

Theorem 2.2.1 *In above setting, we have*

$$\frac{\partial}{\partial v}|_{t=0}(AJ(\alpha_t)(\theta)) = \int_{Y_0} i_{\bar{v}} dH_\alpha \wedge \omega_0 + \int_{Y_0} h_{\alpha_0} \wedge i_{\bar{v}}(p^* h_\theta),$$

where $i_{\bar{v}}$ is the contraction against the vector field $\bar{v}$.

Proof: We have seen that the topological Abel-Jacobi mapping is independent of the choice of Σ and Γ, so we can assume that the trivialization σ also trivializes the family of

Σ_t and Γ_t. So

$$
\begin{aligned}
AJ(\alpha_t)(\theta) &= \int_{\Gamma_t} h_\theta + \int_{Y_t} B_t \wedge d\omega_t \\
&= \int_{\Gamma_t} h_\theta - \int_{\Sigma_t} \omega_t + \int_{Y_t} h_{\alpha_t} \wedge \omega_t.
\end{aligned}
$$

Use $L_{\bar{v}}$ to denote the Lie derivative in $\bar{v}$ direction, we have the Cartan-Lie formula

$$
L_{\bar{v}} = di_{\bar{v}} + i_{\bar{v}}d.
$$

Now we compute the derivative of each term.

$$
\begin{aligned}
\frac{\partial}{\partial v}|_{t=0} \int_{\Gamma_t} h_\theta &= \int_{\Gamma_0} L_{\bar{v}} p^* h_\theta \\
&= \int_{\Gamma_0} di_{\bar{v}} p^* h_\theta + \int_{\Gamma_0} i_{\bar{v}} dp^* h_\theta \\
&= \int_{\Gamma_0} di_{\bar{v}} p^* h_\theta \\
&= \int_{\Sigma_0} i_{\bar{v}} p^* h_\theta.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial v}|_{t=0} \int_{\Sigma_t} \omega_t &= \int_{\Sigma_0} L_{\bar{v}} \Omega \\
&= \int_{\Sigma_0} di_{\bar{v}} \Omega + \int_{\Gamma_0} i_{\bar{v}} d\Omega \\
&= \int_{\Sigma_0} i_{\bar{v}} d\Omega \\
&= \int_{\Sigma_0} i_{\bar{v}} p^* h_\theta.
\end{aligned}
$$

So the derivatives of the first two terms cancel each other. For the third term, we have

$$
\begin{aligned}
\frac{\partial}{\partial v}|_{t=0} \int_{Y_t} h_{\alpha_t} \wedge \omega_t &= \int_{Y_0} L_{\bar{v}}(H_\alpha \wedge \Omega) \\
&= \int_{Y_0} L_{\bar{v}} H_\alpha \wedge \omega_0 + \int_{Y_0} h_{\alpha_0} \wedge L_{\bar{v}}\Omega \\
&= \int_{Y_0} d i_{\bar{v}} H_\alpha \wedge \omega_0 + \int_{Y_0} i_{\bar{v}} d H_\alpha \wedge \omega_0 + \int_{Y_0} h_{\alpha_0} \wedge d i_{\bar{v}}\Omega + \int_{Y_0} h_{\alpha_0} \wedge i_{\bar{v}} d\Omega \\
&= \int_{Y_0} i_{\bar{v}} d H_\alpha \wedge \omega_0 + \int_{Y_0} h_{\alpha_0} \wedge i_{\bar{v}}(p^* H_\theta).
\end{aligned}
$$

□

In particular, if α_0 is a Hodge class, i.e. a class of type $(n-1, n-1)$, and the tangent vector is of type $(0,1)$, the second term vanishes since the integrand is of holomorphic degree larger than the antiholomorphic degree. Moreover, if the tangent vector is also tangent to the Hodge locus, then $i_{\bar{v}} dH_\alpha$ is of type $(n-1, n-1)$ or $(n, n-2)$, so the first term also vanishes for type reason. Hence we obtain

Corollary 2.2.1 *When restricted to the Hodge locus, the topological Abel-Jacobi mapping is holomorphic.*

Moreover, we need the following proposition for later use

Proposition 2.2.2 *The topological Abel-Jacobi mapping is real analytic.*

Proof: Recall from the first section that, when we take

$$h_\theta|_{Y_t} = dd^c \gamma_t$$

for some differential form γ_t on Y_t, the topological Abel-Jacobi image is given by

$$A_{\alpha_t}(h_\theta) = \int_{\Gamma_t} h_\theta - \int_{\Sigma_t} d^c \gamma_t.$$

Notice we can embed X into a projective space such that the family Y_t is a family of hyperplane sections. In this way, the Fubini-Study metric on the projective space induces a real analytic family of metrics on Y_t. By Kuranishi theory, the trivialization of the family can be taken to be real analytic. In particular, the families of Γ_t and Σ_t are real analytic. On the other hand, harmonic forms are always real analytic, so we can take γ_t to be a real analytic family. So the two terms both vary real analytically when t changes. $\square$

CHAPTER 3

Topological Jacobi inversion

3.1 Boundary behavior of the topological Abel-Jacobi mapping

In this section, we want to study the behavior of the topological Abel-Jacobi mapping along the boundary of $\mathcal{U}$. We will focus on the one-dimensional case, which will be important in the next section. Assume we have a family of ample hypersurfaces $\mathscr{Y}$ over one-dimensional disc Δ, and for $t \in \Delta^*$ the hypersurface Y_t is smooth, while Y_0 is a union of smooth hypersurfaces $Y_{0,i}$ with simple normal crossing. Assume $\mathscr{Y}$ is smooth. For any $\alpha_t \in H^{2n-2}(Y_t, \mathbb{Z})_{van}$ of finite monodromy, by applying a finite base change, we can assume α_t is an invariant cycle class, so by the local invariant cycle theorem, it is the restriction of a class on $\mathscr{Y}$. Choosing a topological cycle Σ on $\mathscr{Y}$ representing this class, and write its restriction to Y_t as Σ_t for every t. Moreover we assume that every Σ_t is a differential cycle and that Σ_0 is disjoint from the singularities of Y_0. These assumptions will be automatically satisfied in the applications in next section. Clearly Σ_0 and α_0 decompose into sums of $\Sigma_{0,i}$ and $\alpha_{0,i}$ with respect to the irreducible components decomposition. For $t \neq 0$, α_t is a local section of $\mathcal{U}$ over Δ, and the topological Abel-Jacobi mapping is defined for each α_t. The topological Abel-Jacobi image of α_0 is defined to be the sum of the images of $\alpha_{0,i}$. Note that over the disc Δ, the topological Abel-Jacobi mapping is defined separately for $t \neq 0$ and $t = 0$. However, we have the following theorem:

Theorem 3.1.1 *The topological Abel-Jacobi mapping is continuous for the family of α_t over Δ, i.e. in the primitive Jacobian $J^{2n-1}(X)_{prim}$,*

$$\lim_{t\to 0} AJ(\alpha_t) = AJ(\alpha_0).$$

Proof: Fix $\theta \in F^n H^{2n-1}(X,\mathbb{C})_{prim}$, recall that when we take h_θ to be the harmonic representative of θ,

$$AJ(\alpha_t)(\theta) = \int_{B_t} h_\theta + \int_{\Gamma_t} h_\theta.$$

By our choice of Σ_t, we can take Γ_t to be a continuous family of chains in X, so the second term certainly varies continuously. We need to prove that the first term also varies continuously, i.e. for any given $\varepsilon > 0$, we want to show there exists $\delta > 0$, such that for any $t \in \Delta$, $0 < |t| < \delta$, we have

$$|\int_{B_t} h_\theta - \int_{B_0} h_\theta| < \varepsilon.$$

The fact that the restriction of the primitive class θ to Y_t vanishes for every t implies that the pullback of θ to $\mathscr{Y}$ is cohomologically trivial, hence the pullback of h_θ is exact, written as $d\gamma$. Then we have for $t \in \Delta^*$

$$h_\theta|_{Y_t} = d\gamma_t,$$

and also

$$h_\theta|_{Y_{0,i}} = d\gamma_{0,i},$$

where γ_t and $\gamma_{0,i}$ are the restrictions of γ to Y_t and $Y_{0,i}$.

Take I_0 to be a tubular neighborhood in Y_0 of the singularities, which does not intersect with Σ_0. Let $Y_0^{in} := Y_0 \backslash I_0$, note that Y_0^{in} is a smooth open complex manifold. In [4], Clemens defined a flow on $\mathscr{Y}$, such that for arbitrary I_0, Y_0^{in} deforms smoothly to open submanifolds Y_t^{in} of Y_t under this flow. In particular, this gives us diffeomorphisms $\varphi_t : Y_t^{in} \to Y_0^{in}$, where $\varphi_0 = id$. Let $I_t := Y_t \backslash Y_t^{in}$, we can take Σ_t to be the image of Σ_0 under the diffeomorphism.

