A Priori Estimates for Two-Dimensional Water Waves with Angled Crests

by

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Abstract

We consider the two-dimensional water wave problem in the case where the free interface of the fluid meets a vertical wall at a possibly non-trivial angle; our problem also covers interfaces with angled crests. We assume that the fluid is inviscid, incompressible, and irrotational, with no surface tension and with air density zero. We construct a low-regularity energy and prove a closed energy estimate for this problem. Our work differs from earlier work in that, in our case, only a degenerate Taylor stability criterion holds, with $\frac{-\partial P}{\partial \mathbf{n}} \geqslant 0$, instead of the strong Taylor stability criterion $\frac{-\partial P}{\partial \mathbf{n}} \geqslant c > 0$.

Chapter 1

Introduction

1.1 Water Wave Problems

A class of water wave problems concerns the dynamics of the free surface separating a zero-density region (air) from an incompressible fluid (water) either of infinite depth or bounded by a fixed rigid boundary (e.g., the ocean bottom or the coast), under the influence of gravity.

Let $\Omega(t)$ be the fluid region, $\Sigma(t)$ be the free surface between the fluid and the air, and Υ (if it exists) be the fixed rigid boundary, for time $t \geqslant 0$; thus $\partial(\Omega(t)) = \Sigma(t) \cup \Upsilon$. We will henceforth assume that the fluid is not only incompressible but also irrotational, and we will neglect surface tension and viscosity. Assume that the fluid density is 1. If the gravity field is $-\mathbf{j}$, the governing equations of motion are

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\mathbf{j} - \nabla P \quad \text{on } \Omega(t) \tag{1}$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{on } \Omega(t) \tag{2}$$

$$\operatorname{curl} \mathbf{v} = 0 \quad \text{on } \Omega(t) \tag{3}$$

$$P = 0 \quad \text{on } \Sigma(t) \tag{4}$$

$$(1, \mathbf{v}) \text{ is tangent to the free surface } (t, \Sigma(t)) \tag{5}$$

$$\mathbf{v} \text{ is tangent to the fixed boundary } \Upsilon \text{ (if it exists),} \tag{6}$$

where $\mathbf{v}$ is the fluid velocity and P is the fluid pressure. We also assume that

$$\mathbf{v}(x, y) \to 0 \text{ as } y \to -\infty. \tag{7}$$

An important quantity governing the stability of these problems is $\frac{-\partial P}{\partial \mathbf{n}}$, where $\mathbf{n}$ is the outward unit normal vector to $\Omega(t)$ along $\Sigma(t)$.

1.2   Previous Research and Our Program

The study of water waves dates back centuries; early mathematical works include those by Stokes [Sto47], Levi-Civita [LC25], and Taylor [Tay50]. [Nal74], [Yos82], and [Cra85] obtained early local-in-time existence results for the two-dimensional water wave problem for small data. In 1997, [Wu97] proved, for the infinite depth two-dimensional water wave problem (1)-(5) and (7) ($\Upsilon = \emptyset$), that the Taylor stability criterion $\frac{-\partial P}{\partial \mathbf{n}} \geqslant c > 0$ always holds and that the problem is locally well-posed in Sobolev spaces $H^s, s > 4$, for arbitrary data. [Wu99] proved a similar result in three dimensions. Since then, there have been numerous results on local well-posedness in both two and three dimensions, for the water wave equations with nonzero vorticity [CL00] [Lin05], with a fixed bottom [Lan05], and with nonzero surface tension [AM05] [CS07]. [ABZar] proved local well-posedness of the problem for interfaces in $C^{3/2+\varepsilon}$.[1] All of these works assume the strong Taylor stability criterion $\frac{-\partial P}{\partial \mathbf{n}} \geqslant c > 0$. In addition, in the past several years [Wu09], [Wu11], [GMS12], [IP13], [AD13], and [HIT14] have produced results showing almost global or global well-posedness in two and three dimensions.

All of these results assume either infinitely deep water or else a fixed bottom that is a positive distance away from the free interface. In actual oceans, of course, the free surface does intersect with the rigid boundary, e.g., on the coast. It is important, therefore, to study the water wave problem in this setting. In this dissertation, we will begin such a program, by considering the two-dimensional water wave equations in the presence of a vertical wall that interacts with the free interface.

In addition to addressing the interaction of the free surface with the rigid vertical wall, our research covers singularities in water waves away from the rigid boundary. Indeed, by Schwarz reflection, a model of waves making a non-trivial angle with a rigid vertical wall corresponds to a symmetric angled crest in the middle of the ocean. (See Figure 1.) More generally, our work applies to water waves with very low regularity, including those with

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[1]A recent preprint, [ABZ14], improves on these results using Strichartz estimates, reducing the required smoothness of the interface to a fraction under $C^{3/2}$. The recent work of [HIT14] also has a low-regularity result that improves on [ABZar] for two-dimensional water waves. Both [ABZ14] and [HIT14] work in the setting where the strong Taylor stability criterion $-\frac{\partial P}{\partial \mathbf{n}} \geqslant c > 0$ holds.
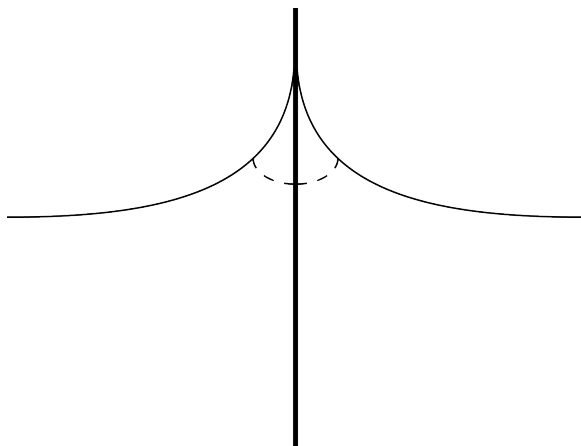
Figure 1: Symmetric Angled Crests. Under a Schwarz reflection, a non-trivial angle at a vertical wall corresponds to a symmetric angled crest in the middle of water.

angled crests.[2] Our work differs from the low-regularity result of [ABZar], which assumes that the strong Taylor stability criterion $\frac{-\partial P}{\partial \mathbf{n}} \geqslant c > 0$ holds and considers surfaces in $C^{3/2+\varepsilon}$. In our case, we allow surfaces that are not even in C^1 where only a degenerate Taylor stability criterion $\frac{-\partial P}{\partial \mathbf{n}} \geqslant 0$ holds.[3] Our research relates to recent work of Wu [Wu12], who constructed a class of convection-dominated self-similar solutions of the two-dimensional water wave equation with such angled crests. Our work indicates that the water wave equations (1)-(7) admit solutions with this type of singularity.

1.3   The Problem

For simplicity, we consider a rigid boundary consisting of two vertical walls, with water of infinite depth in between the walls. Working in two dimensions $(x, y) \in \mathbb{R}^2$, we assume that our fixed walls are at $x = 0, 1$, with the fluid region $\Omega_0(t) \subset [0, 1] \times (-\infty, c)$ for some $c < \infty$.

We assume that the water surface is flat at the corner $x = 0$ (i.e., that the surface makes a right angle with the vertical wall), but we allow a non-trivial (i.e., not necessarily 90°) angle at $x = 1$.[4]

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[2]We note that these angled crests don't have to be symmetric.

[3]In [ABZar], Alazard et al. consider the water wave problem in the presence of a bottom bounded away from the interface. In this framework, the strong Taylor stability condition doesn't necessarily hold, so it is assumed. In our case, the only rigid boundary is the vertical wall and we can prove that the degenerate Taylor stability condition always holds. The degeneracy occurs at singularities in the interface and when the water wave meets the wall with a non-trivial angle.

[4]We note that our problem can be viewed as an idealized contact angle problem.
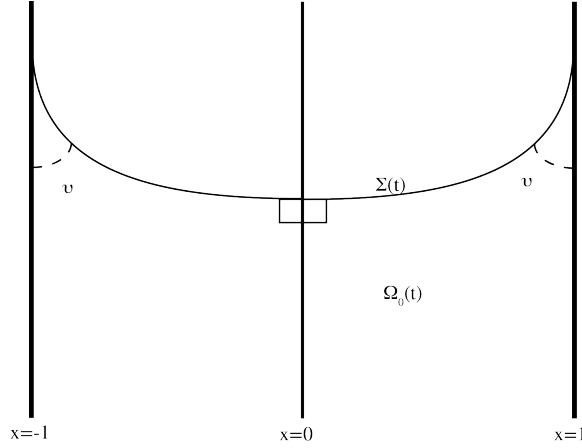
Figure 2: The Geometric Framework. We begin with solid walls at $x = 0, 1$, with a possibly non-trivial angle ν at $x = 1$ and a right angle at $x = 0$. We then symmetrize the problem using Schwarz reflection, so there are non-trivial angles ν at the walls $x = \pm 1$.

This angle will be of fundamental importance for the remainder of the dissertation. We will use ν to denote this angle. (See Figure 2.) We will henceforth say that the water surface is *flat* or the angle is *trivial* at the corner if $\nu = 90° = \frac{\pi}{2}$; otherwise, we will say that the angle at the corner is *non-trivial*.[5] As our terminology suggests, if the angle is trivial, then existing techniques ought to apply, so long as there are no singularities in the middle of the free surface. What's novel about our work is that it extends to the non-trivial regime, and allows singularities in the middle of the free surface.

For the remainder of the paper, we will often frame the problem in terms of singularities at the vertical wall—i.e., a non-trivial angle ν at $x = \pm 1$. Nevertheless, the energy estimate we prove applies more generally to singularities—e.g., waves with angled crests—in the middle of the free surface, away from the walls.

We begin by symmetrizing the problem via a Schwarz reflection to expand the domain across the y-axis, using the fact that (6) implies that $v_1(0, y) \equiv 0$. Precisely, for $\mathbf{v} = (v_1, v_2)$ and $x \in [0, 1]$, we define

$$\mathbf{v}(-x, y) = (-v_1(x, y), v_2(x, y)). \tag{8}$$

It is easy to check that equations (1) through (7) continue to hold in the expanded domain, with $v_1(0, y)$ continuing to be zero in what is now considered part of the water. Henceforth, we shall work exclusively in this reflected domain $\Omega(t)$. $\Omega(t) \subset [-1, 1] \times (-\infty, c)$ is symmetric

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[5]We will see in §3.7.1 (183) that the angle ν must be no more than $\frac{\pi}{2}$. Therefore a non-trivial angle ν is necessarily $< \frac{\pi}{2}$.

with respect to the y-axis, with solid vertical walls Υ at $x = -1, 1$. When we say the *wall* we will mean $x \equiv \pm 1$, and when we say the *corners*, we will mean the corner of the water surface at $x = \pm 1$. (See Figure 2.)

The non-trivial angle the water wave makes with the wall and the singularities in the surface introduce significant technical challenges.

In [Wu97], a key result was proving that the Taylor sign condition $\frac{-\partial P}{\partial \mathbf{n}} \geqslant c > 0$ always holds as long as the surface is smooth and non-self-intersecting. In the situation where $\frac{-\partial P}{\partial \mathbf{n}} < 0$, [Ebi87] has shown that the free boundary problem is not well-posed, a fact which is not surprising, since it is the positive sign of $\frac{-\partial P}{\partial \mathbf{n}}$ that gives the hyperbolicity of the equation. In our situation, however, we will have that $\frac{-\partial P}{\partial \mathbf{n}} = 0$ at the wall when there is a non-trivial angle and at the points on the surface where there are singularities (e.g., angled crests). In this case, the water wave equation is degenerate hyperbolic, and it is harder to construct an energy that can be closed.[6]

A second challenge comes from the potential asymmetry of certain quantities at the left and right boundaries of the domain. Thanks to the Schwarz reflection, we can now treat our problem as being periodic. The non-trivial angle of the free surface with the wall introduces a fundamental asymmetry: if the surface is sloping downwards near $x = -1$, it is sloping upwards near $x = 1$. Therefore, the value of certain quantities at $x = -1$ will not agree with their values at $x = 1$. This leads to problems both with integration by parts and with handling estimates of various commutators involving the (periodic) Hilbert transform. This challenge actually proves to be a blessing in disguise, because it directs us to precisely the quantities that *are* well-behaved as periodic functions, quantities which turn out to behave very well when others have singularities.

A third challenge also proves to have a silver lining. [Wu97] relied on the Riemann mapping to flatten out the fluid domain. When the angle of the free surface with the wall is not 90°, or when there is an angled crest in the middle of the free surface, the Riemann mapping has a singularity. Instead of avoiding the Riemann mapping, we embrace it, since it gives us heuristics to determine which quantities we might or might not expect to be finite. We used these heuristics, and more importantly the self-similar solution constructed in [Wu12], to guide us in the construction of our energy.

In this dissertation, we prove an a priori estimate for solutions of the water wave equation in this framework. We follow the general approach of Wu's earlier papers [Wu97] and [Wu99], in reducing the water wave problem to an equation on the free surface, differentiating with

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[6]We remark that in this case the water wave equation is also degenerately dispersive.

respect to time to get a quasilinear equation, and then using the fact that the velocity is antiholomorphic to express the nonlinearities in terms of commutators involving the Hilbert transform. The novelty is that we introduce a new energy that relies on two different singular weights and controls some natural holomorphic quantities. Our estimates *do not* depend on any positive lower bound for $\frac{-\partial P}{\partial \mathbf{n}}$.[7] Substantial technical difficulties exist due to the very low regularity involved in the energy and the non-existence of a positive lower bound of $\frac{-\partial P}{\partial \mathbf{n}}$. Nevertheless, we have overcome these challenges.

Our energy inequality is a crucial step towards proving local well-posedness in this framework. This will be the focus of an upcoming paper by Wu.

We mention that our energy is finite for the self-similar waves constructed in [Wu12].

1.4 Outline of the Dissertation

In the next section, §1.5, we present some of the notations and conventions used in the dissertation. In §1.6 we introduce the function spaces and norms we will use. Then, in §2, we introduce the Lagrangian framework for the water wave problem, following [Wu97]. In particular, in §2.4, we discuss some of the boundary challenges mentioned above, and introduce a special weighted derivative that we will use in our energies.

In §3, we introduce the Riemann mapping and use it to derive a form of the equation in "Riemannian coordinates." We also introduce the Hilbert transform in these coordinates and prove various technical results about the transform.

In §4, we present various technical results that we will need for the proof.

Finally, in §5, we define the energy we will use (in §5.1) and state our result, the a priori inequality (in §5.3). We offer some discussion in §5.2 about our choice of energy. We begin the proof in §5.4, and then in §5.5 we outline the remainder of the proof, which takes up §6 through §10.

In §11 we discuss the strength of the energy. In §11.1, we give a characterization of the energy in terms of the velocity and the free surface. In §11.2, we offer a general discussion of the heuristics that the Riemann mapping gives about singularities for the equation, including what they suggest about the angle ν.

We have two appendices that might be useful to the reader. In §A, we provide an overview of the notation used in the dissertation, with cross references to where everything

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[7]We will show that in our framework $\frac{-\partial P}{\partial \mathbf{n}} \geqslant 0$. In fact, a positive lower bound on $\frac{-\partial P}{\partial \mathbf{n}}$ is roughly equivalent to a non-singular Riemann mapping; see section §3.7.1.

was initially defined. In §B, we list various quantities controlled by the energy, again with cross references.

1.5   Notation and Conventions

We will define most of our notations throughout the text, as we introduce our various quantities. Here, we list some general conventions and notations, and give references to the sections where we introduce substantial amounts of new notation. The appendix §A presents a summary of all of the important notations in the dissertation.

In §1.3 above, we introduced the basic geometry and defined ν, the angle the free surface makes with the wall. In §1.6 we define the function spaces and norms we will use. In §2 we introduce the basic notations for our various quantities, in what will be Lagrangian coordinates, with basic variables α and β. In §3, we introduce the analogues of those quantities in Riemannian coordinates, where our basic variables will be α' and β'. In §3.2, we will introduce a special notation for partial derivatives in these coordinates. Except as mentioned there, an expression $f_x(x,y)$ means $\partial_x f(x,y)$. (We will also occasionally use the notation f'; this is always the spatial derivative in whatever coordinates we are using.) To save space, we sometimes omit the variables; when we do so, $f_\alpha := f(\alpha,t), f_\beta := f(\beta,t)$, etc. Finally, §5.1, we introduce and define our energy.

We will often use commutators. We define

$$[A,B] := AB - BA. \tag{9}$$

When we use such notation for $[f,\mathbb{H}]g$, where f and g are functions and $\mathbb{H}$ is the Hilbert transform, f will be treated as a multiplication operator: thus $[f,\mathbb{H}]g = f(\mathbb{H}g) - \mathbb{H}(fg)$.

We will use $\Re(x+iy) = x$ and $\Im(x+iy) = y$ to represent the real and imaginary parts, respectively, of a complex number.

Compositions are always in terms of the spatial valuables. For example, for $f = f(\alpha,t)$, $g = g(\alpha,t)$, we define $f \circ g = f \circ g(\alpha,t) := f(g(\alpha,t),t)$.

We will use C as a placeholder to refer to a universal constant, possibly varying from line to line. We will also often use the notation $f \lesssim g$, which means that there exists some universal constant C such that $f \leqslant Cg$.

Once we have reduced the water wave equations to an equation in one spatial dimension, we will primarily be working with the spatial domain $I := [-1,1]$. We will often speak of the

"boundary"; this refers to what happens at ± 1. We write $f|_\partial := f(1) - f(-1)$; therefore, $f|_\partial = 0$ if $f(1) = f(-1)$.

We will use

$$\fint_I f := \frac{1}{|I|} \int_I f(x)dx = \frac{1}{2} \int_{-1}^1 f(x)dx \tag{10}$$

for the mean of a function f. Here, and elsewhere for other integrals, we will often drop the subscript I when there is no risk of ambiguity.

We will use the following notation as an abbreviation for a type of higher-order Calderon commutator:

$$[f, g; h](\alpha') := \frac{1}{2i} \int \frac{f(\alpha') - f(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \frac{g(\alpha') - g(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} h(\beta')d\beta'. \tag{11}$$

We will at several points have long series of identities or inequalities. When we say "on the RHS" of an equation block with a string of multiple equalities or inequalities, we mean all the terms on the right hand side of the last equality or inequality sign in the string. Similarly, when we say "on the LHS," we mean all the terms to the left of the very first equality or inequality sign in the string of equalities and inequalities. We have tried to avoid saying "on the nth line" when any of the previous mathematical "lines" splits into more than one typographic line, but if we have, "line" refers to the mathematical, not typographic, line.

We have tried to give extensive cross references for each time we use a result or estimate. We tend to refer to equation numbers, rather than propositions, since it seems that these will be easier to find as cross references. When we refer to an equation number as part of a proposition, we are of course referring to the whole proposition, including any conditions assumed.

When we are deriving estimates, we sometimes use the cross references within our equations, e.g.:

$$f \leqslant g \tag{12}$$
$$\leqslant h$$

and

$$h \lesssim j + f$$
$$\lesssim j + (12) \tag{13}$$
$$\lesssim j + h.$$

This means (12) is used to obtain (13). We hope this will help the reader locate the previous estimate or estimates.

In several of our more complicated estimates, we will split terms up $f = I + II$ and then $I = I_1 + I_2$, $I_1 = I_{11} + I_{12}$, etc. Such notation will be *local to each chapter*. There is an ambiguity between the use of I as a placeholder, its use as the identity operator, and its use as $I := [-1, 1]$. It should be clear from the context which one is being used.

1.6 Function Spaces and Norms

We introduce here the function spaces and norms we will use. We work with functions $f(\cdot, t)$ defined on $I = [-1, 1]$. Except when necessary to avoid ambiguity, we neglect to write the time variable.

We say $f \in C^k(J)$, $J = (-1, 1)$ or $[-1, 1]$, if for every $0 \leqslant l \leqslant k$, $\partial_x^l f$ is a continuous function on the interval J. We say $f \in C^k(S^1)$ (i.e., periodic C^k) if for every $0 \leqslant l \leqslant k$, $\partial_x^l f \in C^0[-1, 1]$ and $\partial_x^l f(1) = \partial_x^l f(-1)$. ($\partial_x^l f$ at the end points 1 or -1 is defined to be either the left- or right-sided derivative.) Note in particular that saying $f \in C^0(S^1)$ implies that $f|_\partial = 0$.

Let $p \geqslant 1$. We define our L^p spaces by the norms

$$\|f\|_{L^p} := \|f\|_{L^p(I)} := \left(\int_{[-1,1]} |f|^p \right)^{1/p} \tag{14}$$

(and analogously for $p = \infty$). Note that $f \in C^1(-1, 1)$ and $f' \in L^p(I)$ doesn't imply $f(1) = f(-1)$.

We will sometimes deal with weighted L^p spaces. We write

$$\|f\|_{L^p(\omega)} = \|f\|_{L^p(\omega dx)} := \left(\int_I |f|^p \, \omega(x) dx \right)^{1/p} \tag{15}$$

for weights $\omega \geqslant 0$.

Whenever we write L^p, we will be referring to $L^p(I)$, in the spatial variable. Similarly, whenever we write an integral, if the domain is not specified, it is $I = [-1, 1]$. When we prove our estimates for the Hilbert transform in terms of the estimates for the transform on $\mathbb{R}$, we will use $L^p(I)$ and $L^p(\mathbb{R})$ to distinguish domains. For weighted L^p spaces, we always write $L^p(\omega)$, where ω is the weight function.

Note that if $f \in C^0(S^1)$, with $f' \in L^p$, then if we extend f periodically to $(-3, 3)$, the weak derivative of the extended function is the periodic extension of f' to $(-3, 3)$ and is in

$L^p(-3, 3)$.[8]

We define the homogeneous half-derivative space $\dot{H}^{1/2}$ by the norm

$$\|f\|_{\dot{H}^{1/2}} := \left(\frac{\pi}{8} \iint_{I \times I} \frac{|f(\alpha') - f(\beta')|^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} d\alpha' d\beta' \right)^{1/2}. \tag{18}$$

We do so initially for $f \in C^k(S^1)$ for large enough k, and then complete the norm. By Hardy's inequality (217) below, the definition (18) above makes sense for $f \in C^0(S^1)$ with $f' \in L^2$.

For our Hilbert transform in the periodic setting, we will need to define the principal value, which we do using the following distance function. For $x, y \in I$, we define

$$d_{S^1}(x, y) := \min(|x - y|, |x - y + 2|, |x - y - 2|). \tag{19}$$

This is a periodic distance that ensures that if, say, x is close to -1 and y is close to 1 then $d_{S^1}(x, y)$ is small. We then write

$$I_\varepsilon = I_\varepsilon(\alpha') := \{\beta' \in I : d_{S^1}(\alpha', \beta') > \varepsilon\} \tag{20}$$

and

$$(I \times I)_\varepsilon := \{(\alpha', \beta') \in I \times I : d_{S^1}(\alpha', \beta') > \varepsilon\}. \tag{21}$$

We then define

$$\mathrm{pv} \int_I f(\alpha', \beta') d\beta' := \lim_{\varepsilon \to 0} \int_{I_\varepsilon(\alpha')} f(\alpha', \beta') d\beta'. \tag{22}$$

In almost all circumstances, except where there is ambiguity, we will simply write I_ε instead

[8]To see this, let $f \in C^0(\mathbb{R})$, and assume that f' is the weak derivative of f on $(0, \infty)$ and on $(-\infty, 0)$ and that f' is in $L^p(\mathbb{R})$. Let g be the distributional derivative of f on $\mathbb{R}$. Then $\int_\mathbb{R} g\varphi = -\int_\mathbb{R} f\varphi'$ for all $\varphi \in C^\infty$ compactly supported. It suffices to show that $g = f'$ on $\mathbb{R}$. For any $\varepsilon > 0$, let $\psi_\varepsilon(x) = \psi(\varepsilon^{-1}x)$ be a C^∞ function, supported in $(-\varepsilon, \varepsilon)$, and equal to 1 on $(-\varepsilon/2, \varepsilon/2)$. Then

$$\int_\mathbb{R} f'\varphi = \int_\mathbb{R} f'\varphi\psi_\varepsilon + \int_0^\infty f'\varphi(1 - \psi_\varepsilon) + \int_{-\infty}^0 f'\varphi(1 - \psi_\varepsilon). \tag{16}$$

The first term goes to 0 as $\varepsilon \to 0$ since $f' \in L^p$. The second term is

$$\int_0^\infty f'\varphi(1 - \psi_\varepsilon) = -\int_0^\infty f\varphi'(1 - \psi_\varepsilon) + \int_0^\varepsilon f\varphi\varepsilon^{-1}\psi'(\varepsilon^{-1}\cdot). \tag{17}$$

As $\varepsilon \to 0$ this goes to $-\int_0^\infty f\varphi' - f(0)\varphi(0)$ because f is continuous. Similarly, as $\varepsilon \to 0$, the third term goes to $-\int_{-\infty}^0 f\varphi' + f(0)\varphi(0)$. This proves $g = f'$ on $\mathbb{R}$.

of $I_\varepsilon(\alpha')$, since it will be clear from the integral which is the variable of integration.

