TOPICS IN OPTIMAL STOPPING AND FUNDAMENTAL THEOREM OF ASSET PRICING

by

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ABSTRACT

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In this thesis, we investigate several problems in optimal stopping and fundamental theorem of asset pricing (FTAP).

In Chapter II, we study the controller-stopper problems with jumps. By a backward induction, we decompose the original problem with jumps into controller-stopper problems without jumps. Then we apply the decomposition result to indifference pricing of American options under multiple default risk.

In Chapters III and IV, we consider zero-sum stopping games, where each player can adjust her own stopping strategies according to the other's behavior. We show that the values of the games and optimal stopping strategies can be characterized by corresponding Dynkin games. We work in discrete time in Chapter III and continuous time in Chapter IV.

In Chapter V, we analyze an optimal stopping problem, in which the investor can peek ε amount of time into the future before making her stopping decision. We characterize the solution of this problem by a path-dependent reflected backward stochastic differential equation. We also obtain the order of the value as $\varepsilon \searrow 0$.

In Chapters VI-VIII, we investigate arbitrage and hedging under non-dominated model uncertainty in discrete time, where stocks are traded dynamically and liquid European-style options are traded statically. In Chapter VI we obtain the FTAP and hedging dualities under some convex and closed portfolio constraints. In Chapter VII we study arbitrage and super-hedging in the case when the liquid options are quoted with bid-ask spreads. In Chapter VIII we investigate the dualities for sub and super-hedging prices of American options. Note that for these three chapters, since we work in the frameworks lacking dominating measures, many classical tools in probability theory cannot be applied.

In Chapter IX, we consider arbitrage, hedging, and utility maximization in a given model, where stocks are available for dynamic trading, and both European and American options are available for static trading. Using a separating hyperplane argument, we get the result of FTAP, which implies the dualities of hedging prices. Then the hedging dualities lead to the duality for the utility maximization.

CHAPTER I

Introduction

This thesis is concentrated on two topics: optimal stopping (including Chapters II-V, VIII, and IX), and fundamental theorem of asset pricing (FTAP) and hedging duality (including Chapters VI-IX).

Optimal stopping plays an important role in the field of financial mathematics, such as fundamental theorem of asset pricing (FTAP), hedging, utility maximization, and pricing derivatives when American-type options are involved. For the general theory of optimal stopping and its applications, we refer to [54, 71, 76] and the references therein. The most commonly used approach for solving classical optimal stoping problems is to find the Snell envelopes of the underlying processes (see e.g, [70, 76]). However, there are still lots of specific optimal stopping problems of interest, which either require very technical verifications when using this method, or cannot be directly solved by the Snell envelope idea. In the first topic of this thesis, we consider several such problems of optimal stopping. Apart from Snell envelope, we shall use various methods to address these problems.

The arbitrage and hedging have been studied extensively in the field of financial mathematics in the classical setup, i.e, when there is a single physical measure and only stocks are available for dynamic trading. We refer to [24, 34, 43] and the references therein. Recently, there has been a lot of work on this topic in a setup where liquid options are also available for static trading, and/or the market is subject to model uncertainty/independency (see e.g., [1, 17, 19, 28, 32, 35–37, 45, 65]). Compared to the classical framework, it is more practical to study the arbitrage and hedging in this new setup. One reason is that nowadays the volume of options in the financial market is so large that it is not reasonable to ignore the impact of the liquid options. Moreover, estimation of parameters (e.g., volatilities) often ends up with confidence intervals instead of points. These intervals will lead to a set of probability measures which represents the model uncertainty. For the second topic of this thesis, we investigate several problems on arbitrage and hedging where stocks are traded dynamically and options are traded statically (semi-static trading strategies). In particular, most of our work for this topic is done in the framework of model uncertainty. It is worth noting that since the set of probability measures which represents the model uncertainty may not have a reference measure in general, many classical tools in probability theory cannot be applied.

1.1 Outline of the thesis

In Chapter II, we consider controller-stopper problems where the controlled processes can have jumps. We assume that there are at most n jumps. Using a backward induction, we decompose the original problem with jumps into several controller-stopper problems without jumps. Then we study the indifference pricing of an American option under multiple default risk. The backward induction leads to a system of reflected backward stochastic differential equations (RBSDEs). We show that there exists a solution to the RBSDE system, and the solution characterize the indifference price of the American option. This chapter is based on [15]. Parts of the work have been presented at the Financial/Actuarial Mathematics Seminar, University of Michigan, December 10, 2012.

In Chapter III, we consider the zero-sum stopping games

$$\overline{B} := \inf_{\boldsymbol{\rho}\in\mathbb{T}^{ii}} \sup_{\tau\in\mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}(\tau),\tau)] \quad \text{and} \quad \underline{A} := \sup_{\boldsymbol{\tau}\in\mathbb{T}^{i}} \inf_{\rho\in\mathcal{T}} \mathbb{E}[U(\rho,\boldsymbol{\tau}(\rho))]$$

on a filtered probability space $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{t=0,\dots,T})$, where $T\in\mathbb{N}$ is the time horizon in discrete time, $U(s,t)$ is $\mathcal{F}_{s\vee t}$-measurable, $\mathcal{T}$ is the set of stopping times, and $\mathbb{T}^i$ and $\mathbb{T}^{ii}$ are sets of mappings from $\mathcal{T}$ to $\mathcal{T}$ satisfying certain non-anticipativity conditions. We convert the problems into a corresponding Dynkin game, and show that $\overline{B}=\underline{A}=V$, where V is the value of the Dynkin game. We also get the optimal $\boldsymbol{\rho}\in\mathbb{T}^{ii}$ and $\boldsymbol{\tau}\in\mathbb{T}^{i}$ for $\overline{B}$ and $\underline{A}$ respectively. This chapter is based on [16]. Parts of the work have been presented at the Financial/Actuarial Mathematics Seminar, University of Michigan, December 10, 2014; Trading and Portfolio Theory, University of Chicago, November 11-12, 2014.

In Chapter IV, we extend the results in Chapter III to the continuous-time case. That is, we consider the stopping games

$$\overline{G} := \inf_{\boldsymbol{\rho}} \sup_{\tau\in\mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}(\tau),\tau)] \quad \text{and} \quad \underline{G} := \sup_{\boldsymbol{\tau}} \inf_{\rho\in\mathcal{T}} \mathbb{E}[U(\rho,\boldsymbol{\tau}(\rho))]$$

on a filtered probability space $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{0\leq t\leq T})$, where $T\in(0,\infty)$ is the time horizon in continuous time, and $\boldsymbol{\rho},\boldsymbol{\tau}:\mathcal{T}\mapsto\mathcal{T}$ satisfy certain non-anticipativity conditions. We show that $\overline{G}=\underline{G}$ by converting these problems into a corresponding Dynkin game. Compared to the discrete-time case, there are noticeable differences in the continuous-time results regarding the types of the non-anticipativity and the existence of optimal $\boldsymbol{\rho}$ and $\boldsymbol{\tau}$. This chapter is based on [6]. Parts of the work have been presented at the Financial/Actuarial Mathematics Seminar, University of Michigan, December 10, 2014; Trading and Portfolio Theory, University of Chicago, November 11-12, 2014.

In Chapter V, we consider the optimal stopping problem

$$v^{(\varepsilon)} := \sup_{\tau \in \mathcal{T}} \mathbb{E} B_{(\tau-\varepsilon)^+}$$

posed by Shiryaev at the International Conference on Advanced Stochastic Optimization Problems organized by the Steklov Institute of Mathematics in September 2012. Here $T > 0$ is a fixed time horizon, $(B_t)_{0 \le t \le T}$ is the Brownian motion, $\varepsilon \in [0, T]$ is a constant, and $\mathcal{T}$ is the set of stopping times taking values in $[0, T]$. As a first observation, $v^{(\varepsilon)}$ is characterized by a path dependent RBSDE. Furthermore, for large enough ε we obtain an explicit expression for $v^{(\varepsilon)}$, and for small ε we have lower and upper bounds. Then we get the asymptotic order of $v^{(\varepsilon)}$ as $\varepsilon \searrow 0$, which is the main result of this chapter. As a byproduct, we also obtain Lévy's modulus of continuity result in the L^1 sense. This chapter is based on [5].

In Chapter VI, we consider the FTAP and hedging prices of options under non-dominated model uncertainty and portfolio constrains in discrete time. First we show that no arbitrage holds if and only if there exists some family of probability measures such that any admissible portfolio value process is a local super-martingale under these measures. Then we get the non-dominated optional decomposition with constraints. From this decomposition, we get the dualities of the sub- and super-hedging prices of European and American options. Finally, we add liquid options into the market, and get the FTAP and duality of super-hedging prices with semi-static trading strategies. This chapter is based on [7]. Parts of the work have been presented at the SIAM Conference on Financial Mathematics and Engineering, November 13-15, 2014; the Financial Mathematics Seminar, Princeton University, September 11, 2014; Labex Louis Bachelier SIAM SMAI Conference on Financial

Mathematics Advanced Modeling and Numerical Methods, Paris, June 17-20, 2014; the Financial/Actuarial Mathematics Seminar, University of Michigan, March 26, 2014.

In Chapter VII, we consider the FTAP and super-hedging using semi-static trading strategies under model uncertainty in discrete time. We assume that the stocks are liquid and trading in them does not incur transaction costs, but that the options are less liquid and their prices are quoted with bid-ask spreads. We work on the notion of robust no arbitrage in the quasi-surely sense, and show that robust no arbitrage holds if and only if there exists a certain class of martingale measures which correctly price the options for static trading. Moreover, the super-hedging price is given by the supremum of the expectation over all the measures in this class. This chapter is based on [14].

In Chapter VIII, we consider the hedging prices of American options using semi-static trading strategies under model uncertainty in discrete time. First, we obtain the duality of sub-hedging prices as well as the existence of an optimal sub-hedging strategy. We also discuss the exchangeability of the sup and inf in the dual representation. Next, we get the results of duality and the existence of an optimal strategy for super-hedging. We also compare several alternative definitions and argue why our choice is more reasonable. Finally, assuming that the path space is compact, we construct a discretization of the path space and demonstrate the convergence of the hedging prices at the optimal rate. This chapter is based on [11]. Parts of the work have been presented at the Financial/Actuarial Mathematics Seminar, University of Michigan, January 29, 2014.

In Chapter IX, we consider a financial model where stocks are available for dynamic trading, and both European and American options are available for static trading. We assume that the American options are infinitely divisible, and can only be bought but not sold. We first get the FTAP with semi-static trading strategies. Using the FTAP result, we further get the dualities for the hedging prices of European and American options. Based on the hedging dualities, we also get the duality for the utility maximization involving semi-static trading strategies. This chapter is based on [8].

CHAPTER II

On controller-stopper problems with jumps and their applications to indifference pricing of American options

2.1 Introduction

The problem of pricing American options and the very closely related stochastic control problem of a controller and stopper either cooperating or playing a zero-sum game has been analyzed extensively for continuous processes. In particular, [52] considers the super-hedging problem; [12, 55–57] consider the controller-stopper problems, and [64] resolves the indifference pricing problem using the results of [55]. We will consider the above problems in the presence of jumps in the state variables.

The stochastic control problems in the above setup can be solved by Hamilton-Jacobi-Bellman integro-differential equations in the Markovian setup, or by Reflected Backward Stochastic Differential Equations (RBSDEs) with jumps, generalizing the results of [47], which we will call the *global approach*. We prefer to use an alternative approach in which we convert the problem with jumps into a sequence of problems without jumps à la [9], which uses this result for linear pricing of American options, and [72] which uses this approach to solve indifference pricing problems for European-style optimal control problems with jumps under a conditional density hypothesis.

One may wonder what the local approach we propose brings over the global approach in financial applications. Indeed, in the second part of the chapter, where we give an application of the decomposition results of controller-stopper games to indifference pricing of American options, one may use the methods in [33, 64] to convert the original problem into a dual problem over martingale measures which could be represented as a solution of an RBSDE with jumps or integro-PDEs for a non-linear free boundary problem. Compared to this global approach, what we propose has several advantages:

(a) Our method tells us how to behave optimally between jumps. For instance, our stopping times are not hitting times. They are hitting times of certain levels between jumps. But these levels change as the jumps occur. This tells us how the investor reacts to defaults and changes her stopping strategies. However, the global method can provide little insight into the impact of jumps on the optimal strategies.

(b) Like in [50, 72], our decomposition approach allows us to formulate the optimal investment problems where the portfolio constraint set can be updated after each default time, depending on the past defaults, which is financially relevant. Nevertheless, in the global approach the admissible set of strategies has to be fixed in the beginning.

(c) The decomposition result is useful in the analysis of Backward stochastic differential equations (BSDEs) with jumps. For example, [58] uses the decomposition result of [72] to construct a solution to BSDEs with jumps. Similar decomposition results were used earlier by [30] in understanding the structure of control problems in a piece-wise deterministic setting. Also, see [10] for example for the application of the decomposition idea to the solution of a quickest change detection problem.

Following the setup in [50, 72] we also assume that there are at most n jumps. Assuming the number of jumps is finite is not restrictive for financial modeling purposes. We think of jumps representing default events. The jumps in our framework have both predictable and totally inaccessible parts. That is, we are in the hybrid default modeling framework considered by [41, 51, 72] and following these papers we make the assumption that the joint distribution of jump times and marks has a conditional density. For a more precise formulation see the standing assumption in Section 2.3.

In this jump-diffusion model, we give a decomposition of the controller-stopper problem into controller-stopper problems with respect to the Brownian filtration, which are determined by a backward induction. We apply this decomposition method to indifference pricing of American options under multiple jump risk, extending the results of [72]. The solution of this problem leads to a system of reflected backward stochastic differential equations (RBSDEs). We show that there exists a solution to this RBSDE system and the solution provides a characterization of the value function, which can be thought of as an extension of [46].

Our first result, see Theorem 2.2.1 and Proposition 2.2.3, is a decomposition result for stopping times of the global filtration (the filtration generated by the Brownian motion and jump times and marks). Next, in Section 2.3, we show that the expectation of an optional process with jumps can be computed by a backward induction, where each step is an expectation with respect to the Brownian filtration. In Section 2.4, we consider the controller-stopper problems with jumps and decompose the original problem into controller-stopper problems with respect to the Brownian filtration. Finally, we apply our decomposition result to obtain the indifference buying/selling price of American options with jump/default risk in Section 2.5 and characterize the optimal trading strategies and the optimal stopping times in Theorem 2.5.4 and Theorem 2.5.8, which resolves a saddle point problem, which is an important and difficult problem in the controller-stopper games.

Since we work with optional processes (because our optimization problem contains a state variable with unpredictable jumps), we cannot directly rely on the decomposition result of [49, Lemma 4.4 and Remark 4.5] or the corresponding result in [48] (which is for predictable processes and the filtrations involved are right-continuous) from the classical theory of enlargement of filtrations. (See also [73, Chapter 6] for an exposition of this theory in English.) It is well known in the theory of enlargement of filtrations that for a right-continuous enlargement, a decomposition for optional process is not true in general; the remark on page 318 of [3] gives a counter example. See also the introduction of the recent paper by [79]. This is because in the case of optional processes the monotone class argument used in the proof of [49, Lemma 4.4] does not work for the right-continuously enlarged filtration. The phenomenon described here is in fact a classical example demonstrating the well-known exchangeability problem between intersection and the supremum of σ-algebras. In our problem we work in an enlarged filtration which is not right-continuous. This allows to get optional decomposition results with respect to the enlarged filtration. On the other hand, since the enlargement is not right-continuous, no classical stochastic calculus tools can be used to solve the problem anymore. Therefore, our approach gives an important contribution to the stochastic optimization literature. Also, as opposed to [49] we consider a progressive enlargement with several jumps and jump marks. On the other hand, our decomposition of the controller-stopper problems into control-stopper problems in the smaller filtration can be viewed as a non-linear extension of the classical decomposition formulas due to Jeulin [49].

In the rest of this section we will introduce the probabilistic setup and notation that we will use in the rest of the chapter.

2.1.1 Probabilistic setup

As in [72], we start with $(\Omega, \mathbb{F}, \mathbb{P})$ corresponding to the jump-free probability space, where $\mathbb{F} = (\mathcal{F}_t)_{t=0}^{\infty}$ is the filtration generated by the Brownian motion, satisfying the usual conditions. We assume that there are at most n jumps. Define $\Delta_0 = \emptyset$ and

$$\Delta_k = \left\{(\theta_1, \ldots, \theta_k) : \ 0 \le \theta_1 \ldots \le \theta_k\right\}, \quad k = 1, \ldots, n,$$

which represents the space of first k jump times. For $k = 1, \ldots, n$, let e_k be the k-th jump mark taking values in some Borel subset E of $\mathbb{R}^{\hat{d}}$. For $k = 0, \ldots, n$, let $\mathcal{D}^k$ be the filtration generated by the first k jump times and marks, i.e.,

$$\mathcal{D}_t^k = \vee_{i=1}^k \big(\sigma(1_{\{\zeta_i \le s\}}, \ s \le t) \vee \sigma(\ell_i 1_{\{\zeta_i \le s\}}, \ s \le t)\big).$$

Let

$$\mathcal{G}^k = \mathcal{F} \vee \mathcal{D}^k, \quad k = 0, \ldots, n.$$

Denote by $\mathbb{G}^k = (\mathcal{G}_t^k)_{t=0}^{\infty}$ for $k = 0, \ldots, n$, and $\mathbb{G} = \mathbb{G}^n$. (One should note that these filtrations are not necessarily right continuous. When we look at the supremum of two σ algebras, the resulting σ algebra does not have to be right continuous. This is due to the famous exchangeability problem between the intersection and the supremum of two σ algebras.) Then $(\Omega, \mathbb{G}^k, \mathbb{P})$ is the probability space including at most the first k jumps, $k = 0, \ldots, n$. Let $(\Omega, \mathbb{G}, \mathbb{P}) = (\Omega, \mathbb{G}^n, \mathbb{P})$ which we refer to as the global probability space. Note that for $k = 0, \ldots, n$, we may characterize each element in Ω as $(\omega_1, \theta_1, \ldots, \theta_k, e_1, \ldots, e_k)$, when the random variable we consider is $\mathcal{G}_\infty^k$-measurable, where ω_1 is viewed as the Brownian motion argument and $\mathcal{G}_\infty^k = \cup_{t=0}^{\infty} \mathcal{G}_t^k$, see [22, page 76].

Next we will introduce some notation that will be used in the rest of the chapter.

2.1.2 Notation

- For any $(\theta_1, \dots, \theta_k) \in \Delta_k,\ (\ell_1, \dots, \ell_k) \in E^k$, we denote by

$$\boldsymbol{\theta}_k = (\theta_1, \dots, \theta_k), \quad \boldsymbol{\ell}_k = (\ell_1, \dots, \ell_k), \qquad k = 1, \dots, n.$$

  We also denote by $\boldsymbol{\zeta}_k = (\zeta_1, \dots, \zeta_k)$, and $\boldsymbol{\ell}_k = (\ell_1, \dots, \ell_k)$. From now on, for $k = 1, \dots, n$, we use $\theta_k, \boldsymbol{\theta}_k, e_k, \boldsymbol{e}_k$ to represent given fixed numbers or vectors, and $\zeta_k, \boldsymbol{\zeta}_k, \ell_k, \boldsymbol{\ell}_k$ to represent random jump times or marks.

- $\mathcal{P}_{\mathbb{F}}$ is the σ-algebra of $\mathbb{F}$-predictable measurable subsets on $\mathbb{R}_+ \times \Omega$, i.e., the σ-algebra generated by the left-continuous $\mathbb{F}$-adapted processes.

- $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$ is the set of indexed $\mathbb{F}$-predictable processes $Z^k(\cdot)$, i.e., the map $(t, \omega, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \to Z_t^k(\omega, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$ is $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable, for $k = 1, \dots, n$. We also denote $\mathcal{P}_{\mathbb{F}}$ as $\mathcal{P}_{\mathbb{F}}(\Delta_0, E^0)$.

- $\mathcal{O}_{\mathbb{F}}$(resp.$\mathcal{O}_{\mathbb{G}}$) is the σ-algebra of $\mathbb{F}$(resp.$\mathbb{G}$)-optional measurable subsets on $\mathbb{R}_+ \times \Omega$, i.e., the σ-algebra generated by the right-continuous $\mathbb{F}$(resp.$\mathbb{G}$)-adapted processes.

- $\mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$ is the set of indexed $\mathbb{F}$-adapted processes $Z^k(\cdot)$, i.e., the map $(t, \omega, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \to Z_t^k(\omega, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$ is $\mathcal{O}_{\mathbb{F}} \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable, for $k = 1, \dots, n$. We also denote $\mathcal{O}_{\mathbb{F}}$ as $\mathcal{O}_{\mathbb{F}}(\Delta_0, E^0)$.

- For any $\mathcal{G}_\infty^k$-measurable random variable X, we sometimes denote it as $X = X(\omega_1, \boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) = X(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k)$. Given $\boldsymbol{\zeta}_k = \boldsymbol{\theta}_k,\ \boldsymbol{\ell}_k = \boldsymbol{e}_k$, we denote X as $X = X(\omega_1, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = X(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$. Similar notations apply for any $\mathbb{G}^k$-adapted process $(Z_t)_{t \geq 0}$ and its stopped version Z_τ, where τ is a $\mathbb{G}^k$-stopping time.

- For $T \in [0, \infty]$, $\Delta_k(T) := \Delta_k \cap [0, T]^k$.

- $\mathcal{S}_c^\infty[t, T] := \left\{ Y :\ \mathbb{F}\text{-adapted continuous, } ||Y||_{\mathcal{S}_c^\infty[t,T]} := \operatorname*{ess\,sup}_{(s,\omega) \in [t,T] \times \Omega} |Y_s(\omega)| < \infty \right\}$.

- $\mathcal{S}^\infty_c(\Delta_k(T),E^k) := \Big\{Y^k \in \mathcal{O}_{\mathbb{F}}(\Delta_k, E^k) : \ Y^k$ is continuous, and $||Y^k||_{\mathcal{S}^\infty_c(\Delta_k(T),E^k)}$
$$:= \sup_{(\boldsymbol{\theta}_k,\boldsymbol{e}_k)\in\Delta_k(T)\times E^k} ||Y^k(\boldsymbol{\theta}_k,\boldsymbol{e}_k)||_{\mathcal{S}^\infty_c[\theta_k,T]} < \infty\Big\}, \quad k=0,\ldots,n.$$
- $\mathbf{L}^2_W[t,T] := \Big\{Z : \ \mathbb{F}\text{-predictable}, \ \mathbb{E}\Big[\int_t^T |Z_s|^2 ds\Big] < \infty\Big\}.$
- $\mathbf{L}^2_W(\Delta_k(T),E^k) := \Big\{Z^k \in \mathcal{P}_{\mathbb{F}}(\Delta_k, E^k) : \ \mathbb{E}\Big[\int_{\theta_k}^T |Z^k_t(\boldsymbol{\theta}_k,\boldsymbol{e}_k)|^2 dt\Big] < \infty, \ \forall(\boldsymbol{\theta}_k,\boldsymbol{e}_k) \in$
$$\Delta_k(T)\times E^k\Big\}, \quad k=0,\ldots,n.$$
- $\mathbf{A}[t,T] := \big\{K : \ \mathbb{F}\text{-adapted continuous increasing}, \ \ K_t = 0, \ \mathbb{E}K_T^2 < \infty\big\}.$
- $\mathbf{A}(\Delta_k(T),E^k) := \big\{K^k : \ \ \forall(\boldsymbol{\theta}_k,\boldsymbol{e}_k) \in \Delta_k(T)\times E^k, \quad K^k(\boldsymbol{\theta}_k,\boldsymbol{e}_k) \ \in \mathbf{A}[\theta_k,T]\big\},$ $k=0,\ldots,n.$
- We use $\mathbf{eq}(H,f)_{s\le t\le T}$ to represent the RBSDE
$$\begin{cases} Y_t = H_T - \displaystyle\int_t^T f(r,Y_r,Z_r)dr + \int_t^T Z_r dW_r + (K_T - K_t), & s\le t\le T,\\ Y_t \ge H_t, \qquad s\le t\le T, & \\ \displaystyle\int_s^T (Y_t - H_t)dK_t = 0, & \end{cases}$$
and $\mathbf{EQ}(\mathcal{H},\mathfrak{f})_{s\le t\le T}$ to represent the RBSDE
$$\begin{cases} \mathcal{Y}_t = \mathcal{H}_T + \displaystyle\int_t^T \mathfrak{f}(r,\mathcal{Y}_r,\mathcal{Z}_r)dr - \int_t^T \mathcal{Z}_r dW_r + (\mathcal{K}_T - \mathcal{K}_t), & s\le t\le T,\\ \mathcal{Y}_t \ge \mathcal{H}_t, \qquad s\le t\le T, & \\ \displaystyle\int_s^T (\mathcal{Y}_t - \mathcal{H}_t)d\mathcal{K}_t = 0. & \end{cases}$$

2.2 Decomposition of $\mathbb{G}$-stopping times

Theorem 2.2.1 and Proposition 2.2.3, on the decomposition $\mathbb{G}$-stopping times, are the main results of this section.

Theorem 2.2.1. *τ is a $\mathbb{G}$-stopping time if and only if it has the decomposition:*

$$\tau = \tau^0 1_{\{\tau^0<\zeta_1\}} + \sum_{k=1}^{n-1} \tau^k(\boldsymbol{\zeta}_k,\boldsymbol{\ell}_k) 1_{\{\tau^0\ge\zeta_1\}\cap\ldots\cap\{\tau^{k-1}\ge\zeta_k\}\cap\{\tau^k<\zeta_{k+1}\}} + \tau^n(\boldsymbol{\zeta}_n,\boldsymbol{\ell}_n)1_{\{\tau^0\ge\zeta_1\}\ldots\cap\{\tau^{n-1}\ge\zeta_n\}}, \tag{2.2.1}$$

for some $(\tau^0, \ldots, \tau^n)$*, where* τ^0 *is an* $\mathbb{F}$*-stopping time, and* $\tau^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k)$ *is a* $\mathbb{G}^k$*-stopping time satisfying*

$$\tau^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) \geq \zeta_k, \; k = 1, \ldots, n. \tag{2.2.2}$$

Proof. If τ has the decomposition (2.2.1), then

$$\begin{aligned} \{\tau \leq t\} \;\; = \;\; & \bigcup_{k=1}^{n-1} \Big(\{\tau^0 \geq \zeta_1\} \cap \ldots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\} \cap \{\tau^k \leq t\} \Big) \\ & \cup \Big(\{\tau^0 < \zeta_1\} \cap \{\tau^0 \leq t\} \Big) \cup \Big(\{\tau^0 \geq \zeta_1\} \cap \ldots \{\tau^{n-1} \geq \zeta_n\} \cap \{\tau^n \leq t\} \Big). \end{aligned}$$

For $k = 1, \ldots, n$, since $\{\tau^k < \zeta_{k+1}\} \in \mathcal{G}_{\tau^k}$, and

$$\{\tau^{i-1} \geq \zeta_i\} \in \mathcal{G}_{\zeta_i} \subset \mathcal{G}_{\zeta_k} \subset \mathcal{G}_{\tau^k}, \quad i = 1, \ldots, k,$$

we have

$$\{\tau^0 \geq \zeta_1\} \cap \ldots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\} \cap \{\tau^k \leq t\} \in \; \mathcal{G}_t.$$

Similarly we can show $\{\tau^0 < \zeta_1\} \cap \{\tau^0 \leq t\} \in \mathcal{G}_t$ and

$$\{\tau^0 \geq \zeta_1\} \cap \ldots \{\tau^{n-1} \geq \zeta_n\} \cap \{\tau^n \leq t\} \;\; \in \mathcal{G}_t.$$

If τ is a $\mathbb{G}$-stopping time, we will proceed in 3 steps to show that it has the decomposition (2.2.1).

Step 1: We will show that for any discretely valued $\mathbb{G}$-stopping time

$$\tau = \sum_{1 \leq i \leq \infty} a_i 1_{A_i},$$

where $0 \leq a_1 < a_2 < \ldots < a_\infty = \infty$ and $(A_i \in \mathcal{G}_{a_i})_{1 \leq i \leq \infty}$ is a partition of Ω, there exists a $\mathbb{G}^k$-stopping time $\tau^k = \tau^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k)$, such that

$$\tau 1_{\{\tau < \zeta_{k+1}\}} = \tau^k 1_{\{\tau < \zeta_{k+1}\}} \quad \text{and} \quad \{\tau < \zeta_{k+1}\} = \{\tau^k < \zeta_{k+1}\}, \tag{2.2.3}$$

for $k = 0, \dots, n-1$. First, we have

$$\{\tau < \zeta_{k+1}\} = \bigcup_{1 \le i \le \infty} \big(\{\tau < \zeta_{k+1}\} \cap \{A_i\}\big) = \bigcup_{1 \le i \le \infty} \big(\{a_i < \zeta_{k+1}\} \cap \{A_i\}\big).$$

To complete Step 1, we need the following lemma:

Lemma 2.2.2. *For $i = 1, \dots, \infty$, and $A_i \in \mathcal{G}_{a_i}$, there exists $\tilde{A}_i \in \mathcal{G}^k_{a_i}$, such that*

$$\{a_i < \zeta_{k+1}\} \cap \tilde{A}_i = \{a_i < \zeta_{k+1}\} \cap A_i. \tag{2.2.4}$$

Moreover, $(\tilde{A}_i)_{1 \le i \le \infty}$ can be chosen to be mutually disjoint.

Proof of Lemma 2.2.2. Since for $j \ge k+1$,

$$\begin{aligned}
&\big(\sigma(1_{\{\zeta_j \le s\}},\ s \le a_i) \vee \sigma(\ell_j 1_{\{\zeta_j \le s\}},\ s \le a_i)\big) \cap \{a_i < \zeta_{k+1}\} \\
&= \sigma\Big(\{\zeta_j \le s\},\ \big(\{\ell \in C\} \cap \{\zeta_j \le t\}\big) \cup \{\zeta_j > t\},\ \ s,t \le a_i,\ C \in \mathcal{B}(E)\Big) \cap \{a_i < \zeta_{k+1}\} \\
&= \sigma\Bigg(\{\zeta_j \le s\} \cap \{a_i < \zeta_{k+1}\},\ \Big(\big(\{\ell \in C\} \cap \{\zeta_j \le t\}\big) \cup \{\zeta_j > t\}\Big) \cap \{a_i < \zeta_{k+1}\}, \\
&\qquad s,t \le a_i,\ C \in \mathcal{B}(E)\Bigg) \\
&= \big\{\emptyset,\ \{a_i < \zeta_{k+1}\}\big\},
\end{aligned}$$

we have that

$$\begin{aligned}
&\mathcal{G}_{a_i} \cap \{a_i < \zeta_{k+1}\} \\
&= \Bigg(\mathcal{F}_{a_i} \vee \Big(\vee_{j=1}^n \big(\sigma(1_{\{\zeta_j \le s\}},\ s \le a_i) \vee \sigma(\ell_j 1_{\{\zeta_j \le s\}},\ s \le a_i)\big)\Big)\Bigg) \cap \{a_i < \zeta_{k+1}\} \\
&= \Bigg(\big(\mathcal{F}_{a_i} \cap \{a_i < \zeta_{k+1}\}\big) \vee \Big(\vee_{j=1}^n \big(\sigma(1_{\{\zeta_j \le s\}},\ s \le a_i) \vee \sigma(\ell_j 1_{\{\zeta_j \le s\}},\ s \le a_i)\big) \cap \{a_i < \zeta_{k+1}\}\Big)\Bigg) \\
&= \Bigg(\big(\mathcal{F}_{a_i} \cap \{a_i < \zeta_{k+1}\}\big) \vee \Big(\vee_{j=1}^k \big(\sigma(1_{\{\zeta_j \le s\}},\ s \le a_i) \vee \sigma(\ell_j 1_{\{\zeta_j \le s\}},\ s \le a_i)\big) \cap \{a_i < \zeta_{k+1}\}\Big)\Bigg) \\
&= \Bigg(\mathcal{F}_{a_i} \vee \Big(\vee_{j=1}^k \big(\sigma(1_{\{\zeta_j \le s\}},\ s \le a_i) \vee \sigma(\ell_j 1_{\{\zeta_j \le s\}},\ s \le a_i)\big)\Big)\Bigg) \cap \{a_i < \zeta_{k+1}\} \\
&= \mathcal{G}^k_{a_i} \cap \{a_i < \zeta_{k+1}\},
\end{aligned}$$

which proves the existence result in Lemma 2.2.2. Now suppose $(\bar{A}_i \in \mathcal{G}^k_{a_i})_{1\le i\le\infty}$ are the sets such that (2.2.4) holds. Define $\tilde{A}_1 = \bar{A}_1$, $\tilde{A}_\infty = \emptyset$, and

$$\tilde{A}_{m+1} = \bar{A}_{m+1} \setminus \bigcup_{j=1}^{m} \bar{A}_j, \quad m = 1, 2, \ldots$$

Since for $i \neq j$, $\big(\bar{A}_i \cap \{a_i < \zeta_{k+1}\}\big) \cap \big(\bar{A}_j \cap \{a_j < \zeta_{k+1}\}\big) = \emptyset$, we have for $m = 1, 2, \ldots,$

$$\begin{aligned}
\bar{A}_{m+1} \cap \{a_{m+1} < \zeta_{k+1}\} &\supset \tilde{A}_{m+1} \cap \{a_{m+1} < \zeta_{k+1}\} \\
&= \big(\bar{A}_{m+1} \cap \{a_{m+1} < \zeta_{k+1}\}\big) \setminus \bigcup_{j=1}^{m} \big(\bar{A}_j \cap \{a_{m+1} < \zeta_{k+1}\}\big) \\
&\supset \big(\bar{A}_{m+1} \cap \{a_{m+1} < \zeta_{k+1}\}\big) \setminus \bigcup_{j=1}^{m} \big(\bar{A}_j \cap \{a_j < \zeta_{k+1}\}\big) \\
&= \big(\bar{A}_{m+1} \cap \{a_{m+1} < \zeta_{k+1}\}\big).
\end{aligned}$$

Therefore, $\tilde{A}_{m+1} \cap \{a_{m+1} < \zeta_{k+1}\} = \bar{A}_{m+1} \cap \{a_{m+1} < \zeta_{k+1}\}$, and thus $(\tilde{A}_i \in \mathcal{G}^k_{a_i})_{1\le i\le\infty}$ are the disjoint sets such that (2.2.4) holds. This completes the proof of Lemma 2.2.2.

$\square$

Now let us continue with the proof of Theorem 2.2.1. From Lemma 2.2.2, we have

$$\{\tau < \zeta_{k+1}\} = \bigcup_{1\le i\le\infty} \big(\{a_i < \zeta_{k+1}\} \cap \tilde{A}_i\big),$$

where $(\tilde{A}_i \in \mathcal{G}^k_{a_i})_{1\le i\le\infty}$ are disjoint sets such that (2.2.4) holds. Define $\mathbb{G}^k$-stopping time

$$\tau^k = \sum_{1\le i\le\infty} a_i 1_{\tilde{A}_i}.$$

Since

$$\tilde{A}_i \cap \{\tau < \zeta_{k+1}\} = \tilde{A}_i \cap \bigcup_{1\le j\le\infty} \big(\{a_j < \zeta_{k+1}\} \cap \tilde{A}_j\big) = \{a_i < \zeta_{k+1}\} \cap \tilde{A}_i = \{\tau < \zeta_{k+1}\} \cap A_i,$$

we have

$$\tau^k 1_{\{\tau<\zeta_{k+1}\}} = \sum_{1\le i\le\infty} a_i 1_{\tilde{A}_i \cap \{\tau<\zeta_{k+1}\}} = \sum_{1\le i\le\infty} a_i 1_{A_i \cap \{\tau<\zeta_{k+1}\}} = \tau 1_{\{\tau<\zeta_{k+1}\}}.$$

Also,

$$\{\tau < \zeta_{k+1}\} = \bigcup_{1\le i\le\infty} \Big(\{a_i < \zeta_{k+1}\} \cap A_i\Big) = \bigcup_{1\le i\le\infty} \Big(\{a_i < \zeta_{k+1}\} \cap \tilde{A}_i\Big) = \{\tau^k < \zeta_{k+1}\}.$$

Step 2: We will show that for any $\mathbb{G}$-stopping time τ, there exists a $\mathbb{G}^k$-stopping time τ^k, such that (2.2.3) holds. Define the $\mathbb{G}$-stopping times

$$\tau_m := \sum_{j=0}^{\infty} \frac{j+1}{2^m} \cdot 1_{\{\frac{j}{2^m} \le \tau < \frac{j+1}{2^m}\}} + \infty \cdot 1_{\{\tau=\infty\}}, \quad m = 1, 2, \dots$$

By Step 1, there exists a $\mathbb{G}^k$-stopping time τ_m^k, such that

$$\tau_m^k 1_{\{\tau_m<\zeta_{k+1}\}} = \tau_m 1_{\{\tau_m<\zeta_{k+1}\}} \quad \text{and} \quad \{\tau_m < \zeta_{k+1}\} = \{\tau_m^k < \zeta_{k+1}\}. \tag{2.2.5}$$

Define $\tau^k := \limsup_{m\to\infty} \tau_m^k$. Since $\tau_m \searrow \tau$, by taking "lim sup" on both side of (2.2.5), we have (2.2.3).

Step 3: From Step 2, we know that for any $\mathbb{G}$-stopping time τ, there exists $\tau^0,\ \tau^1, \dots, \tau^{n-1}$ being $\mathbb{F}$, $\mathbb{G}^1, \dots,\ \mathbb{G}^{n-1}$-stopping times respectively, such that (2.2.3) holds, for $k = 0, \dots, n-1$. Let $\tau^n := \tau$, then we have

$$\begin{aligned}
\tau &= \tau 1_{\{\tau<\zeta_1\}} + \sum_{k=1}^{n-1} \tau 1_{\{\zeta_k\le\tau<\zeta_{k+1}\}} + \tau 1_{\{\zeta_n\le\tau\}} \\
&= \tau^0 1_{\{\tau<\zeta_1\}} + \sum_{k=1}^{n-1} \tau^k 1_{\{\zeta_k\le\tau<\zeta_{k+1}\}} + \tau^n 1_{\{\zeta_n\le\tau\}} \\
&= \tau^0 1_{\{\tau<\zeta_1\}} + \sum_{k=1}^{n-1} \tau^k 1_{\{\tau\ge\zeta_1\}\cap\ldots\cap\{\tau\ge\zeta_k\}\cap\{\tau<\zeta_{k+1}\}} + \tau^n 1_{\{\tau\ge\zeta_1\}\cap\ldots\cap\{\tau\ge\zeta_n\}} \\
&= \tau^0 1_{\{\tau^0<\zeta_1\}} + \sum_{k=1}^{n-1} \tau^k 1_{\{\tau^0\ge\zeta_1\}\cap\ldots\cap\{\tau^{k-1}\ge\zeta_k\}\cap\{\tau^k<\zeta_{k+1}\}} + \tau^n 1_{\{\tau^0\ge\zeta_1\}\cap\ldots\cap\{\tau^{n-1}\ge\zeta_n\}}.
\end{aligned}$$

We will modify the decomposition so that it satisfies (2.2.2). For $k = 1, \dots, n$, define $\mathbb{G}^k$-stopping time

$$\tilde{\tau}^k = \begin{cases} \tau^k, & \tau^k \ge \zeta_k, \\ \zeta_k, & \tau^k < \zeta_k. \end{cases}$$

and let $\tilde{\tau}^0 := \tau^0$. Then for $k = 1, \dots, n$, $\tilde{\tau}^k \geq \zeta_k$, and

$$\{\tilde{\tau}^k < \zeta_{k+1}\} = \{\tau^k < \zeta_{k+1}\} = \{\tau < \zeta_{k+1}\}, \quad k = 0, \dots, n-1.$$

For $k = 1, \dots, n-1$, since $\{\zeta_k \leq \tau < \zeta_{k+1}\} \subset \{\tau = \tau^k\}$, we have

$$\begin{aligned}
&\{\tau^0 \geq \zeta_1\} \cap \ldots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\} \\
&\quad = \{\zeta_k \leq \tau < \zeta_{k+1}\} = \{\zeta_k \leq \tau < \zeta_{k+1}\} \cap \{\tau = \tau^k\} \subset \{\tau^k \geq \zeta_k\}.
\end{aligned}$$

Also $\{\tau \geq \zeta_n\} \subset \{\tau = \tau_n\}$ implies

$$\{\tau^0 \geq \zeta_1\} \cap \ldots \cap \{\tau^{n-1} \geq \zeta_n\} = \{\tau \geq \zeta_n\} = \{\tau \geq \zeta_n\} \cap \{\tau = \tau^n\} \subset \{\tau^n \geq \zeta_n\}.$$

Therefore, we have

$$\begin{aligned}
\tau &= \tau^0 1_{\{\tau^0 < \zeta_1\}} + \sum_{k=1}^{n-1} \tau^k 1_{\{\tau^0 \geq \zeta_1\} \cap \ldots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\}} + \tau^n 1_{\{\tau^0 \geq \zeta_1\} \cap \ldots \cap \{\tau^{n-1} \geq \zeta_n\}} \\
&= \tilde{\tau}^0 1_{\{\tau^0 < \zeta_1\}} + \sum_{k=1}^{n-1} \tilde{\tau}^k 1_{\{\tau^0 \geq \zeta_1\} \cap \ldots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\}} + \tilde{\tau}^n 1_{\{\tau^0 \geq \zeta_1\} \cap \ldots \cap \{\tau^{n-1} \geq \zeta_n\}} \\
&= \tilde{\tau}^0 1_{\{\tilde{\tau}^0 < \zeta_1\}} + \sum_{k=1}^{n-1} \tilde{\tau}^k 1_{\{\tilde{\tau}^0 \geq \zeta_1\} \cap \ldots \cap \{\tilde{\tau}^{k-1} \geq \zeta_k\} \cap \{\tilde{\tau}^k < \zeta_{k+1}\}} + \tilde{\tau}^n 1_{\{\tilde{\tau}^0 \geq \zeta_1\} \cap \ldots \cap \{\tilde{\tau}^{n-1} \geq \zeta_n\}}.
\end{aligned}$$

This ends the proof of Theorem 2.2.1. $\square$

In the rest of the chapter, we will use the notation $\tau \sim (\tau^0, \dots, \tau^n)$ for the $\mathbb{G}$-stopping time τ if it has the decomposition from (2.2.1). The next result shows that the decomposition of τ in (2.2.1) is unique, in the sense that the terms in the sum of τ's representation are the same even for different $(\tau^0, \dots, \tau^n)$'s in the representation. (Note that one can modify the stopping times τ^i after the jump times ζ_{i+1}.)

Proposition 2.2.3. *Let $\tau \sim (\tau^0, \dots, \tau^n)$ be a $\mathbb{G}$-stopping time. Then $\{\tau^0 < \zeta_1\} = \{\tau < \zeta_1\}$, $\{\tau^0 \geq \zeta_1\} \cap \ldots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\} = \{\zeta_k \leq \tau < \zeta_{k+1}\}$ for $k = 1, \dots, n-1$, and $\{\tau^0 \geq \zeta_1\} \cap \ldots \cap \{\tau^{n-1} \geq \zeta_n\} = \{\zeta_n \leq \tau\}$. Therefore,*

$$\tau = \tau^0 1_{\{\tau < \zeta_1\}} + \sum_{k=1}^{n-1} \tau^k 1_{\{\zeta_k \leq \tau < \zeta_{k+1}\}} + \tau^n 1_{\{\zeta_n \leq \tau\}}.$$

Proof. Let $A_0 := \{\tau^0 < \zeta_1\}$, $A_n := \{\tau^0 \geq \zeta_1\} \cap \ldots \cap \{\tau^{n-1} \geq \zeta_n\}$, and

$$A_k := \{\tau^0 \geq \zeta_1\} \cap \ldots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\}, \quad k = 1, \ldots, n-1.$$

Let $B_0 := \{\tau < \zeta_1\}$, $B_n := \{\zeta_n \leq \tau\}$, and $B_k := \{\zeta_k \leq \tau < \zeta_{k+1}\},\ k = 1, \ldots, n-1$. In the set A_i, we have $\tau = \tau^i$, which implies $\zeta_i \leq \tau < \zeta_{i+1}$, and thus $A_i \subset B_i$, for $i = 1, \ldots, n-1$. Similarly, $A_0 \subset B_0$ and $A_n \subset B_n$. Since $(A_i)_{i=0}^n$ and $(B_i)_{i=0}^n$ are mutually disjoint respectively, and $\Omega = \bigcup_{i=0}^n A_i = \bigcup_{i=0}^n B_i$, we have $A_i = B_i,\ i = 0, \ldots, n$. □

The last proposition generalizes the decomposition result given in [30, Theorem (A2.3), page 261] (also see [20, Theorem T33, page 308]) from the stopping times of piecewise deterministic Markov processes to the stopping times of jump diffusions.

Proposition 2.2.4. *Let $T > 0$ be a constant. τ is an $\mathbb{G}$-stopping time satisfying $\tau \leq T$ if and only if τ has the decomposition* (2.2.1), *with $\tau^0 \leq T$ and $\{\zeta_k \leq T\} = \{\tau^k \leq T\},\ k = 1, \ldots, n$.*

Proof. If τ has the decomposition, then on the set $\{\tau^0 \geq \zeta_1\} \cap \ldots \cap \{\tau^{k-1} \geq \zeta_k\}$, we have

$$T \geq \tau^0 \geq \zeta_1 \;\Rightarrow\; T \geq \tau^1 \;\Rightarrow\; T \geq \zeta_2 \;\Rightarrow \ldots \Rightarrow\; T \geq \tau^{k-1} \;\Rightarrow\; T \geq \zeta_k \;\Rightarrow T \geq \tau^k,$$

For $k = 1, \ldots, n$. Thus $\tau \leq T$.

Conversely, let $\tau \sim (\tau^0, \ldots, \tau^n)$ be a $\mathbb{G}$-stopping time satisfying $\tau \leq T$. Let $\tilde{\tau}^0 := \tau^0$, and

$$\tilde{\tau}^k := \begin{cases} \tau^k \wedge T, & \zeta_k \leq T, \\ \tau^k, & \zeta_k > T. \end{cases}$$

for $k = 0, \ldots, n$. It can be shown that $\tilde{\tau}^k$ is a $\mathbb{G}^k$-stopping time. Then for $k = 1, \ldots, n-1$,

$$\zeta_k \leq \tau < \zeta_{k+1} \;\Rightarrow\; \tau^k = \tau \leq T \;\Rightarrow\; \tilde{\tau}^k = \tau^k.$$

Similarly, $\zeta_n \le \tau \;\Rightarrow\; \tilde{\tau}^n = \tau^n$. Therefore,

$$\tau = \tilde{\tau}^0 1_{\{\tau<\zeta_1\}} + \sum_{k=1}^{n-1} \tilde{\tau}^k 1_{\{\zeta_k \le \tau < \zeta_{k+1}\}} + \tilde{\tau}^n 1_{\{\zeta_n \le \tau\}}.$$

Easy to see $\tilde{\tau}^k \ge \zeta_k$ and $\{\zeta_k \le T\} = \{\tilde{\tau}^k \le T\},\ k = 1, \dots, n$. It remains to show $A_i = B_i, i = 0, \dots, n$, where $A_0 := \{\tau^0 < \zeta_1\}$, $A_n := \{\tau^0 \ge \zeta_1\} \cap \dots \cap \{\tau^{n-1} \ge \zeta_n\}$,

$$A_k := \{\tau^0 \ge \zeta_1\} \cap \dots \cap \{\tau^{k-1} \ge \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\}, \quad k = 0, \dots, n-1,$$

and $B_0 := \{\tilde{\tau}^0 < \zeta_1\}$, $B_n := \{\tilde{\tau}^0 \ge \zeta_1\} \cap \dots \cap \{\tilde{\tau}^{n-1} \ge \zeta_n\}$,

$$B_k := \{\tilde{\tau}^0 \ge \zeta_1\} \cap \dots \cap \{\tilde{\tau}^{k-1} \ge \zeta_k\} \cap \{\tilde{\tau}^k < \zeta_{k+1}\}, \quad k = 0, \dots, n-1.$$

Easy to see $A_0 = B_0$ and $A_n \supset B_n$. Now for $k = 1, \dots, n-1$,

$$\begin{aligned} &\{\tau^0 \ge \zeta_1\} \cap \dots \cap \{\tau^{k-1} \ge \zeta_k\} \cap \{\tilde{\tau}^k < \zeta_{k+1}\} \\ &\subset \{\tau^0 \ge \zeta_1\} \cap \dots \cap \{\tau^{k-1} \ge \zeta_k\} \cap \Big(\{\tau^k < \zeta_{k+1}\} \cup \{T < \zeta_{k+1}\}\Big). \end{aligned}$$

Since

$$\{\tau^0 \ge \zeta_1\} \cap \dots \cap \{\tau^{k-1} \ge \zeta_k\} \cap \{T < \zeta_{k+1}\} \cap \{\tau^k \ge \zeta_{k+1}\} = \emptyset,$$

we have

$$\{\tau^0 \ge \zeta_1\}\cap\dots\cap\{\tau^{k-1} \ge \zeta_k\}\cap\{T < \zeta_{k+1}\} \subset \{\tau^0 \ge \zeta_1\}\cap\dots\cap\{\tau^{k-1} \ge \zeta_k\}\cap\{\tau^k < \zeta_{k+1}\}.$$

Hence, for $k = 1, \dots, n-1$,

$$\begin{aligned} B_k &\subset \{\tau^0 \ge \zeta_1\} \cap \dots \cap \{\tau^{k-1} \ge \zeta_k\} \cap \{\tilde{\tau}^k < \zeta_{k+1}\} \\ &= \{\tau^0 \ge \zeta_1\} \cap \dots \cap \{\tau^{k-1} \ge \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\} = A_k \end{aligned}$$

Since $\bigcup_{k=0}^n A_k = \bigcup_{k=0}^n B_k = \Omega$, and $(A_k)_{k=0}^n$ and $(B_k)_{k=0}^n$ are mutually disjoint respectively, we have $A_k = B_k,\ k = 0, \dots, n$. □

2.3 Decomposition of expectations of $\mathbb{G}$-optional processes

The main result in this section is Theorem 2.3.3, which shows that the expectation of a stopped $\mathbb{G}$-optional process can be calculated using a backward induction, where each step is an expectation with respect to the Brownian filtration.

Standing Assumption: For the rest of the chapter, we assume there exists a conditional probability density function $\alpha \in \mathcal{O}_{\mathbb{F}}(\Delta_n, E^n)$, such that

$$(2.3.1) \qquad \mathbb{P}\big[(\zeta_1, \dots, \zeta_n, \ell_1, \dots, \ell_n) \in d\theta_1 \dots d\theta_n de_1 \dots de_n | \mathcal{F}_t\big]$$
$$= \alpha_t(\theta_1, \dots, \theta_n, e_1, \dots, e_n) d\theta_1 \dots d\theta_n \eta(de_1) \dots \eta(de_n), \quad \text{a.s.,}$$

where $d\theta_k$ is the Lebesgue measure, and $\eta(de_k)$ is some probability measure which may depend on $(\boldsymbol{\theta}_{k-1}, \boldsymbol{e}_{k-1})$ (e.g., transition kernel), for $k = 1, \dots, n$. We also assume that the map $t \to \alpha_t$ is right continuous and

$$(2.3.2) \qquad \mathbb{E}\left[\int_{E^n} \int_{\Delta_n} \sup_{t \geq 0} \alpha_t(\boldsymbol{\theta}_n, \boldsymbol{e}_n) d\theta_1 \dots d\theta_n \eta(de_1) \dots \eta(de_n)\right] < \infty.$$

Following [72], let us set $\alpha_t^n(\boldsymbol{\theta}_n, \boldsymbol{e}_n) = \alpha_t(\boldsymbol{\theta}_n, \boldsymbol{e}_n)$, and

$$(2.3.3) \quad \alpha_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = \int_E \int_t^\infty \alpha_t^{k+1}(\boldsymbol{\theta}_k, \theta_{k+1}, \boldsymbol{e}_k, e_{k+1})\, d\theta_{k+1} \eta(de_{k+1}), \quad k = 0, \dots, n-1.$$

Note that $\alpha = 0$ when $\theta_1, \dots, \theta_n$ are not in an ascending order. As a result, for $k = 0, \dots, n-1$,

$$\alpha_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = \int_{E^k} \int_t^\infty \int_{\theta_{k+1}}^\infty \dots \int_{\theta_{n-1}}^\infty \alpha_t(\boldsymbol{\theta}_n, \boldsymbol{e}_n)\ d\theta_n \dots d\theta_{k+1} \eta(de_n) \dots \eta(de_{k+1}).$$

Hence $\mathbb{P}[\zeta_1 > t | \mathcal{F}_t] = \alpha_t^0$, and for $k = 1, \dots, n-1$,

$$\mathbb{P}[\zeta_{k+1} > t | \mathcal{F}_t] = \int_{E^k} \int_{\Delta_k} \alpha_t^k(\theta_1, \dots, \theta_k, e_1, \dots, e_k)\ d\theta_1 \dots d\theta_k \eta(de_1) \dots \eta(de_k).$$

Therefore, α^k can be interpreted as the survival density of ζ_{k+1}.

Let us recall the following lemma from [72].

Lemma 2.3.1. *Any process $Z = (Z_t)_{t\geq 0}$ is $\mathbb{G}$-optional if and only if it has the decomposition:*

$$Z_t = Z_t^0 1_{\{t<\zeta_1\}} + \sum_{k=1}^{n-1} Z_t^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) 1_{\{\zeta_k \leq t < \zeta_{k+1}\}} + Z_t^n(\boldsymbol{\zeta}_n, \boldsymbol{\ell}_n) 1_{\{\zeta_n \leq t\}}, \tag{2.3.4}$$

for some $Z^k \in \mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$, for $k = 0, \ldots, n$. A similar decomposition result holds for any $\mathbb{G}$-predictable process.

We will use the notation $Z \sim (Z^0, \ldots, Z^n)$ for the $\mathbb{G}$-optional (resp. predictable) process Z from the decomposition (2.3.4). Let $Z \sim (Z^0, \ldots, Z^n)$ be a $\mathbb{G}$-optional process, and $\tau \sim (\tau^0, \ldots, , \tau^n)$ be a $\mathbb{G}$-stopping time. Then from Lemma 2.3.1 and Proposition 2.2.3, Z_τ has the decomposition:

$$Z_\tau = Z_{\tau^0}^0 1_{\{\tau<\zeta_1\}} + \sum_{k=1}^{n-1} Z_{\tau^k}^k 1_{\{\zeta_k \leq \tau < \zeta_{k+1}\}} + Z_{\tau^n}^n 1_{\{\zeta_n \leq \tau\}}. \tag{2.3.5}$$

The following lemma will be used for the rest of the chapter:

Lemma 2.3.2. *$\tau^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k)$ is a $\mathbb{G}^k$-stopping time satisfying $\tau^k \geq \zeta_k$ if and only if for any fixed $(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k \times E^k$, $\tau^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$ is an $\mathbb{F}$-stopping time satisfying $\tau^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \geq \theta_k$ and $\tau^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$ is measurable with respect to $(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$.*

Proof. If $\tau^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$ is an $\mathbb{F}$-stopping time satisfying $\tau^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \geq \theta_k$ and is measurable with respect to $(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$, then $1_{\{\tau^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \leq t\}} \cdot 1_{\{\theta_k \leq t\}} \in \mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$. By Lemma 2.3.1 (here $n = k$), $1_{\{\tau^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) \leq t\}} = 1_{\{\tau^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) \leq t\}} \cdot 1_{\{\zeta_k \leq t\}}$ is a $\mathbb{G}^k$-optional process. Then $\{\tau^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) \leq t\} = \left\{1_{\{\tau^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) \leq t\}} = 1\right\} \in \mathcal{G}_t^k$. Hence, $\tau^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k)$ is a $\mathbb{G}^k$-stopping time. Conversely, if $\tau^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k)$ is a $\mathbb{G}^k$-stopping time, then the $\mathbb{G}^k$-optional process $1_{\{\tau^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) \leq t\}}$ has the representation from Lemma 2.3.1. Thus, for fixed $(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$, $1_{\{\tau^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \leq t\}}$ is $\mathbb{F}$-optional, which implies that $\tau^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$ is an $\mathbb{F}$-stopping time. $\square$

The following theorem is the main result of this section.

Theorem 2.3.3. *Let $Z \sim (Z^0, , \dots, Z^n)$ be a nonnegative (or bounded), right continuous $\mathbb{G}$-optional process, and $\tau \sim (\tau^0, \dots, \tau^n)$ be a finite $\mathbb{G}$-stopping time satisfying $\tau \le T$, where $T \in [0, \infty]$ is a constant. The expectation $\mathbb{E}\big[Z_\tau\big]$ can be computed by a backward induction as*

$$\mathbb{E}\big[Z_\tau\big] = J_0,$$

where $J_0, \dots, J_n$ are given by

$$
\begin{aligned}
(2.3.6)\quad J_n(\boldsymbol{\theta}_n, \boldsymbol{e}_n) &= \mathbb{E}\Big[Z^n_{\tau^n}\alpha^n_{\tau^n}(\boldsymbol{\theta}_n, \boldsymbol{e}_n)\Big|\mathcal{F}_{\theta_n}\Big], \quad (\boldsymbol{\theta}_n, \boldsymbol{e}_n) \in \Delta_n(T) \times E^n,\\
(2.3.7)\quad J_k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) &= \mathbb{E}\Bigg[Z^k_{\tau^k}\alpha^k_{\tau^k}(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\\
&\quad + \int_{\theta_k}^{\tau^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\wedge T}\int_E J_{k+1}(\boldsymbol{\theta}_k, \theta_{k+1}, \boldsymbol{e}_k, e_{k+1})\eta(de_{k+1})d\theta_{k+1}\Big|\mathcal{F}_{\theta_k}\Bigg],
\end{aligned}
$$

$(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$, for $k = 0, \dots, n-1$.

Proof. For the sake of simplicity, let us assume $n = 2$. Using (2.3.6) and (2.3.7), plugging J_2 into J_1, and then J_1 into J_0, we obtain

$$
\begin{aligned}
J_0 &= \mathbb{E}\Big[Z^0_{\tau^0}\alpha^0_{\tau_0}\Big] + \mathbb{E}\Big[\int_0^{\tau^0\wedge T}\int_E \mathbb{E}\Big[Z^1_{\tau^1(\theta_1,e_1)}\cdot\alpha^1_{\tau^1(\theta_1,e_1)}\Big|\mathcal{F}_{\theta_1}\Big]\eta(de_1)d\theta_1\Big]\\
&\quad + \mathbb{E}\Bigg[\int_0^{\tau_0\wedge T}\int_E \mathbb{E}\Big[\int_0^{\tau_1(\theta_1,e_1)\wedge T}\int_E \mathbb{E}\Big[Z^2_{\tau^2(\theta_1,\theta_2,e_1,e_2)}\cdot\alpha^2_{\tau^2}\Big|\mathcal{F}_{\theta_2}\Big]\eta(de_2)d\theta_2\Big|\mathcal{F}_{\theta_1}\Big]\eta(de_1)d\theta_1\Bigg].
\end{aligned}
$$

On the right side of the equation above, let us denote the fist term by I, the second term by II, and the third term by III. We can show that

$$
\begin{aligned}
\mathrm{I} &= \mathbb{E}\Big[\int_{E^2}\int_{\Delta_2} Z^0_{\tau^0}\cdot 1_{\{\theta_1>\tau^0\}}\cdot\alpha_{\tau^0}(\theta_1,\theta_2,e_1,e_2)\ d\theta_1 d\theta_2\eta(de_1)\eta(de_2)\Big],\\
\mathrm{II} &= \mathbb{E}\Big[\int_{E^2}\int_{\Delta_2} Z^1_{\tau^1(\theta_1,e_1)}\cdot 1_{\{\theta_1\le T\}}\cdot 1_{\{\tau^0\ge\theta_1\}\cap\{\tau^1(\theta_1,e_1)<\theta_2\}}\cdot\alpha_{\tau^1}\ d\theta_1 d\theta_2\eta(de_1)\eta(de_2)\Big],\\
\mathrm{III} &= \mathbb{E}\Big[\int_{E^2}\int_{\Delta_2} Z^2_{\tau 2(\theta_1,\theta_2,e_1,e_2)}\cdot 1_{\{\theta_1,\theta_2\le T\}}\cdot 1_{\{\tau^0\ge\theta_1\}\cap\{\tau^1(\theta_1,e_1)\ge\theta_2\}}\cdot\alpha_{\tau^2}\ d\theta_1 d\theta_2\eta(de_1)\eta(de_2)\Big].
\end{aligned}
$$

For fixed $(\theta_1, \theta_2, e_1, e_2) \in \Delta_2 \times E^2$, from Proposition 2.2.3, we have $\{\tau^0 \ge \theta_1\} \cap \{\tau^1 < \theta_2\} = \{\theta_1 \le \tau < \theta_2\} \subset \{\theta_1 \le T\}$, and $\{\tau^0 \ge \theta_1\} \cap \{\tau^1 \ge \theta_2\} = \{\theta_2 \le \tau\} \subset \{\theta_1, \theta_2 \le$

$T\}$. Hence,

$$Z_\tau(\theta_1,\theta_2,e_1,e_2) = Z^0_{\tau^0}\cdot 1_{\{\tau^0<\theta_1\}} + Z^1_{\tau^1}\cdot 1_{\{\tau^0\ge\theta_1\}}\cdot 1_{\{\tau^1<\theta_2\}} + Z^2_{\tau^2}\cdot 1_{\{\tau^0\ge\theta_1\}}\cdot 1_{\{\tau^1\ge\theta_2\}}$$

$$= Z^0_{\tau^0}\cdot 1_{\{\tau^0<\theta_1\}} + Z^1_{\tau^1}\cdot 1_{\{\theta_1\le T\}}\cdot 1_{\{\tau^0\ge\theta_1\}\cap\{\tau^1<\theta_2\}} + Z^2_{\tau^2}\cdot 1_{\{\theta_1,\theta_2\le T\}}\cdot 1_{\{\tau^0\ge\theta_1\}\cap\{\tau^1\ge\theta_2\}}.$$

Therefore, we have

$$J_0 = \mathrm{I}+\mathrm{II}+\mathrm{III} = \mathbb{E}\Big[\int_{E^2}\int_{\Delta_2} Z_\tau(\theta_1,\theta_2,e_1,e_2)\cdot\alpha_\tau(\theta_1,\theta_2,e_1,e_2)\ d\theta_1 d\theta_2\eta(de_1)\eta(de_2)\Big].$$

We will show in two steps that $J_0=\mathbb{E}[Z_\tau]$.