Note that B_0 is a current on Y_0 satisfying

$$dB_0 = \Sigma_{\alpha_0} - \Sigma_0,$$

so in particular, we have

$$\int_{B_0} h_\theta = \int_{B_0} d\gamma_0 = \sum_i (\int_{Y_{0,i}} h_{\alpha_{0,i}} \wedge \gamma_{0,i} - \int_{\Sigma_t} \gamma_{0,i}).$$

As on each component $h_{\alpha_{0,i}} \wedge \gamma_{0,i}$ is a smooth form, we see that we can take I_0 to be sufficiently small, so

$$|\int_{B_0} h_\theta - (\int_{Y_0^{in}} h_{\alpha_0} \wedge \gamma_0 - \int_{\Sigma_0} \gamma_0)| = |\int_{I_0} h_{\alpha_0} \wedge \gamma_0| < \frac{\varepsilon}{4}.$$

Similarly for $t \neq 0$,

$$\int_{B_t} h_\theta = \int_{B_t} d\gamma_t = \int_{Y_t} h_{\alpha_t} \wedge \gamma_t - \int_{\Sigma_t} \gamma_t.$$

Now γ_t is the restriction of the global form γ defined $\mathscr{Y}$, in order to give a uniform bound for $\int_{I_t} h_{\alpha_t} \wedge \gamma_t$, we only need to show the norm of $h_{\alpha_t}|_{I^t}$ uniformly converges to 0, when the volume of I_0 (hence the volume of I_t) goes to 0.

Note that by our assumption, α_t is the restriction of some class of $\mathscr{Y}$, taking any smooth form h on $\mathscr{Y}$ representing this global class, we see that the restriction $h|_{Y_t} = h_{\alpha_t}+$ exact form. As a consequence, we see the norm of h_{α_t} over I_t is bounded by the norm of $h|_{I_t}$, so we only need to show that the norm of $h|_{I_t}$ goes to 0 uniformly, when the volume of I_0 goes to 0.

The point is that h is defined over $\mathscr{Y}$, so we can also restrict it to each component of Y_0. As the Kähler metrics are taken to be the restricted ones, the norm of a continuous family of forms is still continuous. So the norm of $h|_{I_t}$ goes to 0 for every $t \in \Delta$. By possibly shrinking Δ we have this property over a compact set containing $0 \in \Delta$, so the uniform convergence follows from the Arzela-Ascoli theorem. Now by taking a sufficiently small

I_0, we can find $\delta_1 > 0$, such that for any $t \in \Delta$, $0 < |t| < \delta_1$, we have

$$|\int_{B_t} h_\theta - (\int_{Y_t^{in}} h_{\alpha_t} \wedge \gamma_t - \int_{\Sigma_t} \gamma_t)| = \int_{Y_t^{in}} h_{\alpha_t} \wedge \gamma_t < \frac{\varepsilon}{4}.$$

Now take I_0 to be sufficiently small such that the two inequalities above both hold. Note also that the family of cycles Σ_t is away from the singularities of Y_0, so over the support the family of differential forms γ_t is continuous, hence $\int_{\Sigma_t} \gamma_t$ is continuous at $t = 0$. So we have $\delta_2 > 0$, such that for any $t \in \Delta$, $0 < |t| < \delta_2$,

$$|\int_{\Sigma_t} \gamma_t - \int_{\Sigma_0} \tilde{\gamma}_0| < \frac{\varepsilon}{4}.$$

Now we have the following lemma, which deals with the integral involving the harmonic forms:

Lemma 3.1.2 *In the above setting, given any continuous family of differential forms γ_t on Y_t^{in}, the integral $\int_{Y_t^{in}} h_{\alpha_t} \wedge \gamma_t$ is continuous at $t = 0$.*

Admitting this lemma for the moment, we see that there exists $\delta_3 > 0$ such that, for any $t \in \Delta$ with $0 < |t| < \delta_3$,

$$|\int_{Y_t^{in}} h_{\alpha_t} \wedge \gamma_t - \int_{Y_0^{in}} h_{\alpha_0} \wedge \tilde{\gamma}_0| < \frac{\varepsilon}{4}.$$

Now take $\delta = min\{\delta_1, \delta_2, \delta_3\}$, combining the inequalities above we get what we want. $\square$

Proof of the lemma: the point is to show that h_{α_t} is a continuous family at $t = 0$. As the Kähler metric on Y_t^{in} is restricted from the Kähler metric of X, we see the Hodge $*$-operator and hence the norm of differential forms on Y_t^{in} varies continuously at $t = 0$.

Now recall we have diffeomorphisms $\varphi_t : Y_t^{in} \to Y_0^{in}$, where $\varphi_0 = id$. Consider the form $\varphi_{t,*}(h_{\alpha_t})$ since this represents the same cohomology class as h_{α_0}, we see $\varphi_{t,*}(h_{\alpha_t}) = h_{\alpha_0} + d\eta_t$.

In the proof of this lemma, all the norms $\|\cdot\|$ are taken over the open sets Y_t^{in}. We see

$$\|\varphi_{t,*}(h_{\alpha_t})\| = \|h_{\alpha_0} + d\eta_t\| = \|h_{\alpha_0}\| + \|d\eta_t\| \geq \|h_{\alpha_0}\|.$$

Taking limit for $t \to 0$, we see

$$\underline{\lim}_{t\to 0} \|\varphi_{t,*}(h_{\alpha_t})\| \geq \|h_{\alpha_0}\|.$$

On the other hand, we have

$$h_{\alpha_t} = \varphi_t^* \varphi_{t,*}(h_{\alpha_t}) = \varphi_t^*(h_{\alpha_0}) + d\varphi_t^*(\eta_t),$$

so

$$\|\varphi_t^*(h_{\alpha_0})\| = \|h_{\alpha_t} - d\varphi_t^*(\eta_t)\| \geq \|h_{\alpha_t}\|.$$

Taking limit we have

$$\underline{\lim}_{t\to 0} \|\varphi_t^*(h_{\alpha_0})\| \geq \overline{\lim}_{t\to 0} \|h_{\alpha_t}\|.$$

By the fact that φ_t is a continuous family of diffeomorphisms, $\varphi_0 = id$, and the continuity of the norm, we see that

$$\overline{\lim}_{t\to 0} \|h_{\alpha_t}\| = \overline{\lim}_{t\to 0} \|\varphi_{t,*}(h_{\alpha_t})\|,$$

and

$$\underline{\lim}_{t\to 0} \|\varphi_t^*(h_{\alpha_0})\| = \|h_{\alpha_0}\|.$$

Combining the inequalities above, we get

$$\|h_{\alpha_0}\| \geq \overline{\lim}_{t\to 0} \|\varphi_{t,*}(h_{\alpha_t})\| \geq \underline{\lim}_{t\to 0} \|\varphi_{t,*}(h_{\alpha_t})\| \geq \|h_{\alpha_0}\|,$$

hence

$$\lim_{t\to 0} \|\varphi_{t,*}(h_{\alpha_t})\| = \|h_{\alpha_0}\|.$$

In particular, we see

$$\lim_{t\to 0} d\eta_t = 0,$$

and

$$\lim_{t\to 0} \varphi_{t,*}(h_{\alpha_t}) = h_{\alpha_0}.$$

Now

$$\int_{Y_t^{in}} h_{\alpha_t} \wedge \gamma_t = \int_{Y_0^{in}} \varphi_{t,*}(h_{\alpha_t}) \wedge \varphi_{t,*}(\gamma_t),$$

note that $\varphi_{0,*}(\gamma_0) = \gamma_0$, so $\varphi_{t,*}(\gamma_t)$ is a continuous family. By what we proved we see the integral is continuous at $t = 0$. □

3.2 Topological Jacobi inversion

In this section, we prove a Jacobi inversion theorem for the topological Abel-Jacobi mapping. In order to see this, it is easier to work with the real structure of the primitive intermediate Jacobian. Let

$$J^{2n-1}(X)_{prim} = \frac{H^{2n-1}(X, \mathbb{R})^*_{prim}}{H_{2n-1}(X, \mathbb{Z})_{prim}/torsion}.$$

Now the topological Abel-Jacobi mapping $AJ(\alpha)$ of $\alpha \in H^{2n-2}(Y, \mathbb{Z})_{van}$ is still given by the real current

$$A_\alpha = -d^* G\Sigma_\alpha + B + \Gamma,$$

paired with real differential forms representing real primitive cohomology classes of X. Note that this is the real part of the complex Abel-Jacobi mapping defined in previous sections.