Chapter 2

The Lagrangian Framework

2.1 The Geometry and the Framework

We now introduce a *Lagrangian* parametrization of the free surface $\Sigma(t)$. Since we are in two dimensions, we will often work in complex coordinates $(x, y) = x + iy$. We determine a parametrization

$$z(\alpha, t) = x(\alpha, t) + iy(\alpha, t) \tag{23}$$

for $\alpha \in I := [-1, 1]$ by

$$z_t(\alpha, t) = \mathbf{v}(z(\alpha, t), t). \tag{24}$$

We choose our initial parametrization to be in arc length coordinates:

$$|z_\alpha(\alpha, 0)| \equiv 1. \tag{25}$$

Note that in these Lagrangian coordinates z_{tt} is the material derivative forming the LHS of the Euler equation (1). Along the free surface, therefore, the Euler equation (1) is

$$z_{tt} + i = -\nabla P. \tag{26}$$

The incompressibility and irrotationality conditions (2) and (3) imply that the conjugate velocity is holomorphic; therefore $\overline{z}_t$ is the boundary value of a holomorphic function in the fluid region. Through the remainder of the dissertation, except when there's a risk of confusion, we will slightly abuse notation and say that a function f on the surface is "holomorphic" (or anti-holomorphic); what we mean, precisely, is that it is the boundary

value of a function that is holomorphic (or anti-holomorphic) in the fluid region.[9]

2.2 Derivation of the Surface Equation

By condition (4), we know that ∇P is orthogonal to the free surface. Since z_α is tangent to the free surface, iz_α is normal. We can therefore rewrite our main equation as

$$z_{tt} + i = i\mathfrak{a}z_\alpha, \tag{27}$$

where

$$\mathfrak{a} = -\frac{\partial P}{\partial \mathbf{n}} \frac{1}{|z_\alpha|} \in \mathbb{R}, \tag{28}$$

for $\frac{\partial P}{\partial \mathbf{n}}$ the outward-facing normal derivative. Although z will not of course remain an arc length parametrization, we will show in §6.2 that, so long as our energy is finite,

$$0 < c_1 \leqslant |z_\alpha| \leqslant c_2 < \infty. \tag{29}$$

Therefore, for the Taylor stability criterion, we can focus on $\mathfrak{a}$ instead of $\frac{-\partial P}{\partial \mathbf{n}}$, and we will henceforth refer to either $\mathfrak{a}$ or $\frac{-\partial P}{\partial \mathbf{n}}$ as the *Taylor coefficient*. Recall that in [Wu97] it was essential to show the *strong Taylor sign criterion*, which we can state in terms of $\mathfrak{a}$ as $\mathfrak{a} \geqslant c_0 > 0$. Here we will not be lucky, since we will only have $\mathfrak{a} \geqslant 0$.

Indeed, since ∇P is orthogonal to the free surface, we have, treating complex numbers as vectors in $\mathbb{R}^2$,

$$0 = \langle -\nabla P, z_\alpha \rangle = \langle (z_{tt} + i), z_\alpha \rangle, \tag{30}$$

so

$$x_{tt}x_\alpha = -y_\alpha(y_{tt} + 1). \tag{31}$$

Therefore,

$$\frac{y_\alpha}{x_\alpha} = \frac{-x_{tt}}{y_{tt} + 1}. \tag{32}$$

Along the wall, we know that $x_t \equiv 0$ by (6), so we have $x_{tt} \equiv 0$. Observe that the angle ν

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[9] As will be clear from §3.3, it's important that these holomorphic functions be periodic as well (and, usually, that they go to zero as $y \to \infty$). When we speak of something being "holomorphic" in this sense, we implicitly are requiring that it be periodic as well.

between the water surface and the wall $x = -1$ is given by

$$\tan \nu = \frac{x_\alpha}{-y_\alpha}, \tag{33}$$

so

$$\nu = \arctan\left(\frac{x_\alpha}{-y_\alpha}\right) = \arctan\left(\frac{y_{tt} + i}{x_{tt}}\right). \tag{34}$$

If $(y_{tt} + i)/x_{tt}$ is infinite, then the angle must be $90°$. Therefore, the only possible way for the angle of the water surface with the wall to be other than a right angle is if the numerator $y_{tt} + 1$ is zero at the corner. This implies that $\mathfrak{a} = 0$ at the corner.

We will prove below, in §3.7.1, that $\mathfrak{a} \geqslant 0$.[10] Because $\mathfrak{a}$ is non-negative, we can therefore conclude that

$$\mathfrak{a} = \frac{|z_{tt} + i|}{|z_\alpha|}. \tag{35}$$

We note that (27) and (35) imply that

$$\begin{aligned}
\frac{z_{tt} + i}{|z_{tt} + i|} &= i\frac{z_\alpha}{|z_\alpha|} \\
\frac{\bar{z}_{tt} - i}{|\bar{z}_{tt} - i|} &= -i\frac{\bar{z}_\alpha}{|z_\alpha|}.
\end{aligned} \tag{36}$$

2.3   The Quasilinear Equation

We henceforth focus on the equations on the free surface.[11]   As in [Wu97] and following works, we differentiate the basic equation (27) with respect to time and take conjugates, turning it into the quasilinear equation

$$\bar{z}_{ttt} + i\mathfrak{a}\bar{z}_{t\alpha} = -i\mathfrak{a}_t\bar{z}_\alpha, \tag{37}$$

where we continue to have $\bar{z}_t$ the boundary value of a holomorphic function. This is the basic equation we will work with throughout the dissertation.

The holomorphicity of $\bar{z}_t$ implies that $\mathfrak{a}\partial_\alpha$ can be written as $\mathfrak{a}\partial_\alpha\mathfrak{H}$, where $\mathfrak{H}$ is the Hilbert transform with respect to the free surface $\Sigma(t)$.[12]   Because $i\partial_\alpha\mathfrak{H}$ is a nonnegative operator, a

---

[10]Equation (185) in §3.7.1 uses the Riemann mapping to give a precise correspondence between points where $\mathfrak{a} = 0$ and points where there is a singularity on the interface.

[11]We may solve for the velocity on $\Omega(t)$ from its boundary values (including the condition that it goes to zero as $y \to -\infty$), and then solve for the pressure from the velocity.

[12]This Hilbert transform has kernel $\frac{1}{2i}z_\beta(\beta, t)\cot(\frac{\pi}{2}(z(\alpha, t) - z(\beta, t)))$.

natural positive energy can be defined.

The holomorphicity of $\overline{z}_t$ also implies that the RHS is lower-order. In [Wu99] and [Wu09], coordinate-independent formulas for the RHS were derived, using the invertibility of the double-layer potential. We will instead follow the original approach of [Wu97], relying on the Riemann mapping version of the equation to derive the RHS. We do so in §3.7.2.

2.4   Boundary Behavior and the Derivative D_α

Because we are working on a compact domain, we will have to worry about boundary terms when integrating by parts.[13] Indeed, it will be critical that various potentially problematic boundary terms either disappear or can be controlled. For this reason, we need to take care in specifying what the boundary properties of our various functions are.

Our hope is that we are dealing with functions $f(\alpha, t)$ that satisfy

$$f|_\partial := f(1,t) - f(-1,t) = 0. \tag{38}$$

The key property of our basic function $\overline{z}_t = x_t - iy_t$, which has real part odd and imaginary part even, is that the real part x_t is nonetheless zero at the corners, by (6). Therefore,

$$\overline{z}_t|_\partial = 0. \tag{39}$$

Because this boundary difference is constant over time, (39) implies that

$$\overline{z}_{tt}|_\partial = 0. \tag{40}$$

Also, this doesn't change under conjugation, so

$$z_t|_\partial = 0 \tag{41}$$

and

$$z_{tt}|_\partial = 0. \tag{42}$$

Unfortunately, periodic boundary behavior is not preserved under spatial differentiation.

---

[13]The issue arises as well in certain commutators of the Hilbert transform, where a failure of boundary agreement corresponds to a delta function in the derivative, when treating the domain as periodic.

Indeed, although the operator

$$\partial_\alpha = x_\alpha \partial_x + y_\alpha \partial_y \tag{43}$$

simply flips parity (because x_α is even and y_α is odd, and ∂_x flips parity while ∂_y preserves parity), there's no reason for it to preserve the periodicity. Thus, $\partial_\alpha \bar{z}_t$ should have real part even and imaginary part that's odd but not necessarily zero at the boundary.

The key observation is that the conjugate velocity $\bar{\mathbf{v}}$ is both holomorphic and periodic in the domain $\Omega(t)$, since $\mathbf{v}(1, y) = \mathbf{v}(-1, y)$ for all y. For such periodic and holomorphic functions, ∂_y preserves both periodicity and holomorphicity.

Therefore, if we choose a spatial derivative to apply to $\bar{z}_t$ on $\Sigma(t)$ that corresponds to ∂_y inside the domain, we will preserve periodic boundary behavior and holomorphicity. Note that for holomorphic functions $\partial_z = -i\partial_y$. We therefore look for the derivative on the surface $\Sigma(t)$ that corresponds to ∂_z inside the domain.

This derivative is

$$D_\alpha := \frac{1}{z_\alpha} \partial_\alpha. \tag{44}$$

Indeed, if $g(\alpha, t) = G(z(\alpha, t), t)$, and G is holomorphic, then

$$\partial_\alpha g = (G_z \circ z) z_\alpha \tag{45}$$

so

$$D_\alpha g = (\partial_z G) \circ z = (-i\partial_y G) \circ z. \tag{46}$$

Thus $D_\alpha^k g$ is the boundary value of the function $\partial_z^k G$ and therefore is periodic for any $k \geqslant 1$, so long as G is periodic and holomorphic.

We may therefore conclude that

$$D_\alpha^k \bar{z}_t \big|_\partial = 0, \text{ for any } k \geqslant 0. \tag{47}$$

This suggests using D_α as the main derivative for our energy. In addition to preserving periodic boundary behavior and preserving holomorphicity, it transforms well under the Riemann mapping, to be discussed in the next chapter.

We note that (47) implies that

$$\partial_t D_\alpha^k \bar{z}_t \big|_\partial = 0, \tag{48}$$

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$$\left.\overline{D_\alpha^k \overline{z}_t}\right|_\partial = 0, \tag{49}$$

and

$$\left.\overline{\partial_t D_\alpha^k \overline{z}_t}\right|_\partial = 0. \tag{50}$$

We caution that this periodic boundary behavior does *not* necessarily hold for $D_\alpha^k \overline{z}_{tt}$ or $D_\alpha^k z_t$, because the holomorphicity is crucial and disappears under differentiation by t or conjugation.[14]

We do, however, have

$$\Re D_\alpha z_t|_\partial = 0. \tag{51}$$

This will be used in §8.2. Here, the $\Re$ is the savior. Note that $\Re D_\alpha \overline{z}_t$ is even and $\Im D_\alpha \overline{z}_t$ is odd; this is because D_α flips the parity of the real and imaginary parts, and $\Re \overline{z}_t$ is odd and $\Im \overline{z}_t$ is even. Therefore, taking conjugates, we know that $\Re \frac{z_{t\alpha}}{\overline{z}_\alpha}$ is even and $\Im \frac{z_{t\alpha}}{\overline{z}_\alpha}$ is odd. We write

$$
\begin{aligned}
\Re(D_\alpha z_t) &= \Re\left(\frac{\overline{z}_\alpha}{z_\alpha}\frac{z_{t\alpha}}{\overline{z}_\alpha}\right) \\
&= \left(\Re\frac{\overline{z}_\alpha}{z_\alpha}\right)\left(\Re\frac{z_{t\alpha}}{\overline{z}_\alpha}\right) - \left(\Im\frac{\overline{z}_\alpha}{z_\alpha}\right)\left(\Im\frac{z_{t\alpha}}{\overline{z}_\alpha}\right).
\end{aligned}
\tag{52}
$$

We calculate that

$$
\begin{aligned}
\frac{\overline{z}_\alpha}{z_\alpha} &= \frac{x_\alpha - iy_\alpha}{x_\alpha + iy_\alpha}\frac{x_\alpha - iy_\alpha}{x_\alpha - iy_\alpha} \\
&= \frac{x_\alpha^2 - y_\alpha^2 - 2ix_\alpha y_\alpha}{x_\alpha^2 + y_\alpha^2}.
\end{aligned}
\tag{53}
$$

Because x_α is even and y_α is odd, we conclude that $\Re\frac{\overline{z}_\alpha}{z_\alpha}$ is even and $\Im\frac{\overline{z}_\alpha}{z_\alpha}$ is odd. Therefore, each of the terms on the RHS of (52) are even. This concludes the proof of (51).

We will defer the remaining discussions of boundary properties to §3.4 in the chapter on the Riemann mapping.

---

[14]Indeed, for g the boundary value of an antiholomorphic function, $D_\alpha g = i\frac{\overline{z}_\alpha}{z_\alpha}\partial_y g$, as can be calculated using the antiholomorphic Cauchy-Riemann equations.

Chapter 3

The Riemann Mapping Version

We now introduce a version of the water wave equations using the Riemann mapping to flatten out the curved free interface.[15] The Riemann mapping version of the equations present certain advantages. Because we are working on a flat domain, our Hilbert transform is now the traditional Hilbert transform $\mathbb{H}$ defined by

$$\mathbb{H}f(\alpha') := \frac{1}{2i} \, \mathrm{pv} \int_I \cot(\frac{\pi}{2}(\alpha' - \beta'))f(\beta')d\beta'. \tag{54}$$

(The cotangent kernel appears instead of the $\frac{1}{x-y}$ kernel because we are on a periodic domain.) Because $\mathbb{H}f \in i\mathbb{R}$ for f real-valued, this allows us to invert the operator $(I - \mathbb{H})$ on purely real (resp., purely imaginary) functions by taking real (resp., imaginary) parts. This is far simpler than the approach in Lagrangian coordinates used by [Wu99] in three dimensions,[16] where the Hilbert transform $\mathfrak{H}$ is complex-valued, and $(I - \mathfrak{H})$ must be inverted using the double-layer potential.

The challenge with using the Riemann mapping in our problem is that the non-trivial angle at the corner and the singularity in the middle of the interface creates singularities in the Riemann mapping. This differs from previous work (e.g., [Wu97]), where there were no such singularities. On the other hand, the Riemann mapping can be helpful in that it gives us heuristic information about the nature of our singularities. We discuss this further in §11.2.

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[15]The idea of using Riemann mapping to study the well-posedness of 2-d water waves dates back to [Wu97].
[16]Note that there is no Riemann mapping in three dimensions.
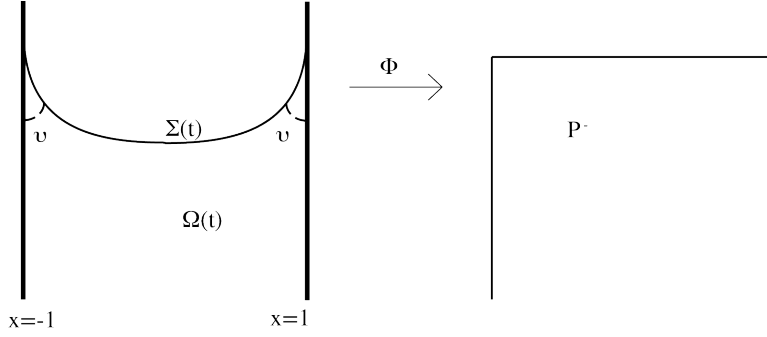
Figure 3: The Riemann Mapping. The Riemann mapping on the reflected domain.

3.1 The Riemann Mapping

We begin by defining our mapping.

Proposition 1 (Riemann Mapping Theorem). *Let $\Omega_0(t) \subset [0,1] \times (-\infty, c)$ be the unreflected fluid domain. Then there exists a unique Riemann mapping $\Phi_0 : \mathring{\Omega}_0(t) \to \mathring{P}_0^- := \{(x,y) : x \in (0,1), y < 0\}$ that extends continuously to the boundaries and sends the two upper corners to $(0,0)$ and $(1,0)$ and ∞ to ∞ (i.e., $\Im \lim_{y \to -\infty} \Phi_0(x + iy) = -\infty$).*

We have uniqueness by specifying where three points go. The continuity of the extension is due to Carathéodory; for full details, see [GK02], Theorem 13.2.3.

We further extend this Riemann mapping by Schwarz reflection to a (unique) map

$$\Phi : \Omega(t) \to P^- := \{(x,y) : x \in [-1,1], y \leqslant 0\} \tag{55}$$

on the reflected domain including boundaries; henceforth, this will be our definition of the Riemann mapping Φ. (See Figure 3.)

3.2 The Riemannian Coordinates and Notation

We now define the Riemann mapping parametrization of our problem. We previously defined our Lagrangian parametrization $z(\alpha, t) : I \to \Sigma(t)$. We define a change of coordinates

$$
\begin{aligned}
&h(\alpha, t) := \Phi(z(\alpha, t), t) : I \to I \\
&h(\alpha, t) : \alpha \mapsto \alpha'.
\end{aligned}
\tag{56}
$$

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We will use α' and β' for the variables in the Riemannian coordinates (on the flattened domain) and α and β for the variables in the Lagrangian coordinates (on the curved domain).

We define

$$Z(\alpha', t) := z \circ h^{-1}(\alpha', t) = z(h^{-1}(\alpha', t), t), \tag{57}$$

where h^{-1} is defined by

$$h(h^{-1}(\alpha', t), t) = \alpha'. \tag{58}$$

We observe that in the new coordinates

$$\partial_{\alpha'}(f \circ h^{-1}) = \frac{f_\alpha}{h_\alpha} \circ h^{-1}. \tag{59}$$

We write

$$Z_t := z_t \circ h^{-1}; Z_{tt} := z_{tt} \circ h^{-1} \tag{60}$$

and

$$Z_{,\alpha'} = \partial_{\alpha'} Z = \frac{z_\alpha}{h_\alpha} \circ h^{-1}; Z_{t,\alpha'} = \frac{z_{t\alpha}}{h_\alpha} \circ h^{-1}; Z_{tt,\alpha'} := \frac{z_{tt\alpha}}{h_\alpha} \circ h^{-1}, \tag{61}$$

and likewise for conjugates. (Observe that when using the subscript notation, a $_t$ subscript always refers to the time derivative in Lagrangian coordinates and precedes any $_{,\alpha'}$, which refers to a derivative in Riemannian coordinates.)

Observe that $Z = z \circ h^{-1} = \Phi^{-1}$. Therefore

$$Z_{,\alpha'}(\alpha', t) = \partial_{\alpha'}(\Phi^{-1}(\alpha', t)) \tag{62}$$

and

$$\frac{1}{Z_{,\alpha'}} = \Phi_z \circ Z. \tag{63}$$

Also, since $\Phi(\Phi^{-1}(\alpha', t), t) = \alpha'$, we have $(\Phi_z \circ Z) \cdot (\Phi^{-1})_t + \Phi_t \circ Z = 0$, and therefore

$$(\Phi^{-1})_t = (-Z_{,\alpha'})(\Phi_t \circ Z). \tag{64}$$

We write

$$\mathcal{A} := (\mathfrak{a} h_\alpha) \circ h^{-1} \tag{65}$$

and

$$\mathcal{A}_t := (\mathfrak{a}_t h_\alpha) \circ h^{-1}. \tag{66}$$

Observe that our derivative $D_\alpha := \frac{1}{z_\alpha}\partial_\alpha$ behaves well under the change of variables:

$$(D_\alpha f) \circ h^{-1} = \frac{1}{Z_{,\alpha'}}\partial_{\alpha'}(f \circ h^{-1}). \tag{67}$$

We therefore define

$$D_{\alpha'} := \frac{1}{Z_{,\alpha'}}\partial_{\alpha'}. \tag{68}$$

3.3 The Hilbert Transform $\mathbb{H}$

We now introduce the Hilbert transform in the Riemannian coordinates.

The Hilbert transform $Hf(x) := \text{pv} \int \frac{f(y)}{x-y}dy$ on $\mathbb{R}$ relates the boundary values of the real and imaginary parts of a holomorphic function on the upper half-plane. We will use a slight variant of this Hilbert transform. Our Hilbert transform works in the periodic rather than infinite domain, so the $\frac{1}{x-y}$ kernel is replaced by a cotangent kernel. We also normalize our Hilbert transform differently: we choose an imaginary normalization of our Hilbert transform $\mathbb{H}$ so that $\mathbb{H}f = f + c$ for holomorphic functions, and we have a different sign because the physical nature of the water wave problems means we work in the lower rather than upper half-plane.

We define our Hilbert transform for $\alpha' \in [-1, 1]$ by

$$\mathbb{H}f(\alpha') := \frac{1}{2i}\text{pv}\int_I \cot(\frac{\pi}{2}(\alpha' - \beta'))f(\beta')d\beta' = \frac{1}{2i}\lim_{\varepsilon \to 0}\int_{I_\varepsilon}\cot(\frac{\pi}{2}(\alpha' - \beta'))f(\beta')d\beta'. \tag{69}$$

(Recall from §1.6 that $I_\varepsilon = \{\beta' \in I : d_{S^1}(\alpha', \beta') > \varepsilon\}$, where d_{S^1} is a periodic distance for I.)

We initially define this for functions $f \in C^0(S^1) \cap C^1[-1, 1]$. Then,

$$\begin{aligned}
\frac{1}{2i}\text{pv}\int_I \cot(\frac{\pi}{2}(\alpha' - \beta'))f(\beta')d\beta' &= \frac{1}{2i}\text{pv}\int_I \cot(\frac{\pi}{2}(\alpha' - \beta'))(f(\beta') - f(\alpha'))d\beta' \\
&= \frac{1}{2i}\int_I \cot(\frac{\pi}{2}(\alpha' - \beta'))(f(\beta') - f(\alpha'))d\beta',
\end{aligned} \tag{70}$$

where the integral makes sense without the principal value. (We've used the fact that $\int_{I_\varepsilon}\cot(\frac{\pi}{2}(\alpha' - \beta'))d\beta' = 0$ for any α' in the first line.) We then extend our definition to L^p by the $L^p \to L^p$ boundedness of $\mathbb{H}$ for $1 < p < \infty$:

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Proposition 2. *Let $1 < p < \infty$. Then there exists $C_p > 0$ such that for all $f \in L^p$*

$$\|\mathbb{H}f\|_{L^p} \leqslant C_p \|f\|_{L^p}. \tag{71}$$

Proof. This is a classical result. We present a proof for the sake of completeness.

We begin by proving the result for $f \in C^0(S^1) \cap C^1[-1, 1]$. Then by the density of $C^0(S^1) \cap C^1[-1, 1]$ in L^p the result holds for all $f \in L^p$.

We could prove our estimate directly for our cotangent kernel, but instead we will simply show how it follows from the result on $\mathbb{R}$, via the classical identity

$$\frac{\pi}{2} \cot(\frac{\pi}{2}x) = \sum_{l \in \mathbb{Z}} \frac{1}{x + 2l}. \tag{72}$$

Here, and in any remaining use of these sums, the infinite summation is summable when added symmetrically:

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} \frac{1}{x + 2k} &= \lim_{N \to \infty} \sum_{|k| \leqslant N} \frac{1}{x + 2k} \\
&= \lim_{N \to \infty} \left(\frac{1}{x} + \sum_{1 \leqslant k \leqslant N} \frac{1}{x + 2k} + \frac{1}{x - 2k} \right) \\
&= \lim_{N \to \infty} \left(\frac{1}{x} + \sum_{1 \leqslant k \leqslant N} \frac{2x}{x^2 - 4k^2} \right).
\end{aligned} \tag{73}$$

We write

$$\begin{aligned}
\mathbb{H}f(\alpha') &= \frac{1}{2i} \operatorname{pv} \int_I \cot(\frac{\pi}{2}(\alpha' - \beta')) f(\beta') d\beta' \\
&= \frac{1}{2i} \operatorname{pv} \int_I \frac{\pi}{2} \sum_{l \in \mathbb{Z}} \frac{1}{(\alpha' - \beta') + 2l} f(\beta') d\beta'.
\end{aligned} \tag{74}$$

We now split into the cases $|l| \leqslant 1$ and $|l| > 1$. For $l = 0$, we estimate

$$\begin{aligned}
\left\| \operatorname{pv} \int_I \frac{1}{(\alpha' - \beta')} f(\beta') d\beta' \right\|_{L^p(I)} &= \left\| \operatorname{pv} \int_{\mathbb{R}} \frac{1}{(\alpha' - \beta')} \chi_I(\beta') f(\beta') \right\|_{L^p(I)} \\
&\leqslant \left\| \operatorname{pv} \int_{\mathbb{R}} \frac{1}{(\alpha' - \beta')} \chi_I(\beta') f(\beta') \right\|_{L^p(\mathbb{R})} \\
&\lesssim \|\chi_I f\|_{L^p(\mathbb{R})} \\
&= \|f\|_{L^p(I)},
\end{aligned} \tag{75}$$

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where we have used the L^p boundedness of the Hilbert transform H on $\mathbb{R}$.

For $|l| = 1$, by periodical extending f to $(-3, 3)$ we have the same estimate.

For $|l| > 1$, by adding symmetrically, we have

$$
\begin{aligned}
\sum_{|l|>1} \frac{1}{(\alpha' - \beta') + 2l} &= \sum_{l>1} \left(\frac{1}{(\alpha' - \beta') + 2l} + \frac{1}{(\alpha' - \beta') - 2l} \right) \\
&= \sum_{l>1} \left(\frac{2(\alpha' - \beta')}{(\alpha' - \beta')^2 - 4l^2} \right) =: k(\alpha', \beta').
\end{aligned}
\tag{76}
$$

Observe that since $|\alpha' - \beta'| \leqslant 2$, there exists a constant $C > 0$ so that

$$
|k(\alpha', \beta')| \leqslant C
\tag{77}
$$

for any α', β'. Therefore,

$$
\left| \int_I f(\beta') k(\alpha', \beta') d\beta' \right| \leqslant C \int_I |f(\beta')|
\tag{78}
$$

and so

$$
\left\| \int_I f(\beta') k(\alpha', \beta') d\beta' \right\|_{L^p(I)} \leqslant \|f\|_{L^p(I)}
\tag{79}
$$

by Jensen's. Adding up the parts gives the inequality. $\qquad\square$

We observe that for any constant c,

$$
\mathbb{H}c = 0.
\tag{80}
$$

We now present an important proposition characterizing the boundary value of periodic holomorphic functions on $[-1, 1] \times (-\infty, 0)$.

Proposition 3. *Let $g \in L^p$ for some $p > 1$. Then*

$$
(I - \mathbb{H})g = c_0
\tag{81}
$$

if and only if g is the boundary value of a holomorphic function G on $[-1, 1] \times (-\infty, 0)$ satisfying $G(-1, y) = G(1, y)$ for all $y < 0$ and $G(x, y) \to c_0$ as $y \to -\infty$. Moreover, $c_0 = \fint_I g$.

Proof. We reduce this to the classical result for bounded domains.

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Let U be a C^1 bounded domain, and let ∂U be parametrized by $\zeta(\alpha)$, for $\alpha \in [-1, 1]$. Let $f : \partial U \to \mathbb{C}$, $f \in L^p(\partial U)$ for some $p > 1$. It is a well-known result that f is the boundary value of a holomorphic function in U if and only if

$$f(\zeta(\alpha)) = \frac{1}{\pi i} \operatorname{pv} \int \frac{f(\zeta(\beta))\zeta'(\beta)}{\zeta(\beta) - \zeta(\alpha)} d\beta. \tag{82}$$

(Here the RHS is the Hilbert transform for the domain U.)