Step 1: If $\tau=\sum_{k=0}^{\infty} a_k 1_{A_k}$, where $0\le a_0<a_1\ldots<\infty$, and $A_k\in\mathcal{G}_{a_k},\ k=0,1,\ldots,$ then

$$\begin{aligned}
&\mathbb{E}[Z_\tau] = \sum_{k=0}^{\infty}\mathbb{E}\big[Z_{a_k}1_{A_k}\big]\\
&= \sum_{k=0}^{\infty}\mathbb{E}\left[\int_{E^2}\int_{\Delta_2} Z_{a_k}(\theta_1,\theta_2,e_1,e_2)1_{A_k}(\theta_1,\theta_2,e_1,e_2)\alpha_{a_k}(\theta_1,\theta_2,e_1,e_2)d\theta_1 d\theta_2\eta(de_1)\eta(de_2)\right]\\
&= \mathbb{E}\left[\int_{E^2}\int_{\Delta_2}\left(\sum_{k=0}^{\infty} Z_{a_k}(\theta_1,\theta_2,e_1,e_2)1_{A_k}(\theta_1,\theta_2,e_1,e_2)\alpha_{a_k}(\theta_1,\theta_2,e_1,e_2)\right)d\theta_1 d\theta_2\eta(de_1)\eta(de_2)\right]\\
&= \mathbb{E}\Big[\int_{E^2}\int_{\Delta^2} Z_\tau(\theta_1,\theta_2,e_1,e_2)\cdot\alpha_\tau(\theta_1,\theta_2,e_1,e_2)\ d\theta_1 d\theta_2\eta(de_1)\eta(de_2)\Big],
\end{aligned}$$

where the second equality above follows from [72, Proposition 2.1].

Step 2: In general, let τ be any finite $\mathbb{G}$-stopping time. Define

$$\tau_m := \sum_{j=0}^{\infty}\frac{j+1}{2^m}\cdot 1_{\{\frac{j}{2^m}\le\tau<\frac{j+1}{2^m}\}},\quad m=1,2,\ldots$$

For fixed $N\in(0,\infty)$, Step 1 implies that

$$\mathbb{E}\big[Z_{\tau^m}\wedge N\big] = \mathbb{E}\Big[\int_{E^2}\int_{\Delta^2}\big(Z_{\tau^m}\big(\theta_1,\theta_2,e_1,e_2\big)\wedge N\big)\cdot\alpha_{\tau^m}(\theta_1,\theta_2,e_1,e_2)\ d\theta_1 d\theta_2\eta(de_1)\eta(de_2)\Big].$$

Thanks to (2.3.2) and the right continuity of Z_t and α_t, by sending $m\to\infty$, we get

$$\mathbb{E}\big[Z_\tau\wedge N\big] = \mathbb{E}\Big[\int_{E^2}\int_{\Delta^2}\big(Z_\tau\big(\theta_1,\theta_2,e_1,e_2\big)\wedge N\big)\cdot\alpha_\tau(\theta_1,\theta_2,e_1,e_2)\ d\theta_1 d\theta_2\eta(de_1)\eta(de_2)\Big].$$

Then letting $N\to\infty$, the result follows. $\square$

Remark 2.3.4. When the Brownian motion and the jumps are independent, (2.3.2) and the right continuity of α_t in the Standing Assumption trivially holds. In this case, Theorem 2.3.3 still holds if the assumption of the right continuity of Z_t is removed. In fact, it follows directly from the expectation under the product probability measure that

$$J_0 = \mathbb{E}\left[\int_{E^2}\int_{\Delta_2} Z_\tau(\theta_1,\theta_2,e_1,e_2)\cdot\alpha(\theta_1,\theta_2,e_1,e_2)\ d\theta_1 d\theta_2 \eta(de_1)\eta(de_2)\right] = \mathbb{E}[Z_\tau].$$

The same applies for Theorem 2.4.2 and Proposition 2.4.4.

2.4 Decomposition of $\mathbb{G}$-controller-stopper problems

Theorem 2.4.2 and Proposition 2.4.4 are the main results for this section, which decompose the global $\mathbb{G}$-controller-stopper problems into a backward induction, where each step is a controller-stopper problem with respect to the Brownian filtration.

A control is a $\mathbb{G}$-predictable process $\pi \sim (\pi^0,\dots,\pi^n)$, where $\pi^k \in \mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$ is valued in a set A^k in some Euclidian space, for $k = 0,\dots,n$. We denote by $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k; A^k)$ the set of elements in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$ valued in A^k, $k = 0,\dots,n$. We require that all the $\mathbb{G}$-stopping times we consider here are valued in $[0,T]$, where $T \in (0,\infty]$ is a given constant. A trading strategy is a pair of a control and a $\mathbb{G}$-stopping time. We will use the notation $(\pi,\tau) \sim (\pi^k,\tau^k)_{k=0}^n$ for the trading strategy if $\pi \sim (\pi^0,\dots,\pi^n)$ and $\tau \sim (\tau^0,\dots,\tau^n)$. A trading strategy $(\pi,\tau) \sim (\pi^k,\tau^k)_{k=0}^n$ is admissible, if for $k = 0,\dots,n$, $(\pi^k,\tau^k) \in \mathcal{A}^k \times \mathcal{T}^k$, where $\mathcal{A}^k$ is some seperable metric space of $\mathcal{P}(\Delta_k, E^k; A^k)$, and $\mathcal{T}^k$ is some set of finite $\mathbb{G}^k$-stopping times. By Proposition 2.2.4, we let $\mathcal{T}^k$ be such that for any $\tau^k \in \mathcal{T}^k$, $\tau^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \le T$ whenever $\theta_k \le T$. Note that $\mathcal{A}^k$ and $\mathcal{T}^k$ may depend on each other in general. We denote the set of admissible trading strategies by $\mathcal{A}_{\mathbb{G}} \times \mathcal{T}_{\mathbb{G}}$.

The following lemma will be used for the measurable selection issue later on.

Lemma 2.4.1. *For $k = 0, \dots, n$, define the metric on $\mathcal{T}^k$ in the following way:*

$$\rho(\tau_1^k, \tau_2^k) := \mathbb{E}\Big[\int_0^\infty e^{-t}\big|1_{\{\tau_1^k \le t\}} - 1_{\{\tau_2^k \le t\}}\big| dt\Big], \qquad \tau_1^k, \tau_2^k \in \mathcal{T}^k.$$

Then $\mathcal{T}^k$ is a separable metric space.

Proof. Since for any $\mathbb{G}^k$-stopping time τ^k, $e^{-t}1_{\{\tau^k \le t\}}$ is a $\mathbb{G}^k$-adapted process in $L^1([0,\infty) \times \Omega)$, the conclusion follows from the separability of L^1, see [78]. □

Following [72], we describe the formulation of a stopped controlled state process as follows:

- Controlled state process between jumps:

$$(x, \pi^k) \in \mathbb{R}^d \times \mathcal{A}^k \longmapsto X^{k,x,\pi^k} \in \mathcal{O}_{\mathbb{F}}(\Delta_k, E^k), \quad k = 0, \dots, n,$$

  such that

$$X_0^{0,x,\pi^0} = x, \qquad X_{\theta_k}^{k,\beta,\pi^k}(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = \beta, \quad \forall \beta \ \mathcal{F}_{\theta_k}\text{-measurable}.$$

- Jumps of controlled state process: we have a collection of maps Γ^k on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d \times A^{k-1} \times E$, for $k = 1, \dots, n$, such that

$$(t, \omega, x, a, e) \mapsto \Gamma^k(\omega, x, a, e) \quad \text{is } \mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A^{k-1}) \otimes \mathcal{B}(E)\text{-measurable}$$

- Global controlled state process:

$$\big(x, \pi \sim (\pi^0, \dots, \pi^n)\big) \in \mathbb{R}^d \times \mathcal{A}_{\mathbb{G}} \longmapsto X^{x,\pi} \in \mathcal{O}_{\mathbb{G}},$$

  where

$$X_t^{x,\pi} = \bar{X}_t^0 1_{\{t < \zeta_1\}} + \sum_{k=1}^{n-1} \bar{X}_t^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) 1_{\{\zeta_k \le t < \zeta_{k+1}\}} + \bar{X}_t^n(\boldsymbol{\zeta}_n, \boldsymbol{\ell}_n) 1_{\{\zeta_n \le t\}}, \tag{2.4.1}$$

  with $(\bar{X}^0, \dots, \bar{X}^n) \in \mathcal{O}_{\mathbb{F}}(\Delta_0, E^0) \times \dots \times \mathcal{O}_{\mathbb{F}}(\Delta_n, E^n)$ with initial data

$$\bar{X}^0 = X^{0,x,\pi^0},$$

$$\bar{X}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = X^{k,\Gamma^k_{\theta_k}(\bar{X}^{k-1}_{\theta_k}, \pi^{k-1}_{\theta_k}, e_k), \pi^k}(\boldsymbol{\theta}_k, \boldsymbol{e}_k), \quad k = 1, \dots, n.$$

- Stopped global controlled state process: given a trading strategy $(\pi,\tau) \sim (\pi^k,\tau^k)_{k=0}^n$ in $\mathcal{A}_{\mathbb{G}} \times \mathcal{T}_{\mathbb{G}}$, let $X^{x,\pi}$ be the process from (2.4.1), then the stopped controlled state process is:

$$(2.4.2) \quad X_\tau^{x,\pi} = \bar{X}_{\tau^0}^0 1_{\{\tau<\zeta_1\}} + \sum_{k=1}^{n-1} \bar{X}_{\tau^k}^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) 1_{\{\zeta_k \le \tau < \zeta_{k+1}\}} + \bar{X}_{\tau^n}^n(\boldsymbol{\zeta}_n, \boldsymbol{\ell}_n) 1_{\{\zeta_n \le \tau\}}.$$

Assume $U \sim (U^0, \dots, U^n)$ is bounded (nonnegative, nonpositive), $\mathcal{O}_{\mathbb{G}} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable which gives the terminal payoff U_t at time t . Consider the two types of the controller-stopper problems:

$$(2.4.3) \qquad V^0(x) = \sup_{\tau \in \mathcal{T}_{\mathbb{G}}} \sup_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}\big[U_\tau(X_\tau^{x,\pi})\big], \quad x \in \mathbb{R}^d,$$

$$(2.4.4) \qquad \mathfrak{V}^0(x) = \sup_{\pi \in \mathcal{A}_{\mathbb{G}}} \inf_{\tau \in \mathcal{T}_{\mathbb{G}}} \mathbb{E}\big[U_\tau(X_\tau^{x,\pi})\big], \quad x \in \mathbb{R}^d.$$

We require that for any $x \in \mathbb{R}^d$ and admissible control π, the map $t \to U_t(X_t^{x,\pi})$ is right continuous.

The following theorem provides a decomposition for calculating V^0 in (2.4.3). Its proof is similar to the proof of [72, Theorem 4.1].

Theorem 2.4.2. *Define value functions* $(\bar{V}^k)_{k=0}^n$ *as*

$$\bar{V}^n(x, \boldsymbol{\theta}_n, \boldsymbol{e}_n) = \operatorname*{ess\,sup}_{\tau^n \in \mathcal{T}^n} \operatorname*{ess\,sup}_{\pi^n \in \mathcal{A}^n} \mathbb{E}\Big[U_{\tau^n}^n(X_{\tau^n}^{n,x,\pi^n}, \boldsymbol{\theta}_n, \boldsymbol{e}_n) \cdot \alpha_{\tau^n}^n(\boldsymbol{\theta}_n, \boldsymbol{e}_n) \Big| \mathcal{F}_{\theta_n}\Big],$$

$(\boldsymbol{\theta}_n, \boldsymbol{e}_n) \in \Delta_n(T) \times E^n$, *and*

$$(2.4.5) \quad \bar{V}^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k) = \operatorname*{ess\,sup}_{\tau^k \in \mathcal{T}^k} \operatorname*{ess\,sup}_{\pi^k \in \mathcal{A}^k} \mathbb{E}\Big[U_{\tau^k}^k(X_{\tau^k}^{k,x,\pi^k}, \boldsymbol{\theta}_k, \boldsymbol{e}_k) \cdot \alpha_{\tau^k}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$$
$$+ \int_{\theta_k}^{\tau^k} \int_E \bar{V}^{k+1}\Big(\Gamma_{\theta_k}^{k+1}(X_{\theta_{k+1}}^{k,x,\pi^k}, \pi_{\theta_{k+1}}^k, e_{k+1}), \boldsymbol{\theta}_k, \theta_{k+1}, \boldsymbol{e}_k, e_{k+1}\Big) \eta(de_{k+1}) d\theta_{k+1} \Big| \mathcal{F}_{\theta_k}\Big],$$

$(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$, *for* $k = 0, \dots, n-1$. *Then* $V^0(x) = \bar{V}^0(x)$.

Remark 2.4.3. In Equation (2.4.5), the first term $U^k_{\tau^k}(X^{k,x,\pi^k}_{\tau^k}, \boldsymbol{\theta}_k, \boldsymbol{e}_k) \cdot \alpha^k_{\tau^k}(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$ can be interpreted as the gain when there are no jumps between θ_k and τ^k, which is measured by the survival density $\alpha^k_{\tau^k}$. The second term

$$\int_{\theta_k}^{\tau^k} \int_E \bar{V}^{k+1}\Big(\Gamma^{k+1}_{\theta_k}(X^{k,x,\pi^k}_{\theta_{k+1}}, \pi^k_{\theta_{k+1}}, e_{k+1}), \boldsymbol{\theta}_k, \theta_{k+1}, \boldsymbol{e}_k, e_{k+1}\Big)\eta(de_{k+1})d\theta_{k+1}$$

can be understood as the gain when there is a jump at time θ_{k+1} between θ_k and τ^k.

Proof of Theorem 2.4.2. For $x \in \mathbb{R}^d$, $(\pi, \tau) \sim (\pi^k, \tau^k)^n_{k=0}$ in $\mathcal{A}_{\mathbb{G}} \times \mathcal{T}_{\mathbb{G}}$, define

$$\begin{aligned} I^n(x, \boldsymbol{\theta}_n, \boldsymbol{e}_n, \pi, \tau) &= \mathbb{E}\Big[U^n_{\tau^n}(X^{n,x,\pi^n}_{\tau^n}, \boldsymbol{\theta}_n, \boldsymbol{e}_n) \cdot \alpha^n_{\tau^n}(\boldsymbol{\theta}_n, \boldsymbol{e}_n)\Big|\mathcal{F}_{\theta_n}\Big], \quad (\boldsymbol{\theta}_n, \boldsymbol{e}_n) \in \Delta_n(T) \times E^n, \\ I^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k, \pi, \tau) &= \mathbb{E}\Big[U^k_{\tau^k}(X^{k,x,\pi^k}_{\tau^k}, \boldsymbol{\theta}_k, \boldsymbol{e}_k) \cdot \alpha^k_{\tau^k}(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \\ &+ \int_{\theta_k}^{\tau^k} \int_E I^{k+1}\Big(\Gamma^{k+1}_{\theta_{k+1}}(X^{k,x,\pi^k}_{\theta_{k+1}}, \pi^k_{\theta_{k+1}}, e_{k+1}), \boldsymbol{\theta}_k, \theta_{k+1}, \boldsymbol{e}_k, e_{k+1}, \pi, \tau\Big)\eta(de_{k+1})d\theta_{k+1}\Big|\mathcal{F}_{\theta_k}\Big], \end{aligned}$$

$(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$, for $k = 0, \ldots, n-1$. Set $\bar{I}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = I^k(\bar{X}^k_{\theta_k}, \boldsymbol{\theta}_k, \boldsymbol{e}_k, \pi, \tau)$, $k = 0, \ldots, n$. From the decomposition (2.4.2), we know that $(\bar{I}^k)^n_{k=0}$ satisfy the backward induction formula:

$$\begin{aligned} \bar{I}^n(\boldsymbol{\theta}_n, \boldsymbol{e}_n) &= \mathbb{E}\Big[U^n_{\tau^n}(\bar{X}^n_{\tau^n}, \boldsymbol{\theta}_n, \boldsymbol{e}_n) \cdot \alpha^n_{\tau^n}(\boldsymbol{\theta}_n, \boldsymbol{e}_n)\Big|\mathcal{F}_{\theta_n}\Big], \\ \bar{I}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) &= \mathbb{E}\Big[U^k_{\tau_k}(\bar{X}^k_{\tau^k}, \boldsymbol{\theta}_k, \boldsymbol{e}_k) \cdot \alpha^k_{\tau^k}(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \\ &\qquad + \int_{\theta_k}^{\tau^k} \int_E \bar{I}^{k+1}(\boldsymbol{\theta}_k, \theta_{k+1}, \boldsymbol{e}_k, e_{k+1})\ \eta(de_{k+1})d\theta_{k+1}\Big|\mathcal{F}_{\theta_k}\Big]. \end{aligned}$$

From Theorem 2.3.3 we have that

$$\bar{I}^0 = I^0 = \mathbb{E}\big[U_\tau(X^{x,\pi}_\tau)\big]. \tag{2.4.6}$$

Define the value function processes

$$V^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k) := \operatorname*{ess\,sup}_{\tau \in \mathcal{A}_{\mathbb{G}}} \operatorname*{ess\,sup}_{\pi \in \mathcal{T}_{\mathbb{G}}}\ I^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k, \pi, \tau), \tag{2.4.7}$$

for $k=0,\dots,n,\ x\in\mathbb{R}^d,\ (\boldsymbol{\theta}_k,\boldsymbol{e}_k)\in\Delta_k(T)\times E^k$. Observe that V^0 defined in (2.4.7) is consistent with its definition in (2.4.3) from (2.4.6). Then it remains to show that $\bar{V}^k=V^k$ for $k=0,\dots,n$. For $k=n$, since $I^n(x,\boldsymbol{\theta}_n,\boldsymbol{e}_n,\pi,\tau)$ in fact only depends on (π^n,τ^n), we immediately have $\bar{V}^n=V^n$. Now assume $\bar{V}^{k+1}=V^{k+1}$, for $0\le k\le n-1$. Then for any $(\pi,\tau)\sim(\pi^k,\tau^k)_{k=0}^n$ in $\mathcal{A}_{\mathbb{G}}\times\mathcal{T}_{\mathbb{G}}$,

$$
\begin{aligned}
&I^k(x,\boldsymbol{\theta}_k,\boldsymbol{e}_k,\pi,\tau)\\
&\le\mathbb{E}\Bigg[U^k_{\tau^k}(X^{k,x,\pi^k}_{\tau^k},\boldsymbol{\theta}_k,\boldsymbol{e}_k)\cdot\alpha^k_{\tau^k}(\boldsymbol{\theta}_k,\boldsymbol{e}_k)\\
&\qquad+\int_{\theta_k}^{\tau^k}\int_E V^{k+1}\Big(\Gamma^{k+1}_{\theta_{k+1}}(X^{k,x,\pi^k}_{\theta_{k+1}},\pi^k_{\theta_{k+1}},e_{k+1}),\boldsymbol{\theta}_k,\theta_{k+1},\boldsymbol{e}_k,e_{k+1}\Big)\ \eta(de_{k+1})d\theta_{k+1}\Big|\mathcal{F}_{\theta_k}\Bigg]\\
&\le\bar{V}^k(x,\boldsymbol{\theta}_k,\boldsymbol{e}_k),
\end{aligned}
$$

which implies that $V^k\le\bar{V}^k$.

Conversely, given $x\in\mathbb{R}^d$ and $(\boldsymbol{\theta}_k,\boldsymbol{e}_k)\in\Delta_k(T)\times E^k$, let us prove $V^k(x,\boldsymbol{\theta}_k.\boldsymbol{e}_k)\ge\bar{V}^k(x,\boldsymbol{\theta}_k,\boldsymbol{e}_k)$. Fix $(\pi^k,\tau^k)\in\mathcal{A}^k\times\mathcal{T}^k$ and the associated controlled process X^{k,x,π^k}, from the definition of V^{k+1}, we have that for any $\omega\in\Omega$ and $\epsilon>0$, there exists $(\pi^{\omega,\epsilon},\tau^{\omega,\epsilon})\in\mathcal{A}_{\mathbb{G}}\times\mathcal{T}_{\mathbb{G}}$, such that it is an $\epsilon e^{-\theta_{k+1}}$-optimal trading strategy for $V^{k+1}(\cdot,\boldsymbol{\theta}_k,\boldsymbol{e}_k)$ at $\big(\omega,\Gamma^{k+1}_{\theta_{k+1}}(X^{k,x,\pi^k}_{\theta_{k+1}},\pi^k_{\theta_{k+1}},e_{k+1})\big)$. By the separability of the set of admissible trading strategies from Lemma 2.4.1, one can use a measurable selection argument (e.g., see [81]) to find $(\pi^\epsilon,\tau^\epsilon)\sim(\pi^{\epsilon,k},\tau^{\epsilon,k})_{k=0}^n$ in $\mathcal{A}_{\mathbb{G}}\times\mathcal{T}_{\mathbb{G}}$, such that $\pi^\epsilon_t(\omega)=\pi^{\omega,\epsilon}_t(\omega),\ dt\otimes d\mathbb{P}$-a.e. and $\tau^\epsilon(\omega)=\tau^{\omega,\epsilon}(\omega)$, a.s., and thus

$$
\begin{aligned}
&V^{k+1}\big(\Gamma^{k+1}_{\theta_{k+1}}(X^{k,x,\pi}_{\theta_{k+1}},\pi^k_{\theta_{k+1}},e_{k+1}),\boldsymbol{\theta}_k,\theta_{k+1},\boldsymbol{e}_k,e_{k+1}\big)-\epsilon e^{-\theta_{k+1}}\\
&\qquad\le I^{k+1}\big(\Gamma^{k+1}_{\theta_{k+1}}(X^{k,x,\pi}_{\theta_{k+1}},\pi^k_{\theta_{k+1}},e_{k+1}),\boldsymbol{\theta}_k,\theta_{k+1},\boldsymbol{e}_k,e_{k+1},\pi^\epsilon,\tau^\epsilon\big),\quad\text{a.s.}
\end{aligned}
$$

Consider the admissible trading strategy $(\tilde{\pi}^\epsilon,\tilde{\tau}^\epsilon)$ with the decomposition

$$\tilde{\pi}^\epsilon\sim(\pi^{\epsilon,0},\dots,\pi^{\epsilon,k-1},\pi^k,\pi^{\epsilon,k+1},\dots,\pi^{\epsilon,n}),$$

$$\tilde{\tau}^\epsilon\sim(\tau^{\epsilon,0},\dots,\tau^{\epsilon,k-1},\tau^k,\tau^{\epsilon,k+1},\dots,\tau^{\epsilon,n}).$$

Since $I^{k+1}(x, \boldsymbol{\theta}_{k+1}, \boldsymbol{e}_{k+1}, \pi, \tau)$ depends on $(\pi, \tau) \sim (\pi^j, \tau^j)_{j=0}^n$ only through their last components $(\pi^j, \tau^j)_{j=k+1}^n$, we have

$$
\begin{aligned}
& V^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k) \\
& \geq I^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k, \tilde{\pi}^\epsilon, \tilde{\tau}^\epsilon) \\
& = \mathbb{E}\Big[U^k_{\tau^k}(X^{k,x,\pi^k}_{\tau^k}, \boldsymbol{\theta}_k, \boldsymbol{e}_k) \cdot \alpha^k_{\tau^k}(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \\
& \quad + \int_{\theta_k}^{\tau^k} \int_E I^{k+1}\Big(\Gamma^{k+1}_{\theta_{k+1}}(X^{k,x,\pi^k}_{\theta_{k+1}}, \pi^k_{\theta_{k+1}}, e_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{e}_{k+1}, \tilde{\pi}^\epsilon, \tilde{\tau}^\epsilon\Big)\eta(de_{k+1})d\theta_{k+1}\Big|\mathcal{F}_{\theta_k}\Big] \\
& \geq \mathbb{E}\Big[U^k_{\tau^k}(X^{k,x,\pi^k}_{\tau^k}, \boldsymbol{\theta}_k, \boldsymbol{e}_k) \cdot \alpha^k_{\tau^k}(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \\
& \quad + \int_{\theta_k}^{\tau^k} \int_E \bar{V}^{k+1}\Big(\Gamma^{k+1}_{\theta_{k+1}}(X^{k,x,\pi^k}_{\theta_{k+1}}, \pi^k_{\theta_{k+1}}, e_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{e}_{k+1}\Big)\eta(de_{k+1})d\theta_{k+1}\Big|\mathcal{F}_{\theta_k}\Big] - \epsilon.
\end{aligned}
$$

Therefore, $V^k \geq \bar{V}^k$, from which the claim of the theorem follows. □

Now let us consider the value function $\mathfrak{V}_0$ in (2.4.4). We have the following result:

Proposition 2.4.4. *Define value functions* $(\bar{\mathfrak{V}}^k)_{k=0}^n$ *as*

$$
\bar{\mathfrak{V}}^n(x, \boldsymbol{\theta}_n, \boldsymbol{e}_n) = \operatorname*{ess\,sup}_{\pi^n \in \mathcal{A}^n} \operatorname*{ess\,inf}_{\tau^n \in \mathcal{T}^n} \mathbb{E}\Big[U^n_{\tau^n}(X^{n,x,\pi^n}_{\tau^n}, \boldsymbol{\theta}_n, \boldsymbol{e}_n) \cdot \alpha^n_{\tau^n}(\boldsymbol{\theta}_n, \boldsymbol{e}_n)\Big|\mathcal{F}_{\theta_n}\Big],
$$

$(\boldsymbol{\theta}_n, \boldsymbol{e}_n) \in \Delta_n(T) \times E^n$, *and*

$$
\begin{aligned}
\bar{\mathfrak{V}}^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k) &= \operatorname*{ess\,sup}_{\pi^k \in \mathcal{A}^k} \operatorname*{ess\,inf}_{\tau^k \in \mathcal{T}^k} \mathbb{E}\Big[U^k_{\tau^k}(X^{k,x,\pi^k}_{\tau^k}, \boldsymbol{\theta}_k, \boldsymbol{e}_k) \cdot \alpha^k_{\tau^k}(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \\
&\quad + \int_{\theta_k}^{\tau^k} \int_E \bar{\mathfrak{V}}^{k+1}\Big(\Gamma^{k+1}_{\theta_k}(X^{k,x,\pi^k}_{\theta_{k+1}}, \pi^k_{\theta_{k+1}}, e_{k+1}), \boldsymbol{\theta}_k, \theta_{k+1}, \boldsymbol{e}_k, e_{k+1}\Big)\eta(de_{k+1})d\theta_{k+1}\Big|\mathcal{F}_{\theta_k}\Big],
\end{aligned}
$$

$(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$, *for* $k = 0, \ldots, n-1$. *Then* $\mathfrak{V}_0(x) = \bar{\mathfrak{V}}^0(x)$.

Proof. Given $\pi \sim (\pi^0, \ldots, \pi^n)$ in $\mathcal{A}_{\mathbb{G}}$, define

$$
\tilde{\mathfrak{V}}^n(x, \boldsymbol{\theta}_n, \boldsymbol{e}_n, \pi) = \operatorname*{ess\,inf}_{\tau^n \in \mathcal{T}^n} \mathbb{E}\Big[U^n_{\tau^n}(X^{n,x,\pi^n}_{\tau^n}, \boldsymbol{\theta}_n, \boldsymbol{e}_n) \cdot \alpha^n_{\tau^n}(\boldsymbol{\theta}_n, \boldsymbol{e}_n)\Big|\mathcal{F}_{\theta_n}\Big],
$$

$(\boldsymbol{\theta}_n, \boldsymbol{e}_n) \in \Delta_n(T) \times E^n$, and

$$
\tilde{\mathfrak{V}}^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k, \pi) = \operatorname*{ess\,inf}_{\tau^k \in \mathcal{T}^k} \mathbb{E}\Big[U^k_{\tau^k}(X^{k,x,\pi^k}_{\tau^k}, \boldsymbol{\theta}_k, \boldsymbol{e}_k) \cdot \alpha^k_{\tau^k}(\boldsymbol{\theta}_k, \boldsymbol{e}_k)
$$

$$+\int_{\theta_k}^{\tau^k}\int_E \tilde{\mathfrak{V}}^{k+1}\Big(\Gamma^{k+1}_{\theta_k}(X^{k,x,\pi^k}_{\theta_{k+1}},\pi^k_{\theta_{k+1}},e_{k+1}),\boldsymbol{\theta}_k,\theta_{k+1},\boldsymbol{e}_k,e_{k+1},\pi\Big)\eta(de_{k+1})d\theta_{k+1}\Big|\mathcal{F}_{\theta_k}\Big],$$

$(\boldsymbol{\theta}_k,\boldsymbol{e}_k)\in\Delta_k(T)\times E^k$, for $k=0,\dots,n-1$. From Theorem 2.4.2, we have

$$\tilde{\mathfrak{V}}^0(x,\pi)=\inf_{\tau\in\mathcal{T}_{\mathbb{G}}}\mathbb{E}\;U_\tau(X^{x,\pi}_\tau).$$

Define

$$\mathfrak{V}^k(x,\boldsymbol{\theta}_k,\boldsymbol{e}_k):=\operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{\mathbb{G}}}\tilde{\mathfrak{V}}^k(x,\boldsymbol{\theta}_k,\boldsymbol{e}_k,\pi),\;(\boldsymbol{\theta}_k,\boldsymbol{e}_k)\in\Delta_k(T)\times E^k,\;k=0,\dots,n.$$

Then the definition for $\mathfrak{V}^0$ above is consistent with (2.4.4). Following the proof of Theorem 2.4.2 we can show $\mathfrak{V}^k=\bar{\mathfrak{V}}^k,\;k=0,\dots,n.$ □

2.5 Application to indifference pricing of American options

In this section, we apply our decomposition method to indifference pricing of American options under multiple default risk. The main results are Theorem 2.5.4 and Theorem 2.5.8, which provide the RBSDE characterization of the indifference prices.

2.5.1 Market model

The model we will use here is similar to that in [50]. Let $T\in(0,\infty)$ be the finite time horizon. We assume in the market, there exists at most n default events. Let $\zeta_1,\dots,\zeta_n$ and $\ell_1,\dots,\ell_n$ represent the random default times and marks respectively, with α defined in (2.3.1) as the probability density. For any time t, if $\zeta_k\le t<\zeta_{k+1},\;k=1,\dots,n-1$ ($t<\zeta_1$ for $k=0$ and $t\ge\zeta_n$ for $k=n$), we say the underlying processes are in the k-default scenario.

We consider a portfolio of d-asset with a value process defined by a d-dimensional $\mathbb{G}$-optional process $S\sim(S^0,\dots,S^n)$ from (2.3.5), where $S^k(\boldsymbol{\theta}_k,\boldsymbol{e}_k)\in\mathcal{O}_{\mathbb{F}}(\Delta_k,E^k)$ is valued in $\mathbb{R}^d_+$, representing the asset value in the k-default scenario, given the past default times $\boldsymbol{\zeta}_k = \boldsymbol{\theta}_k$ and the associated marks $\boldsymbol{\ell}_k = \boldsymbol{e}_k$, for $k = 0, \ldots n$. Suppose the dynamics of the indexed process S^k is given by

$$dS_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = S_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) * \left(b_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)dt + \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)dW_t\right), \quad t \geq \theta_k, \tag{2.5.1}$$

where W is an m-dimensional $(\mathbb{P}, \mathbb{F})$-Brownian motion, $m \geq d$, b^k and σ^k are indexed processes in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$, valued respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$. Here, for $x = (x_1, \ldots, x_d)' \in \mathbb{R}^d$, and $y = (y_1, \ldots, y_d)' \in \mathbb{R}^{d\times q}$, the expression $x * y$ denotes the vector $(x_1y_1, \ldots, x_dy_d)' \in \mathbb{R}^{d\times q}$. Equation (2.5.1) can be viewed as an asset model with change of regimes after default events, with coefficient b^k, σ^k depending on the past default information. We make the usual no-arbitrage assumption that there exists an indexed risk premium process $\lambda^k \in \mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$, such that for all $(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k \times E^k$,

$$\sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = b_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k), \quad t \geq 0. \tag{2.5.2}$$

Moreover, each default time θ_k may induce a jump in the asset portfolio, which will be formalized by considering a family of indexed processes $\gamma^k \in \mathcal{P}(\Delta_k, E^k, E)$, valued in $[-1, \infty)$, for $k = 0, \ldots, n-1$. For $(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k \times E^k$ and $e_{k+1} \in E$, $\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k, e_{k+1})$ represents the relative vector jump size on the d assets at time $t = \theta_{k+1} \geq \theta_k$ with a mark e_{k+1}, given the past default events $(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) = (\boldsymbol{\theta}_k, \boldsymbol{e}_k)$. In other words, we have:

$$S_{\theta_{k+1}}^{k+1}(\boldsymbol{\theta}_{k+1}, \boldsymbol{e}_{k+1}) = S_{\theta_{k+1}^-}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) * \Big(\mathbf{1}_d + \gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k, e_{k+1})\Big), \tag{2.5.3}$$

where $\mathbf{1}_d$ is the vector in $\mathbb{R}^d$ with all components equal to 1.

Remark 2.5.1. It is possible that after default times, some assets may not be traded any more. Now suppose that after k defaults, there are $\bar{d}$ assets still tradable, where $0 \leq \bar{d} \leq d$. Then without loss of generality, we may assume $b^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = \left(\bar{b}(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\ 0\right)$, $\sigma^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = \left(\bar{\sigma}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\ 0\right)$, $\gamma^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k, e) = \left(\bar{\gamma}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k, e)\ 0\right)$, where

$\bar{b}(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$, $\bar{\sigma}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$, $\bar{\gamma}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k, e)$ are $\mathbb{F}$-predictable processes valued respectively in $\mathbb{R}^{\bar{d}}, \mathbb{R}^{\bar{d}^k \times m}, \mathbb{R}^{\bar{d}}$. In this case, we shall also assume that the volatility matrix $\bar{\sigma}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$ is of full rank. we can then define the risk premium

$$\lambda^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = \bar{\sigma}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)' \left(\bar{\sigma}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\bar{\sigma}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)'\right)^{-1} \bar{b}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k),$$

which satisfies (2.5.2).

An American option of maturity T is modeled by a $\mathbb{G}$-optional process $R \sim (R^0, \ldots, R^n)$ from (2.3.4), where $R^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$ is continuous with respect to t, and represents the payoff if the option is exercised at time $t \in [\theta_k, T]$ in the k-default scenario, given the past default events $(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) = (\boldsymbol{\theta}_k, \boldsymbol{e}_k)$, for $k = 0, \ldots, n$.

A control in the d-asset portfolio is a $\mathbb{G}$-predictable process $\pi \sim (\pi^0, \ldots, \pi^n)$, where $\pi^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$ is valued in a closed set A^k of $\mathbb{R}^d$ containing the zero element, and represents the amount invested continuously in the d assets in the k-default scenario, given the past default information $(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) = (\boldsymbol{\theta}_k, \boldsymbol{e}_k)$. An exercise time is a $\mathbb{G}$-stopping time $\tau \sim (\tau^0, \ldots, \tau^n)$ satisfying $\tau \leq T$, with the decomposition from Proposition 2.2.4. A trading strategy is a pair of a control and an exercise time.

For a trading strategy $(\pi, \tau) \sim (\pi^k, \tau^k)_{k=0}^n$, we have the corresponding wealth process $\mathfrak{X} \sim (\mathfrak{X}^0, \ldots, \mathfrak{X}^n)$, where $\mathfrak{X}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$, representing the wealth controlled by $\pi^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$ in the price process $S^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$, given the past default events $(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) = (\boldsymbol{\theta}_k, \boldsymbol{e}_k)$. From (2.5.1) we have

$$d\mathfrak{X}^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = \pi^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k)' \left(b^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k)dt + \sigma^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)dW_t\right), \quad t \geq \theta_k.$$

Moreover, each default time induces a jump in the asset price process, and then also on the wealth process. From (2.5.3), we have

$$\mathfrak{X}^{k+1}_{\theta_{k+1}}(\boldsymbol{\theta}_{k+1}, \boldsymbol{e}_{k+1}) = \mathfrak{X}^k_{\theta^-_{k+1}}(\boldsymbol{\theta}_k, \boldsymbol{e}_k) + \pi^k_{\theta_{k+1}}(\boldsymbol{\theta}_k, \boldsymbol{e}_k)'\gamma^k_{\theta_{k+1}}(\boldsymbol{\theta}_k, \boldsymbol{e}_k, e_{k+1}). \tag{2.5.4}$$

2.5.2 Indifference price

Let U be an exponential utility with risk aversion coefficient $p > 0$:

$$U(x) = -\exp(-px), \quad x \in \mathbb{R},$$

which describes an investor's preference. We will consider two cases. The first case is that the investor can trade the d-assets portfolio following control π, associated to a wealth process $\mathfrak{X} = \mathfrak{X}^{x,\pi}$ with initial capital $\mathfrak{X}_{0^-} = x$. Besides, she holds an American option and can choose to exercise it at any time τ, $\tau \leq T$, to get payoff R_τ. So the maximum utility she can get (or as close as she want, if not attainable) is:

$$V^0(x) = \sup_\tau \sup_\pi \mathbb{E}\left[U(\mathfrak{X}^{x,\pi}_\tau + R_\tau)\right]. \tag{2.5.5}$$

We call $\bar{c}$ the indifference buying price of the American option, if

$$U(x) = V^0(x - \bar{c}).$$

The second case is that the investor trades the d-asset portfolio following control π, while shorting an American option. So she has to deliver the payoff R_τ at some exercise time τ, which is chosen by the holder of the option. By considering the worst scenario, the maximum utility she can get (or as close as she want) is:

$$\mathfrak{V}^0(x) = \sup_\pi \inf_\tau \mathbb{E}\left[U(\mathfrak{X}^{x,\pi}_\tau - R_\tau)\right]. \tag{2.5.6}$$

In this case, we call $\underline{c}$ the indifference selling price of the American option, if

$$U(x) = \mathfrak{V}^0(x + \underline{c}).$$

2.5.3 Indifference buying price

In this sub-section, we will focus on the problem (2.5.5). Theorem 2.5.4 is the main result for this sub-section.

Definition 2.5.2. (Admissible trading strategy) A trading strategy $(\pi, \tau) \sim (\pi^k, \tau^k)_{k=0}^n$ is admissible, if for any $(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$, under the control π^k,

(a) $\int_{\theta_k}^{\tau^k} |\pi_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)' b_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)| dt + \int_{\theta_k}^{\tau^k} |\pi_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)' \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)|^2 dt < \infty$, a.s., $k = 0, \dots, n$,

(b) the family $\left\{U(\mathfrak{X}_{\tau \wedge \tau^k}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)) : \ \tau \text{ is any } \mathbb{F}-\text{stopping time valued in } [\theta_k, T]\right\}$ is uniformly integrable, i.e., $U(\mathfrak{X}_{\cdot \wedge \tau^k}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k))$ is of class (D), for $k = 0, \dots, n$,

(c) $\mathbb{E}\left[\int_{\theta_k}^{\tau^k} \int_E (-U)\left(\mathfrak{X}_s^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) + \pi_s^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)' \gamma_s^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k, e)\right) \eta(de) ds\right] < \infty$, for $k = 0, \dots, n-1$.

The notation $\mathcal{A}_{\mathbb{G}}$, $\mathcal{T}_{\mathbb{G}}$, $\mathcal{A}^k$ and $\mathcal{T}^k$ from Section 2.4 are now specified by the above definition. From Theorem 2.4.2, V^0 in (2.5.5) can be calculated by the following backward induction:

$$V^n(x, \boldsymbol{\theta}_n, \boldsymbol{e}_n) = \operatorname*{ess\,sup}_{\tau^n \in \mathcal{T}^n} \operatorname*{ess\,sup}_{\pi^n \in \mathcal{A}^n} \mathbb{E}\Big[U(\mathfrak{X}_{\tau^n}^{n,x} + H_{\tau^n}^n) | \mathcal{F}_{\theta_n}\Big], \tag{2.5.7}$$

$(\boldsymbol{\theta}_n, \boldsymbol{e}_n) \in \Delta_n(T) \times E^n$, and

$$\begin{aligned} V^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k) = \operatorname*{ess\,sup}_{\tau^k \in \mathcal{T}^k} \operatorname*{ess\,sup}_{\pi^k \in \mathcal{A}^k} \mathbb{E}\Big[& U(\mathfrak{X}_{\tau^k}^{k,x} + H_{\tau^k}^k) \\ & + \int_{\theta_k}^{\tau^k} \int_E V^{k+1}(\mathfrak{X}_{\theta_{k+1}}^{k,x} + \pi_{\theta_{k+1}}^k \cdot \gamma_{\theta_{k+1}}^k(e_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{e}_{k+1}) \eta(de_{k+1}) d\theta_{k+1} | \mathcal{F}_{\theta_k}\Big], \end{aligned} \tag{2.5.8}$$

$(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$, for $k = 0, \dots, n-1$, where

$$H^k := R^k - \frac{1}{p} \ln \alpha^k,$$

in which α^k is given by (2.3.3).

Backward recursive system of RBSDEs

Following [46], we expect the value function to be of the following form:

$$V^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k) = U\left(x + Y_{\theta_k}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\right), \tag{2.5.9}$$

where $Y^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$ is an $\mathbb{F}$-adapted process, satisfying the RBSDE $\mathbf{eq}(H^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k), f^k)_{\theta_k \le t \le T}$, with f^k defined as

$$f^k(t, y, z, \boldsymbol{\theta}_k, \boldsymbol{e}_k) = \inf_{\pi \in A^k} g^k(\pi, t, y, z, \boldsymbol{\theta}_k, \boldsymbol{e}_k), \tag{2.5.10}$$

where

$$\begin{aligned}
g^k(\pi, t, y, z, \boldsymbol{\theta}_k, \boldsymbol{e}_k) &= \frac{p}{2}\left|z - \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)'\pi\right|^2 - b_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)'\pi \\
&\qquad + \frac{1}{p}U(-y)\int_E U\left(\pi \cdot \gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k, e) + Y_t^{k+1}(\boldsymbol{\theta}_k, t, \boldsymbol{e}_k, e)\right)\eta(de) \\
&= -\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \cdot z - \frac{1}{2p}|\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)|^2 + \frac{p}{2}\left|z + \frac{1}{p}\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) - \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)'\pi\right|^2 \\
&\quad + \frac{1}{p}U(-y)\int_E U\big(\pi \cdot \gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k, e) + Y_t^{k+1}(\boldsymbol{\theta}_k, t, \boldsymbol{e}_k, e)\big)\eta(de),
\end{aligned}$$

for $k = 0, \dots, n-1$, and

$$\begin{aligned}
g^n(\pi, t, y, z, \boldsymbol{\theta}_n, \boldsymbol{e}_n) &= \frac{p}{2}\left|z - \sigma_t^n(\boldsymbol{\theta}_n, \boldsymbol{e}_n)'\pi\right|^2 - b_t^n(\boldsymbol{\theta}_n, \boldsymbol{e}_n)'\pi \\
&= -\lambda_t^n(\boldsymbol{\theta}_n, \boldsymbol{e}_n) \cdot z - \frac{1}{2p}|\lambda_t^n(\boldsymbol{\theta}_n, \boldsymbol{e}_n)|^2 + \frac{p}{2}\left|z + \frac{1}{p}\lambda_t^n(\boldsymbol{\theta}_n, \boldsymbol{e}_n) - \sigma_t^n(\boldsymbol{\theta}_n, \boldsymbol{e}_n)'\pi\right|^2.
\end{aligned}$$

In the next two subsections, we will show that: (a) The backward recursive system of RBSDEs admits a solution; (b) The solution characterizes the values of (V^k), i.e., (2.5.9) holds.

Existence to the recursive system of RBSDEs

We make the following boundedness assumptions **(HB)**:

(i) The risk premium is bounded uniformly with respect to its indices: there exists a constant $C > 0$, such that for any $k = 0, \dots, n$, $(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$, $t \in [\theta_k, T]$,

$$|\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)| \le C, \quad \text{a.s.}$$

(ii) The indexed random variables $(H_t^k)_k$ are bounded uniformly in time and their indices: there exists a constant $C > 0$ such that for any $k = 0, \ldots, n$, $(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$, $t \in [\theta_k, T]$,

$$|H_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)| \leq C, \quad \text{a.s.}$$

Theorem 2.5.3. *Under **(HB)**, there exists a solution* $(Y^k, Z^k, K^k)_{k=0}^n \in \prod_{k=0}^n \mathcal{S}_c^\infty$ $(\Delta_k(T), E^k) \times \mathbf{L}_W^2$ $(\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ *to the recursive system of indexed RBSDEs* $\boldsymbol{eq}(H^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k), f^k)_{\theta_k \leq t \leq T}$*,* $k = 0, \ldots, n$*.*

Proof. We prove the result by a backward induction on $k = 0, \ldots, n$. The positive constant C may vary from line to line, but is always independent of $(t, \omega, \boldsymbol{\theta}_k, \boldsymbol{e}_k)$. We will often omit the dependence of $(t, \omega, y, z, \boldsymbol{\theta}_k, \boldsymbol{e}_k)$ in related functions.

(a) For $k = n$. Under **(HB)**, $|f^n| \leq C(|z|^2 + 1)$. By [59, Theorem 1], there exists a solution $\big(Y^n(\boldsymbol{\theta}_n, \boldsymbol{e}_n), Z^n(\boldsymbol{\theta}_n, \boldsymbol{e}_n), K^n\ (\boldsymbol{\theta}_n, \boldsymbol{e}_n)\big) \in \mathcal{S}_c^\infty[\theta_n, T] \times \mathbf{L}_W^2[\theta_n, T] \times \mathbf{A}[\theta_n, T]$ for $\mathbf{eq}(H^n, f^n)_{\theta_n \leq t \leq T}$, satisfying $|Y^n| \leq C$. Moreover, the measurability of (Y^n, Z^n) with respect to $(\boldsymbol{\theta}_n, \boldsymbol{e}_n)$ follows from the measurability of H^n and f^n (see [58, Appendix C] and use the fact that the solution to the RBSDE can be eventually approximated by the solutions to BSDEs). Therefore, $(Y^n, Z^n, K^n) \in \mathcal{S}_c^\infty(\Delta_n(T), E^n) \times \mathbf{L}_W^2(\Delta_n(T), E^n) \times \mathbf{A}(\Delta_n(T), E^n)$.

(b) For $k \in \{0, 1, \ldots, n-1\}$. Assume there exists $(Y^{k+1}, Z^{k+1}, K^{k+1}) \in \mathcal{S}_c^\infty(\Delta_{k+1}(T), E^{k+1}) \times \mathbf{L}_W^2(\Delta_{k+1}(T), E^{k+1}) \times \mathbf{A}(\Delta_{k+1}(T), E^{k+1})$ satisfying $\mathbf{eq}(H^{k+1}, f^{k+1})$. Since $Y^{k+1} \in \mathcal{P}_{\mathbb{F}}(\Delta_{k+1},\ E^{k+1})$, the generator in (2.5.10) is well defined. In order to overcome the technical difficulties coming from the exponencial term in $U(-y)$, we first consider the truncated generator

$$f^{k,N}(t, y, z, \boldsymbol{\theta}_k, \boldsymbol{e}_k) = \inf_{\pi \in A^k} g^k(\pi, t, N \wedge y, z, \boldsymbol{\theta}_k, \boldsymbol{e}_k).$$

Then there exists a positive constant C_N independent of $(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$, such that $|f^{k,N}| \leq C_N(1+z^2)$. Applying [59, Theorem 1], there exists a solution $(Y^{k,N}, Z^{k,N}, K^{k,N}) \in \mathcal{S}_c^\infty(\Delta_k(T), E^k) \times \mathbf{L}_W^2(\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ to $\mathbf{eq}(H^k, f^{k,N})$.

Now we will show that $Y^{k,N}$ has a uniform upper bound. Consider the generator

$$\bar{f}^k(t, y, z, \boldsymbol{\theta}_k, \boldsymbol{e}_k) := -\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \cdot z - \frac{1}{2p}|\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)|^2,$$

which satisfies the Lipschitz condition in (y, z), uniformly in (t, ω). Then by [40, Theorem 5.2], there exists a unique solution $\big(\bar{Y}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k), \bar{Z}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k), \bar{K}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\big) \in \mathcal{S}_c^\infty[\theta_k, T] \times \mathbf{L}_W^2[\theta_k, T] \times \mathbf{A}[\theta_k, T]$ satisfying $|\bar{Y}^k| \leq C$ (see [59, Theorem 1] for the boundedness). Applying in [59, Lemma 2.1(comparison)], we get $Y^{k,N} \leq \bar{Y}^k$. Hence, $Y^{k,N}$ has a uniform upper bound independent of N and $(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$. Therefore, for N large enough, we can remove "N" in the truncated generator $f^{k,N}$, i.e., $(Y^{k,N}, Z^{k,N}, K^{k,N})$ solves $\mathbf{eq}(H^k, f^k)$ for large enough N. $\square$

RBSDE characterization by verification theorem

Theorem 2.5.4. *The value functions $(V^k)_{k=0}^n$, defined in (2.5.7) and (2.5.8), are given by*

$$V^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k) = U\big(x + Y_{\theta_k}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\big), \tag{2.5.11}$$

for $\forall x \in \mathbb{R}$, $(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$, where $(Y^k, Z^k, K^k)_{k=0}^n \in \prod_{k=0}^n \mathcal{S}_c^\infty(\Delta_k(T), E^k) \times \mathbf{L}_W^2(\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ is a solution of the RBSDE system $\boldsymbol{eq}(H^k, f^k)$, $k = 0, \dots, n$. Moreover, there exists an optimal trading strategy $(\pi, \tau) \sim (\hat{\pi}^k, \hat{\tau}^k)_{k=0}^n$ described by:

$$\hat{\pi}_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \operatorname*{arg\,min}_{\pi \in A^k} g^k\big(\pi, t, Y_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k), Z_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k), \boldsymbol{\theta}_k, \boldsymbol{e}_k\big),$$

for $t \in [\theta_k, T]$, and

$$\hat{\tau}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) := \inf\big\{t \geq \theta_k : \ Y_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = H_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\big\}, \tag{2.5.12}$$

for $(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$, *a.s.*, $k = 0, \dots, n$.

Proof. **Step 1: We will show**

$$U(x + Y^k_{\theta_k}(\boldsymbol{\theta}_k, \boldsymbol{e}_k)) \geq V^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k), \quad k = 0, \dots, n. \tag{2.5.13}$$

Let $(Y^k, Z^k, K^k) \in \mathcal{S}^\infty_c(\Delta_k(T), E^k) \times \mathbf{L}^2_W(\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ be a solution of the RBSDE system. For $(\nu^k, \tau^k) \in \mathcal{A}^k \times \mathcal{T}^k$, $x \in \mathbb{R}$, $(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$ and $t \geq \theta_k$, and define

$$\begin{aligned}
\xi^k_t(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k, \nu^k) \quad &:= \quad U(\mathfrak{X}^{k,x}_t + Y^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k)) + \int_{\theta_k}^t \int_E U\Big(\mathfrak{X}^{k,x}_r + \nu^k_r \cdot \gamma^k_r(\boldsymbol{\theta}_k, e_k, e) \\
&\qquad + Y^{k+1}_r(\boldsymbol{\theta}_k, r, \boldsymbol{e}_k, e)\Big)\eta(de)dr, \quad k = 0, \dots, n-1, \\
\xi^n_t(x, \boldsymbol{\theta}_n, \boldsymbol{e}_n, \nu^n) \quad &:= \quad U(\mathfrak{X}^{n,x}_t + Y^n_t(\boldsymbol{\theta}_n, \boldsymbol{e}_n)).
\end{aligned}$$

Applying Itô's formula, we get for $k = 0, \dots, n$,

$$\begin{aligned}
\xi^k_t(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k, \nu^k) = pU\left(\mathfrak{X}^{k,x}_t + Y^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\right)\Big[\big(-f^k(t, Y^k_t, Z^k_t, \boldsymbol{\theta}_k, \boldsymbol{e}_k) \\
+ g^k(\nu^k_t, t, Y^k_t, Z^k_t, \boldsymbol{\theta}_k, \boldsymbol{e}_k)\big)dt + dK^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k) + (Z^k_t - \sigma^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k)'\nu^k_t) \cdot dW_t\Big].
\end{aligned}$$

$f^k(\cdot) = \inf_{\pi \in A^k} g^k(\pi, \cdot)$ implies $\left\{\xi^k_s(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k, \nu^k)\right\}_{\theta_k \leq t \leq T}$ is a local super-martingale, for $k = 0, \dots, n$. Since Y^k and Y^{k+1} are essentially bounded, and $\xi^k_{t \wedge \tau^k \wedge \rho_m}(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k, \nu^k)$ is uniformly integrable, by considering a localizing sequence of stopping times, we can show $\left\{\xi^k_{t \wedge \tau^k}(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k, \nu^k)\right\}_{\theta_k \leq t \leq T}$ is a super-martingale. Consider when $k = n$. Since $Y^n \geq H^n$, we have

$$U\left(x + Y^n_{\theta_n}(\boldsymbol{\theta}_n, \boldsymbol{e}_n)\right) \geq \mathbb{E}\left[U\left(\mathfrak{X}^{n,x}_{\tau^n} + H^n_{\tau^n}(\boldsymbol{\theta}_n, \boldsymbol{e}_n)\right)|\mathcal{F}_{\theta_n}\right]. \tag{2.5.14}$$

Therefore, (2.5.13) holds for $k = n$. Similarly, it holds for $k = 0, \dots, n-1$.

Step 2: $\int_{\theta_k}^{\cdot} Z^k_s(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \cdot dW_s$ **is a BMO-martingale.** Apply Itô's formula to

$\exp(-qY_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k))$ with $q > p$ and any $\mathbb{F}$-stopping time τ valued in $[\theta_k, T]$,

$$
\begin{aligned}
\frac{1}{2}q(q-p)\mathbb{E}&\left[\int_\tau^T \exp\left(-qY_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\right)|Z_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)|^2 dt \Big| \mathcal{F}_\tau\right] \\
&= q\mathbb{E}\left[\int_\tau^T \exp\left(-qY_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\right)\left(f^k(t, Y_t^k, Z_t^k, \boldsymbol{\theta}_k, \boldsymbol{e}_k) - \frac{p}{2}|Z_t^k|^2\right) dt \Big| \mathcal{F}_\tau\right] \\
&\quad + \mathbb{E}\left[\exp\left(-qY_T^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\right) - \exp\left(-qY_\tau^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\right)\Big| \mathcal{F}_\tau\right] \\
&\quad - q\mathbb{E}\left[\int_\tau^T \exp\left(-qY_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\right) dK_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\Big| \mathcal{F}_\tau\right].
\end{aligned}
$$

Since $|f^k(t, y, z, \boldsymbol{\theta}_k, \boldsymbol{e}_k)| \leq \frac{p}{2}|z|^2 - CU(-y)$, $dK^k \geq 0$ and Y^k is bounded, we have

$$
\begin{aligned}
\frac{1}{2}q(q-p)\mathbb{E}&\left[\int_\tau^T \exp\left(-qY_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\right)|Z_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)|^2 dt \Big| \mathcal{F}_\tau\right] \\
&\leq qC\mathbb{E}\left[\int_\tau^T \exp\left(-qY_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\right) dt \Big| \mathcal{F}_\tau\right] + C.
\end{aligned}
$$

By choosing q large enough, we have

$$
\mathbb{E}\left[\int_\tau^T \left|Z_s^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\right|^2 ds \Big| \mathcal{F}_\tau\right] \leq C,
$$

which implies $\int_{\theta_k}^{\cdot} Z_s^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \cdot dW_s$ is a BMO-martingale.

Step 3: Adimissibility of $(\hat{\pi}^k, \hat{\tau}^k)$. For $k = 0, \ldots, n$, define function $\hat{g}^k$ by

$$
\hat{g}^k(\pi, t, \omega, \boldsymbol{\theta}_k, \boldsymbol{e}_k) = g^k\big(\pi, t, Y_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k), Z_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k), \boldsymbol{\theta}_k, \boldsymbol{e}_k\big).
$$

We can show that the map $(\pi, t, \omega, \boldsymbol{\theta}_k, \boldsymbol{e}_k) \to \hat{g}^k(\pi, t, \omega, \boldsymbol{\theta}_k, \boldsymbol{e}_k)$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E_k)$-measurable. Now for $k = 0, \ldots, n$, $(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$, if either $\sigma^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = 0$ or $\gamma^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k, e) = 0$, then the continuous function $\pi \to \hat{g}^k(\pi, t, \omega, \boldsymbol{\theta}_k, \boldsymbol{e}_k)$ attains trivially its infimum of $\hat{g}^k$ when $\pi = 0$. Otherwise, $\sigma^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$ and $\gamma^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k, e)$ are in the form $\sigma^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = (\bar{\sigma}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k), 0)$, $\gamma^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = (\bar{\gamma}(\boldsymbol{\theta}_k, \boldsymbol{e}_k), 0)$ for some full rank matrix $\bar{\sigma}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$. In this case, we let $(\bar{\pi}, 0) = (\sigma^k)' \cdot \pi$, then we get

$$
\begin{aligned}
\bar{g}^k(\bar{\pi}, t, \omega, \boldsymbol{\theta}_k, \boldsymbol{e}_k) := \hat{g}^k(\pi, t, \omega, \boldsymbol{\theta}_k, \boldsymbol{e}_k) &= \frac{p}{2}\left|Z_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) + \frac{1}{p}\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) - \bar{\pi}\right|^2 \\
&\quad + \frac{1}{p}U(-Y_t^k)\int_E U\left(((\bar{\sigma}^k)')^{-1} \cdot \bar{\pi} \cdot \bar{\gamma}_t^k(e) + Y_t^{k+1}(\boldsymbol{\theta}_k, t, \boldsymbol{e}_k, e)\right)\eta(de),
\end{aligned}
$$

for $k = 0, \dots, n-1$, and

$$\bar{g}^n(\bar{\pi}, t, \omega, \boldsymbol{\theta}_n, \boldsymbol{e}_n) := \hat{g}^n(\pi, t, \omega, \boldsymbol{\theta}_n, \boldsymbol{e}_n) = \frac{p}{2}\left| Z_t^n(\boldsymbol{\theta}_n, \boldsymbol{e}_n) + \frac{1}{p}\lambda_t^n(\boldsymbol{\theta}_n, \boldsymbol{e}_n) - \bar{\pi} \right|^2.$$

Since

$$\bar{g}^k(0, t, \omega, \boldsymbol{\theta}_k, \boldsymbol{e}_k) < \liminf_{|\bar{\pi}| \to \infty} \bar{g}^k(\bar{\pi}, t, \omega, \boldsymbol{\theta}_k, \boldsymbol{e}_k),$$

the continuous function $\bar{\pi} \to \bar{g}^k(\bar{\pi}, t, \omega, \boldsymbol{\theta}_k, \boldsymbol{e}_k)$ attains its infimum over the closed set $(\sigma_t^k)' A^k$, and thus the function $\pi \to \hat{g}^k(\pi, t, \omega, \boldsymbol{\theta}_k, \boldsymbol{e}_k)$ attains its infimum over $A^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$. For $k = 0, \dots, n$, using a measurable selection argument (see [81]), one can show that there exists $\hat{\pi}^k \in \mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$, such that

$$\hat{\pi}_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \underset{\pi \in A^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)}{\arg\min}\ \hat{g}^k(\pi, t, \boldsymbol{\theta}_k, \boldsymbol{e}_k), \quad \theta_k \le t \le T, \quad \text{a.s.}$$

Consider $\hat{\tau}^k$ defined in (2.5.12). For $k = 0, \dots, n$, define $\tilde{\tau}^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k)$ as

$$\tilde{\tau}^k := \Big(\inf\{t \ge \zeta_k :\ Y^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) = H^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k)\} \wedge T \Big) \cdot 1_{\{\zeta_k \le T\}} + \zeta_k \cdot 1_{\{\zeta_k > T\}}.$$

We can show that $\tilde{\tau}^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k)$ is a $\mathbb{G}^k$ stopping time satisfying $\tilde{\tau}^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) \ge \zeta_k$ and $\{\tilde{\tau}^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) \le T\} = \{\zeta_k \le T\}$. And given $(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) = (\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$, $\tilde{\tau}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = \hat{\tau}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$. Now we will show that $(\hat{\pi}^k, \hat{\tau}^k)_{k=0}^n$ is admissible in the sense of Definition 2.5.2.

(a) Since $\hat{g}^k(\hat{\pi}_t^k, t, \boldsymbol{\theta}_k, \boldsymbol{e}_k) \le \hat{g}^k(0, t, \boldsymbol{\theta}_k, \boldsymbol{e}_k)$, there exists a constant $C > 0$, such that

$$|\sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)' \hat{\pi}_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)| \le C(1 + |Z_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)|), \quad \theta_k \le t \le T, \quad \text{a.s.},$$

for all $(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k,\ k = 0, \dots, n$. Since $Z^k \in \mathbf{L}_W^2(\Delta_k, E^k)$ and because of **(HB)**(i), $(\hat{\pi}^k, \hat{\tau}^k)_{k=0}^n$ satisfies condition (a) in Definition 2.5.2.

(b) Denote by $\hat{\mathfrak{X}}^{k.x}$ the wealth process controlled by $\hat{\pi}^k$, starting from x at time θ_k. We have

$$f^k(t, Y_t^k, Z_t^k, \boldsymbol{\theta}_k, \boldsymbol{e}_k) = g^k(\hat{\pi}_t^k, t, Y_t^k, Z_t^k, \boldsymbol{\theta}_k, \boldsymbol{e}_k),$$

for $k=0,\dots,n$. Then for $\theta_k \le t \le T$,

$$U(\hat{\mathfrak{X}}_t^{k,x} + Y_t^k) = U(x + Y_{\theta_k}^k)\ \mathcal{E}_t^k\left(p(Z^k - (\sigma^k)'\hat{\pi}^k)\right) R_t^k,$$

where

$$\mathcal{E}_t^k\left(p(Z^k - (\sigma^k)'\hat{\pi}^k)\right) = \exp\left(p\int_{\theta_k}^t (Z_s^k - (\sigma_s^k)'\hat{\pi}_s^k)\cdot dW_s - \frac{p^2}{2}\int_{\theta_k}^t |Z_s^k - (\sigma_s^k)'\hat{\pi}_s^k|^2 ds\right),$$

for $k=0,\dots,n$, and

$$R_t^k = \exp\left(pK_t^k - \int_{\theta_k}^t U(-Y_s^k)\int_E U\left(\hat{\pi}_t^k\cdot\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{e}_k) + Y_t^{k+1}(\boldsymbol{\theta}_k,t,\boldsymbol{e}_k,e)\right)\eta(de)ds\right),$$

for $k=0,\dots,n-1$ and $R_t^n = \exp(pK_t^n)$. From Step 2, $\int_{\theta_k}^{\cdot} p(Z^k - (\sigma^k)\hat{\pi}^k)\cdot dW$ is a BMO-martingale and hence $\mathcal{E}_{\cdot\wedge\hat{\tau}^k}^k\left(p(Z^k - (\sigma^k)'\hat{\pi}^k)\right)$ is of class (D). Moreover, since U is nonpositive and $K_t^k = 0$ when $t \le \hat{\tau}^k$, we have $|R_{\cdot\wedge\hat{\tau}^k}| \le 1$, and thus $U(\hat{\mathfrak{X}}_{t\wedge\hat{\tau}^k}^{k,x} + Y_{t\wedge\hat{\tau}^k}^k)$ is of class (D). So is $U(\hat{\mathfrak{X}}_{\cdot\wedge\hat{\tau}^k}^{k,x})$ since Y^k is essentially bounded.