We first recall a theorem in [20]. Consider the monodromy action of the fundamental group $\pi_1(\mathbb{P}_{sm})$ on $\mathcal{H}^{2n-2}(\mathbb{Q})_{van}$. Given $\alpha \in H^{2n-2}(Y_0, \mathbb{Q})_{van}$ and $g \in G = \pi_1(\mathbb{P}_{sm}, 0)$, such that α is invariant under the action of g, we define the tube mapping to $H^{2n-1}(X, \mathbb{Q})_{prim}$ as follow. Choose a topological cycle Σ representing α, and a closed path γ representing g. When α is transported along γ, it moves through a one-dimensional family of hypersurfaces, and in the process, traces out a $(2n-1)$-chain. This is a $(2n-1)$-cycle, since $g\alpha = \alpha$, and defines a well defined element in $H^{2n-1}(X, \mathbb{Q})_{prim}$. We obtain the tube mapping

$$T : \{g \in G, \alpha \in H^{2n-2}(Y_0, \mathbb{Q})_{van} \mid g\alpha = \alpha\} \to H^{2n-1}(X, \mathbb{Q})_{prim}.$$

In [20], Schnell proved the following theorem:

Theorem 3.2.1 *If $H^{2n-2}(Y_0, \mathbb{Q})_{van} \neq 0$, then the tube mapping is surjective.*

In this paper we will work with the integral tube mapping

$$T : \{g \in G, \alpha \in H^{2n-2}(Y_0, \mathbb{Z})_{van} \mid g\alpha = \alpha\} \to H^{2n-1}(X, \mathbb{Z})_{prim}.$$

As a consequence of Schnell's theorem, the image of this mapping is cofinite, i.e. the quotient of $H^{2n-1}(X, \mathbb{Z})_{prim}$ by $T(\{g \in G, \alpha \in H^{2n-2}(Y_0, \mathbb{Z})_{van} \mid g\alpha = \alpha\})$ is a finite group.

Now we study the relationship between the integral tube mapping and the topological Abel-Jacobi mapping. For a fixed $\alpha \in H^{2n-2}(Y_0, \mathbb{Z})_{van}$, the topological Abel-Jacobi mapping induces a morphism on the fundamental groups

$$AJ_* : \pi_1(\mathcal{U}, \alpha) \to \pi_1(J^{2n-1}(X)_{prim}) \cong H_{2n-1}(X, \mathbb{Z})_{prim} \cong H^{2n-1}(X, \mathbb{Z})_{prim}.$$

Note that by the projection from $\mathcal{U}$ to $\mathbb{P}_{sm}$, an element in $\pi_1(\mathcal{U}, \alpha)$ can be realized as an element $g \in \pi_1(\mathbb{P}_{sm}, 0)$ such that $g\alpha = \alpha$. Hence the tube mapping is also defined for $\pi_1(\mathcal{U}, \alpha)$, and we will show these two mappings actually coincide (up to a sign). To see this, choose a topological cycle Σ_0 on Y_0 representing α, and a topological chain Γ_0 on X

whose boundary is Σ_0. Assume $\gamma : [0,1] \to \mathbb{P}_{sm}$ is a loop representing g, and Γ_0 and Σ_0 are transported to Γ_t and Σ_t along γ. Now

$$\begin{aligned} AJ_*(g) &= (-d^*G\Sigma_\alpha + B_1 + \Gamma_1) - (-d^*G\Sigma_\alpha + B_0 + \Gamma_0) \\ &= (B_1 - B_0) + (\Gamma_1 - \Gamma_0). \end{aligned}$$

Since $B_1 - B_0$ is supported on Y_0, the corresponding class in $H^{2n-1}(X,\mathbb{Z})_{prim}$ is represented by $\Gamma_1 - \Gamma_0$. On the other hand, consider the cylinder traced out by the family of Γ_t, the boundary of it is $\Gamma_1 - \Gamma_0 + T(g,\alpha)$, so up to a sign the tube mapping coincides with the induced morphism of the topological Abel-Jacobi mapping on the fundamental groups.

In the following discussion, we will make use of different linear series. Notations will be similar, with the corresponding linear series specified, e.g. $\mathbb{P}_{sm}(\mathscr{L})$, $\mathcal{H}^{2n-2}(\mathscr{L},\mathbb{Z})_{van}$, $\mathcal{U}(\mathscr{L})$. Note that all the linear series are taken to be complete. We have the following theorem, which is an analogue of the classical Jacobi inversion theorem.

Theorem 3.2.2 *There exists a complete linear series $\mathscr{L}$, and a connected component $\mathcal{U}_0$ of $\mathcal{U}(\mathscr{L})$, such that*

$$AJ : \mathcal{U}_0 \to J^{2n-1}(X)_{prim}$$

is surjective.

Proof: Choose an ample linear series $\mathscr{K}$. As a consequence of Theorem 3.2.1, given a point $0 \in \mathbb{P}_{sm}(\mathscr{K})$, there exist $\alpha_i \in H^{2n-2}(Y_0,\mathbb{Z})_{van}$ and $g_i \in \pi_1(\mathbb{P}_{sm}(\mathscr{K}),0)$, such that $g_i\alpha_i = \alpha_i$, and the images of (α_i, g_i) under the tube mapping generate a cofinite subgroup of $H^{2n-1}(X,\mathbb{Z})_{prim}$. Here $i = 1,2,\ldots,d$, where $d := dim\ H^{2n-1}(X,\mathbb{R})_{prim}$. According to the identification of the tube mapping with the induced morphism of the topological Abel-Jacobi mapping on the fundamental groups, we see $AJ_*(\alpha_i, g_i)$ generate $H^{2n-1}(X,\mathbb{R})_{prim}$, where (α_i, g_i) is understood to be an element of $\pi_1(\mathcal{U}(\mathscr{K}),\alpha_i)$.

In order to translate this property on the fundamental groups to a local statement, we need the following lemma:

Lemma 3.2.3 *Consider a differential manifold M, and a smooth morphism to a compact torus $f : M \to A = V/\Lambda$. Assume that there exist points $p_i \in M$ and $g_i \in \pi_1(M, p_i)$ such that $f_*(g_i) \in \pi_1(A, f(p_i)) = \Lambda \subset V$ generate V. Now consider the morphism*

$$f_* : TM \to TA \cong V \times A \to V,$$

where the first morphism is the pushforward of the tangent vectors, and the last morphism is the projection to V. Then we can find $x_i \in M$, and $v_i \in T_{x_i}M$, such that $f_*(v_i)$ generate V.

Proof of the lemma: Choose smooth morphisms $\gamma_i : \mathbb{S}^1 \to M$ representing g_i. Without loss of generality, we can assume all the loops pass through the origin $0 \in A$. Now consider the images of $(f_* \circ \gamma_{i*})(T\mathbb{S}^1)$ in V, it suffices to show that the images generate V. If not, we can assume the image is contained in a proper subspace W of V. In this case, we see the image $(f \circ \gamma_i)(\mathbb{S}^1)$, as a compact set in A, is the projection of a loop in W to A. As a consequence, $f_*(g_i)$ is contained in W, and this contradicts with the assumption that $f_*(g_i)$ generate V. $\square$

By this lemma, we can choose $\alpha_i \in H^{2n-2}(Y_i, \mathbb{Z})_{van}$ and $v_i \in T\mathcal{U}(\mathscr{K})$ such that $AJ_*(v_i) \in H^{2n-1}(X, \mathbb{R})_{prim}$ generate $H^{2n-1}(X, \mathbb{R})_{prim}$, where

$$AJ_* : T\mathcal{U} \to TJ^{2n-1}(X)_{prim} \cong J^{2n-1}(X)_{prim} \times H^{2n-1}(X, \mathbb{R})_{prim} \to H^{2n-1}(X, \mathbb{R})_{prim}.$$

Now take $\mathscr{L} = d\mathscr{K}$, and consider the singular hypersurface $H_0 := \bigcup_{i=1}^{d} Y_i$. This corresponds to a point $p_0 \in \mathbb{P}_{deg} := \prod_{i=1}^{d} \mathbb{P}_{sm}(\mathscr{K}) \subset \mathbb{P}(\mathscr{L})$, and over $\mathbb{P}_{deg}$, we have the local system $\bigoplus_{i=1}^{d} \mathcal{H}^{2n-2}(\mathbb{Z})_{van}$. Thanks to the Gauss-Manin connection, this local system is also associated with a complex manifold $\mathcal{U}_{deg}$. The topological Abel-Jacobi mapping for this local system is defined to be the sum of the topological Abel-Jacobi mapping over each components, i.e.

$$AJ : \mathcal{U}_{deg} \to J^{2n-1}(X)_{prim},$$
$$(\beta_1, \beta_2, \ldots, \beta_d) \mapsto \sum_{i=1}^{d} AJ(\beta_i).$$

Now over the point p_0, consider the class $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ and tangent vectors of $\mathcal{U}_{deg}$ given by the pushforward of v_i under the natural embedding of $\mathcal{U}(\mathscr{K})$ into $\mathcal{U}_{deg}$. By our construction, the topological Abel-Jacobi mapping over $\mathcal{U}_{deg}$ is a local submersion at the point $\bar{\alpha}$.