Let Ψ be the holomorphic function from $(-1, 1) \times (-\infty, 0)$ to the unit disc given by

$$\Psi(\xi) = e^{-i\pi\xi}, \tag{83}$$

with the inverse

$$\Psi^{-1}(\zeta) = \frac{i}{\pi} \log \zeta. \tag{84}$$

(This is the Riemann mapping from $(-1, 1) \times (-\infty, 0)$ to the unit disc minus a slit; things will be well-defined on this slit since G is periodic.)

The function g is the boundary value of a holomorphic function G on $[-1, 1] \times (-\infty, 0)$ satisfying the periodic boundary and limit conditions if and only if $(g \circ \Psi^{-1})$ is the boundary value of a holomorphic function on the disc with $(G \circ \Psi^{-1})(0) = c_0$.

Consider a parametrization of the unit circle by

$$\zeta(\alpha') = e^{-i\pi\alpha'}. \tag{85}$$

By (82), $g \circ \Psi^{-1}$ is the boundary value of a holomorphic function on the unit disc if and only if

$$\left(g \circ \Psi^{-1}\right)(\zeta(\alpha')) = \frac{1}{\pi i} \operatorname{pv} \int_I \frac{\left(g \circ \Psi^{-1}\right)(\zeta(\beta'))\zeta'(\beta')}{\zeta(\beta') - \zeta(\alpha')} d\beta'. \tag{86}$$

We expand this out

$$
\begin{aligned}
g(\alpha') &= g\left(\frac{i}{\pi}\log(e^{-i\pi\alpha'})\right) \\
&= \left(g \circ \Psi^{-1}\right)(\zeta(\alpha')) \\
&= \frac{1}{\pi i}\,\mathrm{pv}\int_I \frac{\left(g \circ \Psi^{-1}\right)(\zeta(\beta'))\zeta'(\beta')}{\zeta(\beta') - \zeta(\alpha')}d\beta' \\
&= \frac{1}{\pi i}\,\mathrm{pv}\int_I g(\beta')\frac{-i\pi e^{-i\pi\beta'}}{e^{-i\pi\beta'} - e^{-i\pi\alpha'}}d\beta' \\
&= -\,\mathrm{pv}\int_I g(\beta')\frac{e^{-i\pi\beta'}}{e^{-i\pi\beta'} - e^{-i\pi\alpha'}}d\beta' \\
&= \mathrm{pv}\int_I g(\beta')\frac{1}{1 - e^{i\pi(\beta'-\alpha')}}d\beta' \\
&= \mathrm{pv}\int_I g(\beta')\frac{1 - \cos(\pi\,(\beta' - \alpha')) + i\sin(\pi(\beta' - \alpha'))}{2(1 - \cos(\pi(\beta' - \alpha')))}d\beta' \\
&= \frac{1}{2}\int_I g(\beta')d\beta' + \frac{i}{2}\,\mathrm{pv}\int g(\beta')\frac{1}{\tan(\frac{\pi}{2}(\beta' - \alpha'))}d\beta' \\
&= \fint_I g + \frac{1}{2i}\,\mathrm{pv}\int_I g(\beta')\cot(\frac{\pi}{2}(\alpha' - \beta'))d\beta' \\
&= \fint_I g + \mathbb{H}g,
\end{aligned}
\tag{87}
$$

where we have used the trigonometric identities

$$
\frac{1}{1 - e^{ix}} = \frac{1 - \cos(x) + i\sin(x)}{2(1 - \cos(x))}
\tag{88}
$$

and

$$
\frac{\sin(x)}{(1 - \cos(x))} = \frac{1}{\tan(x/2)}.
\tag{89}
$$

By the mean value theorem, $c_0 = \fint_I g$. $\qquad\square$

Proposition 4. *Let $f \in L^p$ for some $p > 1$. Then*

$$
\mathbb{H}^2 f = f - \fint f.
\tag{90}
$$

In particular, $\mathbb{H}^2 = I$ on mean-zero functions.

Proof. By replacing f with $f - \fint f$, it suffices to prove $\mathbb{H}^2 = I$ when f is mean zero. This

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follows from the Fourier series representation of $\mathbb{H}$, since $\widehat{\mathbb{H}f} = -\operatorname{sgn}(n) \cdot \hat{f}$, where sgn is the signum function.[17] $\square$

Proposition 5 (Adjoint of $\mathbb{H}$). *Let $f \in L^p$, $g \in L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. Then*

$$\int f(\mathbb{H}g) = -\int (\mathbb{H}f)g. \tag{91}$$

Proof. Formally, this is immediate by Fubini; the only subtlety is handling the principal value condition correctly. We begin by assuming a priori that $f, g \in C^1(S^1)$. Then

$$\begin{aligned}
\int f(\alpha')(\mathbb{H}g(\alpha'))d\alpha' &= \frac{1}{2i} \int f(\alpha') \operatorname{pv} \int \cot(\frac{\pi}{2}(\alpha' - \beta'))g(\beta')d\beta' d\alpha' \\
&= \lim_{\varepsilon \to 0} \frac{1}{2i} \int_I \int_{I_\varepsilon(\alpha')} f(\alpha') \cot(\frac{\pi}{2}(\alpha' - \beta'))g(\beta')d\beta' d\alpha' \\
&= \lim_{\varepsilon \to 0} \frac{1}{2i} \int_I \int_{I_\varepsilon(\beta')} f(\alpha') \cot(\frac{\pi}{2}(\alpha' - \beta'))g(\beta')d\alpha' d\beta' \\
&= \frac{1}{2i} \int g(\beta') \operatorname{pv} \int \cot(\frac{\pi}{2}(\alpha' - \beta'))f(\alpha')d\alpha' d\beta' \\
&= -\int g(\beta')(\mathbb{H}f(\beta'))d\beta'.
\end{aligned} \tag{92}$$

Here, to justify the interchange of limits and integration, we use Lebesgue dominated convergence theorem: because $f \in L^\infty$ and $g \in C^1(S^1)$, $\left| f(\alpha') \int_{I_\varepsilon(\alpha')} g(\beta') \cot(\frac{\pi}{2}(\alpha' - \beta'))d\beta' \right| \leqslant C > 0$ uniform in ε,[18] and similarly when we change the limits the second time, with f and g reversed. Note that there were no issues in applying Fubini because that was done when $d_{S^1}(\alpha', \beta') > \varepsilon$.

We conclude by using the denseness of $C^1(S^1)$ in L^p and L^q to generalize to arbitrary $f \in L^p$, $g \in L^q$. $\square$

We define the following projection operators:

$$\mathbb{P}_A f := \frac{(I - \mathbb{H})}{2} f; \mathbb{P}_H f := \frac{(I + \mathbb{H})}{2} f \tag{93}$$

We will refer to $\mathbb{P}_A$ as the "antiholomorphic projection" and $\mathbb{P}_H$ as the "holomorphic projection." $\mathbb{P}_H$ (respectively $\mathbb{P}_A$) projects any mean-zero function onto a part that is the boundary value of a periodic holomorphic (respectively, antiholomorphic) function in $[-1, 1] \times (-\infty, 0)$

---

[17]See, for example, [Kat04], for details.

[18]We again use the trick of rewriting $g(\beta') = g(\beta') - g(\alpha')$ and then difference quotient estimates.

with limit 0 as $y \to -\infty$. Note that if a function is the boundary value of an antiholomorphic function in $[-1, 1] \times (-\infty, 0)$, it is the boundary value of a holomorphic function in the *upper* space $[-1, 1] \times (0, \infty)$.

Proposition 6. *Let $f \in L^p$ for some $p > 1$. Then*

$$\mathbb{P}_A^2 f = \mathbb{P}_A f - \frac{1}{4} \int f, \mathbb{P}_H^2 f = \mathbb{P}_H f - \frac{1}{4} \int f \tag{94}$$

and

$$\mathbb{P}_A \mathbb{P}_H f = \frac{1}{4} \int f; \mathbb{P}_H \mathbb{P}_A f = \frac{1}{4} \int f. \tag{95}$$

Proof. We can prove (94) easily by (90):

$$\mathbb{P}_A^2 f = \frac{(I^2 + \mathbb{H}^2) - 2\mathbb{H}}{4} f = \mathbb{P}_A f - \frac{1}{4} \int f. \tag{96}$$

The proof for $\mathbb{P}_H$ is analogous.

To prove (95) we again expand and use (90):

$$\mathbb{P}_A \mathbb{P}_H f = \frac{(I^2 - \mathbb{H}^2)}{4} f = \frac{1}{4} \int f, \tag{97}$$

and analogously for $\mathbb{P}_H \mathbb{P}_A$. $\qquad \square$

Proposition 7. *Let $f \in L^p, g \in L^q$ for $\frac{1}{p} + \frac{1}{q} = 1, 1 < p, q < \infty$. Then*

$$\int (\mathbb{P}_A f) g = \int f (\mathbb{P}_H g). \tag{98}$$

Proof. This is immediate by the adjointness property (91). $\qquad \square$

Proposition 8. *Let $f \in L^p, g \in L^q$ for $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} < 1$. Then*

$$\mathbb{P}_A \left\{ (\mathbb{P}_H f)(\mathbb{P}_H g) \right\} = \frac{1}{2} \int \left\{ (\mathbb{P}_H f)(\mathbb{P}_H g) \right\} = \frac{1}{2} \int \left\{ f (\mathbb{P}_A \mathbb{P}_H g) \right\} = \frac{1}{8} \left(\int f \right) \left(\int g \right). \tag{99}$$

Proof. Note that our assumptions ensure that the product $(\mathbb{P}_H f)(\mathbb{P}_H g) \in L^p$ for some $p > 1$. By (95) and Proposition 3, $\mathbb{P}_H f$ and $\mathbb{P}_H g$ are the boundary values of holomorphic functions. Therefore, their product is the boundary value of a holomorphic function, so by Proposition 3 again, we have the first equality. The second equality holds by (98), and the third by (95). $\qquad \square$

We also note the following commutators:

Proposition 9. *Let $f \in C^0[-1, 1] \cap C^1(-1, 1)$ with $f' \in L^p$ for some $p > 1$.[19] Then*

$$if \ f|_\partial = 0, \ then \ [\partial_{\alpha'}, \mathbb{H}]f = 0. \tag{100}$$

If $f|_\partial \neq 0$, then

$$[\partial_{\alpha'}, \mathbb{H}]f = -\frac{1}{2i} \cot(\frac{\pi}{2}(\alpha' - \beta'))f(\beta')\Big|_\partial \tag{101}$$

in $L^2(\omega^2)$ for any weight ω satisfying $\omega \in L^\infty, \partial_{\alpha'}\omega \in L^2$.

This holds in particular for $\omega = \frac{1}{|Z_{,\alpha'}|}$, since we will show that $\frac{1}{Z_{,\alpha'}} \in L^\infty, \partial_{\alpha'}\frac{1}{Z_{,\alpha'}} \in L^2$ in our energy.

Proof. Formally, these follow immediately by integration by parts. The only subtlety is in justifying the formal derivation rigorously.

We begin by showing (100). We show this assuming initially that $f \in C^2(S^1)$. By change of variables and periodicity, we can write

$$\mathbb{H}f = \frac{1}{2i} \int_I \cot(\frac{\pi}{2}\beta')(f(\alpha' - \beta') - f(\alpha'))d\beta'. \tag{102}$$

We have

$$\partial_{\alpha'}\mathbb{H}f = \frac{1}{2i} \lim_{h \to 0} \int_I \cot(\frac{\pi}{2}\beta')\frac{(f(\alpha' - \beta' + h) - f(\alpha' - \beta')) - (f(\alpha' + h) - f(\alpha'))}{h}d\beta'. \tag{103}$$

If we can bring the limit inside the integral we will be done, since that is $\mathbb{H}\partial_{\alpha'}f$. Therefore, it suffices to show that the integrand is bounded on I, independent of h, so that we may use the Lebesgue dominated convergence theorem. We rewrite the integrand as

$$\frac{(f(\alpha' - \beta' + h) - f(\alpha' + h))\cot(\frac{\pi}{2}\beta')) - (f(\alpha' - \beta') - f(\alpha'))\cot(\frac{\pi}{2}\beta')}{h}$$
$$= \frac{g(\alpha', \beta', h) - g(\alpha', \beta', 0)}{h}, \tag{104}$$

where

$$g(x, y, z) := (f(x - y + z) - f(x + z))\cot(\frac{\pi}{2}y). \tag{105}$$

---

[19]Recall from §1.6 that $f' \in L^p$ allows the possibility for f to differ at ± 1.

By the mean value difference quotient estimate for complex-valued functions (see (213) below), (104) is universally bounded by $\|g_z(\alpha', \beta', \cdot)\|_{L^\infty}$. We estimate

$$
\begin{aligned}
|g_z| &= \left| \frac{(f'(x-y+z) - f'(x+z))}{y} y \cot(\frac{\pi}{2}y) \right| \\
&\lesssim \|f''\|_{L^\infty} ,
\end{aligned}
\tag{106}
$$

where we have used the fact that $y \cot(\frac{\pi}{2}y)$ is in $L^\infty(-1,1)$. This completes the proof that $[\partial_{\alpha'}, \mathbb{H}]f = 0$ for $f \in C^2(S^1)$. By distribution theory, this holds for $(C^2(S^1))'$, where derivatives are interpreted periodically. If $f \in C^0(S^1)$ with $f' \in L^p$, the periodic derivative coincides with the derivative on I, so $[\partial_{\alpha'}, \mathbb{H}]f = 0$ in the non-periodic sense.[20]

To show (101), we begin by rewriting $f(\alpha') = (f - \frac{f|_\partial}{2}\alpha') + \frac{f|_\partial}{2}\alpha'$. Because we have shown (100), it remains only to show that

$$
[\partial_{\alpha'}, \mathbb{H}]\alpha' = -\frac{1}{2i} \cot(\frac{\pi}{2}(\alpha' - \beta'))\beta' \Big|_\partial
\tag{107}
$$

in $L^2(\omega^2)$.

Note that $[\partial_{\alpha'}, \mathbb{H}]\alpha' = \partial_{\alpha'}\mathbb{H}\alpha' - \mathbb{H}(\partial_{\alpha'}\alpha') = \partial_{\alpha'}\mathbb{H}\alpha'$. (Here we are working non-periodically, so there are no delta functions and therefore $\partial_{\alpha'}\alpha' = 1$ and $\mathbb{H}1 = 0$.)

Let φ be a smooth test function in $L^2(-1,1)$ such that $\|\varphi\|_{L^2} \leqslant 1$ and φ is compactly

---

[20]We remark that this may also be shown using Fourier series, since both $\partial_{\alpha'}$ and $\mathbb{H}$ are multiplier operators.

supported in $(-1, 1)$ (so it is zero in a neighborhood of ± 1). Then

$$
\begin{aligned}
\int (\varphi\omega\partial_{\alpha'}\mathbb{H}\alpha')d\alpha' &= \int \left(\varphi\omega\partial_{\alpha'}\frac{1}{2i}\lim_{\varepsilon\to 0}\int_{|\alpha'-\beta'|>\varepsilon} \cot(\frac{\pi}{2}(\alpha'-\beta'))\beta'd\beta' \right) d\alpha' \\
&= \int \left(\varphi\omega\partial_{\alpha'}\frac{1}{2i}\lim_{\varepsilon\to 0}\int_{|\alpha'-\beta'|>\varepsilon} \cot(\frac{\pi}{2}(\alpha'-\beta'))(\beta'-\alpha')d\beta' \right) d\alpha' \\
&= -\int \left((\partial_{\alpha'}(\varphi\omega))\frac{1}{2i}\lim_{\varepsilon\to 0}\int_{|\alpha'-\beta'|>\varepsilon} \cot(\frac{\pi}{2}(\alpha'-\beta'))(\beta'-\alpha')d\beta' \right) d\alpha' \\
&= -\lim_{\varepsilon\to 0}\int \left((\partial_{\alpha'}(\varphi\omega))\frac{1}{2i}\int_{|\alpha'-\beta'|>\varepsilon} \cot(\frac{\pi}{2}(\alpha'-\beta'))(\beta'-\alpha')d\beta' \right) d\alpha' \\
&= \lim_{\varepsilon\to 0}\int \left(\varphi\omega\partial_{\alpha'}\frac{1}{2i}\int_{|\alpha'-\beta'|>\varepsilon} \cot(\frac{\pi}{2}(\alpha'-\beta'))(\beta'-\alpha')d\beta' \right) d\alpha' \\
&= \lim_{\varepsilon\to 0}\int \left(\varphi\omega\partial_{\alpha'}\frac{1}{2i}\int_{|\alpha'-\beta'|>\varepsilon} \cot(\frac{\pi}{2}(\alpha'-\beta'))\beta'd\beta' \right) d\alpha' \\
&= \lim_{\varepsilon\to 0}\int \left(\varphi\omega\frac{1}{2i}\int_{|\alpha'-\beta'|>\varepsilon} \partial_{\alpha'}\cot(\frac{\pi}{2}(\alpha'-\beta'))\beta'd\beta' \right) d\alpha' \\
&\quad + \lim_{\varepsilon\to 0}\int \left(\varphi\omega\frac{1}{2i}\left(-\cot(\frac{\pi}{2}(\alpha'-\beta'))\beta' \right)\Big|_{\alpha'-\varepsilon}^{\alpha'+\varepsilon} \right) d\alpha' \\
&= \lim_{\varepsilon\to 0}\int \left(\varphi\omega\frac{1}{2i}\int_{|\alpha'-\beta'|>\varepsilon} \left(-\partial_{\beta'}\cot(\frac{\pi}{2}(\alpha'-\beta')) \right)\beta'd\beta' \right) d\alpha' \\
&\quad + \lim_{\varepsilon\to 0}\int \left(\varphi\omega\frac{1}{2i}\left(-\cot(\frac{\pi}{2}(\alpha'-\beta'))\beta' \right)\Big|_{\alpha'-\varepsilon}^{\alpha'+\varepsilon} \right) d\alpha' \\
&= \lim_{\varepsilon\to 0}\int \left(\varphi\omega\frac{1}{2i}\int_{|\alpha'-\beta'|>\varepsilon} \cot(\frac{\pi}{2}(\alpha'-\beta'))(\partial_{\beta'}\beta')d\beta' \right) d\alpha' \\
&\quad - \lim_{\varepsilon\to 0}\int \left(\varphi\omega\frac{1}{2i}\left(\cot(\frac{\pi}{2}(\alpha'-\beta'))\beta' \right)\Big|_{-1}^{\alpha'-\varepsilon} + \left(\cot(\frac{\pi}{2}(\alpha'-\beta'))\beta' \right)\Big|_{\alpha'+\varepsilon}^{1} \right) d\alpha' \\
&\quad + \lim_{\varepsilon\to 0}\int \left(\varphi\omega\frac{1}{2i}\left(-\cot(\frac{\pi}{2}(\alpha'-\beta'))\beta' \right)\Big|_{\alpha'-\varepsilon}^{\alpha'+\varepsilon} \right) d\alpha' \\
&= \lim_{\varepsilon\to 0}\int \left(\varphi\omega\frac{1}{2i}\underbrace{\int_{|\alpha'-\beta'|>\varepsilon} \cot(\frac{\pi}{2}(\alpha'-\beta'))d\beta'}_{0} \right) d\alpha' \\
&\quad - \lim_{\varepsilon\to 0}\int \left(\varphi\omega\frac{1}{2i}\left\{ \cot(\frac{\pi}{2}(\alpha'-\beta'))\beta'\Big|_{\partial} \right\} \right) d\alpha' \\
&= \int \left(\varphi\omega\frac{1}{2i}\left\{ -\cot(\frac{\pi}{2}(\alpha'-\beta'))\beta'\Big|_{\partial} \right\} \right) d\alpha'.
\end{aligned}
$$

$$(108)$$

Here, to justify pulling the $\lim_{\varepsilon \to 0}$ outside the outer integral, we use Cauchy-Schwarz; the fact that $\partial_{\alpha'}(\varphi\omega) \in L^2$; and the fact that, because of φ's compact support in $(-1, 1)$, we know that $|\alpha' - \beta'| \leqslant c < 2$, so $\int_{|\alpha'-\beta'|>\varepsilon} \cot(\frac{\pi}{2}(\alpha' - \beta'))(\beta' - \alpha')d\beta'$ is uniformly bounded (for each fixed φ), since $x \cot(\frac{\pi}{2}x) \in C^0(-1, 1)$. This gives the result. $\square$

Finally, we will need the following proposition.

Proposition 10. *Suppose* $(I - \mathbb{H})f = \fint f$, *with* $f \in L^\infty$, *and let* g *be an arbitrary function in* L^p *for some* $p > 1$. *Then*

$$[f, \mathbb{H}]g = \mathbb{H}[f, \mathbb{H}]g - \fint (fg) + \left(\fint f\right)\left(\fint g\right). \tag{109}$$

Proof. We begin by noting that $f = \mathbb{P}_H f + \frac{1}{2}\fint f$. Observe that

$$\begin{aligned}[f, \mathbb{H}] g &= f\mathbb{H}g - \mathbb{H}(fg) \\ &= f(I + \mathbb{H})g - (I + \mathbb{H})(fg).\end{aligned} \tag{110}$$

Therefore,

$$\begin{aligned}(I - \mathbb{H})[f, \mathbb{H}]g &= (I - \mathbb{H})\left\{f(I + \mathbb{H})g - (I + \mathbb{H})(fg)\right\} \\ &= (I - \mathbb{H})\left\{f(I + \mathbb{H})g\right\} - (I - \mathbb{H})(I + \mathbb{H})(fg) \\ &= (I - \mathbb{H})\left\{\left(\mathbb{P}_H f + \frac{1}{2}\left(\fint f\right)\right)(I + \mathbb{H})g\right\} - (I - \mathbb{H})(I + \mathbb{H})(fg) \\ &= (I - \mathbb{H})\left\{(\mathbb{P}_H f)(I + \mathbb{H})g\right\} + \frac{1}{2}\left(\fint f\right)(I - \mathbb{H})(I + \mathbb{H})g \\ &\quad - (I - \mathbb{H})(I + \mathbb{H})(fg) \\ &= \frac{1}{2}\left(\fint f\right)\left(\fint g\right) + \frac{1}{2}\left(\fint f\right)\left(\fint g\right) - \fint (fg) \\ &= \left(\fint f\right)\left(\fint g\right) - \fint (fg)\end{aligned} \tag{111}$$

by (95) and (99). $\square$

3.4 Boundary Properties and the Riemann Mapping

In Riemannian coordinates, we will need to check that certain functions have periodic boundary behavior. By applying change of coordinates to (39), (40), (41), (42), (47), and (49), we have

$$\overline{Z}_t\big|_\partial = \overline{Z}_{tt}\big|_\partial = Z_t\big|_\partial = Z_{tt}\big|_\partial = 0 \tag{112}$$

and

$$D_{\alpha'}^k \overline{Z}_t\big|_\partial = \overline{D_{\alpha'}^k \overline{Z}_t}\Big|_\partial = 0 \text{ for } k \geqslant 0. \tag{113}$$

We list a few other boundary properties that we will need in the following. Some rely on quantities introduced in §3.7 and could properly be put there.

We also have

$$\frac{1}{Z_{,\alpha'}}\bigg|_\partial = 0; \tag{114}$$

this follows from (63) and the symmetry of the Riemann mapping Φ.

We also can state the following:

$$\frac{1}{Z_{,\alpha'}}(\pm 1) \equiv 0 \text{ or } \left\{ \text{ the angle } \nu \text{ is } 90° \text{ and } \frac{z_\alpha}{|z_\alpha|}\bigg|_\partial = 0 \right\}. \tag{115}$$

This holds because $\frac{1}{Z_{,\alpha'}} = 0$ when $\nu < \frac{\pi}{2}$; otherwise, $\nu = \frac{\pi}{2}$ and so $y_\alpha(\pm 1) \equiv 0$, and thus $z_\alpha\big|_\partial = 0$.[21]

We will at one point need the fact that

$$\left(\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'}\right)\big|_\partial = \left\{\overline{Z}_{ttt} + (Z_{tt} + i)D_{\alpha'}\overline{Z}_t\right\}\big|_\partial = 0. \tag{116}$$

This follows from the fact that each of the factors $\overline{Z}_{ttt}, (Z_{tt} + i)$, and $D_{\alpha'}\overline{Z}_t$ satisfies the periodic boundary conditions.

We will also at one point need the fact that

$$\frac{h_{t\alpha}}{h_\alpha} \circ h^{-1}\bigg|_\partial = 0. \tag{117}$$

This follows because h and therefore h_t are odd, and ∂_α flips parity.

We note that functions f satisfying $f\big|_\partial = 0$ form an algebra; any product of them will

---

[21]We will see below at (183) that $\nu \leqslant \frac{\pi}{2}$.

retain this behavior. In particular, we will often use

$$\frac{Z_t}{Z_{,\alpha'}}\bigg|_{\partial} = 0, \tag{118}$$

which follows from (112) and (114).