(c) Because $dK_t^k = 0$ when $t \le \hat{\tau}^k$, the process $\xi_{\cdot\wedge\hat{\tau}^k}^k(x,\boldsymbol{\theta}_k,\boldsymbol{e}_k,e)$ defined in Step 1 under control $\hat{\pi}^k$ is a local martingale. By considering a localizing $\mathbb{F}$-stopping time sequence $(\rho_m)_m$ valued in $[\theta_k,T]$, we obtain:

$$\begin{aligned}
&\mathbb{E}\left[\int_{\theta_k}^{\hat{\tau}^k\wedge\rho_m}\int_E (-U)\left(\hat{\mathfrak{X}}_t^{k,x} + \hat{\pi}_t^k\cdot\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{e}_k,e) + Y_t^{k+1}(\boldsymbol{\theta}_k,t,\boldsymbol{e}_k,e)\right)\eta(de)dt\right]\\
&= \mathbb{E}\left[U(\hat{\mathfrak{X}}_{\hat{\tau}^k\wedge\rho_m}^{k,x} + Y_{\hat{\tau}^k\wedge\rho_m}^k) - U(x + Y_{\theta_k}^k)\right] \le \mathbb{E}\left[-U(x+Y_{\theta_k}^k)\right],
\end{aligned}$$

By Fatou's lemma, we get Condition (c) in Definition 2.5.2 holds.

Step 4: We will show (2.5.11) **holds and** $(\hat{\pi}^k,\hat{\tau}^k)_{k=0}^n$ **is an optimal trading strategy.** Consider when $k=n$. By the admissibility of $(\hat{\pi}^n,\hat{\tau}^n)$, the local martingale $\xi_{t\wedge\hat{\tau}^n}$ under the control $\hat{\pi}^n$ is a martingale. Thus,

$$U(x + Y_{\theta_n}^n) = \mathbb{E}\left[U(\hat{\mathfrak{X}}_{\hat{\tau}^n}^{n,x} + H_{\hat{\tau}^n}^n)\middle|\mathcal{F}_{\theta_n}\right].$$

Along with (2.5.14) this results in

$$V^n(x,\boldsymbol{\theta}_n,\boldsymbol{e}_n) = \operatorname*{ess\,sup}_{\tau^n\in\mathcal{T}^n}\operatorname*{ess\,sup}_{\pi^n\in\mathcal{A}^n}\mathbb{E}\left[U(\mathfrak{X}^{n,x}_{\tau^n}+H^n_{\tau^n}(\boldsymbol{\theta}_n,\boldsymbol{e}_n)\middle|\mathcal{F}_{\theta_n}\right] \le U(x+Y^n_{\theta_n}(\boldsymbol{\theta}_n,\boldsymbol{e}_n))$$
$$= \mathbb{E}\left[U(\hat{\mathfrak{X}}^{n,x}_{\hat{\tau}^n}+H^n_{\hat{\tau}^n}(\boldsymbol{\theta}_n,\boldsymbol{e}_n))\middle|\mathcal{F}_{\theta_n}\right] \le V^n(x,\boldsymbol{\theta}_n,\boldsymbol{e}_n),$$

which implies (2.5.11) for $k=n$ and the optimality of $(\hat{\pi}^n,\hat{\tau}^n)$. We can show (2.5.11) and the optimality of $(\hat{\pi}^k,\hat{\tau}^k)$ for $k=0,\dots,n-1$, similarly using (2.5.8). □

2.5.4 Indifference selling price

In this sub-section, we consider the problem (2.5.6), and Theorem 2.5.8 is the main result.

Definition 2.5.5. (Admissible trading strategy) A trading strategy $(\pi,\tau)\sim(\pi^k,\tau^k)^n_{k=0}$ is admissible, if for any $(\boldsymbol{\theta}_k,\boldsymbol{e}_k)\in\Delta_k(T)\times E^k$, under the control π^k,

(a) $\displaystyle\int_{\theta_k}^T |\pi^k_t(\boldsymbol{\theta}_k,\boldsymbol{e}_k)'b^k_t(\boldsymbol{\theta}_k,\boldsymbol{e}_k)|dt + \int_{\theta_k}^T |\pi^k_t(\boldsymbol{\theta}_k,\boldsymbol{e}_k)'\sigma^k_t(\boldsymbol{\theta}_k,\boldsymbol{e}_k)|^2dt < \infty$, a.s., $k=0,\dots,n$,

(b) the family $\Big\{U(\mathfrak{X}^k_\tau(\boldsymbol{\theta}_k,\boldsymbol{e}_k)) : \tau \text{ is any } \mathbb{F}-\text{stopping time valued in } [\theta_k,T]\Big\}$ is uniformly integrable, i.e., $U(\mathfrak{X}^k(\boldsymbol{\theta}_k,\boldsymbol{e}_k))$ is of class (D), for $k=0,\dots,n$,

(c) $\displaystyle\mathbb{E}\left[\int_{\theta_k}^T\int_E(-U)\left(\mathfrak{X}^k_s(\boldsymbol{\theta}_k,\boldsymbol{e}_k)+\pi^k_s(\boldsymbol{\theta}_k,\boldsymbol{e}_k)'\gamma^k_s(\boldsymbol{\theta}_k,\boldsymbol{e}_k,e)\right)\eta(de)ds\right]<\infty$, for $k=0,\dots,n-1$.

Remark 2.5.6. Unlike in Definition 2.5.2, the admissible trading strategy here is in fact independent of stopping times. This is because the investor cannot choose when to stop.

Backward recursive system of RBSDEs

We decompose $\mathfrak{V}^0$ in (2.5.6) into a backward induction as before:

$$\mathfrak{V}^n(x,\boldsymbol{\theta}_n,\boldsymbol{e}_n) = \operatorname*{ess\,sup}_{\pi^n\in\mathcal{A}^n}\operatorname*{ess\,inf}_{\tau^n\in\mathcal{T}^n}\mathbb{E}\Big[U(\mathfrak{X}^{n,x}_{\tau^n}-\mathcal{H}^n_{\tau^n})|\mathcal{F}_{\theta_n}\Big], \tag{2.5.15}$$

$(\boldsymbol{\theta}_n, \boldsymbol{e}_n) \in \Delta_n(T) \times E^n$, and

$$(2.5.16) \quad \mathfrak{V}^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k) = \operatorname*{ess\,sup}_{\pi^k \in \mathcal{A}^k} \operatorname*{ess\,inf}_{\tau^n \in \mathcal{T}^k} \mathbb{E}\Big[U(\mathfrak{X}^{k,x}_{\tau^k} - \mathcal{H}^k_{\tau^k})$$
$$+ \int_{\theta_k}^{\tau^k} \int_E \mathfrak{V}^{k+1}\Big(\mathfrak{X}^{k,x}_{\theta_{k+1}} + \pi^k_{\theta_{k+1}} \cdot \gamma^k_{\theta_{k+1}}(e_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{e}_{k+1}\Big)\eta(de_{k+1})d\theta_{k+1}|\mathcal{F}_{\theta_k}\Big],$$

$(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$, for $k = 0, \ldots, n-1$, where

$$\mathcal{H}^k = R^k + \frac{1}{p}\ln \alpha^k, \quad k = 0, \ldots, n.$$

Consider

$$\mathfrak{V}^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k) = U\big(x - \mathcal{Y}^k_{\theta_k}(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\big), \quad k = 0, \ldots, n,$$

where $\{\mathcal{Y}^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\}_{k=0}^n$ satisfies the RBSDE $\mathbf{EQ}(\mathcal{H}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k), \mathfrak{f}^k)_{\theta_k \le t \le T}$, with $\mathfrak{f}^k$ defined as

$$\mathfrak{f}^k(t, y, z, \boldsymbol{\theta}_k, \boldsymbol{e}_k) = \inf_{\pi \in A^k} \mathfrak{g}^k(\pi, t, y, z, \boldsymbol{\theta}_k, \boldsymbol{e}_k),$$

where

$$\mathfrak{g}^k(\pi, t, y, z, \boldsymbol{\theta}_k, \boldsymbol{e}_k) = \frac{p}{2}\left|z - \sigma^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k)'\pi\right|^2 - b^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k)'\pi$$
$$+ \frac{1}{p}U(y)\int_E U\left(\pi \cdot \gamma^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k, e) - \mathcal{Y}^{k+1}_t(\boldsymbol{\theta}_k, t, \boldsymbol{e}_k, e)\right)\eta(de)$$

for $k = 0, \ldots, n-1$, and

$$\mathfrak{g}^n(\pi, t, y, z, \boldsymbol{\theta}_n, \boldsymbol{e}_n) = \frac{p}{2}\left|z - \sigma^n_t(\boldsymbol{\theta}_n, \boldsymbol{e}_n)'\pi\right|^2 - b^n_t(\boldsymbol{\theta}_n, \boldsymbol{e}_n)'\pi.$$

Existence to the recursive system of RBSDEs

We will make the same boundedness assumption as **(HB)** in Section 2.5.3 except that we will replace H^k with $\mathcal{H}^k$. Let us denote this assumption by (**HB'**).

Theorem 2.5.7. *Under **(HB')**, there exists a solution* $(\mathcal{Y}^k, \mathcal{Z}^k, \mathcal{K}^k)_{k=0}^n \in \prod_{k=0}^n \mathcal{S}^\infty_c$ $(\Delta_k(T), E^k) \times \mathbf{L}^2_W\ (\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ *to the recursive system of indexed RBSDEs* $\boldsymbol{EQ}(\mathcal{H}^k, \mathfrak{f}^k),\ k = 0, \ldots, n.$

Proof. We prove the result by a backward induction on $k = 0, \dots, n$

For $k = n$. Using the same argument as in the proof of Theorem 2.5.3, we can show that there exists a solution $(\mathcal{Y}^n, \mathcal{Z}^n, \mathcal{K}^n) \in \mathcal{S}_c^\infty(\Delta_n(T), E^n) \times \mathbf{L}_W^2(\Delta_n(T), E^n) \times \mathbf{A}(\Delta_n(T), E^n)$ to $\mathbf{EQ}(\mathcal{H}^n, \mathfrak{f}^n)$.

For $k \in \{0, 1, \dots, n-1\}$. Assume there exists $(\mathcal{Y}^{k+1}, \mathcal{Z}^{k+1}, \mathcal{K}^{k+1}) \in \mathcal{S}_c^\infty(\Delta_{k+1}(T), E^{k+1}) \times \mathbf{L}_W^2\ (\Delta_{k+1}(T), E^{k+1}) \times \mathbf{A}(\Delta_{k+1}(T), E^{k+1})$ satisfying $\mathbf{EQ}(\mathcal{H}^{k+1}, \mathfrak{f}^{k+1})$. Consider the truncated generator

$$\mathfrak{f}^{k,N}(t, y, z, \boldsymbol{\theta}_k, \boldsymbol{e}_k) = \inf_{\pi \in A^k} \mathfrak{g}^k(\pi, t, -N \vee y, z, \boldsymbol{\theta}_k, \boldsymbol{e}_k).$$

Then there exists some constant $C_N > 0$, independent of $(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$, such that $|\mathfrak{f}^{k,N}| \le C_N(1 + z^2)$. Hence, there exists a solution $(\mathcal{Y}^{k,N}, \mathcal{Z}^{k,N}, \mathcal{K}^{k,N}) \in \mathcal{S}_c^\infty(\Delta_k(T), E^k) \times \mathbf{L}_W^2(\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ to $\mathbf{EQ}(\mathcal{H}^k, \mathfrak{f}^{k,N})$. By Assumption **(HB')**, $\mathcal{Y}^{k,N} \ge \mathcal{H}^k \ge -C$, where $C > 0$ is a constant independent of N and $(\boldsymbol{\theta}_k, \boldsymbol{e}_k)$. Therefore, for N large enough, $(\mathcal{Y}^{k,N}, \mathcal{Z}^{k,N}, \mathcal{K}^{k,N})$ also solves $\mathbf{EQ}(\mathcal{H}^k, \mathfrak{f}^k)$. $\square$

RBSDE characterization by verification theorem

Theorem 2.5.8. *The value functions* $(\mathfrak{V}^k)_{k=0}^n$ *defined in* (2.5.15) *and* (2.5.16), *are given by*

$$\mathfrak{V}^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k) = U(x - \mathcal{Y}_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)), \tag{2.5.17}$$

for $\forall x \in \mathbb{R}$, $(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k \times E^k$, *where* $(\mathcal{Y}^k, \mathcal{Z}^k, \mathcal{K}^k)_{k=0}^n \in \prod_{k=0}^n \mathcal{S}_c^\infty\ (\Delta_k(T), E^k) \times \mathbf{L}_W^2(\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ *is a solution of the system of RBSDEs* $\boldsymbol{EQ}(\mathcal{H}^k, \mathfrak{f}^k)$, $k = 0, \dots, n$. *Moreover, there exists a saddle point* $(\pi, \tau) \sim (\hat{\pi}^k, \hat{\tau}^k)_{k=0}^n$ *described by:*

$$\hat{\pi}_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \arg\min_{\pi \in A^k} \mathfrak{g}^k\big(\pi, t, \mathcal{Y}_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k), \mathcal{Z}_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k), \boldsymbol{\theta}_k, \boldsymbol{e}_k\big),$$

for $t \in [\theta_k, T]$, *and*

$$\hat{\tau}^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) := \inf\left\{t \ge \theta_k :\ \mathcal{Y}_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k) = \mathcal{H}_t^k(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\right\}, \tag{2.5.18}$$

for $(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$, *a.s.,* $k = 0, \dots, n$. *More specifically, for any admissible trading strategy* $(\pi, \tau) \sim (\pi^k, \tau^k)_{k=0}^n$,

$$\mathbb{E}\left[U(\mathfrak{X}^{n,x}_{\hat{\tau}^n} - \mathcal{H}^n_{\hat{\tau}^n})\big|\mathcal{F}_{\theta_n}\right] \le \mathbb{E}\left[U(\hat{\mathfrak{X}}^{n,x}_{\hat{\tau}^n} - \mathcal{H}^n_{\hat{\tau}^n})\big|\mathcal{F}_{\theta_n}\right] \le \mathbb{E}\left[U(\hat{\mathfrak{X}}^{n,x}_{\tau^n} - \mathcal{H}^n_{\tau^n})\big|\mathcal{F}_{\theta_n}\right],$$

and similar inequalities hold for $k = 0, \dots, n-1$, *where* $\hat{\mathfrak{X}}^{k,x}$ *is the wealth process under control* $\hat{\pi}^k$, $k = 0, \dots, n$.

Proof. We follow the steps in the proof of Theorem 2.5.4.

Step 1: We will show for $(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$**,**

$$U(x - \mathcal{Y}^k_{\theta_k}(\boldsymbol{\theta}_k, \boldsymbol{e}_k, \nu^k)) \ge \mathfrak{V}^k(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k), \quad k = 0, \dots, n. \tag{2.5.19}$$

Let $(\mathcal{Y}^k, \mathcal{Z}^k, \mathcal{K}^k) \in \mathcal{S}^\infty_c(\Delta_k(T), E^k) \times \mathbf{L}^2_W(\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ be a solution of the RBSDE system. For $\nu^k \in \mathcal{A}^k$, $\forall x \in \mathbb{R}$, $(\boldsymbol{\theta}_k, \boldsymbol{e}_k) \in \Delta_k(T) \times E^k$, define $(\xi^k)_{k=0}^n$ as:

$$\begin{aligned} \xi^k_t(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k, \nu^k) &:= U\big(\mathfrak{X}^{k,x}_t - \mathcal{Y}^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k)\big) \\ &+ \int_{\theta_k}^t \int_E U\left(\mathfrak{X}^{k,x}_r + \nu^k_r \cdot \gamma^k_r(\boldsymbol{\theta}_k, e_k, e) - \mathcal{Y}^{k+1}_r(\boldsymbol{e}_k, r, \boldsymbol{e}_k, e)\right)\eta(de)dr, \end{aligned} \tag{2.5.20}$$

for $k = 0, \dots, n-1$, and

$$\xi^n_t(x, \boldsymbol{\theta}_n, \boldsymbol{e}_n, \nu^n) := U\big(\mathfrak{X}^{n,x}_t - \mathcal{Y}^n_t(\boldsymbol{\theta}_n, \boldsymbol{e}_n)\big). \tag{2.5.21}$$

Applying Itô's formula, we obtain, for $k = 0, \dots, n$,

$$\begin{aligned} d\xi^k_t(x, \boldsymbol{\theta}_k, \boldsymbol{e}_k, \nu^n) &= pU(\mathfrak{X}^{k,x}_t - \mathcal{Y}^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k))\Big[\big(- \mathfrak{f}^k(t, \mathcal{Y}^k_t, \mathcal{Z}^k_t, \boldsymbol{\theta}_k, \boldsymbol{e}_k) \\ &+ \mathfrak{g}^k(\nu^k_t, t, \mathcal{Y}^k_t, \mathcal{Z}^k_t, \boldsymbol{\theta}_k, \boldsymbol{e}_k)\big)dt - d\mathcal{K}^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k) + (\mathcal{Z}^k_t - \sigma^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k)'\nu^k_t) \cdot dW_t\Big], \end{aligned}$$

Define $\hat{\tau}^k$ as in (2.5.18), then $d\mathcal{K}^k_{t\wedge\hat{\tau}^k} = 0$, $\theta_k \le t \le T$. Therefore, $(\xi^k_{t\wedge\hat{\tau}^k})_{\theta_k \le t \le T}$ is a local super-martingale. By introducing a localizing sequence of stopping times $(\rho_m)_m$, and then letting $m \to \infty$, we can show for $k = 0, \dots, n$,

$$\xi^k_{t\wedge\tilde{\tau}^k} \ge \mathbb{E}\left[\xi^k_{s\wedge\tilde{\tau}^k}\Big|\mathcal{F}_t\right], \quad \theta_k \le t \le s \le T.$$

In particular,

$$U\big(x - \mathcal{Y}^n_{\theta_n}(\boldsymbol{\theta}_n, \boldsymbol{e}_n)\big) = \xi^n_{\theta_n} \geq \mathbb{E}\left[\xi^n_{\tilde{\tau}^n}\middle|\mathcal{F}_{\theta_n}\right] = \mathbb{E}\Big[U(\mathfrak{X}^{n,x}_{\tilde{\tau}^n} - \mathcal{H}^n_{\tilde{\tau}^n})\Big|\mathcal{F}_{\theta_n}\Big]. \tag{2.5.22}$$

Hence,

$$U\big(x - \mathcal{Y}^n_{\theta_k}(\boldsymbol{\theta}_n, \boldsymbol{e}_n)\big) \geq \operatorname*{ess\,inf}_{\tau^n \in \mathcal{T}^n} \mathbb{E}[U(\mathfrak{X}^{n,x}_{\tau^n} - \mathcal{H}^n_{\tau^n})|\mathcal{F}_{\theta_k}].$$

for any $\nu^n \in \mathcal{A}^n$. So (2.5.19) follows for $k = n$. Similarly, it holds for $k = 0, \ldots, n-1$.

Steps 2&3: Similar to the proof of Theorem 2.5.4.

Step 4: We will show (2.5.17) **holds and** $(\pi, \tau) \sim (\hat{\pi}^k, \hat{\tau}^k)^n_{k=0}$ **is a saddle point.** Under the admissible control $\hat{\pi}^k$, the dynamics of $(\xi^k)_k$ defined in (2.5.20) and (2.5.21) are given by

$$d\xi^k_t(x, \boldsymbol{\theta}, \boldsymbol{e}, \hat{\pi}^k) = pU(\mathfrak{X}^{k,x}_t - \mathcal{Y}^k_t)\Big[-d\mathcal{K}^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k) + (\mathcal{Z}^k_t - \sigma^k_t(\boldsymbol{\theta}_k, \boldsymbol{e}_k)'\nu^k_t) \cdot dW_t\Big],$$

for $k = 0, \ldots, n$. By the uniform integrality of ξ^k_t, we know ξ^k_t is a sub-martingale. Consider when $k = n$. For any $\mathbb{F}$-stopping time τ^n valued in $[\theta_n, T]$,

$$U(x - \mathcal{Y}^n_{\theta_n}) \leq \mathbb{E}[U(\hat{\mathfrak{X}}^{n,x}_{\tau^n} - \mathcal{Y}^n_{\tau^n})|\mathcal{F}_{\theta_n}] \leq \mathbb{E}[U(\hat{\mathfrak{X}}^{n,x}_{\tau^n} - \mathcal{H}^n_{\tau^n})|\mathcal{F}_{\theta_n}], \tag{2.5.23}$$

Therefore, we have

$$U(x - \mathcal{Y}^n_{\theta_n}) \leq \operatorname*{ess\,inf}_{\tau^n \in \mathcal{T}^n} \mathbb{E}\Big[U(\hat{\mathfrak{X}}^{n,x}_{\tau^n} - \mathcal{H}^n_{\tau^n})\big|\mathcal{F}_{\theta_n}\Big] \leq \operatorname*{ess\,sup}_{\pi^n \in \mathcal{A}^n} \operatorname*{ess\,inf}_{\tau^n \in \mathcal{T}^n} \mathbb{E}\Big[U(\mathfrak{X}^{n,x}_{\tau^n} - \mathcal{H}^n_{\tau^n})\big|\mathcal{F}_{\theta_n}\Big].$$

Now, the last equation along with (2.5.19) implies that (2.5.17) holds for $k = n$.

By the definition and admissibility of $\hat{\pi}^n$, we can show that under control $\hat{\pi}^n$, $\xi^n_{t\wedge\hat{\tau}^n}$ is a martingale. Thus from (2.5.23) we have

$$\begin{aligned}
&\mathbb{E}\left[U(\hat{\mathfrak{X}}^{n,x}_{\hat{\tau}^n} - \mathcal{H}^n_{\hat{\tau}^n})\big|\mathcal{F}_{\theta_n}\right] = \mathbb{E}\left[U(\hat{\mathfrak{X}}^{n,x}_{\hat{\tau}^n} - \mathcal{Y}^n_{\hat{\tau}^n})\big|\mathcal{F}_{\theta_n}\right] = U(x - \mathcal{Y}^n_{\theta_n}) \\
&\leq \mathbb{E}\left[U(\hat{\mathfrak{X}}^{n,x}_{\tau^n} - \mathcal{Y}^n_{\tau^n})\big|\mathcal{F}_{\theta_n}\right] \leq \mathbb{E}\left[U(\hat{\mathfrak{X}}^{n,x}_{\tau^n} - \mathcal{H}^n_{\tau^n})\big|\mathcal{F}_{\theta_n}\right].
\end{aligned}$$

And from (2.5.22) we have

$$\mathbb{E}\left[U(\hat{\mathfrak{X}}^{n,x}_{\hat{\tau}^n} - \mathcal{H}^n_{\hat{\tau}^n})\middle|\mathcal{F}_{\theta_n}\right] = \mathbb{E}\left[U(\hat{\mathfrak{X}}^{n,x}_{\hat{\tau}^n} - \mathcal{Y}^n_{\hat{\tau}^n})\middle|\mathcal{F}_{\theta_n}\right] = U(x - \mathcal{Y}^n_{\theta_n})$$
$$\geq \mathbb{E}\left[U(\mathfrak{X}^{n,x}_{\hat{\tau}^n} - \mathcal{Y}^n_{\hat{\tau}^n})\middle|\mathcal{F}_{\theta_n}\right] = \mathbb{E}\left[U(\mathfrak{X}^{n,x}_{\hat{\tau}^n} - \mathcal{H}^n_{\hat{\tau}^n})\middle|\mathcal{F}_{\theta_n}\right]$$

Thus, $(\hat{\pi}^n, \hat{\tau}^n)$ is a saddle point. Similarly, it can be shown that the corresponding conclusions hold for $k = 0, \dots, n-1$ using (2.5.16). $\square$

CHAPTER III

On zero-sum optimal stopping games in discrete time

3.1 Introduction

On a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t=0,\dots,T})$, let us consider

$$\overline{C} := \inf_{\rho} \sup_{\tau} \mathbb{E}U(\rho, \tau) \quad \text{and} \quad \underline{C} := \sup_{\tau} \inf_{\rho} \mathbb{E}U(\rho, \tau), \tag{3.1.1}$$

where $U(s,t)$ is $\mathbb{F}_{s\vee t}$-measurable and ρ, τ are $\mathbb{F}$-stopping times taking values in $\{0, \dots, T\}$. When $U(s,t) = f_s 1_{\{s<t\}} + g_t 1_{\{s\geq t\}}$ in which f_t and g_t are bounded $\mathbb{F}$-adapted processes, the problem above is said to be a Dynkin game (see, e.g., [66, Chapter VI-6]). It is well-known that if $f \geq g$ then $\overline{C} = \underline{C}$.

However, it may fail that $\overline{C} = \underline{C}$ in general even for some other natural choices of U. Consider $U(s,t) = |f_s - f_t|$. This means in the game (3.1.1), Player "inf" tries to match Player "sup". Let $f_t = t,\ t = 0, \dots, T$ and the problem becomes deterministic. It is easy to see that $\overline{C} = \lceil T/2 \rceil > 0 = \underline{C}$. So the game is not fair.

On the other hand, when playing game (3.1.1), Players "inf" and "sup" can adjust their stopping strategies according to each other's stopping behavior. Therefore, it is more reasonable to incorporate a stopping strategy that can be adjusted according to the other's behavior. That is, we consider the stopper-stopper problem

$$\inf_{\boldsymbol{\rho}} \sup_{\tau} \mathbb{E}[U(\boldsymbol{\rho}(\tau), \tau)] \quad \text{and} \quad \sup_{\boldsymbol{\tau}} \inf_{\rho} \mathbb{E}[U(\rho, \boldsymbol{\tau}(\rho))],$$

where $\rho, \tau \in \mathcal{T}$, and $\boldsymbol{\rho}(\cdot), \boldsymbol{\tau}(\cdot): \mathcal{T} \mapsto \mathcal{T}$ satisfy certain non-anticipativity conditions, where $\mathcal{T}$ is the set of stopping times.

One possible definition of non-anticipative stopping strategies (we denote the collection of them as $\mathbb{T}^i$) would be that, $\boldsymbol{\rho} \in \mathbb{T}^i$, if $\boldsymbol{\rho}: \mathcal{T} \mapsto \mathcal{T}$ satisfies

$$\text{either} \quad \boldsymbol{\rho}(\sigma_1) = \boldsymbol{\rho}(\sigma_2) \leq \sigma_1 \wedge \sigma_2 \quad \text{or} \quad \boldsymbol{\rho}(\sigma_1) \wedge \boldsymbol{\rho}(\sigma_2) > \sigma_1 \wedge \sigma_2, \quad \forall \sigma_1, \sigma_2 \in \mathcal{T}.$$

That is, $\boldsymbol{\rho} = \boldsymbol{\rho}(\tau)$ can be adjusted according to the previous (but not current) behavior of τ. However, using this definition, it may be the case that

$$\overline{A} := \inf_{\boldsymbol{\rho} \in \mathbb{T}^i} \sup_{\tau \in \mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}(\tau), \tau)] \neq \underline{A} := \sup_{\boldsymbol{\tau} \in \mathbb{T}^i} \inf_{\rho \in \mathcal{T}} \mathbb{E}[U(\rho, \boldsymbol{\tau}(\rho))].$$

Below is an example.

Example 3.1.1. Let $T = 1$ and $U(s,t) = |f_s - f_t| = 1_{\{s \neq t\}}$ with $f_t = t,\ t = 0, 1$. Then there are only two elements, $\boldsymbol{\rho}^0$ and $\boldsymbol{\rho}^1$, in $\mathbb{T}^i$, with $\boldsymbol{\rho}^0(0) = \boldsymbol{\rho}^0(1) = 0$ and $\boldsymbol{\rho}^1(0) = \boldsymbol{\rho}^1(1) = 1$. It can be shown that $\overline{A} = 1$ and $\underline{A} = 0$.

Another possible definition of non-anticipative stopping strategies (we denote the collection as $\mathbb{T}^{ii}$) would be that, $\boldsymbol{\rho} \in \mathbb{T}^{ii}$, if $\boldsymbol{\rho}: \mathcal{T} \mapsto \mathcal{T}$ satisfies

$$\text{either} \quad \boldsymbol{\rho}(\sigma_1) = \boldsymbol{\rho}(\sigma_2) < \sigma_1 \wedge \sigma_2 \quad \text{or} \quad \boldsymbol{\rho}(\sigma_1) \wedge \boldsymbol{\rho}(\sigma_2) \geq \sigma_1 \wedge \sigma_2, \quad \forall \sigma_1, \sigma_2 \in \mathcal{T}.$$

That is, $\boldsymbol{\rho} = \boldsymbol{\rho}(\tau)$ can be adjusted according to both the previous and the current behavior of τ. However, under this definition, it may be the case that

$$\overline{B} := \inf_{\boldsymbol{\rho} \in \mathbb{T}^{ii}} \sup_{\tau \in \mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}(\tau), \tau)] \neq \underline{B} := \sup_{\boldsymbol{\tau} \in \mathbb{T}^{ii}} \inf_{\rho \in \mathcal{T}} \mathbb{E}[U(\rho, \boldsymbol{\tau}(\rho))].$$

We still use Example 3.1.1 as an example.

Example 3.1.2. Let $T = 1$ and $U(s,t) = |f_s - f_t| = 1_{\{s \neq t\}}$ with $f_t = t,\ t = 0, 1$. Then in this case $\mathbb{T}^{ii}$ is the set of all the maps from $\mathcal{T}$ to $\mathcal{T}$. By letting $\boldsymbol{\rho}(0) = 0$ and $\boldsymbol{\rho}(1) = 1$, we have that $\overline{B} = 0$. By Letting $\boldsymbol{\tau}(0) = 1$ and $\boldsymbol{\tau}(1) = 0$, we have that $\underline{B} = 1$.

Observe that $\overline{A} = \underline{B}$ and $\underline{A} = \overline{B}$ in Examples 3.1.1 and 3.1.2. In fact it is by no means a coincidence as we will see later in this chapter. That is, we always have

$$\overline{B} = \inf_{\boldsymbol{\rho} \in \mathbb{T}^{ii}} \sup_{\tau \in \mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}(\tau), \tau)] = \sup_{\boldsymbol{\tau} \in \mathbb{T}^{i}} \inf_{\rho \in \mathcal{T}} \mathbb{E}[U(\rho, \boldsymbol{\tau}(\rho))] = \underline{A}.$$

An intuitive reason for using both $\mathbb{T}^{ii}$ for $\boldsymbol{\rho}$ and $\mathbb{T}^{i}$ for $\boldsymbol{\tau}$ above is that, in order to let the game be fair, at each time period we designate the same player (here we choose "sup") to act first. (Note that both "to stop" and "not to stop" are actions.) So this player ("sup") can only take advantage of the other's ("inf's") previous behavior (as opposed to "inf" taking advantage of "sup's" current behavior in addition).

In this chapter, we analyze the problems associated to $\overline{B}$ and $\underline{A}$. We show that these problems can be converted into a corresponding Dynkin game, and that $\overline{B} = \underline{A} = V$, where V is the value of the Dynkin game. We also provide the optimal $\boldsymbol{\rho}(\cdot) \in \mathbb{T}^{ii}$ and $\boldsymbol{\tau}(\cdot) \in \mathbb{T}^{i}$ for $\overline{B}$ and $\underline{A}$ respectively.

The rest of the chapter is organized as follows. In the next section, we introduce the setup and the main result. We provide two examples in Section 3.3. In Section 3.4, we give the proof of the main result. Finally we give some insight for the corresponding problems in continuous time in Section 3.5.

3.2 The setup and the main result

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathbb{F} = (\mathcal{F}_t)_{t=0,\dots,T}$ be the filtration enlarged by P-null sets, where $T \in \mathbb{N}$ is the time horizon in discrete time. Let $U : \{0, \dots, T\} \times \{0, \dots, T\} \times \Omega \mapsto \mathbb{R}$, such that $U(s,t,\cdot) \in \mathcal{F}_{s \vee t}$. For simplicity, we assume that U is bounded. Denote $\mathbb{E}_t[\cdot]$ for $\mathbb{E}[\cdot|\mathcal{F}_t]$. We shall often omit "a.s." when a property holds outside a P-null set. Let $\mathcal{T}_t$ be the set of $\mathbb{F}$-stopping times taking values in $\{t \dots, T\}$, and $\mathcal{T} := \mathcal{T}_0$. We define the stopping strategies of Type I and Type II as follows:

Definition 3.2.1. $\boldsymbol{\rho}$ is a stopping strategy of Type I (resp. II), if $\boldsymbol{\rho}: \mathcal{T} \mapsto \mathcal{T}$ satisfies the "non-anticipativity" condition of Type I (resp. II), i.e., for any $\sigma_1, \sigma_2 \in \mathcal{T}$,

$$\text{either} \quad \boldsymbol{\rho}(\sigma_1) = \boldsymbol{\rho}(\sigma_2) \leq \text{(resp. } <) \ \sigma_1 \wedge \sigma_2 \quad \text{or} \quad \boldsymbol{\rho}(\sigma_1) \wedge \boldsymbol{\rho}(\sigma_2) > \text{(resp. } \geq) \ \sigma_1 \wedge \sigma_2. \tag{3.2.1}$$

Denote $\mathbb{T}^i$ (resp. $\mathbb{T}^{ii}$) as the set of stopping strategies of Type I (resp. II).

Remark 3.2.2. We can treat $\mathcal{T}$ as a subset of $\mathbb{T}^i$ and $\mathbb{T}^{ii}$ (i.e., each $\tau \in \mathcal{T}$ can be treated as the map with only one value τ). Hence we have $\mathcal{T} \subset \mathbb{T}^i \subset \mathbb{T}^{ii}$.

Consider the problem

$$\overline{B} := \inf_{\boldsymbol{\rho} \in \mathbb{T}^{ii}} \sup_{\tau \in \mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}(\tau), \tau)] \quad \text{and} \quad \underline{A} := \sup_{\boldsymbol{\tau} \in \mathbb{T}^i} \inf_{\rho \in \mathcal{T}} \mathbb{E}[U(\rho, \boldsymbol{\tau}(\rho))]. \tag{3.2.2}$$

We shall convert this problem into a Dynkin game. In order to do so, let us introduce the following two processes that will represent the payoffs in the Dynkin game.

$$V_t^1 := \operatorname*{ess\,inf}_{\rho \in \mathcal{T}_t} \mathbb{E}_t[U(\rho, t)], \quad t = 0, \dots, T, \tag{3.2.3}$$

and

$$V_t^2 := \max \left\{ \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t+1}} \mathbb{E}_t[U(t, \tau)], V_t^1 \right\}, \quad t = 0, \dots, T-1, \tag{3.2.4}$$

and $V_T^2 = U(T, T)$. Observe that

$$V_t^1 \leq V_t^2, \quad t = 0, \dots, T. \tag{3.2.5}$$

By the classic optimal stopping theory, there exist an optimizer $\rho_u(t) \in \mathcal{T}_t$ for V_t^1, and an optimizer $\tau_u(t) \in \mathcal{T}_{t+1}$ for $\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t+1}} \mathbb{E}_t[U(t, \tau)]$, $t = 0, \dots, T-1$. We let $\rho_u(T) = \tau_u(T) = T$ for convenience.

Define the corresponding Dynkin game as follows:

$$V := \inf_{\rho \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[V_\tau^1 1_{\{\tau \leq \rho\}} + V_\rho^2 1_{\{\tau > \rho\}}\right] = \sup_{\tau \in \mathcal{T}} \inf_{\rho \in \mathcal{T}} \mathbb{E}\left[V_\tau^1 1_{\{\tau \leq \rho\}} + V_\rho^2 1_{\{\tau > \rho\}}\right],$$

where the second equality above follows from (3.2.5). Moreover, there exists a saddle point (ρ_d, τ_d) described by

$$\rho_d := \inf\{s \geq 0: \ V_s = V_s^2\} \quad \text{and} \quad \tau_d := \inf\{s \geq 0: \ V_s = V_s^1\}, \tag{3.2.6}$$

where

$$V_t := \operatorname*{ess\,inf}_{\rho\in\mathcal{T}_t} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}_t\left[V_\tau^1 1_{\{\tau\leq\rho\}} + V_\rho^2 1_{\{\tau>\rho\}}\right] = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \operatorname*{ess\,inf}_{\rho\in\mathcal{T}_t} \mathbb{E}_t\left[V_\tau^1 1_{\{\tau\leq\rho\}} + V_\rho^2 1_{\{\tau>\rho\}}\right].$$

That is,

$$V = \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[V_\tau^1 1_{\{\tau\leq\rho_d\}} + V_{\rho_d}^2 1_{\{\tau>\rho_d\}}\right] = \inf_{\rho\in\mathcal{T}} \mathbb{E}\left[V_{\tau_d}^1 1_{\{\tau_d\leq\rho\}} + V_\rho^2 1_{\{\tau_d>\rho\}}\right].$$

Below is the main result of this chapter.

Theorem 3.2.3. *We have that*

$$\overline{B} = \underline{A} = V.$$

Besides, there exists $\boldsymbol{\rho}^* \in \mathbb{T}^{ii}$ *and* $\tau^* : \mathbb{T}^{ii} \mapsto \mathcal{T}$ *described by*

$$\boldsymbol{\rho}^*(\tau) = \rho_d 1_{\{\tau>\rho_d\}} + \rho_u(\tau) 1_{\{\tau\leq\rho_d\}}, \quad \tau\in\mathcal{T}, \tag{3.2.7}$$

and

$$\tau^* = \tau^*(\boldsymbol{\rho}) := \tau_d 1_{\{\tau_d\leq\boldsymbol{\rho}(\tau_d)\}} + \tau_u(\boldsymbol{\rho}(\tau_d)) 1_{\{\tau_d>\boldsymbol{\rho}(\tau_d)\}}, \quad \boldsymbol{\rho}\in\mathbb{T}^{ii}, \tag{3.2.8}$$

such that

$$\overline{B} = \sup_{\tau\in\mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}^*(\tau), \tau)] = \inf_{\boldsymbol{\rho}\in\mathbb{T}^{ii}} \mathbb{E}[U(\boldsymbol{\rho}(\tau^*), \tau^*)].$$

Similarly, there exists $\boldsymbol{\tau}^{**} \in \mathbb{T}^i$ *and* $\rho^{**} : \mathbb{T}^i \mapsto \mathcal{T}$ *described by*

$$\boldsymbol{\tau}^{**}(\rho) = \tau_d 1_{\{\rho\geq\tau_d\}} + \tau_u(\rho) 1_{\{\rho<\tau_d\}}, \quad \rho\in\mathcal{T}, \tag{3.2.9}$$

and

$$\rho^{**} = \rho^{**}(\boldsymbol{\tau}) := \rho_d 1_{\{\rho_d<\boldsymbol{\tau}(\rho_d)\}} + \rho_u(\boldsymbol{\tau}(\rho_d)) 1_{\{\rho_d\geq\boldsymbol{\tau}(\rho_d)\}}, \quad \boldsymbol{\tau}\in\mathbb{T}^i,$$

such that

$$\underline{A} = \inf_{\rho \in \mathcal{T}} \mathbb{E}[U(\rho, \boldsymbol{\tau}^{**}(\rho))] = \sup_{\boldsymbol{\tau} \in \mathbb{T}^i} \mathbb{E}[U(\rho^{**}, \boldsymbol{\tau}(\rho^{**}))].$$

Remark 3.2.4. In the definition (3.2.8) $\tau^*(\cdot)$ is a map of $\boldsymbol{\rho}$ instead of a stopping time. But once the outside $\boldsymbol{\rho}$ is given, $\tau^*(\boldsymbol{\rho})$ would become a stopping time, and thus this shall cause no problem in our definition of τ^*. (To convince oneself, one may think of $\inf_x \sup_y f(x,y) = \inf_x f(x, y^*(x))$.) We shall often simply write τ^* and omit its dependence of $\boldsymbol{\rho}$.

Corollary 3.2.5.

$$\overline{B} = \mathbb{E}[U(\boldsymbol{\rho}^*(\tau^*), \tau^*)].$$

(Here $\tau^* = \tau^*(\boldsymbol{\rho}^*)$ as we indicated in Remark 3.2.4.) Moreover,

(3.2.10)

$$\boldsymbol{\rho}^*(\tau^*) = \rho_d 1_{\{\tau_d > \rho_d\}} + \rho_u(\tau_d) 1_{\{\tau_d \leq \rho_d\}} \quad \textit{and} \quad \tau^*(\boldsymbol{\rho}^*) = \tau_d 1_{\{\tau_d \leq \rho_d\}} + \tau_u(\rho_d) 1_{\{\tau_d > \rho_d\}}.$$

Similar results hold for $\underline{A}$.

Proof. By (3.2.7),

$$\boldsymbol{\rho}^*(\tau_d) = \rho_d 1_{\{\tau_d > \rho_d\}} + \rho_u(\tau_d) 1_{\{\tau_d \leq \rho_d\}}.$$

If $\tau_d > \rho_d$, then $\boldsymbol{\rho}^*(\tau_d) = \rho_d < \tau_d$, which implies that $\{\tau_d > \rho_d\} \subset \{\tau_d > \boldsymbol{\rho}^*(\tau_d)\}$. If $\tau_d \leq \rho_d$, then $\boldsymbol{\rho}^*(\tau_d) = \rho_u(\tau_d) \geq \tau_d$, which implies that $\{\tau_d \leq \rho_d\} \subset \{\tau_d \leq \boldsymbol{\rho}^*(\tau_d)\}$. Therefore, $\{\tau_d > \rho_d\} = \{\tau_d > \boldsymbol{\rho}^*(\tau_d)\}$ and $\{\tau_d \leq \rho_d\} = \{\tau_d \leq \boldsymbol{\rho}^*(\tau_d)\}$. Hence we have that

$$\tau^*(\boldsymbol{\rho}^*) = \tau_d 1_{\{\tau_d \leq \rho_d\}} + \tau_u(\boldsymbol{\rho}^*(\tau_d)) 1_{\{\tau_d > \rho_d\}} = \tau_d 1_{\{\tau_d \leq \rho_d\}} + \tau_u(\rho_d) 1_{\{\tau_d > \rho_d\}},$$

where the second equality follows from that $\boldsymbol{\rho}^*(\tau_d) = \rho_d$ on $\{\tau_d > \rho_d\}$.

Now if $\tau_d \leq \rho_d$, then $\tau^* = \tau_d \leq \rho_d$, and thus $\{\tau_d \leq \rho_d\} \subset \{\tau^* \leq \rho_d\}$. If $\tau_d > \rho_d$, then $\tau^* = \tau_u(\rho_d) > \rho_d$ since $\tau_u(t) \geq t + 1$ if $t < T$, and thus $\{\tau_d > \rho_d\} \subset \{\tau^* > \rho_d\}$.

Therefore, $\{\tau_d \le \rho_d\} = \{\tau^* \le \rho_d\}$ and $\{\tau_d > \rho_d\} = \{\tau^* > \rho_d\}$. Hence we have that

$$\boldsymbol{\rho}^*(\tau^*) = \rho_d 1_{\{\tau_d > \rho_d\}} + \rho_u(\tau^*) 1_{\{\tau_d \le \rho_d\}} = \rho_d 1_{\{\tau_d > \rho_d\}} + \rho_u(\tau_d) 1_{\{\tau_d \le \rho_d\}},$$

where the second equality follows from that $\tau^* = \tau_d$ on $\{\tau_d \le \rho_d\}$. $\square$

3.3 Examples

In this section we provide two examples within the setup of Section 3.2. The first example shows that in the classical Dynkin game one does not need to use non-anticipative stopping strategies. The second example is a relevant problem from mathematical finance in which our results can be applied. This problem is on determining the optimal exercise strategy when one trades two different American options in different directions.

3.3.1 Dynkin game using non-anticipative stopping strategies

Let

$$U(s,t) = f_s 1_{\{s<t\}} + g_t 1_{\{s \ge t\}},$$

where $(f_t)_t$ and $(g_t)_t$ are $\mathbb{F}$-adapted, satisfying $f \ge g$. Then we have that

$$V_t^1 = g_t,\ t = 0, \dots, T, \quad \text{and} \quad V_t^2 = f_t,\ t = 0, \dots, T-1.$$

Then by Theorem 3.2.3 we have that

$$\begin{aligned}
&\inf_{\boldsymbol{\rho} \in \mathbb{T}^{ii}} \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[f_{\boldsymbol{\rho}(\tau)} 1_{\{\boldsymbol{\rho}(\tau) < \tau\}} + g_\tau 1_{\{\boldsymbol{\rho}(\tau) \ge \tau\}}\right] = \sup_{\boldsymbol{\tau} \in \mathbb{T}^i} \inf_{\rho \in \mathcal{T}} \mathbb{E}\left[f_\rho 1_{\{\rho < \boldsymbol{\tau}(\rho)\}} + g_{\boldsymbol{\tau}(\rho)} 1_{\{\rho \ge \boldsymbol{\tau}(\rho)\}}\right] \\
&= \sup_{\tau \in \mathcal{T}} \inf_{\rho \in \mathcal{T}} \mathbb{E}\left[f_\rho 1_{\{\rho < \tau\}} + g_\tau 1_{\{\rho \ge \tau\}}\right] = \inf_{\rho \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[f_\rho 1_{\{\rho < \tau\}} + g_\tau 1_{\{\rho \ge \tau\}}\right].
\end{aligned}$$

Besides, by the property of U, the $\boldsymbol{\rho}^*$ and $\boldsymbol{\tau}^{**}$ defined in (3.2.7) and (3.2.9) can w.l.o.g. be written as

$$\boldsymbol{\rho} = \rho_d \quad \text{and} \quad \boldsymbol{\tau} = \tau_d.$$

Therefore, in the Dynkin game, using non-anticipative stopping strategies is the same as using a usual stopping time.

Remark 3.3.1. In this example we let $\boldsymbol{\rho} \in \mathbb{T}^{ii}$ and $\boldsymbol{\tau} \in \mathbb{T}^{i}$. The same conclusion holds if we let $\boldsymbol{\rho} \in \mathbb{T}^{i}$ and $\boldsymbol{\tau} \in \mathbb{T}^{ii}$ instead.

3.3.2 A robust utility maximization problem

Let

$$U(t,s) = \mathcal{U}(f_t - g_s),$$

where $\mathcal{U} : \mathbb{R} \mapsto \mathbb{R}$ is a utility function, and f and g are adapted to $\mathbb{F}$. Consider

$$\overline{\mathcal{V}} := \sup_{\boldsymbol{\rho} \in \mathbb{T}^{ii}} \inf_{\tau \in \mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}(\tau), \tau)].$$

This problem can be interpreted as the one in which an investor longs an American option f and shorts an American option g, and the goal is to choose an optimal stopping strategy to maximize the utility according to the stopping behavior of the holder of g. Here we assume that the maturities of f and g are the same (i.e., T). This is without loss of generality. Indeed for instance, if the maturity of f is $\hat{t} < T$, then we can define $f(t) = f(\hat{t})$ for $t = \hat{t}+1, \ldots, T$.

3.4 Proof of Theorem 3.2.3

We will only prove the results for $\overline{B}$, since the proofs for $\underline{A}$ are similar.

Lemma 3.4.1. *For any $\sigma \in \mathcal{T}$, $\rho_u(\sigma) \in \mathcal{T}$ and $\tau_u(\sigma) \in \mathcal{T}$.*

Proof. Take $\sigma \in \mathcal{T}$. Then for $t \in \{0, \ldots, T\}$

$$\{\rho_u(\sigma) \le t\} = \cup_{i=0}^{t} (\{\sigma = i\} \cap \{\rho_u(i) \le t\}) \in \mathcal{F}_t.$$

□

Lemma 3.4.2. *$\boldsymbol{\rho}^*$ defined in* (3.2.7) *is in $\mathbb{T}^{ii}$ and τ^* defined in* (3.2.8) *is a map from $\mathbb{T}^{ii}$ to $\mathcal{T}$.*

Proof. Take $\tau \in \mathcal{T}$. We have that

$$\begin{aligned}\{\boldsymbol{\rho}^*(\tau) \le t\} &= (\{\tau > \rho_d\} \cap \{\rho_d \le t\}) \cup (\{\tau \le \rho_d\} \cap \{\rho_u(\tau) \le t\}) \\ &= (\{\tau > \rho_d\} \cap \{\rho_d \le t\}) \cup (\{\tau \le \rho_d\} \cap \{\tau \le t\} \cap \{\rho_u(\tau) \le t\}) \in \mathcal{F}_t.\end{aligned}$$

Hence $\boldsymbol{\rho}^*(\tau) \in \mathcal{T}$. Similarly we can show that $\tau^*(\boldsymbol{\rho}) \in \mathcal{T}$ for any $\boldsymbol{\rho} \in \mathbb{T}^{ii}$.

It remains to show that $\boldsymbol{\rho}^*$ satisfies the non-anticipative condition of Type II in (3.2.1). Take $\tau_1, \tau_2 \in \mathcal{T}$. If $\boldsymbol{\rho}^*(\tau_1) < \tau_1 \wedge \tau_2 \le \tau_1$, then $\tau_1 > \rho_d$ and thus $\boldsymbol{\rho}^*(\tau_1) = \rho_d < \tau_1 \wedge \tau_2 \le \tau_2$, which implies $\boldsymbol{\rho}^*(\tau_2) = \rho_d = \boldsymbol{\rho}^*(\tau_1) < \tau_1 \wedge \tau_2$. If $\boldsymbol{\rho}^*(\tau_1) \ge \tau_1 \wedge \tau_2$, then if $\boldsymbol{\rho}^*(\tau_2) < \tau_1 \wedge \tau_2$ we can use the previous argument to get that $\boldsymbol{\rho}^*(\tau_1) = \boldsymbol{\rho}^*(\tau_2) < \tau_1 \wedge \tau_2$ which is a contradiction, and thus $\boldsymbol{\rho}^*(\tau_2) \ge \tau_1 \wedge \tau_2$. □

Lemma 3.4.3.

$$\overline{B} \le \sup_{\tau \in \mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}^*(\tau), \tau)] \le V.$$

Proof. Recall $\boldsymbol{\rho}^*$ defined in (3.2.7) and ρ_d defined in (3.2.6). We have that

$$\begin{aligned}\overline{B} &\le \sup_{\tau \in \mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}^*(\tau), \tau)] \\ &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[U(\rho_d, \tau) 1_{\{\rho_d < \tau\}} + U(\rho_u(\tau), \tau) 1_{\{\rho_d \ge \tau\}}\right] \\ &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[1_{\{\rho_d < \tau\}} \mathbb{E}_{\rho_d}[U(\rho_d, \tau)] + 1_{\{\rho_d \ge \tau\}} \mathbb{E}_\tau[U(\rho_u(\tau), \tau)]\right] \\ &\le \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[1_{\{\rho_d < \tau\}} V^2_{\rho_d} + 1_{\{\rho_d \ge \tau\}} V^1_\tau\right] \\ &= V.\end{aligned}$$

□

Lemma 3.4.4.

$$\overline{B} \ge \inf_{\boldsymbol{\rho} \in \mathbb{T}^{ii}} \mathbb{E}[U(\boldsymbol{\rho}(\tau^*), \tau^*)] \ge V.$$

Proof. Take $\boldsymbol{\rho} \in \mathbb{T}^{ii}$. Recall τ^* defined in (3.2.8). By the non-anticipativity condition of Type II in (3.2.1),

$$\text{either} \quad \boldsymbol{\rho}(\tau^*) = \boldsymbol{\rho}(\tau_d) < \tau_d \wedge \tau^* \quad \text{or} \quad \boldsymbol{\rho}(\tau^*), \boldsymbol{\rho}(\tau_d) \geq \tau_d \wedge \tau^*.$$

Therefore,

$$\text{if} \quad \boldsymbol{\rho}(\tau_d) \geq \tau_d, \quad \textit{then} \quad \boldsymbol{\rho}(\tau^*) \geq \tau_d \wedge \tau^* = \tau_d = \tau^*,$$

and

$$\text{(3.4.1) if} \quad \boldsymbol{\rho}(\tau_d) < \tau_d, \quad \textit{then} \quad \tau^* = \tau_u(\boldsymbol{\rho}(\tau_d)) > \boldsymbol{\rho}(\tau_d) \quad \Longrightarrow \quad \boldsymbol{\rho}(\tau_d) < \tau^* \wedge \tau_d$$
$$\Longrightarrow \quad \boldsymbol{\rho}(\tau_d) = \boldsymbol{\rho}(\tau^*),$$

where in (3.4.1) we used the fact that $\tau_u(t) \geq t+1$ if $t < T$ (in the first conclusion).

Besides, if $\tau_d > \boldsymbol{\rho}(\tau_d)$, then

$$V^1_{\boldsymbol{\rho}(\tau_d)} < V_{\boldsymbol{\rho}(\tau_d)} \leq V^2_{\boldsymbol{\rho}(\tau_d)},$$

which implies that

$$V^2_{\boldsymbol{\rho}(\tau_d)} = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t+1}} \mathbb{E}_{\boldsymbol{\rho}(\tau_d)}[U(\boldsymbol{\rho}(\tau_d), \tau)] = \mathbb{E}_{\boldsymbol{\rho}(\tau_d)}[U(\boldsymbol{\rho}(\tau_d), \tau_u(\boldsymbol{\rho}(\tau_d)))].$$

Now we have that

$$\begin{aligned}
&\sup_{\tau \in \mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}(\tau), \tau))] \\
&\geq \mathbb{E}[U(\boldsymbol{\rho}(\tau^*), \tau^*)] \\
&= \mathbb{E}\left[U(\boldsymbol{\rho}(\tau^*), \tau^*) 1_{\{\tau_d \leq \boldsymbol{\rho}(\tau_d)\}} + U(\boldsymbol{\rho}(\tau^*), \tau^*) 1_{\{\tau_d > \boldsymbol{\rho}(\tau_d)\}}\right] \\
&= \mathbb{E}\left[U(\boldsymbol{\rho}(\tau^*), \tau_d) 1_{\{\tau_d \leq \boldsymbol{\rho}(\tau_d)\}} + U(\boldsymbol{\rho}(\tau_d), \tau_u(\boldsymbol{\rho}(\tau_d))) 1_{\{\tau_d > \boldsymbol{\rho}(\tau_d)\}}\right] \\
&= \mathbb{E}\left[1_{\{\tau_d \leq \boldsymbol{\rho}(\tau_d)\}} \mathbb{E}_{\tau_d}[U(\boldsymbol{\rho}(\tau^*), \tau_d)] + 1_{\{\tau_d > \boldsymbol{\rho}(\tau_d)\}} \mathbb{E}_{\boldsymbol{\rho}(\tau_d)}[U(\boldsymbol{\rho}(\tau_d), \tau_u(\boldsymbol{\rho}(\tau_d)))]\right] \\
&\geq \mathbb{E}\left[1_{\{\tau_d \leq \boldsymbol{\rho}(\tau_d)\}} V^1_{\tau_d} + 1_{\{\tau_d > \boldsymbol{\rho}(\tau_d)\}} V^2_{\boldsymbol{\rho}(\tau_d)}\right] \\
&\geq \inf_{\rho \in \mathcal{T}} \mathbb{E}\left[1_{\{\tau_d \leq \rho\}} V^1_{\tau_d} + 1_{\{\tau_d > \rho\}} V^2_{\rho}\right] \\
&= V,
\end{aligned}$$

where the fifth inequality follows from the definition of V^1 in (3.2.3) and the fact that $\boldsymbol{\rho}(\tau^*) \geq \tau_d$ on $\{\boldsymbol{\rho}(\tau_d) \geq \tau_d\}$. As this holds for arbitrary $\boldsymbol{\rho} \in \mathbb{T}^{ii}$, the conclusion follows. □

Proof of Theorem 3.2.3. This follows from Lemmas 3.4.1-3.4.4. □

3.5 Some insight into the continuous-time version

We can also consider the continuous time version of the stopper-stopper problem. If we want to follow the argument in Section 3.4, there are mainly two technical parts we might need to handle as opposed to the discrete-time case, which are as follows.

- We may need to show that V^1 and V^2 defined in (3.2.3) and (3.2.4) have RCLL modifications.
- On an intuitive level, the optimizers (or choose to be ε-optimizers in continuous time) $\rho_u(\cdot)$ and $\tau_u(\cdot)$ are maps from $\mathcal{T}$ to $\mathcal{T}$. Yet this may not be easy to prove in continuous time, as opposed to the argument in Lemma 3.4.2.

In order to address the two points above, we may have to assume some continuity of U in (s, t) (maybe also in ω). On the other hand, with such continuity, there will essentially be no difference between using stopping strategies of Type I and using stopping strategies of Type II, as opposed to the discrete-time case (see Examples 3.1.1 and 3.1.2).

In the next chapter, we will extent our results to continuous time case. We shall use the theory of optimal stopping in a general framework, which can help us avoid the two technical difficulties listed above.

CHAPTER IV

On a stopping game in continuous time

4.1 Introduction

On a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T})$, we consider the zero-sum optimal stopping games

$$\overline{G} := \inf_{\boldsymbol{\rho}} \sup_{\tau \in \mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}(\tau), \tau)] \quad \text{and} \quad \underline{G} := \sup_{\boldsymbol{\tau}} \inf_{\rho \in \mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}(\tau), \tau)]$$

in continuous time, where $U(s,t)$ is $\mathcal{F}_{s \vee t}$-measurable, $\mathcal{T}$ is the set of stopping times, and $\boldsymbol{\rho}, \boldsymbol{\tau} : \mathcal{T} \mapsto \mathcal{T}$ satisfy certain non-anticipativity conditions. In order to avoid the technical difficulties stemming from the verification of path regularity of some related processes (whether they are right continuous and have left limits), we work within the general framework of optimal stopping developed in [60–62]. We convert the problems into a corresponding Dynkin game, and show that $\overline{G} = \underline{G} = V$, where V is the value of the Dynkin game. This result extends the one in Chapter III to the continuous-time case and can be viewed as an application of the results in [61], which weakens the usual path regularity assumptions on the reward processes.

It is worth noting that in Chapter III two different types of non-anticipativity conditions are imposed for $\overline{G}$ and $\underline{G}$ respectively, for otherwise it can be the case that $\overline{G} \neq \underline{G}$. Now in the continuous-time case, we still have this inequality in general (see Remark 2.1). But by assuming U is right continuous along stopping times in the sense of expectation as in [62], we are able to show that there is no essential difference between the two types of non-anticipativity conditions.

The rest of the chapter is organized as follows. In the next section, we introduce the setup and the main result. In Section 4.3, we give the proof of the main result. In Section 4.4, we briefly discuss about the existence of optimal stopping strategies.

4.2 The setup and the main result

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtrated probability space, where $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ is the filtration satisfying the usual conditions with $T \in (0,\infty)$ the time horizon in continuous time. Let $\mathcal{T}_t$ and $\mathcal{T}_{t+}$ be the set of $\mathbb{F}$-stopping times taking values in $[t,T]$ and $(t,T]$ respectively, $t\in[0,T)$. Denote $\mathcal{T}_T := \mathcal{T}_{T+} := \{T\}$ and $\mathcal{T} := \mathcal{T}_0$. We shall often omit "a.s." when a property holds outside a P-null set. Recall the definition of admissible families of random variables, e.g., in [62].

Definition 4.2.1. A family $\{X(\sigma),\ \sigma\in\mathcal{T}\}$ is admissible if for all $\sigma\in\mathcal{T}$, $X(\sigma)$ is a bounded $\mathcal{F}_\sigma$-measurable random variable, and for all $\sigma_1,\sigma_2\in\mathcal{T}$, $X(\sigma_1)=X(\sigma_2)$ on $\{\sigma_1=\sigma_2\}$.

Definition 4.2.2. A family $\{Y(\rho,\tau),\ \rho,\tau\in\mathcal{T}\}$ is biadmissible if for all $\rho,\tau\in\mathcal{T}$, $Y(\rho,\tau)$ is an $\mathcal{F}_{\rho\vee\tau}$-measurable bounded random variable, and for all $\rho_1,\rho_2,\tau_1,\tau_2\in\mathcal{T}$, $Y(\rho_1,\tau_1)=Y(\rho_2,\tau_2)$ on $\{\rho_1=\rho_2\}\cap\{\tau_1=\tau_2\}$.

Let us also recall the two types of stopping strategies defined in Chapter III.

Definition 4.2.3. $\boldsymbol{\rho}$ is a stopping strategy of Type I (resp. II), if $\boldsymbol{\rho}:\ \mathcal{T}\mapsto\mathcal{T}$ satisfies the "non-anticipativity" condition of Type I (resp. II), i.e., for any $\sigma_1,\sigma_2\in\mathcal{T}$,

$$\text{either}\quad \boldsymbol{\rho}(\sigma_1)=\boldsymbol{\rho}(\sigma_2)\leq \text{(resp. } <\text{)}\ \sigma_1\wedge\sigma_2\quad\text{or}\quad \boldsymbol{\rho}(\sigma_1)\wedge\boldsymbol{\rho}(\sigma_2) > \text{(resp. } \geq\text{)}\ \sigma_1\wedge\sigma_2. \tag{4.2.1}$$

Denote by $\mathbb{T}^i$ (resp. $\mathbb{T}^{ii}$) the set of stopping strategies of Type I (resp. II).

Below is an interesting property for the non-anticipative stopping strategies of Type I (but not Type II).

Proposition 4.2.4. *For any* $\boldsymbol{\rho} \in \mathbb{T}^i$,

$$\boldsymbol{\rho}(\boldsymbol{\rho}(T)) = \boldsymbol{\rho}(T).$$

Proof. Since

$$\boldsymbol{\rho}(\boldsymbol{\rho}(T)) \wedge \boldsymbol{\rho}(T) \leq \boldsymbol{\rho}(T) = \boldsymbol{\rho}(T) \wedge T,$$

by (4.2.1) we have that

$$\boldsymbol{\rho}(\boldsymbol{\rho}(T)) = \boldsymbol{\rho}(T) \leq \boldsymbol{\rho}(T) \wedge T.$$

□

Let $\{U(\rho,\tau),\ \rho,\tau \in \mathcal{T}\}$ be an biadmissible family. Consider the optimal stopping games

$$\overline{A} := \inf_{\boldsymbol{\rho}\in\mathbb{T}^i} \sup_{\tau\in\mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}(\tau),\tau)] \quad \text{and} \quad \underline{A} := \sup_{\boldsymbol{\tau}\in\mathbb{T}^i} \inf_{\rho\in\mathcal{T}} \mathbb{E}[U(\rho,\boldsymbol{\tau}(\rho))].$$

and

$$\overline{B} := \inf_{\boldsymbol{\rho}\in\mathbb{T}^{ii}} \sup_{\tau\in\mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}(\tau),\tau)] \quad \text{and} \quad \underline{B} := \sup_{\boldsymbol{\tau}\in\mathbb{T}^{ii}} \inf_{\rho\in\mathcal{T}} \mathbb{E}[U(\rho,\boldsymbol{\tau}(\rho))].$$

We shall convert the problems into a corresponding Dynkin game. In order to do so, let us introduce two families of random variables that will represent the payoffs in the Dynkin game.

$$V^1(\tau) := \operatorname*{ess\,inf}_{\rho\in\mathcal{T}_\tau} \mathbb{E}_\tau[U(\rho,\tau)], \quad \tau \in \mathcal{T} \tag{4.2.2}$$

and

$$V^2(\rho) := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\rho} \mathbb{E}_\rho[U(\rho,\tau)], \quad \rho \in \mathcal{T}, \tag{4.2.3}$$

where $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$. Observe that

$$V^1(\sigma) \leq U(\sigma,\sigma) \leq V^2(\sigma), \quad \sigma \in \mathcal{T}.$$

Define the corresponding Dynkin game as follows:

$$\overline{V} := \inf_{\rho\in\mathcal{T}} \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[V^1(\tau)1_{\{\tau\leq\rho\}} + V^2(\rho)1_{\{\tau>\rho\}}\right], \tag{4.2.4}$$

$$\underline{V} := \sup_{\tau\in\mathcal{T}} \inf_{\rho\in\mathcal{T}} \mathbb{E}\left[V^1(\tau)1_{\{\tau\leq\rho\}} + V^2(\rho)1_{\{\tau>\rho\}}\right]. \tag{4.2.5}$$

Recall the (uniform) right continuity in expectation along stopping times defined in, e.g., [62].