The singular hypersurface $H_0 = \bigcup_{i=1}^{d} Y_i \in \mathbb{P}_{deg}$, locally in $\mathscr{L}$, can be deformed into smooth hypersurfaces in $\mathbb{P}_{sm}(\mathscr{L})$. Now for every $i = 1, \ldots, d$, take topological cycles Σ_i on Y_i representing α_i. Since α_i are vanishing cohomology classes, we can assume Σ_i to be disjoint from the intersection of Y_i with other Y_j's. Take I to be a tubular neighborhood in H_0 of the intersection of different Y_i and Y_j, which does not intersect with any Σ_i, and let $H_0^{in} := H_0 \backslash I$. H_0^{in} is a smooth open complex manifold, and locally in $\mathscr{L}$, it deforms smoothly to open submanifolds of smooth hypersurfaces in $\mathbb{P}_{sm}(\mathscr{L})$. The topological cycles Σ_i deform to topological cycles on those smooth hypersurfaces, and all of the deformations represent vanishing cohomology classes, since the corresponding cohomology classes in X do not change. Note that this works for any singular hypersurfaces in $\mathbb{P}_{deg}$ and any points in $\mathcal{U}_{deg}$, so this gives us a way to realize $\mathcal{U}_{deg}$ as a submanifold of the boundary of $\mathcal{U}(\mathscr{L})$.

Concretely, let the hypersurfaces Y_i be defined by equations $f_i = 0$. Then H_0 is simply the hypersurface defined by $F := \prod f_i = 0$. For a tangent direction v along $\mathbb{P}_{deg}$, we can choose hypersurfaces $Y_i(s)$ defined by functions $f_i(s)$, both depending on the parameter $s \in \Delta$, such that the infinitesimal deformation at $s = 0$ represents v. Note that the hypersurfaces given by $F(s) := \prod f_i(s)$ are contained in $\mathbb{P}_{deg}$. Now consider the family of hypersurfaces $H(s,t)$ defined by functions $G(s,t) = F(s) + tE$, where $s, t \in \Delta$, and E is a function of the same degree as F. For a generic choice of E, we can assume that $H(s,t) \in \mathbb{P}_{sm}$ for $t \neq 0$.

The family $H(0,t)$ is an example of the deformation in last paragraph. Also, by taking the derivative with respect to s, a tangent vector v along $\mathbb{P}_{deg}$ can be lifted to a vector field $v(t)$ in $T\mathbb{P}|_{H(0,t)}$.

Now we want to show the derivatives of the topological Abel-Jacobi mapping along such a vector field $v(t)$, defined seperately for $\mathcal{U}_{deg}$ and $\mathcal{U}(\mathscr{L})$, is in fact continuous on the union of these two manifolds. To see this, assume we have a family of hypersurfaces H_t over a disc Δ, where H_0 is as before and all the other fibers are smooth. Let $\mathscr{H}$ to denote the total space. So Δ is embedded in $\mathbb{P}(\mathscr{L})$, and the v_i are tangent vectors of $\mathbb{P}_{deg}$ at $0 \in \Delta$. As in the proof of Theorem 3.1.1, take a differential form Ω on $\mathscr{H}$ such that $d\Omega = h_\theta|_{\mathscr{H}}$, and H_{α_i} to be the family of harmonic forms. Now recall when we deform H_0 in $\mathbb{P}_{deg}$ in direction v, the derivative of the topological Abel-Jacobi mapping is given by

$$\sum_i \Big(\int_{Y_i} i_{\bar{v}} dH_{\alpha_i} \wedge \Omega|_{Y_i} + \int_{Y_i} h_{\alpha_i} \wedge i_{\bar{v}}(h_\theta|_{Y_i})\Big),$$

where $\bar{v}$ is a lifting of v, and h_θ is the harmonic representative of $\theta \in H^{2n-1}(X, \mathbb{R})_{prim}$.

Extend v_i to a tangent vector field $v_i(t)$ of $\mathbb{P}(\mathscr{L})$ over Δ. Since $\mathbb{P}_{sm}$ is open in $\mathbb{P}(\mathscr{L})$, we see for $t \in \Delta^*$, H_t deforms to a smooth hypersurface in direction $v_i(t)$, so the derivative of the topological Abel-Jacobi mapping is given by

$$\int_{H_t} i_{\bar{v}} dH_{\alpha_t} \wedge \Omega|_{H_t} + \int_{H_t} h_{\alpha_t} \wedge i_{\bar{v}_i(t)}(h_\theta|_{H_t}).$$

As in the proof of Theorem 3.1.1, take $H_t^{in} = H_t \backslash I_t$. Since $v_i(t)$ is a continuous vector field, we can take lifting $\bar{v}_i(t)$ to H_t^{in}, such that this family of lifting is continuous at $t = 0$. On one hand, the forms under integration remain bounded on I_t, so we can take sufficiently small I_t such that the integration on I_t is bounded by arbitrarily small constant. On the other hand, the family of forms $\Omega|_{H_t^{in}}$ and $i_{\bar{v}_i(t)}(h_\theta|_{H_t^{in}})$ is clearly continuous, so by Lemma 3.1.2, the integral

$$\int_{H_t^{in}} i_{\bar{v}} dH_{\alpha_t} \wedge \Omega|_{H_t^{in}} \text{ and } \int_{H_t^{in}} h_{\alpha_t} \wedge i_{\bar{v}_i(t)}(h_\theta|_{H_t^{in}})$$

are both continuous at $t = 0$. So we see the derivatives of the topological Abel-Jacobi mapping along $v_i(t)$ are continuous.

Now since the topological Abel-Jacobi mapping is a local submersion at the point $\bar{\alpha}$, we see it is a local submersion at a generic point in the intersection of $\mathcal{U}(\mathscr{L})$ and Δ. Pick such a point p, by the fact that the topological Abel-Jacobi mapping over $\mathcal{U}(\mathscr{L})$ is real analytic, we see there is a surjection from a neighborhood of $p \in \mathcal{U}(\mathscr{L})$ to a neighborhood of $AJ(p) \in J^{2n-1}(X)_{prim}$. Thanks to the group structures on $\mathcal{U}(\mathscr{L})$ and $J^{2n-1}(X)_{prim}$, when we take a multiple mp of p for $m \gg 0$, a neighborhood of mp will map surjectively to $J^{2n-1}(X)_{prim}$. So the connected component $\mathcal{U}_0$ containing mp is what we need. $\square$

CHAPTER 4

Holomorphic relative Abel-Jacobi mapping

4.1 Recall: Abel-Jacobi mapping for Calabi-Yau threefolds and Hodge loci

In this chapter, we will study a relative version of the Abel-Jacobi mapping. The idea originates from an approach to Griffiths' Abel-Jacobi mapping for Calabi-Yau threefolds, given by Clemens and Voisin [5]. To motivate the whole construction, we review this story from [5].

Assume X_0 is a Calabi-Yau threefold, i.e. X_0 is simply connected and $\omega_{X_0} \cong O_{X_0}$. It is known that the infinitesimal deformations of X_0 are unobstructed. Each line bundle $\mathscr{L}_0$ on X_0 deforms canonically with X_0 since $H^{0,1}(X_0) = 0$. Here we assume that $\mathscr{L}_0$ is ample, and fix a smooth hypersurface $Y_0 \in |\mathscr{L}_0|$. Following [5], let X' be a local deformation space of X_0, and U' a local deformation space of the pair (X_0, Y_0). Note that $H^0(X, \omega_X)$ is one-dimensional, and we consider the space $\tilde{X}'$ of pairs (X, ω), where ω is a holomorphic $(3,0)$-form. $\tilde{U}'$ is the space of triples (X, Y, ω).