Proposition 11. *Let $f \in L^2, f' \in L^2$ with $f|_{\partial} = 0$. Then*

$$(\mathbb{H}f)|_{\partial} = 0. \tag{119}$$

Proof. By the L^2 boundedness of $\mathbb{H}$ and (100), $\mathbb{H}f \in L^2, (\mathbb{H}f)' \in L^2$. From Sobolev embeddings we know that $\mathbb{H}f$ is in $C^0[-1, 1]$. Now, for $f \in C^1[-1, 1] \cap C^0(S^1)$, we have, for all $\alpha' \in I$,

$$\mathbb{H}f(\alpha') = \frac{1}{2i} \int \cot(\frac{\pi}{2}(\alpha' - \beta'))(f(\beta') - f(\alpha'))d\beta'. \tag{120}$$

The periodicity of $\cot(\frac{\pi}{2}(\alpha' - \beta'))$ in α' gives (119). Our result for general f follows from the density of $C^1[-1, 1] \cap C^0(S^1)$ in $H^1[-1, 1] \cap C^0(S^1)$ (where $H^1[-1, 1]$ is defined by norm $\|f\|_{H^1} = \|f\|_{L^2} + \|f'\|_{L^2}$) and the L^2 boundedness of $\mathbb{H}$. $\qquad \square$

From this proposition, we can conclude that

$$\mathbb{H}\frac{Z_t}{Z_{,\alpha'}}\bigg|_{\partial}, \ \mathbb{P}_A\frac{Z_t}{Z_{,\alpha'}}\bigg|_{\partial}, \ \mathbb{P}_H\frac{Z_t}{Z_{,\alpha'}}\bigg|_{\partial} = 0. \tag{121}$$

To apply the proposition, we rely on (i) the fact that $\frac{Z_t}{Z_{,\alpha'}} \in L^2$ and $\partial_{\alpha'}\frac{Z_t}{Z_{,\alpha'}} \in L^2$, which will be true in our regime, since $D_{\alpha'}Z_t \in L^\infty, \partial_{\alpha'}\frac{1}{Z_{,\alpha'}} \in L^2$ when our energy is finite and we make the a priori assumption that $\frac{1}{Z_{,\alpha'}} \in C^0$ and $Z_t \in C^0$; and (ii) the boundary condition (118).

Finally, we note a boundary property that will come up in §3.6. Suppose $G(z, t)$ is a periodic holomorphic function in $\Omega(t)$. Then

$$G_t(z(\alpha, t), t)|_{\partial} = G_t(z(h^{-1}(\alpha', t), t), t)\big|_{\partial} = 0. \tag{122}$$

Now, let $F(z(\alpha, t), t) = \bar{z}_t(\alpha, t)$ be the conjugate of the velocity. Then

$$F_t \circ Z|_{\partial} = F_t(z(h^{-1}(\alpha', t), t), t)\big|_{\partial} = 0. \tag{123}$$

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We will show in (192) in §3.7.2 that

$$F_{tz} \circ z = D_\alpha(\overline{z}_{tt} - (D_\alpha \overline{z}_t)z_t). \tag{124}$$

Note that this implies, by (122) and (113), that

$$F_{tz} \circ z|_\partial = D_\alpha((\overline{z}_{tt} - (D_\alpha \overline{z}_t)z_t))|_\partial = 0. \tag{125}$$

We similarly have

$$F_{ttz} \circ Z|_\partial = 0 \tag{126}$$

and

$$D_{\alpha'}^2 (F_{tz} \circ Z)\big|_\partial = F_{tzzz} \circ Z|_\partial = 0. \tag{127}$$

3.5   A Priori Smoothness Assumptions

For the remainder of the dissertation, we will make certain a priori smoothness assumptions on the various quantities we are working with. What we assume is that all functions are reasonably well-behaved locally within $(-1,1)$ (e.g., C^0 or C^1); that those functions that don't have singularities (due to a non-trivial angle ν) are well-behaved at the boundaries (and are periodic to the extent that it follows from §3.4); and that those functions that do have singularities (due to a non-trivial angle ν) at the boundaries are at least in some L^p space for $p > 1$. See §11.2 for heuristic discussions of which functions have singularities.[22] Thus, for example, we can assume that $D_{\alpha'}^k \overline{Z}_t \in C^0(S^1) \cap C^1(-1,1)$, since $D_{\alpha'}^k$ preserves all good properties, but we only assume that $\partial_{\alpha'} D_{\alpha'}^k \overline{Z}_t \in L^p \cap C^0(-1,1)$ for some $p > 1$, since $\partial_{\alpha'}$ introduces singularities and does not preserve periodic boundary behavior.

   We have phrased the following in terms of singularities at the corner. For solutions with singularities (e.g., angled crests) in the middle of the free surface, the a priori smoothness assumptions can be adjusted accordingly.

   We emphasize that we use such a priori assumptions only to justify our formal arguments; what really matters are the norms that appear in our actual estimates, all of which we show are finite under our energy.

---

[22]We note that those heuristics indicate that if $f_\alpha \in L^\infty$, then $\partial_{\alpha'}(f \circ h^{-1}) \in L^{1+\varepsilon}$ for some ε sufficiently small, so long as the angle $\nu > 0$.

3.6 Holomorphic Functions and What Disappears Under $(I - \mathbb{H})$

In this section, we note which of the functions we are dealing with are holomorphic—more precisely, are the boundary values of periodic holomorphic functions—and which, further, have mean zero and thus disappear under $(I - \mathbb{H})$.

Recall from (81) that for $f \in L^p, p > 1$, $(I - \mathbb{H})f = c_0$ iff f is the boundary value of a periodic holomorphic function going to c_0 as $y \to -\infty$, and, moreover, $c_0 = \fint f$. Therefore, to show that $(I-\mathbb{H})$ of various functions disappears, it suffices to show that they are boundary values of periodic holomorphic functions and to show that their means are zero.

We rely fundamentally on the following facts:

First, we have that the conjugate velocity is holomorphic, and goes to zero as $y \to -\infty$ by (7), so

$$(I - \mathbb{H})\overline{Z}_t = 0. \tag{128}$$

Then we have three identities about the Riemann mapping. Recall that

$$Z_{,\alpha'} = \partial_{\alpha'}\Phi^{-1}(\alpha', t) \tag{129}$$

and

$$\frac{1}{Z_{,\alpha'}} = \Phi_z \circ Z. \tag{130}$$

Both of these are clearly holomorphic. Therefore, we have

$$(I - \mathbb{H})\frac{1}{Z_{,\alpha'}} = \fint \frac{1}{Z_{,\alpha'}} \tag{131}$$

and

$$(I - \mathbb{H})Z_{,\alpha'} = 1. \tag{132}$$

The mean $\fint Z_{,\alpha'} = 1$ by the fundamental theorem of calculus, since $Z(1,t) = 1, Z(-1,t) = -1$ for all time.

Finally, we have

$$(I - \mathbb{H})\{\Phi_t \circ Z\} = \fint_I \Phi_t \circ Z. \tag{133}$$

Here, we have $\Phi_t \circ \Phi^{-1}$ is holomorphic because it is the limit of holomorphic functions, and we know that Φ_t is periodic by the Schwarz reflection.

From these facts, we will be able to deduce everything else that we need, from the following principles:

- If $(I - \mathbb{H})f = c$ and $f|_\partial = 0$ then $(I - \mathbb{H})\partial_{\alpha'}f = 0$, since $[\partial_{\alpha'}, \mathbb{H}]f = 0$ by (100).

- If $(I - \mathbb{H})f = 0$ and $(I - \mathbb{H})g = c(t)$ then $(I - \mathbb{H})(fg) = 0$. Indeed, fg is the boundary value of the product of two periodic holomorphic functions and therefore is holomorphic and periodic. Since one of the factors goes to 0 as $y \to -\infty$, the product also goes to 0 as $y \to -\infty$.

- If $(I - \mathbb{H})f = 0$ and $f|_\partial = 0$, then $(I - \mathbb{H})D_{\alpha'}f = 0$, since $(I - \mathbb{H})\partial_{\alpha'}f = 0$ and $(I - \mathbb{H})\frac{1}{Z_{,\alpha'}} = c$.

- Therefore, if $G(z,t)$ is a periodic holomorphic function on $\Omega(t)$ going to zero as $y \to -\infty$, that is, if

$$(I - \mathbb{H})G(z(h^{-1}(\alpha', t), t), t) = (I - \mathbb{H})(G \circ Z) = 0, \tag{134}$$

  then $(I - \mathbb{H})G_z(z(h^{-1}(\alpha', t), t), t) = 0$, since $G_z \circ Z = D_{\alpha'}(G \circ Z)$.

- If $G(z,t)$ is a periodic holomorphic function on $\Omega(t)$ going to zero as $y \to -\infty$, so $(I - \mathbb{H})(G \circ Z) = 0$, then $(I - \mathbb{H})(G_t \circ Z) = 0$.[23] $G_t \circ Z$ is holomorphic, since it is the limit of holomorphic functions. It remains to show that $\fint G_t \circ Z = 0$. Note that $\fint G \circ Z = 0$ for all time. Therefore, using (64) and the fact $Z = \Phi^{-1}$, we have

$$
\begin{aligned}
0 &= \frac{d}{dt}\fint G(Z(\alpha', t), t) \\
&= \fint G_t(Z(\alpha', t), t) + \fint (G_z \circ Z) \cdot (\Phi^{-1})_t(\alpha', t) \\
&= \fint G_t \circ Z - \fint (\partial_{\alpha'}(G \circ Z)) \cdot (\Phi_t \circ \Phi^{-1}).
\end{aligned} \tag{135}
$$

It therefore suffices to show that the second integral is zero. To do this, we use $(I -$

---

[23]Note that this argument does not apply to $\Phi_t \circ Z$ itself, because Φ is not periodic.

$\mathbb{H}) \partial_{\alpha'} (G \circ Z) = 0$, integration by parts, the adjoint property (98), and (133):

$$
\begin{aligned}
\oint (\partial_{\alpha'} (G \circ Z)) \Phi_t \circ \Phi^{-1} &= \oint \{\mathbb{P}_H (\partial_{\alpha'} (G \circ Z))\} \, \Phi_t \circ \Phi^{-1} \\
&= \oint (\partial_{\alpha'} (G \circ Z)) \left(\mathbb{P}_A (\Phi_t \circ \Phi^{-1}) \right) \\
&= - \oint (G \circ Z) \partial_{\alpha'} \left(\mathbb{P}_A (\Phi_t \circ \Phi^{-1}) \right) \\
&= 0.
\end{aligned}
\tag{136}
$$

3.6.1 Identities

We now present the various identities. These all follow from the above principles and the fact that these quantities have periodic boundary behavior.

$$
(I - \mathbb{H}) \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} = 0 \tag{137}
$$

$$
(I - \mathbb{H}) D_{\alpha'} \overline{Z}_t = 0 \tag{138}
$$

$$
(I - \mathbb{H}) D_{\alpha'}^2 \overline{Z}_t = 0 \tag{139}
$$

$$
(I - \mathbb{H}) D_{\alpha'}^3 \overline{Z}_t = 0 \tag{140}
$$

$$
(I - \mathbb{H}) \overline{Z}_{t,\alpha} = 0 \tag{141}
$$

$$
(I - \mathbb{H}) \partial_{\alpha'} D_{\alpha'} \overline{Z}_t = 0 \tag{142}
$$

$$
(I - \mathbb{H}) \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t = 0 \tag{143}
$$

$$
(I - \mathbb{H}) \left\{ \frac{1}{Z_{,\alpha'}} D_{\alpha'}^k \overline{Z}_t \right\} = 0, k \leqslant 2. \tag{144}
$$

We will write F for the conjugate velocity, so F is a periodic holomorphic function on $\Omega(t)$ and $\overline{z}_t(\alpha, t) = F(z(\alpha, t), t)$. By (128),

$$
(I - \mathbb{H}) \{F \circ Z\} = 0. \tag{145}
$$

We will need the following statements about F:

$$
(I - \mathbb{H}) \{F_t \circ Z\} = 0 \tag{146}
$$

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$$(I - \mathbb{H}) \{F_{tt} \circ Z\} = 0 \tag{147}$$

$$(I - \mathbb{H}) \{Z_{,\alpha'}(F_t \circ Z)\} = 0 \tag{148}$$

$$(I - \mathbb{H}) \{Z_{,\alpha'}(F_{tt} \circ Z)\} = 0 \tag{149}$$

$$(I - \mathbb{H})D_{\alpha'}(F_t \circ Z) = 0 \tag{150}$$

$$(I - \mathbb{H})D_{\alpha'}^2(F_t \circ Z) = 0 \tag{151}$$

$$(I - \mathbb{H})D_{\alpha'}^2(F_{tt} \circ Z) = 0. \tag{152}$$

Recall that we will show at (192) in §3.7.2 that

$$F_{tz} \circ z = D_\alpha(\overline{z}_{tt} - (D_\alpha \overline{z}_t)z_t). \tag{153}$$

We begin with the basic identity

$$(I - \mathbb{H})D_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) = 0. \tag{154}$$

This gives:

$$(I - \mathbb{H})\partial_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) = 0 \tag{155}$$

$$(I - \mathbb{H})\partial_{\alpha'}D_{\alpha'}^2(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) = 0 \tag{156}$$

$$(I - \mathbb{H})\left\{ \left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'}D_{\alpha'}^2(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) \right\} = 0. \tag{157}$$

We also have

$$(I - \mathbb{H})\partial_{\alpha'}(\Phi_t \circ Z) = 0 \tag{158}$$

$$(I - \mathbb{H}) \left\{ (\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t)\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \right\} = 0 \tag{159}$$

$$(I - \mathbb{H}) \left\{ (\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t) \left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \right)^2 \right\} = 0 \tag{160}$$

$$(I - \mathbb{H})\partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2\overline{Z}_t \right) = 0 \tag{161}$$

$$(I - \mathbb{H}) \left\{ \left(\mathbb{P}_H \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right) \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2\overline{Z}_t \right) \right\} = 0 \tag{162}$$

$$(I - \mathbb{H})\left\{\left(\mathbb{P}_H\left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right)\right)^2 D_{\alpha'}^2 \overline{Z}_t\right\} = 0 \tag{163}$$

$$(I - \mathbb{H})\left\{(\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t)\left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}}\right)(I + \mathbb{H})D_{\alpha'}Z_t\right\} = 0. \tag{164}$$

Finally, we have one, rare, equation where the mean is not zero.[24] Using (99),

$$
\begin{aligned}
(I - \mathbb{H})&\left\{[(I + \mathbb{H})D_{\alpha'}Z_t]\,\mathbb{P}_H\left[(\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t)\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right)\right]\right\} \\
&= \frac{1}{2}\left(\fint D_{\alpha'}Z_t\right)\left(\fint (\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t)\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right)\right).
\end{aligned}
\tag{165}
$$

3.6.2 Mean Conditions

We have implicitly in the preceding section shown that various quantities are mean-zero, but we don't use that fact other than in those identities. We will at one point, though, require an explicit mean-zero condition, to use a variant of the Sobolev inequality. This is that

$$\fint (D_{\alpha'}\overline{Z}_t)^2 d\alpha' = 0. \tag{166}$$

We note that $\fint (D_{\alpha'}\overline{Z}_t) = 0$ because $(I - \mathbb{H})D_{\alpha'}\overline{Z}_t = 0$. Therefore, $D_{\alpha'}\overline{Z}_t$ is the boundary value of a periodic holomorphic function going to 0 as $y \to -\infty$, so its square will also be the boundary value of a periodic holomorphic function going to 0 as $y \to -\infty$.

3.7 The Riemann Mapping Version of the Equation

We now derive the Riemann mapping version of the equation.

We follow the approach of [Wu97], although we work with the real and imaginary parts together instead of separating $Z_t = X_t + iY_t$ into real and imaginary parts.

In this section, we present a full derivation of the RHS of the quasilinear equation in Riemannian coordinates. We don't end up using this derivation directly for our energy estimates, which involve more derivatives on the RHS; instead, we will have to do those estimates by hand, in later sections.[25] Rather, the derivation in this section proves useful

---

[24]By adjointness, we can write the second mean as $\fint (\mathbb{P}_H \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t)\frac{Z_t}{Z_{,\alpha'}} = \int Z_t D_{\alpha'}^3 \overline{Z}_t$.

[25]Because we are using weighted derivatives $D_{\alpha'}$ (to ensure periodic boundary behavior and prevent singularity) we have to take more care in applying derivatives to the RHS than [Wu97] does. A particular source of difficulty is that our derivative $D_{\alpha'}$ is not purely real, so inverting $(I - \mathbb{H})$ is more subtle.

because it provides several key formulas, especially one that allows us to estimate the crucial quantity $\frac{\mathfrak{a}_t}{\mathfrak{a}}$. Still, it seems most natural to motivate the following calculations by writing as if our goal were to derive the full quasilinear Riemann mapping version of the equations, rather than just to get a technical formula for $\frac{\mathfrak{a}_t}{\mathfrak{a}}$.

We note, also, that this section serves as an introduction to the techniques that we will use in later sections to derive advantageous commutator formulas for various quantities that we will need to control.

We begin as in the Lagrangian case by differentiating the main equation (27) with respect to t to get

$$z_{ttt} - i\mathfrak{a}z_{t\alpha} = i\mathfrak{a}_t z_\alpha. \tag{167}$$

We now precompose both sides with h^{-1} to get an equation in the flattened Riemann mapping coordinates:

$$Z_{ttt} - i\mathcal{A}Z_{t,\alpha'} = i\mathcal{A}_t Z_{,\alpha'}. \tag{168}$$

(Recall that $\mathcal{A} = (\mathfrak{a}h_\alpha) \circ h^{-1}$ and $\mathcal{A}_t = (\mathfrak{a}_t h_\alpha) \circ h^{-1}$.)

We now seek formulas for $\mathcal{A}$ and the RHS nonlinearity $\mathcal{A}_t$. We will be working under the boundary conditions of §3.4 and the a priori assumptions of §3.5.

3.7.1 $\mathcal{A}$ and the quantity A_1

We first derive a formula for $\mathcal{A}$. Historically, in [Wu97], this derivation showed that the Taylor positivity condition would automatically hold. There is now [Wu99] a direct proof of this using basic elliptic theory without requiring the Riemann mapping. For our purposes, though, this original derivation here will be crucial because it introduces a quantity, A_1, that compares the degeneracy of $\frac{-\partial P}{\partial \mathbf{n}}$ directly with that of the Riemann mapping and therefore the geometry of the surface.

We begin with the conjugated form of our initial equation (27) and precompose with h^{-1}:

$$\overline{Z}_{tt} - i = -i\mathcal{A}\overline{Z}_{,\alpha'}. \tag{169}$$

The key observation is that $\overline{Z}_{tt} - i$ is in some sense close to holomorphic, since $\overline{Z}_t$ is holomorphic. Therefore, if we apply $(I - \mathbb{H})$ to this, it should be "small" in some sense. If the RHS were purely imaginary, we would be able to invert $(I - \mathbb{H})$ by taking imaginary parts, and thus get a formula for the RHS. The RHS is not purely real or imaginary, but it is if we

multiply both sides by the holomorphic $Z_{,\alpha'}$:

$$Z_{,\alpha'}(\overline{Z}_{tt} - i) = -i\mathcal{A} |Z_{,\alpha'}|^2. \tag{170}$$

We will apply $(I - \mathbb{H})$ to both sides of (170). Before we do so, we must expand out $\overline{Z}_{tt}$. Let

$$F(z(\alpha, t), t) := \overline{z}_t(\alpha, t); \tag{171}$$

note that this is the boundary value of a periodic holomorphic function. (We will use this expansion several times in the sequel, always with this definition of F.) By the chain rule,

$$\overline{z}_{tt} = \frac{d}{dt}F(z(\alpha, t), t) = (F_z \circ z)z_t + (F_t \circ z). \tag{172}$$

Recall from §2.4 that $\partial_z = D_\alpha$ for holomorphic functions. Therefore,

$$F_z \circ z = \frac{\overline{z}_{t\alpha}}{z_\alpha}, \tag{173}$$

and thus

$$\overline{z}_{tt} = \frac{\overline{z}_{t\alpha}}{z_\alpha}z_t + F_t \circ z. \tag{174}$$

We precompose with h^{-1}:

$$\overline{Z}_{tt} = \left(\frac{\overline{Z}_{t,\alpha'}}{Z_{,\alpha'}}\right) Z_t + F_t \circ Z. \tag{175}$$

We can now write our equation (170) as

$$\overline{Z}_{t,\alpha'}Z_t + Z_{,\alpha'}(F_t \circ Z) - iZ_{,\alpha'} = -i\mathcal{A} |Z_{,\alpha'}|^2. \tag{176}$$

We apply $(I - \mathbb{H})$ to both sides. By (141), we have

$$(I - \mathbb{H})Z_t\overline{Z}_{t,\alpha'} = [Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}. \tag{177}$$

Therefore, by (148), (132), and (177), applying $(I - \mathbb{H})$ to each side of (176) gives

$$[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'} - i = (I - \mathbb{H})\left(-i\mathcal{A} |Z_{,\alpha'}|^2\right). \tag{178}$$

We now take imaginary parts of both sides:

$$\mathcal{A}\,|Z_{,\alpha'}|^2 = \Im\left(-[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}\right) + 1. \tag{179}$$

We define

$$A_1 := \mathcal{A}\,|Z_{,\alpha'}|^2 = \Im\left(-[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}\right) + 1. \tag{180}$$

This is the same A_1 as that in [Wu97]. It's easy to see that $\Im\left(-[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}\right)$ is non-negative, by integration by parts. Indeed, if $Z_t = X_t + iY_t$, then

$$
\begin{aligned}
\Im(-[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}) &= -\Im\frac{1}{2i}\int (Z_t(\alpha') - Z_t(\beta'))\cot(\frac{\pi}{2}(\alpha' - \beta'))\overline{Z}_{t,\beta'}d\beta' \\
&= \frac{1}{2}\int ((X_t(\alpha') - X_t(\beta'))X_{t,\beta'} + (Y_t(\alpha') - Y_t(\beta'))Y_{t,\beta'})\cot(\frac{\pi}{2}(\alpha' - \beta'))d\beta' \\
&= \frac{1}{2}\int \frac{1}{2}\left\{-\partial_{\beta'}\left[(X_t(\alpha') - X_t(\beta'))^2 + (Y_t(\alpha') - Y_t(\beta'))^2\right]\right\}\cot(\frac{\pi}{2}(\alpha' - \beta'))d\beta' \\
&= \frac{1}{2}\int \frac{1}{2}\frac{\pi}{2}\frac{(X_t(\alpha') - X_t(\beta'))^2 + (Y_t(\alpha') - Y_t(\beta'))^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}d\beta' \\
&\geqslant 0.
\end{aligned}
\tag{181}
$$

We conclude that

$$A_1 \geqslant 1. \tag{182}$$

Notice by (170) and (180) that

$$\frac{1}{Z_{,\alpha'}} = i\frac{\overline{Z}_{tt} - i}{A_1}. \tag{183}$$

Therefore $Z_{,\alpha'} \neq 0$, for otherwise $Z_{tt} = \infty$. This shows that a non-trivial angle ν at the corner must be $< \frac{\pi}{2}$.[26]

Because

$$A_1 \circ h = \frac{\mathfrak{a}\,|z_\alpha|^2}{h_\alpha} \tag{184}$$

and

$$\frac{-\partial P}{\partial \mathbf{n}} = |z_\alpha|\,\mathfrak{a} = \frac{A_1 \circ h}{|Z_{,\alpha'} \circ h|} \tag{185}$$

the estimate $A_1 \geqslant 1$ gave [Wu97] a strictly positive lower bound for the Taylor coefficient

---

[26]Similarly, this shows that the interior (i.e., within the water) angle of crests in the middle of the free surface cannot be over π.

$\frac{-\partial P}{\partial \mathbf{n}}$. Indeed, $\frac{1}{Z_{,\alpha'}} = \Phi_z \circ Z$, and in the setup without singularities at the corner or in the middle of the free surface, it is easy to see from the Riemann mapping that $0 < c_0 \leqslant |\Phi_z \circ Z| \leqslant C_0 < \infty$. In our situation, $\frac{1}{Z_{,\alpha'}} \to 0$ at the corner if the angle $\nu < \frac{\pi}{2}$.[27] Control of A_1, however, is still very useful; if we can control A_1 above and below, we know that the degeneracy of $\frac{-\partial P}{\partial \mathbf{n}}$ corresponds precisely to the degeneracy of the Riemann mapping. Similarly, this shows that the degeneracy of $\mathfrak{a}$ corresponds to that of h_α; we will use these quantities (rather than $\frac{-\partial P}{\partial \mathbf{n}}$) in the following. In any case, since $\frac{1}{|Z_{,\alpha'}|} \geqslant 0$, this proves that the degenerate Taylor stability condition $\frac{-\partial P}{\partial \mathbf{n}} \geqslant 0$ always holds.

3.7.2 $\mathcal{A}_t$

Now we seek a formula for the nonlinearity on the RHS of (168), $\mathcal{A}_t$. We begin by conjugating (168):

$$\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'} = -i\mathcal{A}_t\overline{Z}_{,\alpha'}. \tag{186}$$

As in the previous section, we will hope that the LHS is close to holomorphic and apply $(I - \mathbb{H})$ to both sides and then invert. Here we start by multiplying both sides by $Z_{,\alpha'}$, since we want the RHS to be purely imaginary:

$$Z_{,\alpha'}(\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'}) = -i\mathcal{A}_t |Z_{,\alpha'}|^2. \tag{187}$$

We once again carefully expand the LHS. As before, let $F(z(\alpha,t),t) = \overline{z}_t(\alpha,t)$. Again, we have

$$\overline{z}_{tt} = (F_z \circ z)z_t + F_t \circ z, \tag{188}$$

so

$$\begin{aligned}
\overline{z}_{ttt} &= (F_{zz} \circ z)z_t^2 + (F_{zt} \circ z)z_t + (F_z \circ z)z_{tt} + (F_{tz} \circ z)z_t + F_{tt} \circ z \\
&= (F_{zz} \circ z)z_t^2 + 2(F_{tz} \circ z)z_t + (F_z \circ z)z_{tt} + F_{tt} \circ z.
\end{aligned} \tag{189}$$

We now solve for $F_z \circ z$, $F_{zz} \circ z$ and $F_{tz} \circ z$. Since $\partial_z = D_\alpha$ on holomorphic functions

$$F_z \circ z = \frac{\overline{z}_{t\alpha}}{z_\alpha} = D_\alpha \overline{z}_t \tag{190}$$

---

[27]Similarly, $\frac{1}{Z_{,\alpha'}} = 0$ at angled crests in the middle of the free surface.

and

$$F_{zz} \circ z = D_\alpha^2 \overline{z}_t. \tag{191}$$

We solve for $F_{tz} \circ z$ by applying $\partial_z = D_\alpha$ to (188):

$$F_{tz} \circ z = D_\alpha \left(\overline{z}_{tt} - (D_\alpha \overline{z}_t) z_t \right). \tag{192}$$

Therefore, by substituting (190), (191), and (192) into (189), we get

$$\overline{z}_{ttt} = (D_\alpha^2 \overline{z}_t) z_t^2 + 2 z_t D_\alpha \left(\overline{z}_{tt} - (D_\alpha \overline{z}_t) z_t \right) + (D_\alpha \overline{z}_t) z_{tt} + F_{tt} \circ z. \tag{193}$$

Precomposing with h^{-1}, we have

$$\overline{Z}_{ttt} = (D_{\alpha'}^2 \overline{Z}_t) Z_t^2 + 2 Z_t D_{\alpha'} (\overline{Z}_{tt} - (D_{\alpha'} \overline{Z}_t) Z_t) + (D_{\alpha'} \overline{Z}_t) Z_{tt} + F_{tt} \circ Z. \tag{194}$$

We now go back to (187), substituting in (194) to get

$$Z_{,\alpha'} \left((D_{\alpha'}^2 \overline{Z}_t) Z_t^2 + 2 Z_t D_{\alpha'} (\overline{Z}_{tt} - (D_{\alpha'} \overline{Z}_t) Z_t) + (D_{\alpha'} \overline{Z}_t) Z_{tt} + F_{tt} \circ Z + i \mathcal{A} \overline{Z}_{t,\alpha'} \right) \\ = -i \mathcal{A}_t \left| Z_{,\alpha'} \right|^2. \tag{195}$$

We simplify, distributing the $Z_{,\alpha'}$ and then using the identity $Z_{tt} + i = i\mathcal{A} Z_{,\alpha'}$ on the last term:

$$(\partial_{\alpha'} D_{\alpha'} \overline{Z}_t) Z_t^2 + 2 Z_t \partial_{\alpha'} (\overline{Z}_{tt} - (D_{\alpha'} \overline{Z}_t) Z_t) + \overline{Z}_{t,\alpha'} Z_{tt} + Z_{,\alpha'} (F_{tt} \circ Z) + (Z_{tt} + i) \overline{Z}_{t,\alpha'} \\ = -i \mathcal{A}_t \left| Z_{,\alpha'} \right|^2. \tag{196}$$

We now apply $(I - \mathbb{H})$ to both sides. Various terms will disappear on the LHS and others will turn into commutators, due to holomorphicity. We do this term by term:

- The first term will become $[Z_t^2, \mathbb{H}] \partial_{\alpha'} D_{\alpha'} \overline{Z}_t$ by (142).