Definition 4.2.5. An admissible family $\{X(\sigma),\ \sigma \in \mathcal{T}\}$ is said to be right continuous along stopping times in expectation (RCE) if for any $\sigma \in \mathcal{T}$ and any $(\sigma_n)_n \subset \mathcal{T}$ with $\sigma_n \searrow \sigma$, one has

$$\mathbb{E}[X(\sigma)] = \lim_{n\to\infty} \mathbb{E}[X(\sigma_n)].$$

Definition 4.2.6. A biadmissible family $\{Y(\rho,\tau),\ \rho,\tau \in \mathcal{T}\}$ is said to be uniformly right continuous along stopping times in expectation (URCE) if for any $\rho,\tau \in \mathcal{T}$ and any $(\rho_n)_n, (\tau_n)_n \subset \mathcal{T}$ with $\rho_n \searrow \rho$ and $\tau_n \searrow \tau$, one has

$$\lim_{n\to\infty} \sup_{\rho\in\mathcal{T}} \mathbb{E}|Y(\rho,\tau) - Y(\rho,\tau_n)| = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup_{\tau\in\mathcal{T}} \mathbb{E}|Y(\rho,\tau) - Y(\rho_n,\tau)| = 0$$

Below is the main result of this chapter.

Theorem 4.2.7. *Assume the biadmissible family $\{U(\rho,\tau),\ \rho,\tau \in \mathcal{T}\}$ is URCE. We have that*

$$\overline{A} = \underline{A} = \overline{B} = \underline{B} = \overline{V} = \underline{V}.$$

Remark 4.2.8. Without the right continuity assumption of U, it may fail that $\overline{A} = \underline{A}$ or $\overline{B} = \underline{B}$, even for some natural choices of U. For example. let $U(s,t) = |f(s)-f(t)|$, where

$$f(t) = \begin{cases} 0\ , & 0 \le t \le T/2, \\ 1\ , & T/2 < t \le T. \end{cases}$$

Then the problems related to $\overline{A}, \underline{A}, \overline{B}, \underline{B}$ become deterministic.

Let us first show that $\overline{A} = 1$. Take $\boldsymbol{\rho} \in \mathbb{T}^i$. If $\boldsymbol{\rho}(T) \le T/2$, then by taking $\tau = T$ we have that $\overline{A} = 1$. Otherwise $\boldsymbol{\rho}(T) > T/2$, and we take $\tau = T/2$; Then by the non-anticipativity condition (4.2.1), we have that $\boldsymbol{\rho}(T/2) \wedge \boldsymbol{\rho}(T) > (T/2) \wedge T = T/2$, which implies $\overline{A} = 1$. Next, consider $\underline{A}$. For any $\boldsymbol{\tau} \in \mathbb{T}^i$, by Proposition 4.2.4 $\boldsymbol{\tau}(\boldsymbol{\tau}(T)) = \boldsymbol{\tau}(T)$. Then letting $\rho = \boldsymbol{\tau}(T)$ we have that $\underline{A} = 0$. Therefore, $\overline{A} \ne \underline{A}$.

Now by taking $\boldsymbol{\rho}(\tau) = \tau$ we have that $\overline{B} = 0$. Let us consider $\underline{B}$. Let $\boldsymbol{\tau} \in \mathbb{T}^{ii}$ defined as

$$\boldsymbol{\tau}(\rho) = \begin{cases} T\ , & 0 \le \rho \le T/2, \\ T/2\ , & T/2 < \rho \le T. \end{cases}$$

Then for any $\rho \in \mathcal{T}$, $U(\rho, \boldsymbol{\tau}(\rho)) = 1$ and thus $\underline{B} = 1$. Therefore, $\overline{B} \ne \underline{B}$.

4.2.1 A sufficient condition for U to be URCE

Let $W : [0,T] \times [0,T] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be $\mathcal{B}([0,T]) \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$-measurable. Assume that W satisfies the Lipschitz condition, i.e., there exists some $L \in (0, \infty)$ such that

$$|W(s_1, t_1, x_1, y_1) - W(s_2, t_2, x_2, y_2)| \le L(|s_1 - s_2| + |t_1 - t_2| + |x_1 - x_2| + |y_1 - y_2|).$$

Let $f = (f_t)_{0 \le t \le T}$ and $g = (g_t)_{0 \le t \le T}$ be two bounded and right continuous $\mathbb{F}$-progressively measurable processes.

Proposition 4.2.9. *The family $\{U(\rho, \tau) := W(\rho, \tau, f_\rho, g_\tau),\ \rho, \tau \in \mathcal{T}\}$ is biadmissible and URCE.*

Proof. The biadmissibility is easy to check. Let us check U satisfies URCE: For any $\rho, \tau \in \mathcal{T}$ and any $(\tau_n)_n \subset \mathcal{T}$ with $\tau_n \searrow \tau$, we have that

$$\lim_{n\to\infty} \sup_{\rho\in\mathcal{T}} \mathbb{E}|U(\rho,\tau) - U(\rho,\tau_n)| \leq L \lim_{n\to\infty} \mathbb{E}\left[|\tau - \tau_n| + |f_\tau - f_{\tau_n}|\right] = 0.$$

□

4.2.2 An application

Let $U(t,s) = \mathcal{U}(f_t - g_s)$, where $\mathcal{U} : \mathbb{R} \mapsto \mathbb{R}$ is a utility function, and f and g are two right continuous progressively measurable process. Consider

$$\overline{\mathbb{B}} := \sup_{\boldsymbol{\rho}\in\mathbb{T}^{ii}} \inf_{\tau\in\mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}(\tau),\tau)].$$

This problem can be interpreted as the one in which an investor longs an American option f and shorts an American option g, and the goal is to choose an optimal stopping strategy to maximize the utility according to the stopping behavior of the holder of g. Here we assume that the maturities of f and g are the same (i.e., T). This is without loss of generality. Indeed for instance, if the maturity of f is $\hat{t} < T$, then we can define $f(t) = f(\hat{t})$ for $t \in (\hat{t}, T]$.

4.3 Proof of Theorem 4.2.7

We will only prove that $\overline{A} = \overline{V} = \underline{V}$, and the proof we provide in this section also works for $\underline{A}, \overline{B}$ and $\underline{B}$. Throughout this section, we assume that the biadmissible family $\{U(\rho,\tau),\ \rho,\tau \in \mathcal{T}\}$ is URCE.

Lemma 4.3.1. $\overline{V} = \underline{V}$.

Proof. The argument below (2.2) in [62] shows that $\{V^1(\tau),\ \tau \in \mathcal{T}\}$ and $\{V^2(\rho),\ \rho \in \mathcal{T}\}$ are admissible. By [62, Theorem 2.2], V^1 and V^2 are RCE because U is assumed to be URCE. Then by [61, Theorem 3.6] we have that $\overline{V} = \underline{V}$. □

Remark 4.3.2. It follows from the construction in [61] that when U is URCE then, the common value of $\overline{V}$ and $\underline{V}$ does not change if we replace $\{\tau \le \rho\}$ and $\{\tau > \rho\}$ in (4.2.4) and (4.2.5) with $\{\tau < \rho\}$ and $\{\tau \ge \rho\}$ respectively. In the rest of the chapter we will also use this replaced version when necessary without pointing this out.

Lemma 4.3.3. *For any $\varepsilon > 0$ and $\tau \in \mathcal{T}$, there exists $\rho_\tau \in \mathcal{T}_{\tau+}$, such that*

$$\mathbb{E}|\mathbb{E}_\tau[U(\rho_\tau, \tau)] - V^1(\tau)| < \varepsilon.$$

A similar result holds for V^2.

Proof. First let us show that

$$V^1(\tau) = \operatorname*{ess\,inf}_{\rho \in \mathcal{T}_{\tau+}} \mathbb{E}_\tau[U(\rho, \tau)].$$

Obviously $V^1(\tau) \le \operatorname{ess\,inf}_{\rho \in \mathcal{T}_{\tau+}} \mathbb{E}_\tau[U(\rho, \tau)]$. To show the reverse inequality, let us first fix $\tau \in \mathcal{T}$. For any $\rho_0 \in \mathcal{T}_\tau$, take $\rho_n = (\rho_0 + (T - \rho_0)/n) \wedge T \in \mathcal{T}_{\tau+}$, $n = 1, 2, \ldots$. Then $\rho_n \searrow \rho_0$. By the URCE assumption of U, $\mathbb{E}|U(\rho_n, \tau) - U(\rho, \tau)| \to 0$. Hence, there exists a subsequence $(n_k)_k$ such that $U(\rho_{n_k}, \tau) \to U(\rho_0, \tau)$ a.s.. Therefore,

$$\operatorname*{ess\,inf}_{\rho \in \mathcal{T}_{\tau+}} \mathbb{E}_\tau[U(\rho, \tau)] \le \lim_{k \to \infty} \mathbb{E}_\tau[U(\rho_{n_k}, \tau)] = \mathbb{E}_\tau[U(\rho_0, \tau)]$$

By the arbitrariness of ρ_0, we have that $V^1(\tau) \ge \operatorname{ess\,inf}_{\rho \in \mathcal{T}_{\tau+}} \mathbb{E}_\tau[U(\rho, \tau)]$.

Next, fix $\tau \in \mathcal{T}$. Since the family $\{\mathbb{E}_\tau[U(\rho, \tau)] : \rho \in \mathcal{T}_{\tau+}\}$ is closed under pairwise minimization, by, e.g., [54, Theorem A.3], there exists $(\rho_n) \in \mathcal{T}_{\tau+}$ such that

$$\lim_{n \to \infty} \mathbb{E}_\tau[U(\rho_n, \tau)] = \operatorname*{ess\,inf}_{\rho \in \mathcal{T}_{\tau+}} \mathbb{E}_\tau[U(\rho, \tau)] = V^1(\tau).$$

Since U and $V^1(\tau)$ are bounded, we have that

$$\lim_{n \to \infty} \mathbb{E}|\mathbb{E}_\tau[U(\rho_n, \tau)] - V^1(\tau)| = 0,$$

which implies the result. $\square$

Lemma 4.3.4. $\overline{A} \leq \overline{V}$.

Proof. Take $\varepsilon > 0$. Let $\rho_\varepsilon \in \mathcal{T}$ be an ε-optimizer of $\overline{V}$, i.e.,

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}\left[1_{\{\rho_\varepsilon \leq \tau\}} V^2(\rho_\varepsilon) + 1_{\{\rho_\varepsilon > \tau\}} V^1(\tau)\right] < \overline{V} + \varepsilon.$$

For any $\tau \in \mathcal{T}$, by Lemma 4.3.3 there exists $\rho_\varepsilon^1(\tau) \in \mathcal{T}_{\tau+}$ such that

$$\mathbb{E}|\mathbb{E}_\tau[U(\rho_\varepsilon^1(\tau), \tau) - V^1(\tau)| < \varepsilon.$$

Define $\boldsymbol{\rho}_\varepsilon$ as

$$\boldsymbol{\rho}_\varepsilon(\tau) := \rho_\varepsilon 1_{\{\tau \geq \rho_\varepsilon\}} + \rho_\varepsilon^1(\tau) 1_{\{\tau < \rho_\varepsilon\}}, \quad \tau \in \mathcal{T}. \tag{4.3.1}$$

Let us show that $\boldsymbol{\rho}_\varepsilon$ is in $\mathbb{T}^i$. First, for any $\tau \in \mathcal{T}$, $\boldsymbol{\rho}_\varepsilon(\tau) \in \mathcal{T}$ since for any $t \in [0, T]$,

$$\begin{aligned} \{\boldsymbol{\rho}_\varepsilon(\tau) \leq t\} &= (\{\tau \geq \rho_\varepsilon\} \cap \{\rho_\varepsilon \leq t\}) \cup \left(\{\tau < \rho_\varepsilon\} \cap \{\rho_\varepsilon^1(\tau) \leq t\}\right) \\ &= (\{\tau \geq \rho_\varepsilon\} \cap \{\rho_\varepsilon \leq t\}) \cup \left(\{\tau < \rho_\varepsilon\} \cap \{\tau \leq t\} \cap \{\rho_\varepsilon^1(\tau) \leq t\}\right) \in \mathcal{F}_t. \end{aligned}$$

Then let us show that $\boldsymbol{\rho}_\varepsilon$ satisfies the non-anticipativity condition of Type I in (4.2.1). Take $\tau_1, \tau_2 \in \mathcal{T}$. Assume that $\boldsymbol{\rho}_\varepsilon(\tau_1) \leq \tau_1 \wedge \tau_2 \leq \tau_1$. If $\tau_1 < \rho_\varepsilon \leq T$, then $\boldsymbol{\rho}_\varepsilon(\tau_1) = \rho_\varepsilon^1(\tau_1) > \tau_1$, contradiction. Hence $\tau_1 \geq \rho_\varepsilon$, and thus $\boldsymbol{\rho}_\varepsilon(\tau_1) = \rho_\varepsilon \leq \tau_1 \wedge \tau_2 \leq \tau_2$, which implies $\boldsymbol{\rho}_\varepsilon(\tau_2) = \rho_\varepsilon = \boldsymbol{\rho}_\varepsilon(\tau_1) \leq \tau_1 \wedge \tau_2$. Assume that $\boldsymbol{\rho}_\varepsilon(\tau_1) > \tau_1 \wedge \tau_2$. If $\boldsymbol{\rho}_\varepsilon(\tau_2) \leq \tau_1 \wedge \tau_2$ then we can use the previous argument to get that $\boldsymbol{\rho}_\varepsilon(\tau_1) = \boldsymbol{\rho}_\varepsilon(\tau_2) \leq \tau_1 \wedge \tau_2$ which is a contradiction, and thus $\boldsymbol{\rho}_\varepsilon(\tau_2) > \tau_1 \wedge \tau_2$.

We have that

$$
\begin{aligned}
\overline{A} &\leq \sup_{\tau\in\mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}_\varepsilon(\tau),\tau)] \\
&= \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[U(\rho_\varepsilon,\tau)1_{\{\rho_\varepsilon\leq\tau\}} + U(\rho_\varepsilon^1(\tau),\tau)1_{\{\rho_\varepsilon>\tau\}}\right] \\
&= \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[1_{\{\rho_\varepsilon\leq\tau\}}\mathbb{E}_{\rho_\varepsilon}[U(\rho_\varepsilon,\tau)] + 1_{\{\rho_\varepsilon>\tau\}}\mathbb{E}_\tau[U(\rho_\varepsilon^1(\tau),\tau)]\right] \\
&\leq \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[1_{\{\rho_\varepsilon\leq\tau\}}V^2(\rho_\varepsilon) + 1_{\{\rho_\varepsilon>\tau\}}V^1(\tau)\right] + \varepsilon \\
&\leq \overline{V} + 2\varepsilon.
\end{aligned}
$$

□

Remark 4.3.5. Once we show Theorem 4.2.7, we can see that $\boldsymbol{\rho}_\varepsilon \in \mathbb{T}^i \subset \mathbb{T}^{ii}$ defined in (4.3.1) is a 2ε-optimizer for $\overline{A}$ and $\overline{B}$.

Lemma 4.3.6. $\overline{A} \geq \underline{V}$.

Proof. Fix $\varepsilon > 0$. Let $\tau_\varepsilon \in \mathcal{T}$ be an ε-optimizer for $\underline{V}$. For any $\rho \in \mathcal{T}$, by Lemma 4.3.3 there exists $\tau_\varepsilon^2(\rho) \in \mathcal{T}_{\rho+}$ such that

$$\mathbb{E}|\mathbb{E}_\tau[U(\rho,\tau_\varepsilon^2(\rho)) - V^2(\rho)| < \varepsilon$$

For any $\boldsymbol{\rho} \in \mathbb{T}^i$, define $\tau_{\boldsymbol{\rho}}$ as

$$\tau_{\boldsymbol{\rho}} := \tau_\varepsilon 1_{\{\tau_\varepsilon\leq\boldsymbol{\rho}(\tau_\varepsilon)\}} + \tau_\varepsilon^2(\boldsymbol{\rho}(\tau_\varepsilon))1_{\{\tau_\varepsilon>\boldsymbol{\rho}(\tau_\varepsilon)\}}.$$

Using a similar argument as in the proof of Lemma 4.3.4, we can show that $\tau_{\boldsymbol{\rho}}$ is a stopping time.

Since $\mathbb{T}^i \subset \mathbb{T}^{ii}$, and also in order to let our proof also fit for $\overline{B}$, we shall only use the non-anticipativity condition of Type II for $\boldsymbol{\rho}$ in (4.2.1), although $\boldsymbol{\rho} \in \mathbb{T}^i$. By (4.2.1) w.r.t. Type II,

$$\text{either}\quad \boldsymbol{\rho}(\tau_{\boldsymbol{\rho}}) = \boldsymbol{\rho}(\tau_\varepsilon) < \tau_\varepsilon \wedge \tau_{\boldsymbol{\rho}} \quad\text{or}\quad \boldsymbol{\rho}(\tau_{\boldsymbol{\rho}}) \wedge \boldsymbol{\rho}(\tau_\varepsilon) \geq \tau_\varepsilon \wedge \tau_{\boldsymbol{\rho}}.$$

Therefore,

$$\text{if} \quad \tau_\varepsilon \leq \boldsymbol{\rho}(\tau_\varepsilon), \quad \textit{then} \quad \boldsymbol{\rho}(\tau_{\boldsymbol{\rho}}) \geq \tau_\varepsilon \wedge \tau_{\boldsymbol{\rho}} = \tau_\varepsilon = \tau_{\boldsymbol{\rho}},$$

and

$$\text{if} \quad \tau_\varepsilon > \boldsymbol{\rho}(\tau_\varepsilon), \quad \textit{then} \quad \tau_{\boldsymbol{\rho}} = \tau_\varepsilon^2(\boldsymbol{\rho}(\tau_\varepsilon)) > \boldsymbol{\rho}(\tau_\varepsilon) \implies \boldsymbol{\rho}(\tau_\varepsilon) < \tau_{\boldsymbol{\rho}} \wedge \tau_\varepsilon \implies \boldsymbol{\rho}(\tau_\varepsilon) = \boldsymbol{\rho}(\tau_{\boldsymbol{\rho}}).$$

Now we have that

$$\begin{aligned}
&\sup_{\tau \in \mathcal{T}} \mathbb{E}[U(\boldsymbol{\rho}(\tau), \tau))] \\
&\geq \mathbb{E}[U(\boldsymbol{\rho}(\tau_{\boldsymbol{\rho}}), \tau_{\boldsymbol{\rho}})] \\
&= \mathbb{E}\left[U(\boldsymbol{\rho}(\tau_{\boldsymbol{\rho}}), \tau_{\boldsymbol{\rho}}) 1_{\{\tau_\varepsilon \leq \boldsymbol{\rho}(\tau_\varepsilon)\}} + U(\boldsymbol{\rho}(\tau_{\boldsymbol{\rho}}), \tau_{\boldsymbol{\rho}}) 1_{\{\tau_\varepsilon > \boldsymbol{\rho}(\tau_\varepsilon)\}}\right] \\
&= \mathbb{E}\left[U(\boldsymbol{\rho}(\tau_{\boldsymbol{\rho}}), \tau_\varepsilon) 1_{\{\tau_\varepsilon \leq \boldsymbol{\rho}(\tau_\varepsilon)\}} + U(\boldsymbol{\rho}(\tau_\varepsilon), \tau_\varepsilon^2(\boldsymbol{\rho}(\tau_\varepsilon))) 1_{\{\tau_\varepsilon > \boldsymbol{\rho}(\tau_\varepsilon)\}}\right] \\
&= \mathbb{E}\left[1_{\{\tau_\varepsilon \leq \boldsymbol{\rho}(\tau_\varepsilon)\}} \mathbb{E}_{\tau_\varepsilon}[U(\boldsymbol{\rho}(\tau_{\boldsymbol{\rho}}), \tau_\varepsilon)] + 1_{\{\tau_\varepsilon > \boldsymbol{\rho}(\tau_\varepsilon)\}} \mathbb{E}_{\boldsymbol{\rho}(\tau_\varepsilon)}[U(\boldsymbol{\rho}(\tau_\varepsilon), \tau_\varepsilon^2(\boldsymbol{\rho}(\tau_\varepsilon)))]\right] \\
&\geq \mathbb{E}\left[1_{\{\tau_\varepsilon \leq \boldsymbol{\rho}(\tau_\varepsilon)\}} V^1(\tau_\varepsilon) + 1_{\{\tau_\varepsilon > \boldsymbol{\rho}(\tau_\varepsilon)\}} V^2(\boldsymbol{\rho}(\tau_\varepsilon))\right] - \varepsilon \\
&\geq \inf_{\rho \in \mathcal{T}} \mathbb{E}\left[1_{\{\tau_\varepsilon \leq \rho\}} V^1(\tau_\varepsilon) + 1_{\{\tau_\varepsilon > \rho\}} V^2(\rho)\right] - \varepsilon \\
&\geq \underline{V} - 2\varepsilon,
\end{aligned}$$

where the fifth inequality follows from the definition of V^1 in (4.2.2) and the fact that $\boldsymbol{\rho}(\tau_{\boldsymbol{\rho}}) \geq \tau_\varepsilon$ on $\{\tau_\varepsilon \leq \boldsymbol{\rho}(\tau_\varepsilon)\}$. By the arbitrariness of $\boldsymbol{\rho} \in \mathbb{T}^i$ and ε, the conclusion follows. □

Proof of Theorem 4.2.7. This follows from Lemmas 4.3.1, 4.3.4 and 4.3.6. □

4.4 Existence of optimal stopping strategies

If we impose a strong left continuity assumptions on U in addition (see e.g. [60–62]), we would get the existence of the optimal stopping strategies for $\overline{B}$ and $\underline{B}$. For example let us consider $\overline{B}$. Indeed, the left and right continuity of U would imply the required left and right continuity of V^1 and V^2, as well as the existence of an optimal stopping time $\rho_0^1(\tau) \in \mathcal{T}_\tau$ for $V^1(\tau)$. The continuity of V^1 and V^2 would further imply the existence of an optimal stopping time ρ_0 for $\overline{V}$ (see [61]). Then define

$$\boldsymbol{\rho}_0(\tau) := \rho_0 1_{\{\tau \geq \rho_0\}} + \rho_0^1(\tau) 1_{\{\tau < \rho_0\}}, \quad \tau \in \mathcal{T}.$$

Following the proof of Lemma 4.3.4, one can show that $\boldsymbol{\rho}_0 \in \mathbb{T}^{ii}$ is optimal for $\overline{B}$. One should note that in this case $\boldsymbol{\rho}_0$ may not be in $\mathbb{T}^i$ as opposed to $\boldsymbol{\rho}_\varepsilon$ define in (4.3.1), this is because here it is possible that $\rho_0(\tau) = \tau$ on $\{\tau < T\}$.

On the other hand, the existence of optimal stopping strategies for $\overline{A}$ and $\underline{A}$ may fail in general even if U is quite regular. For example, let $U(s,t) = |s-t|$. By taking $\boldsymbol{\rho}(\tau) = \tau$ we have that $\overline{B} = 0$ which is equal to $\overline{A}$ by Theorem 4.2.7. Now assume there exists some optimal $\hat{\boldsymbol{\rho}} \in \mathbb{T}^i$ for $\overline{A}$. That is,

$$\sup_\tau |\hat{\boldsymbol{\rho}}(\tau) - \tau| = \overline{A} = 0.$$

Then we have that $\boldsymbol{\rho}(\tau) = \tau$ for any $\tau \in [0,T]$, which contradicts with the non-anticipativity condition of Type I by letting $\sigma_1 \neq \sigma_2$ in (4.2.1).

CHAPTER V

On an optimal stopping problem of an insider

5.1 Introduction

In this chapter we consider Shiryaev's optimal stopping problem:

$$v^{(\varepsilon)} = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E} B_{(\tau-\varepsilon)^+}, \tag{5.1.1}$$

where $T > 0$ is a fixed time horizon, $(B_t)_{0 \le t \le T}$ is the Brownian motion, $\varepsilon \in [0, T]$ is a constant, and $\mathcal{T}_{\varepsilon,T}$ is the set of stopping times taking values in $[\varepsilon, T]$. This can be thought of a problem of an insider in which she is allowed to peek ε into the future for the payoff before making her stopping decision.

We show that $v^{(\varepsilon)}$ is the solution of a corresponding path dependent reflected backward stochastic differential equation (RBSDEs). This is essentially an existence result, and it shows that an optimal stopping time exists. But the main advantage of using an RBSDE representation is that we can easily get the continuity of $v^{(\varepsilon)}$ with respect to ε from the stability of the RBSDEs. However, we want to compute the function as explicitly as possible, and the RBSDE representation of the problem does not help. This is because the problem is path dependent (one of the state variables would have be an entire path of length ε), and there is no numerical result available so far that can cover our case.

In fact, we will observe that $v^{(\varepsilon)} = \sqrt{\frac{2(T-\varepsilon)}{\pi}}$ if $\varepsilon \in [T/2, T]$, while as far as we know there is no explicit solution for $v^{(\varepsilon)}$ if $\varepsilon \in (0, T/2)$. But for smaller ε, there are only lower and upper bounds available. As the main result of this chapter, we provide the asymptotic behavior of $v^{(\varepsilon)}$ as $\varepsilon \searrow 0$ (see Theorem 5.3.1). As a byproduct, we also get Lévy's modulus of continuity theorem in the L^1 sense as opposed to the almost-surely sense (compare Corollary 5.3.4 and, e.g., [53, Theorem 9.25, page 114]).

5.2 First observations

Let $T > 0$ and let $\{B_t, t \in [0, T]\}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$ be the natural filtration augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$. We aim at the problem (5.1.1). But for the sake of generality, let us first look at the more general optimal stopping problem of an insider:

$$w = \sup_{\tau \in \mathcal{T}_{\varepsilon,T}} \mathbb{E}\left[\sum_{i=1}^{n} \phi^i_{(\tau - \varepsilon^i)+}\right], \tag{5.2.1}$$

where $(\phi^i_t)_{0 \le t \le T}$ is continuous and progressively measurable, $\varepsilon^i \in [0, T]$, $i = 1, \dots, n$, are given constants, and $\mathcal{T}_{\varepsilon,T}$ is the set of stopping times that lie between a constant $\varepsilon \in [0, T]$ and T. Observe that $\tau - \varepsilon^i$ is not a stopping time with respect to $\mathbb{F}$ for $\varepsilon^i > 0$. The solution to (5.2.1) is described by the following result:

Proposition 5.2.1. *Assume* $\mathbb{E}\big[\sup_{0 \le t \le T}(\xi^+_t)^2\big] < \infty$, *where* $\xi_t = \sum_{i=1}^n \phi^i_{(t-\varepsilon^i)+}$, $0 \le t \le T$. *Then the value defined in* (5.2.1) *can be calculated using a reflected backward stochastic differential equation (RBSDE). More precisely,* $w = \mathbb{E}Y_\varepsilon$, *for any* $\varepsilon \in [0, T]$, *where* $(Y_t)_{0 \le t \le T}$ *satisfies the RBSDE*

$$\begin{aligned} &\xi_t \le Y_t = \xi_T - \int_t^T Z_s dW_s + (K_T - K_t),\ 0 \le t \le T, \\ &\int_0^T (Y_t - \xi_t) dK_t = 0, \end{aligned} \tag{5.2.2}$$

Moreover, there exists an optimal stopping time $\hat{\tau}$ described by

$$\hat{\tau} = \inf\{t \in [\varepsilon, T] : Y_t = \xi_t\}.$$

Remark 5.2.2. One should note that the optimal stopping problem we are considering is path dependent (i.e. not of Markovian type) and therefore one would not be able to write down a classical free boundary problem corresponding to (5.1.1).

We prefer to use an RBSDE representation of the value function instead of directly using the representation directly from the classical optimal stopping theory because we want to use the stability result, which we will state in Corollary 5.2.3, associated with the former.

Proof of Proposition 5.2.1. For any $\tau \in \mathcal{T}_{\varepsilon,T}$,

$$\mathbb{E}\xi_\tau = \mathbb{E}[\mathbb{E}[\xi_\tau|\mathcal{F}_\varepsilon]] \leq \mathbb{E}\Big[\operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{\varepsilon,T}} \mathbb{E}[\xi_\sigma|\mathcal{F}_\varepsilon]\Big].$$

Therefore,

$$w = \sup_{\tau\in\mathcal{T}_{\varepsilon,T}} \mathbb{E}\xi_\tau \leq \mathbb{E}\Big[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{\varepsilon,T}} \mathbb{E}[\xi_\tau|\mathcal{F}_\varepsilon]\Big]. \tag{5.2.3}$$

By [40, Theorem 5.2] there exists a unique solution (Y, Z, K) to the RBSDE in (5.2.2). Then by Proposition 2.3 (and its proof) in [40] we have

$$\sup_{\tau\in\mathcal{T}_{\varepsilon,T}} \mathbb{E}\xi_\tau \geq \mathbb{E}\xi_{\hat{\tau}} = \mathbb{E}Y_{\hat{\tau}} = \mathbb{E}Y_\varepsilon = \mathbb{E}\Big[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{\varepsilon,T}} \mathbb{E}[\xi_\tau|\mathcal{F}_\varepsilon]\Big].$$

Along with (5.2.3) the last inequality completes the proof. □

Now let us get back to Shiryaev's problem (5.1.1). As a corollary of Proposition 5.2.1, we have the following result for $v^{(\varepsilon)}$, $\varepsilon \in [0, T]$.

Corollary 5.2.3. *The value defined in* (5.1.1) *can be calculated using an RBSDE. More precisely, $v^\varepsilon = Y_0$ almost surely, where $(Y_t)_{0\leq t\leq T}$ satisfies the RBSDE* (5.2.2) *with ξ defined as $\xi_t = B_{(t-\varepsilon)^+},\ 0 \le t \le T$. Moreover, there exists an optimal stopping time $\tilde{\tau}$ described by*

$$\tilde{\tau} = \inf\{t \ge 0 : Y_t = B_{(t-\varepsilon)^+}\} \ge \varepsilon 1_{\{\varepsilon < T\}}, \quad a.s.. \tag{5.2.4}$$

Furthermore, the function $\varepsilon \to v^{(\varepsilon)}$, $\varepsilon \in [0, T]$, is a continuous function.

Proof. By Proposition 5.2.1 $v^{(\varepsilon)} = Y_0$ a.s., and $\tilde{\tau}$ defined in (5.2.4) is optimal. Besides, the continuity of $\varepsilon \to v^{(\varepsilon)}$, $\varepsilon \in [0, T]$ is a direct consequence of the stability of RBSDEs indicated by [40, Proposition 3.6]. Observe that for $\varepsilon \in (0, T)$ and $t \in [0, \varepsilon]$, $Y_t \ge \mathbb{E}[Y_\varepsilon | \mathcal{F}_t] > 0 = B_{(t-\varepsilon)^+}$ a.s.. Hence we have that $\tilde{\tau} \ge \varepsilon 1_{\{\varepsilon < T\}}$ a.s.. □

Remark 5.2.4. In the above result, since for any $\delta \in [0, \varepsilon]$

$$v^{(\varepsilon)} = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E} B_{(\tau-\varepsilon)^+} = \sup_{\tau \in \mathcal{T}_{\delta,T}} \mathbb{E} B_{(\tau-\varepsilon)^+},$$

we can conclude from Proposition 5.2.1 that $v^{(\varepsilon)} = \mathbb{E} Y_\delta$, which implies that $(Y_t)_{t\in[0,\varepsilon]}$ is a martingale.

Next, we will make some observations about the magnitude of the function $\varepsilon \to v^{(\varepsilon)}$:

Remark 5.2.5. Observe that for $\varepsilon \in (0, T)$, insider's value defined in (5.1.1) is strictly greater than 0 (and hence does strictly better than a stopper which does not posses the insider information):

$$v^{(\varepsilon)} \ge \mathbb{E}\Big[\max_{0 \le t \le \varepsilon \wedge (T-\varepsilon)} B_t\Big] = \sqrt{\frac{2}{\pi}\left(\varepsilon \wedge (T-\varepsilon)\right)} > v^{(0)} = 0,$$

which shows that there is an incentive for waiting. We also have an upper bound

$$v^{(\varepsilon)} \le \mathbb{E}\Big[\max_{0 \le t \le T} B_t\Big] = \sqrt{\frac{2T}{\pi}}.$$

In fact when $\varepsilon \in [T/2, T]$, $v^{(\varepsilon)}$ can be explicitly determined as

$$v^{(\varepsilon)} = \mathbb{E}\left[\max_{0 \le t \le T-\varepsilon} B_t\right] = \sqrt{\frac{2\,(T-\varepsilon)}{\pi}}, \quad \varepsilon \in [T/2, T].$$

and we have a strict lower bound for $\varepsilon \in [0, T/2)$

$$v^{(\varepsilon)} > \mathbb{E}\left[\max_{0\le t\le\varepsilon} B_t\right] = \sqrt{\frac{2\varepsilon}{\pi}}, \quad \varepsilon \in [0, T/2).$$

5.3 Asymptotic behavior of $v^{(\varepsilon)}$ as $\varepsilon \searrow 0$

The following theorem states that the order of $v^{(\varepsilon)}$ defined in (5.1.1) is $\sqrt{2\varepsilon\ln(1/\varepsilon)}$ as $\varepsilon \searrow 0$, which is the same as Levy's modulus for Brownian motion. Notice that

$$v^{(\varepsilon)} = \sup_{\tau\in\mathcal{T}_{\varepsilon,T}} \mathbb{E}[B_{\tau-\varepsilon} - B_\tau].$$

Theorem 5.3.1.

$$\lim_{\varepsilon\searrow 0} \frac{v^{(\varepsilon)}}{\sqrt{2\varepsilon\ln(1/\varepsilon)}} = 1. \tag{5.3.1}$$

In order to prepare the proof of the theorem, we will need two lemmas.

Lemma 5.3.2.

$$\liminf_{\varepsilon\searrow 0} \frac{v^{(\varepsilon)}}{\sqrt{2\varepsilon\ln(1/\varepsilon)}} \ge 1.$$

Proof. Let $d \in (0,1)$ be a constant, and define $\tau^* \in \mathcal{T}_{\varepsilon,T}$

$$\tau^* := \inf\{n\varepsilon :\ B_{(n-1)\varepsilon} - B_{n\varepsilon} \ge d\sqrt{2\varepsilon\ln(1/\varepsilon)},\ n = 1, \dots, [T/\varepsilon] - 1\} \wedge T.$$

Then

$$\begin{aligned}
&\sup_{\tau\in\mathcal{T}_{\varepsilon,T}} \mathbb{E}[B_{\tau-\varepsilon} - B_\tau] \\
&\ge \mathbb{E}[B_{\tau^*-\varepsilon} - B_{\tau^*}] \\
&= \mathbb{E}\left[(B_{\tau^*-\varepsilon} - B_{\tau^*})\, 1_{\{\tau^*\le\varepsilon[T/\varepsilon]-\varepsilon\}}\right] + \mathbb{E}\left[(B_{\tau^*-\varepsilon} - B_{\tau^*})\, 1_{\{\tau^*>\varepsilon[T/\varepsilon]-\varepsilon\}}\right] \\
&\ge d\sqrt{2\varepsilon\ln(1/\varepsilon)}\, P(\tau^* \le \varepsilon[T/\varepsilon] - \varepsilon) + \mathbb{E}\left[(B_{T-\varepsilon} - B_T)\, 1_{\{\tau^*>\varepsilon[T/\varepsilon]-\varepsilon\}}\right] \\
&= d\sqrt{2\varepsilon\ln(1/\varepsilon)}\, P(\tau^* \le \varepsilon[T/\varepsilon] - \varepsilon).
\end{aligned}$$

We have that

$$\begin{aligned}
P(\tau^* \le \varepsilon[T/\varepsilon]-\varepsilon) &= 1 - P\left(B_{(n-1)\varepsilon} - B_{n\varepsilon} < d\sqrt{2\varepsilon\ln(1/\varepsilon)},\ n = 1,\dots,[T/\varepsilon]-1\right)\\
&= 1 - \left[P\left(B_\varepsilon - B_0 < d\sqrt{2\varepsilon\ln(1/\varepsilon)}\right)\right]^{[T/\varepsilon]-1}\\
&= 1 - \left[1 - \int_{d\sqrt{2\varepsilon\ln(1/\varepsilon)}}^{\infty} \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}\,dx\right]^{[T/\varepsilon]-1}\\
&= 1 - (1-\alpha)^{\frac{1}{\alpha}([T/\varepsilon]-1)\alpha},
\end{aligned}$$

(5.3.2)

where

$$\alpha := \int_{d\sqrt{2\varepsilon\ln(1/\varepsilon)}}^{\infty} \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}\,dx = \frac{1}{2d\sqrt{\pi\ln(1/\varepsilon)}}\varepsilon^{d^2}(1+o(1)) \to 0,$$

by, e.g., [53, (9.20), page 112]. Since $d \in (0,1)$, $([T/\varepsilon]-1)\alpha \to \infty$, and thus

$$P(\tau^* \le \varepsilon[T/\varepsilon]-\varepsilon) \to 1, \quad \varepsilon \searrow 0.$$

Therefore,

$$\liminf_{\varepsilon\searrow 0} \frac{v^{(\varepsilon)}}{\sqrt{2\varepsilon\ln(1/\varepsilon)}} \ge \liminf_{\varepsilon\searrow 0} \left[d\,P(\tau^* \le \varepsilon[T/\varepsilon]-\varepsilon)\right] = d.$$

Then (5.3.1) follows by letting $d \nearrow 1$. $\square$

Lemma 5.3.3. *The family*

$$\left\{\frac{\sup_{\varepsilon\le t\le T}|B_{t-\varepsilon}-B_t|}{\sqrt{2\varepsilon\ln(1/\varepsilon)}} :\ \varepsilon \in \left(0, \frac{T\wedge 1}{2}\right]\right\}$$

is uniformly integrable.

Proof. Since

$$\begin{aligned}
\frac{\sup_{\varepsilon\le t\le T}|B_{t-\varepsilon}-B_t|}{\sqrt{2\varepsilon\ln(1/\varepsilon)}} &\le \frac{2\max_{1\le n\le[T/\varepsilon]+1}\sup_{(n-1)\varepsilon\le t,t'\le n\varepsilon}|B_t - B_{t'}|}{\sqrt{2\varepsilon\ln(1/\varepsilon)}}\\
&\le \frac{4\max_{1\le n\le[T/\varepsilon]+1}\sup_{(n-1)\varepsilon\le t\le n\varepsilon}|B_t - B_{(n-1)\varepsilon}|}{\sqrt{2\varepsilon\ln(1/\varepsilon)}},
\end{aligned}$$

it suffices to show that the family

$$\left\{ M_\varepsilon := \frac{\max_{1\le n\le [T/\varepsilon]+1} \sup_{(n-1)\varepsilon\le t\le n\varepsilon} |B_t - B_{(n-1)\varepsilon}|}{\sqrt{\varepsilon \ln(1/\varepsilon)}} : \ \varepsilon \in \left(0, \frac{T\wedge 1}{2}\right] \right\}$$

is uniformly integrable. For $a \ge 0$,

$$P(M_\varepsilon \le a) = \left[P\left(\sup_{0\le t\le \varepsilon} |B_t| \le a\sqrt{\varepsilon \ln(1/\varepsilon)} \right) \right]^{[T/\varepsilon]+1}.$$

Hence the density of M_ε, f_ε, satisfies that for $a \ge 0$,

$$\begin{aligned} f_\varepsilon(a) &\le ([T/\varepsilon]+1) \left[P\left(\sup_{0\le t\le \varepsilon} |B_t| \le a\sqrt{\varepsilon \ln(1/\varepsilon)} \right) \right]^{[T/\varepsilon]} \sqrt{\frac{8}{\pi}} \sqrt{\ln(1/\varepsilon)} e^{-\frac{\ln(1/\varepsilon)}{2} a^2} \\ &\le \frac{4T\sqrt{\ln(1/\varepsilon)}}{\varepsilon} e^{-\frac{\ln(1/\varepsilon)}{2} a^2}, \end{aligned}$$

where for the first inequality we use, e.g., [53, (8.3), page 96], and the fact that the density of $\sup_{0\le t\le\varepsilon} |B_t|$ is no greater than twice the density of $\sup_{0\le t\le\varepsilon} B_t$. Then we have that for $N > 0$,

$$\begin{aligned} &\mathbb{E}\left[M_\varepsilon 1_{\{M_\varepsilon > N\}} \right] = \int_N^\infty x f_\varepsilon(x) dx \le \frac{4T\sqrt{\ln(1/\varepsilon)}}{\varepsilon} \int_N^\infty x e^{-\frac{\ln(1/\varepsilon)}{2} x^2} dx \\ &= \frac{4T\varepsilon^{\frac{N^2}{2}-1}}{\sqrt{\ln(1/\varepsilon)}} \le \frac{T}{2^{\frac{N^2}{2}-3}\sqrt{\ln 2}}, \end{aligned}$$

i.e.,

$$\lim_{N\to\infty} \sup_{\varepsilon \in (0, \frac{T\wedge 1}{2}]} \mathbb{E}\left[M_\varepsilon 1_{\{M_\varepsilon > N\}} \right] = 0.$$

$\square$

Now let us turn to the proof of Theorem 5.3.1.

Proof of Theorem 5.3.1.

$$\begin{aligned} \limsup_{\varepsilon\searrow 0} \frac{\sup_{\tau\in\mathcal{T}_{\varepsilon,T}} \mathbb{E}[B_{\tau-\varepsilon} - B_\tau]}{\sqrt{2\varepsilon \ln(1/\varepsilon)}} &\le \limsup_{\varepsilon\searrow 0} \mathbb{E}\left[\frac{\sup_{\varepsilon\le t\le T} |B_{t-\varepsilon} - B_t|}{\sqrt{2\varepsilon \ln(1/\varepsilon)}} \right] \\ &\le \mathbb{E}\left[\limsup_{\varepsilon\searrow 0} \frac{\sup_{\varepsilon\le t\le T} |B_{t-\varepsilon} - B_t|}{\sqrt{2\varepsilon \ln(1/\varepsilon)}} \right] \\ &\le 1, \end{aligned}$$

where we apply Lemma 5.3.3 for the second inequality, and use Levy's modulus of continuity for Brownian motion (see, e.g., [53, Theorem 9.25, page 114]) for the third inequality. Together with (5.3.1), the conclusion follows. □

Using the above proof, we can actually show the following result, which is Lévy's modulus of continuity result in the L^1 sense, as opposed to the almost-surely sense (see, e.g., [53, Theorem 9.25, page 114]).

Corollary 5.3.4.

$$\lim_{\varepsilon \searrow 0} \frac{\sup_{\tau \in \mathcal{T}_{\varepsilon,T}} \mathbb{E}[B_{\tau-\varepsilon} - B_\tau]}{\sqrt{2\varepsilon \ln(1/\varepsilon)}} = \lim_{\varepsilon \searrow 0} \mathbb{E}\left[\frac{\sup_{\varepsilon \le t \le T}(B_{t-\varepsilon} - B_t)}{\sqrt{2\varepsilon \ln(1/\varepsilon)}}\right]$$
$$= \lim_{\varepsilon \searrow 0} \mathbb{E}\left[\frac{\sup_{\varepsilon \le t \le T} |B_{t-\varepsilon} - B_t|}{\sqrt{2\varepsilon \ln(1/\varepsilon)}}\right] = 1.$$

CHAPTER VI

On arbitrage and duality under model uncertainty and portfolio constraints

6.1 Introduction

We consider the fundamental theorem of asset pricing (FTAP) and hedging prices of European and American options under the non-dominated model certainty framework of [19] with *convex closed portfolio constraints* in discrete time. We first show that no arbitrage in the quasi-sure sense is equivalent to the existence of a set of probability measures; under each of these measures any admissible portfolio value process is a local super-martingale. Then we get the non-dominated version of optional decomposition under portfolio constraints. From this optional decomposition, we get the duality of super- and sub-hedging prices of European and American options. We also show that the optimal super-hedging strategies exist. Finally, we add options to the market and get the FTAP and duality of super-hedging prices of European options by using semi-static trading strategies (i.e., strategies dynamically trading in stocks and statically trading in options).

Our results generalize the ones in [43, Section 9] to a non-dominated model-uncertainty set-up, and extend the results in [19] to the case where portfolio constraints are involved. These conclusions are general enough to cover many interesting models with the so-called *delta constraints*; for example, when shorting stocks is not allowed, or some stocks enter or leave the market at certain periods.

Compared to [43, Section 9], the main difficulty in our setting is due to the fact that the set of probability measures does not admit a dominating measure. We use the measurable selection mechanism developed in [19] to overcome this difficulty, i.e., first get the FTAP and super-hedging result in one period, and then "measurably" glue each period together to get multiple-period versions. It is therefore of crucial importance to get the one-period results. In [19], Lemma 3.3 serves as a fundamental tool to show the FTAP and super-hedging result in one-period model, whose proof relies on an induction on the number of stocks and a separating hyperplane argument. While in our set-up, both the induction and separating argument do not work due to the presence of constraints. In this chapter, we instead use a finite covering argument to overcome the difficulty stemming from constraints. Another major difference from [19] is the proof for the existence of optimal super-hedging strategy in multiple period, which is a direct consequence of Theorem 2.2 there. A key step in the proof of Theorem 2.2 is modifying the trading strategy to the one with fewer "rank" yet still giving the same portfolio value. However, this approach fails to work in our set-up, since the modification may not be admissible anymore due to the portfolio constrains. In our chapter, we first find the optimal static trading strategy of options, and then find the optimal dynamical trading strategy of stocks by optional decomposition with constraints. Optional decomposition also helps us obtain the duality results for the American options.

We work within the no-arbitrage framework of [19], in which there is said to be an arbitrage when there exists a trading strategy whose gain is quasi surely non-negative and strictly positive with positive probability under an admissible measure. In this framework we are given a model and the non-dominated set of probability measures comes from estimating the parameters of the model. Since estimating results in confidence intervals for the parameters we end up with a set of non-dominated probability measures.

There is another no-arbitrage framework which was introduced by Acciaio et al. [1]. In that framework, there is said to be an arbitrage if the gain from trading is strictly positive for all scenarios. Under the framework of [1], the model uncertainty is in fact part of the model itself and the user of that model does not have confidence in her ability to estimate the parameters. The choice between the framework of [1] and the framework of [19] is a modeling issue.

Our assumptions mainly contain two parts: (1) the closedness and convexity of the related control sets (see Assumptions 6.2.1, 6.3.1, 6.4.1 and 6.5.1), and (2) some measurability assumptions (see the set-up of Section 6.3.1 and Assumptions 6.3.1 and 6.5.1). The first part is almost necessary (see Example 6.2.5), and can be easily verified in many interesting cases (see e.g., Example 6.2.1). The second part is the analyticity of some relevant sets, which we make in order to apply measurable selection results and perform dynamic programming principle type arguments. Analyticity (which is a measurability concept more general than Borel measurability, so in particular every Borel set is analytic) is a minimal assumption one can have in order to have a dynamic programming principle and this goes well back to Blackwell. These concepts are covered by standard textbooks on measure theory, see e.g. [23]. See also [18] for applications in stochastic control theory and the references therein. In Section 6.3.3, we provide some general and easily verifiable sufficient conditions for Assumptions 6.3.1(iii) and 6.5.1(ii), as well as Examples 6.3.7 and 6.3.8.

The rest of the chapter is organized as follows: We show the FTAP in one period and in multiple periods in Sections 6.2 and 6.3 respectively. In Section 6.4, we get the super-hedging result in one period. In Section **??**, we provide the non-dominated optional decomposition with constraints in multiple periods. Then starting from the optional decomposition, we analyze the sub- and super-hedging prices of European and American options in multiple periods in Section 6.6. In Section 6.7, we add options to the market, and study the FTAP and super-hedging using semi-static trading strategies in multiple periods. Finally in the appendix, we provide the proofs of Lemmas 6.3.2, 6.3.3, 6.5.4 and 6.5.6; these proofs are with a lot of technicalities and can be safely skipped in the first reading.

We devote the rest of this section to frequently used notation and concepts in the chapter.

6.1.1 Frequently used notation and concepts

- $\mathfrak{P}(\Omega)$ denotes set of all the probability measures on $(\Omega, \mathcal{B}(\Omega))$, where Ω is some polish space, and $\mathcal{B}(\Omega)$ denotes its Borel σ-algebra. $\mathfrak{P}(\Omega)$ is endowed with the topology of weak convergence.
- $\Delta S_t(\omega, \cdot) = S_{t+1}(\omega, \cdot) - S_t(\omega),\ \omega \in \Omega_t := \Omega^t$ (t-fold Cartesian product of Ω). We may simply write ΔS when there is only one period (i.e., $t = 0$).
- Let $\mathcal{P} \subset \mathfrak{P}(\Omega_t)$. A property holds $\mathcal{P} - q.s.$ if and only if it holds P-a.s. for any $P \in \mathcal{P}$. A set $A \in \Omega_t$ is $\mathcal{P}$-polar if $\sup_{P \in \mathcal{P}} P(A) = 0$.
- Let $\mathcal{P} \subset \mathfrak{P}(\Omega)$. $\mathrm{supp}_{\mathcal{P}}(\Delta S)$ is defined as the smallest closed subset $A \subset \mathbb{R}^d$ such that $\Delta S \in A\ \mathcal{P} - q.s.$. Define $N(\mathcal{P}) := \{H \in \mathbb{R}^d :\ H\Delta S = 0,\ \mathcal{P} - q.s.\}$ and $N^\perp(\mathcal{P}) := \mathrm{span}(\mathrm{supp}_{\mathcal{P}}(\Delta S)) \subset \mathbb{R}^d$. Then $N^\perp(\mathcal{P}) = (N(\mathcal{P}))^\perp$ by [68, Lemma 2.6]. Denote $N(P) = N(\{P\})$ and $N^\perp(P) = N^\perp(\{P\})$.
- For $\mathcal{H} \subset \mathbb{R}^d$, $\mathcal{H}(\mathcal{P}) := \{H :\ H \in \mathrm{proj}_{N^\perp(\mathcal{P})}(\mathcal{H})\}$. Denote $\mathcal{H}(P) = \mathcal{H}(\{P\})$.
- For $\mathcal{H} \subset \mathbb{R}^d$, $\mathcal{C}_{\mathcal{H}}(\mathcal{P}) := \{cH :\ H \in \mathcal{H}(\mathcal{P}),\ c \geq 0\}$. Denote $\mathcal{C}_{\mathcal{H}}(P) = \mathcal{C}_{\mathcal{H}}(\{P\})$.

- $\mathfrak{C}_{\mathcal{H}} := \{cH \in \mathbb{R}^d : \ H \in \mathcal{H}, \ c \geq 0\}$, where $\mathcal{H} \subset \mathbb{R}^d$.
- $(H \cdot S)_t = \sum_{i=0}^{t-1} H_i(S_{i+1} - S_i)$.
- $\mathbb{R}^* := [-\infty, \infty]$.
- $|| \cdot ||$ represents the Euclidean norm.
- $E_P|X| := E_P|X^+| - E_P|X^-|$, and by convention $\infty - \infty = -\infty$. Similarly the conditional expectation is also defined in this extended sense.
- $L^0_+(\mathcal{P})$ is the space of random variables X on the corresponding topological space satisfying $X \geq 0$ $\mathcal{P} - q.s.$, and $L^1(\mathcal{P})$ is the space of random variables X satisfying $\sup_{P\in\mathcal{P}} E_P|X| < \infty$. Denote $L^0_+(P) = L^0_+(\{P\})$, and $L^1(P) = L^1(\{P\})$. Similar definitions holds for L^0, L^1_+ and L^∞. We shall sometimes omit $\mathcal{P}$ or P in L^0_+, L^1, etc., when there is no ambiguity.
- We say NA($\mathcal{P}$) holds, if for any $H \in \mathcal{H}$ satisfying $(H \cdot S)_T \geq 0$, $\mathcal{P} - q.s.$, then $(H \cdot S)_T = 0$, $\mathcal{P} - q.s.$, where $\mathcal{H}$ is some admissible control set of trading strategies for stocks. Denote NA(P) for NA($\{P\}$).
- We write $Q \lll \mathcal{P}$, if there exists some $P \in \mathcal{P}$ such that $Q \ll P$.
- Let $(X, \mathcal{G})$ be a measurable space and Y be a topological space. A mapping Φ from X to the power set of Y is denoted by $\Phi : \ X \twoheadrightarrow Y$. We say Φ is measurable (resp. Borel measurable), if

  (6.1.1)
  $$\{x \in X : \ \Phi(x) \cap A \neq \emptyset\} \in \mathcal{G}, \ \forall \text{ closed (resp. Borel measurable) } A \subset Y.$$

  Φ is closed (resp. compact) valued if $\Phi(x) \subset Y$ is closed (resp. compact) for all $x \in X$. We refer to [2, Chapter 18] for these concepts.
- A set of random variables A is $\mathcal{P} - q.s.$ closed, if $(a_n)_n \subset A$ convergent to some a $\mathcal{P} - q.s.$ implies $a \in A$.

- For $\Phi: \ X \twoheadrightarrow Y$, $\mathrm{Gr}(\Phi) := \{(x,y) \in X \times Y: \ y \in \Phi(x)\}$.
- Let X be a Polish space. A set $A \subset X$ is analytic if it is the image of a Borel subset of another Polish space under a Borel measurable mapping. A function $f: X \mapsto \mathbb{R}^*$ is upper (resp. lower) semianalytic if the set $\{f > c\}$ (resp. $\{f < c\}$) is analytic. "u.s.a." (resp. "l.s.a.") is short for upper (resp. lower) semianalytic.
- Let X be a polish space. The σ-algebra $\cap_{P \in \mathfrak{P}(X)} \mathcal{B}(X)^P$ is called the universal completion of $\mathcal{B}(X)$, where $\mathcal{B}(X)^P$ is the P-completion of $\mathcal{B}(X)$. A set $A \subset X$ is universally measurable if $A \in \cap_{P \in \mathfrak{P}(X)} \mathcal{B}(X)^P$. A function f is universally measurable if $f \in \cap_{P \in \mathfrak{P}(X)} \mathcal{B}(X)^P$. "u.m." is short for universally measurable.
- Let X and Y be some Borel spaces and $U: \ X \twoheadrightarrow Y$. Then u is a u.m. selector of U, if $u: \ X \mapsto Y$ is u.m. and $u(\cdot) \in U(\cdot)$ on $\{U \neq \emptyset\}$.

6.2 The FTAP in one period

We derive the FTAP for one-period model in this section. Theorem 6.2.2 is the main result of this section.

6.2.1 The set-up and the main result

Let $\mathcal{P}$ be a set of probability measures on a Polish space Ω, which is assumed to be convex. Let $S_0 \in \mathbb{R}^d$ be the initial stock price, and Borel measurable $S_1: \ \Omega \mapsto \mathbb{R}^d$ be the stock price at time $t = 1$. Denote $\Delta S = S_1 - S_0$. Let $\mathcal{H} \subset \mathbb{R}^d$ be the set of admissible trading strategies. We assume $\mathcal{H}$ satisfies the following conditions:

Assumption 6.2.1. *$\mathcal{C}_{\mathcal{H}}(\mathcal{P})$ is (i) convex, and (ii) closed.*

Example 6.2.1. Let $\mathcal{H} := \prod_{i=1}^d [\underline{a}^i, \overline{a}^i]$ for some $\underline{a}^i, \overline{a}^i \in \mathbb{R}$ with $\underline{a}^i \leq \overline{a}^i, \ i = 1 \dots, d$. Then $\mathcal{H}$ satisfies Assumption 6.2.1 for any $\mathcal{P} \subset \mathfrak{P}(\Omega)$. Indeed, $\mathcal{H} \subset \mathbb{R}^d$ is a bounded, closed, convex set with finitely many vertices, and so is $\mathcal{H}(\mathcal{P})$. Hence the generated cone $\mathcal{C}_{\mathcal{H}}(\mathcal{P})$ is convex and closed.

Define

$$\mathcal{Q} := \{Q \in \mathfrak{P}(\Omega): \ Q \lll \mathcal{P}, \ E_Q|\Delta S| < \infty \text{ and } E_Q[H\Delta S] \leq 0, \ \forall H \in \mathcal{H}\}.$$

The following is the main result of this section:

Theorem 6.2.2. *Let Assumption 6.2.1 hold. Then $NA(\mathcal{P})$ holds if and only if for any $P \in \mathcal{P}$, there exists $Q \in \mathcal{Q}$ dominating P.*

6.2.2 Proof for Theorem 6.2.2

Let us first prove the following lemma, which is the simplified version of Theorem 6.2.2 when $\mathcal{P}$ consists of a single probability measure.

Lemma 6.2.3. *Let $P \in \mathfrak{P}(\Omega)$ and Assumption 6.2.1 w.r.t. $\mathcal{C}_{\mathcal{H}}(P)$ hold. Then $NA(P)$ holds if and only if there exists $Q \sim P$, such that $E_Q|\Delta S| < \infty$ and $E_Q[H\Delta S] \leq 0$, for any $H \in \mathcal{H}$.*

Proof. Sufficiency is obvious. We shall prove the necessity in two steps. W.l.o.g. we assume that $E_P|\Delta S| < \infty$ (see e.g., [19, Lemma 3.2]).

Step 1: In this step, we will show that $K - L^0_+$ is closed in L^0, where

$$K := \{H\Delta S: \ H \in \mathcal{C}_{\mathcal{H}}(P)\}.$$

Let $X_n = H_n\Delta S - Y_n \xrightarrow{P} X$, where $H_n \in \mathcal{C}_{\mathcal{H}}(P)$ and $Y_n \geq 0$. Without loss of generality, assume $X_n \to X$, P-a.s.. If $(H_n)_n$ is not bounded, then let $0 < ||H_{n_k}|| \to \infty$ and we have that

$$\frac{H_{n_k}}{||H_{n_k}||}\Delta S = \frac{X_{n_k}}{||H_{n_k}||} + \frac{Y_{n_k}}{||H_{n_k}||} \geq \frac{X_{n_k}}{||H_{n_k}||}.$$

Taking limit on both sides along a further sub-sequence, we obtain that $H\Delta S \geq 0$ P-a.s. for some $H \in \mathbb{R}^d$ with $||H|| = 1$. Since $\mathcal{C}_{\mathcal{H}}(P)$ is closed, $H\Delta S \in \mathcal{C}_{\mathcal{H}}(P)$. By NA$(P)$, $H\Delta S = 0$ P-a.s., which implies $H \in N(P) \cap N^{\perp}(P) = \{0\}$. This contradicts $||H|| = 1$. Therefore, $(H_n)_n$ is bounded, and thus there exists a subsequence $(H_{n_j})_j$ convergent to some $H' \in \mathcal{C}_{\mathcal{H}}(P)$. Then

$$0 \leq Y_{n_j} = H_{n_j}\Delta S - X_{n_j} \to H'\Delta S - X =: Y, \quad P\text{-a.s..}$$

Then $X = H'\Delta S - Y \in K - L^0_+$.

Step 2: From Step 1, we know that $K' := (K - L^0_+) \cap L^1$ is a closed, convex cone in L^1, and contains $-L^{\infty}_+$. Also by NA(P), $K' \cap L^1_+ = \{0\}$. Then Kreps-Yan theorem (see e.g., [43, Theorem 1.61]) implies the existence of $Q \sim P$ with $dQ/dP \in L^{\infty}_+(P)$, such that $E_Q[H\Delta S] \leq 0$ for any $H \in \mathcal{H}$. □

Remark 6.2.4. The FTAP under a single probability measure with constraints is analyzed in [43, Chapter 9]. However, although the idea is quite insightful, the result there is not correct: what we need is the closedness of the generated cone $\mathcal{C}_{\mathcal{H}}(P)$ instead of the closedness of $\mathcal{H}(P)$. (In this sense, our result is different from [29]; in [29] it is the closedness of the corresponding projection that matters.) Below is a counter-example to [43, Theorem 9.9].

Example 6.2.5. Consider the one-period model: there are two stocks S^1 and S^2 with the path space $\{(1,1)\} \times \{(s,0) : \ s \in [1,2]\}$; let

$$\mathcal{H} := \{(h_1, h_2) : \ h_1^2 + (h_2 - 1)^2 \leq 1\}.$$

be the set of admissible trading strategies; let P be a probability measure on this path space such that S^1_1 is uniformly distributed on $[1,2]$. It is easy to see that NA(P) holds, and $\mathcal{H}$ satisfies the assumptions (a), (b) and (c) on [43, page 350]. Let

$H = (h_1, h_2)$ such that $H\Delta S = 0$, P-a.s. Then $h_1(S_1^1 - 1) = h_2$, P-a.s., which implies $h_1 = h_2 = 0$. By [43, Remark 9.1], $\mathcal{H}$ also satisfies assumption (d) on [43, page 350].

Now suppose [43, Theorem 9.9] holds, then there exists $Q \sim P$, such that

$$E_Q[H\Delta S] \leq 0, \quad \forall H \in \mathcal{H}. \tag{6.2.1}$$

Since $Q \sim P$, $E_Q(S_1^1 - 1) > 0$. Take $(h_1, h_2) \in \mathcal{H}$ with $h_1, h_2 > 0$ and $h_2/h_1 < E_Q(S_1^1 - 1)$. Then

$$h_1 E_Q(S_1^1 - 1) - h_2 > 0,$$

which contradicts (6.2.1).

In fact, it is not hard to see that in this example,

$$\mathcal{C}_{\mathcal{H}}(P) = \{(h_1, h_2) : h_2 > 0 \text{ or } h_1 = h_2 = 0\}$$

is not closed.