For each pair $(X_{x'}, Y_{u'})$, we have the vanishing cohomology group $H^2(Y_{u'}, \mathbb{Z})_{van}$ and the relative homology group $H_3(X_{x'}, Y_{u'}, \mathbb{Z})$. Over U', consider the local system $H^2_{Y,\mathbb{Z},van}$ and $H_{3,X,Y,\mathbb{Z}}$ with fibers $H^2(Y_{u'}, \mathbb{Z})_{van}$ and $H_3(X_{x'}, Y_{u'}, \mathbb{Z})$ respectively. Using the Gauss-Manin connection, a vanishing class $\alpha \in H^2(Y_0, \mathbb{Z})_{van}$ can be extended to a local flat section $\alpha_{u'}$ of $H^2_{Y,\mathbb{Z},van}$, and this lifts to a local section $\tilde{\alpha}_{u'}$ of $H_{3,X,Y,\mathbb{Z}}$. More concretely, if $\alpha_{u'}$ are represented by 2-cycles $\Sigma_{u'}$, then $\tilde{\alpha}_{u'}$ are represented by differentiable 3-chains $\Gamma_{u'}$ such that

$$\partial\Gamma_{u'} = \Sigma_{u'}.$$

Now consider the Hodge bundle $\mathcal{H}^3_X$ and $\mathcal{H}^3_{X,Y}$ over U', which are bundles with fiber $H^3(X_{x'}, \mathbb{C})$ and $H^3(X_{x'}, Y_{u'}, \mathbb{C})$ respectively. There is a natural bundle mapping

$$f : \mathcal{H}^3_{X,Y} \to \mathcal{H}^3_X,$$

induced by the inclusion operator on cycles. Since $(3, 0)$-forms restrict to zero on any hypersurface, we see that a section $\omega_{\tilde{u}'}$ of $F^3\mathcal{H}^3_X$ canonically lifts to a section ω_{rel} of $\mathcal{H}^3_{X,Y}$. Using the pairing between $H_3(X_{x'}, Y_{u'})$ and $H^3(X_{x'}, Y_{u'})$, we can define the potential function

$$\begin{aligned} \Phi_{BN} : \tilde{U}' &\to \mathbb{C} \\ \tilde{u}' &\to < \omega_{rel}(\tilde{u}'), \tilde{\alpha}(\tilde{u}') > . \end{aligned}$$

More concretely, we have

$$\Phi_{BN}(\tilde{u}') = \int_{\Gamma_{u'}} \omega_{\tilde{u}'}.$$

This function is well-defined up to the choice of the lifting $\tilde{\alpha}_{u'}$. For two different choices, the difference is an element in $H_3(X_{x'}, \mathbb{Z})$, so the function is well defined up to periods of $\omega_{\tilde{u}'}$. Since the section U_{rel} is holomorphic and the lifting $\tilde{\alpha}_{u'}$ is flat, the function Φ_{BN} is holomorphic. We will first study the relationship between the relative differential $d_{\tilde{U}'/\tilde{X}'}\Phi_{BN}$ and the Hodge loci.

Recall the following lemma in [5]:

Lemma 4.1.1

$$\nabla_{\tilde{U}'/\tilde{X}'} U_{rel} \in \Omega^1_{\tilde{U}'/\tilde{X}'} \otimes i(F^2\mathcal{H}^2_Y),$$

Furthermore, the morphism

$$\nabla_{\tilde{U}'/\tilde{X}'} U_{rel} : T_{\tilde{U}'/\tilde{X}'} \to i(F^2\mathcal{H}_Y^2)$$

is surjective.

In particular,

$$\begin{aligned}\nabla_{\tilde{U}'/\tilde{X}'} \Phi_{BN} &= < \nabla_{\tilde{U}'/\tilde{X}'} U_{rel}, \tilde{\alpha} > \\ &= < i(\mu), \tilde{\alpha} > \\ &= < \mu, \partial\tilde{\alpha} > \\ &= < \mu, \alpha > .\end{aligned}$$

where μ is a local section of $F^2\mathcal{H}_Y^2$, and the last intersection pairing is taken over $\mathcal{H}_Y^2$. Note that by the surjectivity lemma, $(\nabla_{\tilde{U}'/\tilde{X}'} \Phi_{BN})|_{u'} = 0$ if and only if $\alpha_{u'}$ is of type $(1, 1)$.

The Hodge locus U'_h is defined by

$$U'_h := \{u' \in U' | \alpha_{u'} \in F^1H^2(Y_{u'})\},$$

and we use $\tilde{U}'_h$ to denote the inverse image of U'_h in $\tilde{U}'$. By the argument above, we have the following analogue of the result in [5]:

Theorem 4.1.2 *The Hodge locus $\tilde{U}'_h$ is defined by the vanishing of the relative differential $d_{\tilde{U}'/\tilde{X}'} \Phi_{BN}$.*

Now we want to show that the differential of the potential function gives information on the Abel-Jacobi mapping. Fix a local section ω of $F^3\mathcal{H}_X^3$. By applying the Gauss-Manin connection, we have the following isomorphism([5], [8]):

$$\nabla\omega : T_{\tilde{X}'} \cong F^2\mathcal{H}_X^3$$

and dually

$$\Omega_{\tilde{X}'} \cong (F^2\mathcal{H}^3_X)^*.$$

Then for any vanishing algebraic class α on Y_0, we have the potential function Φ_{BN}. The differential $d\Phi_{BN} \in \Omega_{\tilde{X}'}$ can be identified with an element in $F^2H^3(X_0)^*$ via the isomorphism above, and it projection to

$$J^3(X_0)_{prim} \cong \frac{F^2H^3(X_0)^*}{H_3(X_0,\mathbb{Z})}$$

is well defined, independent of the choice of the liftings and ω. The following theorem shows that Griffiths' Abel-Jacobi image of an algebraic cycle agrees with the above construction [5].

Theorem 4.1.3 *For a vanishing class α of type $(1,1)$ on Y_0, the differential*

$$d\Phi_{BN}|_{\{X_0,Y_0,\omega\}} \in \Omega_{\tilde{X}',0} \cong F^2H^3(X_0)^* \to J^3(X_0)_{prim}$$

maps to the Abel-Jacobi image of α.

4.2 Relative Abel-Jacobi mapping

For a smooth projective variety X of dimension $2n-1$, which is ample in a fixed smooth variety W of dimension $2n$, we will consider the vanishing cohomology $H^{2n-1}(X)_{van}$, which is the kernel of the pushforward $H^{2n-1}(X) \to H^{2n+1}(W)$. For an ample smooth hypersurface Y in X, define the subspace $H^{2n-1}(X,Y)_W$ of $H^{2n-1}(X,Y,\mathbb{C})$ to be the preimage of $H^{2n-1}(X,\mathbb{C})_{van}$ under the coboundary morphism of cohomology groups. Similarly we can define a space $H_{2n-1}(X,Y)_W$ in the relative homology, and there is a natural pairing between these two subspaces, as the restriction of the pairing between the relative homology and cohomology. We have short exact sequences of mixed Hodge structures

$$\begin{tikzcd}
0 \arrow[r] & H^{2n-2}(Y)_{van} \arrow[r, "i"] \arrow[d, equal] & H^{2n-1}(X,Y)_W \arrow[r, "f"] \arrow[d, hook] & H^{2n-1}(X)_{van} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & H^{2n-2}(Y)_{van} \arrow[r, "i"] & H^{2n-1}(X,Y) \arrow[r, "f"] & H^{2n-1}(X)_{prim} \arrow[r] & 0.
\end{tikzcd}$$

In this section, we will make use of the relative Jacobian $J(X,Y)_W$, defined by

$$J(X,Y)_W := \frac{F^n H^{2n-1}(X,Y)_W^*}{H_{2n-1}(X,\mathbb{Z})_{van}},$$

where we use the natural inclusion

$$H_{2n-1}(X,\mathbb{Z})_{van} \to H^{2n-1}(X,\mathbb{C})_{van}^* \to H^{2n-1}(X,Y,\mathbb{C})_W^* \to F^n H^{2n-1}(X,Y)_W^*.$$

$J(X,Y)_W$ is naturally an open complex manifold. with the complex structure induced from that of $F^n H^{2n-1}(X,Y)_W^*$.

Now if we dualize the first short exact sequence, we get

$$0 \to H_{2n-1}(X,\mathbb{Z})_{van} \to H^{2n-1}(X,Y,\mathbb{Z})_W^* \to H_{2n-2}(Y,\mathbb{Z})_{van} \to 0.$$

For an element $\alpha \in H_{2n-2}(Y,\mathbb{Z})_{van}$, we can choose a lifting $\tilde{\alpha} \in H^{2n-1}(X,Y,\mathbb{Z})_W^*$, which can be viewed as an element in $F^n H^{2n-1}(X,Y)_W^*$ via the natural morphism

$$H^{2n-1}(X,Y,\mathbb{Z})_W^* \to F^n H^{2n-1}(X,Y)_W^*.$$

Now for two different liftings, they differ by an element in $H_{2n-1}(X,\mathbb{Z})_{van}$, so we see that $\tilde{\alpha}$ is in fact a well defined point in the relative Jacobian $J(X,Y)_W$.