- The second term becomes $2[Z_t, \mathbb{H}] \partial_{\alpha'} (\overline{Z}_{tt} - (D_{\alpha'} \overline{Z}_t) Z_t)$ (155).

- The third term and the first part of the fifth term become $2[Z_{tt}, \mathbb{H}] \overline{Z}_{t,\alpha}$ by (141).

- The fourth term disappears, by (149).

- The second part of the fifth term disappears by (141).

We get

$$[Z_t^2, \mathbb{H}]\partial_{\alpha'}D_{\alpha'}\overline{Z}_t + 2[Z_t, \mathbb{H}]\partial_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) + 2[Z_{tt}, \mathbb{H}]\overline{Z}_{t,\alpha}$$
$$= (I - \mathbb{H})\left\{-i\mathcal{A}_t\, |Z_{,\alpha'}|^2\right\}. \tag{197}$$

We could continue working with this form, but two integrations by parts will give us a nicer equation for getting our estimates. We take the first term and the second part of the second term and integrate by parts both terms:

$$\left[Z_t^2, \mathbb{H}\right]\partial_{\alpha'}D_{\alpha'}\overline{Z}_t - 2[Z_t, \mathbb{H}]\partial_{\alpha'}\left((D_{\alpha'}\overline{Z}_t)Z_t\right)$$
$$= \frac{1}{i}\int Z_t(\beta')Z_{t,\beta'}(\beta')\cot(\frac{\pi}{2}(\alpha' - \beta'))D_{\beta'}\overline{Z}_t(\beta')d\beta'$$
$$- \frac{1}{2i}\int \frac{\pi}{2}\frac{Z_t^2(\alpha') - Z_t^2(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}D_{\beta'}\overline{Z}_t d\beta'$$
$$- \frac{1}{i}\int Z_t(\beta')Z_{t,\beta'}(\beta')\cot(\frac{\pi}{2}(\alpha' - \beta'))D_{\beta'}\overline{Z}_t d\beta'$$
$$+ \frac{1}{i}\int \frac{\pi}{2}\frac{Z_t(\alpha') - Z_t(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}(D_{\beta'}\overline{Z}_t)Z_t(\beta')d\beta'. \tag{198}$$
$$= -\frac{1}{2i}\int \frac{\pi}{2}\frac{Z_t^2(\alpha') - Z_t^2(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}D_{\beta'}\overline{Z}_t d\beta'$$
$$+ \frac{1}{i}\int \frac{\pi}{2}\frac{Z_t(\alpha') - Z_t(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}(D_{\beta'}\overline{Z}_t)Z_t(\beta')d\beta'$$
$$= -\frac{1}{2i}\int \frac{\pi}{2}\frac{(Z_t(\alpha') - Z_t(\beta'))^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}D_{\beta'}\overline{Z}_t(\beta')d\beta'.$$

This is a type of higher-order Calderon commutator, which we write as $-\frac{\pi}{2}[Z_t, Z_t; D_{\alpha'}\overline{Z}_t]$ (see (11)). We therefore can rewrite (197) as:

$$-i(I - \mathbb{H})\left\{\mathcal{A}_t\, |Z_{,\alpha'}|^2\right\} = 2[Z_t, \mathbb{H}]\overline{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\overline{Z}_{t,\alpha} - \frac{\pi}{2}[Z_t, Z_t; D_{\alpha'}\overline{Z}_t]. \tag{199}$$

Taking imaginary parts, we get

$$\mathcal{A}_t\, |Z_{,\alpha'}|^2 = -\Im\left(2[Z_t, \mathbb{H}]\overline{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\overline{Z}_{t,\alpha} - \frac{\pi}{2}[Z_t, Z_t; D_{\alpha'}\overline{Z}_t]\right). \tag{200}$$

3.7.3 Applications of the Riemann Derivation

We could now plug (179) and (200) into (168) to get the quasilinear equation in Riemannian coordinates. Instead, though, we will focus on a different quantity. Observe that dividing (200) by (179) we have

$$\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} = \frac{\mathcal{A}_t}{\mathcal{A}} = \frac{\mathcal{A}_t \, |Z_{,\alpha'}|^2}{\mathcal{A} \, |Z_{,\alpha'}|^2} = \frac{-\Im\left(2[Z_t, \mathbb{H}]\overline{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\overline{Z}_{t,\alpha'} - \frac{\pi}{2}[Z_t, Z_t; D_{\alpha'}\overline{Z}_t]\right)}{A_1}. \quad (201)$$

If we want to control $\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty}$, then, because $A_1 \geqslant 1$ by (182), it suffices to control the numerator of the RHS (201), which we will be able to do.

Chapter 4

Technical Details

We now present assorted technical results that we will need in our dissertation. None of the results here are original.

4.1 Sobolev Inequalities and the Peter-Paul Trick

We present here the one-dimensional Sobolev inequality we will use in our proof. Our version is a slight variant of the standard inequality: it includes weighted L^2 spaces and allows a factor of ε on the highest-order term.

Proposition 12 (Weighted Sobolev Inequality with ε). *Let $\varepsilon > 0$. Then for all $f \in C^1(-1,1) \cap L^2(\frac{1}{\omega}) \cap L^2$ with $f' \in L^2(\omega)$,*

$$\|f\|_{L^\infty} \lesssim \frac{1}{\varepsilon} \|f\|_{L^2(\frac{1}{\omega})} + \varepsilon \|f'\|_{L^2(\omega)} + \|f\|_{L^2} \tag{202}$$

for any weight $\omega \geqslant 0$.

Furthermore,

$$\fint f^2 = 0 \Rightarrow \|f\|_{L^\infty} \lesssim \frac{1}{\varepsilon} \|f\|_{L^2(\frac{1}{\omega})} + \varepsilon \|f'\|_{L^2(\omega)}. \tag{203}$$

The second version, (203), will give us slightly more flexibility. The condition $\fint f^2 = 0$ will actually follow from $\fint f = 0$ for f the boundary value of a periodic holomorphic function on $[-1,1] \times (-\infty, 0)$ (see the discussion for (166)), and so is not so constraining.

Proof. Let $x \in (-1, 1)$. Then for any $u \in C^1(-1, 1) \cap L^1(-1, 1)$,

$$
\begin{aligned}
\left| u(x) - \frac{1}{2} \int_{-1}^{1} u(y) dy \right| &= \left| \frac{1}{2} \int_{-1}^{1} (u(x) - u(y)) dy \right| \\
&\leqslant \frac{1}{2} \int_{-1}^{1} |u(x) - u(y)|\, dy \\
&= \frac{1}{2} \int_{-1}^{1} \left| \int_{x}^{y} u'(t) dt \right| dy \\
&\leqslant \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} |u'(t)|\, dt dy \\
&= \|u'\|_{L^1} .
\end{aligned}
\tag{204}
$$

Letting $u = f^2$, we get

$$
\left| f^2(x) - \frac{1}{2} \int_{-1}^{1} f^2(y) dy \right| \leqslant \left| \int_{-1}^{1} (f^2)' \right| = 2 \left| \int_{-1}^{1} f f' \right| = 2 \left| \int_{-1}^{1} \frac{f}{\sqrt{\omega}} (f' \sqrt{\omega}) \right| .
\tag{205}
$$

Therefore, applying Cauchy-Schwarz and Cauchy's inequality, we get

$$
\|f^2\|_{L^\infty} \lesssim \frac{1}{2} \int_{-1}^{1} |f|^2 + \frac{1}{\varepsilon^2} \left(\int_{-1}^{1} |f|^2 \frac{1}{\omega} \right) + \varepsilon^2 \left(\int_{-1}^{1} |f'|^2 \omega \right) .
\tag{206}
$$

Taking square roots and adjusting gives inequality (202).

To get (203), we note that in (205) if $\fint f^2 = 0$, then the second term on the LHS is zero, and the result follows. $\qquad \square$

We will once use the following "Peter-Paul" trick[28] with the weighted Sobolev inequality with ε.

Proposition 13 (Peter-Paul Trick). *Let $f \in C^1(-1, 1)$. Suppose*

$$
\|D_\alpha f\|_{L^2(\mu)} \leqslant c_1 + c_2 \|f\|_{L^\infty} ,
\tag{207}
$$

where $\mu \geqslant 0$ is a weight and $D_\alpha := \frac{1}{z_\alpha} \partial_\alpha$. Suppose further that $\|f\|_{L^\infty} < \infty$. Then

$$
\|f\|_{L^\infty} \lesssim c_2 \|f\|_{L^2(\frac{|z_\alpha|^2}{\mu})} + c_1 + \|f\|_{L^2} ,
\tag{208}
$$

where the constant implicit in $\lesssim$ is universal.

---

[28]So-called because one takes from Peter to give to Paul.

Proof. By the weighted Sobolev inequality (202) with $\omega = \frac{\mu}{|z_\alpha|^2}$ and then (207),

$$
\begin{aligned}
\|f\|_{L^\infty} &\leqslant C\left(\frac{1}{\varepsilon}\|f\|_{L^2(\frac{|z_\alpha|^2}{\mu})} + \varepsilon\|f'\|_{L^2(\frac{\mu}{|z_\alpha|^2})} + \|f\|_{L^2}\right) \\
&= C\left(\frac{1}{\varepsilon}\|f\|_{L^2(\frac{|z_\alpha|^2}{\mu})} + \varepsilon\|D_\alpha f\|_{L^2(\mu)} + \|f\|_{L^2}\right) \\
&\leqslant C\left(\frac{1}{\varepsilon}\|f\|_{L^2(\frac{|z_\alpha|^2}{\mu})} + \varepsilon(c_1 + c_2\|f\|_{L^\infty}) + \|f\|_{L^2}\right).
\end{aligned}
\tag{209}
$$

We choose $\varepsilon \leqslant \min(\frac{1}{2Cc_2}, 1)$, so $Cc_2\varepsilon \leqslant \frac{1}{2}$. Subtracting $Cc_2\varepsilon\|f\|_{L^\infty}$ from both sides, we get

$$
\frac{1}{2}\|f\|_{L^\infty} \leqslant C\left(\frac{1}{\varepsilon}\|f\|_{L^2(\frac{|z_\alpha|^2}{\mu})} + \varepsilon c_1 + \|f\|_{L^2}\right),
\tag{210}
$$

which gives desired inequality. $\qquad\square$

4.2   Derivatives and Complex-Valued Functions

Because our functions will be complex-valued, and we will often be looking at derivatives of angular and modular parts of these functions, we note here a few elementary facts about such functions.

Let $f(\alpha) = r(\alpha)e^{i\theta(\alpha)}$, with r, θ real-valued functions. Then

$$
\underbrace{|\partial_\alpha|f||}_{|r_\alpha|} \leqslant \underbrace{|f'|}_{|r_\alpha e^{i\theta} + ir\theta_\alpha e^{i\theta}|}
\tag{211}
$$

and

$$
\underbrace{\left|\partial_\alpha \frac{f}{|f|}\right|}_{|i\theta_\alpha e^{i\theta}|} \leqslant \underbrace{\left|\frac{f'}{|f|}\right|}_{\left|\frac{r_\alpha e^{i\theta} + i\theta_\alpha r e^{i\theta}}{r}\right|}.
\tag{212}
$$

Note that the mean value theorem doesn't apply for complex-valued functions of a real-variable. Luckily, we still have the basic L^∞ estimate for difference quotients. Indeed, if $f = u + iv$, then

$$
\left|\frac{f(\alpha) - f(\beta)}{\alpha - \beta}\right| \leqslant \left|\frac{u(\alpha) - u(\beta)}{\alpha - \beta}\right| + \left|\frac{v(\alpha) - v(\beta)}{\alpha - \beta}\right| \leqslant \|u_\alpha\|_{L^\infty} + \|v_\alpha\|_{L^\infty} \leqslant 2\|f'\|_{L^\infty}.
\tag{213}
$$

A more subtle complication arises from the fact that we will care about controlling both Γf and $\Gamma \overline{f}$, for different combinations of differential operators Γ. If we were using only pure spatial and time derivatives ∂_α and ∂_t, then there would be no problem, but our weighted derivative $D_\alpha = \frac{1}{z_\alpha}\partial_\alpha$ poses a problem. Indeed, $D_\alpha^2 z = D_\alpha 1 = 0$, but there is no reason to expect $D_\alpha^2 \overline{z} = D_\alpha \frac{\overline{z}_\alpha}{z_\alpha}$ to be zero. We therefore present carefully the relevant facts:

$$\left|\partial_\alpha^k f\right| = \left|\partial_\alpha^k \overline{f}\right|, k \geqslant 0 \text{ (indeed, conjugation commutes with } \partial_\alpha) \tag{214}$$

$$\left|\partial_t^k f\right| = \left|\partial_t^k \overline{f}\right|, k \geqslant 0 \text{ (indeed, conjugation commutes with } \partial_t) \tag{215}$$

$$|D_\alpha f| = \left|D_\alpha \overline{f}\right|. \tag{216}$$

It turns out we can control $D_\alpha^2 f$ by $D_\alpha^2 \overline{f}$ in the appropriate weighted L^2 spaces by our energy. Because this relies on the specific terms of the energy, we will postpone this until §6.1.

4.3 Hardy's Inequality

We will often use Hardy's inequality to control the L^2 norm of a difference quotient.

Proposition 14 (Hardy's Inequality). *Let $f \in C^0(S^1) \cap C^1(-1,1)$ (and so $f|_\partial = 0$), with $f' \in L^2$. Then there exists $C > 0$ independent of f such that for any $\alpha' \in I$,*

$$\left|\int_I \frac{(f(\alpha') - f(\beta'))^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}d\beta'\right| \leqslant C \|f'\|_{L^2}^2 \tag{217}$$

and

$$\left|\int_I (f(\alpha') - f(\beta'))^2 \cot^2(\frac{\pi}{2}(\alpha' - \beta'))d\beta'\right| \leqslant C \|f'\|_{L^2}^2. \tag{218}$$

Proof. By the fundamental theorem of calculus, for any $\alpha', \beta' \in [-1, 1]$,

$$\frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} = \frac{1}{\alpha' - \beta'}\int_{\beta'}^{\alpha'} f'(x)dx. \tag{219}$$

(Observe that we have assumed sufficient regularity for this to be the case. We have C^1 regularity inside $(-1, 1)$, so the only issue is if α' or β' is ± 1, which is handled by the continuity of f in the whole domain and $f' \in L^2$.)

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Therefore,

$$\left(\int_I \left| \frac{(f(\alpha') - f(\beta'))^2}{(\alpha' - \beta')^2} \right| d\beta' \right)^{1/2} = \left(\int \left| \frac{1}{\alpha' - \beta'} \int_{\beta'}^{\alpha'} f'(x) dx \right|^2 \right)^{1/2}. \tag{220}$$

Recall the classical Hardy inequality, which states in its more traditional form that if $G(x) = \frac{1}{x} \int_0^x g(t) dt$, then $\|G\|_{L^p(\mathbb{R})} \leqslant C_p \|g\|_{L^p(\mathbb{R})}$ for $1 < p < \infty$. This implies that

$$\left| \int_I \frac{(f(\alpha') - f(\beta'))^2}{(\alpha' - \beta')^2} d\beta' \right| \leqslant C \|f'\|_{L^2}^2. \tag{221}$$

To replace the $(\alpha' - \beta')^2$ with $\sin^2(\frac{\pi}{2}(\alpha' - \beta'))$, we use the following classical identity:

$$\frac{1}{\sin^2(\frac{\pi}{2}\alpha')} = \left(\frac{2}{\pi} \right)^2 \sum_{l \in \mathbb{Z}} \frac{1}{(\alpha' + 2l)^2}, \tag{222}$$

which can be obtained by differentiating (72). We begin by rewriting the LHS of (217) using this identity:

$$\int_I \frac{(f(\alpha') - f(\beta'))^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} d\beta' = \left(\frac{2}{\pi} \right)^2 \int_I \sum_{l \in \mathbb{Z}} \frac{(f(\alpha') - f(\beta'))^2}{(\alpha' - \beta' + 2l)^2} d\beta'. \tag{223}$$

We split the integral into the cases where $|l| \leqslant 1$ and where $|l| \geqslant 2$. For $l = 0$, we have the result by (221); by periodic extension of f, so long as $f|_\partial = 0$, we have the same result for $|l| = 1$. (If $f|_\partial \neq 0$ we would have a delta function in the derivative of the extended function f.)

For $|l| \geqslant 2$, the denominator $|\alpha' - \beta' + 2l| \geqslant 2|l| - 2$, and so the infinite sum

$$\sum_{l \in \mathbb{Z}, |l| > 2} \frac{1}{(\alpha' - \beta' + 2l)^2} =: l(\alpha', \beta') \tag{224}$$

is bounded by a universal constant. Therefore,

$$\left| \int_I (f(\alpha') - f(\beta'))^2 l(\alpha', \beta') d\beta \right| \leqslant \int_I \left| \frac{(f(\alpha') - f(\beta'))^2}{(\alpha' - \beta')^2} (\alpha' - \beta')^2 l(\alpha', \beta') d\beta' \right| \\ \leqslant C \|f'\|_{L^2}^2 \|l(\alpha', \beta')\|_{L^\infty}, \tag{225}$$

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which gives (217). Together with the boundedness of cosine, this implies (218). $\qquad\square$

4.4 Commutator Estimates

We present here the basic commutator estimates we will rely on to prove our energy estimate. Several of these estimates control quantities of the form $[f, \mathbb{H}]g'$ by something involving f' and g; they thus reduce the amount of regularity required on g, at the expense of further regularity on f.

For many of these estimates, we must pay close attention to the boundary conditions. Recall that $f \in C^0(S^1)$ implies that $f|_{\partial} = 0$. Many of these estimates do not hold if this periodic boundary condition is removed. In the remainder of the paper, when we use these results, we do not always explicitly cite these boundary conditions, but they are always met, by the results in §3.4.[29]

The first main estimate is a version of Calderon's classical commutator estimate.

Proposition 15 ($L^\infty \times L^2$ Estimate [Cal65]). *There exists a constant $C > 0$ such that for any $f \in C^1[-1,1] \cap C^0(S^1)$, $g \in C^0(S^1) \cap C^1(-1,1)$ with $g' \in L^p$ for some $p > 1$[30] (and so $f|_{\partial} = g|_{\partial} = 0$),*

$$\|[f, \mathbb{H}]\partial_{\alpha'} g\|_{L^2} \leqslant C \|f'\|_{L^\infty} \|g\|_{L^2}. \tag{226}$$

Proof. We reduce the periodic result here to the classical result on $\mathbb{R}$ by [Cal65].[31]

We begin by integrating by parts[32]:

$$
\begin{aligned}
[f, \mathbb{H}] \partial_{\alpha'} g &= \frac{1}{2i} \int (f(\alpha') - f(\beta')) \cot(\frac{\pi}{2}(\alpha' - \beta')) \partial_{\beta'} g(\beta') d\beta' \\
&= \frac{1}{2i} \int f_{\beta'}(\beta') \cot(\frac{\pi}{2}(\alpha' - \beta')) g(\beta') d\beta' - \frac{1}{2i} \int \frac{\pi}{2} \frac{f(\alpha') - f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} g(\beta') d\beta' \quad (227) \\
&= \mathbb{H}(f'g) - \frac{1}{2i} \int \frac{\pi}{2} \frac{f(\alpha') - f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} g(\beta') d\beta'.
\end{aligned}
$$

Observe that the periodic boundary conditions ensure that the boundary term disappears. The estimate for the first term is immediate by boundedness of $\mathbb{H}$ and Hölder's inequality,

---

[29]We remark that the difficulties posed by non-periodic boundary conditions likewise appear for singularities in the middle of the free surface away from the corners. By handling the problem correctly at the corners, we have automatically handled the problem correctly in the middle of the free surface.

[30]We assume $g' \in L^p$ only to ensure $[f, \mathbb{H}]g'$ is well-defined.

[31]The result was later extended by [CM78].

[32]Technically, we must include the principal values, so the last integral is well-defined.

so it suffices to focus on the second term.

We return to thinking of the cotangent kernel in terms of the infinite summation, using the identity (222). We must therefore control

$$\int_I (f(\alpha') - f(\beta')) \sum_{l \in \mathbb{Z}} \frac{1}{(\alpha' - \beta' + 2l)^2} g(\beta') d\beta'. \tag{228}$$

We split this into two cases. The main term will consist of the case when $l = 0$.

For the main term, we have

$$\int_I \frac{f(\alpha') - f(\beta')}{(\alpha' - \beta')^2} g(\beta') d\beta'. \tag{229}$$

The work of [Cal65] implies that such an integral over $\mathbb{R}$ is bounded in $L^2(\mathbb{R})$ by $\|f'\|_{L^\infty(\mathbb{R})}$ $\|g\|_{L^2(\mathbb{R})}$. To apply this result in $\mathbb{R}$, we must extend f and g from I to $\mathbb{R}$. We may assume without loss of generality that $f(\pm 1) = 0$, and then we extend f to the function $\chi_I f$; note that this is still Lipschitz continuous. We extend g by $g\chi_I$. Then

$$
\begin{aligned}
\left\| \int_I \frac{f(\alpha') - f(\beta')}{(\alpha' - \beta')^2} g(\beta') d\beta' \right\|_{L^2(I)} &= \left\| \int_I \frac{\chi_I f(\alpha') - \chi_I f(\beta')}{(\alpha' - \beta')^2} \chi_I(\beta') g(\beta') d\beta' \right\|_{L^2(I)} \\
&\leqslant \left\| \int_I \frac{\chi_I f(\alpha') - \chi_I f(\beta')}{(\alpha' - \beta')^2} \chi_I(\beta') g(\beta') d\beta' \right\|_{L^2(\mathbb{R})} \\
&= \left\| \int_\mathbb{R} \frac{\chi_I f(\alpha') - \chi_I f(\beta')}{(\alpha' - \beta')^2} \chi_I(\beta') g(\beta') d\beta' \right\|_{L^2(\mathbb{R})} \\
&\lesssim \|(\chi_I f)'\|_{L^\infty(\mathbb{R})} \|\chi_I g\|_{L^2(\mathbb{R})} \\
&\lesssim \|f'\|_{L^\infty(I)} \|g\|_{L^2(I)}.
\end{aligned}
\tag{230}
$$

We now consider the remainder term, for $|l| > 0$. We control $|l| = 1$ identically to the main term, by periodicity. Therefore, we may assume $|l| > 1$, which implies that $|\alpha' - \beta' + 2l| \geqslant 2 > 0$.

Our kernel is now

$$l(\alpha', \beta') := \sum_{|l|>1} \frac{1}{(\alpha' - \beta' + 2l)^2}. \tag{231}$$

Since $|\alpha' - \beta' + 2l| \geqslant 2$, our kernel $l(\alpha', \beta')$ is summable and universally bounded. We write

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the remaining terms as

$$\left\|\int_I \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'}(\alpha' - \beta')l(\alpha', \beta')g(\beta')d\beta'\right\|_{L^2} \leqslant C \|f'\|_{L^\infty} \|g\|_{L^2} \tag{232}$$

by the difference quotient estimate (213) and Hölder's inequality. $\qquad\square$

We now present the other main estimate. Observe that for this estimate we don't require $g|_\partial$ to be 0.

Proposition 16 ($L^2 \times L^\infty$ Estimate). *There exists a constant $C > 0$ such that for any $f \in C^0(S^1) \cap C^1(-1, 1)$ with $f' \in L^2$, $g \in C^0[-1, 1]$ with $g' \in L^p$ for some $p > 1$[33] (so $f|_\partial = 0$, though possibly $g|_\partial \neq 0$),*

$$\|[f, \mathbb{H}]\partial_{\alpha'}g\|_{L^2} \leqslant C \|f'\|_{L^2} \|g\|_{L^\infty}. \tag{233}$$

Proof. This result is the periodic modification of a result from [Wu09], which in turn is a consequence of the $T(b)$ theorem [DJS85].