Lemma 6.2.6. *Let Assumption 6.2.1(ii) hold. Then there exists $P'' \in \mathcal{P}$, such that $N^\perp(P'') = N^\perp(\mathcal{P})$ and $NA(P'')$ holds.*

Proof. Denote $\mathbb{H} := \{H \in \mathcal{C}_{\mathcal{H}}(\mathcal{P}) : \ ||H|| = 1\}$. For any $H \in \mathbb{H} \subset N^\perp(\mathcal{P})$, by $NA(\mathcal{P})$, there exists $P_H \in \mathcal{P}$, such that $P_H(H\Delta S < 0) > 0$. It can be further shown that there exists $\varepsilon_H > 0$, such that for any $H' \in B(H, \varepsilon_H)$,

$$P_H(H'\Delta S < 0) > 0, \tag{6.2.2}$$

where $B(H, \varepsilon_H) := \{H'' \in \mathbb{R}^d : \ ||H'' - H|| < \varepsilon_H\}$. Indeed, there exists some $\delta > 0$ such that $P_H(H\Delta S < -\delta) > 0$. Then there exists some $M > 0$, such that $P_H(H\Delta S < -\delta, \ ||\Delta S|| < M) > 0$. Taking $\varepsilon_H := \delta/M$, we have that for any $H' \in B(H, \varepsilon_H)$, $P_H(H'\Delta S < 0, \ ||\Delta S|| < M) > 0$, which implies (6.2.2).

Because $\mathbb{H} \subset \cup_{H \in \mathbb{H}} B(H, \varepsilon_H)$ and $\mathbb{H}$ is compact from Assumption 6.2.1, there exists a finite cover of $\mathbb{H}$, i.e., $\mathbb{H} \subset \cup_{i=1}^n B(H_i, \varepsilon_{H_i})$. Let $P' = \sum_{i=1}^n a_i P_{H_i}$, with

$\sum_{i=1}^{n} a_i = 1$ and $a_i > 0,\ i = 1, \ldots, n$. Then $P' \in \mathcal{P}$, and $P'(H\Delta S < 0) > 0$ for any $H \in \mathbb{H}$.

Obviously, $N^{\perp}(P') \subset N^{\perp}(\mathcal{P})$. If $N^{\perp}(P') = N^{\perp}(\mathcal{P})$, then let $P'' = P'$. Otherwise, take $H \in N^{\perp}(\mathcal{P}) \cap N(P')$. Then there exists $R_1 \in \mathcal{P}$, such that $R_1(H\Delta S \neq 0) > 0$. Let $R_1' = (P' + R_1)/2$. Then $P' \ll R_1' \in \mathcal{P}$, and thus $N^{\perp}(R_1') \supset N^{\perp}(P')$. Since $H \in N(P') \setminus N(R_1')$, we have that $N^{\perp}(R_1') \supsetneqq N^{\perp}(P')$. If $N^{\perp}(R_1') \subsetneqq N^{\perp}(\mathcal{P})$, then we can similarly construct $R_2' \in \mathcal{P}$, such that $R_2' \gg R_1'$ and $N^{\perp}(R_2') \supsetneqq N^{\perp}(R_1')$. Since $N^{\perp}(\mathcal{P})$ is a finite dimensional vector space, after finite such steps, we can find such $P'' \in \mathcal{P}$ dominating P' with $N^{\perp}(P'') = N^{\perp}(\mathcal{P})$. For any $H \in \mathbb{H}$, $P''(H\Delta S < 0) > 0$ since $P'' \gg P'$. This implies that $\mathrm{NA}(P'')$ holds. $\square$

Proof of Theorem 6.2.2. *Sufficiency.* If not, there exists $H \in \mathcal{H}$ and $P \in \mathcal{P}$, such that $H\Delta S \geq 0,\ P - a.s.$ and $P(H\Delta S > 0) > 0$. Take $Q \in \mathcal{Q}$ with $Q \gg P$. Then $E_Q[H\Delta S] \leq 0$, which contradicts $H\Delta S \geq 0\ Q - a.s.$ and $Q(H\Delta S > 0) > 0$.

Necessity. Take $P \in \mathcal{P}$. By Lemma 6.2.6 there exists $P'' \in \mathcal{P}$ such that $N^{\perp}(P'') = N^{\perp}(\mathcal{P})$ and $\mathrm{NA}(P'')$ holds. Let $R := (P + P'')/2 \in \mathcal{P}$. Then $N^{\perp}(R) = N^{\perp}(P'') = N^{\perp}(\mathcal{P})$, and thus $\mathcal{C}_{\mathcal{H}}(R) = \mathcal{C}_{\mathcal{H}}(\mathcal{P})$ which is convex and closed by Assumption 6.2.1. Besides, $\mathrm{NA}(P'')$ implies that for any $H \in \mathcal{C}_{\mathcal{H}}(R) \setminus \{0\} = \mathcal{C}_{\mathcal{H}}(P'') \setminus \{0\}$, $P''(H\Delta S < 0) > 0$, and thus $R(H\Delta S < 0) > 0$ since $R \gg P''$. This shows that $\mathrm{NA}(R)$ holds. From Lemma 6.2.3, there exists $Q \sim R \gg P$, such that $E_Q|\Delta S| < \infty$ and $E_Q[H\Delta S] \leq 0$ for any $H \in \mathcal{H}$. $\square$

6.3 The FTAP in multiple periods

We derive the FTAP in multiple periods in this section, and Theorem 6.3.1 is our main result. We will reduce it to a one-step problem and apply Theorem 6.2.2.

6.3.1 The set-up and the main result

We use the set-up in [19]. Let $T \in \mathbb{N}$ be the time Horizon and let Ω be a Polish space. For $t \in \{0, 1, \ldots, T\}$, let $\Omega_t := \Omega^t$ be the t-fold Cartesian product, with the convention that Ω_0 is a singleton. We denote by $\mathcal{F}_t$ the universal completion of $\mathcal{B}(\Omega_t)$, and we shall often treat Ω_t as a subspace of Ω_T. For each $t \in \{0, \ldots, T-1\}$ and $\omega \in \Omega_t$, we are given a nonempty convex set $\mathcal{P}_t(\omega) \subset \mathfrak{P}(\Omega)$ of probability measures. Here $\mathcal{P}_t$ represents the possible models for the t-th period, given state ω at time t. We assume that for each t, the graph of $\mathcal{P}_t$ is analytic, which ensures by the Jankov-von Neumann Theorem (see, e.g., [18, Proposition 7.49]) that $\mathcal{P}_t$ admits a u.m. selector, i.e., a u.m. kernel $P_t : \ \Omega_t \to \mathfrak{P}(\Omega)$ such that $P_t(\omega) \in \mathcal{P}_t(\omega)$ for all $\omega \in \Omega_t$. Let

$$\mathcal{P} := \{P_0 \otimes \ldots \otimes P_{T-1} : \ P_t(\cdot) \in \mathcal{P}_t(\cdot), \ t = 0, \ldots, T-1\},$$

where each P_t is a u.m. selector of $\mathcal{P}_t$, and for $A \in \Omega_T$

$$P_0 \otimes \ldots \otimes P_{T-1}(A) = \int_\Omega \ldots \int_\Omega 1_A(\omega_1, \ldots, \omega_T) P_{T-1}(\omega_1, \ldots, \omega_{T-1}; d\omega_T) \ldots P_0(d\omega_1).$$

Let $S_t = (S_t^1, \ldots, S_t^d) : \Omega_t \to \mathbb{R}^d$ be Borel measurable, which represents the price at time t of a stock S that can be traded dynamically in the market.

For each $t \in \{0, \ldots, T-1\}$ and $\omega \in \Omega_t$, we are given a set $\mathcal{H}_t(\omega) \subset \mathbb{R}^d$, which is thought as the set of admissible controls for the t-th period, given state ω at time t. We assume for each t, graph$(\mathcal{H}_t)$ is analytic, and thus admits a u.m. selector; that is, an $\mathcal{F}_t$-measurable function $H_t(\cdot) : \ \Omega_t \mapsto \mathbb{R}^d$, such that $H_t(\omega) \in \mathcal{H}_t(\omega)$. We introduce the set of admissible portfolio controls $\mathcal{H}$:

$$\mathcal{H} := \left\{(H_t)_{t=0}^{T-1} : \ H_t \text{ is a u.m. selector of } \mathcal{H}_t, \ t = 0, \ldots, T-1\right\}.$$

[0] In order not to burden the reader with further notation we prefer use the same notation $\mathcal{P}$ for the set of probability measures in one-period models and multi-period models. We will do the same for other sets of probability measures that appear later in the chapter and also for the set of admissible strategies.

Then for any $H \in \mathcal{H}$, H is an adapted process. We make the following assumptions on $\mathcal{H}$.

Assumption 6.3.1.

(i) $0 \in \mathcal{H}_t(\omega)$, *for* $\omega \in \Omega_t$, $t = 0, \ldots, T-1$.

(ii) $\mathcal{C}_{\mathcal{H}_t(\omega)}(\mathcal{P}_t(\omega))$ *is closed and convex, for* $\omega \in \Omega_t$, $t = 0, \ldots, T-1$.

(iii) The set

$$\Psi_{\mathcal{H}_t} := \{(\omega, Q) \in \Omega_t \times \mathfrak{P}(\Omega) : \; E_Q|\Delta S_t(\omega, \cdot)| < \infty,$$

$$\text{and } E_Q[y\Delta S_t(\omega, \cdot)] \leq 0, \; \forall y \in \mathcal{H}_t(\omega)\}$$

is analytic, for $t = 0, \ldots, T-1$.

Define

$$\mathcal{Q} := \{Q \in \mathfrak{P}(\Omega_T) : \; Q \lll \mathcal{P}, \; E_Q[|\Delta S_t| \, |\mathcal{F}_t] < \infty \; Q\text{-a.s. for}$$
$$t = 0, \ldots, T-1, \; H \cdot S \text{ is a } Q\text{-local-supermartingale } \forall H \in \mathcal{H}\}. \tag{6.3.1}$$

Below is the main theorem of this section:

Theorem 6.3.1. *Under Assumption 6.3.1, NA*$(\mathcal{P})$ *holds if and only if for each* $P \in \mathcal{P}$, *there exists* $Q \in \mathcal{Q}$ *dominating* P.

6.3.2 Proof of Theorem 6.3.1

We will first provide some auxiliary results. The following lemma essentially says that if there is no arbitrage in T periods, then there is no arbitrage in any period. It is parallel to [19, Lemma 4.6]. Our proof shall mainly focuses on the difference due to the presence of constraints and we put the proof in the appendix.

Lemma 6.3.2. *Let* $t \in \{0, \ldots, T-1\}$. *Then the set*

$$N_t := \{\omega \in \Omega_t : \; NA(\mathcal{P}_t(\omega)) \text{ fails } \} \tag{6.3.2}$$

is u.m., and if Assumption 6.3.1(i) and NA($\mathcal{P}$) hold, then N_t is $\mathcal{P}$-polar.

The lemma below is a measurable version of Theorem 6.2.2. It is parallel to [19, Lemma 4.8]. We provide its proof in the appendix.

Lemma 6.3.3. *Let $t \in \{0, \dots, T-1\}$, let $P(\cdot): \ \Omega_t \mapsto \mathfrak{P}(\Omega)$ be Borel, and let $\mathcal{Q}_t: \ \Omega_t \twoheadrightarrow \mathfrak{P}(\Omega)$,*

$$\mathcal{Q}_t(\omega) := \{Q \in \mathfrak{P}(\Omega): \ Q \lll \mathcal{P}_t(\omega), \ E_Q|\Delta S_t(\omega, \cdot)| < \infty,$$

$$E_Q[y\Delta S_t(\omega, \cdot)] \le 0, \ \forall y \in \mathcal{H}_t(\omega)\}.$$

If Assumption 6.3.1(ii)(iii) holds, then $\mathcal{Q}_t$ has an analytic graph and there exist u.m. mappings $Q(\cdot), \hat{P}(\cdot): \ \Omega_t \to \mathfrak{P}(\Omega)$ such that

$$\begin{aligned}
&P(\omega) \ll Q(\omega) \ll \hat{P}(\omega) \quad \textit{for all } \omega \in \Omega_t,\\
&\hat{P}(\omega) \in \mathcal{P}_t(\omega) \quad \textit{if } P(\omega) \in \mathcal{P}_t(\omega),\\
&Q(\omega) \in \mathcal{Q}_t(\omega) \quad \textit{if NA}(\mathcal{P}_t(\omega)) \textit{ holds and } P(\omega) \in \mathcal{P}_t(\omega).
\end{aligned}$$

Proof of Theorem 6.3.1*.* Using Lemmas 6.3.2 and 6.3.3, we can perform the same glueing argument Bouchard and Nutz use in the proof of [19, Theorem 4.5], and thus we omit it here. □

6.3.3 Sufficient conditions for Assumption 6.3.1(iii)

By [18, Proposition 7.47], the map $(\omega, Q) \mapsto \sup_{y \in \mathcal{H}_t(\omega)} E_Q[y\Delta S_t(\omega, \cdot)]$ is u.s.a., which does not necessarily imply the analyticity of $\Psi_{\mathcal{H}_t}$ as the complement of an analytic set may fail to be analytic. Therefore we provide some sufficient conditions for Assumption 6.3.1(iii) below.

Definition 6.3.4. We call $\mathfrak{H}_t: \ \Omega_t \twoheadrightarrow \mathbb{R}^d$ a stretch of $\mathcal{H}_t$, if for any $\omega \in \Omega_t$, $\mathfrak{C}_{\mathfrak{H}_t(\omega)} = \mathfrak{C}_{\mathcal{H}_t(\omega)}$.

It is easy to see that for any stretch $\mathfrak{H}_t$ of $\mathcal{H}_t$,

$$\Psi_{\mathcal{H}_t} = \Psi_{\mathfrak{H}_t} = \{(\omega, Q) \in \Omega_t \times \mathfrak{P}(\Omega) : \ E_Q|\Delta S_t(\omega, \cdot)| < \infty, \ \sup_{y \in \mathfrak{H}_t(\omega)} y E_Q[\Delta S_t(\omega, \cdot)] \leq 0\}.$$

Therefore, in order to show $\Psi_{\mathcal{H}_t}$ is analytic, it suffices to show that there exists a stretch $\mathfrak{H}_t$ of $\mathcal{H}_t$, such that the map $\varphi_{\mathfrak{H}_t} : \ \Omega_t \times \mathfrak{P}(\Omega) \mapsto \mathbb{R}^*$

$$\varphi_{\mathfrak{H}_t}(\omega, Q) = \sup_{y \in \mathfrak{H}_t(\omega)} y E_Q[\Delta S_t(\omega, \cdot)] \tag{6.3.3}$$

is l.s.a. on $\mathcal{J} := \{(\omega, Q) \in \Omega_t \times \mathfrak{P}(\Omega) : \ E_Q|\Delta S_t(\omega, \cdot)| < \infty\}$.

Proposition 6.3.5. *If there exists a measurable (w.r.t. $\mathcal{B}(\mathbb{R}^d)$) stretch $\mathfrak{H}_t$ of $\mathcal{H}_t$ with nonempty compact values, then $\varphi_{\mathfrak{H}_t}$ is Borel measurable, and thus $\Psi_{\mathcal{H}_t}$ is Borel measurable.*

Proof. The conclusion follows directly from [2, Theorem 18.19]. □

Proposition 6.3.6. *If there exists a stretch $\mathfrak{H}_t$ of $\mathcal{H}_t$ satisfying*

(i) graph($\mathfrak{H}_t$) is Borel measurable,

(ii) there exists a countable set $(y_n)_n \subset \mathbb{R}^d$, such that for any $\omega \in \Omega_t$ and $y \in \mathfrak{H}_t(\omega)$, there exist $(y_{n_k})_k \subset (y_n)_n \cap \mathfrak{H}_t$ converging to y,

then $\varphi_{\mathfrak{H}_t}$ is Borel measurable, and thus $\Psi_{\mathcal{H}_t}$ is Borel measurable.

Proof. Define function $\phi : \ \mathbb{R}^d \times \mathcal{J} \mapsto \mathbb{R}^*$,

$$\phi(y, \omega, Q) = \begin{cases} y E_Q[\Delta S_t(\omega, \cdot)] & \text{if } y \in \mathfrak{H}_t(\omega), \\ -\infty & \text{otherwise.} \end{cases}$$

It can be shown by a monotone class argument that ϕ is Borel measurable. So the function $\varphi : \ \mathcal{J} \mapsto \mathbb{R}$

$$\varphi(\omega, Q) = \sup_n \phi(y_n, \omega, Q)$$

is Borel measurable. It remains to show that $\varphi = \varphi_{\mathfrak{H}_t}$. It is easy to see that $\varphi \geq \varphi_{\mathfrak{H}_t}$. Conversely, take $(\omega, Q) \in \mathcal{J}$. Then $\phi(y_n, \omega, Q) = y_n E_Q[\Delta S(\omega, \cdot)] \leq \varphi_{\mathfrak{H}_t}(\omega, Q)$ if $y_n \in \mathfrak{H}_t(\omega)$, and $\phi(y_n, \omega, Q) = -\infty < \varphi_{\mathfrak{H}_t}(\omega, Q)$ if $y_n \notin \mathfrak{H}_t(\omega)$; i.e., $\varphi(\omega, Q) = \sup_n \phi(y_n, \omega, Q) \leq \varphi_{\mathfrak{H}_t}(\omega, Q)$. □

Example 6.3.7. Let $\underline{a}_t^i,\ \overline{a}_t^i:\ \Omega_t \mapsto \mathbb{R}$ be Borel measurable, with $\underline{a}_t^i < \overline{a}_t^i,\ i = 1, \dots, d$. Let

$$\mathcal{H}_t(\omega) = \prod_{i=1}^{d} [\underline{a}_t^i(\omega), \overline{a}_t^i(\omega)], \quad \omega \in \Omega_t.$$

Then both Propositions 6.3.5 and 6.3.6 hold with $\mathfrak{H}_t = \mathcal{H}_t$ and $(y_n)_n = \mathbb{Q}^d$.

Example 6.3.8. Let $d = 1$ and $\mathcal{H}_t$ be such that for any $\omega \in \Omega_t$, $\mathcal{H}_t(\omega) \subset (0, \infty)$. We assume that graph$(\mathcal{H}_t)$ is analytic, but not Borel. Then $\mathcal{H}_t$ itself does not satisfy the assumptions in Proposition 6.3.5 or 6.3.6. Now let $\mathfrak{H}_t(\omega) = [1, 2],\ \omega \in \Omega_t$. Then $\mathfrak{H}_t$ is a stretch of $\mathcal{H}_t$, and $\mathfrak{H}_t$ satisfies the assumptions in Propositions 6.3.5 and 6.3.6 with $(y_n)_n = \mathbb{Q}$.

6.4 Super-hedging in one period

6.4.1 The set-up and the main result

We use the set-up in Section 6.2. Let f be a u.m. function. Define the super-hedging price

$$\pi^{\mathcal{P}}(f) := \inf\{x :\ \exists H \in \mathcal{H},\ \text{s.t. } x + H \cdot S \geq f,\ \mathcal{P} - q.s.\}.$$

We also denote $\pi^P(f) = \pi^{\{P\}}(f)$. We further assume:

Assumption 6.4.1. *$\mathcal{H}(\mathcal{P})$ is convex and closed.*

Remark 6.4.1. It is easy to see that if $\mathcal{H}(\mathcal{P})$ is convex, then $\mathcal{C}_{\mathcal{H}}(\mathcal{P})$ is convex.

Define

$$\mathfrak{Q} := \{Q \in \mathfrak{P}(\Omega) :\ Q \lll \mathcal{P},\ E_Q|\Delta S| < \infty,\ A^Q := \sup_{H \in \mathcal{H}} E_Q[H\Delta S] < \infty\}.$$

Below is the main result of this section.

Theorem 6.4.2. *Let Assumptions 6.2.1(ii) & 6.4.1 and NA($\mathcal{P}$) hold. Then*

$$\pi^{\mathcal{P}}(f) = \sup_{Q\in\mathfrak{Q}}(E_Q[f] - A^Q). \tag{6.4.1}$$

Besides, $\pi^{\mathcal{P}}(f) > -\infty$ *and there exists* $H \in \mathcal{H}$ *such that* $\pi^{\mathcal{P}}(f) + H\Delta S \geq f\ \mathcal{P}-q.s.$.

6.4.2 Proof of Theorem 6.4.2

We first provide two lemmas.

Lemma 6.4.3. *Let NA($\mathcal{P}$) hold. If* $\mathcal{H}(\mathcal{P})$ *and* $\mathcal{C}_{\mathcal{H}}(\mathcal{P})$ *are closed, then*

$$\pi^{\mathcal{P}}(f) = \sup_{P\in\mathcal{P}} \pi^P(f).$$

Proof. It is easy to see that $\pi^{\mathcal{P}}(f) \geq \sup_{P\in\mathcal{P}} \pi^P(f)$. We shall prove the reverse inequality. If $\pi^{\mathcal{P}}(f) > \sup_{P\in\mathcal{P}} \pi^P(f)$, then there exists $\varepsilon > 0$ such that

$$\alpha := \pi^{\mathcal{P}}(f) \wedge \frac{1}{\varepsilon} - \varepsilon > \sup_{P\in\mathcal{P}} \pi^P(f). \tag{6.4.2}$$

By Lemma 6.2.6 there exists $P'' \in \mathcal{P}$, such that $N^{\perp}(P'') = N^{\perp}(\mathcal{P})$ and NA(P'') holds.

Moreover, we have that the set

$$A_\alpha := \{H \in \mathcal{H}(\mathcal{P}):\ \alpha + H\Delta S \geq f,\ P'' - \text{a.s.}\}$$

is compact. In order to prove this claim take $(H_n)_n \subset A_\alpha$. If $(H_n)_n$ is not bounded, w.l.o.g. we assume $0 < ||H_n|| \to \infty$; then

$$\frac{\alpha}{||H_n||} + \frac{H_n}{||H_n||}\Delta S \geq \frac{f}{||H_n||}. \tag{6.4.3}$$

Since $\mathcal{C}_{\mathcal{H}}(\mathcal{P})$ is closed, there exist some $H \in \mathcal{C}_{\mathcal{H}}(\mathcal{P}) = \mathcal{C}_{\mathcal{H}}(P'')$ with $||H|| = 1$ such that $H_{n_k}/||H_{n_k}|| \to H$. Taking the limit along $(n_k)_k$, we have $H\Delta S \geq 0$ P''-a.s.

NA(P'') implies $H\Delta S = 0$ P''-a.s. So $H \in \mathcal{C}_{\mathcal{H}}(P'') \cap N(P'') = \{0\}$, which contradicts $||H|| = 1$. Thus $(H_n)_n$ is bounded, and there exists $H'' \in \mathbb{R}^d$, such that $(H_{n_j})_j \to H''$. Since $\mathcal{H}(\mathcal{P})$ is closed, $H'' \in \mathcal{H}(\mathcal{P})$, which further implies $H'' \in A_\alpha$.

For any $H \in A_\alpha$, since $\alpha < \pi^{\mathcal{P}}(f)$ by (6.4.2), there exist $P_H \in \mathcal{P}$ such that

$$P_H(\alpha + H\Delta S < f) > 0.$$

It can be further shown that there exists $\delta_H > 0$, such that for any $H' \in B(H, \delta_H)$,

$$P_H(\alpha + H'\Delta S < f) > 0.$$

Since $A_\alpha \subset \cup_{H \in A_\alpha} B(H, \delta_H)$ and A_α is compact, there exists $(H_i)_{i=1}^n \subset A_\alpha$, such that $A_\alpha \subset \cup_{i=1}^n B(H_i, \delta_{H_i})$. Let

$$P' := \sum_{i=1}^n a_i P_{H_i} + a_0 P'' \in \mathcal{P},$$

where $\sum_{i=0}^n a_i = 1$ and $a_i > 0,\ i = 0, \dots, n$. Then it is easy to see that for any $H \in \mathcal{H}(\mathcal{P}) = \mathcal{H}(P'') = \mathcal{H}(P')$,

$$P'(\alpha + H\Delta S < f) > 0,$$

which implies that

$$\alpha \leq \pi^{P'}(f) \leq \sup_{P \in \mathcal{P}} \pi^P(f),$$

which contradicts (6.4.2). $\square$

Lemma 6.4.4. *Let NA$(\mathcal{P})$ hold. If $\mathcal{H}(\mathcal{P})$ and $\mathcal{C}_{\mathcal{H}}(\mathcal{P})$ are closed, then the set*

$$K(\mathcal{P}) := \{H\Delta S - X :\ H \in \mathcal{H},\ X \in L^0_+(\mathcal{P})\} \tag{6.4.4}$$

is $\mathcal{P} - q.s.$ closed.

Proof. Let $W^n = H^n\Delta S - X^n \in K(\mathcal{P}) \to W\ \mathcal{P}-q.s.$, where w.l.o.g. $H^n \in \mathcal{H}(\mathcal{P})$ and $X^n \in L^0_+(\mathcal{P}),\ n = 1, 2, \dots$ If $(H^n)_n$ is not bounded, then without loss of generality, $0 < ||H^n|| \to \infty$. Consider

$$\frac{W^n}{||H^n||} = \frac{H^n}{||H^n||}\Delta S - \frac{X^n}{||H^n||}. \tag{6.4.5}$$

As $(H^n/||H^n||)_n$ is bounded, there exists some subsequence $(H^{n_k}/||H^{n_k}||)_k$ converging to some $H \in \mathbb{R}^d$ with $||H|| = 1$. Taking the limit in (6.4.5) along $(n_k)_k$, we get that $H\Delta S \geq 0\ \mathcal{P} - q.s.$. Because $(H^{n_k}/||H^{n_k}||)_k \in \mathcal{C}_{\mathcal{H}}(\mathcal{P})$ and $\mathcal{C}_{\mathcal{H}}(\mathcal{P})$ is closed, $H \in \mathcal{C}_{\mathcal{H}}(\mathcal{P})$. Hence $H\Delta S = 0\ \mathcal{P} - q.s.$ by NA($\mathcal{P}$). Then $H \in \mathcal{C}_{\mathcal{H}}(\mathcal{P}) \cap N(\mathcal{P}) = \{0\}$, which contradicts $||H|| = 1$.

Therefore, $(H^n)_n$ is bounded and there exists some subsequence $(H^{n_j})_j$ converging to some $H' \in \mathbb{R}^d$. Since $\mathcal{H}(\mathcal{P})$ is closed, $H' \in \mathcal{H}(\mathcal{P})$. Let $X := H'\Delta S - W \in L^0_+(\mathcal{P})$, then $W = H'\Delta S - X \in K(\mathcal{P})$. $\square$

Proof of Theorem 6.4.2. We first show that $\pi^{\mathcal{P}}(f) > -\infty$ and the optimal superhedging strategy exists. If $\pi^{\mathcal{P}}(f) = \infty$ then we are done. If $\pi^{\mathcal{P}}(f) = -\infty$, then for any $n \in \mathbb{N}$, there exists $H^n \in \mathcal{H}$ such that

$$H^n\Delta S \geq f + n \geq (f + n) \wedge 1, \quad \mathcal{P} - q.s.$$

By Lemma 6.4.4, there exists some $H \in \mathcal{H}$ such that $H\Delta S \geq 1\ \mathcal{P} - q.s.$, which contradicts NA($\mathcal{P}$). If $\pi^{\mathcal{P}}(f) \in (-\infty, \infty)$, then for any $n \in \mathbb{N}$, there exists some $\tilde{H}^n \in \mathcal{H}$, such that $\pi^{\mathcal{P}}(f) + 1/n + \tilde{H}^n\Delta S \geq f$. Lemma 6.4.4 implies that there exists some $\tilde{H} \in \mathcal{H}$, such that $\pi^{\mathcal{P}}(f) + \tilde{H}\Delta S \geq f$.

By Lemma 6.4.3,

(6.4.6)
$$\pi^{\mathcal{P}}(f)=\sup_{P\in\mathcal{P}}\pi^{P}(f)=\sup_{Q\in\mathcal{Q}}\pi^{Q}(f)=\sup_{Q\in\mathcal{Q}}\sup_{\substack{Q'\in\mathfrak{Q},\\ Q'\sim Q}}(E_{Q'}[f]-A^{Q'})\leq\sup_{Q\in\mathfrak{Q}}(E_Q[f]-A^Q]),$$

where we apply Theorem 6.2.2 for the second equality, and [43, Proposition 9.23] for the third equality. Conversely, if $\pi^{\mathcal{P}}(f)=\infty$, then we are done. Otherwise let $x>\pi^{\mathcal{P}}(f)$, and there exist $H\in\mathcal{H}$, such that $x+H\Delta S\geq f$ $\mathcal{P}-q.s.$. Then for any $Q\in\mathfrak{Q}$,

$$x\geq E_Q[f]-E_Q[H\Delta S]\geq E_Q[f]-A^Q.$$

By the arbitrariness of x and Q, we have that

$$\pi^{\mathcal{P}}(f)\geq\sup_{Q\in\mathfrak{Q}}(E_Q[f]-A^Q),$$

which together with (6.4.6) implies (6.4.1). □

6.5 Optional decomposition in multiple periods

6.5.1 The set-up and the main result

We use the set-up in Section 6.3. In addition, let $f:\ \Omega_T\mapsto\mathbb{R}$ be u.s.a. We further assume:

Assumption 6.5.1.

(i) For $t\in\{0,\dots,T-1\}$ and $\omega\in\Omega_t$, $(\mathcal{H}_t(\omega))(\mathcal{P}_t(\omega))$ is convex and closed;

(ii) the map $A_t(\omega,Q):\ \Omega_t\times\mathfrak{P}(\Omega)\mapsto\mathbb{R}^*$,

$$A_t(\omega,Q)=\sup_{y\in\mathcal{H}_t(\omega)}yE_Q[\Delta S_t(\omega,\cdot)]$$

is l.s.a. on the set $\{(\omega,Q):\ E_Q|\Delta S_t(\omega,\cdot)|<\infty\}$.

Remark 6.5.1. Observe that $\Psi_{\mathcal{H}_t}$ defined in Assumption 6.3.1 satisfies

$$\Psi_{\mathcal{H}_t}=\{(\omega,Q)\in\Omega_t\times\mathfrak{P}(\Omega):\ E_Q|\Delta S_t(\omega,\cdot)|<\infty,\ A_t(\omega,Q)\leq 0\}. \tag{6.5.1}$$

Therefore, Assumption 6.5.1(ii) implies Assumption 6.3.1(iii).

Remark 6.5.2. If Proposition 6.3.5 or 6.3.6 hold with $\mathfrak{H}_t = \mathcal{H}_t$, then since $A_t = \varphi_{\mathfrak{H}_t}$ ($\varphi_{\mathfrak{H}_t}$ is defined in (6.3.3)), Assumption 6.5.1(ii) holds. See Example 6.3.7 for a case when this holds.

For any $Q \in \mathfrak{P}(\Omega_T)$, there are Borel kernels $Q_t : \ \Omega_t \mapsto \mathfrak{P}(\Omega)$ such that $Q = Q_0 \otimes \ldots \otimes Q_{T-1}$. For $E^Q[|\Delta S_t| \ |\mathcal{F}_t] < \infty$ Q-a.s., define $A_t^Q(\cdot) := A_t(\cdot, Q_t(\cdot))$ for $t = 0, \ldots, T-1$, and

$$B_t^Q := \sum_{i=0}^{t-1} A_i^Q, \quad t = 1, \ldots, T$$

and set $B_0^Q = 0$. Let

$$\mathfrak{Q} := \{Q \in \mathfrak{P}(\Omega_T) : \ Q \lll \mathcal{P}, \ E_Q[|\Delta S_t| \, |\mathcal{F}_t] < \infty \ Q\text{-a.s. for all } t, \text{ and } B_T^Q < \infty \ Q\text{-a.s.}\}.$$

Then it is not difficult to see that $\mathcal{Q} \subset \mathfrak{Q}$, where $\mathcal{Q}$ is defined in (6.3.1).[1] Also if for each $t \in \{0, \ldots, T-1\}$ and $\omega \in \Omega_t$, $\mathcal{H}_t(\omega)$ is a convex cone, then $\mathfrak{Q} = \mathcal{Q}$. Below is the main result of this section.

Theorem 6.5.3. *Let Assumptions 6.3.1 & 6.5.1 and NA($\mathcal{P}$) hold. Let V be an adapted process such that V_t is u.s.a. for $t = 1, \ldots, T$. Then the following are equivalent:*

(i) $V - B^Q$ is a Q-local-supermartingale for each $Q \in \mathfrak{Q}$.

(ii) There exists $H \in \mathcal{H}$ and an adapted increasing process C with $C_0 = 0$ such that

$$V_t = V_0 + (H \cdot S)_t - C_t, \quad \mathcal{P} - q.s.$$

[1] A rigorous argument is as follows. Let $Q = Q_0 \otimes \ldots \otimes Q_{T-1} \in \mathcal{Q}$, where Q_t is a Borel kernels, $0 \le t \le T-1$. It can be shown by a monotone class argument that the map $(\omega, y, Q') \mapsto yE_{Q'}[\Delta S(\omega, \cdot)]$ is Borel measurable for $(\omega, y, Q') \in \Omega_t \times \mathbb{R}^d \times \mathfrak{P}(\Omega)$. Hence the map $(\omega, y) \mapsto yE_{Q_t(\omega)}[\Delta S(\omega, \cdot)]$ is Borel measurable for $(\omega, y) \in \Omega_t \times \mathbb{R}^d$. Since Graph($\mathcal{H}_t$) is analytic, by [18, Proposition 7.50] there exists a u.m. selector $H_t^n(\cdot) \in \mathcal{H}_t(\cdot)$, such that

$$A_t^Q(\omega) \wedge n - 1/n \le H_t^n(\omega) E_{Q_t(\omega)}[\Delta S_t(\omega, \cdot)] \le 0, \text{ for } Q\text{-a.s. } \omega \in \Omega_t,$$

where the second inequality follows from the local-supermartingale property of $H^n \cdot S$ with $H^n = (0, \ldots, 0, H_t^n, 0 \ldots, 0) \in \mathcal{H}$. Sending $n \to \infty$ we get that $A_t^Q \le 0$ Q-a.s. for $t = 0, \ldots, T-1$, and thus $Q \in \mathfrak{Q}$.

6.5.2 Proof of Theorem 6.5.3

We first provide three lemmas for the proof of Theorem 6.5.3. We shall prove Lemmas 6.5.4 & 6.5.6 in the appendix.

Lemma 6.5.4. *Let Assumption 6.5.1(ii) hold, and define* $\mathfrak{Q}_t:\ \Omega_t \twoheadrightarrow \mathfrak{P}(\Omega)$ *by*

$$\mathfrak{Q}_t(\omega) := \{Q \in \mathfrak{P}(\Omega):\ Q \lll \mathcal{P}_t(\omega),\ E_Q|\Delta S_t(\omega,\cdot)| < \infty,\ A_t(\omega,Q) < \infty\}. \tag{6.5.2}$$

Then $\mathfrak{Q}_t$ *has an analytic graph.*

The following lemma, which is a measurable version of Theorem 6.4.2, is parallel to [19, Lemma 4.10]. Given Theorem 6.4.2, the proof of this lemma follows exactly the argument of [19, Lemma 4.10], and thus we omit it here.

Lemma 6.5.5. *Let NA*$(\mathcal{P})$ *and Assumption 6.5.1 hold, and let* $t \in \{0,\dots,T-1\}$ *and* $\hat{f}:\ \Omega_t \times \Omega \mapsto \mathbb{R}^*$ *be u.s.a.. Then*

$$\mathcal{E}_t(\hat{f}):\ \Omega_t \mapsto \mathbb{R}^*, \quad \mathcal{E}_t(\hat{f})(\omega) := \sup_{Q \in \mathfrak{Q}_t(\omega)} \left(E_Q[\hat{f}(\omega,\cdot)] - A_t(\omega,Q)\right)$$

is u.s.a.. Besides, there exists a u.m. function $y(\cdot):\ \Omega_t \mapsto \mathbb{R}^d$ *with* $y(\cdot) \in \mathcal{H}_t(\cdot)$*, such that*

$$\mathcal{E}_t(\hat{f})(\omega) + y(\omega)\Delta S_t(\omega,\cdot) \geq \hat{f}(\omega,\cdot) \quad \mathcal{P}_t(\omega) - q.s.$$

for all $\omega \in \Omega_t$ *such that NA*$(\mathcal{P}_t(\omega))$ *holds and* $\hat{f}(\omega,\cdot) > -\infty$ $\mathcal{P}_t(\omega) - q.s.$.

Lemma 6.5.6. *Let Assumptions 6.3.1 & 6.5.1 and NA*$(\mathcal{P})$ *hold. Recall* $\mathfrak{Q}_t$ *defined in* (6.5.2)*. We have that*

$$\mathfrak{Q} = \left\{Q_0 \otimes \ldots \otimes Q_{T-1}:\ Q_t(\cdot) \text{ is a u.m. selector of } \mathfrak{Q}_t,\ t = 0,\dots,T-1\right\}.$$

Proof of Theorem 6.5.3. (ii) $\Longrightarrow$ (i): For any $Q \in \mathfrak{Q}$,

$$V_{t+1} = V_t + H_t\Delta S_t - (C_{t+1}^Q - C_t^Q) \leq V_t + H_t\Delta S_t,\ Q\text{-a.s.}.$$

Hence,

$$E_Q[V_{t+1}|\mathcal{F}_t] \le V_t + H_t E_Q[\Delta S_t|\mathcal{F}_t] \le V_t + A_t^Q = V_t + B_{t+1}^Q - B_t^Q,$$

i.e.,

$$E_Q[V_{t+1} - B_{t+1}^Q|\mathcal{F}_t] \le V_t - B_t^Q.$$

(i) $\Longrightarrow$ (ii): We shall first show that

$$\mathcal{E}_t(V_{t+1}) \le V_t, \quad \mathcal{P} - q.s. \tag{6.5.3}$$

Let $Q = Q_1 \otimes \ldots \otimes Q_{T-1} \in \mathfrak{Q}$ and $\varepsilon > 0$. The map $(\omega, Q) \to E_Q[V_{t+1}(\omega,\cdot)] - A_t(\omega, Q)$ is u.s.a., and graph$(\mathfrak{Q}_t)$ is analytic. As a result, by [18, Proposition 7.50] there exists a u.m. selector $Q_t^\varepsilon : \Omega_t \mapsto \mathfrak{P}(\Omega)$, such that $Q_t^\varepsilon(\cdot) \in \mathfrak{Q}_t(\cdot)$ on $\{\mathfrak{Q}_t \neq \emptyset\}$ (whose complement is a Q-null set), and

$$E_{Q_t^\varepsilon(\cdot)}[V_{t+1}] - A_t(\cdot, Q_t^\varepsilon(\cdot)) \ge \mathcal{E}_t(V_{t+1}) \wedge \frac{1}{\varepsilon} - \varepsilon, \quad Q\text{-a.s.}$$

Define

$$Q' = Q_1 \otimes \ldots \otimes Q_{t-1} \otimes Q_t^\varepsilon \otimes Q_{t+1} \otimes Q_{T-1}.$$

Then $Q' \in \mathfrak{Q}$ by Lemma 6.5.6. Therefore,

$$E_{Q'}[V_{t+1} - B_{t+1}^{Q'}|\mathcal{F}_t] \le V_t - B_t^{Q'}, \quad Q'\text{-a.s.}$$

Noticing that $Q = Q'$ on Ω_t, we have

$$V_t \ge E_{Q'}[V_{t+1}|\mathcal{F}_t] - A_t^{Q'} = E_{Q_t^\varepsilon(\cdot)}[V_{t+1}] - A_t(\cdot, Q_t^\varepsilon(\cdot)) \ge \mathcal{E}_t(V_{t+1}) \wedge \frac{1}{\varepsilon} - \varepsilon, \quad Q\text{-a.s..}$$

By the arbitrariness of ε and Q, we have (6.5.3) holds.

By Lemma 6.5.5, there exists a u.m. function $H_t : \Omega_t \mapsto \mathbb{R}^d$ such that

$$\mathcal{E}_t(V_{t+1})(\omega) + H_t(\omega)\Delta S_{t+1}(\omega,\cdot) \ge V_{t+1}(\omega,\cdot) \quad \mathcal{P}_t(\omega) - q.s.$$

for $\omega \in \Omega_t \setminus N_t$. Fubini's theorem and (6.5.3) imply that

$$V_t + H_t \Delta S_t \geq V_{t+1} \quad \mathcal{P} - q.s..$$

Finally, by defining $C_t := V_0 + (H \cdot S)_t - V_t$, the conclusion follows. □

6.6 Hedging European and American options in multiple periods

6.6.1 Hedging European options

Let $f : \Omega_T \mapsto \mathbb{R}$ be a u.s.a. function, which represents the payoff of a European option. Define the super-hedging price

$$\pi(f) := \inf\{x : \exists H \in \mathcal{H}, \text{ s.t. } x + (H \cdot S)_T \geq f, \ \mathcal{P} - q.s.\}.$$

Theorem 6.6.1. *Let Assumptions 6.3.1 & 6.5.1 and NA($\mathcal{P}$) hold. Then the super-hedging price is given by*

$$\pi(f) = \sup_{Q \in \mathfrak{Q}} \left(E_Q[f] - E_Q[B_T^Q] \right). \tag{6.6.1}$$

Moreover, $\pi(f) > -\infty$ and there exists $H \in \mathcal{H}$, such that $\pi(f) + (H \cdot S)_T \geq f$ $\mathcal{P} - q.s.$.

Proof. It is easy to see that $\pi(f) \geq \sup_{Q \in \mathfrak{Q}} (E_Q[f] - E_Q[B_T^Q])$. We shall show the reverse inequality. Define $V_T = f$ and

$$V_t = \mathcal{E}_t(V_{t+1}), \ t = 0, \dots, T-1.$$

Then V_t is u.s.a. by Lemma 6.5.5 for $t = 1, \dots, T$. It is easy to see that $(V_t - B_t^Q)_t$ is a Q-local-supermartingale for each $Q \in \mathfrak{Q}$. Then by Theorem 6.5.3, there exists $H \in \mathcal{H}$, such that

$$V_0 + (H \cdot S)_T \geq V_T = f, \quad \mathcal{P} - q.s.$$

Hence $V_0 \geq \pi(f)$. It remains to show that

$$V_0 \leq \sup_{Q \in \mathfrak{Q}} \left(E_Q[f] - E_Q[B_T^Q] \right). \tag{6.6.2}$$

First assume that f is bounded from above. Then by [18, Proposition 7.50], Lemma 6.5.4 and Lemma 6.5.5, we can choose a u.m. ε optimizer Q_t^ε for $\mathcal{E}_t$ in each time period. Define $Q^\varepsilon := Q_0^\varepsilon \otimes \ldots \otimes Q_{T-1}^\varepsilon \in \mathfrak{Q}$,

$$V_0 = \mathcal{E}_0 \circ \ldots \circ \mathcal{E}_{T-1}(f) \leq E_{Q^\varepsilon}[f - B_T^{Q^\varepsilon}] + T\varepsilon \leq \sup_{Q \in \mathfrak{Q}} E_Q[f - B_T^Q] + T\varepsilon,$$

which implies (6.6.2).

In general let f be any u.s.a. function. Then we have

$$\mathcal{E}_0 \circ \ldots \circ \mathcal{E}_{T-1}(f \wedge n) \leq \sup_{Q \in \mathfrak{Q}} \left(E_Q[f \wedge n] - E_Q[B_T^Q] \right).$$

Obviously the limit of the right hand side above is $\sup_{Q \in \mathfrak{Q}} \left(E_Q[f] - E_Q[B_T^Q] \right)$. To conclude that the limit of the left hand side is $\mathcal{E}_0 \circ \ldots \circ \mathcal{E}_{T-1}(f)$, it suffices to show that for any $t \in \{0, \ldots, T-1\}$, and $\mathcal{F}_{t+1}$-measurable functions $v^n \nearrow v$,

$$\gamma := \sup_n \mathcal{E}_t(v^n) = \mathcal{E}_t(v), \ \mathcal{P} - q.s..$$

Indeed, for $\omega \in \Omega_t \setminus N_t$, by Theorem 6.4.2 $v^n(\omega) - \gamma(\omega) \in K(\mathcal{P}(\omega))$, where N_t and $K(\cdot)$ are defined in (6.3.2) and (6.4.4) respectively. Since $K(\mathcal{P}(\omega))$ is closed by Lemma 6.4.4, $v(\omega) - \gamma(\omega) \in K(\mathcal{P}(\omega))$, which implies $\gamma(\omega) \geq \mathcal{E}_t(v)(\omega)$ by Theorem 6.4.2.

Finally, using a backward induction we can show that $V_t > -\infty$ $\mathcal{P} - q.s.$, $t = 0, \ldots, T-1$ by Lemma 6.3.2 and Theorem 6.4.2. In particular, $\pi(f) = V_0 > -\infty$. □

Corollary 6.6.2. *Let Assumption 6.5.1 and NA($\mathcal{P}$) hold. Assume that for any $t \in \{0, \ldots, T-1\}$ and $\omega \in \Omega_t$, $\mathcal{H}_t(\omega)$ is a convex cone containing the origin. Then*

$$\pi(f) = \sup_{Q \in \mathcal{Q}} E_Q[f].$$

Proof. This follows from (6.5.1) and that $\mathfrak{Q} = \mathcal{Q}$ and $B_T^Q = 0$ for any $Q \in \mathcal{Q}$. □

6.6.2 Hedging American options

We consider the sub- and super-hedging prices of an American option in this subsection. The same problems are analyzed in Chapter VIII but without portfolio constraints. The analysis here is essentially the same, so we only provide the results and the main ideas for their proofs. For more details and discussion see Chapter VIII.

For $t \in \{0, \dots, T-1\}$ and $\omega \in \Omega_t$, define

$$\mathfrak{Q}^t(\omega) := \{Q_t(\omega) \otimes \dots \otimes Q_{T-1}(\omega, \cdot) : \ Q_i \text{ is a u.m. selector of } \mathfrak{Q}_i, \ i = t, \dots, T-1\}.$$

In particular $\mathfrak{Q}^0 = \mathfrak{Q}$. Assume graph$(\mathfrak{Q}^t)$ is analytic. Let $\mathcal{T}$ be the set of stopping times with respect to the raw filtration $(\mathcal{B}(\Omega_t))_t$, and let $\mathcal{T}_t \subset \mathcal{T}$ be the set of stopping times that are no less than t.

Let $\mathfrak{f} = (f_t)_t$ be the payoff of the American option. Assume that $\mathfrak{f}_t \in \mathcal{B}(\Omega_t), \ t = 1, \dots, T$, and $\mathfrak{f}_\tau \in L^1(Q)$ for any $\tau \in \mathcal{T}$ and $Q \in \mathfrak{Q}$. Define the sub-hedging price:

$$\underline{\pi}(\mathfrak{f}) := \sup\{x : \ \exists (H, \tau) \in \mathcal{H} \times \mathcal{T}, \text{ s.t. } \mathfrak{f}_\tau + (H \cdot S)_\tau \geq x, \ \mathcal{P} - q.s.\},$$

and the super-hedging price:

$$\overline{\pi}(\mathfrak{f}) := \inf\{x : \ \exists H \in \mathcal{H}, \text{ s.t. } x + (H \cdot S)_\tau \geq \mathfrak{f}_\tau, \ \mathcal{P} - q.s., \ \forall \tau \in \mathcal{T}\}.$$

Theorem 6.6.3. *(i) The sub-hedging price is given by*

$$\underline{\pi}(\mathfrak{f}) = \sup_{\tau \in \mathcal{T}} \inf_{Q \in \mathfrak{Q}} E_Q[\mathfrak{f}_\tau + B_T^Q]. \tag{6.6.3}$$

(ii) For $t \in \{1, \dots, T-1\}$, assume that the map

$$\phi_t : \ \Omega_t \times \mathfrak{P}(\Omega_{T-t}) \mapsto \mathbb{R}^*, \ \phi_t(\omega, Q) = \sup_{\tau \in \mathcal{T}_t} E_Q\left[\mathfrak{f}_\tau(\omega, \cdot) - \sum_{i=t}^{\tau-1} A_i^Q(\omega, \cdot)\right]$$

is u.s.a. Then

$$\overline{\pi}(\mathfrak{f}) = \sup_{\tau\in\mathcal{T}} \sup_{Q\in\mathfrak{Q}} E_Q[\mathfrak{f}_\tau - B_\tau^Q], \tag{6.6.4}$$

and there exists $H \in \mathcal{H}$*, such that* $\overline{\pi}(f) + (H\cdot S)_\tau \geq \mathfrak{f}_\tau, \mathcal{P} - q.s.,\ \forall \tau \in \mathcal{T}$*.*

Proof. (i) We first show that

$$\underline{\pi}(f) = \sup\{x:\ \exists (H,\tau) \in \mathcal{H}\times\mathcal{T},\ \text{s.t. } \mathfrak{f}_\tau + (H\cdot S)_T \geq x,\ \mathcal{P} - q.s.\} =: \beta.$$

For any $x < \underline{\pi}(f)$, there exists $(H,\tau) \in \mathcal{H}\times\mathcal{T}$, such that $\mathfrak{f}_\tau + (H\cdot S)_\tau \geq x\ \mathcal{P} - q.s.$. Define $H' := (H_t 1_{\{t<\tau\}})_t$. For $t = 0,\dots,T-1$, since $\{t<\tau\} \in \mathcal{B}(\Omega_t)$, $H_t'(\cdot)$ is u.m.; besides, $H_t'(\cdot)$ is equal to either $H_t(\cdot) \in \mathcal{H}_t(\cdot)$ or $0 \in \mathcal{H}_t(\cdot)$. Hence $H' \in \mathcal{H}$. Then $\mathfrak{f}_\tau + (H'\cdot S)_T = \mathfrak{f}_\tau + (H\cdot S)_\tau \geq x\ \mathcal{P} - q.s$, which implies $x \leq \beta$, and thus $\underline{\pi}(f) \leq \beta$.

Conversely, for $x < \beta$, there exists $(H,\tau) \in \mathcal{H}\times\mathcal{T}$, such that $\mathfrak{f}_\tau + (H\cdot S)_T \geq x\ \mathcal{P} - q.s.$ Then we also have that $\mathfrak{f}_\tau + (H\cdot S)_\tau \geq x\ \mathcal{P} - q.s.$. To see this, let us define $D := \{\mathfrak{f}_\tau + (H\cdot S)_\tau < x\}$ and $H' := (H_t 1_{\{t\geq\tau\}\cap D})_t \in \mathcal{H}$. We get that

$$(H'\cdot S)_T = [(H\cdot S)_T - (H\cdot S)_\tau]1_D \geq 0\ \mathcal{P} - q.s.,\ \text{and } (H'\cdot S)_T > 0\ \mathcal{P} - q.s. \text{ on } D.$$

NA($\mathcal{P}$) implies D is $\mathcal{P}$-polar. Therefore $x \leq \underline{\pi}(f)$, and thus $\beta \leq \underline{\pi}(f)$.

It can be shown that

$$\underline{\pi}(\mathfrak{f}) = \beta = \sup_{\tau\in\mathcal{T}} \sup\{x:\ \exists H \in \mathcal{H}:\ \mathfrak{f}_\tau + (H\cdot S)_T \geq x,\ \mathcal{P} - q.s.\} = \sup_{\tau\in\mathcal{T}} \inf_{Q\in\mathfrak{Q}} E_Q[\mathfrak{f}_\tau + B_T^Q],$$

where we apply Theorem 6.6.1 for the last equality above.

(ii) Define

$$V_t:\ \Omega_t \mapsto \mathbb{R}^*, \quad V_t = \sup_{Q\in\mathfrak{Q}^t} \sup_{\tau\in\mathcal{T}_t} E_Q\left[\mathfrak{f}_\tau(\omega,\cdot) - \sum_{i=t}^{\tau-1} A_i^Q(\omega,\cdot)\right].$$

It can be shown that V_t is u.s.a. for $t = 1,\dots,T$ and $(V_t - B_t^Q)_t$ is a Q-supermartingale for each $Q \in \mathfrak{Q}$. By Theorem 6.5.3, there exists $H \in \mathcal{H}$ such that

$$V_0 + (H\cdot S)_\tau \geq \mathfrak{f}_\tau, \mathcal{P} - q.s.,\ \forall \tau \in \mathcal{T}.$$

Therefore, $\sup_{\tau\in\mathcal{T}}\sup_{Q\in\mathfrak{Q}} E_Q[\mathfrak{f}_\tau - B_\tau^Q] = V_0 \le \overline{\pi}(\mathfrak{f})$. The reverse inequality is easy to see. $\square$

Remark 6.6.4. In (6.6.3) and (6.6.4), the penalization terms are B_T^Q and B_τ^Q respectively. In fact, similar to the argument in (i) above, one can show that

$$\begin{aligned}
\hat{\pi}(f) &:= \inf\{x:\ \forall\tau\in\mathcal{T},\ \exists H\in\mathcal{H},\ \text{s.t. } x+(H\cdot S)_\tau \ge \mathfrak{f}_\tau,\ \mathcal{P}-q.s.\}\\
&= \sup_{\tau\in\mathcal{T}}\inf\{x:\ \exists H\in\mathcal{H},\ \text{s.t. } x+(H\cdot S)_\tau \ge \mathfrak{f}_\tau,\ \mathcal{P}-q.s.\}\\
&= \sup_{\tau\in\mathcal{T}}\inf\{x:\ \exists H\in\mathcal{H},\ \text{s.t. } x+(H\cdot S)_T \ge \mathfrak{f}_\tau,\ \mathcal{P}-q.s.\}\\
&= \sup_{\tau\in\mathcal{T}}\sup_{Q\in\mathfrak{Q}} E_Q[\mathfrak{f}_\tau - B_T^Q]
\end{aligned} \tag{6.6.5}$$

Even though the definition of $\hat{\pi}(f)$ is less useful for super-hedging since the stopping time should not be known in advance, it suggests that B_T^Q comes from knowing τ in advance (compare $\underline{\pi}(f)$ and $\hat{\pi}(f)$). It is also both mathematically and financially meaningful that $\hat{\pi}(f)\le\overline{\pi}(f)$. However, it is interesting that when B^Q vanishes (e.g., when $\mathcal{H}_t(\cdot)$ is a cone), then $\hat{\pi}(f)=\overline{\pi}(f)$.

6.7 FTAP and super-hedging in multiple periods with options

Let us use the set-up in Section 6.3. In addition, let $g=(g^1,\dots,g^e):\ \Omega_T\mapsto\mathbb{R}^e$ be Borel measurable, and each g^i is seen as an option which can and only can be traded at time $t=0$ without constraints. Without loss of generality we assume the price of each option is 0. In this section, we say NA$(\mathcal{P})^g$ holds if for any $(H,h)\in\mathcal{H}\times\mathbb{R}^e$,

$$(H\cdot S)_T + hg \ge 0\ \mathcal{P}-q.s. \implies (H\cdot S)_T + hg = 0\ \mathcal{P}-q.s..$$

Obviously NA$(\mathcal{P})^g$ implies NA$(\mathcal{P})$.

Definition 6.7.1. $f:\ \Omega_T\mapsto\mathbb{R}$ is replicable (by stocks and options), if there exists some $x\in\mathbb{R},\ h\in\mathbb{R}^e$ and $H\in\mathcal{H}$, such that

$$x+(H\cdot S)_T+hg=f \quad \text{or} \quad x+(H\cdot S)_T+hg=-f.$$

Let

$$\mathcal{Q}_g := \{Q \in \mathcal{Q} : \ E_Q[g] = 0\}.$$

Below is the main result of this section:

Theorem 6.7.2. *Let assumptions in Corollary 6.6.2 hold. Also assume that g^i is not replicable by stocks and other options, and $g^i \in L^1(\mathcal{Q})$, $i = 1, \dots, e$. Then we have the following.*

(i) NA$(\mathcal{P})^g$ holds if and only if for each $P \in \mathcal{P}$, there exists $Q \in \mathcal{Q}_g$ dominating P.

(ii) Let NA$(\mathcal{P})^g$ holds. Let $f : \ \Omega_T \mapsto \mathbb{R}$ be Borel measurable such that $f \in L^1(\mathcal{Q})$. Then

(6.7.1)

$$\pi(f) := \inf\{x \in \mathbb{R} : \ \exists (H, h) \in \mathcal{H} \times \mathbb{R}^e \ s.t. \ x + (H \cdot S)_T + hg \geq f, \ \mathcal{P}-q.s.\} = \sup_{Q \in \mathcal{Q}_g} E_Q[f].$$

Moreover, there exists $(H, h) \in \mathcal{H} \times \mathbb{R}^e$, such that $\pi(f) + (H \cdot S)_T + hg \geq f$ $\mathcal{P}-q.s.$.

(iii) Assume in addition $\mathcal{H} = -\mathcal{H}$. Let NA$(\mathcal{P})^g$ hold and $f : \ \Omega_T \mapsto \mathbb{R}$ be Borel measurable satisfying $f \in L^1(\mathcal{Q}_g)$. Then the following are equivalent:

(a) f is replicable;

(b) The mapping $Q \mapsto E_Q[f]$ is a constant on $\mathcal{Q}_g$;

(c) For all $P \in \mathcal{P}$ there exists $Q \in \mathcal{Q}_g$ such that $P \ll Q$ and $E_Q[f] = \pi(f)$.

Moreover, the market is complete[2] if and only if $\mathcal{Q}_g$ is a singleton.

Proof. We first show the existence of an optimal super-hedging strategy in (ii). It can be shown that

$$\begin{aligned}
\pi(f) &= \inf_{h \in \mathbb{R}^e} \inf\{x \in \mathbb{R} : \ \exists H \in \mathcal{H} \text{ s.t. } x + (H \cdot S)_T \geq f - hg, \ \mathcal{P} - q.s.\} \\
&= \inf_{h \in \mathbb{R}^e} \sup_{Q \in \mathcal{Q}} E_Q[f - hg],
\end{aligned}$$

[2]That is, for any Borel measurable function $f : \ \Omega_T \mapsto \mathbb{R}$ satisfying $f \in L^1_g(\mathcal{Q})$, f is replicable.

where we apply Corollary 6.6.1 for the second equality above.

We claim that 0 is a relative interior point of the convex set

$$\mathcal{I} := \{E_Q[g] : \ Q \in \mathcal{Q}\}.$$

If not, then there exists some $h \in \mathbb{R}^e$ with $h \neq 0$, such that $E_Q[hg] \leq 0$ for any $Q \in \mathcal{Q}$. Then the super-hedging price of hg using S, $\pi^0(hg)$, satisfies $\pi^0(hg) \leq 0$ by Corollary 6.6.2. Hence by Theorem 6.6.1 there exists $H \in \mathcal{H}$, such that $(H \cdot S)_T \geq hg \ \mathcal{P} - q.s.$. As the price of hg is 0, NA$(\mathcal{P})^g$ implies that

$$(H \cdot S)_T - hg = 0 \ \mathcal{P} - q.s.,$$

which contradicts the assumption that each g^i cannot be replicated by S and the other options, as $h \neq 0$. Hence we have shown that 0 is a relative interior point of $\mathcal{I}$.

Define $\phi : \ \mathbb{R}^e \mapsto \mathbb{R}$,

$$\phi(h) = \sup_{Q \in \mathcal{Q}} E_Q[f - hg],$$

and observe that

$$\pi(f) = \inf_{h \in \mathbb{R}^e} \phi(h) = \inf_{h \in \mathrm{span}(\mathcal{I})} \phi(h).$$

We will now show that there exists a compact set $\mathbb{K} \subset \mathrm{span}(\mathcal{I})$, such that

$$\pi(f) = \inf_{h \in \mathbb{K}} \phi(h). \tag{6.7.2}$$

In order to do this, we will show that for any h outside a particular ball will satisfy $\phi(h) \geq \phi(0)$, which establishes the claim.

Now, since 0 is a relative interior point of $\mathcal{I}$, there exists $\gamma > 0$, such that

$$B_\gamma := \{v \in \mathrm{span}(\mathcal{I}) : \ ||v|| \leq \gamma\} \subset \mathcal{I}.$$

Consider the ball $\mathbb{K} := \{h \in \mathrm{span}(\mathcal{I}) : \ ||h|| \leq 2\sup_{Q \in \mathcal{Q}} E_Q|f|/\gamma\}$. Then for any $h \in \mathrm{span}(\mathcal{I}) \setminus \mathbb{K}$, there exists $Q \in \mathcal{Q}$ such that $-hE_Q[g] > 2\sup_{Q \in \mathcal{Q}} E_Q|f|$ (pick Q

s.t. $E_Q[g]$ is in the same direction as $-h$ and lies on the circumference of B_γ). This implies that

$$\phi(h) \geq \sup_{Q\in\mathcal{Q}} E_Q[-hg] - \sup_{Q\in\mathcal{Q}} E_Q|f| > \sup_{Q\in\mathcal{Q}} E_Q|f| = \phi(0).$$

Since such h are suboptimal, it follows that

$$\pi(f) = \inf_{h\in\mathbb{K}} \phi(h).$$

On the other hand, observe that

$$|\phi(h)-\phi(h')| \leq \sup_{Q\in\mathcal{Q}} |E_Q[f-hg]-E_Q[f-h'g]| \leq \sup_{Q\in\mathcal{Q}} E|(h-h')g| \leq ||h-h'|| \sup_{Q\in\mathcal{Q}} E_Q[||g||],$$

i.e. ϕ is continuous (in fact Lipschitz). Hence there exists some $h^* \in \mathbb{K} \subset \mathbb{R}^e$, such that

$$\begin{aligned}
\pi(f) &= \inf_{h\in\mathbb{R}^e} \sup_{Q\in\mathcal{Q}} E_Q[f-hg] \\
&= \sup_{Q\in\mathcal{Q}} E_Q[f-h^*g] \\
&= \inf\{x\in\mathbb{R}: \ \exists H\in\mathcal{H} \text{ s.t. } x + H\cdot S \geq f - h^*g, \ \mathcal{P}-q.s.\}.
\end{aligned}$$

Then by Theorem 6.6.1 there exists $H^* \in \mathcal{H}$, such that $\pi(f) + (H^*\cdot S)_T \geq f - h^*g \ \mathcal{P} - q.s.$.

Next let us prove (i) and (6.7.1) in (ii) simultaneously by induction. For $e = 0$, (i) and (6.7.1) hold by Theorem 6.2.2 and Corollary 6.6.2. Assume for $e = k$ (i) and (6.7.1) hold and we consider $e = k+1$. We first consider (i). Let $\pi^k(g^{k+1})$ be the super-hedging price of g^{k+1} using stocks S and options $g' := (g^1, \dots, g^k)$. By induction hypothesis, we have

$$\pi^k(g^{k+1}) = \sup_{Q\in\mathcal{Q}_{g'}} E_Q[g^{k+1}].$$

Recall that the price of g^{k+1} is 0. Then $\mathrm{NA}(\mathcal{P})^g$ implies $\pi^k(g^{k+1}) \geq 0$. If $\pi^k(g^{k+1}) = 0$, then there exists $(H,h) \in \mathcal{H} \times \mathbb{R}^k$, such that $(H \cdot S)_T + hg' - g^{k+1} \geq 0$ $\mathcal{P} - q.s.$. Then by $\mathrm{NA}(\mathcal{P})^g$,

$$(H \cdot S)_T + hg' - g^{k+1} = 0, \quad \mathcal{P} - q.s.,$$

which contradicts the assumption that g^{k+1} cannot be replicated by S and g'. Therefore, $\pi^k(g^{k+1}) > 0$. Similarly $\pi^k(-g^{k+1}) > 0$. Thus we have

$$\inf_{Q \in \mathcal{Q}_{g'}} E_Q[g^{k+1}] < 0 < \sup_{Q \in \mathcal{Q}_{g'}} E_Q[g^{k+1}].$$

Then there exists $Q_-, Q_+ \in \mathcal{Q}_{g'}$ satisfying

$$E_{Q_-}[g^{k+1}] < 0 < E_{Q_+}[g^{k+1}]. \tag{6.7.3}$$

Then for any $P \in \mathcal{P}$, let $Q \in \mathcal{Q}_{g'}$ dominating P. Let

$$Q' := \lambda_- Q_- + \lambda Q + \lambda_+ Q_+.$$

By choosing some appropriate $\lambda_-, \lambda, \lambda_+ > 0$ with $\lambda_- + \lambda + \lambda_+ = 1$, we have $P \ll Q' \in \mathcal{Q}_g$, where $g = (g^1, \dots, g^{k+1})$.