Definition 4.2.1 *The above morphism*

$$H_{2n-2}(Y,\mathbb{Z})_{van} \to J(X,Y)_W$$

is defined to be the relative Abel-Jacobi mapping.

In the rest of this chapter, we will give an alternate construction of this mapping, when X satisfies certain surjectivity condition on the Gauss-Manin connections. We will also show that this condition holds when X is a section of a sufficiently high power of an ample line bundle on W.

4.3 Surjectivity of the Gauss-Manin connection

When X is a smooth projective variety of dimension $n-1$, recall that we have the (graded) Gauss-Manin connection

$$\bar{\nabla} : H^{n-p,p-1}(X)_{van} \times H^1(X, T_X) \to H^{n-p-1,p}(X)_{van}.$$

In this section we want to study a surjectivity condition on the Gauss-Manin connection, and show that this holds for sufficiently ample hypersurfaces in any given projective smooth variety W of dimension n. Here being sufficiently ample means that for any ample line bundle, sections of a sufficiently high power of the line bundle will satisfy the given property. Recall the following lemma in [10]:

Lemma 4.3.1 *For a sufficiently ample hypersurface X in a smooth projective variety W of dimension n, we have the surjective morphisms*

$$\bar{\alpha}_p : H^0(W, K_W(pX)) \to H^{n-p,p-1}(X)_{van}.$$

The morphisms $\bar{\alpha}_p$ are constructed as an analogue of the residue maps [10, 23]. For hypersurfaces X in W, the infinitesimal deformation space contains the sections $H^0(W, \mathcal{O}_W(X))$ of the normal bundle as a subspace, and we have the following important description of the Gauss-Manin connection [3]:

Theorem 4.3.2 *The Gauss-Manin connection*

$$\bar{\nabla} : H^{n-p,p-1}(X)_{van} \times H^0(W, \mathcal{O}_W(X)) \to H^{n-p-1,p}(X)_{van}$$

can be described as follows. For $P \in H^0(W, K_W(pX))$ and $H \in H^0(W, \mathcal{O}_W(X))$, we have

$$\bar{\nabla}(\bar{\alpha}_p(P))(H) = -p\bar{\alpha}_{p+1}(PH).$$

By this theorem, in order to show that a sufficiently ample hypersurface X satisfies the surjectivity condition of the Gauss-Manin connection, we only need to prove that the multiplication morphism

$$H^0(W, K_W(pX)) \times H^0(W, \mathcal{O}_W(X)) \to H^0(W, K_W((p+1)X))$$

is surjective.

Consider the product variety $W \times W$, and the projection morphisms $p_1, p_2 : W \times W \to W$. For the diagonal $\triangle$, we have the short exact sequence:

$$0 \to \mathcal{I}_\triangle \to \mathcal{O}_{W\times W} \to \mathcal{O}_\triangle \to 0.$$

Tensor with the sheaf $p_1^* K_W(pX) \otimes p_2^* \mathcal{O}_W(X)$ and take the long exact sequence of cohomology, we see

$$H^0(W, K_W(pX)) \times H^0(W, \mathcal{O}_W(X)) \to H^0(W, K_W((p+1)X)) \to H^1(W \times W, \mathcal{I}_\triangle \otimes p_1^* K_W(pX) \otimes p_2^* \mathcal{O}_W(X)).$$

When X is sufficiently ample, by Serre asymptotic vanishing theorem, $H^1(W \times W, \mathcal{I}_\triangle \otimes p_1^* K_W(pX) \otimes p_2^* \mathcal{O}_W(X)) = 0$, so we obtain:

Proposition 4.3.3 *For a sufficiently ample hypersurface X in a smooth projective variety*

W of dimension n, Gauss-Manin connections

$$\bar{\nabla} : H^{n-p,p-1}(X)_{van} \times H^1(X, T_X) \to H^{n-p-1,p}(X)_{van}$$

are surjective.

4.4 General construction

Let X_0 be a sufficiently ample hypersurface in a smooth projection variety W of dimension $2n$. Note that by the results in Section 4.3, we proved that the Gauss-Manin connections

$$\bar{\nabla} : H^{2n-p,p-1}(X_0)_{van} \times H^1(X_0, T_{X_0}) \to H^{2n-p-1,p}(X_0)_{van}$$

are surjective for $p \geq 1$. Also fix a smooth ample hypersurface Y_0 in X_0. Similarly, we use X' to denote a local deformation space of X_0, and U' for a local deformation space of the pair (X_0, Y_0). So there is a morphism $U' \to X'$. Now consider the Hodge bundles $\mathcal{H}^{2n-1}_{X,van}$ and $\mathcal{H}^{2n-1}_{X,Y,W}$ over U', associated to the local systems with fibers $H^{2n-1}(X_{x'}, \mathbb{Z})_{van}$ and $H^{2n-1}(X_{x'}, Y_{u'}, \mathbb{Z})_W$. We get the natural bundle morphism

$$f : \mathcal{H}^{2n-1}_{X,Y,W} \to \mathcal{H}^{2n-1}_{X,van}.$$

Now use $\tilde{X}'$ to denote the total space of the subbundle $F^{2n-1}H^{2n-1}_{X,van}$, i.e. the total space of the pair $(X_{x'}, \omega)$, where $x' \in X'$ and $\omega \in H^{2n-1,0}(X_{x'})_{van}$. Similarly we have $\tilde{U}'$ for the subbundle $F^{2n-1}H^{2n-1}_{X,Y,W}$, consisting of triples $(X_{x'}, Y_{u'}, \omega)$, where $\omega \in F^{2n-1}H^{2n-1}(X_{x'})_{van}$.

Over U', we have the local systems $H^{2n-2}_{Y,\mathbb{Z},van}$ and $H_{2n-1,X,Y,\mathbb{Z},W}$ with fibers $H^{2n-2}(Y_{u'}, \mathbb{Z})_{van}$ and $H_{2n-1}(X_{x'}, Y_{u'}, \mathbb{Z})_W$ respectively. Also we have the Hodge bundle $\mathcal{H}^{2n-2}_Y$, and morphism

$$i : \mathcal{H}^{2n-2}_Y \to \mathcal{H}^{2n-1}_{X,Y,W}.$$

Fix a vanishing class $\alpha \in H^{2n-2}(Y_0, \mathbb{Z})_{van}$, it induces a local flat section $\alpha_{u'}$ of $H^{2n-2}_{Y,\mathbb{Z},van}$, which can be lifted to a local section $\tilde{\alpha}_{u'}$ of $H_{2n-1,X,Y,\mathbb{Z},W}$. More concretely, if $\alpha_{u'}$ are represented by $(2n-2)$-cycles $\Sigma_{u'}$, then $\tilde{\alpha}_{u'}$ are represented by differentiable $(2n-1)$-chains $\Gamma_{u'}$ such that

$$\partial\Gamma_{u'} = \Sigma_{u'}.$$

Now we also use $\mathcal{H}^{2n-1}_{X,Y,W}$ to denote the pull back of the bundle $\mathcal{H}^{2n-1}_{X,Y,W}$ to $\tilde{U}'$. Since any holomorphic $(2n-1,0)$-form restricts to zero on a hypersurface, $\tilde{U}'$ canonically lifts to a section U_{rel} of the pullback bundle $\mathcal{H}^{2n-1}_{X,Y,W}$. Using the paring between $H_{2n-1}(X_{x'}, Y_{u'})_W$ and $H^{2n-1}(X_{x'}, Y_{u'})_W$, we can define the potential function over $\tilde{U}'$

$$\Phi_{BN} =< U_{rel}, \tilde{\alpha} > .$$

More concretely, we have

$$\Phi_{BN}(\tilde{u}') = \int_{\Gamma_{u'}} \omega_{\tilde{u}'}.$$

This function is well-defined up to the choice of the lifting $\tilde{\alpha}_{u'}$. For two different choices, the difference is an element in $H_{2n-1}(X_{x'}, \mathbb{Z})_{van}$, so the function is well defined up to periods of $\omega_{\tilde{u}'}$. Since the section U_{rel} is holomorphic and the lifting $\tilde{\alpha}_{u'}$ is flat, the function Φ_{BN} is holomorphic.

Now Use u_0 to denote the point $(X_0, Y_0, 0) \in \tilde{U}'$. Consider the (k+1)-st order differential operators $D_{k+1}(\tilde{U}')|_{u_0}$ over $\tilde{U}'$ supported at u_0, we have the natural symbol morphism composed with the push forward

$$S_k : D_{k+1}(\tilde{U}')|_{u_0} \to Sym^{k+1} T\tilde{U}'|_{u_0} \to Sym^{k+1} TU'|_{(X_0,Y_0)}.$$

Use V_k to denote the (infinitely dimensional) vector space $\ker(S_k)$.