As in the proof of (226), we integrate by parts,

$$
\begin{aligned}
[f, \mathbb{H}]\partial_{\alpha'}g = \mathbb{H}(f'g) &- \frac{1}{2i}\,\mathrm{pv}\int \frac{\pi}{2}\frac{f(\alpha') - f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}g(\beta')d\beta \\
&+ \frac{1}{2i}(f(\alpha') - f(\beta'))\cot(\frac{\pi}{2}(\alpha' - \beta'))g(\beta')\Big|_\partial,
\end{aligned}
\tag{234}
$$

where we now have a boundary term because we didn't place periodic boundary assumptions on g.

We control the first term by the L^2 boundedness of $\mathbb{H}$ and Hölder. We control the last term by Hardy's inequality (218).

We are therefore left with the second term. We handle this as we did in the proof of Proposition 15, by expanding the formula. Now the second estimate of Proposition 3.2 in [Wu09], instead of the result of [Cal65], proves this estimate for integrals of this form, over $\mathbb{R}$. Although the statement in [Wu09] requires $f' \in C^0(\mathbb{R})$, in fact the proof only requires $f' \in L^2$, which we have; without loss of generality, we may assume $f(-1) = f(1) = 0$ and we extend f from I to $\mathbb{R}$ by defining it to be zero elsewhere. For the remainder term (232), we use Hardy's inequality (in the original form, without a trigonometric kernel) instead of the difference quotient estimate. $\qquad\square$

---

[33]We require this only to ensure that $[f, \mathbb{H}]g'$ is well-defined.

Proposition 17 ($L^2 \times L^\infty$ *Estimate Variant*). *There exists a constant $C > 0$ such that for any $f \in C^0(S^1) \cap C^1(-1, 1)$ with $f' \in L^2$, $g \in C^0[-1, 1]$ (and so $f|_\partial = 0$ but possibly $g|_\partial \neq 0$),*

$$\left\| \text{pv} \int \frac{(f(\alpha') - f(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} g(\beta') d\beta' \right\|_{L^2} \leqslant C \|f'\|_{L^2} \|g\|_{L^\infty} . \tag{235}$$

Proof. This is one of the terms in (234) in Proposition 16 above, which we controlled there.[34]

$\square$

We present here a variant of the $L^2 \times L^\infty$ estimate, where we replace L^∞ with $\dot{H}^{1/2}$ (whose norm is defined at (18)).

Proposition 18. *There exists a constant $C > 0$ such that for any $f, g \in C^1(-1, 1) \cap C^0(S^1)$ with $f' \in L^2$ and $g' \in L^p$ for some $p > 1$ (and so $f|_\partial = g|_\partial = 0$),*

$$\|[f, \mathbb{H}]\partial_{\alpha'} g\|_{L^2} \leqslant C \|f'\|_{L^2} \|g\|_{\dot{H}^{1/2}} . \tag{236}$$

Proof. We integrate by parts, first rewriting $\partial_{\beta'} g(\beta') = \partial_{\beta'}(g(\beta') - g(\alpha'))$:

$$\begin{aligned}
[f, \mathbb{H}]\partial_{\alpha'} g &= \frac{1}{2i} \int f_{\beta'}(\beta') \cot(\frac{\pi}{2}(\alpha' - \beta'))(g(\beta') - g(\alpha'))d\beta' \\
&\quad - \frac{1}{2i} \int \frac{\pi}{2} \frac{f(\alpha') - f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} (g(\beta') - g(\alpha'))d\beta',
\end{aligned} \tag{237}$$

where there is no boundary term because of the periodic boundary conditions.

For the first term, we apply Cauchy-Schwarz:

$$\begin{aligned}
&\left| \int f_{\beta'}(\beta') \cot(\frac{\pi}{2}(\alpha' - \beta'))(g(\beta') - g(\alpha'))d\beta' \right| \\
&\leqslant \|f'\|_{L^2} \left(\int |g(\alpha') - g(\beta')|^2 \left| \cot^2(\frac{\pi}{2}(\alpha' - \beta')) \right| d\beta' \right)^{1/2} .
\end{aligned} \tag{238}$$

Taking L^2 of this in α' and using the boundedness of cosine to replace $\cot^2$ with $\frac{1}{\sin^2}$, we get the needed estimate.

---

[34] Note that we do not need the $g' \in L^p$ condition of that proposition, which was used to make sure $\mathbb{H}g'$ was well-defined.

For the second term, we use Cauchy-Schwarz:

$$\left\| \int \frac{f(\alpha') - f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}(g(\beta') - g(\alpha'))d\beta' \right\|_{L^2_{\alpha'}}$$

$$\leqslant \left(\int \int \frac{|f(\alpha') - f(\beta')|^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}d\beta' \int \frac{|g(\alpha') - g(\beta')|^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}d\beta'd\alpha' \right)^{1/2}, \tag{239}$$

and then we use Hardy's inequality (217) on f to get our inequality (236). $\qquad\square$

There's also another, easier $\dot{H}^{1/2}$ estimate:

Proposition 19. *There exists a constant $C > 0$ such that for any $f \in \dot{H}^{1/2}, g \in L^2$ (and so $f|_\partial = 0$),*

$$\|[f, \mathbb{H}]g\|_{L^2} \leqslant C \|f\|_{\dot{H}^{1/2}} \|g\|_{L^2}. \tag{240}$$

Proof. This is immediate by Cauchy-Schwarz and the boundedness of cosine. $\qquad\square$

Proposition 20. *There exists a constant $C > 0$ such that for any $f, g \in C^1(-1, 1) \cap C^0(S^1)$ with $f', g' \in L^2$ and $h \in L^2$ (and so $f|_\partial, g|_\partial = 0$),*

$$\|[f, g; h]\|_{L^2} := \left\| \frac{1}{2i} \int \frac{f(\alpha') - f(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \frac{g(\alpha') - g(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))}h(\beta')d\beta' \right\|_{L^2} \tag{241}$$

$$\leqslant C \|f'\|_{L^2} \|g'\|_{L^2} \|h\|_{L^2}.$$

Proof. By Cauchy-Schwarz,

$$\left| \int \frac{f(\alpha') - f(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \frac{g(\alpha') - g(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))}h(\beta')d\beta' \right|$$

$$\leqslant \left(\int \left| \frac{f(\alpha') - f(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \right|^2 d\beta' \right)^{1/2} \left(\int \left| \frac{g(\alpha') - g(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))}h(\beta') \right|^2 d\beta' \right)^{1/2}. \tag{242}$$

Now we take the L^2 of this in the α' variable. By Hardy's inequality (217), we control the f factor by $\|f'\|_{L^2}$, and are left with

$$\left(\int \int \left| \frac{g(\alpha') - g(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))}h(\beta') \right|^2 d\beta'd\alpha' \right)^{1/2}. \tag{243}$$

Applying Fubini and then using Hardy's inequality (217) once more gives the result. $\qquad\square$

We also rely on the following easy L^∞ estimate.

Proposition 21. *There exists a constant $C > 0$ such that for any $f \in C^1(-1,1) \cap C^0(S^1)$ with $f' \in L^2$, $g \in L^2$ (and so $f|_\partial = 0$),*

$$\| [f, \mathbb{H}]g \|_{L^\infty} \leqslant C \, \|f'\|_{L^2} \, \|g\|_{L^2} \,. \tag{244}$$

Proof. Estimate (244) holds by Cauchy-Schwarz and Hardy's inequality (218). □

Proposition 22. *There exists a constant $C > 0$ such that for any $f, g \in C^1(-1,1) \cap C^0(S^1)$ with $f', g' \in L^2$ and $h \in L^2$ (and so $f|_\partial, g|_\partial = 0$),*

$$\| \partial_{\alpha'}[f, [g, \mathbb{H}]]h \|_{L^2} \lesssim \|f'\|_{L^2} \, \|g'\|_{L^2} \, \|h\|_{L^2} \,. \tag{245}$$

Proof. We differentiate[35]:

$$\partial_{\alpha'} \frac{1}{2i} \int (f(\alpha') - f(\beta'))(g(\alpha') - g(\beta')) \cot(\frac{\pi}{2}(\alpha' - \beta'))h(\beta')d\beta'$$

$$= f'[g, \mathbb{H}]h + g'[f, \mathbb{H}]h - \frac{1}{2i} \int \frac{\pi}{2} \frac{(f(\alpha') - f(\beta'))(g(\alpha') - g(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} h(\beta')d\beta'. \tag{246}$$

We control the first two terms by Hölder and then (244). We control the last term by (241). □

At one point, we require an estimate on a higher-order Calderon commutator.

Proposition 23 (Higher-Order Calderon Commutator [CM78])**.** *There exists a constant $C > 0$ such that for any $f \in C^1[-1,1] \cap C^0(S^1)$ and $h \in C^0(S^1) \cap C^1(-1,1)$ with $h' \in L^p$ for some $p > 1$ (and so $f|_\partial, h|_\partial = 0$),*

$$\| [f, f; \partial_{\alpha'}h] \|_{L^2} \leqslant C \, \|f'\|_{L^\infty}^2 \, \|h\|_{L^2} \,. \tag{247}$$

Proof. The proof is entirely analogous to the proof of (226), and now follows from the work of [CM78], which extends the original result of [Cal65] used for (226) to allow two difference quotient factors, instead of one. To move from $\mathbb{R}$ to our compact domain, we do the same infinite summation argument, with the only difference being that instead of $\partial_{\alpha'} \cot(\frac{\pi}{2}\alpha') = c \sum \frac{1}{(\alpha'+2l)^2}$ we have $\partial_{\alpha'} \frac{1}{\sin^2(\frac{\pi}{2}\alpha')} = c \sum \frac{1}{(\alpha'+2l)^3}$. □

---

[35] To differentiate under the integral sign, we first assume that $f, g \in C^1[-1,1] \cap C^0(S^1)$. Then by the density of $C^0[-1,1]$ in L^2 we can reduce to assuming $f', g' \in L^2$.

4.5 The Half-Derivative Space $\dot{H}^{1/2}$

We show here that the following.

Proposition 24. *Let $f \in C^1(-1,1) \cap C^0(S^1)$ with $f' \in L^2$. Then*

$$(I - \mathbb{H})f = \fint f \Rightarrow \|f\|_{\dot{H}^{1/2}}^2 = \int i(\partial_{\alpha'} f)\overline{f} d\alpha'. \tag{248}$$

Proof. We begin by rewriting the RHS of (248) as

$$\int i(\partial_{\alpha'} \mathbb{H} f)\overline{f} d\alpha', \tag{249}$$

using $(I - \mathbb{H})f = \fint f$, where the mean term disappears under the derivative. We now expand this out. To be absolutely rigorous with interchanging limits, etc., our derivation is slightly convoluted. We begin as usual by introducing the difference quotient:

$$\begin{aligned}
\int i(\partial_{\alpha'} \mathbb{H} f)\overline{f} d\alpha' &= \int \frac{i}{2i} \overline{f}(\alpha') \partial_{\alpha'} \lim_{\varepsilon \to 0} \int_{I_\varepsilon} \cot(\frac{\pi}{2}(\alpha' - \beta'))f(\beta')d\beta' d\alpha' \\
&= \frac{1}{2} \int \overline{f}(\alpha') \partial_{\alpha'} \lim_{\varepsilon \to 0} \int_{I_\varepsilon} \cot(\frac{\pi}{2}(\alpha' - \beta'))(f(\beta') - f(\alpha'))d\beta' d\alpha'.
\end{aligned} \tag{250}$$

Now we want to move the $\lim_{\varepsilon \to 0}$ outside the outer integral. To do this, we first integrate by parts, so that we can use the Lebesgue dominated convergence theorem, and then integrate by parts back. On integrating by parts, we have

$$\int i(\partial_{\alpha'} \mathbb{H} f)\overline{f} d\alpha' = -\frac{1}{2} \int (\partial_{\alpha'} \overline{f}(\alpha')) \lim_{\varepsilon \to 0} \int_{I_\varepsilon} \cot(\frac{\pi}{2}(\alpha' - \beta'))(f(\beta') - f(\alpha'))d\beta' d\alpha'. \tag{251}$$

(Note that there are no boundary terms by the periodicity of cotangent and f.) By Hardy's inequality (218), the outer integrand is uniformly bounded in ε by $|f'| \, \|f'\|_{L^2} \in L^1$, so we may use the Lebesgue dominated convergence theorem to pull the limit outside the outer integral. We do this and then integrate by parts back (where, once again, there are no

boundary terms):

$$\int i(\partial_{\alpha'}\mathbb{H}f)\overline{f}d\alpha' = -\frac{1}{2}\lim_{\varepsilon\to0}\int(\partial_{\alpha'}\overline{f}(\alpha'))\int_{I_\varepsilon}\cot(\frac{\pi}{2}(\alpha'-\beta'))(f(\beta')-f(\alpha'))d\beta'd\alpha'$$
$$= \frac{1}{2}\lim_{\varepsilon\to0}\int\overline{f}(\alpha')\partial_{\alpha'}\int_{I_\varepsilon}\cot(\frac{\pi}{2}(\alpha'-\beta'))(f(\beta')-f(\alpha'))d\beta'd\alpha'$$
$$= \frac{1}{2}\lim_{\varepsilon\to0}\int\overline{f}(\alpha')\int_{I_\varepsilon}\partial_{\alpha'}\left(\cot(\frac{\pi}{2}(\alpha'-\beta'))(f(\beta')-f(\alpha'))\right)d\beta'd\alpha'+\partial \quad (252)$$
$$= -\frac{1}{2}\lim_{\varepsilon\to0}\int\overline{f}(\alpha')\int_{I_\varepsilon}\frac{\pi}{2}\frac{f(\beta')-f(\alpha')}{\sin^2(\frac{\pi}{2}(\alpha'-\beta'))}d\beta'd\alpha'+\partial,$$

where[36]

$$\partial = \frac{1}{2}\lim_{\varepsilon\to0}\int\overline{f}(\alpha')\cot(\frac{\pi}{2}(\alpha'-(\alpha'-\varepsilon)))(f(\alpha'-\varepsilon)-f(\alpha'))d\alpha'$$
$$- \frac{1}{2}\lim_{\varepsilon\to0}\int\overline{f}(\alpha')\cot(\frac{\pi}{2}(\alpha'-(\alpha'+\varepsilon)))(f(\alpha'+\varepsilon)-f(\alpha'))d\alpha'$$
$$= \lim_{\varepsilon\to0}\frac{1}{2}\int\overline{f}(\alpha')\left(\cot(\frac{\pi}{2}\varepsilon)(f(\alpha'-\varepsilon)-f(\alpha'))-\cot(-\frac{\pi}{2}\varepsilon)(f(\alpha'+\varepsilon)-f(\alpha'))\right)d\alpha' \quad (253)$$
$$= \lim_{\varepsilon\to0}\frac{1}{2}\int\overline{f}(\alpha')\left(\cot(\frac{\pi}{2}\varepsilon)(f(\alpha-\varepsilon)+f(\alpha'+\varepsilon)-2f(\alpha'))\right)d\alpha'$$
$$= -G'(0)+G'(0)=0$$

for

$$G(x)=\frac{1}{\pi}\int\overline{f}(\alpha')f(\alpha'+x)d\alpha'. \quad (254)$$

We were able to pull the derivative inside the integral in the third line of (252) because of the principal value, and there was no term $\int_{I_\varepsilon}\cot(\frac{\pi}{2}(\alpha'-\beta'))f'(\alpha')d\beta'$ in the last line of (252) because that is zero. By the same derivation, interchanging α' and β', we have

$$\int i(\partial_{\alpha'}\mathbb{H}f)\overline{f}d\alpha' = \int i(\partial_{\beta'}\mathbb{H}f)\overline{f}d\beta'$$
$$= -\frac{1}{2}\lim_{\varepsilon\to0}\int\overline{f}(\beta')\int_{I_\varepsilon}\frac{\pi}{2}\frac{f(\alpha')-f(\beta')}{\sin^2(\frac{\pi}{2}(\beta'-\alpha'))}d\alpha'd\beta' \quad (255)$$
$$= \frac{1}{2}\lim_{\varepsilon\to0}\int\overline{f}(\beta')\int_{I_\varepsilon}\frac{\pi}{2}\frac{f(\beta')-f(\alpha')}{\sin^2(\frac{\pi}{2}(\alpha'-\beta'))}d\alpha'd\beta'.$$

---

[36]Technically, this has to be adjusted slightly if $\alpha'-\varepsilon<-1$ or $\alpha+\varepsilon>1$; recall our definition of I_ε at (20). By the periodicity of f, this reduces to what's below.

We may now apply Fubini to (252) and (255), separately, and then average the two, noting that because of the principal value there are no issues with applying Fubini. We get

$$
\begin{aligned}
\int i(\partial_{\alpha'}\mathbb{H}f)\overline{f}d\alpha' &= \frac{\pi}{8}\lim_{\varepsilon\to 0}\iint_{(I\times I)_\varepsilon} \frac{(f(\beta')-f(\alpha'))\left(\overline{f}(\beta')-\overline{f}(\alpha')\right)}{\sin^2(\frac{\pi}{2}(\alpha'-\beta'))}d\beta'd\alpha' \\
&= \frac{\pi}{8}\lim_{\varepsilon\to 0}\iint_{(I\times I)_\varepsilon} \frac{|f(\beta')-f(\alpha')|^2}{\sin^2(\frac{\pi}{2}(\alpha'-\beta'))}d\beta'd\alpha' \\
&= \frac{\pi}{8}\iint \frac{|f(\beta')-f(\alpha')|^2}{\sin^2(\frac{\pi}{2}(\alpha'-\beta'))}d\beta'd\alpha' \\
&= \|f\|_{\dot{H}^{1/2}}^2,
\end{aligned}
\tag{256}
$$

where we've used the monotone convergence theorem to remove the principal value limit. $\quad\square$

4.6 Commutator Identities

We include here for reference the various commutator identities that are necessary.

First, we calculate the basic commutators to interchange time differentiation and our special weighted spatial derivative.

$$
[\partial_t, D_\alpha] = -(D_\alpha z_t)D_\alpha
\tag{257}
$$

$$
\begin{aligned}
\left[\partial_t, D_\alpha^2\right] &= \partial_t D_\alpha^2 - D_\alpha\partial_t D_\alpha + D_\alpha\partial_t D_\alpha - D_\alpha^2\partial_t \\
&= [\partial_t, D_\alpha]D_\alpha + D_\alpha[\partial_t, D_\alpha] \\
&= -(D_\alpha z_t)D_\alpha^2 - D_\alpha((D_\alpha z_t)D_\alpha) \\
&= -2(D_\alpha z_t)D_\alpha^2 - (D_\alpha^2 z_t)D_\alpha.
\end{aligned}
\tag{258}
$$

Now we calculator the commutators $[\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha^k]$, which we will need to control for $k = 1, 2$.

$$
\begin{aligned}
\left[\partial_t^2, D_\alpha\right] &= \partial_t^2\frac{1}{z_\alpha}\partial_\alpha - \frac{1}{z_\alpha}\partial_\alpha\partial_t^2 \\
&= -\partial_t\frac{z_{t\alpha}}{z_\alpha^2}\partial_\alpha + \partial_t\frac{1}{z_\alpha}\partial_\alpha\partial_t - \frac{1}{z_\alpha}\partial_a\partial_t^2 \\
&= -\frac{z_{tt\alpha}}{z_\alpha^2}\partial_\alpha + \frac{z_{t\alpha}2z_{t\alpha}}{z_\alpha^3}\partial_\alpha - \frac{z_{t\alpha}}{z_\alpha^2}\partial_\alpha\partial_t - \frac{z_{t\alpha}}{z_\alpha^2}\partial_\alpha\partial_t + \frac{1}{z_\alpha}\partial_\alpha\partial_t^2 - \frac{1}{z_\alpha}\partial_\alpha\partial_t^2 \\
&= (-D_\alpha z_{tt})D_\alpha + 2(D_\alpha z_t)^2 D_\alpha - 2(D_\alpha z_t)D_\alpha\partial_t.
\end{aligned}
\tag{259}
$$

60

To calculate $[i\mathfrak{a}\partial_\alpha, D_\alpha]$, we use $i\mathfrak{a}z_\alpha = z_{tt} + i$ (27) to rewrite $i\mathfrak{a}\partial_\alpha = i\mathfrak{a}z_\alpha D_\alpha = (z_{tt} + i)D_\alpha$. Therefore

$$[i\mathfrak{a}\partial_\alpha, D_\alpha] = [(z_{tt} + i)D_\alpha, D_\alpha] = -(D_\alpha z_{tt})D_\alpha. \tag{260}$$

Adding (259) and (260), we conclude that

$$\left[\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha\right] = (-2D_\alpha z_{tt})D_\alpha + 2(D_\alpha z_t)^2 D_\alpha - 2(D_\alpha z_t)D_\alpha\partial_t. \tag{261}$$

Because $[(\partial_t^2 + i\mathfrak{a}\partial_\alpha), D_\alpha^2] = [(\partial_t^2 + i\mathfrak{a}\partial_\alpha), D_\alpha]D_\alpha + D_\alpha[(\partial_t^2 + i\mathfrak{a}\partial_\alpha), D_\alpha]$, we have

$$
\begin{aligned}
\left[(\partial_t^2 + i\mathfrak{a}\partial_\alpha), D_\alpha^2\right] = {} & (-2D_\alpha z_{tt})D_\alpha^2 + 2(D_\alpha z_t)^2 D_\alpha^2 - 2(D_\alpha z_t)D_\alpha\partial_t D_\alpha - (2D_\alpha^2 z_{tt})D_\alpha \\
& - (2D_\alpha z_{tt})D_\alpha^2 + 4(D_\alpha z_t)(D_\alpha^2 z_t)D_\alpha + 2(D_\alpha z_t)^2 D_\alpha^2 \\
& - 2(D_\alpha^2 z_t)D_\alpha\partial_t - 2(D_\alpha z_t)D_\alpha^2\partial_t \\
= {} & (-4D_\alpha z_{tt})D_\alpha^2 + 4(D_\alpha z_t)^2 D_\alpha^2 - 2(D_\alpha z_t)D_\alpha\partial_t D_\alpha - (2D_\alpha^2 z_{tt})D_\alpha \\
& + 4(D_\alpha z_t)(D_\alpha^2 z_t)D_\alpha - 2(D_\alpha^2 z_t)D_\alpha\partial_t - 2(D_\alpha z_t)D_\alpha^2\partial_t.
\end{aligned}
\tag{262}
$$

Chapter 5

The Energy and the Theorem

5.1   Definition of the Energy

We now introduce the energy for which we will prove an a priori inequality. We consider a general equation of the form

$$(\partial_t^2 + i\mathfrak{a}\partial_\alpha)\theta = G_\theta, \tag{263}$$

with the constraint that θ is the boundary value of a periodic holomorphic function in $\Omega(t)$ and has mean zero. Our base case is (37), written in this form as

$$(\partial_t^2 + i\mathfrak{a}\partial_\alpha)\overline{z}_t = -i\mathfrak{a}_t\overline{z}_\alpha, \tag{264}$$

with $\theta = \overline{z}_t$ and $G_{\overline{z}_t} = -i\mathfrak{a}_t\overline{z}_a$. Higher-order cases will come from applying D_α^k to (264); in those cases, $\theta = D_\alpha^k\overline{z}_t$ with

$$G_\theta = D_\alpha^k(-i\mathfrak{a}_t\overline{z}_\alpha) + [\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha^k]\overline{z}_t. \tag{265}$$

A natural energy for (263) would be

$$\int |\theta_t|^2\,d\alpha + \Re\int i\,(\mathfrak{a}\partial_\alpha\theta)\,\overline{\theta}d\alpha. \tag{266}$$

Here, the fact that θ is holomorphic allows the second term to be rewritten as $\int i\,(\mathfrak{a}\mathfrak{H}\partial_\alpha\theta)\,\overline{\theta}d\alpha$, where $\mathfrak{H}$ is the Hilbert transform for the curved domain Ω. Because $i\mathfrak{H}\partial_\alpha$ is a positive operator—in flattened coordinates, it corresponds to $|D| = \sqrt{-\Delta}$—the second term will be non-negative up to an error term (due to the coefficient $\mathfrak{a}$).

We will, instead, use two variants of this energy, differing primarily in that we multiply or divide by the roughly equivalent singular weights $\mathfrak{a}$ and $\frac{h_\alpha}{A_1 \circ h}$.[37] (From our calculation in §3.7, we saw that $\mathfrak{a} \approx h_\alpha$, so long as $|z_\alpha|$ is bounded above and below and A_1 is bounded above.) In earlier works, where there was a strict bound $\mathfrak{a} \geqslant c_0 > 0$ (in addition to the upper bound we continue to have), all of these energies were essentially equivalent, but here the singularity $\mathfrak{a} = 0$ makes an important difference, and so the choice of the weights is critical.

We note that we are phrasing everything here in Lagrangian coordinates. With a change of variables, we can easily switch to the Riemannian coordinates. We will often express our basic quantities in the Lagrangian coordinates when we need to take a time derivative but use Riemannian coordinates when we need to estimate terms, since that gives us access to the easily invertible $(I - \mathbb{H})$ operator.

Our first energy differs from (266) primarily by a multiplicative factor of h_α:

$$
\begin{aligned}
E_{a,\theta}(t) &:= \int_I |\theta_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha + \Re \int_I (i\mathfrak{a}\partial_\alpha \theta)\overline{\theta}\frac{h_\alpha}{A_1 \circ h} d\alpha \\
&\quad + \Re \int_I \left(i\frac{1}{\overline{z}_\alpha}\partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha + \int_I |\theta|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha.
\end{aligned}
\tag{267}
$$

Here, the $A_1 \circ h$ appears for technical reasons, which will be explained below in §5.2. The first two terms are the main terms. The third term is a correction to the second term, to be explained below, and the fourth term is lower-order.

For our second energy, E_b, we divide through by $\mathfrak{a}$ in the main terms:

$$
E_{b,\theta}(t) := \int_I \frac{1}{\mathfrak{a}} |\theta_t|^2 \, d\alpha + \int_I (i\partial_\alpha \theta)\overline{\theta}d\alpha + \int_I \frac{(A_1 \circ h)}{\mathfrak{a}} |\theta|^2 \, d\alpha.
\tag{268}
$$

Here, the first two integrals are the primary terms; the last is lower-order. It's easy to check, by integration by parts, that the second term is purely real.