Next consider (6.7.1) in (ii). Denote the super-hedging price $\pi^k(\cdot)$ when using S and g', and $\pi(\cdot)$ when using S and g, which is consistent with the definition in (6.7.1). It is easy to see that

$$\pi(f) \geq \sup_{Q \in \mathcal{Q}_g} E_Q[f], \tag{6.7.4}$$

and we focus on the reverse inequality. It suffices to show that

$$\exists Q_n \in \mathcal{Q}_{g'}, \text{ s.t. } E_{Q_n}[g^{k+1}] \to 0 \text{ and } E_{Q_n}[f] \to \pi(f). \tag{6.7.5}$$

Indeed, if (6.7.5) holds, then we define

$$Q'_n := \lambda_-^n Q_- + \lambda^n Q_n + \lambda_+^n Q_+, \quad \text{s.t. } E_{Q'_n}[g^{k+1}] = 0, \text{ i.e., } Q'_n \in \mathcal{Q}_g,$$

where Q_+, Q_- are from (6.7.3) and $\lambda_-^n, \lambda^n, \lambda_+^n \in [0,1]$ such that $\lambda_-^n + \lambda^n + \lambda_+^n = 1$. Since $E_{Q_n}[g^{k+1}] \to 0$, we can choose $\lambda_\pm^n \to 0$. Then $E_{Q'_n}[f] \to \pi(f)$, which implies $\pi(f) \leq \sup_{Q \in \mathcal{Q}_g} E_Q[f]$.

So let us concentrate on proving (6.7.5). By a translation, we may w.l.o.g. assume $\pi(f) = 0$. Thus if (6.7.5) fails, we have

$$0 \notin \overline{\{E_Q[(g^{k+1}, f)] : \ Q \in \mathcal{Q}_{g'}\}} \subset \mathbb{R}^2.$$

Then there exists a separating vector $(y, z) \in \mathbb{R}^2$ with $||(y,z)|| = 1$ such that

$$\sup_{Q \in \mathcal{Q}_{g'}} E_Q[yg^{k+1} + zf] < 0. \tag{6.7.6}$$

By the induction hypothesis, we have that

$$0 > \sup_{Q \in \mathcal{Q}_{g'}} E_Q[yg^{k+1} + zf] = \pi^k(yg^{k+1} + zf) \geq \pi(yg^{k+1} + zf) = \pi(zf).$$

Obviously from the above $z \neq 0$. If $z > 0$, then by positive homogeneity $\pi(f) < 0$, contradicting the assumption $\pi(f) = 0$. Hence $z < 0$. Take $Q'' \in \mathcal{Q}_g \subset \mathcal{Q}_{g'}$. Then by (6.7.6) $0 > E_{Q''}[yg^{k+1} + zf] = E_{Q''}[zf]$, and thus $E_{Q''}[f] > 0 = \pi(f)$, which contradicts (6.7.4).

Finally, let us prove (iii). It is easy to see that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c). Now let (c) hold. Let $(H, h) \in \mathcal{H} \times \mathbb{R}^e$ such that $\pi(f) + (H \cdot S)_T + hg \geq f \ \mathcal{P} - q.s.$ If there exists $P \in \mathcal{P}$ satisfying

$$P\{\pi(f) + (H \cdot S)_T + hg > f\} > 0,$$

then by choosing a $Q \in \mathcal{Q}_g$ that dominates P, we would have that $\pi(f) > E_Q[f] = \pi(f)$, contradiction. Hence $\pi(f) + H \cdot S + hg = f \ \mathcal{P} - q.s.$, i.e., f is replicable.

If the market is complete, then by letting $f = 1_A$, we know that $Q \mapsto Q(A)$ is constant on $\mathcal{Q}$ for every $A \in \mathcal{B}(\Omega)$ by (b). As any probability measure is uniquely determined by its value on $\mathcal{B}(\Omega)$, we know that $\mathcal{Q}$ is a singleton. Conversely, if $\mathcal{Q}$ is a singleton, then (b) holds, and thus the market is complete by (a). $\square$

VI.A Proofs of Some Technical Results

VI.A.1 Proof of Lemma 6.3.2

Proof. Fix $t \in \{0, \dots, T-1\}$ and let

$$\text{(VI.A.1)} \quad \Lambda^\circ(\omega) := \{y \in \mathbb{R}^d : \ yv \geq 0, \text{ for all } v \in \operatorname{supp}_{\mathcal{P}(\omega)}(\Delta S_t(\omega, \cdot))\}, \quad \omega \in \Omega_t.$$

It could be easily shown that

$$N_t^c = \{\omega \in \Omega_t : \ \Lambda^\circ_{\mathcal{H}}(\omega) \subset -\Lambda^\circ(\omega)\},$$

where $\Lambda^\circ_{\mathcal{H}} = \Lambda^\circ \cap \mathcal{H}_t$. For any $P \in \mathfrak{P}(\Omega_t)$, by [19, (4.5)], there exists a Borel-measurable mapping $\Lambda^\circ_P : \ \Omega_t \twoheadrightarrow \mathbb{R}^d$ with non-empty closed values such that $\Lambda^\circ_P = \Lambda^\circ$ P-a.s.. This implies that the graph(Λ°_P) is Borel (see [2, Theorem 18.6]). Then it can be shown directly from the definition (6.1.1) that $\Lambda^\circ_{\mathcal{H},P} := \Lambda^\circ_P \cap \mathcal{H}_t$ is u.m. Thanks to the closedness of $-\Lambda^\circ$, the set

$$N^c_{t,P} = \{\omega : \ \Lambda^\circ_{\mathcal{H},P}(\omega) \subset -\Lambda^\circ(\omega)\} = \cap_{y \in \mathbb{Q}^d}\{\omega : \ \operatorname{dist}(y, \Lambda^\circ_{\mathcal{H},P}(\omega)) \geq \operatorname{dist}(y, -\Lambda^\circ(\omega))\}$$

is u.m. Therefore, there exists a Borel measurable set $\tilde{N}^c_{t,P}$, such that $\tilde{N}^c_{t,P} = N^c_{t,P} = N^c_t$ P-a.s. Thus N^c_t is u.m. by [18, Lemma 7.26].

It remains to show that N_t is $\mathcal{P}$-polar. If not, then there exists $P_* \in \mathcal{P}$ such that $P_*(N_t) > 0$. Similar to the argument above, there exists a map $\Lambda^\circ_* : \ \Omega_t \twoheadrightarrow \mathbb{R}^d$ with a Borel measurable graph(Λ°_*), such that

$$\text{(VI.A.2)} \qquad \Lambda^\circ_* = \Lambda^\circ \ P_*\text{-a.s..}$$

Let

$$\Phi(\omega) := \{(y, P) \in (\Lambda^\circ_* \cap \mathcal{H}_t)(\omega) \times \mathcal{P}_t(\omega) : \ E_P[y\Delta S_t(\omega, \cdot)] > 0\}, \quad \omega \in \Omega_t.$$

Then $N_t = \{\Phi \neq \emptyset\}$ P_*-a.s. by (6.3.2), (VI.A.1) and (VI.A.2). It is easy to see that (with a slight abuse of notation)

$$\begin{aligned} \text{graph}(\Phi) &= [\text{graph}(\mathcal{P}_t) \times \mathbb{R}^d] \cap [\mathfrak{P}(\Omega) \times \text{graph}(\Lambda^\circ_*)] \\ &\quad \cap \{E_P[y\Delta S_t(\omega, \cdot)] > 0\} \cap [\mathfrak{P}(\Omega) \times \text{graph}(\mathcal{H}_t)] \end{aligned}$$

is analytic. Therefore, by the Jankov-von Neumann Theorem [18, Proposition 7.49], there exists a u.m. selector (y, P) such that $(y(\cdot), P(\cdot)) \in \Phi(\cdot)$ on $\{\Phi \neq \emptyset\}$. As $N_t = \{\Phi \neq \emptyset\}$ $P_* - a.s.$, y is P_*-a.s. an arbitrage on N_t. Redefine $y = 0$ on $\{y \notin \Lambda^\circ \cap \mathcal{H}_t\}$, and P to be any u.m. selector of $\mathcal{P}_t$ on $\{\Phi = \emptyset\}$. (Here we redefine y on $\{y \notin \Lambda^\circ \cap \mathcal{H}_t\}$ instead of $\{\Phi \neq \emptyset\}$ in order to make sure that $y(\cdot) \in \Lambda^\circ(\cdot)$ so that $y\Delta S_t \geq 0$ $\mathcal{P} - q.s.$.) So we have that $y(\cdot) \in \mathcal{H}_t(\cdot)$, $P(\cdot) \in \mathcal{P}_t(\cdot)$, $y\Delta S_t \geq 0$ $\mathcal{P} - q.s.$, and

$$P(\omega)\{y(\omega)\Delta S_t(\omega, \cdot) > 0\} > 0 \quad \text{for } P_*\text{-a.s. } \omega \in N_t. \tag{VI.A.3}$$

Now define $H = (H_0, \dots, H_{T-1}) \in \mathcal{H}$ satisfying

$$H_t = y, \text{ and } H_s = 0, \; s \neq t.$$

Also define

$$P^* = P_*|_{\Omega_t} \otimes P \otimes P_{t+1} \otimes \ldots \otimes P_{T-1} \in \mathcal{P},$$

where P_s is any u.m. selector of $\mathcal{P}_s$, $s = t+1, \dots, T-1$. Then $(H \cdot S)_T \geq 0$ $\mathcal{P} - q.s.$, and $P^*\{(H \cdot S)_T > 0\} > 0$ by (VI.A.3), which contradicts NA($\mathcal{P}$). $\square$

VI.A.2 Proof of Lemma 6.3.3

Proof. Let

$$\Phi(\omega) := \{(R, \hat{R}) \in \mathfrak{P}(\Omega) \times \mathfrak{P}(\Omega) : \; P(\omega) \ll R \ll \hat{R}\}, \quad \omega \in \Omega_t,$$

which has an analytic graph as shown in the proof of [19, Lemma 4.8]. Consider $\Xi:\ \Omega_t \twoheadrightarrow \mathfrak{P}(\Omega)\times\mathfrak{P}(\Omega)$,

$$\begin{aligned}\Xi(\omega) :=&\{(Q,\hat{P})\in\mathfrak{P}(\Omega)\times\mathfrak{P}(\Omega):\ E_Q|\Delta S_t(\omega,\cdot)|<\infty,\\ &E_Q[y\Delta S_t(\omega,\cdot)]\le 0,\ \forall y\in\mathcal{H}_t(\omega),\ P(\omega)\ll Q\ll \hat{P}\in\mathcal{P}_t(\omega)\}.\end{aligned}$$

Recall the analytic set $\Psi_{\mathcal{H}_t}$ defined Assumption 6.3.1(iii). We have that

$$\mathrm{graph}(\Xi)=[\Psi_{\mathcal{H}_t}\times\mathfrak{P}(\Omega)]\cap[\mathfrak{P}(\Omega)\times\mathrm{graph}(\mathcal{P}_t)]\cap\mathrm{graph}(\Phi)$$

is analytic. As a result, we can apply the Jankov-von Neumann Theorem [18, Proposition 7.49] to find u.m. selectors $Q(\cdot),\hat{P}(\cdot)$ such that $(Q(\cdot),\hat{P}(\cdot))\in\Xi(\cdot)$ on $\{\Xi\neq\emptyset\}$. We set $Q(\cdot):=\hat{P}(\cdot):=P(\cdot)$ on $\{\Xi=\emptyset\}$. By Theorem 6.2.2, if Assumption 6.3.1(ii) and $\mathrm{NA}(\mathcal{P}_t(\omega))$ hold, and $P(\omega)\in\mathcal{P}_t(\omega)$, then $\Xi(\omega)\neq\emptyset$. So our construction satisfies the conditions stated in the lemma.

It remains to show that $\mathrm{graph}(\mathcal{Q}_t)$ is analytic. Using the same argument for Ξ, but omitting the lower bound $P(\cdot)$, we see that the map $\tilde{\Xi}:\ \Omega_t\twoheadrightarrow\mathfrak{P}(\Omega)\times\mathfrak{P}(\Omega)$,

$$\begin{aligned}\tilde{\Xi}(\omega) :=&\{(Q,\hat{P})\in\mathfrak{P}(\Omega)\times\mathfrak{P}(\Omega):\ E_Q|\Delta S_t(\omega,\cdot)|<\infty,\\ &E_Q[y\Delta S_t(\omega,\cdot)]\le 0,\ \forall y\in\mathcal{H}_t(\omega),\ Q\ll \hat{P}\in\mathcal{P}_t(\omega)\}\end{aligned}$$

has an analytic graph. Since $\mathrm{graph}(\mathcal{Q}_t)$ is the image of $\mathrm{graph}(\tilde{\Xi})$ under the canonical projection $\Omega_t\times\mathfrak{P}(\Omega)\times\mathfrak{P}(\Omega)\to\Omega_t\times\mathfrak{P}(\Omega)$, it is also analytic. □

VI.A.3 Proof of Lemma 6.5.4

Proof. Similar to the argument in [19, Lemma 4.8], we can show that the set

$$J:=\{(P,Q)\in\mathfrak{P}(\Omega)\times\mathfrak{P}(\Omega):\ Q\ll P\}$$

is Borel measurable. Thus, for $\Xi:\ \Omega_t\twoheadrightarrow\mathfrak{P}(\Omega)$

$$\Xi(\omega)=\{Q\in\mathfrak{P}(\Omega):\ Q\lll\mathcal{P}_t(\omega)\},$$

graph(Ξ) is analytic since it is the projection of the analytic set

$$[\Omega_t \times J] \cap [\text{graph}(\mathcal{P}_t) \times \mathfrak{P}(\Omega)]$$

onto $\Omega_t \times \mathfrak{P}(\Omega)$. By Assumption 6.5.1(ii), the function $\hat{A}: \ \Omega_t \times \mathfrak{P}(\Omega) \mapsto \mathbb{R}^*$,

$$\hat{A}(\omega, Q) = A(\omega, Q) 1_{\{E_Q|\Delta S_t(\omega,\cdot)|<\infty\}} + \infty 1_{\{E_Q|\Delta S_t(\omega,\cdot)|=\infty\}}$$

is l.s.a. As a result,

$$\text{graph}(\mathfrak{Q}_t) = \text{graph}(\Xi) \cap \{\hat{A} < \infty\}$$

is analytic. □

VI.A.4 Proof of Lemma 6.5.6

Proof. Denote the right side above by $\mathfrak{R}$. Let $R = Q_0 \otimes \ldots \otimes Q_{T-1} \in \mathfrak{R}$. Without loss of generality, we can assume that $Q_t: \ \Omega_t \mapsto \mathfrak{P}(\Omega)$ is Borel measurable and $Q_t(\cdot) \in \mathfrak{Q}_t(\cdot)$ on $\{\mathfrak{Q}_t \neq \emptyset\}$ $Q^{t-1} := Q_0 \otimes \ldots \otimes Q_{t-1}$-a.s., $t = 1, \ldots, T-1$. For $\omega \in \Omega_t, \ t = 0, \ldots, T-1$, let

$$\Phi_t(\omega) := \{(Q, P) \in \mathfrak{P}(\Omega) \times \mathfrak{P}(\Omega) : \ Q_t(\omega) = Q \ll P \in \mathcal{P}_t(\omega)\}.$$

Similar to the argument in the proof of [19, Lemma 4.8], it can be shown that graph(Φ) is analytic, and thus there exists u.m. selectors $\hat{Q}_t(\cdot), \hat{P}_t(\cdot)$, such that $(\hat{Q}_t(\cdot), \hat{P}_t(\cdot)) \in \Phi(\cdot)$ on $\{\Phi_t \neq \emptyset\}$. We shall show by an induction that for $t = 0, \ldots, T-1$,

$$\Phi_t \neq \emptyset \text{ for } t = 0, \text{ and } \{\Phi_t = \emptyset\} \text{ is a } Q^{t-1}\text{-null set for } t = 1, \ldots T-1,$$

and there exists a universally selector of $\mathcal{P}_t$ which we denote by $P_t(\cdot): \ \Omega_t \mapsto \mathfrak{P}(\Omega)$ such that

$$Q^t = \hat{Q}_0 \otimes \ldots \otimes \hat{Q}_t \ll P_0 \otimes \ldots \otimes P_t.$$

Then by setting $t = T-1$, we know $R = Q^{T-1} \in \mathfrak{Q}$. It is easy to see that the above holds for $t = 0$. Assume it holds for $t = k < T-1$. Then $\{\Phi_{k+1} = \emptyset\} \subset \{Q_{k+1}(\cdot) \notin \mathfrak{Q}_{k+1}(\cdot)\}$ is a Q^k-null set by Lemma 6.3.2 and the induction hypothesis. As a result, $\hat{Q}_{k+1} = Q_{k+1}$ Q^k-a.s., which implies that $Q^{k+1} = \hat{Q}_0 \otimes \ldots \otimes \hat{Q}_{k+1}$. Setting $P_{k+1} = \hat{P}_{k+1} 1_{\{\Phi \neq \emptyset\}} + \tilde{P}_{k+1} 1_{\{\Phi = \emptyset\}}$, where $\tilde{P}_{k+1}(\cdot)$ is any u.m. selector of $\mathcal{P}_{k+1}$, we have that $P_0 \otimes \ldots \otimes P_{k+1} \in \mathcal{P}^{k+1}$. Since $Q_{k+1}(\omega) \ll P_{k+1}(\omega)$ for Q^k-a.s. $\omega \in \Omega_k$, together with the induction hypothesis, we have that $Q^{k+1} \ll P_0 \otimes \ldots \otimes P_{k+1}$. Thus we finish the proof for the induction.

Conversely, for any $R \in \mathfrak{Q}$, we may write $R = Q_0 \otimes \ldots \otimes Q_{T-1}$, where $Q_t : \Omega_t \mapsto \mathfrak{P}(\Omega)$ is some Borel kernel, $t = 0, \ldots, T-1$. Then $Q_t(\omega) \in \mathfrak{Q}_t(\omega)$ for Q^{t-1}-a.s. $\omega \in \Omega_{t-1}$. Thanks to the analyticity of graph$(\mathfrak{Q}_t)$, we can modify $Q_t(\cdot)$ on a Q^{t-1}-null set, such that the modification $\hat{Q}_t(\cdot)$ is u.m. and $\hat{Q}_t(\cdot) \in \mathfrak{Q}_t(\cdot)$ on $\{\mathfrak{Q}_t \neq \emptyset\}$. Using a forward induction of this modification, we have that $R = \hat{Q}_0 \otimes \ldots \otimes \hat{Q}_{T-1} \in \mathfrak{R}$. $\square$

CHAPTER VII

Fundamental theorem of asset pricing under model uncertainty and transaction costs in hedging options

7.1 Introduction

We consider a discrete time financial market in which stocks are traded dynamically and options are available for static hedging. We assume that the dynamically traded asset is liquid and trading in them does not incur transaction costs, but that the options are less liquid and their prices are quoted with a bid-ask spread. (The more difficult problem with transaction costs on a dynamically traded asset is analyzed in [4, 37].) As in [19] we do not assume that there is a single model describing the asset price behavior but rather a collection of models described by the convex collection $\mathcal{P}$ of probability measures, which does not necessarily admit a dominating measure. One should think of $\mathcal{P}$ as being obtained from calibration to the market data. We have a collection rather than a single model because generally we do not have point estimates but a confidence intervals for the parameters of our models. Our first goal is to obtain a criteria for deciding whether the collection of models represented by $\mathcal{P}$ is viable or not. Given that $\mathcal{P}$ is viable we would like to obtain the range of prices for other options written on the dynamically traded assets. The dual elements in these result are martingale measures that price the hedging options correctly (i.e. consistent with the quoted prices). As in classical transaction costs literature, we need to replace the no arbitrage condition by the stronger *robust no arbitrage* condition, as we shall see in Section 7.2. In Section 7.3 we will make the additional assumption that the hedging options with non-zero spread are *non-redundant* (see Definition 7.3.1). We will see that under this assumption no arbitrage and robust no arbitrage are equivalent. Our main results are Theorems 7.2.4 and 7.3.4.

7.2 Fundamental theorem with robust no arbitrage

Let $S_t = (S_t^1, \dots, S_t^d)$ be the prices of d traded stocks at time $t \in \{0, 1, \dots, T\}$ and $\mathcal{H}$ be the set of all predictable $\mathbb{R}^d$-valued processes, which will serve as our trading strategies. Let $g = (g^1, \dots, g^e)$ be the payoff of e options that can be traded only at time zero with bid price $\underline{g}$ and ask price $\overline{g}$, with $\overline{g} \geq \underline{g}$ (the inequality holds component-wise). We assume S_t and g are Borel measurable, and there are no transaction costs in the trading of stocks.

Definition 7.2.1 (No arbitrage and robust no arbitrage)**.** We say that condition NA($\mathcal{P}$) holds if for all $(H, h) \in \mathcal{H} \times \mathbb{R}^e$,

$$H \cdot S_T + h^+(g - \overline{g}) - h^-(g - \underline{g}) \geq 0 \quad \mathcal{P} - \text{quasi-surely (-q.s.)}^1$$

implies

$$H \cdot S_T + h^+(g - \overline{g}) - h^-(g - \underline{g}) = 0 \quad \mathcal{P}\text{-q.s.},$$

where $h^{\pm}$ are defined component-wise and are the usual positive/negative part of h.[2]

We say that condition RNA($\mathcal{P}$) holds if there exists $\underline{g}', \overline{g}'$ such that $[\underline{g}', \overline{g}'] \subseteq ri[\underline{g}, \overline{g}]$ and NA($\mathcal{P}$) holds if g has bid-ask prices $\underline{g}', \overline{g}'$.[3]

[1] A set is $\mathcal{P}$-polar if it is P-null for all $P \in \mathcal{P}$. A property is said to hold $\mathcal{P}$-q.s. if it holds outside a $\mathcal{P}$-polar set.
[2] When we multiply two vectors, we mean their inner product.
[3] "ri" stands for relative interior. $[\underline{g}', \overline{g}'] \subseteq ri[\underline{g}, \overline{g}]$ means component-wise inclusion.

Definition 7.2.2 (Super-hedging price)**.** For a given a random variable f, its super-hedging price is defined as

$$\pi(f) := \inf\{x \in \mathbb{R} : \exists\,(H,h) \in \mathcal{H}\times\mathbb{R}^e,\ \text{s.t. } x+H\cdot S_T+h^+(g-\overline{g})-h^-(g-\underline{g}) \geq f\ \mathcal{P}\text{-q.s.}\}.$$

Any pair $(H,h) \in \mathcal{H} \times \mathbb{R}^e$ in the above definition is called a semi-static hedging strategy.

Remark 7.2.3. **(1)** Let $\hat{\pi}(g^i)$ and $\hat{\pi}(-g^i)$ be the super-hedging prices of g^i and $-g^i$, where the hedging is done using stocks and options excluding g^i. RNA($\mathcal{P}$) implies either

$$-\hat{\pi}(-g^i) \leq \underline{g}^i = \overline{g}^i \leq \hat{\pi}(g^i)$$

or

$$-\hat{\pi}(-g^i) \leq (\overline{g}')^i < \overline{g}^i \quad \text{and} \quad \underline{g}^i < (\underline{g}')^i \leq \hat{\pi}(g^i) \tag{7.2.1}$$

where $\underline{g}', \overline{g}'$ are the more favorable bid-ask prices in the definition of robust no arbitrage. The reason for working with robust no arbitrage is to be able to have the strictly inequalities in (7.2.1) for options with non-zero spread, which turns out to be crucial in the proof of the closedness of the set of hedgeable claims in (7.2.3) (hence the existence of an optimal hedging strategy), as well as in the construction of a dual element (see (7.2.6)).

(2) Clearly RNA($\mathcal{P}$) implies NA($\mathcal{P}$), but the converse is not true. For example, assume in the market there is no stock, and there are only two options: $g_1(\omega) = g_2(\omega) = \omega,\ \omega \in \Omega := [0,1]$. Let $\mathcal{P}$ be the set of probability measures on Ω, $\underline{g}_1 = \overline{g}_1 = 1/2$, $\underline{g}_2 = 1/4$ and $\overline{g}_2 = 1/2$. Then NA($\mathcal{P}$) holds while RNA($\mathcal{P}$) fails.

For $b, a \in \mathbb{R}^e$, let

$$\mathcal{Q}^{[b,a]} := \{Q \lll \mathcal{P} : Q \text{ is a martingale measure and } E_Q[g] \in [b,a]\}$$

where $Q \lll \mathcal{P}$ means $\exists P \in \mathcal{P}$ such that $Q \ll P$.[4] Let $\mathcal{Q}_\varphi^{[b,a]} := \{Q \in \mathcal{Q} : E_Q[\varphi] < \infty\}$. When $[b,a] = [\underline{g}, \overline{g}]$, we drop the superscript and simply write $\mathcal{Q}, \mathcal{Q}_\varphi$. Also define

$$\mathcal{Q}^s := \{Q \lll \mathcal{P} : Q \text{ is a martingale measure and } E_Q[g] \in ri[\underline{g}, \overline{g}]\}$$

and $\mathcal{Q}_\varphi^s := \{Q \in \mathcal{Q}^s : E_Q[\varphi] < \infty\}$.

Theorem 7.2.4. *Let $\varphi \geq 1$ be a random variable such that $|g^i| \leq \varphi$ $\forall i = 1, \dots, e$. The following statements hold:*

(a) (Fundamental Theorem of Asset Pricing): The following statements are equivalent

(i) RNA($\mathcal{P}$) holds.

(ii) There exists $[\underline{g}', \overline{g}'] \subseteq ri[\underline{g}, \overline{g}]$ such that $\forall P \in \mathcal{P}$, $\exists Q \in \mathcal{Q}_\varphi^{[\underline{g}', \overline{g}']}$ such that $P \ll Q$.

(b) (Super-hedging) Suppose RNA($\mathcal{P}$) holds. Let $f : \Omega \to \mathbb{R}$ be Borel measurable such that $|f| \leq \varphi$. The super-hedging price is given by

$$\pi(f) = \sup_{Q \in \mathcal{Q}_\varphi^s} E_Q[f] = \sup_{Q \in \mathcal{Q}_\varphi} E_Q[f] \in (-\infty, \infty], \tag{7.2.2}$$

and there exists $(H, h) \in \mathcal{H} \times \mathbb{R}^e$ such that

$$\pi(f) + H \cdot S_T + h^+(g - \overline{g}) - h^-(g - \underline{g}) \geq f \quad \mathcal{P}\text{-q.s.}.$$

Proof. It is easy to show (ii) in (a) implies that NA($\mathcal{P}$) holds for the market with bid-ask prices $\underline{g}', \overline{g}'$, Hence RNA($\mathcal{P}$) holds for the original market. The rest of our proof consists two parts as follows.

[4] $E_Q[g] \in [b,a]$ means $E_Q[g^i] \in [b^i, a^i]$ for all $i = 1, \dots, e$.

Part 1: $\pi(f) > -\infty$ and the existence of an optimal hedging strategy in (b). Once we show that the set

$$\mathcal{C}_g := \{H \cdot S_T + h^+(g - \overline{g}) - h^-(g - \underline{g}) :\ (H, h) \in \mathcal{H} \times \mathbb{R}^e\} - \mathcal{L}^0_+ \tag{7.2.3}$$

is $\mathcal{P} - q.s.$ closed (i.e., if $(W^n)_{n=1}^\infty \subset \mathcal{C}_g$ and $W^n \to W\ \mathcal{P} - q.s.$, then $W \in \mathcal{C}_g$), the argument used in the proof of [19, Theorem 2.3] would conclude the results in part 1. We will demonstrate the closedness of $\mathcal{C}_g$ in the rest of this part.

Write $g = (u, v)$, where $u = (g^1, \ldots, g^r)$ consists of the hedging options without bid-ask spread, i.e, $\underline{g}^i = \overline{g}^i$ for $i = 1, \ldots, r$, and $v = (g^{r+1}, \ldots, g^e)$ consists of those with spread, i.e., $\underline{g}^i < \overline{g}^i$ for $i = r + 1, \ldots, e$, for some $r \in \{0, \ldots, e\}$. Denote $\underline{u} := (\underline{g}^1, \ldots, \underline{g}^r)$ and similarly for $\underline{v}$ and $\overline{v}$. Define

$$\mathcal{C} := \{H \cdot S_T + \alpha(u - \underline{u}) :\ (H, \alpha) \in \mathcal{H} \times \mathbb{R}^r\} - \mathcal{L}^0_+.$$

Then $\mathcal{C}$ is $\mathcal{P} - q.s.$ closed by [19, Theorem 2.2].

Let $W^n \to W\ \mathcal{P} - q.s.$ with

$$W^n = H^n \cdot S_T + \alpha^n(u - \underline{u}) + (\beta^n)^+(v - \overline{v}) - (\beta^n)^-(v - \underline{v}) - U^n \in \mathcal{C}_g, \tag{7.2.4}$$

where $(H^n, \alpha^n, \beta^n) \in \mathcal{H} \times \mathbb{R}^r \times \mathbb{R}^{e-r}$ and $U^n \in \mathcal{L}^0_+$. If $(\beta^n)_n$ is not bounded, then by passing to subsequence if necessary, we may assume that $0 < ||\beta^n|| \to \infty$ and rewrite (7.2.4) as

$$\frac{H^n}{\beta^n} \cdot S_T + \frac{\alpha^n}{||\beta^n||}(u - \underline{u}) \geq \frac{W^n}{||\beta^n||} - \left(\frac{\beta^n}{||\beta^n||}\right)^+ (v - \overline{v}) + \left(\frac{\beta^n}{||\beta^n||}\right)^- (v - \underline{v}) \in \mathcal{C},$$

where $||\cdot||$ represents the sup-norm. Since $\mathcal{C}$ is $\mathcal{P} - q.s.$ closed, the limit of the right hand side above is also in $\mathcal{C}$, i.e., there exists some $(H, \alpha) \in \mathcal{H} \times \mathbb{R}^r$, such that

$$H \cdot S_T + \alpha(u - \underline{u}) \geq -\beta^+(v - \overline{v}) + \beta^-(v - \underline{v}), \quad \mathcal{P} - a.s.,$$

where β is the limit of $(\beta^n)_n$ along some subsequence with $||\beta|| = 1$. NA($\mathcal{P}$) implies that

$$H \cdot S_T + \alpha(u - \underline{u}) + \beta^+(v - \overline{v}) - \beta^-(v - \underline{v}) = 0, \quad \mathcal{P} - a.s.. \tag{7.2.5}$$

As $\beta =: (\beta_{r+1}, \ldots, \beta_e) \neq 0$, we assume without loss of generality (w.l.o.g.) that $\beta_e \neq 0$. If $\beta_e < 0$, then we have from (7.2.5) that

$$\underline{g}^e + \frac{H}{\beta_e^-} \cdot S_T + \frac{\alpha}{\beta_e^-}(u - \underline{u}) + \sum_{i=r+1}^{e-1} \left[\frac{\beta_i^+}{\beta_e^-}(g^i - \overline{g}^i) - \frac{\beta_i^-}{\beta_e^-}(g^i - \underline{g}^i) \right] = g^e, \quad \mathcal{P} - a.s..$$

Therefore $\hat{\pi}(g^e) \leq \underline{g}_e$, which contradicts the robust no arbitrage property (see (7.2.1)) of g^e. Here $\hat{\pi}(g^e)$ is the super-hedging price of g^e using S and g excluding g^e. Similarly we get a contradiction if $\beta_e > 0$.

Thus $(\beta^n)_n$ is bounded, and has a limit $\beta \in \mathbb{R}^{e-r}$ along some subsequence $(n_k)_k$. Since by (7.2.4)

$$H^n \cdot S_T + \alpha^n(u - \underline{u}) \geq W^n - (\beta^n)^+(v - \overline{v}) + (\beta^n)^-(v - \underline{v}) \in \mathcal{C},$$

the limit of the right hand side above along $(n_k)_k$, $W - \beta^+(v - \overline{v}) + \beta^-(v - \underline{v})$, is also in $\mathcal{C}$ by its closedness, which implies $W \in \mathcal{C}_g$.

Part 2: $(i) \Rightarrow (ii)$ **in part (a) and** (7.3.3) **in part (b).** We will prove the results by an induction on the number of hedging options, as in [19, Theorem 5.1]. Suppose the results hold for the market with options $g^1, \ldots, g^e$. We now introduce an additional option $f \equiv g^{e+1}$ with $|f| \leq \varphi$, available at bid-ask prices $\underline{f} < \overline{f}$ at time zero. (When the bid and ask prices are the same for f, then the proof is identical to [19].)

$(i) \implies (ii)$ in (a): Let $\pi(f)$ be the super-hedging price when stocks and $g^1, \ldots, g^e$ are available for trading. By RNA($\mathcal{P}$) and (7.3.3) in part (b) of the induction hypothesis, we have

$$\overline{f} > \overline{f}' \geq -\pi(-f) = \inf_{Q \in \mathcal{Q}_\varphi^s} E_Q[f] \quad \text{and} \quad \underline{f} < \underline{f}' \leq \pi(f) = \sup_{Q \in \mathcal{Q}_\varphi^s} E_Q[f] \tag{7.2.6}$$

where $[\underline{f}', \overline{f}'] \subseteq (\underline{f}, \overline{f})$ comes from the definition of robust no arbitrage. This implies that there exists $Q_+, Q_- \in \mathcal{Q}^s_\varphi$ such that $E_{Q_+}[f] > \underline{f}''$ and $E_{Q_-}[f] < \overline{f}''$ where $\underline{f}'' = \frac{1}{2}(\underline{f} + \underline{f}')$, $\overline{f}'' = \frac{1}{2}(\overline{f} + \overline{f}')$. By (a) of induction hypothesis, there exists $[b, a] \subseteq ri[\underline{g}, \overline{g}]$ such that for any $P \in \mathcal{P}$, we can find $Q_0 \in \mathcal{Q}^{[b,a]}_\varphi$ satisfying $P \ll Q_0 \lll \mathcal{P}$. Define

$$\underline{g}' = \min(b, E_{Q_+}[g], E_{Q_-}[g]), \quad \text{and} \quad \overline{g}' = \max(a, E_{Q_+}[g], E_{Q_-}[g])$$

where the minimum and maximum are taken component-wise. We have $[b, a] \subseteq [\underline{g}', \overline{g}'] \subseteq ri[\underline{g}, \overline{g}]$ and $Q_+, Q_- \in \mathcal{Q}^{[\underline{g}', \overline{g}']}_\varphi$.

Now, let $P \in \mathcal{P}$. (a) of induction hypothesis implies the existence of a $Q_0 \in \mathcal{Q}^{[b,a]}_\varphi \subseteq \mathcal{Q}^{[\underline{g}', \overline{g}']}_\varphi$ satisfying $P \ll Q_0 \lll \mathcal{P}$. Define

$$Q := \lambda_- Q_- + \lambda_0 Q_0 + \lambda_+ Q_+.$$

Then $Q \in \mathcal{Q}^{[\underline{g}', \overline{g}']}_\varphi$ and $P \ll Q$. By choosing suitable weights $\lambda_-, \lambda_0, \lambda_+ \in (0, 1)$, $\lambda_- + \lambda_0 + \lambda_+ = 1$, we can make $E_Q[f] \in [\underline{f}'', \overline{f}''] \subseteq ri[\underline{f}, \overline{f}]$.

(7.3.3) in (b): Let ξ be a Borel measurable function such that $|\xi| \leq \varphi$. Write $\pi'(\xi)$ for its super-hedging price when stocks and $g^1, \ldots, g^e, f \equiv g^{e+1}$ are traded, $\mathcal{Q}'_\varphi := \{Q \in \mathcal{Q}_\varphi : E_Q[f] \in [\underline{f}, \overline{f}]\}$ and $\mathcal{Q}'^s_\varphi := \{Q \in \mathcal{Q}^s_\varphi : E_Q[f] \in (\underline{f}, \overline{f})\}$. We want to show

$$\pi'(\xi) = \sup_{Q \in \mathcal{Q}'^s_\varphi} E_Q[\xi] = \sup_{Q \in \mathcal{Q}'_\varphi} E_Q[\xi]. \tag{7.2.7}$$

It is easy to see that

$$\pi'(\xi) \geq \sup_{Q \in \mathcal{Q}'_\varphi} E_Q[\xi] \geq \sup_{Q \in \mathcal{Q}'^s_\varphi} E_Q[\xi] \tag{7.2.8}$$

and we shall focus on the reverse inequalities. Let us assume first that ξ is bounded from above, and thus $\pi'(\xi) < \infty$. By a translation we may assume $\pi'(\xi) = 0$.

First, we show $\pi'(\xi) \le \sup_{Q \in \mathcal{Q}'_\varphi} E_Q[\xi]$. It suffices to show the existence of a sequence $\{Q_n\} \subseteq \mathcal{Q}_\varphi$ such that $\lim_n E_{Q_n}[f] \in [\underline{f}, \overline{f}]$ and $\lim_n E_{Q_n}[\xi] = \pi'(\xi) = 0$. (See page 30 of [19] for why this is sufficient.) In other words, we want to show that

$$\overline{\{E_Q[(f,\xi)] : Q \in \mathcal{Q}_\varphi\}} \cap \big([\underline{f}, \overline{f}] \times \{0\}\big) \neq \emptyset.$$

Suppose the above intersection is empty. Then there exists a vector $(y, z) \in \mathbb{R}^2$ with $|(y,z)| = 1$ that strictly separates the two closed, convex sets, i.e. there exists $b \in \mathbb{R}$ s.t.

$$\sup_{Q \in \mathcal{Q}_\varphi} E_Q[yf + z\xi] < b \quad \text{and} \quad \inf_{a \in [\underline{f}, \overline{f}]} ya > b. \tag{7.2.9}$$

It follows that

$$y^+\underline{f} - y^-\overline{f} + \pi'(z\xi) \le \pi'(yf + z\xi) \le \pi(yf + z\xi) = \sup_{Q \in \mathcal{Q}_\varphi} E_Q[yf + z\xi] < b < y^+\underline{f} - y^-\overline{f}, \tag{7.2.10}$$

where the first inequality is because one can super-replicate $z\xi = (yf + z\xi) + (-yf)$ from initial capital $\pi'(yf + z\xi) - y^+\underline{f} + y^-\overline{f}$, the second inequality is due to the fact that having more options to hedge reduces hedging cost, and the middle equality is by (b) of induction hypothesis. The last two inequalities are due to (7.2.9).

It follows from (7.2.10) that $\pi'(z\xi) < 0$. Therefore, we must have that $z < 0$, otherwise $\pi'(z\xi) = z\pi'(\xi) = 0$ (since the super-hedging price is positively homogenous). Recall that we have proved in part (a) that $\mathcal{Q}'_\varphi \neq \emptyset$. Let $Q' \in \mathcal{Q}'_\varphi \subseteq \mathcal{Q}_\varphi$. The part of (7.2.10) after the equality implies that $yE_{Q'}[f] + zE_{Q'}[\xi] < y^+\underline{f} - y^-\overline{f}$. Since $E_{Q'}[f] \in [\underline{f}, \overline{f}]$, we get $zE_{Q'}[\xi] < y^+(\underline{f} - E_{Q'}[f]) - y^-(\overline{f} - E_{Q'}[f]) \le 0$. Since $z < 0$, $E_{Q'}[\xi] > 0$. But by (7.2.8), $E_{Q'}[\xi] \le \pi'(\xi) = 0$, which is a contradiction.

Next, we show $\sup_{Q \in \mathcal{Q}'_\varphi} E_Q[\xi] \le \sup_{Q \in \mathcal{Q}'^s_\varphi} E_Q[\xi]$. It suffices to show for any $\varepsilon > 0$ and every $Q \in \mathcal{Q}'_\varphi$, we can find $Q^s \in \mathcal{Q}'^s_\varphi$ such that $E_{Q^s}[\xi] > E_Q[\xi] - \varepsilon$. To this end, let $Q' \in \mathcal{Q}'^s_\varphi$ which is nonempty by part (a). Define

$$Q^s := (1-\lambda)Q + \lambda Q'.$$

We have $Q^s \lll \mathcal{P}$ by the convexity of $\mathcal{P}$, and $Q^s \in \mathcal{Q}'^s_\varphi$ if $\lambda \in (0,1]$. Moreover,

$$E_{Q^s}[\xi] = (1-\lambda)E_Q[\xi] + \lambda E_{Q'}[\xi] \to E_Q[\xi] \quad \text{as} \quad \lambda \to 0.$$

So for $\lambda > 0$ sufficiently close to zero, the Q^s constructed above satisfies $E_{Q^s}[\xi] > E_Q[\xi] - \varepsilon$. Hence we have shown that the supremum over $\mathcal{Q}'_\varphi$ and $\mathcal{Q}'^s_\varphi$ are equal. This finishes the proof for upper bounded ξ.

Finally when ξ is not bounded from above, we can apply the previous result to $\xi \wedge n$, and then let $n \to \infty$ and use the closedness of $\mathcal{C}_g$ in (7.2.3) to show that (7.3.3) holds. The argument would be the same as the last paragraph in the proof of [19, Thoerem 3.4] and we omit it here. □

7.3 A sharper fundamental theorem with the non-redundancy assumption

We now introduce the concept of non-redundancy. With this additional assumption we will sharpen our result.

Definition 7.3.1 (Non-redundancy)**.** A hedging option g^i is said to be non-redundant if it is not perfectly replicable by stocks and other hedging options, i.e. there does not exist $x \in \mathbb{R}$ and a semi-static hedging strategy $(H, h) \in \mathcal{H} \times \mathbb{R}^e$ such that

$$x + H \cdot S_T + \sum_{j \neq i} h^j g^j = g^i \ \mathcal{P}\text{-q.s.}.$$

Remark 7.3.2. RNA($\mathcal{P}$) does not imply non-redundancy. For Instance, having only two identical options in the market whose payoffs are in $[c, d]$, with identical bid-ask prices b and a satisfying $b < c$ and $a > d$, would give a trivial counter example where RNA($\mathcal{P}$) holds yet we have redundancy.

Lemma 7.3.3. *Suppose all hedging options with non-zero spread are non-redundant. Then NA($\mathcal{P}$) implies RNA($\mathcal{P}$).*

Proof. Let $g = (g^1, \dots, g^{r+s})$, where $u := (g^1, \dots, g^r)$ consists of the hedging options without bid-ask spread, i.e, $\underline{g}^i = \overline{g}^i$ for $i = 1, \dots, r$, and $(g^{r+1}, \dots, g^{r+s})$ consists of those with bid-ask spread, i.e., $\underline{g}^i < \overline{g}^i$ for $i = r+1, \dots, r+s$. We shall prove the result by induction on s. Obviously the result holds when $s = 0$. Suppose the result holds for $s = k \geq 0$. Then for $s = k+1$, denote $v := (g^{r+1} \dots, g^{r+k})$, $\underline{v} := (\underline{g}^{r+1}, \dots, \underline{g}^{r+k})$ and $\overline{v} := (\overline{g}^{r+1}, \dots, \overline{g}^{r+k})$. Denote $f := g^{r+k+1}$.

By the induction hypothesis, there exists $[\underline{v}', \overline{v}'] \subset (\underline{v}, \overline{v})$ be such that NA($\mathcal{P}$) holds in the market with stocks, options u and options v with any bid-ask prices b and a satisfying $[\underline{v}', \overline{v}'] \subset [b, a] \subset (\underline{v}, \overline{v})$. Let $\underline{v}_n \in (\underline{v}, \underline{v}')$, $\overline{v}_n \in (\overline{v}', \overline{v})$, $\underline{f}_n > \underline{f}$ and $\underline{f}_n < \overline{f}$, such that $\underline{v}_n \searrow \underline{v}$, $\overline{v}_n \nearrow \overline{v}$, $\underline{f}_n \searrow \underline{f}$ and $\overline{f}_n \nearrow \overline{f}$. We shall show that for some n, NA($\mathcal{P}$) holds with stocks, options u, options v with bid-ask prices $\underline{v}_n$ and $\overline{v}_n$, option f with bid-ask prices $\underline{f}_n$ and $\overline{f}_n$. We will show it by contradiction.

If not, then for each n, there exists $(H^n, h^n_u, h^n_v, h^n_f) \in \mathcal{H} \times \mathbb{R}^r \times \mathbb{R}^k \times \mathbb{R}$ such that

$$\text{(7.3.1)} \qquad H^n \cdot S_T + h^n_u(u - \underline{u}) + (h^n_v)^+(v - \overline{v}_n) - (h^n_v)^-(v - \underline{v}_n)$$

$$+(h^n_f)^+(f - \overline{f}_n) - (h^n_f)^-(f - \underline{f}_n) \geq 0, \quad \mathcal{P} - q.s.,$$

and the strict inequality for the above holds with positive probability under some $P_n \in \mathcal{P}$. Hence $h^n_f \neq 0$. By a normalization, we can assume that $|h^n_f| = 1$. By extracting a subsequence, we can w.l.o.g. assume that $h^n_f = -1$ (the argument when assuming $h^n_f = 1$ is similar). If $(h^n_u, h^n_v)_n$ is not bounded, then w.l.o.g. we assume that $0 < c^n := ||(h^n_u, h^n_v)|| \to \infty$. By (7.3.1) we have that

$$\frac{H^n}{c^n} \cdot S_T + \frac{h^n_u}{c^n}(u - \underline{u}) + \frac{(h^n_v)^+}{c^n}(v - \overline{v}_n) - \frac{(h^n_v)^-}{c^n}(v - \underline{v}_n) - \frac{1}{c^n}(f - \underline{f}_n) \geq 0, \ \mathcal{P} - q.s..$$

By [19, Theorem 2.2], there exists $H \in \mathcal{H}$, such that

$$H \cdot S_T + h_u(u - \underline{u}) + h_v^+(v - \overline{v}) - h_v^-(v - \underline{v}) \geq 0, \ \mathcal{P} - q.s.,$$

where (h_u, h_v) is the limit of $(h_u^n/c^n, h_u^n/c^n)$ along some subsequence with $||(h_u, h_v)|| = 1$. NA($\mathcal{P}$) implies that

$$H \cdot S_T + h_u(u - \underline{u}) + h_v^+(v - \overline{v}) - h_v^-(v - \underline{v}) = 0, \ \mathcal{P} - q.s.. \tag{7.3.2}$$

Since $(h_u, h_v) \neq 0$, (7.3.2) contradicts the non-redundancy assumption of (u, v).

Therefore, $(h_u^n, h_v^n)_n$ is bounded, and w.l.o.g. assume it has the limit $(\hat{h}_u, \hat{h}_v)$. Then applying [19, Theorem 2.2] in (7.3.1), there exists $\hat{H} \in \mathcal{H}$ such that

$$\hat{H} \cdot S_T + \hat{h}_u(u - \underline{u}) + \hat{h}_v^+(v - \overline{v}) - \hat{h}_v^-(v - \underline{v}) - (f - \underline{f}) \geq 0, \ \mathcal{P} - q.s..$$

NA($\mathcal{P}$) implies that

$$\hat{H} \cdot S_T + \hat{h}_u(u - \underline{u}) + \hat{h}_v^+(v - \overline{v}) - \hat{h}_v^-(v - \underline{v}) - (f - \underline{f}) = 0, \ \mathcal{P} - q.s.,$$

which contradicts the non-redundancy assumption of f. $\square$

We have the following FTAP and super-hedging result in terms of NA($\mathcal{P}$) instead of RNA($\mathcal{P}$), when we additionally assume the non-redundancy of g.

Theorem 7.3.4. *Suppose all hedging options with non-zero spread are non-redundant. Let $\varphi \geq 1$ be a random variable such that $|g^i| \leq \varphi$ $\forall i = 1, \ldots, e$. The following statements hold:*

(a') (Fundamental Theorem of Asset Pricing): The following statements are equivalent

(i) NA($\mathcal{P}$) holds.

(ii) $\forall P \in \mathcal{P}$, $\exists Q \in \mathcal{Q}_\varphi$ such that $P \ll Q$.

(b') (Super-hedging) Suppose NA($\mathcal{P}$) holds. Let $f : \Omega \to \mathbb{R}$ be Borel measurable such that $|f| \le \varphi$. The super-hedging price is given by

$$\pi(f) = \sup_{Q \in \mathcal{Q}_\varphi} E_Q[f] \in (-\infty, \infty], \tag{7.3.3}$$

and there exists $(H, h) \in \mathcal{H} \times \mathbb{R}^e$ such that

$$\pi(f) + H \cdot S_T + h^+(g - \overline{g}) - h^-(g - \underline{g}) \ge f \quad \mathcal{P}\text{-}q.s..$$

Proof. (a')(ii) $\Longrightarrow$ (a')(i) is trivial. Now if (a')(i) holds, then by Lemma 7.3.3, (a)(i) in Theorem 7.2.4 holds, which implies (a)(ii) holds, and thus (a')(ii) holds. Finally, (b') is implied by Lemma 7.3.3 and Theorem 7.2.4(b). □

Remark 7.3.5. Theorem 7.3.4 generalizes the results of [19] to the case when the option prices are quoted with bid-ask spreads. When $\mathcal{P}$ is the set of all probability measures and the given options are all call options written on the dynamically traded assets, a result with option bid-ask spreads similar to Theorem 7.3.4-(a) had been obtained by [26]; see Proposition 4.1 therein, although the non-redundancy condition did not actually appear. (The objective of [26] was to obtain relationships between the option prices which are necessary and sufficient to rule out semi-static arbitrage and the proof relies on determining the correct set of relationships and then identifying a martingale measure.)

However, the no arbitrage concept used in [26] is different: the author there assumes that there is no *weak arbitrage* in the sense of [32]; see also [1,31].[5] (Recall that a market is said to have weak arbitrage if for any given model (probability measure) there is an arbitrage strategy which is an arbitrage in the classical sense.) The arbitrage concept used here and in [19] is weaker, in that we say that a non-negative wealth ($\mathcal{P}$-q.s.) is an arbitrage even if there is a single P under which the

[5]The no arbitrage assumption in [1] is the model independent arbitrage of [32]. However that paper rules out the model dependent arbitrage by assuming that a superlinearly growing option can be bought for static hedging.

wealth process is a classical arbitrage. Hence our no arbitrage condition is stronger than the one used in [26]. But what we get out from a stronger assumption is the existence of a martingale measure $Q \in \mathcal{Q}_\varphi$ for each $P \in \mathcal{P}$. Whereas [26] only guarantees the existence of only one martingale measure which prices the hedging options correctly.

CHAPTER VIII

On hedging American options under model uncertainty

8.1 Introduction

We consider the problem of pricing and semi-static hedging of American options in the model uncertainty set-up of [19]. In semi-static hedging stocks are traded dynamically and options are traded statically. This formulation is frequently used in the literature since options are less liquid than stocks (see e.g. [32]). In this setting, so far only the super-hedging prices of (path dependent) European options under (non-dominated) model uncertainty were considered: see e.g. [1, 17, 19]. [36] obtained these results for a continuous time financial market. Some results are available on the pricing of American options in the model independent framework without the static hedging in options. See for example [35] for duality results in discrete time set-up, and [13, 39, 67] for similar duality results and in particular the analysis of the related optimal stopping problem.

In this chaper, we consider the problems of sub- and super-hedging of American options using semi-static trading strategies in the model independent set-up of [19]. We first obtain the duality results for both the sub- and super-hedging prices, as well as the existence of the optimal hedging strategies. Then for compact state spaces we show how to discretize it in order to obtain the optimal rate of convergence.

In the first part of this chaper, we focus on the sub-and super-hedging dualities. For the sub-hedging prices we discuss whether the sup and inf in the dual representation can be exchanged. We show that the exchangeability may fail in general unless there is no hedging option. For the super-hedging prices we discuss several alternative definitions. The correct definition involves "non-anticipative" strategies, which is quite different from the one in the classical case when there is no hedging option. As for the existence for the optimal hedging strategies, we first develop a new proof to obtain the existence of an optimal static hedge. Then we use the non-dominated optimal stopping to obtain the optimal trading strategy in the stock for sub-hedging problem, and the optional decomposition for super-hedging.

In the second part of this chaper, we concentrate on how to use hedging prices in the discretized market to approximate the ones in the original market. This approximation is useful for numerical computations since in the discretized market the state space is finite, and thus there exists a dominating measure on it. Our approximation result is a generalization of [35], but in our case the construction of the approximation becomes much more complicated due to the presence of the hedging options. In particular, in contrast to [35], it is not a priori clear that the discretized market is free of arbitrage. We also show how to pick the prices of the hedging options in the discretized market in order to obtain the optimal convergence rate. One should note that, although in [35] the no-arbitrage notions of [?] and [19] coincide (see Appendix VIII.D), in our case they are different since there are hedging options available. We choose to work in the framework of [19].

The rest of the chaper is organized as follows: We obtain the duality results for the sub- and super-hedging prices of American options in Sections 8.2 and 8.3, respectively. In Section 8.4, we discretize the path space and show that hedging prices in the discretized market converge to the original ones. The appendix is devoted to verify some of the statements we make in Sections 8.1, 8.2 and 8.3. Of particular interest in that section is the analysis of the adverse optimal stopping problems for nonlinear expectations in discrete time, which resolves the optimal stopping problems in [35] for more general state spaces (see Appendix VIII.B). This result is useful particularly in showing the existence of the optimal sub-hedging strategy. The existence of the optimal super hedging strategy is a consequence of the non-dominated optional decomposition theorem [19] and the analysis in Appendix VIII.C.

The remainder of this section is devoted to setting up the notation used in the rest of the chaper.

8.1.1 Notation

We use the set-up in [19]. Let $T \in \mathbb{N}$ be the time Horizon and let Ω_1 be a Polish space. For $t \in \{0, 1, \dots, T\}$, let $\Omega_t := \Omega_1^t$ be the t-fold Cartesian product, with the convention that Ω_0 is a singleton. We denote by $\mathcal{F}_t$ the universal completion of $\mathcal{B}(\Omega_t)$ and write $(\Omega, \mathcal{F})$ for $(\Omega_T, \mathcal{F}_T)$. For each $t \in \{0, \dots, T-1\}$ and $\omega \in \Omega_t$, we are given a nonempty convex set $\mathcal{P}_t(\omega) \subset \mathfrak{P}(\Omega_1)$ of probability measures. Here $\mathcal{P}_t$ represents the possible models for the t-th period, given state ω at time t. We assume that for each t, the graph of $\mathcal{P}_t$ is analytic, which ensures that $\mathcal{P}_t$ admits a universally measurable selector, i.e., a universally measurable kernel $P_t : \Omega_t \to \mathfrak{P}(\Omega_t)$ such that $P_t(\omega) \in \mathcal{P}_t(\omega)$ for all $\omega \in \Omega_t$. Let

$$\mathcal{P} := \{P_0 \otimes \dots \otimes P_{T-1} : \; P_t(\cdot) \in \mathcal{P}_t(\cdot), \; t = 0, \dots, T-1\}, \tag{8.1.1}$$

where each P_t is a universally measurable selector of $\mathcal{P}_t$, and for $A \in \Omega$,

$$P_0 \otimes \dots \otimes P_{T-1}(A) = \int_{\Omega_1} \dots \int_{\Omega_1} 1_A(\omega_1, \dots, \omega_T) P_{T-1}(\omega_1, \dots, \omega_{T-1}; d\omega_T) \dots P_0(d\omega_1).$$

Let $S_t : \Omega_t \to \mathbb{R}$ be Borel measure, which represents the price at time t of a stock S that can be traded dynamically in the market. Let $g = (g_1, \dots, g_e) : \Omega \to \mathbb{R}^e$ be Borel measurable, representing the options that can only be traded at the beginning at price 0. Assume NA($\mathcal{P}$) holds, i.e, for all $(H, h) \in \mathcal{H} \times \mathbb{R}^e$,

$$(H \cdot S)_T + hg \geq 0 \quad \mathcal{P} - \text{q.s.} \qquad \text{implies} \qquad (H \cdot S)_T + hg = 0 \quad \mathcal{P} - \text{q.s.},$$

where $\mathcal{H}$ is the set of predictable processes representing trading strategies, $(H \cdot S)_T = \sum_{t=0}^{T-1} H_t(S_{t+1} - S_t)$, and hg denotes the inner product of h and g. Then from [19, FTAP], for all $P \in \mathcal{P}$, there exists $Q \in \mathcal{Q}$ such that $P \ll Q$, where

$$\mathcal{Q} := \{Q \text{ martingale measure}^1 : \ E_Q[g^i] = 0, \ i = 1, \dots, e, \ \text{and } \exists P' \in \mathcal{P} \text{ s.t. } Q \ll P'\}.$$

In the next section we will consider an American option with pay-off stream Φ. We will assume that $\Phi : \{0, \dots, T\} \times \Omega \to \mathbb{R}$ is adapted[2]. Let $\mathcal{T}$ be the set of stopping times with respect to the *raw* filtration $(\mathcal{B}(\Omega_t))_{t=0}^T$, and $\mathcal{T}_t \subset \mathcal{T}$ the set of stopping times that are no less than t. For $t = 0, \dots, T$ and $\omega \in \Omega_t$, define

$$\mathcal{Q}_t(\omega) := \{Q \in \mathfrak{P}(\Omega_1) : \ Q \ll P, \text{ for some } P \in \mathcal{P}_t(\omega), \text{ and } E_Q[\Delta S_{t+1}(\omega, \cdot)] = 0\}.$$

By [19, Lemma 4.8], there exists a universally measurable selector Q_t such that $Q_t(\cdot) \in \mathcal{Q}_t(\cdot)$ on $\{\mathcal{Q}_t \neq \emptyset\}$. Using these selectors we define for $t \in \{0, \dots, T-1\}$ and $\omega \in \Omega_t$,

$$\mathcal{M}_t(\omega) := \{Q_t \otimes \ldots \otimes Q_{T-1} : \ Q_i(\omega, \cdot) \in \mathcal{Q}_i(\omega, \cdot) \text{ on } \{\mathcal{Q}_i(\omega, \cdot) \neq \emptyset\}, \ i = t, \dots, T-1\},$$

which is similar to (8.1.1) but starting from time t instead of time 0. In particular $\mathcal{M}_0 = \mathcal{M}$, where

$$\mathcal{M} := \{Q \text{ martingale measure} : \exists P \in \mathcal{P}, \text{ s.t. } Q \ll P\}. \tag{8.1.2}$$

[1] That is, Q satisfies $E_Q[|S_{t+1}| \ |\mathcal{F}_t] < \infty$ and $E_Q[S_{t+1}|\mathcal{F}_t] = S_t$, Q-a.s. for $t = 0, \dots, T-1$.

[2] Unless otherwise specified the measurability and related concepts (adaptedness, etc) are with respect to the filtration $(\mathcal{F}_t)_{t=0}^T$.

We will assume in the rest of the chapter that the graph of $\mathcal{M}_t$ is analytic, $t = 0, \dots, T-1$. Below we provide a general sufficient condition for the analyticity of graph$(\mathcal{M}_t)$ and leave its proof to Appendix VIII.A.

Proposition 8.1.1. *For $t = 0, \dots, T-1$ and $\omega \in \Omega_t$, define*

$$\mathbb{P}_t(\omega) := \{P_t \otimes \ldots \otimes P_{T-1} : \ P_i(\omega, \cdot) \in \mathcal{P}_i(\omega, \cdot), \ i = t, \dots, T-1\},$$

where each P_i is a universally measurable selector of $\mathcal{P}_i$. If graph($\mathbb{P}_t$) is analytic, then graph($\mathcal{M}_t$) is also analytic.

For any measurable function f and probability measure P, we define the P-expectation of f as $E_P[f] = E_P[f^+] - E_P[f^-]$ with convention $\infty - \infty = -\infty$. We use $|\cdot|$ to denote the sup norm in various cases. For $\omega \in \Omega$ and $t \in \{0, \dots, T\}$, we will use the notation $\omega^t \in \Omega_t$ to denote the path up to time t. For a given function f defined on Ω, let us denote

$$\underline{\mathcal{E}}_\tau(f)(\omega) := \inf_{Q \in \mathcal{M}_{\tau(\omega)}(\omega^{\tau(\omega)})} E_Q[f(\omega^{\tau(\omega)}, \cdot)], \quad \omega \in \Omega,$$

and

$$\overline{\mathcal{E}}_\tau(f)(\omega) := \sup_{Q \in \mathcal{M}_{\tau(\omega)}(\omega^{\tau(\omega)})} E_Q[f(\omega^{\tau(\omega)}, \cdot)], \quad \omega \in \Omega.$$

We use the abbreviations u.s.a. for upper-semianalytic, l.s.c. for lower-semicontinuous, and u.s.c. for upper-semicontinuous.

8.2 Sub-hedging Duality

We define the sub-hedging price of the American option as

(8.2.1)

$$\underline{\pi}(\Phi) := \sup \left\{ x \in \mathbb{R} : \ \exists (H, \tau, h) \in \mathcal{H} \times \mathcal{T} \times \mathbb{R}^e, \ \text{ s.t. } \Phi_\tau + (H \cdot S)_T + hg \geq x, \ \mathcal{P} - q.s. \right\}.$$

Remark 8.2.1. In the above definition, we require the trading in the stock S to be up to time T instead of τ. This is because it is possible that the maturities of some options in g are later than τ. When there is no hedging options involved, for sub-hedging (and in fact also super-hedging) trading S up to time T is equivalent to up to time τ (e.g. see the beginning of the proof of Theorem 6.6.3).

We have the following duality theorem for sub-hedging prices.