By Griffiths transversality, we have the following morphism:

$$\phi^k : V_k \to F^{2n-k-1}H^{2n-1}(X_0, Y_0)_W,$$
$$v \to \nabla_v(U_{rel}).$$

Lemma 4.4.1 *The morphisms ϕ^k are surjective for $0 \le k \le n-1$.*

Proof: In the case $k = 0$, this is clear since we can only differentiate in the $F^{2n-1}H^{2n-1}(X_0, Y_0)_W$ direction.

We have the short exact sequence

$$0 \to H^{2n-2}(Y_0)_{van} \xrightarrow{i} H^{2n-1}(X_0, Y_0)_W \xrightarrow{f} H^{2n-1}(X_0)_{van} \to 0.$$

By Lemma 4.3.3, when we take infinitesimal deformations of Y_0, $i(F^{2n-2}H^{2n-2}(Y_0)_{van})$ is contained in the image of ϕ^1. Then by Section 4.3, when we take higher order derivatives by deforming Y_0, $i(F^{2n-k-1}H^{2n-2}(Y_0)_{van})$ is contained in the image of ϕ^k.

On the other hand,

$$f(\nabla_v(U_{rel}) = \nabla_v(f(U_{rel})),$$

so the image of $f \circ \phi^k$ contains the image of the Gauss-Manin connection ϕ^k on $F^{2n-1}\mathcal{H}^{2n-1}(X)_{van}$. We claim that the image of $f \circ \phi^k$ contains $F^{2n-k-1}H^{2n-1}(X_0)_{van}$. This is clearly true when $k = 0$. Assume that this holds for $k-1$, by our assumption and Section 4.3, the Gauss-Manin connection on the graded piece

$$\bar{\nabla} : H^{2n-k,k-1}(X_0)_{van} \times H^1(X_0, T_{X_0}) \to H^{2n-k-1,k}(X_0)_{van}$$

is surjective. So the image of $f \circ \phi^k$ contains $F^{2n-k-1}H^{2n-1}(X_0)_{van}$, hence ϕ^k is surjective. □

In particular, we see the morphism

$$\phi^{n-1} : V_{n-1} \to F^n H^{2n-1}(X_0, Y_0)_W$$

is a surjection.

Now for a local section α of $H^{2n-2}_{Y,\mathbb{Z},van}$, we can define the potential function Φ_{BN}. The differential of Φ_{BN} of order n induces an element in V^*_{n-1}. Note that for any vector $v \in V_{n-1}$ satisfying

$$\phi^{n-1}(v) = 0,$$

we have

$$d_v(\Phi_{BN}) =< \nabla_v U_{rel}, \tilde{\alpha} >=< \phi(v), \tilde{\alpha} >= 0,$$

so $d\Phi_{BN}$ is the pullback of an element $J(\alpha_0)$ in $F^n H^{2n-1}(X_0, Y_0)^*_W$. By Lemma 4.4.1, this element is unique. Moreover, different choices of the lifting $\tilde{\alpha}$ are differed by elements in $H_{2n-1}(X_0, \mathbb{Z})_W$, so

$$J(\alpha_0) \in J(X, Y)_W = \frac{F^n H^{2n-1}(X, Y)^*_W}{H_{2n-1}(X, \mathbb{Z})_{van}}$$

is a canonically defined element. Note that this is defined by the pairing of $\tilde{\alpha}$ with elements of $F^n H^{2n-1}(X, Y)_W$, so this coincides with the relative Abel-Jacobi mapping defined in Section 4.2.

By the holomorphicity of the Gauss-Manin connection and the potential function Φ_{BN}, we see for the local section α, $J(\alpha)$ is a holomorphic section of the holomorphic family $J(X, Y)_{var}$.

4.5 Extension of the potential function

Let X be a smooth projective variety of dimension $2n - 1$, and $\mathscr{L}$ be a sufficiently ample line bundle over X. We will use $\mathbb{P}$ to denote the linear series $\mathbb{P} := \mathbb{P}(H^0(X, \mathscr{L}))$, and use

$\mathbb{P}_{sm}$ to denote the smooth locus, i.e. the locus of smooth sections of $\mathscr{L}$. Let $\mathbb{P}_{sing} := \mathbb{P} - \mathbb{P}_{sm}$ be the singular locus. As in last section, over $\mathbb{P}_{sm}$ we have the local system $H^{2n-2}_{Y,\mathbb{Z},van}$ of the vanishing cohomology of hypersurfaces, and $\mathcal{U}$ for the associated topological space. For the convenience of our discussion, we will use the following form of the potential function: fix a class $\omega \in F^{2n-1}H^{2n-1}(X)_{prim}$, and let $\tilde{\omega} \in F^{2n-1}H^{2n-1}(X, Y_t)$ be the canonical lifting for $t \in \mathbb{P}_{sm}$. For any $\alpha_t \in H^{2n-2}(Y_t, \mathbb{Z})_{van}$, pick a lifting $\tilde{\alpha}_t \in H^{2n-1}(X, Y_t, \mathbb{Z})$, and we define the potential function over $\mathcal{U}$ by

$$\Phi_{BN}(\alpha_t) =< \tilde{\omega}, \tilde{\alpha}_t > .$$

More concretely, we have

$$\Phi_{BN}(\alpha_t) = \int_{\Gamma_t} \omega,$$

where Γ_t is a chain representing $\tilde{\alpha}_t$, and we denote the harmonic representative of ω by the same letter. Note that this is well defined up to the periods of ω. This is the restriction of the potential function defined in last section to a fixed ω.

In the introduction we recalled Schnell's construction of the extension $\bar{\mathcal{U}}$ of $\mathcal{U}$. Here we want to show that the potential function can be extended to a holomorphic function over an open subset of $\bar{\mathcal{U}}$. It is a standard fact (for example, see [23, Section II.2.1]) that the locus $\mathbb{P}^0_{sing}$ consisting of hypersurfaces with at most one ordinary double point is a smooth dense subvariety in $\mathbb{P}_{sing}$. Since the morphism $\pi : \bar{\mathcal{U}}(K) \to \mathbb{P}$ is holomorphic, we see that the subvariety $\pi^{-1}(\mathbb{P}^0_{sing})$ in $\bar{\mathcal{U}}(K)$ is an open analytic subset. We want to show that the potential function extends to $\pi^{-1}(\mathbb{P}^0_{sing})$.

The question is local, and we introduce the following model: let $\triangle \to \mathbb{P}$ be a holomorphically embedded disc, such that the image intersects with $\mathbb{P}^0_{sing}$ transversely at the origin. Now consider the universal family $\mathscr{Y} \to \triangle$, and the local monodromy operator:

$$\rho : \pi_1(\triangle^*, t) \cong \mathbb{Z} \to Aut(H^{2n-2}(Y_t, \mathbb{Z})),$$

for $t \neq 0$.

Since Y_0 has only one ordinary double point, by the Picard-Lefschetz formula [23, Theorem II.3.16], the vanishing cohomology decomposes into the direct sum of the monodromy invariant subgroup and the subgroup spanned by the local vanishing cycle, over which the monodromy operator has order two. The decomposition is equivariant under the parallel transformation.

Now we have the restriction of the local system $H^{2n-2}_{Y,\mathbb{Z},van}$ to $\triangle^*$. There are two cases:

1) If we start with an invariant class $\alpha_t \in H^{2n-2}(Y_t, \mathbb{Z})$, this gives us a section of $H^{2n-2}_{Y,\mathbb{Z},van}$ over $\triangle^*$ by the parallel transformation. By Schnell's construction, the extended complex space is isomorphic to $\triangle$ in this case, and we just need to show that the potential function remains bounded when t approached the origin.

To see this, recall that we have the local invariant cycle theorem [23, Theorem II.4.18], which says that there exists a class $\alpha \in H^{2n-2}(\mathscr{Y})$, such that $\alpha_t = \alpha|_{Y_t}$. Now we can choose a cycle Σ in $\mathscr{Y}$ representing α, and let $\Sigma_t = \Sigma \cap Y_t$ for any $t \in \triangle$. This is a continuous family, so back in X, we can choose a continuous family of Γ_t whose boundary is Σ_t. Recall over $\triangle^*$, the potential function is given by

$$\Phi_{BN}(\alpha_t) = \int_{\Gamma_t} \omega.$$

Now Γ_t extends continuously to the origin, so the potential function extends continuously to the origin, hence holomorphically.

2) If $\alpha_t \in H^{2n-2}(Y_t, \mathbb{Z})$ is a local vanishing cycle, it gives a two sheeted covering space of $\triangle^*$, and Schnell's extension is the $2:1$ cover of $\triangle$, ramified at the origin.