Our total energy consists primarily of $E_{a,\theta}$ using two D_α derivatives and $E_{b,\theta}$ using one D_α derivative. In addition, we will include one other, lower-order term in our total energy: $|\overline{z}_{tt}(\alpha_0, t) - i|$ for some fixed $\alpha_0 \in I$. Our total energy therefore is

$$
E(t) := E_{a,D_\alpha^2 \overline{z}_t}(t) + E_{b,D_\alpha \overline{z}_t}(t) + |\overline{z}_{tt}(\alpha_0, t) - i|.
\tag{269}
$$

Even though we have now specialized to the case where $\theta = D_\alpha^2 \overline{z}_t$ and $\theta = D_\alpha \overline{z}_t$ for $E_{a,\theta}$ and

---

[37]These weights go to zero at the corner for non-trivial angles, as well as at angled crests in the middle of the free surface.

$E_{b,\theta}$, respectively, we will in the following present some of the proofs about these energies in broader generality. We will refer to $E_{a,\theta}$ and $E_{b,\theta}$ for such generic energies, and use $E_a := E_{a,D_\alpha^2 \bar{z}_t}$ and $E_b := E_{b,D_\alpha \bar{z}_t}$ for the specific energies.

5.2 Discussion of the Energy

In this section, we offer a few comments on our energy.

We begin by discussing $E_{a,\theta}$. Here, what's most significant is the h_α weight. On changing variables to Riemannian coordinates, the first and the fourth terms of $E_{a,\theta}$ become $\|\theta_t \circ h^{-1}\|_{L^2(1/A_1)}^2$ and $\|\theta \circ h^{-1}\|_{L^2(1/A_1)}^2$. Since we will be able to control $\|A_1\|_{L^\infty}$ by our energy, these two terms therefore are equivalent to $\|\theta_t \circ h^{-1}\|_{L^2}$ and $\|\theta \circ h^{-1}\|_{L^2}$.

The second term and third terms in $E_{a,\theta}$ treated together equal $\left\| \frac{1}{Z_{,\alpha'}} (\theta \circ h^{-1}) \right\|_{\dot{H}^{1/2}}$. Indeed, when we use $A_1 \circ h = \frac{\mathfrak{a}|z_\alpha|^2}{h_\alpha}$ and then change variables, these two terms in $E_{a,\theta}$ become

$$
\Re \int (i\mathfrak{a}\partial_\alpha \theta)\bar{\theta} \frac{h_\alpha}{A_1 \circ h} d\alpha + \Re \int_I \left(i\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha
$$
$$
= \Re \int i\frac{h_\alpha^2}{|z_\alpha|^2} (\partial_\alpha \theta)\bar{\theta} d\alpha + \Re \int_I \left(i\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha
$$
$$
= \Re \int i \left(\partial_\alpha \left(\theta \frac{h_\alpha}{z_\alpha} \right) \right) \left(\bar{\theta} \frac{h_\alpha}{\bar{z}_\alpha} \right) d\alpha
$$
$$
= \Re \int i \left(\partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} (\theta \circ h^{-1}) \right) \right) \left(\frac{1}{\overline{Z}_{,\alpha'}} (\bar{\theta} \circ h^{-1}) \right) d\alpha'.
$$
(270)

We showed in §4.5 that this equals to the square of the $\dot{H}^{1/2}$ norm of $\frac{1}{Z_{,\alpha'}}(\theta \circ h^{-1})$. (Note that $\frac{1}{Z_{,\alpha'}}(\theta \circ h^{-1})$ is holomorphic; this explains why we divided through by $A_1 \circ h$ in our definition of $E_{a,\theta}$.) This $\dot{H}^{1/2}$ control will be crucial in allowing us to close our energy equality without loss of derivatives.

We now specialize to $E_a = E_{a,D_\alpha^2 \bar{z}_t}$. By the above discussions,

$$
E_a = \left\| (\partial_t D_\alpha^2 \bar{z}_t) \circ h^{-1} \right\|_{L^2(1/A_1)}^2 + \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\|_{\dot{H}^{1/2}}^2 + \left\| D_{\alpha'}^2 \overline{Z}_t \right\|_{L^2(1/A_1)}^2 .
$$
(271)

We will be able to control the commutator $[\partial_t, D_\alpha^2]$, so the first term corresponds roughly to $\left\| D_{\alpha'}^2 \overline{Z}_{tt} \right\|_{L^2(1/A_1)}$ or $\left\| D_{\alpha'}^2 \overline{Z}_{tt} \right\|_{L^2}$; we make this more precise in §6.1.

Similarly, by using the half-derivative calculation from §4.5, $E_b = E_{b,D_\alpha \bar{z}_t}$ equals

$$E_b = \left\| \partial_t D_\alpha \bar{z}_t \right\|_{L^2(\frac{1}{\mathfrak{a}})}^2 + \left\| D_{\alpha'} \overline{Z}_t \right\|_{\dot{H}^{1/2}}^2 + \left\| D_\alpha \bar{z}_t \right\|_{L^2(\frac{(A_1 \circ h)}{\mathfrak{a}})}^2 . \tag{272}$$

On changing variables to Riemannian coordinates, we see that, modulo a commutator $[\partial_t, D_\alpha]$ and a factor of A_1,

$$\left\| \partial_t D_\alpha \bar{z}_t \right\|_{L^2(\frac{1}{\mathfrak{a}})} \approx \left\| \overline{Z}_{tt,\alpha'} \right\|_{L^2} . \tag{273}$$

We make this more precise in §6.1. Furthermore, due to the additional $A_1 \circ h$ factor on the weight, the lower-order term

$$\left\| D_\alpha \bar{z}_t \right\|_{L^2(\frac{(A_1 \circ h)}{\mathfrak{a}})} = \left\| \overline{Z}_{t,\alpha'} \right\|_{L^2} . \tag{274}$$

We will see in (297) that $\|A_1\|_{L^\infty}$ is bounded by $\left\| \overline{Z}_{t,\alpha'} \right\|_{L^2}^2 + 1$. This explains our choice of this lower order term. We note that A_1 as well as $\|Z_{t,\alpha'}\|_{L^2}$ are invariant with respect to the scaling $\alpha' \sim t^2$ of the water wave equations (1)-(5), (7) (with $\Upsilon = \emptyset$).

We note that the roughly inverse singular weights of E_a and E_b will allow us to use the weighted Sobolev inequality (202) to control the L^∞ norm of various quantities.

Finally, we also include $|\bar{z}_{tt}(\alpha_0, t) - i|$ in our energy since it, along with control of $\left\| \overline{Z}_{tt,\alpha'} \right\|_{L^2}$, will allow us to control the lower-order term $\left\| \overline{Z}_{tt} - i \right\|_{L^\infty}$. We remark that this is a very benign assumption: all we are assuming is that the material derivative of the velocity is finite at a single point on the free sufrace. We never depend on an estimate of $\|Z_t\|_{L^\infty}$, but by a similar argument, so long as the velocity is finite at one point on the free surface, it will be bounded everywhere since $Z_{t,\alpha'} \in L^2$.

In §11 we offer further discussion of what our energy controls: in §11.1 we characterize the energy in terms only of $\overline{Z}_t$ and $\frac{1}{Z_{,\alpha'}}$, and in §11.2 we discuss what angles ν our energy allows.

5.3   The Main Result

We can now state our result.

Theorem 25. *[A Priori Inequality for the Water Wave Equation] There exists a polynomial $p(x, y, z)$ with universal coefficients such that, for any solution of water wave equations (1)*

to (7) satisfying the a priori smoothness assumptions of §3.5 with $E(t) < \infty$ for all $t \in [0, T]$,

$$\frac{d}{dt} E(t) \leqslant p \left(E(t), \|z_\alpha\|_{L^\infty} (t), \left\| \frac{1}{z_\alpha} \right\|_{L^\infty} (t) \right) \tag{275}$$

for all $t \in [0, T]$, where

$$\|z_\alpha\|_{L^\infty} (t) \lesssim \exp \int_0^t \left(E_a^{1/2} + E_a^{1/2} E_b^{1/2} + E_b^{1/2} \right) (\tau) d\tau \tag{276}$$

and

$$\left\| \frac{1}{z_\alpha} \right\|_{L^\infty} (t) \lesssim \exp \int_0^t \left(E_a^{1/2} + E_a^{1/2} E_b^{1/2} + E_b^{1/2} \right) (\tau) d\tau. \tag{277}$$

Here the fixed boundary Υ in the water wave equations (1)-(7) is that specified in §1.3. We do not state precise polynomial $p(E, \|z_\alpha\|_{L^\infty}, \|1/z_\alpha\|_{L^\infty})$, but it can be calculated by carefully combining the estimates we make in the proof. Estimates for the individual parts of the energy E_a, E_b, and $|\bar{z}_{tt}(\alpha_0)|$ are listed in §5.4.[38]

Although we state the a priori inequality for the original equations (1)-(7) on $\Omega(t)$, our proof relies exclusively on the Lagrangian reduction to the equation (27) on the free surface $\Sigma(t)$, along with the accompanying statements about $\bar{z}_t$'s holomorphicity and periodicity.

5.4   The Proof

As is standard for energy inequalities, we begin the proof by differentiating the main components of $E(t)$ in time. For the two main energies, E_a and E_b, we then integrate by parts to arrive at a term $\partial_t^2 \theta + i\mathfrak{a}\partial_\alpha \theta$ and use the basic equation $\partial_t^2 \theta + i\mathfrak{a}\partial_\alpha \theta = G_\theta$ to replace it with G_θ. What remains to be estimated will be G_θ, along with several ancillary terms. We then control these quantities in terms of a polynomial of the energy, along with $\|z_\alpha\|_{L^\infty}$ and $\|1/z_\alpha\|_{L^\infty}$, which we estimate in §6.2.[39]

5.4.1   The Estimate for E_a

We begin by differentiating E_a with respect to t.

---

[38]We remark that we have not distinguished between the higher-order and lower-order parts of E_a and E_b in these differential inequalities, but it would be straightforward to do a refined estimate separating these.

[39]Technically, our polynomial includes square roots of the energy; this may be relaxed to remove the square roots, at the loss of some sharpness.

We will work initially with general θ satisfying $\theta|_\partial = 0$, $(I - \mathbb{H})(\theta \circ h^{-1}) = 0$, and the basic equation (263), and then we will specialize to the $\theta = D_\alpha^2 \bar{z}_t$ in our energy. Note that $\theta|_\partial = 0$ ensures that there is no boundary term in the integration by parts below; $\theta = D_\alpha^2 \bar{z}_t$ satisfies this periodic boundary condition by (47).

We use the fact that $\frac{\mathfrak{a} h_\alpha}{(A_1 \circ h)} = \frac{h_\alpha^2}{|z_\alpha|^2}$ in the following calculation.

$$
\frac{d}{dt} E_{a,\theta}(t) = \int (\theta_{tt}\bar{\theta}_t + \theta_t\bar{\theta}_{tt}) \frac{h_\alpha}{A_1 \circ h} d\alpha + \int |\theta_t|^2 \frac{h_{t\alpha}}{A_1 \circ h} d\alpha - \int |\theta_t|^2 \frac{h_\alpha}{A_1 \circ h} \frac{(A_1 \circ h)_t}{A_1 \circ h} d\alpha
$$
$$
+ \Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha + \underbrace{\Re \int i \frac{h_\alpha^2}{|z_\alpha|^2} \theta_{t\alpha} \bar{\theta} d\alpha}_{-\Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_\alpha \theta_t \bar{\theta} d\alpha + \Re \int \frac{h_\alpha^2}{|z_\alpha|^2} \theta_t \overline{i\theta_\alpha} d\alpha} + \Re \int i \frac{h_\alpha^2}{|z_\alpha|^2} \theta_\alpha \bar{\theta}_t d\alpha
$$
$$
+ \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha + \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) (\theta_t \bar{\theta} + \theta\bar{\theta}_t) h_\alpha d\alpha
$$
$$
+ \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_{t\alpha} d\alpha
$$
$$
+ \int (\theta_t\bar{\theta} + \bar{\theta}_t\theta) \frac{h_\alpha}{A_1 \circ h} + \int |\theta|^2 \frac{h_{t\alpha}}{A_1 \circ h} d\alpha - \int |\theta|^2 \frac{h_\alpha}{A_1 \circ h} \frac{(A_1 \circ h)_t}{A_1 \circ h} d\alpha
$$
$$
= \int 2\Re \left((\theta_{tt} + i\mathfrak{a}\theta_\alpha)\bar{\theta}_t \right) \frac{h_\alpha}{A_1 \circ h} d\alpha
$$
$$
+ \int |\theta_t|^2 \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{(A_1 \circ h)_t}{A_1 \circ h} \right) \frac{h_\alpha}{A_1 \circ h} d\alpha + \int |\theta|^2 \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{(A_1 \circ h)_t}{A_1 \circ h} \right) \frac{h_\alpha}{A_1 \circ h} d\alpha
$$
$$
+ \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha
$$
$$
+ \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) (\theta_t \bar{\theta} + \theta\bar{\theta}_t) h_\alpha d\alpha
$$
$$
+ \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 \frac{h_{t\alpha}}{h_\alpha} h_\alpha d\alpha
$$
$$
+ 2\Re \int \theta_t \bar{\theta} \frac{h_\alpha}{A_1 \circ h} d\alpha
$$
$$
- \Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_\alpha \theta_t \bar{\theta} d\alpha
$$
$$
+ \Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha.
$$

(278)

Now we show how we control each of these terms.

For the first, we replace $\theta_{tt} + i\mathfrak{a}\theta_\alpha$ with the RHS G_θ by the main equation (263) and then

use Cauchy-Schwarz:

$$\int 2\Re\left((\theta_{tt} + i\mathfrak{a}\theta_\alpha)\overline{\theta}_t\right) \frac{h_\alpha}{A_1 \circ h} d\alpha = \int 2\Re\left\{G_\theta\overline{\theta}_t\right\} \frac{h_\alpha}{A_1 \circ h} d\alpha$$

$$\lesssim \left(\int |G_\theta|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha\right)^{1/2} \left(\int |\overline{\theta}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha\right)^{1/2}. \tag{279}$$

The first factor, the RHS of the equation, is the main term to control. For $\theta = D_\alpha^2 \overline{z}_t$,

$$G_\theta = D_\alpha^2(-i\mathfrak{a}_t\overline{z}_\alpha) + [\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha^2]\overline{z}_t. \tag{280}$$

We estimate these terms in §10: the first at (583) and the commutator at (475).

Using Hölder's inequality, we estimate

$$\int |\theta_t|^2 \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{(A_1 \circ h)_t}{A_1 \circ h}\right) \frac{h_\alpha}{A_1 \circ h} d\alpha + \int |\theta|^2 \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{(A_1 \circ h)_t}{A_1 \circ h}\right) \frac{h_\alpha}{A_1 \circ h} d\alpha$$

$$\leqslant \left\|\frac{h_{t\alpha}}{h_\alpha} - \frac{(A_1 \circ h)_t}{A_1 \circ h}\right\|_{L^\infty} \left(\int |\theta_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha + \int |\theta|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha\right)$$

$$\leqslant \left(\left\|\frac{h_{t\alpha}}{h_\alpha}\right\|_{L^\infty} + \left\|\frac{(A_1 \circ h)_t}{A_1 \circ h}\right\|_{L^\infty}\right) E_{a,\theta}. \tag{281}$$

In §7, we will control

$$\left|\Re \int i\left(\frac{1}{\overline{z}_\alpha}\partial_\alpha \frac{h_\alpha}{z_\alpha}\right)_t |\theta|^2 h_\alpha d\alpha\right| \lesssim (387). \tag{282}$$

We estimate

$$\left|\Re \int i\left(\frac{1}{\overline{z}_\alpha}\partial_\alpha \frac{h_\alpha}{z_\alpha}\right)(\theta_t\overline{\theta} + \theta\overline{\theta}_t)h_\alpha d\alpha\right| \lesssim \left\|D_\alpha \frac{h_\alpha}{z_\alpha}\right\|_{L^\infty} \left(\int |\theta_t|^2 h_\alpha d\alpha\right)^{1/2} \left(\int |\theta|^2 h_\alpha d\alpha\right)^{1/2}$$

$$\lesssim \left\|D_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \|A_1\|_{L^\infty} E_{a,\theta}. \tag{283}$$

We estimate

$$\left|\Re \int i\left(\frac{1}{\overline{z}_\alpha}\partial_\alpha \frac{h_\alpha}{z_\alpha}\right)|\theta|^2 \frac{h_{t\alpha}}{h_\alpha} h_\alpha d\alpha\right| \lesssim \left\|D_\alpha \frac{h_\alpha}{z_\alpha}\right\|_{L^\infty} \left\|\frac{h_{t\alpha}}{h_\alpha}\right\|_{L^\infty} \int |\theta|^2 h_\alpha d\alpha$$

$$\lesssim \left\|D_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \left\|\frac{h_{t\alpha}}{h_\alpha}\right\|_{L^\infty} \|A_1\|_{L^\infty} E_{a,\theta}. \tag{284}$$

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We estimate

$$\left| 2\Re \int \theta_t \bar{\theta} \frac{h_\alpha}{A_1 \circ h} d\alpha \right| \leqslant 2 \left(\int |\theta_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \right)^{1/2} \left(\int |\theta|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \right)^{1/2} \tag{285}$$
$$\leqslant 2E_{a,\theta}.$$

We estimate

$$\left| -\Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_\alpha \theta_t \bar{\theta} d\alpha \right| = \left| -\Re \int i \frac{\left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_\alpha}{h_\alpha} (A_1 \circ h) \theta_t \bar{\theta} \frac{h_\alpha}{A_1 \circ h} d\alpha \right|$$
$$\leqslant \|A_1 \circ h\|_{L^\infty} \left\| \frac{\left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_\alpha}{h_\alpha} \right\|_{L^\infty} \int \left| \theta_t \bar{\theta} \frac{h_\alpha}{A_1 \circ h} \right| d\alpha. \tag{286}$$

We observe that

$$\left\| \frac{\left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_\alpha}{h_\alpha} \right\|_{L^\infty} = \left\| \partial_{\alpha'} \frac{1}{|Z_{,\alpha'}|^2} \right\|_{L^\infty} \leqslant 2 \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}. \tag{287}$$

Therefore, using Cauchy-Schwarz to expand the last factor in (286), we have

$$\left| -\Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_\alpha \theta_t \bar{\theta} d\alpha \right| \lesssim \|A_1\|_{L^\infty} \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} E_{a,\theta}. \tag{288}$$

Finally, in §8 we control

$$\Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha \lesssim (433). \tag{289}$$

We now combine these estimates and specialize to $\theta = D_\alpha^2 \bar{z}_t$. Each of the remaining factors we will control in §6; we list the location of the final estimate for each quantity of

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the following in the subscripts. We get

$$
\begin{aligned}
\left| \frac{d}{dt} E_a \right| \lesssim\ & \underbrace{\left\| G_{D_\alpha^2 \bar{z}_t} \right\|_{L^2\left(\frac{h_\alpha}{A_1 \circ h}\right)}}_{\lesssim (583)+(475)} E_a^{1/2} \\
& + \underbrace{\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty}}_{\lesssim (347)} E_a + \underbrace{\left\| \frac{(A_1 \circ h)_t}{A_1 \circ h} \right\|_{L^\infty}}_{\lesssim (350)} E_a \\
& + \underbrace{\left(1 + \left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} \right)}_{\lesssim 1+(347)} \underbrace{\| A_1 \|_{L^\infty}}_{(297)} \underbrace{\left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}}_{\lesssim (367)} E_a \\
& + E_a \\
& + \underbrace{\Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha}_{\lesssim (387)} \\
& + \underbrace{\Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha}_{\lesssim (433)} \,.
\end{aligned}
\tag{290}
$$

5.4.2   The Estimate for E_b

Now we consider our second energy. Once again, we work first with general θ satisfying $\theta|_\partial = 0$, $(I - \mathbb{H})(\theta \circ h^{-1}) = 0$, and the main equation (263). Then we specialize to $\theta = D_\alpha \bar{z}_t$. Note again that $\theta|_\partial = 0$ ensures that there is no boundary term in the integration by parts

below; $\theta = D_\alpha \bar{z}_t$ satisfies this periodic boundary condition by (47).

$$\frac{d}{dt} E_{b,\theta}(t) = \int \frac{1}{\mathfrak{a}} (\theta_{tt} \bar{\theta}_t + \theta_t \bar{\theta}_{tt}) d\alpha - \int \frac{\mathfrak{a}_t}{\mathfrak{a}} \frac{1}{\mathfrak{a}} |\theta_t|^2 d\alpha$$

$$+ \underbrace{\int i\theta_{t\alpha} \bar{\theta} d\alpha}_{= \int \overline{i\theta_\alpha \theta_t} d\alpha} + \int i\theta_\alpha \bar{\theta}_t d\alpha$$

$$+ \int \frac{(A_1 \circ h)}{\mathfrak{a}} (\theta_t \bar{\theta} + \theta \bar{\theta}_t) d\alpha + \int \frac{(A_1 \circ h)_t}{\mathfrak{a}} |\theta|^2 d\alpha - \int \frac{\mathfrak{a}_t}{\mathfrak{a}} \frac{(A_1 \circ h)}{\mathfrak{a}} |\theta|^2 d\alpha$$

$$= 2\Re \left(\int \frac{1}{\mathfrak{a}} \theta_{tt} \bar{\theta}_t d\alpha + \int i\theta_\alpha \bar{\theta}_t d\alpha \right) - \int \frac{\mathfrak{a}_t}{\mathfrak{a}} \frac{1}{\mathfrak{a}} |\theta_t|^2 d\alpha \qquad (291)$$

$$+ \int \frac{A_1 \circ h}{\mathfrak{a}} (\theta_t \bar{\theta} + \theta \bar{\theta}_t) d\alpha + \int \left(\frac{(A_1 \circ h)_t}{(A_1 \circ h)} - \frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \frac{(A_1 \circ h)}{\mathfrak{a}} |\theta|^2 d\alpha$$

$$= 2\Re \int \frac{G_\theta}{\mathfrak{a}} \bar{\theta}_t d\alpha - \int \frac{\mathfrak{a}_t}{\mathfrak{a}} \frac{1}{\mathfrak{a}} |\theta_t|^2 d\alpha$$

$$+ \int \frac{A_1 \circ h}{\mathfrak{a}} (\theta_t \bar{\theta} + \theta \bar{\theta}_t) d\alpha + \int \left(\frac{(A_1 \circ h)_t}{(A_1 \circ h)} - \frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \frac{(A_1 \circ h)}{\mathfrak{a}} |\theta|^2 d\alpha.$$

By Hölder and Cauchy-Schwarz, we conclude that

$$\left| \frac{d}{dt} E_{b,\theta}(t) \right| \lesssim \left\| \frac{G_\theta}{\sqrt{\mathfrak{a}}} \right\|_{L^2} E_{b,\theta}^{1/2} + \left(\|A_1\|_{L^\infty}^{1/2} + \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} + \left\| \frac{(A_1 \circ h)_t}{(A_1 \circ h)} \right\|_{L^\infty} \right) E_{b,\theta}. \qquad (292)$$

For $\theta = D_\alpha \bar{z}_t$, we control $\left\| \frac{G_\theta}{\sqrt{\mathfrak{a}}} \right\|_{L^2}$ in §9, at (435) and (472). We control $\|A_1\|_{L^\infty}$ at (297), $\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty}$ at (327), and $\left\| \frac{(A_1 \circ h)_t}{(A_1 \circ h)} \right\|_{L^\infty}$ at (350).

5.4.3 The Estimate for $|z_{tt}(\alpha_0, t)|$

Finally, we show that we can control $|\bar{z}_{tt}(\alpha_0, t) - i|$.[40] By differentiating with respect to t, we have, by the basic quasilinear equation (37),

$$\frac{d}{dt} |\bar{z}_{tt}(\alpha_0) - i| \leqslant |\bar{z}_{ttt}(\alpha_0)|$$

$$= |i\mathfrak{a}_t(\alpha_0) \bar{z}_\alpha(\alpha_0) + i\mathfrak{a}(\alpha_0) \bar{z}_{t\alpha}(\alpha_0)| \qquad (293)$$

$$\lesssim \left(\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} + \|D_\alpha \bar{z}_t\|_{L^\infty} \right) |\bar{z}_{tt}(\alpha_0) - i|.$$

---

[40] We remark that, if we chose $\alpha_0 = 1$, then so long as we are in the non-trivial angle regime, $\bar{z}_{tt}(\alpha_0) - i \equiv 0$ for all time.

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We control $\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty}$ below at (327) and $\|D_\alpha \bar{z}_t\|_{L^\infty}$ at (301).

5.5   Outline of the Remainder of the Proof

In §6 through §10, we complete the proof of the a priori inequality (275).

In §6, we control various quantities that are necessary for our proof. In §6.1, we carefully list the basic quantities controlled by our energy. In §6.2, we show that $\|z_\alpha\|_{L^\infty}$ and $\|1/z_\alpha\|_{L^\infty}$ are controlled. In §6.3-§6.10, we estimate various other quantities in terms of quantities already controlled in previous parts of §6. These quantities include many of the quantities to be controlled in §5.4 above, and are used in the remainder of the proof. In Appendix §B, we list and give references to all the quantities controlled in §6, which we then use without citation in §7 through §10.

In §7 and §8 we estimate the terms from (282) and (289) in the estimate of $\frac{d}{dt}E_a$ above. Finally, in §9 and §10 we conclude the estimates for $\frac{d}{dt}E_b$ and $\frac{d}{dt}E_a$, respectively, by controlling the G_θ terms, completing the proof.

The basic approach for many of the estimates is to try and use the fact that certain quantities are purely real-valued and others are holomorphic to express problematic terms as commutators involving the Hilbert transform, and then use the commutator estimates from §4.4 to avoid loss of derivatives. The estimates are very tight, and sometimes convoluted and tedious. Among the reasons for the complexity are:

- we have very little regularity to work with;

- we are working with a weighted derivative $D_{\alpha'}$ that has to be commuted with $\mathbb{H}$ and whose complex-valued weight makes inverting $(I - \mathbb{H})$ on real-valued functions more difficult;

- we have no positive lower bound for the Taylor coefficient $\mathfrak{a}$;

- because our estimates are tight, we have to take care in using different estimates for different terms, including treating certain terms as commutators while keeping others in $(I - \mathbb{H})$ form;

- we must take care that the quantities in our commutators have appropriate periodic boundary behavior; and

- we sometimes have to carefully decompose certain quantities into their holomorphic and antiholomorphic projections.