Theorem 8.2.2. *Assume that Φ_t is l.s.a. for $t = 1, \dots, T$. Then*

$$\underline{\pi}(\Phi) = \sup_{\tau \in \mathcal{T}} \inf_{Q \in \mathcal{Q}} E_Q[\Phi_\tau]. \tag{8.2.2}$$

Moreover, if $\sup_{Q\in\mathcal{M}} E_Q[|g|] < \infty$, $\sup_{Q\in\mathcal{M}} E_Q[\max_{0\le t\le T} |\Phi_t|] < \infty$, and for any $h \in \mathbb{R}^e$ and $t \in \{0, \dots, T-1\}$, the maps $\Phi_t + \underline{\mathcal{E}}_t(hg)$ and $\phi:\ \Omega \mapsto \mathbb{R}^e$ defined by

$$\phi = \underline{\mathcal{E}}_t \left(\inf_{\tau \in \mathcal{T}_{t+1}} \underline{\mathcal{E}}_{t+1} \left(\Phi_\tau + \underline{\mathcal{E}}_\tau(hg)\right)\right) \quad \left(\text{or } \phi = \underline{\mathcal{E}}_t \left(\sup_{Q\in\mathcal{M}_{t+1}} \inf_{\tau\in\mathcal{T}_{t+1}} E_Q\left(\Phi_\tau + \underline{\mathcal{E}}_\tau(hg)\right)\right)\right)$$

are Borel measurable, then there exists $(H^*, \tau^*, h^*) \in \mathcal{H} \times \mathcal{T} \times \mathbb{R}^e$, such that

$$\Phi_{\tau^*} + (H^* \cdot S)_T + h^* g \ge \pi(\Phi), \quad \mathcal{P} - q.s. \tag{8.2.3}$$

Proof. For any $\tau \in \mathcal{T}$, define

$$\underline{\pi}(\Phi_\tau) := \sup \left\{ x \in \mathbb{R} :\ \exists (H, h) \in \mathcal{H} \times \mathbb{R}^e, \ \text{ s.t. } \Phi_\tau + (H \cdot S)_T + hg \ge x,\ \mathcal{P} - q.s. \right\}.$$

Since Φ_t is u.s.a. and τ is a stopping time with respect to the raw filtration, it follows that Φ_τ is u.s.a. Then applying [19, Theorem 5.1 (b)], we get

$$\underline{\pi}(\Phi_\tau) = \inf_{Q\in\mathcal{Q}} \mathbb{E}_Q[\Phi_\tau] \quad \Longrightarrow \quad \sup_{\tau\in\mathcal{T}} \underline{\pi}(\Phi_\tau) = \sup_{\tau\in\mathcal{T}} \inf_{Q\in\mathcal{Q}} \mathbb{E}_Q[\Phi_\tau].$$

Since $\underline{\pi}(\Phi) \ge \underline{\pi}(\Phi_\tau)$, $\forall \tau \in \mathcal{T}$, it follows that $\underline{\pi}(\Phi) \ge \sup_{\tau\in\mathcal{T}} \underline{\pi}(\Phi_\tau)$. Therefore, it remains to show that $\underline{\pi}(\Phi) \le \sup_{\tau\in\mathcal{T}} \underline{\pi}(\Phi_\tau)$. For any $\varepsilon > 0$, there exists $x \in (\underline{\pi}(\Phi) \wedge (1/\varepsilon) - \varepsilon, \underline{\pi}(\Phi) \wedge (1/\varepsilon)]$ and $(H^\varepsilon, \tau^\varepsilon, h^\varepsilon) \in \mathcal{H} \times \mathcal{T} \times \mathbb{R}^e$ satisfying

$$\Phi_{\tau^\varepsilon} + (H^\varepsilon \cdot S)_T + h^\varepsilon g \ge x,\ \mathcal{P} - q.s.$$

As a result,

$$\underline{\pi}(\Phi) \wedge \frac{1}{\varepsilon} - \varepsilon < x \leq \underline{\pi}(\Phi_{\tau^\varepsilon}) \leq \sup_{\tau \in \mathcal{T}} \underline{\pi}(\Phi_\tau),$$

from which (8.2.2) follows since ε is arbitrary.

Let us turn to the proof of the existence of the optimal sub-hedging strategies. Similar to the proof above, we can show that

$$\begin{aligned}
\underline{\pi}(\Phi) &= \sup_{h \in \mathbb{R}^e} \sup_{\tau \in \mathcal{T}} \sup\{x : \ \exists H \in \mathcal{H}, \text{ s.t. } \Phi_\tau + (H \cdot S)_T + hg \geq x, \ \mathcal{P} - q.s.\} \\
&= \sup_{h \in \mathbb{R}^e} \sup_{\tau \in \mathcal{T}} \inf_{Q \in \mathcal{M}} E_Q[\Phi_\tau + hg].
\end{aligned}$$

We shall first show in two steps that the optimal h^* exists for the above equations.

Step 1: We claim that 0 is in the relative interior of the convex set $\{E_Q[g], \ Q \in \mathcal{M}\}$. If not, then there exists $h \in \mathbb{R}^e$, such that $E_Q[hg] \leq 0$, for any $Q \in \mathcal{M}$, and moreover there exists $\bar{Q} \in \mathcal{M}$, such that $E_{\bar{Q}}[hg] < 0$. By [19, Theorem 4.9], the super-hedging price of hg (using only the stock) is $\sup_{Q \in \mathcal{M}} E_Q[hg] \leq 0$, and there exists $H \in \mathcal{H}$, such that

$$(H \cdot S)_T \geq hg, \quad \mathcal{P} - q.s.$$

Then $E_{\bar{Q}}[(H \cdot S)_T - hg] > 0$, and thus, for any $P \in \mathcal{P}$ dominating $\bar{Q}$, we have that

$$P((H \cdot S)_T - hg > 0) > 0,$$

which contradicts NA($\mathcal{P}$).

Step 2: Since 0 is a relative interior point of $\{E_Q[g], \ Q \in \mathcal{M}\}$, and

$$\sup_{Q \in \mathcal{M}} E_Q[\max_{0 \leq t \leq T} |\Phi_t|] < \infty,$$

we know that

$$\underline{\pi}(\Phi) = \sup_{h \in \mathbb{R}^e} \sup_{\tau \in \mathcal{T}} \inf_{Q \in \mathcal{M}} E_Q[\Phi_\tau + hg] = \sup_{h \in \mathbb{K}} \sup_{\tau \in \mathcal{T}} \inf_{Q \in \mathcal{M}} E_Q[\Phi_\tau + hg],$$

where $\mathbb{K}$ is a compact subset of $\mathbb{R}^e$. Define the map $\varphi:\ \mathbb{R}^e \mapsto \mathbb{R}$ by

$$\varphi(h) = \sup_{\tau\in\mathcal{T}} \inf_{Q\in\mathcal{M}} E_Q[\Phi_\tau + hg].$$

The function φ is continuous since $|\varphi(h) - \varphi(h')| \leq e|h-h'| \sup_{Q\in\mathcal{M}} E_Q|g|$. Hence, there exists $h^* \in \mathbb{K} \subset \mathbb{R}^e$ such that

$$\underline{\pi}(\Phi) = \sup_{\tau\in\mathcal{T}} \inf_{Q\in\mathcal{M}} E_Q[\Phi_\tau + h^* g] = \sup_{\tau\in\mathcal{T}} \inf_{Q\in\mathcal{M}} E_Q[\Phi_\tau + \underline{\mathcal{E}}_\tau(h^* g)], \tag{8.2.4}$$

where the second equality above follows from [69, Theorem 2.3]. Using the measurability assumptions in the statement of this theorem, we can apply Theorem VIII.B.1, and obtain a $\tau^* \in \mathcal{T}$ that is optimal for (8.2.4), i.e.,

$$\begin{aligned} \underline{\pi}(\Phi) &= \inf_{Q\in\mathcal{M}} E_Q[\Phi_{\tau^*} + \underline{\mathcal{E}}_{\tau^*}(h^* g)] = \sup_{\tau\in\mathcal{T}} \inf_{Q\in\mathcal{M}} E_Q[\Phi_\tau + h^* g] \\ &= \sup\{x:\ \exists H \in \mathcal{H},\ \text{s.t. } \Phi_{\tau^*} + (H\cdot S)_T + h^* g \geq x,\ \mathcal{P} - q.s.\} \end{aligned}$$

Then by [19, Theorem 4.9], there exists a strategy $H^* \in \mathcal{H}$, such that (8.2.3) holds.

□

8.2.1 Exchangeability of the supremum and infimum in (8.2.2)

When there are no options available for static hedging (then $\mathcal{Q} = \mathcal{M}$), $\mathcal{Q}$ is closed under pasting. Using this property we show in Theorem VIII.B.1 and Proposition VIII.B.3 that the order of "inf" and "sup" in (8.2.2) can be exchanged under some reasonable assumptions. These conclusions cover the specific results of [35] which works with a compact path space. (Although, our no arbitrage assumption seems to be different than the one in [35], we verify in Proposition VIII.D.2 that they are the same when there are no options, i.e., $e = 0$.) The same holds true for our super-hedging result in the next section.

In general, $\mathcal{Q}$ may not be stable under pasting due to the distribution constraints imposed by having to price the given options correctly. Then whether the "inf" and "sup" in (8.2.2) can be exchanged is not clear, and in fact may not be possible as the example below demonstrates.

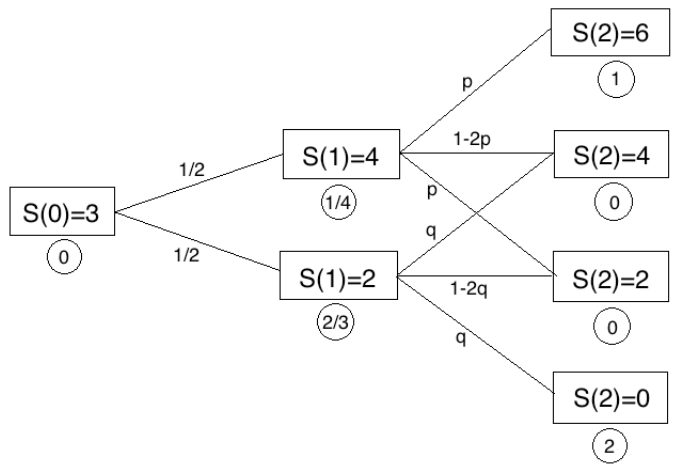

Figure 8.1: A two-period example

Example 8.2.3. We consider a two-period model as described by the figure above. The stock price process is restricted to the finite path space indicated by the figure, where $S(t)$ is the stock price at time t, $t = 0, 1, 2$. Let $\mathcal{P}$ be all the probability measures on this path space. Then each martingale measure $Q \in \mathcal{M}$ can be uniquely characterized by a pair (p, q), $0 \leq p, q \leq 1/2$, as indicated in the figure. Assume there is one European option $g = [S(2) - 3]^+ - 5/6$ that can be traded at price 0. Let Φ be the payoff of a path-independent American option that needs to be hedged. In the figure, the number in each circle right below the rectangle (node) represents the value of Φ when the stock price is at that node.

Each $Q \in \mathcal{Q} \subset \mathcal{M}$ is characterized by (p, q) with the additional condition: $p + q = 2/3$. There are in total 5 stopping strategies: stop at node $S(0) = 3$, or continue to node $S(1) = k$, $k = 2, 4$, then choose either to stop or to continue. It is easy to check that

$$\begin{aligned}
&\sup_{\tau\in\mathcal{T}}\inf_{Q\in\mathcal{Q}} E_Q[\Phi_\tau] \\
&= 0 \vee \frac{11}{24} \vee \left(\frac{1}{8} + \inf_{\substack{0\le p,q\le 1/2\\ p+q=2/3}} q\right) \vee \left(\inf_{\substack{0\le p,q\le 1/2\\ p+q=2/3}} \frac{p}{2} + \frac{1}{3}\right) \vee \left(\inf_{\substack{0\le p,q\le 1/2\\ p+q=2/3}} \left(\frac{p}{2} + q\right)\right) \\
&= \frac{11}{24},
\end{aligned}$$

and

$$\inf_{Q\in\mathcal{Q}}\sup_{\tau\in\mathcal{T}} E_Q[\Phi_\tau] = \inf_{\substack{0\le p,q\le 1/2\\ p+q=2/3}} \left[\frac{1}{2}\left(p \vee \frac{1}{4} + 2q \vee \frac{2}{3}\right) \vee 0\right] = \frac{1}{2} > \sup_{\tau\in\mathcal{T}}\inf_{Q\in\mathcal{Q}} E_Q[\Phi_\tau].$$

8.3 The Super-hedging Duality

We define the super-hedging price as

$$\begin{aligned}
\text{(8.3.1)} \qquad \overline{\pi}(\Phi) := \inf\Big\{&x\in\mathbb{R}:\ \exists (H,h)\in\mathcal{H}'\times\mathbb{R}^e, \\
&\text{s.t. } x + (H\cdot S)_T + hg \ge \Phi_\tau,\ \mathcal{P}-q.s.,\ \forall \tau\in\mathcal{T}\Big\},
\end{aligned}$$

where $\mathcal{H}'$ is the set of processes that have the "non-anticipativity" property, i.e.,

$$\text{(8.3.2)} \qquad \mathcal{H}' := \{H:\mathcal{T}\mapsto\mathcal{H},\ \text{s.t. } H_t(\tau^1) = H_t(\tau^2),\ \forall t < \tau^1\wedge\tau^2\}.$$

In other words, the seller of the American option is allowed to adjust the trading strategy according to the stopping time τ after it is realized.

The following is our duality theorem for the super-hedging prices.

Theorem 8.3.1. *Assume that for $(\omega, P)\in\Omega_T\times\mathfrak{P}(\Omega_{T-t})$,*

$$\text{(8.3.3)} \qquad \textit{the map } (\omega,P)\mapsto \sup_{\tau\in\mathcal{T}_t} E_P[\Phi_\tau(\omega^t,\cdot)] \textit{ is u.s.a.},\ t = 1,\dots,T.$$

Then

$$\text{(8.3.4)} \qquad \overline{\pi}(\Phi) = \inf_{h\in\mathbb{R}^e}\sup_{\tau\in\mathcal{T}}\sup_{Q\in\mathcal{M}} E_Q[\Phi_\tau - hg].$$

Moreover, if $\sup_{Q\in\mathcal{M}} E_Q[|g|] < \infty$ *and* $\sup_{Q\in\mathcal{M}} E_Q[\max_{0\le t\le T}|\Phi_t|] < \infty$*, then there exists* $(H^*, h^*) \in \mathcal{H}' \times \mathbb{R}^e$*, such that*

$$\overline{\pi}(\Phi) + (H^* \cdot S)_T + h^* g \ge \Phi_\tau,\ \mathcal{P} - q.s.,\ \forall \tau \in \mathcal{T}. \tag{8.3.5}$$

Proof. An argument similar to the one used in the proof of Theorem 8.2.2 implies that $\overline{\pi}(\Phi) = \inf_{h\in\mathbb{R}^e} \overline{\pi}(\Phi, h)$, where

$$\overline{\pi}(\Phi, h) = \inf\left\{x \in \mathbb{R} :\ \exists H \in \mathcal{H}',\ \ \text{s.t. } x + (H \cdot S)_T + hg \ge \Phi_\tau,\ \mathcal{P} - q.s.,\ \forall \tau \in \mathcal{T}\right\}.$$

It is easy to see that $\overline{\pi}(\Phi, h) \ge \sup_{\tau\in\mathcal{T}} \sup_{Q\in\mathcal{M}} E_Q[\Phi_\tau - hg]$. In what follows we will demonstrate the reverse inequality. Define

$$V_t = \sup_{\tau\in\mathcal{T}_t} \overline{\mathcal{E}}_t(\Phi_\tau - hg). \tag{8.3.6}$$

Using assumption (8.3.3), we apply Proposition VIII.C.1 to show that V_t is u.s.a., $\mathcal{F}_t$-measurable and a super-martingale under each $Q \in \mathcal{M}$. As a result, we can apply the optional decomposition theorem for the nonlinear expectations [19, Theorem 6.1], which implies that there exists $H' \in \mathcal{H}$, such that for any $\tau \in \mathcal{T}$,

$$V_0 + (H' \cdot S)_\tau \ge V_\tau = \sup_{\rho\in\mathcal{T}_\tau} \overline{\mathcal{E}}_\tau(\Phi_\rho - hg) \ge \Phi_\tau + \overline{\mathcal{E}}_\tau(-hg), \quad \mathcal{P} - q.s. \tag{8.3.7}$$

Let us also define

$$W_t := \overline{\mathcal{E}}_t(-hg).$$

Thanks to Proposition VIII.C.1, we can apply [19, Theorem 6.1] again and get that there exists $H'' \in \mathcal{H}$, such that for any $\tau \in \mathcal{T}$,

$$W_\tau + (H'' \cdot S)_{\tau,T} = \overline{\mathcal{E}}_\tau(-hg) + (H'' \cdot S)_{\tau,T} \ge W_T = -hg, \quad \mathcal{P} - q.s., \tag{8.3.8}$$

where $(H'' \cdot S)_{\tau,T} = \sum_{i=\tau}^{T-1} H''_i[S_{i+1} - S_i]$. Combining (8.3.7) and (8.3.8), we get that

$$V_0 + (H \cdot S)_T + hg \ge \Phi_\tau,\ \forall \tau \in \mathcal{T},\ \mathcal{P} - q.s.,$$

where $H_t = H'_t 1_{\{t<\tau\}} + H''_t 1_{\{t\geq\tau\}}$. Note that H' in (8.3.7) is independent of τ, which implies that H is indeed in $\mathcal{H}'$. Hence, $V_0 = \sup_{\tau\in\mathcal{T}} \sup_{Q\in\mathcal{M}} E_Q[\Phi_\tau - hg] \geq \overline{\pi}(\Phi, h)$.

As in the proof of Theorem 8.2.2, there exists $h^* \in \mathbb{R}^e$ that is optimal for (8.3.4):

$$\overline{\pi}(\Phi) = \sup_{\tau\in\mathcal{T}} \sup_{Q\in\mathcal{M}} E_Q[\Phi_\tau - h^* g] = \overline{\pi}(\Phi, h^*).$$

Also observe from the proof above that there exists $H^* \in \mathcal{H}'$, such that

$$\overline{\pi}(\Phi, h^*) + (H^* \cdot S)_T + h^* g \geq \Phi_\tau, \quad \mathcal{P} - q.s., \ \forall \tau \in \mathcal{T},$$

which implies (8.3.5). □

Proposition 8.3.2 (A sufficient condition on the assumption (8.3.3) of Theorem 8.3.1)**.** *Assume that Φ_t is l.s.c. and bounded from below for $t = 1, \dots, T$. Then for $(\omega, P) \in \Omega_T \times \mathfrak{P}(\Omega_{T-t})$, the map $(\omega, P) \mapsto \sup_{\tau\in\mathcal{T}_t} E_P[\Phi_\tau(\omega^t, \cdot)]$ is l.s.c., and thus u.s.a, $t = 1, \dots, T$.*

Proof. If Φ is uniformly continuous in ω with modulus of continuity ρ, then for $({}^n\omega, P^n) \to (\omega, P)$, we have that

$$\begin{aligned}
&\sup_{\tau\in\mathcal{T}_t} E_{P^n}[\Phi_\tau(({}^n\omega)^t, \cdot)] - \sup_{\tau\in\mathcal{T}_t} E_P[\Phi_\tau(\omega^t, \cdot)] \\
&= \sup_{\tau\in\mathcal{T}_t} E_{P^n}[\Phi_\tau(({}^n\omega)^t, \cdot)] - \sup_{\tau\in\mathcal{T}_t} E_{P^n}[\Phi_\tau(\omega^t, \cdot)] \\
&\quad + \sup_{\tau\in\mathcal{T}_t} E_{P^n}[\Phi_\tau(\omega^t, \cdot)] - \sup_{\tau\in\mathcal{T}_t} E_P[\Phi_\tau(\omega^t, \cdot)] \\
&\geq -\rho(||{}^n\omega - \omega||) + \sup_{\tau\in\mathcal{T}_t} E_{P^n}[\Phi_\tau(\omega^t, \cdot)] - \sup_{\tau\in\mathcal{T}_t} E_P[\Phi_\tau(\omega^t, \cdot)].
\end{aligned} \tag{8.3.9}$$

Noting that the map $P \mapsto \sup_{\tau\in\mathcal{T}_t} E_P[\Phi_\tau(\omega^t, \cdot)]$ is l.s.c. (see e.g., [42, Theorem 1.1]), we know that the map $(P, \omega) \mapsto \sup_{\tau\in\mathcal{T}_t} E_P[\Phi_\tau(\omega^t, \cdot)]$ is l.s.c. by taking the limit in (8.3.9). In general, if Φ_t be l.s.c. and bounded from below, then there exists uniformly continuous functions $(\Phi_t^n)_n$, such that $\Phi_t^n \nearrow \Phi_t$ pointwise (see e.g., [18, Lemma 7.14]),

$t = 1, \dots, T$. Therefore,

$$\sup_{\tau\in\mathcal{T}_t} E_P[\Phi_\tau(\omega^t,\cdot)] = \sup_{\tau\in\mathcal{T}_t}\sup_n E_P[\Phi_\tau^n(\omega^t,\cdot)] = \sup_n\sup_{\tau\in\mathcal{T}_t} E_P[\Phi_\tau^n(\omega^t,\cdot)],$$

which implies that the map $(\omega, P) \mapsto \sup_{\tau\in\mathcal{T}_t} E_P[\Phi_\tau(\omega^t,\cdot)]$ is l.s.c. □

8.3.1 Comparison of several definitions of super-hedging

In the duality result (8.3.4), one would expect that $\overline{\pi}(\Phi) = \sup_{\tau\in\mathcal{T}}\sup_{Q\in\mathcal{Q}} E_Q[\Phi_\tau]$. More precisely, if the orders in (8.3.4) could be exchanged for then we would have

$$\overline{\pi}(\Phi) = \inf_{h\in\mathbb{R}^e}\sup_{\tau\in\mathcal{T}}\sup_{Q\in\mathcal{M}} E_Q[\Phi_\tau - hg] = \sup_{\tau\in\mathcal{T}}\sup_{Q\in\mathcal{M}}\inf_{h\in\mathbb{R}^e} E_Q[\Phi_\tau - hg] = \sup_{\tau\in\mathcal{T}}\sup_{Q\in\mathcal{Q}} E_Q[\Phi_\tau].$$

But the latter is in fact equal to

(8.3.10)

$$\hat{\pi}(\Phi) := \inf\{x\in\mathbb{R}:\ \forall\tau\in\mathcal{T}, \exists(H,h)\in\mathcal{H}\times\mathbb{R}^e,\ \text{s.t. } x+(H\cdot S)_T+hg \ge \Phi_\tau,\ \mathcal{P}-q.s.\}.$$

That is,

$$\hat{\pi}(\Phi) = \sup_{\tau\in\mathcal{T}}\sup_{Q\in\mathcal{Q}} E_Q[\Phi_\tau]. \tag{8.3.11}$$

Since for the definition of $\hat{\pi}$ in (8.3.10) the seller knows the buyer's stopping strategy τ in advance (which is unreasonable for super-hedging), we may expect that in general it is possible $\overline{\pi}(\Phi) > \hat{\pi}(\Phi)$. We shall provide Example 8.3.3 showing $\overline{\pi}(\Phi) > \hat{\pi}(\Phi)$ at the end of this section.

An alternative way to define the super-hedging price is:

$$\tilde{\pi}(\Phi) := \inf\Big\{x\in\mathbb{R}:\ \exists(H,h)\in\mathcal{H}\times\mathbb{R}^e, \quad \text{s.t. } x+(H\cdot S)_T+hg \ge \Phi_\tau,\ \mathcal{P}-q.s.,\ \forall\tau\in\mathcal{T}\Big\}. \tag{8.3.12}$$

However, this definition is not as useful since any reasonable investor would adjust her strategy after observing how the buyer of the option behaves. (In fact, $\mathcal{H}$ can be treated as a subset of $\mathcal{H}'$, and each element in $\mathcal{H}$ is indifferent to stopping strategies used by the buyer, and the non-anticipativity is automatically satisfied.) Due to the fact that for $\tilde{\pi}$ the seller fails to use the information of the realization of τ, it could very well be the case that $\overline{\pi}(\Phi) < \tilde{\pi}(\Phi)$. We shall see in Example 8.3.3 that it is indeed the case.

If $\mathcal{P}$ is the set of all probability measures on a subset Ω' of Ω, then under the definition of (8.3.12), super-hedging the American option is equivalent to super-hedging the lookback option $\max_{t\leq T}\Phi_t$. To wit, suppose for $x \in \mathbb{R}$ and $(H, h) \in \mathcal{H} \times \mathbb{R}^e$, we have that

$$x + (H \cdot S)_T + hg \geq \Phi_\tau, \ \forall s \in \Omega', \ \forall \tau \in \mathcal{T}, \tag{8.3.13}$$

and

$$x + (H \cdot S)_T + hg < \max_{t\leq T}\Phi_t, \quad \text{along some path } s^* = (s_0^* = 1, s_1^*, \ldots, s_T^*) \in \Omega'.$$

Let $t^* = \arg\max_{t\leq T}\Phi_t(s^*)$ and define $\tau^* \in \mathcal{T}$ with the property that $\tau(s^*) = t^*$, i.e., the holder of the American option will stop at time t^* once she observes $(s_0^*, \ldots, s_{t^*}^*)$ happens. Then (8.3.13) does not hold if we take $\tau = \tau^*$ and $s = s^*$. So the super-hedging price under the definition of (8.3.12) is:

$$\tilde{\pi}(\Phi) = \sup_{Q\in\mathcal{Q}} E_Q\left[\max_{t\leq T}\Phi_t\right].$$

Example 8.3.3 below shows that it is possible that $\hat{\pi}(\Phi) < \overline{\pi}(\Phi) < \tilde{\pi}(\Phi)$, which indicates that the super-hedging definitions in (8.3.10) and (8.3.12) are unreasonable.

Example 8.3.3. We will use the set-up in Example 8.2.3. An easy calculation shows that

$$\begin{aligned}
\overline{\pi}(\Phi) &= \inf_{h\in\mathbb{R}} \sup_{Q\in\mathcal{M}} \sup_{\tau\in\mathcal{T}} E_Q[\Phi_\tau - hg] \\
&= \inf_{h\in\mathbb{R}} \sup_{0\le p,q\le 1/2} \left[\frac{p}{2}\vee\frac{1}{8} + q\vee\frac{1}{3} - h\left(\frac{p}{2}+\frac{q}{2}-\frac{1}{3}\right)\right] \\
&= \inf_{h\in\mathbb{R}} \left[\left(\frac{11}{24}+\frac{h}{3}\right)\vee\left(\frac{5}{8}+\frac{h}{12}\right)\vee\left(\frac{7}{12}+\frac{h}{12}\right)\vee\left(\frac{3}{4}-\frac{h}{6}\right)\right] \\
&= \frac{2}{3},
\end{aligned}$$

where the infimum is attained when $h = 1/2$. On the other hand,

$$\tilde{\pi}(\Phi) = \sup_{Q\in\mathcal{Q}} E_Q\left[\max_{t\le T}\Phi_t\right] = \sup_{\substack{0\le p,q\le 1/2\\ p+q=2/3}} \left(\frac{3}{8}p + \frac{2}{3}q + \frac{11}{24}\right) = \frac{41}{48} > \overline{\pi}(\Phi),$$

and

$$\hat{\pi}(\Phi) = \sup_{\tau\in\mathcal{T}} \sup_{Q\in\mathcal{Q}} E_Q[\Phi_\tau] = \sup_{\substack{0\le p,q\le 1/2\\ p+q=2/3}} \left(\frac{p}{2}\vee\frac{1}{8} + q\vee\frac{1}{3}\right) = \frac{5}{8} < \overline{\pi}(\Phi).$$

8.4 Approximating the hedging-prices by discretizing the path space

In this section, we take $\mathcal{P}$ to be the set of all the probability measures on Ω and consider the hedging problems path-wise. We will make the same no-arbitrage assumption and also assume that no hedging option is redundant (see Assumption 8.4.1(ii)). We will discretize the path space to obtain a discretized market, and show that the hedging prices in the discretized market converges to the original ones. We also get the rate of convergence. Theorems 8.4.7 and 8.4.8 are the main results of this section.

We will now collect some notation that will be used in the rest of this section. The meaning of some of the parameters will become clear when they first appear in context.

8.4.1 Notation

- $\Omega = \{1\} \times [a_1, b_1] \times \ldots \times [a_T, b_T]$, where $0 \leq a_T < \ldots < a_1 < 1 < b_1 < \ldots < b_T < \infty$. (This means that the wingspan of the discrete-time model is growing as for example it does in a binomial tree market.)
- $\Omega^n = \Omega \cap \{0, 1/2^n, 2/2^n, \ldots\}^{T+1}$.
- $\mathcal{P}$ all the probability measures on Ω.
- $\mathcal{P}^n$ all the probability measures on Ω^n.
- $\mathcal{Q} := \{\mathbb{Q}$ martingale measure on $\Omega : \ \mathbb{E}_{\mathbb{Q}} g_i = 0, \ i = 1, \ldots, e\}$.
- $\mathcal{Q}^n := \{\mathbb{Q}$ martingale measure on $\Omega^n : \ \mathbb{E}_{\mathbb{Q}} g_i = c_i^n, \ i = 1, \ldots, e\}$.
- $\mathcal{H}$ is the set of trading strategies $H = (H_i)_{i=0}^{T-1}$ consists of functions H_i defined on $\prod_{j=1}^{i} [a_i, b_i], \ i = 0, \ldots, T-1$.
- $\mathcal{H}^n$ is the set of trading strategies $H = (H_i)_{i=0}^{T-1}$ consists of functions H_i defined on $\prod_{j=1}^{i} [a_j^n, b_j^n] \cap \{0, 1/2^n, 2/2^n, \ldots\}^i, \ i = 0, \ldots, T-1$.
- $\mathcal{T}$ is the set of stopping times $\tau : \Omega \to \{0, 1, \ldots, T\}$, i.e., for $k = 0, 1, \ldots, T, \ s^j = (s_0^j, \ldots, s_T^j) \in \Omega, \ j = 1, 2,$

$$\text{if } \tau(s^1) = k, \text{ and } s_i^1 = s_i^2, \ i = 0, \ldots, k, \text{ then } \tau(s^2) = k.$$

- $\mathcal{T}^n$ is the set of stopping times $\tau : \Omega^n \to \{0, 1, \ldots, T\}$.
- $\mathcal{H}' := \{H : \mathcal{T} \mapsto \mathcal{H}, \ \text{s.t.} \ H_t(\tau^1) = H_t(\tau^2), \ \forall t < \tau^1 \wedge \tau^2\}$.
- $\mathcal{H}^{n\prime} := \{H : \mathcal{T}^n \mapsto \mathcal{H}^n, \ \text{s.t.} \ H_t(\tau^1) = H_t(\tau^2), \ \forall t < \tau^1 \wedge \tau^2\}$.
- $|\cdot|$ represents the sup norm in various cases.
- $\mathbb{D} = \cup_n \{0, 1/2^n, 2/2^n, \ldots\}$.

8.4.2 Original market

We restrict the price process, denoted by $S = (S_0, \dots, S_T)$, to take values in some compact set Ω. In other words, we take S to be the canonical process $S_i(s_0, \dots, s_T) = s_i$ for any $(s_0, \dots, s_T) \in \Omega$, and denote by $\{\mathcal{F}_i\}_{i=1,\dots,T}$ the natural filtration generated by S. The options $(g_i)_{i=1}^e$, which can be bought at price 0, and the American option Φ are continuous. We assume that NA($\mathcal{P}$) holds and that no hedging option is redundant, i.e., it cannot be replicated by the stock and other options available for static hedging. Besides, from the structure of Ω, we know that for $H \in \mathcal{H}$, if $(H \cdot S)_T \geq 0,\ \forall s \in \Omega$, then $H \equiv 0$. Thus, we will make the following standing assumption.

Assumption 8.4.1. *(i) g and Φ are continuous. (ii) For any $(H, h) \in \mathcal{H} \times \mathbb{R}^e$, if $h \neq 0$, then there exists $s \in \Omega$, such that along the path s,*

$$(H \cdot S)_T + hg < 0.$$

Example 8.4.1. Consider the market with $\Omega = \{1\} \times [2/3, 4/3] \times [1/3, 5/3]$, with a European option $(S_2 - 1)^+ - 1/5$ that can be traded at price 0. A simple calculation can show that Assumption 8.4.1 is satisfied.

We consider the sub-hedging price $\underline{\pi}(\Phi)$ and the super-hedging price $\overline{\pi}(\Phi)$ with respect to $(\Omega, \mathcal{P})$, i.e.,

$$\underline{\pi}(\Phi) := \sup\Big\{x \in \mathbb{R} :\ \exists (H, \tau, h) \in \mathcal{H} \times \mathcal{T} \times \mathbb{R}^e,$$
$$\text{s.t. } \Phi_\tau + (H \cdot S)_T + hg \geq x,\ \forall s \in \Omega\Big\},$$

and

$$\overline{\pi}(\Phi) := \inf\Big\{x \in \mathbb{R} :\ \exists (H, h) \in \mathcal{H}' \times \mathbb{R}^e,$$
$$\text{s.t. } x + (H \cdot S)_T + hg \geq \Phi_\tau,\ \forall s \in \Omega,\ \forall \tau \in \mathcal{T}\Big\}.$$

Recall that $\underline{\pi}(\Phi)$ and $\overline{\pi}(\Phi)$ satisfy the dualities in (8.2.2) and (8.3.4) respectively.

8.4.3 Discretized market

For simplicity, we assume that $a_i,\ b_i \in \mathbb{D},\ i = 1, \dots, T$, in the notation of Ω, and we always start from n large enough, such that Ω^n has the end points a_i, b_i at each time i. Let $\{c^n = (c_1^n, \dots, c_e^n)\}_n$ be a sequence such that $|c^n| \to 0$. Now for each n, consider the following discretized market: The stock price process takes values in the path space Ω^n, and the options $(g_i)_{i=1}^e$ can be traded at the beginning at price $(c_i^n)_{i=1}^e$.

We consider the sub-hedging price $\underline{\pi}^n(\Phi)$ and the super-hedging price $\overline{\pi}^n(\Phi)$ with respect to $(\Omega^n, \mathcal{P}^n)$, i.e.,

$$\underline{\pi}^n(\Phi) := \sup\big\{x \in \mathbb{R} :\ \exists (H, \tau, h) \in \mathcal{H}^n \times \mathcal{T}^n \times \mathbb{R}^e,$$

$$\text{s.t. } \Phi_\tau + (H \cdot S)_T + hg \geq x,\ \forall s \in \Omega^n\big\},$$

and

$$\overline{\pi}^n(\Phi) := \inf\big\{x \in \mathbb{R} :\ \exists (H, h) \in \mathcal{H}^{n\prime} \times \mathbb{R}^e,$$

$$\text{s.t. } x + (H \cdot S)_T + hg \geq \Phi_\tau,\ \forall s \in \Omega^n,\ \forall \tau \in \mathcal{T}^n\big\}.$$

Recall that $\underline{\pi}^n(\Phi)$ and $\overline{\pi}^n(\Phi)$ satisfy the dualities in (8.2.2) and (8.3.4) respectively.

Remark 8.4.2. Assuming $a_i, b_i \in \mathbb{D}$ and the points in Ω^n is equally spaced is without loss of generality. In fact, as long as $\Omega^n \cap \Omega$ are increasing and $\overline{\cup_n(\Omega^n \cap \Omega)} = \Omega$, we will have the same results with only a little adjustment in the proofs.

8.4.4 Consistency

The following theorem states that for n large enough, the discretized market is well defined, i.e., NA($\mathcal{P}^n$) holds.

Theorem 8.4.3. *For n large enough, NA($\mathcal{P}^n$) holds.*

Proof. If not, then there exists $(H^n, h^n) \in \mathcal{H}^n \times \mathbb{R}^e$, such that

$$(H^n \cdot S)_T + h^n(g - c^n) \geq 0, \quad \forall s \in \Omega^n, \tag{8.4.1}$$

and is strictly positive along some path in Ω^n. Obviously, $h^n \neq 0$, so without loss of generality we will assume that $|h^n| = 1$. On the other hand, since g is continuous on a compact set it is bounded. Then there exists a constant $C > 0$ independent of n, such that

$$(H^n \cdot S)_T > -C. \tag{8.4.2}$$

We will need the following result in order to carry out the proof of the theorem. We preferred to separate this result from the proof of the theorem since it will be used again in the proof of the convergence result.

Lemma 8.4.4. *If $(H^n \cdot S)_T > -C$, then there exists a constant $M = M(C) > 0$ independent of n, such that $|H^n| \leq M$.*

Proof. Let $\alpha := \min_{1 \leq i \leq T}\{a_{i-1} - a_i, b_i - b_{i-1}\} > 0$, with $a_0 := b_0 := 1$. We will prove this by an induction argument. Take the path $(s_0 = 1, s_1 = a_1, s_2 = a_1, \ldots, s_T = a_1)$, then (8.4.2) becomes

$$H_0^n(a_1 - 1) > -C,$$

which implies $H_0^n < C/\alpha$. Similarly, we can show that $H_0^n > -C/\alpha$ by taking the path $(s_0 = 1, s_1 = b_1, s_2 = b_1, \ldots, s_T = b_1)$. Hence, H_0^n is bounded uniformly in n. Now assume there exists $K = K(C) > 0$ independent of n, such that $|H_j^n| \leq K,\ j \leq i - 1 \leq T - 1$. Since Ω^n is uniformly bounded and by the induction hypothesis, we have that

$$\sum_{j=i}^{T-1} H_j^n(s_1, \ldots, s_j)(s_{j+1} - s_j) > -C',$$

where $C' > 0$ only depends on C. For any $(s_1, \ldots, s_i) \in \prod_{j=1}^{i}([a_j, b_j] \cap \{k/2^n,\ k \in \mathbb{N}\})$, by taking the paths $(1, s_1, \ldots, s_i, s_{i+1} = a_{i+1}, \ldots, s_T = a_{i+1})$ and $(1, s_1, \ldots, s_i, s_{i+1} = b_{i+1}, \ldots, s_T = b_{i+1})$, we can show that $|H_i^n(s_1, \ldots, s_i)| \leq C'/\alpha$. $\square$

Proof of Theorem 8.4.3 continued. We proved in Lemma 8.4.4 that $|H^n| \leq M$ for some $M > 0$ independent of n. By a standard selection (using a diagonalization argument, e.g., see [74, Page 307]), we can show that there exists a subsequence $(H^{n_k}, h^{n_k}) \xrightarrow{|\cdot|} (H, h)$, where $H = (H_i)_{i=0}^{T-1}$ consists of functions H_i defined on $\prod_{j=1}^{i}([a_j, b_j] \cap \mathbb{D}),\ i = 0, \ldots, T-1$, with $|H| \leq M$, and $h \in \mathbb{R}^e$ with $|h| = 1$. By taking the limit on both sides of (8.4.1) along (n_k), we have

$$(H \cdot S)_T + hg \geq 0, \quad \forall s \in \Omega \cap \mathbb{D}^{T+1}. \tag{8.4.3}$$

If we can extend the domain of function H from $\Omega \cap \mathbb{D}^{T+1}$ to Ω, such that the inequality (8.4.3) still holds on Ω, we would obtain a contradiction to Assumption 8.4.1 since $h \neq 0$.

Define

$$\tilde{\Omega}_i = \{1\} \times [a_1, b_1] \times \ldots \times [a_i, b_i] \times \big([a_{i+1}, b_{i+1}] \cap \mathbb{D}\big) \times \ldots \times \big([a_T, b_T] \cap \mathbb{D}\big)$$

for $i = 1, \ldots, T-1$. We will do the extension inductively as follows (the notation for H will not be changed during the extension):

(i) For each $s_1 \in [a_1, b_1] \setminus \mathbb{D}$, using the standard selection argument, we can choose $[a_1, b_1] \cap \mathbb{D} \ni s_1^n \to s_1$, such that for any $j \in \{1, \ldots, T-1\}$ and $(s_2, \ldots, s_j) \in \prod_{k=2}^{j}\big([a_k, b_k] \cap \mathbb{D}\big)$, the limit $\lim_{n\to\infty} H(s_1^n, s_2, \ldots, s_j)$ exists. Define

$$H_j(s_1, \ldots, s_j) := \lim_{n\to\infty} H_j(s_1^n, s_2, \ldots, s_j).$$

Then we extend the domain of H to $\tilde{\Omega}_1$. It's easy to check that (8.4.3) still holds on $\tilde{\Omega}_1$.

(ii) In general, assume that we have already extended the domain of H to $\tilde{\Omega}_i$, $i \leq T-2$, such that (8.4.3) holds on it. Then for each $(s_1, \dots, s_i) \in \prod_{j=1}^{i}[a_j, b_j]$ and $s_{i+1} \in [a_{i+1}, b_{i+1}] \setminus \mathbb{D}$, performing the same selection and extension as in (i) (we fix $(s_1, \dots, s_i)$ while doing the selection), we can see that (8.4.3) still holds on $\tilde{\Omega}_{i+1}$.

Therefore, we can extend H to $\tilde{\Omega}_{T-1}$, such that (8.4.3) holds. Clearly, (8.4.3) also holds on Ω. $\square$

8.4.5 Convergence

We shall prove the convergence result for sub-hedging (Theorem 8.4.7). The super-hedging case is similar, and thus we shall only provide the corresponding result (Theorem 8.4.8) without proof.

Lemma 8.4.5. *For $(H^n, \tau^n, h^n) \in \mathcal{H}^n \times \mathcal{T}^n \times \mathbb{R}^e$, if for $x \in \mathbb{R}$*

$$\Phi_{\tau^n} + (H^n \cdot S)_T + h^n(g - c^n) \geq x, \quad \forall s \in \Omega^n, \tag{8.4.4}$$

then $(H^n)_n$ and $(h^n)_n$ are bounded.

Proof. We first show that $(h^n)_n$ are bounded. If not, by extracting a subsequence, we can without loss of generality assume that $0 < \beta < |h^n| \to \infty$. We consider two cases:

(a) $|H^n|/|h^n|$ is not bounded. Then we can rewrite (8.4.4) as

$$\left(\frac{H^n}{|h^n|} \cdot S\right)_T \geq -\frac{h^n}{|h^n|}(g - c^n) + \frac{1}{|h^n|}\Phi_{\tau^n} + \frac{x}{|h^n|}, \ \forall s \in \Omega^n.$$

Since g and Φ are continuous on a compact set, they are bounded. Hence, there exists $C > 0$, such that

$$\left(\frac{H^n}{|h^n|} \cdot S\right)_T \geq -C,$$

which contradicts with Lemma 8.4.4.

(b) $|H^n|/|h^n|$ is bounded. Let us rewrite (8.4.4) as

$$\left(\frac{H^n}{|h^n|}\cdot S\right)_T + \frac{h^n}{|h^n|}(g-c^n) \geq \frac{x+\Phi_{\tau^n}}{|h^n|},\ \forall s\in\Omega^n.$$

Since $(x+\Phi_{\tau^n})/|h^n|\to 0$, we can follow the proof of Theorem 8.4.3 to get a contradiction with Assumption 8.4.1.

Next we show that $(H^n)_n$ is a bounded collection. Let us rewrite (8.4.4) as

$$(H^n\cdot S)_T \geq -\Phi_{\tau^n} - h^n(g-c^n)+x, \quad \forall s\in\Omega^n.$$

Since $(h^n)_n$ and $(g-c^n)_n$ are bounded, then right-hand-side is bounded. Therefore, the conclusion follows from Lemma 8.4.4. □

Proposition 8.4.6. *For n large enough, there exists some $N>0$ independent of n, such that*

$$\underline{\pi}^n(\Phi) = \sup\Big\{x\in\mathbb{R}:\ \exists (H,\tau,h)\in\mathcal{H}^n\times\mathcal{T}^n\times\mathbb{R}^e,\ |H|,|h|\leq N, \\ s.t.\ \Phi_\tau+(H\cdot S)_T+hg\geq x,\ \forall s\in\Omega^n\Big\}. \tag{8.4.5}$$

and

$$\underline{\pi}(\Phi) = \sup\Big\{x\in\mathbb{R}:\ \exists (H,\tau,h)\in\mathcal{H}\times\mathcal{T}\times\mathbb{R}^e,\ |H|,|h|\leq N, \\ s.t.\ \Phi_\tau+(H\cdot S)_T+hg\geq x,\ \forall s\in\Omega\Big\}. \tag{8.4.6}$$

Proof. Let $\underline{x} := \min_{(t,s)\in\{1,\dots,T\}\times\Omega}\Phi(t,s)$. It is easy to see that

$$\underline{\pi}^n(\Phi) = \sup\Big\{x\geq\underline{x}:\ \exists (H,\tau,h)\in\mathcal{H}^n\times\mathcal{T}^n\times\mathbb{R}^e, \\ \text{s.t. } \Phi_\tau+(H\cdot S)_T+hg\geq x,\ \forall s\in\Omega^n\Big\}. \tag{8.4.7}$$

For n large enough, the set

$$\{(H^n, h^n) \in \mathcal{H}^n \times \mathbb{R}^e : \exists \tau \in \mathcal{T}^n, \text{ s.t. } \Phi_\tau + (H \cdot S)_T + hg \geq \underline{x}, \ \forall s \in \Omega^n\}$$

is uniformly bounded in n, which is indicated by Lemma 8.4.5. Since this set of strategies is the largest among the ones we need to consider for sub-hedging, thanks to (8.4.7), there exists a constant $N > 0$, such that for n large enough,

$$\underline{\pi}^n(\Phi) = \sup \big\{ x \geq \underline{x} : \ \exists (H, \tau, h) \in \mathcal{H} \times \mathcal{T} \times \mathbb{R}^e, \ |H^n|, |h^n| \leq N$$

$$\text{s.t. } \Phi_\tau + (H \cdot S)_T + hg \geq x, \ \forall s \in \Omega^n \big\},$$

which implies (8.4.5).

Similarly, we have that the set

$$\{(H, h) \in \mathcal{H} \times \mathbb{R}^e : \exists \tau \in \mathcal{T}, \text{ s.t. } \Phi_\tau + (H \cdot S)_T + hg \geq \underline{x}, \ \forall s \in \Omega\}$$

is bounded. Otherwise, there exists $(H^m, \tau^m, h^m) \in \mathcal{H} \times \mathcal{T} \times \mathbb{R}^e$, such that

$$\Phi_{\tau^m} + (H^m \cdot S)_T + h^m g \geq \underline{x}, \ \forall s \in \Omega \cap \mathbb{D}^{T+1},$$

with $|H^m| + |h^m| \to \infty$. Then we can use a similar argument to the one in the proof of Theorem 8.4.3 to get a contradiction. Now (8.4.6) follows. □

Theorem 8.4.7. *Under Assumption 8.4.1, we have*

$$\lim_{n \to \infty} \underline{\pi}^n(\Phi) = \underline{\pi}(\Phi). \tag{8.4.8}$$

Furthermore, if Φ *and* g *are Lipschitz continuous, then*

$$|\underline{\pi}^n(\Phi) - \underline{\pi}(\Phi)| = O(1/2^n) \tag{8.4.9}$$

by taking $|c^n| = O(1/2^n)$.

Proof. For $x \in (\underline{\pi}(\Phi)-\varepsilon, \underline{\pi}(\Phi)]$, there exists $(H,\tau,h) \in \mathcal{H}\times\mathcal{T}\times\mathbb{R}^e$,with $|H|,|h| \le N$, such that

$$\Phi_\tau + (H\cdot S)_T + hg \ge x,\ \forall s \in \Omega.$$

Hence,

$$\Phi_\tau + (H\cdot S)_T + h(g-c^n) \ge x - eN|c^n|,\ \forall s \in \Omega^n.$$

Therefore,

$$\underline{\pi}(\Phi) - \varepsilon - eN|c^n| \le x - eN|c^n| \le \underline{\pi}^n(\Phi).$$

By letting $\varepsilon \to 0$, we have

$$\underline{\pi}^n(\Phi) \ge \underline{\pi}(\Phi) - eN|c^n|. \tag{8.4.10}$$

On the other hand, for $x^n \in (\underline{\pi}^n(\Phi)-\varepsilon, \underline{\pi}^n(\Phi)]$, there exists $(H^n,\tau^n,h^n) \in \mathcal{H}^n \times \mathcal{T}^n \times \mathbb{R}^e$,with $|H^n|,|h^n| \le N$, such that

$$\Phi_{\tau^n} + (H^n\cdot S)_T + h^n(g-c^n) \ge x^n,\ \forall s \in \Omega^n. \tag{8.4.11}$$

Consider the map $\phi^n : \Omega \to \Omega^n$ given by

$$\phi^n(1,s_1,\dots,s_T) = (1, \lfloor 2^n s_1\rfloor/2^n, \dots, \lfloor 2^n s_T\rfloor/2^n), \quad \forall (1,s_1,\dots,s_T) \in \Omega.$$

Also define $(H,\tau) \in \mathcal{H}\times\mathcal{T}$ as

$$H(s) = H^n(\phi^n(s)) \ \text{ and } \tau(s) = \tau^n(\phi^n(s)) \tag{8.4.12}$$

Since Φ and g are continuous on a compact set, they are uniformly continuous. Also $(H^n,q^n)_n$ are uniformly bounded, and $c^n \to 0$. Then from (8.4.11) we have that for n large enough, the trading strategy (H,τ) defined in (8.4.12) satisfies

$$\Phi_\tau + (H\cdot S)_T + h^n g \ge x^n - \varepsilon,\ \forall s \in \Omega, \tag{8.4.13}$$

by noting that $\phi^n(s) \to s$ uniformly and $\tau(s) = \tau(\phi^n(s))$. Thus, $\underline{\pi}(\Phi) > \underline{\pi}^n(\Phi) - 2\varepsilon$. Combining with (8.4.10), we have (8.4.14).

If Φ and g are Lipschitz continuous, then we have a stronger version of (8.4.13):

$$\Phi_\tau + (H \cdot S)_T + h^n g \geq x^n - eN|c^n| - C/2^n, \ \forall s \in \Omega,$$

where $C > 0$ is a constant only depends on N, e, T and the Lipschitz constants of Φ and g. Hence,

$$\underline{\pi}^n(\Phi) - \varepsilon - eN|c^n| - C/2^n \leq x^n - eN|c^n| - C/2^n \leq \underline{\pi}(\Phi).$$

Letting $\varepsilon \to 0$ and taking $|c^n| = O(1/2^n)$, and combining with (8.4.10), we obtain (8.4.15). □

Similar to the proof of the sub-hedging case, we can show the following convergence result for super-hedging.

Theorem 8.4.8. *Under Assumption 8.4.1, we have*

$$\lim_{n\to\infty} \overline{\pi}^n(\Phi) = \overline{\pi}(\Phi). \tag{8.4.14}$$

Furthermore, if Φ *and* g *are Lipschitz continuous, then*

$$|\overline{\pi}^n(\Phi) - \overline{\pi}(\Phi)| = O(1/2^n) \tag{8.4.15}$$

by taking $|c^n| = O(1/2^n)$.

8.4.6 A suitable construction for c^n and $\mathcal{Q}^n$

In Section 8.4.4 we obtained that as long as $c^n \to 0$, then for n large enough, $\text{NA}(\mathcal{P}^n)$ holds, which implies $\mathcal{Q}^n \neq \emptyset$ (see [1, Theorem 1.3] or [19, FTAP]). The theorem below gives a more specific way to construct c^n, such that $\mathcal{Q}^n \neq \emptyset$ for all n with $\Omega^n \subset \Omega$, when all the hedging options are vanilla. [This analysis would be useful for the consistency, when there are infinitely many options and the marginal distribution of the stock price (at the maturities of the hedging European options) under the martingale measures appearing in the duality are fixed.]

Proposition 8.4.9. *Let $\mu_0, \dots, \mu_T$ be the marginal of a martingale measure on $\mathbb{R}_+^{T+1}$. Then there exist a collection of probability measures $\{\mu_i^n : i = 0, \dots, T,\ n \in \mathbb{N}\}$ on $\mathbb{R}$ such that*

(1) $\mu_i^n \xrightarrow{w} \mu_i,\ i = 0, \dots, T,$

(2) $\mu_i^n(K^n) = 1,\ i = 0, \dots, T,$

(3) For each $n \in \mathbb{N}$, $\mathcal{M}^n \neq \emptyset$,

where $K^n = \{0, 1/2^n, 2/2^n, \dots\}$ and $\mathcal{M}^n$ is the set of martingale measures on $(K^n)^{T+1}$ with marginals $(\mu_i^n)_{i=0}^T$.

Proof. Fix $i \in \{0, \cdots, T\}$. For any $n \in \mathbb{N}$, define a measure μ_i^n on $\{0, 1/2^n, 2/2^n, \cdots\}$ by

$$\mu_i^n(\{0\}) := \int_0^{1/2^n} (1 - 2^n x) d\mu_i(x),$$
$$\mu_i^n(\{k/2^n\}) := \int_{(k-1)/2^n}^{k/2^n} (2^n x + 1 - k) d\mu(x) + \int_{k/2^n}^{(k+1)/2^n} (1 + k - 2^n x) d\mu(x), \quad \forall k \in \mathbb{N}.$$

By construction, we have $\sum_{k \in \mathbb{N} \cup \{0\}} \mu_i^n(\{k/2^n\}) = \int_{\mathbb{R}_+} d\mu_i(x) = 1$. It follows that μ_i^n is a probability measure on $\{0, 1/2^n, 2/2^n, \cdots\}$.

For any function $h : \mathbb{R} \mapsto \mathbb{R}$, consider the piecewise linear function h^n defined by setting $h^n(k/2^n) := h(k/2^n)$ for $k \in \mathbb{N} \cup \{0\}$. We define $h^n(x)$ for $x \in \mathbb{R}_+ \setminus \{0, 1/2^n, 2/2^n, \cdots\}$ using linear interpolation. That is, for any $x \in \mathbb{R}_+$,

$$\begin{aligned} h^n(x) &:= (1 + \lfloor 2^n x \rfloor - 2^n x) h\left(\frac{\lfloor 2^n x \rfloor}{2^n}\right) + (2^n x - \lfloor 2^n x \rfloor) h\left(\frac{1 + \lfloor 2^n x \rfloor}{2^n}\right) \\ &= h\left(\frac{k}{2^n}\right)(1 + k - 2^n x) + h\left(\frac{k+1}{2^n}\right)(2^n x - k), \quad \forall k \in \mathbb{N} \cup \{0\}. \end{aligned}$$

From the above identity and the definition of μ_i^n, we observe that

$$\int_{\mathbb{R}_+} h d\mu_i^n = \int_{\mathbb{R}_+} h^n d\mu_i. \tag{8.4.16}$$

Now, if we take h to be an arbitrary bounded continuous function, then $h^n \to h$ pointwise and the integrals in (8.4.16) are finite. By using (8.4.16) and the dominated convergence theorem, we have $\int_{\mathbb{R}_+} h d\mu_i^n \to \int_{\mathbb{R}_+} h d\mu_i$. This shows that $\mu_i^n \xrightarrow{w} \mu_i$. On the other hand, if we take h to be an arbitrary convex function, then h^n by definition is also convex. Thanks to [80, Theorem 8], the convexity of h^n imply that $\int_{\mathbb{R}_+} h^n d\mu_i$ is nondecreasing in i. We then obtain form (8.4.16) that $\int_{\mathbb{R}_+} h d\mu_i^n$ is nondecreasing in i. Since this holds for any given convex function h, we conclude from [80, Theorem 8] that $\mathcal{M}^n \neq \emptyset$. □

Now we further assume that the finitely many options are vanilla. Take $Q \in \mathcal{Q}$ and let μ_i be the distribution of S_i under Q for $i = 1, \dots, T$. From the theorem above (and the construction of μ_i^n), there exists a martingale measures Q^n supported on Ω^n, with marginals $\mu_i^n \xrightarrow{w} \mu_i$, for $i = 1, \dots, T$. Set

$$c_i^n := \mathbb{E}_{Q^n}[g_i] - \mathbb{E}_Q[g_i], \quad i = 1, \dots, e.$$

Then, we have $c^n \to 0$ by the weak convergence of the marginals, and $\mathcal{Q}^n \neq \emptyset$ for all n with $\Omega^n \subset \Omega$, since $Q^n \in \mathcal{Q}^n$. In addition, if g is Lipschitz continuous, we have that $|c^n| = O(1/2^n)$.

VIII.A Proof of Proposition 8.1.1

Proof of Proposition 8.1.1. Following the proof of Lemma 6.5.6, it can be shown that for $t \in \{0, \dots, T-1\}$ and $\omega \in \Omega_{T-t}$,

$$\mathcal{M}_t(\omega) = \{Q \in \mathfrak{P}(\Omega_{T-t}) : \ Q \ll P \text{ for some } P \in \mathbb{P}_t(\omega),$$

$$(S_k(\omega,\cdot))_{k=t,\dots,T} \text{ is a } Q\text{-martingale}\}.$$

Hence, in order to show the analyticity of graph($\mathcal{M}_t$), it suffices to show that the sets

$$\mathcal{I} := \{(\omega, Q) \in \Omega_t \times \mathfrak{P}(\Omega_{T-t}) :\ Q \ll P \text{ for some } P \in \mathbb{P}_t(\omega)\}$$

and

$$\mathcal{J} := \{(\omega, Q) \in \Omega_t \times \mathfrak{P}(\Omega_{T-t}) :\ (S_k(\omega,\cdot))_{k=t,\dots,T} \text{ is a } Q\text{-martingale}\}$$

are analytic.

Thanks to the analyticity of graph($\mathbb{P}_t$), we can follow the argument in the proof of [19, Lemma 4.8] to show that $\mathcal{I}$ is analytic. Now let us consider $\mathcal{J}$. For $k = t, \dots, T-1$, there exists a countable algebra $(A_i^k)_{i=1}^{\infty}$ generating $\mathcal{F}_k$. Then

$$\mathcal{I} = \bigcap_{k=t}^{T-1} \bigcap_{i=1}^{\infty} \{(\omega, Q) \in \Omega_t \times \mathfrak{P}(\Omega_{T-t}) :\ E_Q[\Delta S_k(\omega,\cdot) 1_{A_i^k}(\omega,\cdot)] = 0\}.$$

By a monotone class argument, we can show that for $(\omega, Q) \in \Omega_t \times \mathfrak{P}(\Omega_{T-t})$, the map

$$(\omega, Q) \mapsto E_Q[\Delta S_k(\omega,\cdot) 1_{A_i^k}(\omega,\cdot)]$$

is Borel measurable (e.g., see the first paragraph in the proof of [69, Theorem 2.3]). Therefore, the set $\mathcal{J}$ is Borel measurable, and in particular it is analytic. $\square$

VIII.B Optimal stopping for adverse nonlinear expectations

In this section, we analyze both the adverse optimal stopping problems for nonlinear expectations. This result is used in Theorem 8.2.2 for showing the existence of the sub hedging strategy. Note that [13, 39, 67] analyze similar problems in continuous time. Instead of referring to these papers directly, we decided to include a short analysis here because it is much simpler to carry it out in discrete time using backward induction.

For each $t \in \{0, \dots, T-1\}$ and $\omega \in \Omega_t$, we are given a nonempty convex set $\mathcal{R}_t(\omega) \subset \mathfrak{P}(\Omega_1)$ of probability measures. We assume that for each t, the graph of $\mathcal{R}_t$ is analytic, and thus admits a universally measurably selector. For $t = 0, \dots, T-1$ and $\omega \in \Omega_t$, define

$$\mathfrak{R}_t(\omega) := \{P_t \otimes \ldots \otimes P_{T-1} : \ P_i(\omega, \cdot) \in \mathcal{R}_i(\omega, \cdot), \ i = t, \dots, T-1\},$$

where each P_i is a universally measurable selector of $\mathcal{R}_i$. We write $\mathcal{R}$ for $\mathfrak{R}_0$ for short. We assume the graph of $\mathfrak{R}_t$ is analytic for $t = 0, \dots, T-1$. Let ξ be a u.s.a. function. For $\omega \in \Omega$, define the nonlinear conditional expectation as

$$\mathcal{E}_t[\xi](\omega) = \sup_{P \in \mathfrak{R}_t(\omega^t)} E_P[\xi(\omega^t, \cdot)].$$

We also write $\mathcal{E}$ for $\mathcal{E}_0$ for short. By [69, Theorem 2.3], we know that the function $\mathcal{E}_t[\xi]$ is u.s.a. and $\mathcal{F}_t$-measurable, and the nonlinear conditional expectation satisfies the tower property, i.e., for $0 \le s < t \le T$, it holds that

$$\mathcal{E}_s \mathcal{E}_t[\xi] = \mathcal{E}_s[\xi]. \tag{VIII.B.1}$$

Moreover, by Galmarino's test (see [69, Lemma 2.5]), it follows that if a function is $\mathcal{F}_t$-measurable, it only depends on the path up to time t. Throughout this section, we will assume that f is an adapted process with respect to the raw filtration $(\mathcal{B}(\Omega_t))_{t=0}^T$.

We consider the optimal stopping problem

$$X := \inf_{\tau \in \mathcal{T}} \mathcal{E}[f_\tau]. \tag{VIII.B.2}$$

and define the upper value process

$$X_t := \inf_{\tau \in \mathcal{T}_t} \mathcal{E}_t[f_\tau], \tag{VIII.B.3}$$

and the lower value process

$$Y_t(\omega) := \sup_{P \in \mathfrak{R}_t(\omega^t)} \inf_{\tau \in \mathcal{T}_t} E_P[f_\tau(\omega^t, \cdot)]. \tag{VIII.B.4}$$

In particular $X = X_0$. We have the following result:

Theorem VIII.B.1. *Assume for* $t = 1, \ldots, T-1$, $\mathcal{E}_t[X_{t+1}]$ *(or* $\mathcal{E}_t[Y_{t+1}]$*) is* $\mathcal{B}(\Omega_t)$*-measurable. Then* $X_t = Y_t$, $t = 0, \ldots, T$. *In particular, the game defined in* (VIII.B.2) *has a value, i.e.,*

$$\inf_{\tau \in \mathcal{T}} \mathcal{E}[f_\tau] = \sup_{P \in \mathcal{R}} \inf_{\tau \in \mathcal{T}} E[f_\tau]. \tag{VIII.B.5}$$

Moreover, there exists an optimal stoping time described by

$$\tau^* = \inf\{t \geq 0 : \ f_t = X_t\}. \tag{VIII.B.6}$$

Proof. We shall prove the result under the Borel measurability assumption for $\mathcal{E}_t[X_{t+1}]$. In fact, it could be seen from the proof later on that the Borel measurability assumption on $\mathcal{E}_t[X_{t+1}]$ is equivalent to that on $\mathcal{E}_t[Y_{t+1}]$.

Step 1: We first show that for $s \in \{0, \ldots, T\}$,

$$X_s = \inf_{\tau \in \mathcal{T}_s} \mathcal{E}_s(f_\tau 1_{\{\tau < t\}} + X_t 1_{\{\tau \geq t\}}), \quad 0 \leq s < t \leq T. \tag{VIII.B.7}$$

We shall prove it by a backward induction. For $s = T-1$, since τ equals either $T-1$ or T, we have from (VIII.B.3) that $X_{T-1} = f_{T-1} \wedge \mathcal{E}_{T-1}(f_T) = f_{T-1} \wedge \mathcal{E}_{T-1}(X_T)$, and thus (VIII.B.7) holds. Assume for $s+1 \in \{0, \ldots, T-1\}$ the corresponding conclusion holds. Let $t \in \{s+1, \ldots, T\}$. For any $\tau \in \mathcal{T}_s$, using the tower property (VIII.B.1) and the definition of X_t in (VIII.B.3), we have that

$$\mathcal{E}_s(f_\tau) = \mathcal{E}_s\left(f_\tau 1_{\{\tau < t\}} + \mathcal{E}_t(f_{\tau \vee t}) 1_{\{\tau \geq t\}}\right) \geq \mathcal{E}_s\left(f_\tau 1_{\{\tau < t\}} + X_t 1_{\{\tau \geq t\}}\right),$$

which implies the inequality "$\geq$" in (VIII.B.7).