By Morse theory and the local structure of vanishing cycles [23, Section II.2.3], there exists a cone C in $\mathscr{Y}$, such that each α_t is represented by $C \cap Y_t \cong S^{2n-2}$, and Y_0 intersects C at the vertex. By the same argument as above, we can choose a continuous family of Γ_t, and show that the potential function extends holomorphically to the origin. Note that in this case, Γ_0 is a cycle, so the value of the potential function at the origin is zero up to periods.

So we prove that the potential function extends to $\pi^{-1}(\mathbb{P}^0_{sing})$.

To extend this even further, let $\bar{\mathcal{U}}_{lf}$ be the locally finite locus of the projection π. By construction, $\bar{\mathcal{U}}_{lf}$ is an open dense normal analytic subspace of $\bar{\mathcal{U}}$. Clearly it contains $\pi^{-1}(\mathbb{P}^0_{sing})$ as a subspace. Since $\mathbb{P}_{sing} - \mathbb{P}^0_{sing}$ is of codimension at least 2 in $\mathbb{P}$, by the local finiteness, we see that $(\pi^{-1}(\mathbb{P}_{sing} - \mathbb{P}^0_{sing})) \cup \bar{\mathcal{U}}_{lf}$ is of codimension at least 2 in $\bar{\mathcal{U}}_{lf}$. By the Hartogs' theorem ([7, Corollary 2.7.8]), the potential function extends to a holomorphic function on $\bar{\mathcal{U}}_{lf}$.

CHAPTER 5

Further questions on the boundary behaviors

From last chapter we see that the holomorphic relative Abel-Jacobi mapping is defined over the complex manifold $\mathcal{U}$ associated to the local system of vanishing cohomology. By the theory of mixed Hodge modules [17, 18], there is a natural extension $\bar{J}$ over $\mathbb{P}$ of the relative Jacobian $J(X, Y)$. It is a natural question to ask whether this mapping extends to a holomorphic morphism from Schnell's extension $\bar{\mathcal{U}}$ to $\bar{J}$. Here we will list several conjectures related to the boundary behaviors of this mapping and a possible way to attack them.

Again use $\mathcal{U}(K)$ to denote the subspace of $\mathcal{U}$ over which the self-intersection is bounded by K. Recall from Section 4.5 that the morphism $\pi : \bar{\mathcal{U}}(K) \to \mathbb{P}$ is the natural projection, and the locus $\mathbb{P}^0_{sing}$ consisting of hypersurfaces with at most one ordinary double point is a smooth dense subvariety in $\mathbb{P}_{sing}$.

Conjecture 5.0.1 *The holomorphic relative Abel-Jacobi mapping extends holomorphically to* $\pi^{-1}(\mathbb{P}^0_{sing})$.

The reason for this should be the simple form of the monodromy operator around a hypersurface with only one ordinary double point. Moreover, we have the following conjecture:

Conjecture 5.0.2 *Let* Δ *be a holomorphically embedded disc in* $\bar{\mathcal{U}}$*, such that* Δ^* *is contained in* $\mathcal{U}$*. Consider the cohomology classes parametrized by* Δ^**, which can be locally* *lifted to the relative cohomology groups. Suppose that the liftings are invariant under the monodromy action of $\pi_1(\Delta^*)$, i.e. we can choose a global lifting over Δ^*. Then the holomorphic relative Abel-Jacobi mapping, restricted to Δ^*, extends holomorphically to the origin.*

When the hypersurface acquires worse singularities, the monodromy is of a more complicated form. In this case, we expect the holomorphic relative Abel-Jacobi mapping extends only as a meromorphic mapping.

Question 5.0.1 *What is the meromorphic locus in $\bar{\mathcal{U}}$ of the extension of the holomorphic relative Abel-Jacobi mapping?*

One possible way to attack these conjectures is to generalize Schnell's construction in [19]. Now consider the image of $\mathcal{U}(K)$ in $J(X, Y)$.

Conjecture 5.0.3 *The closure of the image of $\mathcal{U}(K)$ in $\bar{J}$ is an analytic subspace.*

The point is to prove a similar lemma on the boundary behavior of variation of mixed Hodge structures as in [19]. If this holds, we can extend the holomorphic relative Abel-Jacobi mapping to a meromorphic mapping over $\bar{\mathcal{U}}$.

Given this, we can now mimic Schnell's construction: take the normalization $\bar{\Xi}$ of the closure of the image, and construct the extension of $\mathcal{U}(K)$ as the finite (ramified) cover of $\bar{\Xi}$. It is not clear whether this extension coincides with Schnell's $\bar{\mathcal{U}}$. In Section 1.6, we introduced the normal analytic space $\bar{\Psi}$. There is a natural morphism from $\bar{\Xi}$ to $\bar{\Psi}$, induced by the cohomological mapping:

$$\begin{aligned} H_{2n-2}(Y, \mathbb{Z}) &\to F^n H^{2n-1}(X, Y)^* \to F^n H^{2n-2}(Y)^*, \\ \alpha &\to \quad < \alpha, \cdot > \qquad \to \; < \alpha, \cdot > . \end{aligned}$$

The main question is:

Question 5.0.2 *When is the morphism from $\bar{\Xi}$ to $\bar{\Psi}$ finite?*

In fact, the non-finite locus is closely related to the locus of $\bar{\mathcal{U}}$ over which the mapping fails to extend holomorphically. The study of this locus should give us much information about the corresponding cohomology classes.

BIBLIOGRAPHY

[1] Niels Abel, *Œuvres completes*, 1881.

[2] Patrick Brosnan, Hao Fang, Zhaohu Nie, and Gregory Pearlstein, *Singularities of admissible normal functions*, Inventiones mathematicae (2009).

[3] James Carlson and Phillip Griffiths, *Infinitesimal variation of Hodge structure and the global Torelli problem*, Géométrie algébrique, Angers (Arnaud Beauville, ed.), Sijthoff-Noordhoff, 1980, pp. 51–76.

[4] Herbert Clemens, *Degeneration of Kähler manifolds*, Duke Math. J. **44** (1977), no. 2, 215–290.

[5] Herbert Clemens, *Moduli schemes associated to K-trivial threefolds as gradient schemes*, J. Algebraic Geom. **14** (2005), no. 4, 705–739. MR 2147351 (2006d:14045)

[6] Herbert Clemens and Phillip Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. (2) **95** (1972), 281–356. MR 0302652 (46 #1796)

[7] Jean-Pierre Demailly, *Complex analytic and differential geometry*, OpenContent Book.

[8] Ron Donagi and Eyal Markman, *Cubics, integrable systems, and Calabi-Yau threefolds*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), Israel Math. Conf. Proc., vol. 9, Bar-Ilan Univ., Ramat Gan, 1996, pp. 199–221. MR 1360503 (97f:14039)

[9] Mark Green and Phillip Griffiths, *Algebraic cycles and singularities of normal functions*, Algebraic Cycles and Motives (Grenoble, 2007), Cambridge University Press,.

[10] Phillip Griffiths, *On the periods of certain rational integrals i, ii*, Ann. Math. **90** (1969), 460–541.

[11] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, John Wiley and Sons, 1994.

[12] William Hodge, *The theory and applications of harmonic integrals*, Cambridge University press, 1952.

[13] Carl Jacobi, *Note sur les fonctions abéliennes*, J. Reine Angew. Math. **30** (1846), 183–184.

[14] Solomon Lefschetz, *L'analysis situs et la géométrie algébrique*, Collection de Monographies publiée sous la Direction de M. Émile Borel, 1924.

[15] David Mumford, *Curves and their Jacobians*, Springer-Verlag Berlin Heidelberg, 1977.

[16] Chris Peters and Joseph Steenbrink, *Mixed Hodge structures*, Springer-Verlag Berlin Heidelberg, 2008.

[17] Morihiko Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849–995.

[18] Morihiko Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), no. 2, 211–333.

[19] Christian Schnell, *The extended locus of Hodge classes*, arXiv:1401.7303.

[20] Christian Schnell, *Primitive cohomology and the tube mapping*, Math. Z. **268** (2011), no. 3-4, 1069–1089. MR 2818744 (2012h:14061)

[21] Christian Schnell, *Residues and filtered D-modules*, Math. Ann. **354** (2012), no. 2, 727–763.

[22] Carl Siegel, *Topics in complex function theory*, John Wiley and Sons, 1971.

[23] Claire Voisin, *Hodge theory and complex algebraic geometry*, Cambridge University press, 2002.

[24] Oscar Zariski, *Algebraic surfaces*, Springer-Verlag Berlin Heidelberg, 1935.