Let us turn to the inequality "$\leq$" in (VIII.B.7). By the induction assumption, we have that for $k \geq s+1$,

$$X_k = \inf_{\tau \in \mathcal{T}_k} \mathcal{E}_k(f_\tau 1_{\{\tau < k+1\}} + X_{k+1} 1_{\{\tau \geq k+1\}}) = f_k \wedge \mathcal{E}_k(X_{k+1}). \tag{VIII.B.8}$$

Define

$$\begin{aligned} A_s &:= \{f_s \le \mathcal{E}_s(X_{s+1})\} \ \in \ \mathcal{B}(\Omega_s), \\ A_k &:= \left[\{f_k \le \mathcal{E}_k(X_{k+1})\} \setminus (\cup_{i=s}^{k-1} A_i)\right] = \left[\{f_k = X_k\} \setminus (\cup_{i=s}^{k-1} A_i)\right] \ \in \ \mathcal{B}(\Omega_k), \end{aligned}$$

$k = s+1, \ldots, T$. Note that $A_T = (\cup_{i=s}^{T-1} A_i)^c \in \mathcal{B}(\Omega_{T-1})$. Denoting

$$\bar{\tau} = \sum_{k=s}^{T} k 1_{A_k} \in \mathcal{T}_s. \tag{VIII.B.9}$$

and using the tower property repeatedly, we obtain that

$$\begin{aligned} X_s &\le \mathcal{E}_s(f_{\bar{\tau}}) \\ &= \mathcal{E}_s\left(\sum_{k=s}^{T-2} f_k 1_{A_k} + f_{T-1} 1_{A_{T-1}} + \mathcal{E}_{T-1}(X_T) 1_{(\cup_{i=s}^{T-1} A_i)^c}\right) \\ &= \mathcal{E}_s\left(\sum_{k=s}^{T-2} f_k 1_{A_k} + X_{T-1} 1_{(\cup_{i=s}^{T-2} A_i)^c}\right) \\ &= \mathcal{E}_s\left(\sum_{k=s}^{T-3} f_k 1_{A_k} + f_{T-2} 1_{A_{T-2}} + \mathcal{E}_{T-2}(X_{T-1}) 1_{(\cap_{k=s}^{T-2} A_k)^c}\right) \\ &= \ldots \\ &= \mathcal{E}_s\left(f_s 1_{A_s} + X_{s+1} 1_{A_s^c}\right) \\ &= f_s \wedge \mathcal{E}_s(X_{s+1}). \end{aligned} \tag{VIII.B.10}$$

On the other hand, for $t \in \{s+1, \ldots, T\}$, by (VIII.B.8) and the tower property, we have that

$$\begin{aligned} X_s &\ge \inf_{\tau \in \mathcal{T}_s} \mathcal{E}_s\left(f_\tau 1_{\{\tau < t\}} + X_t 1_{\{\tau \ge t\}}\right) \\ &\ge \inf_{\tau \in \mathcal{T}_s} \mathcal{E}_s\left(f_\tau 1_{\{\tau < t-1\}} + X_{t-1} 1_{\{\tau = t-1\}} + \mathcal{E}_{t-1}(X_t) 1_{\{\tau \ge t\}}\right) \\ &\ge \inf_{\tau \in \mathcal{T}_s} \mathcal{E}_s\left(f_\tau 1_{\{\tau < t-1\}} + X_{t-1} 1_{\{\tau \ge t-1\}}\right) \\ &\ge \ldots \\ &\ge \inf_{\tau \in \mathcal{T}_s} \mathcal{E}_s\left(f_\tau 1_{\{\tau < s+1\}} + X_{s+1} 1_{\{\tau \ge s+1\}}\right) \\ &= f_s \wedge \mathcal{E}_s(X_{s+1}). \end{aligned} \tag{VIII.B.11}$$

Hence, we have (VIII.B.7) holds for s.

Step 2: Define $\hat{\tau} = \sum_{k=0}^{T} k1_{A_k}$, same as $\bar{\tau}$ defined in (VIII.B.9) for $s = 0$. From (VIII.B.10) & (VIII.B.11) in Step 1, we have that $X = \mathcal{E}(f_{\hat{\tau}})$. Noting $A_0 = \{f_0 \leq \mathcal{E}(X_1)\} = \{f_0 = X\}$, we have $\hat{\tau} = \tau^*$.

Step 3: Using (VIII.B.7), we can follow the proof of [67, Lemma 4.11] mutatis mutandis, to show by a backward induction that $X_t = Y_t,\ t = 0, \dots, T$. In particular (VIII.B.5) holds. □

The next remark is concerned with the "sup sup" version of the optimal stopping problem:

Remark VIII.B.2. For the optimal stopping problem

$$Z := \sup_{\tau \in \mathcal{T}} \mathcal{E}[f_\tau],$$

let us define

$$Z_t := \sup_{\tau \in \mathcal{T}_t} \mathcal{E}_t[f_\tau], \quad t = 0, \dots, T.$$

In particular $Z = Z_0$. Following Steps 1 and 2 in the proof of Theorem VIII.B.1, we can show that if $\mathcal{E}_t[Z_{t+1}]$ is $\mathcal{B}(\Omega_t)$-measurable for $t = 1, \dots, T-1$, then

$$Z_t = f_t \vee \mathcal{E}_t(Z_{t+1}), \quad t = 0, \dots, T,$$

and $\tau^{**} := \inf\{t \geq 0 : \ f_t = Z_t\}$ is optimal.

VIII.B.1 An example in which $\mathcal{E}_t[Y_{t+1}]$ is Borel measurable

Let $S = (S_i)_{i=1}^T$ be the canonical process and $\mathcal{R}$ be the set of martingale measures on some compact set $\mathcal{K} \subset \Omega_T$. Assume $\mathcal{R} \neq \emptyset$. Then for $\omega \in \mathcal{K}$, $\mathfrak{R}_t(\omega^t)$ is the set of martingale measures on $\mathcal{K}$ from time t to T given the previous path ω^t. Proposition VIII.B.3 below indicates that the assumption in Theorem VIII.B.1 is satisfied provided f is u.s.c. in ω.

Proposition VIII.B.3. *Assume that f_t is u.s.c. for $t = 1, \dots, T$. Then $\mathcal{E}_t[Y_{t+1}]$ is u.s.c., and thus $\mathcal{B}(\Omega_t)$-measurable, $t = 1, \dots, T$.*

Proof. Since $\mathcal{K}$ is compact, it is easy to check that the set $\{(\omega, P) : \ \omega \in \mathcal{K}, P \in \mathfrak{R}_t(\omega^t)\}$ is closed. By [18, Proposition 7.33], Y_t defined in (VIII.B.4) is u.s.c. Following the proof similar to that of Proposition 8.3.2, it could be shown that for $(\omega, P) \in \Omega_T \times \mathfrak{P}(\Omega_{T-t})$, the map $(\omega, P) \mapsto E_P[Y(\omega^t, \cdot)]$ is u.s.c. Then applying [18, Proposition 7.33] again, we know that $\mathcal{E}_t[Y_{t+1}]$ is u.s.c. $\square$

VIII.C Upper-semianalyticity and the super-martingale property

The result in this section is used in the proof of Theorem 8.3.1. Let us use the setting in Section VIII.B. Let $\phi = (\phi_t)_{t=0}^T$ be an adapted process, and $\mathfrak{g}$ be u.s.a. Define the process $U = (U_t)_{t=0}^T$ as

(VIII.C.1) $$U_t := \sup_{\tau \in \mathcal{T}_t} \mathcal{E}_t[\phi_\tau + \mathfrak{g}].$$

We have the following result.

Proposition VIII.C.1. *Assume for $(\omega, P) \in \Omega_T \times \mathfrak{P}(\Omega_{T-t})$, the map $(\omega, P) \mapsto \sup_{\tau \in \mathcal{T}_t} E_P[\phi_\tau(\omega^t, \cdot)]$ is u.s.a., $t = 1, \dots, T$. Then U_t defined in* (VIII.C.1) *is u.s.a. and $\mathcal{F}_t$-measurable for $t = 1, \dots, T$, and $U = (U_t)_{t=0}^T$ is a super-martingale under each $P \in \mathcal{R}$.*

Proof. Using the fact that the map $(\omega, P) \mapsto E_p[\mathfrak{g}(\omega^t, \cdot)]$ is u.s.a. for $(\omega, P) \in \Omega_T \times \mathfrak{P}(\Omega_{T-t})$ (see the last paragraph on page 8 in [69]), we deduce that the map $(\omega, P) \mapsto \sup_{\tau \in \mathcal{T}_t} E_P[\phi_\tau(\omega^t, \cdot) + \mathfrak{g}(\omega^t, \cdot)]$ is u.s.a. Since $\mathfrak{R}_t(\omega^t)$ is the ω-section of an analytic set, we can apply [18, Proposition 7.47] to conclude that U_t is u.s.a., $t = 1, \dots, T$. As U_t only depends on the path up to time t, it is $\mathcal{F}_t$-measurable.

In the rest of the proof, we shall show that

$$U_t \geq \mathcal{E}_t[U_{t+1}], \tag{VIII.C.2}$$

which will imply the super-martingale property of U under each $P \in \mathcal{R}$. Fix $(t,\omega) \in \{0,\dots,T\} \times \Omega_T$ and let $P = P_t \otimes \dots \otimes P_{T-1} \in \mathfrak{R}_t(\omega^t)$. For any $\varepsilon > 0$, since the map $(\tilde{\omega}, P) \mapsto \sup_{\tau\in\mathcal{T}_t} E_P[\phi_\tau(\omega^t,\tilde{\omega},\cdot) + \mathfrak{g}(\omega^t,\tilde{\omega},\cdot)]$ is u.s.a. for $(\tilde{\omega}, P) \in \Omega_1 \times \mathfrak{P}(\Omega_{T-t-1})$, and $\mathfrak{R}_{t+1}(\omega^t,\tilde{\omega})$ is the $\tilde{\omega}$-section of an analytic set, we can apply theorem [18, Proposition 7.50] and get that there exists a universally measurable selector $P^\varepsilon(\omega^t,\cdot)$, such that $P^\varepsilon(\omega^t,\tilde{\omega}) = P^\varepsilon_{t+1}(\omega^t,\tilde{\omega}) \otimes \dots \otimes P^\varepsilon_{T-1}(\omega^t,\tilde{\omega},\cdot) \in \mathfrak{R}_{t+1}(\omega^t,\tilde{\omega})$, and

$$\begin{aligned}&\left(\sup_{\tilde{P}\in\mathfrak{R}_{t+1}(\omega^t,\tilde{\omega})} \sup_{\tau\in\mathcal{T}_{t+1}} E_{\tilde{P}}[\phi_\tau(\omega^t,\tilde{\omega},\cdot) + \mathfrak{g}(\omega^t,\tilde{\omega},\cdot)] - \varepsilon\right) 1_A + \frac{1}{\varepsilon} 1_{A^c}\\ &\leq \sup_{\tau\in\mathcal{T}_{t+1}} E_{P^\varepsilon(\omega^t,\tilde{\omega})}[\phi_\tau(\omega^t,\tilde{\omega},\cdot) + \mathfrak{g}(\omega^t,\tilde{\omega},\cdot)],\end{aligned}$$

where

$$A = \{\tilde{\omega} \in \Omega_1 : \sup_{\tilde{P}\in\mathfrak{R}_{t+1}(\omega^t,\tilde{\omega})} \sup_{\tau\in\mathcal{T}_{t+1}} E_{\tilde{P}}[\phi_\tau(\omega^t,\tilde{\omega},\cdot) + \mathfrak{g}(\omega^t,\tilde{\omega},\cdot)] < \infty\}.$$

Define

$$P^* := P_t \otimes P^\varepsilon_{t+1} \otimes \dots \otimes P^\varepsilon_{T-1} \in \mathfrak{R}_t(\omega^t).$$

Then we have that

$$\begin{aligned}
&E_P\left[\left(U_{t+1}(\omega^t,\cdot) - \varepsilon\right) 1_A + \frac{1}{\varepsilon} 1_{A^c}\right]\\
&\quad= E_P\left[\left(\sup_{\tilde{P}\in\mathfrak{R}_{t+1}(\omega^t,\tilde{\omega})} \sup_{\tau\in\mathcal{T}_{t+1}} E_{\tilde{P}}[\phi_\tau(\omega^t,\tilde{\omega},\cdot) + \mathfrak{g}(\omega^t,\tilde{\omega},\cdot)] - \varepsilon\right) 1_A + \frac{1}{\varepsilon} 1_{A^c}\right]\\
&\quad\leq E_P\left[\sup_{\tau\in\mathcal{T}_{t+1}} E_{P^\varepsilon(\omega^t,\tilde{\omega})}[\phi_\tau(\omega^t,\tilde{\omega},\cdot) + \mathfrak{g}(\omega^t,\tilde{\omega},\cdot)]\right]\\
&\quad= E_{P^*}\left[\sup_{\tau\in\mathcal{T}_{t+1}} E_{P^\varepsilon(\omega^t,\tilde{\omega})}[\phi_\tau(\omega^t,\tilde{\omega},\cdot) + \mathfrak{g}(\omega^t,\tilde{\omega},\cdot)]\right]\\
&\quad= E_{P^*}\left[\sup_{\tau\in\mathcal{T}_{t+1}} E_{P^\varepsilon(\omega^t,\tilde{\omega})}[\phi_\tau(\omega^t,\tilde{\omega},\cdot)]\right] + E_{P^*}[\mathfrak{g}(\omega^t,\cdot)]\\
&\quad\leq \sup_{\tau\in\mathcal{T}_t} E_{P^*}[\phi_\tau(\omega^t,\cdot)] + E_{P^*}[\mathfrak{g}(\omega^t,\cdot)]\\
&\quad\leq U_t(\omega),
\end{aligned}$$

where the fourth line follows from the fact that $P^* = P$ from time t to $t+1$, the fifth line follows from the tower property as $P^* = P_t \otimes P^\varepsilon$, and the sixth line follows from the classical optimal stopping theory under a single probability measure P^*. As t, ω, P and ε are arbitrary, (VIII.C.2) holds. □

VIII.D No arbitrage when there are no options for static hedging

Let $S = (S_t)_{t=0,\dots,T}$ be the canonical process taking values in some path space $\mathcal{K} \subset \{1\} \times \mathbb{R}^T$, which represents the stock price process. We take $\mathcal{P}$ to be the set of all the probability measures on $\mathcal{K}$. In this secton, we assume that there is no hedging option available, i.e., $e = 0$. Let us first identify the reasonable path spaces:

Definition VIII.D.1. $\mathcal{K} \subset \{1\} \times \mathbb{R}^T$ is called a reasonable path space, if for any $t \in \{0, \dots, T\}$ and $(s_0 = 1, s_1, \dots, s_T) \in \mathcal{K}$,

(i) if $s_t > 0$, then there exists $(s_0, \dots, s_t, , s_{t+1}^i, \dots, s_T^i) \in \mathcal{K},\ i = 1, 2$, such that $s_{t+1}^1 < s_t < s_{t+1}^2$;

(ii) if $s_t = 0$, then $s_k = 0,\ k \geq t+1$.

Obviously, if $\mathcal{K}$ is a reasonable path space, then a martingale measure on $\mathcal{K}$ is easy to construct, and thus the no arbitrage in [1] is satisfied. The following proposition states that NA($\mathcal{P}$) also holds. So the no arbitrage definitions in [1] and [19] in fact coincide in the case when $\mathcal{K}$ is a reasonable path space and $e = 0$.

Proposition VIII.D.2. *If $\mathcal{K}$ is a reasonable path space, then NA($\mathcal{P}$) holds.*

Proof. Let $H = (H_0, \dots, H_{T-1}(s_1, \dots, s_{T-1}))$ be a trading strategy such that

$$(H \cdot S)_T \geq 0, \quad \forall s \in \mathcal{K}. \tag{VIII.D.1}$$

We need to show $(H \cdot S)_T = 0, \forall s \in \mathcal{K}$. It suffices to show that

$$H_k(s_1, \dots, s_k) = 0, \quad \text{for } s_k > 0, \tag{VIII.D.2}$$

for $k = 0, \ldots, T-1$. We shall show (VIII.D.2) by the induction.

Assume $H_0 \neq 0$. Then take $s_1^* > s_0$ if $H_0 < 0$, and take $s_1^* < s_0$ if $H_0 > 0$. In general, for $j = 1, \ldots T-1$, take $s_{j+1}^* \geq s_j^*$ if $H(s_1^*, \ldots, s_j^*) \leq 0$ and $s_{j+1}^* \leq s_j^*$ if $H(s_1^*, \ldots, s_j^*) > 0$. Then $(H \cdot S)_T(s_0, s_1^*, \ldots, s_T^*) < 0$, which contradicts (VIII.D.1). Hence $H_0 = 0$ and (VIII.D.2) holds for $k = 0$.

Assume (VIII.D.2) holds for $k \leq t-1$. Then for any $(s_0, \ldots, s_t)$ with $s_t > 0$, by assumption (ii), we have that $s_i > 0$, $i = 0, \ldots, t-1$, and thus $H_i(s_1, \ldots, s_i) = 0$, $i = 0, \ldots, t-1$ by the induction hypothesis. If $H_t(s_1, \ldots, s_t) \neq 0$, then we can similarly construct $(s_{t+1}^*, \ldots, s_T^*)$ as above, such that $(H \cdot S)_T(s_0, \ldots, s_t, s_{t+1}^*, \ldots, s_T^*) < 0$, which contradicts (VIII.D.1). Hence $H_t(s_1, \ldots, s_t) = 0$ and (VIII.D.2) holds for $k = t$. $\square$

CHAPTER IX

Arbitrage, hedging and utility maximization using semi-static trading strategies with American options

9.1 Introduction

The arbitrage, hedging, and utility maximization problems have been extensively studied in the field of financial mathematics. We refer to [24, 34] and the references therein. Recently, there has been a lot of work on these three topics where stocks are traded dynamically and (European-style) options are traded statically (hedging strategies, see e.g., [32]). For example, [1, 17, 19, 32] analyze the arbitrage and/or super-hedging in the setup of model free or model uncertainty, and [77] studies the utility maximization within a given model. It is worth noting that most of the literature related to semi-static strategies only consider European-style options as to be liquid options, and there are only a few papers incorporating American-style options for static trading. In particular, [21] studies the completeness (in some $\mathbb{L}^2$ sense) of the market where American put options of all the strike prices are available for semi-static trading, and [27] studies the no arbitrage conditions on the price function of American put options where European and American put options are available.

In this chapter, we consider a market model in discrete time consisting of stocks, (path-dependent) European options, and (path-dependent) American options (we also refer to these as hedging options), where the stocks are traded dynamically and European and American options are traded statically. We assume that the American options are infinitely divisible, and we can only buy but not sell American options. We first obtain the fundamental theorem of asset pricing (FTAP) under the notion of robust no arbitrage that is slightly stronger than no arbitrage in the usual sense. Then by the FTAP result, we further get dualities of the sub-hedging prices of European and American options. Using the duality result, we then study the utility maximization problem and get the duality of the value function.

It is crucial to assume the infinite divisibility of the American options just like the stocks and European options. From a financial point of view, it is often the case that we can do strictly better when we break one unit of the American options into pieces and exercise each piece separately. In Section 9.2, we provide a motivating example in which without the divisibility assumption of the American option the no arbitrage condition holds yet there is no equivalent martingale measure (EMM) that prices the hedging options correctly. Moreover, we see in this example that the super-hedging price of the European option is not equal to the supremum of the expectation over all the EMMs which price the hedging options correctly. Mathematically, the infinite divisibility leads to the convexity and closedness of some related set of random variables, which enables us to apply the separating hyperplane argument to obtain the the existence of an EMM that prices the options correctly, as well as the dualities for hedging and utility maximization.

The rest of the chapter is organized as follows. In the next section, we will provide a motivating example. In Section 9.3, we shall introduce the setup and the main results of FTAP, sub-hedging duality and utility maximization duality. These results are proved in Sections 9.4, 9.5 and 9.6, respectively.

9.2 A motivating example

In this section, we shall look at an example of super-hedging of a European option using the stock and the American option. This example will motivate us to consider the divisibility of American options.

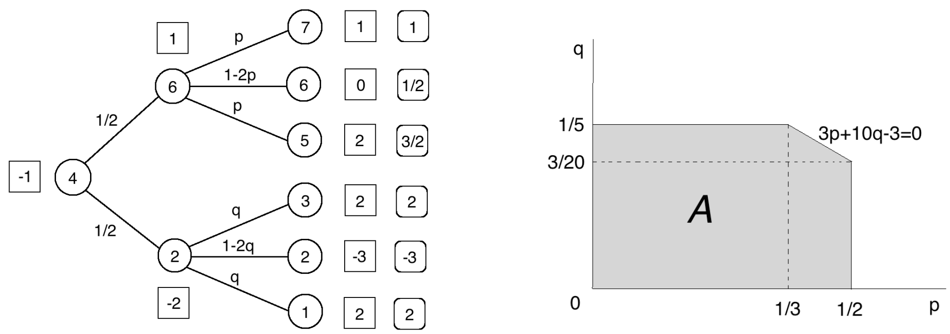

Figure 9.1: A motivating example

Consider a simple model given by the left graph above. The stock prices $S = (S_t)_{t=0,1,2}$, payoffs of the American option $h = (h_t)_{t=0,1,2}$, and payoffs of the European option ψ are indicated by the numbers in the circles, squares with straight corners, and squares with rounded corners, respectively. Let $(\Omega, \mathcal{B}(\Omega))$ be the path space indicated by the left graph above, and let $(\mathcal{F}_t)_{t=0,1,2}$ be the filtration generated by S. Let $\mathbb{P}$ be a probability measure that is supported on Ω. Hence any EMM would be characterized by the pair (p, q) shown in the left graph above with $0 < p, q < 1/2$.

We assume that the American option h can only be bought at time $t = 0$ with price $\bar{h} = 0$. Then in order to avoid arbitrage involving stock S and American option h, we expect that the set

$$\mathcal{Q} := \left\{ \mathbb{Q} \text{ is an EMM} : \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} h_\tau \leq 0 \right\}$$

is not empty, where $\mathcal{T}$ represents the set of stopping times. Equivalently, to avoid arbitrage, the set

$$A := \left\{(p,q) \in \left(0,\frac{1}{2}\right) \times \left(0,\frac{1}{2}\right):\ \left(\frac{1}{2}[(3p)\vee 1] + \frac{1}{2}[(10q-3)\vee(-2)]\right)\vee(-1) \le 0\right\}$$

should be nonempty. In the right graph in Figure 9.1 A is indicated by the shaded area, which shows that $A \neq \emptyset$.

Now consider the super-hedging price $\bar{\pi}(\psi)$ of the European option ψ using semi-static trading strategies. That is,

$$\bar{\pi}(\psi) := \inf\{x :\ \exists (H,c,\tau) \in \mathcal{H}\times\mathbb{R}_+\times\mathcal{T},\ \text{s.t. } x + H\cdot S + ch_\tau \ge \psi,\ \mathbb{P}-\text{a.s.}\},$$

where $\mathcal{H}$ is the set of adapted processes, and $H\cdot S = \sum_{t=0}^{1} H_t(S_{t+1}-S_t)$. One may expect that the super-hedging duality would be given by

$$\bar{\pi}(\psi) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\psi.$$

By calculation,

$$\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\psi = \sup_{(p,q)\in A}\left(\frac{3}{4}p + 5q - \frac{5}{4}\right) = \left(\frac{3}{4}p + 5q - \frac{5}{4}\right)\Bigg|_{\left(\frac{1}{3},\frac{1}{5}\right)} = 0.$$

On the other hand, it can be shown that

$$\begin{aligned}
\bar{\pi}(\psi) &= \inf_{\tau\in\mathcal{T}}\inf_{c\in\mathbb{R}_+}\inf\{x :\ \exists H\in\mathcal{H},\ \text{s.t. } x + H\cdot S \ge \psi - ch_\tau\}\\
&= \inf_{\tau\in\mathcal{T}}\inf_{c\in\mathbb{R}_+}\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[\psi - ch_\tau]\\
&= \frac{1}{8},
\end{aligned}$$

where $\mathcal{M}$ is the set of EMMs. Here we use the classical result of super-hedging for the second line, and the value in the third line can be calculated by brute force since we only have five stopping times.[1] **Therefore, the super hedging price is**

[1] For example, when

$$\tau = \begin{cases} 2, & S_1 = 6,\\ 1, & S_1 = 2,\end{cases}$$

then

$$\inf_{c\in\mathbb{R}_+}\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[\psi - ch_\tau] = \inf_{c\ge 0}\sup_{0<p,q<\frac{1}{2}}\left[\left(\frac{3}{4}-\frac{3}{2}c\right)p + 5q - \frac{5}{4} + c\right] = \frac{13}{8}$$

strictly bigger than the sup over the EMMs $\mathbb{Q} \in \mathcal{Q}$, i.e.,

$$\bar{\pi}(\psi) > \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\psi.$$

As a consequence, if we add ψ into the market, and assume that we can only sell ψ at $t = 0$ with price $\underline{\psi} = 1/16$ $(> 0 = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\psi)$, then the market would **admit no arbitrage, yet there is no** $Q \in \mathcal{Q}$**, such that** $\mathbb{E}_{\mathbb{Q}}[\psi] \geq \underline{\psi}$.

However, observe that $\psi = \frac{1}{2}(h_{\tau_{12}} + h_2)$, where

$$\tau_{12} = \begin{cases} 1, & S_1 = 6, \\ 2, & S_1 = 2. \end{cases}$$

This suggests that if we assume that h is infinitely divisible, i.e., we can break one unit of h into pieces, and exercise each piece separately, then we can show that the super-hedging price of ψ is $\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\psi = 0$. Now if we add ψ into the market with selling price $\underline{\psi} < 0$, then we can find $\mathbb{Q} \in \mathcal{Q}$, such that $\mathbb{E}_{\mathbb{Q}}\psi > \underline{\psi}$.

9.3 Setup and main results

In this section, we first describe the setup of our financial model. In particular, as suggested by the example in the last section, we shall assume that the American options are divisible. Then we shall provide the main results, including Theorem 9.3.4 for FTAP, Theorem 9.3.5 for sub-hedging, and Theorem 9.3.8 for utility maximization.

9.3.1 Setup

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1,\dots,T}, \mathbb{P})$ be a filtered probability space, where $\mathcal{F}$ is assumed to be countably generated, and $T \in \mathbb{N}$ represents the time horizon in discrete time. Let $S = (S_t)_{t=0,\dots,T}$ be an adapted process taking values in $\mathbb{R}^d$ which represents the stock price process. Let $f^i, g^j : \ \Omega \mapsto \mathbb{R}$ be $\mathcal{F}_T$-measurable, representing the payoffs of

European options, $i = 1, \dots, L$ and $j = 1, \dots, M$. We assume that we can buy *and* sell each f^i at time $t = 0$ at price $\bar{f}^i$, and we can only buy but *not* sell each g^j at time $t = 0$ with price $\bar{g}^j$. Let $h^k = (h_t^k)_{t=0,\dots,T}$ be an adapted process, representing the payoff process of an American option, $k = 1, \dots, N$. We assume that we can only buy but *not* sell each h^k at time $t = 0$ with price $\bar{h}^k$. Denote $f = (f^1, \dots, f^L)$ and $\bar{f} = (\bar{f}^1, \dots, \bar{f}^L)$, and similarly for $g, \bar{g}, h$ and $\bar{h}$. For simplicity, we assume that g and h are bounded.

Definition 9.3.1. An adapted process $\eta = (\eta_t)_{t=0,\dots,T}$ is said to be a liquidating strategy, if $\eta_t \geq 0$ for $t = 0, \dots, T$, and

$$\sum_{t=0}^{T} \eta_t = 1, \quad \mathbb{P} - \text{a.s.}.$$

Denote $\mathbb{T}$ as the set of all liquidating strategies.

Remark 9.3.2. Let us also mention the related concept of a randomized stopping time, which is a random variable γ on the enlarged probability space $(\Omega \times [0,1], \mathcal{F} \otimes \mathcal{B}, \mathbb{P} \times \lambda)$, such that $\{\gamma = t\} \in \mathcal{F}_t \otimes \mathcal{B}$ for $t = 0, \dots, T$, where $\mathcal{B}$ is the Borel sigma algebra on $[0,1]$ and λ is the Lebesgue measure. Let $\mathbb{T}'$ be the set of randomized stopping times. For $\gamma \in \mathbb{T}'$, its ω-distribution $\xi = (\xi_t)_{t=0,\dots,T}$ defined by

$$\xi_t(\cdot) = \lambda\{v : \ \gamma(\cdot, v) = t\}, \quad t = 0, \dots, T,$$

is a member in $\mathbb{T}$. There is one-to-one correspondence between $\mathbb{T}$ and $\mathbb{T}'$ (up to a increasing rearrangement). We refer to [38] for these facts.

In spite of the one-to-one correspondence, the paths of a liquidating strategy and a randomized stopping time are quite different. A randomized stopping time is the strategy of flipping a coin to decide which stopping time to use (so the whole unit is liquidated only once), while a liquidating strategy is an exercising flow (so different parts of the whole unit are liquidated at different times).

Because of this difference, Theorem 9.3.4 (FTAP), Theorem 9.3.5 (hedging duality) and Theorem 9.3.8 (utility maximization duality) will not hold if we replace liquidating strategies with randomized stopping times. (For randomized stopping times, one may still consider FTAP and hedging on the enlarged probability space, and the results would be different.) For instance, in the example from last section, unlike liquidating strategies, we cannot merely use h to super-hedge ψ (on the enlarged probability space) via any randomized stopping time. See Remark 9.3.9 for more explanation for the case of utility maximization.

For each $\eta \in \mathbb{T}$ and American option h^k, denote $\eta(h^k)$ as the payoff of h^k by using the liquidating strategy η. That is,

$$\eta(h^k) = \sum_{t=0}^{T} h_t^k \eta_t.$$

For $\mu = (\mu^1, \ldots, \mu^N) \in \mathbb{T}^N$, denote

$$\mu(h) = (\mu^1(h^1), \ldots, \mu^N(h^N)).$$

Let $\mathcal{H}$ be the set of adapted processes which represents the dynamical trading strategies for stocks. Let $(H \cdot S)_t := \sum_{t=0}^{T-1} H_t(S_{t+1} - S_t)$, and denote $H \cdot S$ for $(H \cdot S)_T$ for short. For a semi-static trading strategy $(H, a, b, c, \mu) \in \mathcal{H} \times \mathbb{R}^L \times \mathbb{R}_+^M \times \mathbb{R}_+^N \times \mathbb{T}^N$, the terminal value of the portfolio starting from initial wealth 0 is given by

$$\Phi_{\bar{g},\bar{h}}(H, a, b, c, \mu) := H \cdot S + a(f - \bar{f}) + b(g - \bar{g}) + c(\mu(h) - \bar{h}),$$

where $f - \bar{f} := (f^1 - \bar{f}^1, \ldots, f^L - \bar{f}^L)$, and af represents the inner product of a and f, and similarly for the other related terms. For $(H, a) \in \mathcal{H} \times \mathbb{R}^L$ we shall also use the notation

$$\Phi(H, a) := H \cdot S + a(f - \bar{f})$$

for short. From now on, when we write out the quintuple such as (H, a, b, c, μ), they are by default in $\mathcal{H} \times \mathbb{R}^L \times \mathbb{R}_+^M \times \mathbb{R}_+^N \times \mathbb{T}^N$ unless we specifically point out, and similarly for (H, a).

9.3.2 Fundamental theorem of asset pricing

Definition 9.3.3. We say no arbitrage (NA) holds w.r.t. $\bar{g}$ and $\bar{h}$, if for any (H, a, b, c, μ),

$$\Phi_{\bar{g},\bar{h}}(H, a, b, c, \mu) \geq 0 \quad \mathbb{P}\text{-a.s.} \implies \Phi_{\bar{g},\bar{h}}(H, a, b, c, \mu) = 0 \quad \mathbb{P}\text{-a.s.}.$$

We say robust no arbitrage (RNA) holds, if there exists $\varepsilon_g \in (0, \infty)^M$ and $\varepsilon_h \in (0, \infty)^N$ (from now on we shall use $\varepsilon_g, \varepsilon_h > 0$ for short), such that NA holds w.r.t. $\bar{g} - \varepsilon_g$ and $\bar{h} - \varepsilon_h$.

Define

$$\mathcal{Q} := \{\mathbb{Q} \text{ is an EMM} : \ \mathbb{E}_{\mathbb{Q}} f = \bar{f}, \ \mathbb{E}_{\mathbb{Q}} g < \bar{g}, \ \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} h_\tau < \bar{h}\},$$

where $\mathcal{T}$ is the set of stopping times,

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} h_\tau := (\sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} h_\tau^1, \dots, \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} h_\tau^N),$$

and the expectation and equality/inequality above are understood in a component-wise sense.

Below is the main result of FTAP.

Theorem 9.3.4 (FTAP). *RNA* $\iff \mathcal{Q} \neq \emptyset$.

9.3.3 Sub-hedging

Let $\psi : \ \Omega \mapsto \mathbb{R}$ be $\mathcal{F}_T$-measurable, which represents the payoff of a European option. Let $\phi = (\phi_t)_{t=0,\dots,T}$ be an adapted process, representing the payoff process of an American option. For simplicity, we assume that ψ and ϕ are bounded. Define the sub-hedging price of ψ

$$\pi_{eu}(\psi) := \sup\{x: \ \exists (H,a,b,c,\mu), \ \text{s.t. } \Phi_{\bar{g},\bar{h}}(H,a,b,c,\mu) + \psi \geq x\},$$

and the sub-hedging price of ϕ

$$\pi_{am}(\phi) := \sup\{x: \ \exists (H,a,b,c,\mu) \text{ and } \eta \in \mathbb{T}, \ \text{s.t. } \Phi_{\bar{g},\bar{h}}(H,a,b,c,\mu) + \eta(\phi) \geq x\}.$$

Below is the main result of sub-hedging.

Theorem 9.3.5 (Sub-hedging)**.** *Let RNA hold. Then*

$$\pi_{eu}(\psi) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\psi, \tag{9.3.1}$$

and

$$\pi_{am}(\phi) = \inf_{\mathbb{Q}\in\mathcal{Q}} \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathbb{Q}}\phi_\tau. \tag{9.3.2}$$

Moreover, there exists $(H^*, a^*, b^*, c^*, \mu^*)$ such that

$$\Phi_{\bar{g},\bar{h}}(H^*, a^*, b^*, c^*, \mu^*) + \psi \geq \pi_{eu}(\psi),$$

and there exists $(H^{**}, a^{**}, b^{**}, c^{**}, \mu^{**})$ and $\eta^{**} \in \mathbb{T}$ such that

$$\Phi_{\bar{g},\bar{h}}(H^{**}, a^{**}, b^{**}, c^{**}, \mu^{**}) + \eta^{**}(\phi) \geq \pi_{am}(\phi). \tag{9.3.3}$$

Remark 9.3.6. In fact, from the proof of Theorem 9.3.5 we have that

$$\pi_{am}(\phi) = \sup_{\eta\in\mathbb{T}} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\eta(\phi)] = \inf_{\mathbb{Q}\in\mathcal{Q}} \sup_{\eta\in\mathbb{T}} \mathbb{E}_{\mathbb{Q}}[\eta(\phi)] = \inf_{\mathbb{Q}\in\mathcal{Q}} \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathbb{Q}}\phi_\tau.$$

However, the order of "sup" and "inf" in the duality (9.3.2) cannot be exchanged. That is, it is possible that

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathbb{Q}}\phi_\tau > \sup_{\tau\in\mathcal{T}} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\phi_\tau.$$

We refer to Example 8.2.3 for such an example.

9.3.4 Utility maximization

Let $U : (0, \infty) \mapsto \mathbb{R}$ be a utility function, which is strictly increasing, strictly concave, continuously differentiable, and satisfies the Inada condition

$$\lim_{x \to 0+} U'(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} U'(x) = 0.$$

Consider the utility maximization problem

$$u(x) := \sup_{(H,a,b,c,\mu) \in \mathcal{A}(x)} \mathbb{E}_{\mathbb{P}}[U(x + \Phi_{\bar{g},\bar{h}}(H, a, b, c, \mu))], \quad x > 0,$$

where

$$\mathcal{A}(x) := \{(H, a, b, c, \mu) : \ x + \Phi_{\bar{g},\bar{h}}(H, a, b, c, \mu) > 0, \ \mathbb{P}\text{-a.s.}\}, \quad x > 0.$$

Remark 9.3.7. [44] also studies the utility maximization problem involving the liquidation of a *given* amount of infinitely divisible American options. Unlike the problem in [44], here we also incorporate the stocks and European options, and we need to decide how many shares of American options we need to buy at time $t = 0$. Another difference is that [44] focuses on the primary problem of the utility maximization, while we shall mainly find the duality of the value function u.

Let us define

$$V(y) := \sup_{x>0}[U(x) - xy], \quad y > 0,$$

$$I := -V' = (U')^{-1},$$

and for $x, y > 0$,

$$\begin{aligned}
\mathcal{X}(x) &:= \{X \text{ adapted}: \ X_0 = x, \ X_T = x + \Phi_{\bar{g},\bar{h}}(H, a, b, c, \mu) \geq 0 \\
&\quad \text{for some } (H, a, b, c, \mu)\}, \\
\mathcal{Y}(y) &:= \{Y \geq 0 \text{ adapted}: \ Y_0 = y, \ ((1 + (H \cdot S)_t) Y_t)_{t=0,\dots,T} \\
&\quad \text{is a } \mathbb{P}\text{-super-martingle for any } H \in \mathcal{H} \text{ satisfying} \\
&\quad 1 + H \cdot S \geq 0, \ \mathbb{E}_{\mathbb{P}} X_T Y_T \leq xy \text{ for any } X \in \mathcal{X}(x)\} \\
\mathcal{C}(x) &:= \{p \in \mathbb{L}^0_+: \ p \leq X_T \text{ for some } X \in \mathcal{X}(x)\}, \quad (9.3.4) \\
\mathcal{D}(y) &:= \{q \in \mathbb{L}^0_+: \ q \leq Y_T \text{ for some } Y \in \mathcal{Y}(y)\}, \quad (9.3.5)
\end{aligned}$$

where $\mathbb{L}^0_+$ is the set of random variables that are nonnegative $\mathbb{P}$-a.s.. Then we have that

$$u(x) = \sup_{p \in \mathcal{C}(x)} \mathbb{E}_{\mathbb{P}}[U(p)], \quad x > 0.$$

Let us also define

$$v(y) := \inf_{q \in \mathcal{D}(y)} \mathbb{E}_{\mathbb{P}}[V(q)], \quad y > 0.$$

Below is the main result of utility maximization.

Theorem 9.3.8 (Utility maximization duality)**.** *Let RNA hold. Then we have the following.*

i) $u(x) < \infty$ for any $x > 0$, and there exists $y_0 > 0$ such that $v(y) < \infty$ for any $y > y_0$. Moreover, u and v are conjugate:

$$v(y) = \sup_{x>0}[u(x) - xy], \quad y > 0 \quad \textit{and} \quad u(x) = \inf_{y>0}[v(y) + xy], \quad x > 0.$$

Furthermore, u is continuous differentiable on $(0, \infty)$, v is strictly convex on $\{v < \infty\}$, and

$$\lim_{x \to 0+} u'(x) = \infty \quad \textit{and} \quad \lim_{y \to \infty} v'(y) = 0.$$

ii) If $v(y) < \infty$, then there exists a unique $\hat{q}(y) \in \mathcal{D}(y)$ that is optimal for $v(y)$.

iii) If U has asymptotic elasticity strictly less than 1, i.e.,

$$AE(U) := \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1,$$

Then we have the following.

a) $v(y) < \infty$ for any $y > 0$, and v is continuously differentiable on $(0, \infty)$. u' and $-v'$ are strictly decreasing, and satisfy

$$\lim_{x \to \infty} u'(x) = 0 \quad \text{and} \quad \lim_{y \to 0+} v'(y) = -\infty.$$

Besides, $|AE(u)| \leq |AE(U)| < 1$.

b) There exists a unique $\hat{p}(x) \in \mathcal{C}(x)$ that is optimal for $u(x)$. If $\hat{q}(y) \in \mathcal{D}(y)$ is optimal for $v(y)$, where $y = u'(x)$, then

$$\hat{p}(x) = I(\hat{q}(y)),$$

and

$$\mathbb{E}_{\mathbb{P}}[\hat{p}(x)\hat{q}(y)] = xy.$$

c) We have that

$$u'(x) = \mathbb{E}_{\mathbb{P}}\left[\frac{\hat{p}(x)U'(\hat{p}(x))}{x}\right] \quad \text{and} \quad v'(y) = \mathbb{E}_{\mathbb{P}}\left[\frac{\hat{q}(y)V'(\hat{q}(y))}{y}\right].$$

Remark 9.3.9. We cannot replace the liquidating strategies with randomized stopping times since the two types of strategies yield to very different optimization problems:

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}U(\eta(\phi)) &= \mathbb{E}_{\mathbb{P}}\left[U\left(\sum_{t=0}^{T}\phi_t\eta_t\right)\right], \quad \text{if } \eta \text{ is a liquidating strategy,} \\ \mathbb{E}_{\mathbb{P}\times\lambda}U(\phi_\gamma) &= \mathbb{E}_{\mathbb{P}}\left[\sum_{t=0}^{T}U(\phi_t)\,\eta_t\right], \quad \text{if } \eta \text{ is the } \omega\text{-distribution of } \gamma \in \mathbb{T}'. \end{aligned}$$

9.4 Proof of Theorem 9.3.4

Proof of Theorem 9.3.4. "$\Longleftarrow$": Let $\mathbb{Q} \in \mathcal{Q}$. Then there exists $\varepsilon_g, \varepsilon_h > 0$, such that

$$\mathbb{E}_{\mathbb{Q}} g < \bar{g} - \varepsilon_g \quad and \quad \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} h_\tau < \bar{h} - \varepsilon_h.$$

Thanks to the one-to-one correspondence between $\mathbb{T}$ and $\mathbb{T}'$, we have that for any $\mathbb{Q} \in \mathcal{Q}$,

$$\sup_{\eta \in \mathbb{T}} \mathbb{E}_{\mathbb{Q}}[\eta(h^i)] = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} h^i_\tau, \quad i = 1, \dots, N,$$

see e.g., [38, Proposition 1.5]. Then it is easy to see that NA w.r.t. $\bar{g} - \varepsilon_g, \bar{h} - \varepsilon_h$ holds, and thus RNA holds.

"$\Longrightarrow$": We shall proceed in three steps.

Step 1. Define

$$\mathcal{I} := \{\Phi(H, a) - W : \text{ for some } (H, a) \text{ and } W \in \mathbb{L}^0_+\} \cap \mathbb{L}^\infty,$$

where $\mathbb{L}^\infty$ is the set of bounded random variables. We shall show that $\mathcal{I}$ is sequentially closed under weak star topology in this step.

Let $(X^n)_{n=1}^\infty \subset \mathcal{I}$ such that

$$X^n = \Phi(H^n, a^n) - W^n \xrightarrow{w^*} X \in \mathbb{L}^\infty,$$

where the notation "$\xrightarrow{w^*}$" represents the convergence under the weak star topology. Then there exist $(Y^m)_{m=1}^\infty$ which are convex combinations of $(X^n)_n$, such that $Y^m \to X$ a.s. (see e.g., the argument below Definition 3.1 on page 35 in [75]). Since $\mathcal{I}$ is convex, $(Y^m)_m \subset \mathcal{I}$. By [19, Theorem 2.2], there exists (H, a) and $W \in \mathbb{L}^+_0$ such that

$$\Phi(H, a) - W = X,$$

which implies $X \in \mathcal{I}$.

Step 2. By RNA, there exist $\varepsilon_g, \varepsilon_h > 0$, such that NA holds w.r.t. $\bar{g} - \varepsilon_g$ and $\bar{h} - \varepsilon_h$. Then NA also holds w.r.t. $\bar{g} - \varepsilon_g/2$ and $\bar{h} - \varepsilon_h/2$. Define

$$\mathcal{J} := \left\{ \Phi_{\bar{g}-\frac{1}{2}\varepsilon_g, \bar{h}-\frac{1}{2}\varepsilon_h}(H, a, b, c, \mu) - W : \text{ for some } (H, a, b, c, \mu) \text{ and } W \in \mathbb{L}^0_+ \right\} \cap \mathbb{L}^\infty.$$

We shall show that $\mathcal{J}$ is sequentially closed under weak star topology.

Let $(X^n)_{n=1}^\infty \subset \mathcal{J}$ such that

$$X^n = \Phi_{\bar{g}-\frac{1}{2}\varepsilon_g, \bar{h}-\frac{1}{2}\varepsilon_h}(H^n, a^n, b^n, c^n, \mu^n) - W^n \xrightarrow{w^*} X \in \mathbb{L}^\infty.$$

We consider the following two cases:

$$\liminf_{n\to\infty} ||(b^n, c^n)|| < \infty \quad \text{and} \quad \liminf_{n\to\infty} ||(b^n, c^n)|| = \infty,$$

where $|| \cdot ||$ represents the sup norm.

Case (i) $\liminf_{n\to\infty} ||(b^n, c^n)|| < \infty$. Without loss of generality, assume that $(b^n, c^n) \to (b, c) \in \mathbb{R}^M \times \mathbb{R}^N$. By [38, Theorem 1.1], there exists $\mu \in \mathbb{T}^N$, such that up to a subsequence $\mu^n \xrightarrow{w^*} \mu$ (i.e., $\mu^n_t \xrightarrow{w^*} \mu_t$ for $t = 0, \dots, T$). Since h is bounded,

$$\mu^n(h) \xrightarrow{w^*} \mu(h).$$

Then we have that

$$\begin{aligned} & b^n \left(g - \left(\bar{g} - \frac{1}{2}\varepsilon_g \right) \right) + c^n \left(\mu^n(h) - \left(\bar{h} - \frac{1}{2}\varepsilon_h \right) \right) \\ & \xrightarrow{w^*} b \left(g - \left(\bar{g} - \frac{1}{2}\varepsilon_g \right) \right) + c \left(\mu(h) - \left(\bar{h} - \frac{1}{2}\varepsilon_h \right) \right). \end{aligned}$$

Hence,

$$\Phi(H^n, a^n) - W^n \xrightarrow{w^*} X - b \left(g - \left(\bar{g} - \frac{1}{2}\varepsilon_g \right) \right) + c \left(\mu(h) - \left(\bar{h} - \frac{1}{2}\varepsilon_h \right) \right) \in \mathbb{L}^\infty.$$

Then by Step 1, there exists (H, a) and $W \in \mathbb{L}^0_+$ such that

$$\Phi(H, a) - W = X - b \left(g - \left(\bar{g} - \frac{1}{2}\varepsilon_g \right) \right) + c \left(\mu(h) - \left(\bar{h} - \frac{1}{2}\varepsilon_h \right) \right).$$

Therefore,

$$X = \Phi_{\bar{g}-\frac{1}{2}\varepsilon_g, \bar{h}-\frac{1}{2}\varepsilon_h}(H, a, b, c, \mu) - W \in \mathcal{J}.$$

Case (ii) $\liminf_{n\to\infty} ||(b^n, c^n)|| = \infty$. Without loss of generality, Assume that $d^n := ||(b^n, c^n)|| > 0$ for any n. We have that

$$\frac{X^n}{d^n} = \Phi_{\bar{g}-\frac{1}{2}\varepsilon_g, \bar{h}-\frac{1}{2}\varepsilon_h}\left(\frac{H^n}{d^n}, \frac{a^n}{d^n}, \frac{b^n}{d^n}, \frac{c^n}{d^n}, \mu^n\right) - \frac{W^n}{d^n} \xrightarrow{w^*} 0.$$

Then by Case (i), there exist (H', a', b', c', μ') and $W' \in \mathbb{L}^0_+$, such that

$$\Phi_{\bar{g}-\frac{1}{2}\varepsilon_g, \bar{h}-\frac{1}{2}\varepsilon_h}(H', a', b', c', \mu') - W' = 0.$$

Moreover, $b', c' \geq 0$ and at least one component of (b', c') equals 1. Hence

$$\Phi_{\bar{g}-\varepsilon_g, \bar{h}-\varepsilon_h}(H', a', b', c', \mu') > 0, \quad \mathbb{P}\text{-a.s.},$$

which contradicts NA w.r.t. $\bar{g} - \varepsilon_g$ and $\bar{h} - \varepsilon_h$.

Step 3. Since $\mathcal{J}$ is convex and sequentially closed under weak star topology, it is weak star closed by [25, Corollary 5.12.7]. Apply the theorem below Remark 3.1 on page 34 in [75], we have that there exists an EMM $\mathbb{Q}$ satisfying

$$\mathbb{E}_{\mathbb{Q}} f = \bar{f}, \quad \mathbb{E}_{\mathbb{Q}} g \leq \bar{g} - \varepsilon_g, \quad \text{and} \quad \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} h_\tau \leq \bar{h} - \varepsilon_h.$$

In particular, $\mathcal{Q} \neq \emptyset$. □

9.5 Proof of Theorem 9.3.5

Proof of Theorem 9.3.5. We shall only prove the results for ϕ. The case for ψ is similar, and in fact simpler. Let us first prove (9.3.2). It can be shown that

$$\pi_{am}(\phi) \leq \sup_{\eta \in \mathbb{T}} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\eta(\phi)] \leq \inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{\eta \in \mathbb{T}} \mathbb{E}_{\mathbb{Q}}[\eta(\phi)] = \inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} \phi_\tau.$$

If $\pi_{am}(\phi) < \inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} \phi_\tau$, then take $\bar{\phi} \in \mathbb{R}$ such that

$$\pi_{am}(\phi) < \bar{\phi} < \inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} \phi_\tau. \tag{9.5.1}$$

Now we add ϕ into the market, and we assume that ϕ can only be bought at time $t=0$ with price $\bar{\phi}$. Then since $\bar{\phi} > \pi_{am}(\phi)$, RNA also holds when ϕ is involved. As a consequence, there exists $\mathbb{Q} \in \mathcal{Q}$ such that $\sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathbb{Q}}\phi_\tau < \bar{\phi}$ by Theorem 9.3.4, which contradicts (9.5.1). Therefore, we have that (9.3.2) holds. Similarly we can show that (9.3.1) holds.

Next, let us prove the existence of an optimal sub-hedging strategy for ϕ. It can be shown that

$$\begin{aligned}\pi_{am}(\phi) &= \sup_{b\in\mathbb{R}_+^M, c\in\mathbb{R}_+^N} \sup_{\mu\in\mathbb{T}^N, \eta\in\mathbb{T}} \sup\{x:\ \exists (H,a),\ \text{s.t. } \Phi_{\bar{g},\bar{h}}(H,a,b,c,\mu) + \eta(\phi) \geq x\} \\ &= \sup_{b\in\mathbb{R}_+^M, c\in\mathbb{R}_+^N} \sup_{\mu\in\mathbb{T}^N, \eta\in\mathbb{T}} \inf_{\mathbb{Q}\in\mathcal{Q}_f} \mathbb{E}_{\mathbb{Q}}[b(g-\bar{g}) + c(\mu(h)-\bar{h}) + \eta(\phi)],\end{aligned}$$

where

$$\mathcal{Q}_f := \{\mathbb{Q} \text{ is an EMM}:\ \mathbb{E}_{\mathbb{Q}} f = \bar{f}\},$$

and we apply Superheging Theorem on page 6 in [?] for the second line. We shall proceed in three steps to show the existence of $(H^{**}, a^{**}, b^{**}, c^{**}, \mu^{**})$ and η^{**} for (9.3.3).

Step 1. Consider the map $F: \mathbb{R}_+^M \times \mathbb{R}_+^N \mapsto \mathbb{R}$,

$$F(b,c) = \sup_{\mu\in\mathbb{T}^N, \eta\in\mathbb{T}} \inf_{\mathbb{Q}\in\mathcal{Q}_f} \mathbb{E}_{\mathbb{Q}}[b(g-\bar{g}) + c(\mu(h)-\bar{h}) + \eta(\phi)].$$

Since for $(b,c),(b',c') \in \mathbb{R}_+^M \times \mathbb{R}_+^N$

$$\begin{aligned}|F(b,c) - F(b',c')| &\leq \sup_{\mu\in\mathbb{T}^N, \eta\in\mathbb{T}} \sup_{\mathbb{Q}\in\mathcal{Q}_f} \mathbb{E}_{\mathbb{Q}}[|b-b'||g-\bar{g}| + |c-c'||\mu(h)-\bar{h}|] \\ &\leq K(M+N)||(b,c)-(b',c')||,\end{aligned}$$

where $|b-b'| := (|b^1 - b'^1|, \ldots, |b^M - b'^M|)$ and similar for the other related terms, and $K>0$ is a constant such that

$$||g(\cdot)-\bar{g}||, ||h_t(\cdot)-\bar{h}||, ||\phi_t(\cdot)|| \leq K, \quad \forall (t,\omega) \in \{0,\ldots,T\} \times \Omega.$$

Hence F is continuous.

Step 2. Now take $\mathbb{Q} \in \mathcal{Q} \subset \mathcal{Q}_f$. Let

$$\varepsilon := \min_{1\le i\le M} \left\{\bar{g}^i - \mathbb{E}_{\mathbb{Q}} g^i\right\} \wedge \min_{1\le i\le N} \left\{\bar{h}^i - \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathbb{Q}} h^i_\tau\right\} > 0.$$

Then

$$\sup_{b\in\mathbb{R}^M_+, c\in\mathbb{R}^N_+} F(b,c) \ge F(0,0) \ge -K > -2K \ge \sup_{||(b,c)||>\frac{3K}{\varepsilon}} F(b,c).$$

As a consequence,

$$\sup_{b\in\mathbb{R}^M_+, c\in\mathbb{R}^N_+} F(b,c) = \sup_{||(b,c)||\le\frac{3K}{\varepsilon}} F(b,c).$$

By the continuity of F from Step 1, there exists $(b^{**}, c^{**}) \in \mathbb{R}^M_+ \times \mathbb{R}^N_+$, such that

$$\begin{aligned}
\pi_{am}(\phi) &= \sup_{b\in\mathbb{R}^M_+, c\in\mathbb{R}^N_+} F(b,c) = F(b^{**}, c^{**}) \\
&= \sup_{\mu\in\mathbb{T}^N, \eta\in\mathbb{T}} \inf_{\mathbb{Q}\in\mathcal{Q}_f} \mathbb{E}_{\mathbb{Q}}[b^{**}(g-\bar{g}) + c^{**}(\mu(h)-\bar{h}) + \eta(\phi)].
\end{aligned}$$

Step 3. For any $\mathbb{Q} \in \mathcal{Q}_f$, the map

$$\begin{aligned}
(\mu,\eta) &\mapsto \mathbb{E}_{\mathbb{Q}}[b^{**}(g-\bar{g}) + c^{**}(\mu(h)-\bar{h}) + \eta(\phi)] \\
&\quad = \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\left(b^{**}(g-\bar{g}) + c^{**}(\mu(h)-\bar{h}) + \eta(\phi)\right)\right]
\end{aligned}$$

is continuous under the weak star topology (or Baxter-Chacon topology, see e.g., [38]). Then the map

$$(\mu,\eta) \mapsto \inf_{\mathbb{Q}\in\mathcal{Q}_f} \mathbb{E}_{\mathbb{Q}}[b^{**}(g-\bar{g}) + c^{**}(\mu(h)-\bar{h}) + \eta(\phi)]$$

is upper semi-continuous under the weak star topology. By [38, Theorem 1.1], the set $\mathbb{T}^N \times \mathbb{T}$ is weak star compact. Hence there exists $(\mu^{**}, \eta^{**}) \in \mathbb{T}^N \times \mathbb{T}$, such that

$$\begin{aligned}
\pi_{am}(\phi) &= \sup_{\mu\in\mathbb{T}^N, \eta\in\mathbb{T}} \inf_{\mathbb{Q}\in\mathcal{Q}_f} \mathbb{E}_{\mathbb{Q}}[b^{**}(g-\bar{g}) + c^{**}(\mu(h)-\bar{h}) + \eta(\phi)] \\
&= \inf_{\mathbb{Q}\in\mathcal{Q}_f} \mathbb{E}_{\mathbb{Q}}[b^{**}(g-\bar{g}) + c^{**}(\mu^{**}(h)-\bar{h}) + \eta^{**}(\phi)] \\
&= \sup\{x :\ \exists(H,a),\ \text{s.t. } \Phi_{\bar{g},\bar{h}}(H,a,b^{**},c^{**},\mu^{**}) + \eta^{**}(\phi) \ge x\},
\end{aligned}$$

where we apply the Superhedging Theorem in [19] for the third line. By the same theorem in [19], there exists (H^{**}, a^{**}) such that

$$\Phi_{\bar{g},\bar{h}}(H^{**}, a^{**}, b^{**}, c^{**}, \mu^{**}) + \eta^{**}(\phi) \geq \pi_{am}(\phi).$$

□

9.6 Proof of Theorem 9.3.8

Proof of Theorem 9.3.8. Recall $\mathcal{C}(x)$ defined in (9.3.4) and $\mathcal{D}(x)$ defined in (9.3.5), and denote $\mathcal{C} := \mathcal{C}(1)$ and $\mathcal{D} := \mathcal{D}(1)$. By Theorems 3.1 and 3.2 in [63], it suffices to show that $\mathcal{C}$ and $\mathcal{D}$ have the following properties:

1) $\mathcal{C}(1)$ and $cD(1)$ are convex, solid, and closed in the topology of convergence in measure.

2) For $p \in \mathbb{L}^0_+$,

$$p \in \mathcal{C} \Longleftrightarrow \mathbb{E}_{\mathbb{P}}[pq] \leq 1 \quad \text{for } \forall q \in \mathcal{D}.$$

For $q \in \mathbb{L}^0_+$,

$$q \in \mathcal{D} \Longleftrightarrow \mathbb{E}_{\mathbb{P}}[pq] \leq 1 \quad \text{for } \forall p \in \mathcal{C}.$$

3) $\mathcal{C}$ is bounded in probability and contains the identity function $\mathbb{1}$.

It is easy to see that $\mathcal{C}$ and $\mathcal{D}$ are convex and solid, $\mathbb{E}_{\mathbb{P}}[pq] \leq 1$ for any $p \in \mathcal{C}$ and $q \in \mathcal{D}$, and $\mathcal{C}$ contains the function $\mathbb{1}$. We shall prove the rest of the properties in three parts.

Part 1. We shall show $\mathcal{C}$ is bounded in probability. Take $\mathbb{Q} \in \mathcal{Q}$. Then $d\mathbb{Q}/d\mathbb{P} \in \mathcal{D}$, and

$$\sup_{p \in \mathcal{C}} \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} p\right] = \sup_{p \in \mathcal{C}} \mathbb{E}_{\mathbb{Q}} p \leq 1.$$

Therefore, we have that

$$\begin{aligned}
&\sup_{p\in\mathcal{C}} \mathbb{P}(p > C) \\
&= \sup_{p\in\mathcal{C}} \mathbb{P}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}p > \frac{d\mathbb{Q}}{d\mathbb{P}}C\right) \\
&= \sup_{p\in\mathcal{C}} \left[\mathbb{P}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}p > \frac{d\mathbb{Q}}{d\mathbb{P}}C,\ \frac{d\mathbb{Q}}{d\mathbb{P}} \le \frac{1}{\sqrt{C}}\right) + \mathbb{P}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}p > \frac{d\mathbb{Q}}{d\mathbb{P}}C,\ \frac{d\mathbb{Q}}{d\mathbb{P}} > \frac{1}{\sqrt{C}}\right)\right] \\
&\le \mathbb{P}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \le \frac{1}{\sqrt{C}}\right) + \sup_{p\in\mathcal{C}} \mathbb{P}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}p > \sqrt{C}\right) \\
&\le \mathbb{P}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \le \frac{1}{\sqrt{C}}\right) + \frac{1}{\sqrt{C}} \\
&\to 0, \quad C \to \infty.
\end{aligned}$$

Part 2. We shall show that for $p \in \mathbb{L}^0_+$, if $\mathbb{E}_\mathbb{P}[pq] \le 1$ for any $q \in \mathcal{D}$, then $p \in \mathcal{C}$, and as a consequence, $\mathcal{C}$ is closed under the topology of convergence in measure. Take $p \in \mathbb{L}^0_+$ satisfying $\mathbb{E}_\mathbb{P}[pq] \le 1$ for any $q \in \mathcal{D}$. It is easy to see that for any $\mathbb{Q} \in \mathcal{Q}$, the process $(\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t})_{t=0,\dots,T}$ is in $\mathcal{Y}(1)$. Therefore,

$$\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_\mathbb{Q} p = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_\mathbb{P}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}p\right] \le 1.$$

Thanks to Theorem 9.3.5, there exists (H, a, b, c, μ) such that

$$1 + \Phi_{\bar{g},\bar{h}}(H, a, b, c, \mu) \ge p,$$

which implies that $p \in \mathcal{C}$.

Now let $(p^n)_{n=1}^\infty \subset \mathcal{C}$ such that $p^n \xrightarrow{\mathbb{P}} p$. Then without loss of generality, we assume that $p^n \to p$ a.s.. For any $q \in \mathcal{D}$, we have that

$$\mathbb{E}_\mathbb{P}[pq] \le \liminf_{n\to\infty} \mathbb{E}_\mathbb{P}[p^n q] \le 1.$$

This implies $p \in \mathcal{C}$.

Part 3. We shall show that for $q \in \mathbb{L}^0_+$, if $\mathbb{E}_\mathbb{P}[pq] \le 1$ for any $p \in \mathcal{C}$, then $q \in \mathcal{D}$, and as a consequence, $\mathcal{D}$ is closed under the topology of convergence in measure.

Take $q \in \mathbb{L}^0_+$ satisfying $\mathbb{E}_{\mathbb{P}}[pq] \leq 1$ for any $p \in \mathcal{C}$. Since

$$\mathcal{C} \supset \{p' \in \mathbb{L}^0_+ : \ p' \leq 1 + H \cdot S, \text{ for some } H \in \mathcal{H}\},$$

by [63, Proposition 3.1] there exists a nonnegative adapted process $Y' = (Y'_t)_{t=0,\dots,T}$, such that $q \leq Y'_T$, and for any $H \in \mathcal{H}$ with $1 + H \cdot S \geq 0$, $((1 + (H \cdot S)_t)Y'_t)_{t=0,\dots,T}$ is a $\mathbb{P}$-super-martingale. Now define

$$Y_t = \begin{cases} Y'_t, & t = 0, \dots, T-1, \\ q, & t = T. \end{cases}$$

Then it can be shown that $Y = (Y_t)_{t=0,\dots,T} \in \mathcal{Y}(1)$. Since $q = Y_T$, $q \in \mathcal{D}$. Similar to the argument in Part 2, we can show that $\mathcal{D}$ is closed under the topology of convergence in measure. $\square$

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