Throughout the remaining derivations, we will repeatedly rely on the identity

$$A_1 \circ h = \frac{\mathfrak{a}\, |z_\alpha|^2}{h_\alpha}. \tag{294}$$

Chapter 6

Quantities Controlled by Our Energy

Here we collect together many quantities that are controlled by our energy. In Appendix §B, we give a list of all the quantities controlled in this chapter that are quoted without citation in future chapters or sections within this chapter.

6.1   Basic Quantities Controlled by the Energy

In this section, we present a list of basic quantities controlled by our energy. Because conjugations and commutations of ∂_t with D_α add complexity, we take care to list those estimates as well. Consulting the appendix §B as a reference will be more useful than this section in many cases.

Using E_b and changing variables to Riemannian coordinates, we control

$$
\begin{aligned}
\left\| \overline{Z}_{t,\alpha'} \right\|_{L^2} &= \left(\int \left| \frac{\overline{z}_{t\alpha}}{h_\alpha} \circ h^{-1} \right|^2 d\alpha' \right)^{1/2} \\
&= \left(\int \frac{1}{h_\alpha} \left| \overline{z}_{t\alpha} \right|^2 d\alpha \right)^{1/2} \\
&= \left(\int \frac{(A_1 \circ h)}{\mathfrak{a}} \left| D_\alpha \overline{z}_t \right|^2 d\alpha \right)^{1/2} \\
&\lesssim E_b^{1/2}.
\end{aligned}
\tag{295}
$$

By taking conjugates, we similarly have

$$
\left\| Z_{t,\alpha'} \right\|_{L^2} \lesssim E_b^{1/2}.
\tag{296}
$$

Now we estimate $\|A_1\|_{L^\infty}$. By (180), we have

$$
\begin{aligned}
\|A_1\|_{L^\infty} &\leqslant \left\|[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}\right\|_{L^\infty} + 1 \\
&\lesssim \|Z_{t,\alpha'}\|_{L^2}^2 + 1 \\
&\lesssim E_b + 1
\end{aligned}
\tag{297}
$$

by the commutator estimate (244), and then by (295).

A natural space for quantities controlled by E_a is $L^2\left(\frac{h_\alpha d\alpha}{(A_1 \circ h)}\right)$, or, by a change of variables, $L^2(\frac{d\alpha'}{A_1})$. Thanks to (297) and $\|f\|_{L^2(d\alpha')} \leqslant \|A_1\|_{L^\infty}^{1/2} \|f\|_{L^2\left(\frac{d\alpha'}{A_1}\right)}$, we then have control of the same quantities in $L^2(d\alpha')$.

By (271), E_a directly controls

$$
\left\|D_\alpha^2 \overline{z}_t\right\|_{L^2\left(\frac{h_\alpha}{A_1 \circ h} d\alpha\right)} = \left\|D_{\alpha'}^2 \overline{Z}_t\right\|_{L^2\left(\frac{1}{A_1} d\alpha'\right)} \leqslant E_a^{1/2}
\tag{298}
$$

and

$$
\left\|\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t\right\|_{\dot{H}^{1/2}} = \left\|\overline{\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t}\right\|_{\dot{H}^{1/2}} \lesssim E_a^{1/2}.
\tag{299}
$$

Now we control $\|D_\alpha \overline{z}_t\|_{L^\infty} = \left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty}$. We work in Riemannian coordinates and use the weighted Sobolev inequality (203) with weight $\omega = \frac{1}{|Z_{,\alpha'}|^2}$ (and $\varepsilon = 1$). Note that $\fint (D_{\alpha'}\overline{Z}_t)^2 = 0$ by (166). By the Sobolev inequality, we have

$$
\begin{aligned}
\left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty} &\lesssim \left(\int \left|D_{\alpha'}\overline{Z}_t\right|^2 |Z_{,\alpha'}|^2 \, d\alpha'\right)^{1/2} + \left(\int \left|\partial_{\alpha'}D_{\alpha'}\overline{Z}_t\right|^2 \frac{1}{|Z_{,\alpha'}|^2} d\alpha'\right)^{1/2} \\
&= \left\|\overline{Z}_{t,\alpha'}\right\|_{L^2} + \left\|D_{\alpha'}^2 \overline{Z}_t\right\|_{L^2} \\
&\leqslant E_b^{1/2} + \|A_1\|_{L^\infty}^{1/2} E_a^{1/2} \\
&\lesssim E_a^{1/2} + E_a^{1/2} E_b^{1/2} + E_b^{1/2}.
\end{aligned}
\tag{300}
$$

We conclude that

$$
\begin{aligned}
\|D_\alpha z_t\|_{L^\infty} = \|D_{\alpha'} Z_t\|_{L^\infty} = \|D_\alpha \overline{z}_t\|_{L^\infty} &= \left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty} \\
&\lesssim E_a^{1/2} + E_a^{1/2} E_b^{1/2} + E_b^{1/2}.
\end{aligned}
\tag{301}
$$

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Now we use the commutator (257) to move the ∂_t inside the first term in E_b:

$$\left(\int |D_\alpha \overline{z}_{tt}|^2 \frac{d\alpha}{\mathfrak{a}}\right)^{1/2} \leqslant \left(\int |\partial_t D_\alpha \overline{z}_t|^2 \frac{d\alpha}{\mathfrak{a}}\right)^{1/2} + \left(\int |[\partial_t, D_\alpha]\overline{z}_t|^2 \frac{d\alpha}{\mathfrak{a}}\right)^{1/2}$$

$$\leqslant E_b^{1/2} + \left(\int |D_\alpha z_t|^2 |D_\alpha \overline{z}_t|^2 \frac{d\alpha}{\mathfrak{a}}\right)^{1/2} \tag{302}$$

$$\leqslant E_b^{1/2} + \|D_\alpha z_t\|_{L^\infty} \left(\int |D_\alpha \overline{z}_t|^2 \frac{d\alpha}{\mathfrak{a}}\right)^{1/2}$$

$$\lesssim (1 + E_a^{1/2} + E_a^{1/2} E_b^{1/2} + E_b^{1/2}) E_b^{1/2}.$$

by (301) and $A_1 \geqslant 1$ (182).

By changing variables as in (295), we conclude from (302) that

$$\|Z_{tt,\alpha'}\|_{L^2} = \left\|\overline{Z}_{tt,\alpha'}\right\|_{L^2} \lesssim \|A_1\|_{L^\infty}^{1/2} \left(\int |D_\alpha \overline{z}_{tt}|^2 \frac{d\alpha}{\mathfrak{a}}\right)^{1/2} \tag{303}$$

$$\lesssim (E_b^{1/2} + E_b)(1 + E_a^{1/2} + E_a^{1/2} E_b^{1/2} + E_b^{1/2}).$$

Now we show that for a generic f, we can control $D_\alpha^2 f$ by $D_\alpha^2 \overline{f}$ in $L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)$ norm, at the expense of some lower-order terms. (Recall from §4.2 that there's no reason a priori for $|D_\alpha^2 f|$ to equal $\left|D_\alpha^2 \overline{f}\right|$.) For notational convenience, we define here

$$|D_\alpha| = \frac{1}{|z_\alpha|}\partial_\alpha = \frac{z_\alpha}{|z_\alpha|}D_\alpha. \tag{304}$$

We expand:

$$|D_\alpha^2 f| = \left|\left(\frac{|z_\alpha|}{z_\alpha}\right)^2 |D_\alpha|^2 f + \frac{|z_\alpha|}{z_\alpha}\left(|D_\alpha|\frac{|z_\alpha|}{z_\alpha}\right)|D_\alpha| f\right|$$

$$\leqslant \left||D_\alpha|^2 f\right| + \left|\left(|D_\alpha|\frac{|z_\alpha|}{z_\alpha}\right)|D_\alpha| f\right|$$

$$= \left||D_\alpha|^2 \overline{f}\right| + \left|\left(|D_\alpha|\frac{|z_\alpha|}{z_\alpha}\right)|D_\alpha| \overline{f}\right| \tag{305}$$

$$\leqslant \left|D_\alpha^2 \overline{f}\right| + 2\left|\left(|D_\alpha|\frac{|z_\alpha|}{z_\alpha}\right)|D_\alpha| \overline{f}\right|$$

$$= \left|D_\alpha^2 \overline{f}\right| + 2\left||D_\alpha|\frac{|z_\alpha|}{z_\alpha}\right|\left|D_\alpha \overline{f}\right|.$$

Therefore,

$$\left(\int |D_\alpha^2 f|^2 \frac{h_\alpha d\alpha}{A_1 \circ h}\right)^{1/2} \leqslant \left(\int |D_\alpha^2 \overline{f}|^2 \frac{h_\alpha d\alpha}{A_1 \circ h}\right)^{1/2} + 2\left(\int \left||D_\alpha|\frac{|z_\alpha|}{z_\alpha}\right|^2 |D_\alpha \overline{f}|^2 \frac{h_\alpha d\alpha}{A_1 \circ h}\right)^{1/2}$$

$$\leqslant \left(\int |D_\alpha^2 \overline{f}|^2 \frac{h_\alpha d\alpha}{A_1 \circ h}\right)^{1/2} + 2\left\|D_\alpha \overline{f}\right\|_{L^\infty}\left(\int \left|D_\alpha \frac{|z_\alpha|}{z_\alpha}\right|^2 \frac{h_\alpha d\alpha}{A_1 \circ h}\right)^{1/2}.$$
$$(306)$$

We use (36) to rewrite

$$\left(\int \left|D_\alpha \frac{|z_\alpha|}{z_\alpha}\right|^2 \frac{h_\alpha d\alpha}{A_1 \circ h}\right)^{1/2} = \left(\int \left|D_\alpha \frac{\overline{z}_{tt} - i}{|\overline{z}_{tt} - i|}\right|^2 \frac{h_\alpha d\alpha}{A_1 \circ h}\right)^{1/2}$$

$$\lesssim \left(\int |D_\alpha \overline{z}_{tt}|^2 \frac{h_\alpha}{|\overline{z}_{tt} - i|^2 (A_1 \circ h)} d\alpha\right)^{1/2}$$

$$= \left(\int |D_\alpha \overline{z}_{tt}|^2 \frac{h_\alpha}{\mathfrak{a}^2 |z_\alpha|^2 (A_1 \circ h)} d\alpha\right)^{1/2}$$
$$(307)$$

$$= \left(\int |D_\alpha \overline{z}_{tt}|^2 \frac{1}{\mathfrak{a}(A_1 \circ h)^2} d\alpha\right)^{1/2}$$

$$\leqslant \left(\int |D_\alpha \overline{z}_{tt}|^2 \frac{1}{\mathfrak{a}} d\alpha\right)^{1/2}$$

$$\lesssim (1 + E_a^{1/2} + E_a^{1/2} E_b^{1/2} + E_b^{1/2}) E_b^{1/2},$$

where we used (212), the fact that $A_1 \geqslant 1$ (182), and (302). We plug this into (306):

$$\left(\int |D_\alpha^2 f|^2 \frac{h_\alpha d\alpha}{A_1 \circ h}\right)^{1/2} \lesssim \left(\int |D_\alpha^2 \overline{f}|^2 \frac{h_\alpha d\alpha}{A_1 \circ h}\right)^{1/2}$$
$$(308)$$
$$+ 2\left\|D_\alpha \overline{f}\right\|_{L^\infty} (1 + E_a^{1/2} + E_a^{1/2} E_b^{1/2} + E_b^{1/2}) E_b^{1/2}.$$

We now apply (308) to $f = z_t$, using (301) to control $\|D_\alpha \overline{z}_t\|_{L^\infty}$:

$$\left\|D_{\alpha'}^2 Z_t\right\|_{L^2(\frac{1}{A_1} d\alpha')} = \left(\int |D_\alpha^2 z_t|^2 \frac{h_\alpha d\alpha}{A_1 \circ h}\right)^{1/2}$$

$$\lesssim E_a^{1/2} + (E_a^{1/2} + E_a^{1/2} E_b^{1/2} + E_b^{1/2})(1 + E_a^{1/2} + E_a^{1/2} E_b^{1/2} + E_b^{1/2}) E_b^{1/2}.$$
$$(309)$$

We now control $\|D_\alpha^2 \overline{z}_{tt}\|_{L^2(\frac{h_\alpha d\alpha}{A_1 \circ h})}$. We use the commutator $[\partial_t, D_\alpha^2] = (-2D_\alpha z_t)D_\alpha^2 -$

$(D_\alpha^2 z_t)D_\alpha$ (258) to get

$$
\begin{aligned}
\left\| D_{\alpha'}^2 \overline{Z}_{tt} \right\|_{L^2(\frac{1}{A_1} d\alpha')} &= \left\| D_\alpha^2 \overline{z}_{tt} \right\|_{L^2(\frac{h_\alpha d\alpha}{A_1 \circ h})} \\
&\leqslant \left\| \partial_t D_\alpha^2 \overline{z}_t \right\|_{L^2(\frac{h_\alpha d\alpha}{A_1 \circ h})} + 2 \left\| (D_\alpha z_t)D_\alpha^2 \overline{z}_t \right\|_{L^2(\frac{h_\alpha d\alpha}{A_1 \circ h})} + \left\| (D_\alpha^2 z_t)D_\alpha \overline{z}_t \right\|_{L^2(\frac{h_\alpha d\alpha}{A_1 \circ h})} \\
&\leqslant E_a^{1/2} + 2 \left\| D_\alpha z_t \right\|_{L^\infty} \left\| D_\alpha^2 \overline{z}_t \right\|_{L^2(\frac{h_\alpha d\alpha}{A_1 \circ h})} + \left\| D_\alpha^2 z_t \right\|_{L^2(\frac{h_\alpha d\alpha}{A_1 \circ h})} \left\| D_\alpha \overline{z}_t \right\|_{L^\infty} \\
&\lesssim E_a^{1/2} + (301)(E_a^{1/2} + (309)).
\end{aligned}
$$
(310)

We will also need to control $D_{\alpha'}^2 Z_{tt}$; we delay doing this until later, after we control $\|D_\alpha \overline{z}_{tt}\|_{L^\infty}$.

We will also at one point (in §10.1) need to control

$$
\begin{aligned}
\left\| D_\alpha \partial_t D_\alpha \overline{z}_t \right\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)} &\leqslant \left\| \partial_t D_\alpha D_\alpha \overline{z}_t \right\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)} + \left\| [\partial_t, D_\alpha] D_\alpha \overline{z}_t \right\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)} \\
&\leqslant E_a^{1/2} + \left\| D_\alpha z_t \right\|_{L^\infty} \left\| D_\alpha^2 \overline{z}_t \right\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)} \\
&\leqslant E_a^{1/2}(1 + (301)).
\end{aligned}
$$
(311)

We now control $\|\overline{z}_{tt} - i\|_{L^\infty} = \left\|\overline{Z}_{tt} - i\right\|_{L^\infty}$. Recall from our definition of the energy (269) that the energy includes $|\overline{z}_{tt}(\alpha_0, t) - i|$ for some fixed $\alpha_0 \in I$. Let $\alpha_0' = h(\alpha_0, t)$. Then, by the fundamental theorem of calculus, for arbitrary $\alpha' \in I$,

$$
\begin{aligned}
\left| \overline{Z}_{tt}(\alpha', t) - i \right| &\leqslant \left| \overline{Z}_{tt}(\alpha_0', t) - i \right| + \int_{\alpha_0'}^{\alpha'} \left| \overline{Z}_{tt,\alpha'} \right| d\alpha' \\
&\lesssim \left| \overline{Z}_{tt}(\alpha_0', t) - i \right| + \left\| \overline{Z}_{tt,\alpha'} \right\|_{L^2} \\
&\lesssim \left| \overline{Z}_{tt}(\alpha_0', t) - i \right| + (303).
\end{aligned}
$$
(312)

We conclude that

$$
\begin{aligned}
\|z_{tt} + i\|_{L^\infty} = \|Z_{tt} + i\|_{L^\infty} = \|\overline{z}_{tt} - i\|_{L^\infty} &= \left\| \overline{Z}_{tt} - i \right\|_{L^\infty} \\
&\lesssim |\overline{z}_{tt}(\alpha_0, t) - i| + (303).
\end{aligned}
$$
(313)

Because of (183) and (182), we can also conclude that

$$
\left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \lesssim |\overline{z}_{tt}(\alpha_0, t) - i| + (303).
$$
(314)

We now control $\|D_\alpha \overline{z}_{tt}\|_{L^\infty}$, using the weighted Sobolev inequality (202) in Riemannian

coordinates with weight $\omega = \frac{1}{|Z_{,\alpha'}|^2}$ (and $\varepsilon = 1$).[41] By the Sobolev inequality,

$$
\begin{aligned}
\|D_\alpha z_{tt}\|_{L^\infty} &= \|D_{\alpha'} Z_{tt}\|_{L^\infty} = \|D_\alpha \overline{z}_{tt}\|_{L^\infty} = \|D_{\alpha'} \overline{Z}_{tt}\|_{L^\infty} \\
&\lesssim \|\overline{Z}_{tt,\alpha'}\|_{L^2} + \|D_{\alpha'}^2 \overline{Z}_{tt}\|_{L^2} + \left(\int \left| D_{\alpha'} \overline{Z}_{tt} \right|^2 d\alpha' \right)^{1/2} \\
&\lesssim \left(1 + \|1/Z_{,\alpha'}\|_{L^\infty} \right) \|\overline{Z}_{tt,\alpha'}\|_{L^2} + \|D_{\alpha'}^2 \overline{Z}_{tt}\|_{L^2} \\
&\lesssim (1 + (314))(303) + \|A_1\|_{L^\infty}^{1/2} (310) \\
&\lesssim (1 + (314))(303) + (E_b^{1/2} + 1)(310).
\end{aligned}
\tag{315}
$$

Finally, we use (308), (310), and (315) to control $D_\alpha^2 z_{tt}$ and $D_{\alpha'}^2 Z_{tt}$:

$$
\begin{aligned}
\left\| D_{\alpha'}^2 Z_{tt} \right\|_{L^2(\frac{1}{A_1} d\alpha')} &= \left\| D_\alpha^2 z_{tt} \right\|_{L^2(\frac{h_\alpha d\alpha}{A_1 \circ h})} \\
&\lesssim \left\| D_{\alpha'}^2 \overline{Z}_{tt} \right\|_{L^2(\frac{1}{A_1} d\alpha')} + 2 \left\| D_{\alpha'} \overline{z}_{tt} \right\|_{L^\infty} (1 + E_b^{1/2} + E_a^{1/2} E_b^{1/2} + E_a^{1/2}) E_b^{1/2} \\
&\lesssim (310) + (315)(1 + E_a^{1/2} + E_a^{1/2} E_b^{1/2} + E_b^{1/2}) E_b^{1/2}.
\end{aligned}
\tag{316}
$$

6.2   Maintaining Control of $\|z_\alpha\|_{L^\infty}$ and $\|1/z_\alpha\|_{L^\infty}$

We assume by our parametrization at initial time that $|z_\alpha(\alpha, 0)| \equiv 1$ (25). Therefore, we know that

$$
\|z_\alpha\|_{L^\infty}(0), \|1/z_\alpha\|_{L^\infty}(0) = 1 < \infty.
\tag{317}
$$

These norms will not necessarily remain identically one after initial time, so we will need to control them. We will, of course, be able to do so via the fundamental theorem of calculus, because our energy will control the time derivative of z_α.

We apply the fundamental theorem of calculus to z_α. We have

$$
\begin{aligned}
|z_\alpha(\alpha, t)| &= |z_\alpha(\alpha, 0)| + \int_0^t |z_\alpha(\alpha, \tau)|_\tau \, d\tau \\
&\leqslant |z_\alpha(\alpha, 0)| + \int_0^t |z_\alpha(\alpha, \tau)| \, |D_\alpha \overline{z}_t(\alpha, \tau)| \, d\tau.
\end{aligned}
\tag{318}
$$

---

[41] Note that unlike our proof for $\|D_{\alpha'} \overline{Z}_t\|_{L^\infty}$ at (300) above, we don't necessarily have that $\fint (D_{\alpha'} \overline{Z}_{tt})^2$ is zero, so we get a third term in the Sobolev inequality.

We take the supremum over all α, and conclude that

$$\|z_\alpha\|_{L^\infty}(t) \leqslant \|z_\alpha\|_{L^\infty}(0) + \int_0^t \|z_\alpha\|_{L^\infty}(\tau)\|D_\alpha \overline{z}_t\|_{L^\infty}(\tau)d\tau. \tag{319}$$

Similarly, for $\frac{1}{z_\alpha}$,

$$\begin{aligned}
\left|\frac{1}{z_\alpha(\alpha,t)}\right| &= \left|\frac{1}{z_\alpha(\alpha,0)}\right| + \int_0^t \left|\frac{1}{z_\alpha(\alpha,\tau)}\right|_\tau d\tau \\
&\leqslant \left|\frac{1}{z_\alpha(\alpha,0)}\right| + \int_0^t \left|\frac{1}{z_\alpha(\alpha,\tau)}\right| |D_\alpha \overline{z}_t(\alpha,\tau)| \, d\tau.
\end{aligned} \tag{320}$$

Once again, we take supremum over all α and conclude:

$$\left\|\frac{1}{z_\alpha}\right\|_{L^\infty}(t) \leqslant \left\|\frac{1}{z_\alpha}\right\|_{L^\infty}(0) + \int_0^t \left\|\frac{1}{z_\alpha}\right\|_{L^\infty}(\tau)\|D_\alpha \overline{z}_t\|_{L^\infty}(\tau)d\tau. \tag{321}$$

We now use (301), which controls $\|D_\alpha z_t\|_{L^\infty}$ in terms of the total energy, independent of $\|z_\alpha\|_{L^\infty}$ and $\left\|\frac{1}{z_\alpha}\right\|_{L^\infty}$. By Gronwall's inequality,[42] this implies that

$$\|z_\alpha\|_{L^\infty}(t) \lesssim \exp\left(\int_0^t \left(E_a^{1/2} + E_a^{1/2}E_b^{1/2} + E_b^{1/2}\right)(\tau)d\tau\right) \tag{322}$$

and

$$\left\|\frac{1}{z_\alpha}\right\|_{L^\infty}(t) \lesssim \exp\left(\int_0^t \left(E_a^{1/2} + E_a^{1/2}E_b^{1/2} + E_b^{1/2}\right)(\tau)d\tau\right). \tag{323}$$

6.3   Controlling $\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty}$

We now show that we can control $\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty}$, using (201). Because $A_1 \geqslant 1$ (182), it suffices to control

$$\left\|2[Z_t, \mathbb{H}]\overline{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\overline{Z}_{t,\alpha'} - \frac{\pi}{2}[Z_t, Z_t; D_{\alpha'}\overline{Z}_t]\right\|_{L^\infty}. \tag{324}$$

We control the first two terms by (244):

$$\left\|2[Z_t, \mathbb{H}]\overline{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\overline{Z}_{t,\alpha'}\right\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}\left\|\overline{Z}_{tt,\alpha'}\right\|_{L^2}. \tag{325}$$

---

[42]See [Tao06], Theorem 1.10; the continuity in t of the spatial L^∞ follows from the fact that we assume a priori that $z_{t\alpha} \in C^0$.

We control

$$\left\|\frac{\pi}{2}[Z_t, Z_t; D_{\alpha'}\overline{Z}_t]\right\|_{L^\infty} = \left\|\frac{1}{2i}\frac{\pi}{2}\int\frac{(Z_t(\alpha') - Z_t(\beta'))^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}D_{\beta'}\overline{Z}_t(\beta')d\beta'\right\|_{L^\infty} \tag{326}$$

$$\lesssim \|Z_{t,\alpha'}\|_{L^2}^2\|D_{\alpha'}\overline{Z}_t\|_{L^\infty}$$

by Hölder and then Hardy's inequality (217).

We conclude that

$$\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}(\|Z_{t,\alpha'}\|_{L^2}\|D_{\alpha'}\overline{Z}_t\|_{L^\infty} + \|\overline{Z}_{tt,\alpha'}\|_{L^2}). \tag{327}$$

6.4   Controlling $\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}$

Recall from (183) that

$$\frac{1}{Z_{,\alpha'}} = i\frac{\overline{Z}_{tt} - i}{A_1}. \tag{328}$$

Therefore,

$$\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} = i\frac{\overline{Z}_{tt,\alpha'}}{A_1} - i\frac{\overline{Z}_{tt} - i}{A_1^2}\partial_{\alpha'}A_1. \tag{329}$$

Because $A_1 \geqslant 1$ (182), we can control the first term by $\|\overline{Z}_{tt,\alpha'}\|_{L^2}$. Now we address the second term.

We recall that

$$A_1 = \Im\left(-[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}\right) + 1. \tag{330}$$

Therefore,

$$\partial_{\alpha'}A_1 = \partial_{\alpha'}\Im\frac{-1}{2i}\int(Z_t(\alpha') - Z_t(\beta'))\cot(\frac{\pi}{2}(\alpha' - \beta'))\overline{Z}_{t,\beta'}d\beta'$$

$$= -\underbrace{\Im Z_{t,\alpha'}\mathbb{H}\overline{Z}_{t,\alpha'}}_{0} + \Im\frac{1}{2i}\int\frac{\pi}{2}\frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}\overline{Z}_{t,\beta'}(\beta')d\beta \tag{331}$$

$$= \Im\frac{1}{2i}\int\frac{\pi}{2}\frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}\overline{Z}_{t,\beta'}(\beta')d\beta,$$

where the first term disappears because $\mathbb{H}\overline{Z}_{t,\alpha'} = \overline{Z}_{t,\alpha'}$ (141) and so $Z_{t,\alpha'}\mathbb{H}\overline{Z}_{t,\alpha'}$ is purely real.[43]

---

[43]We ignore here and a few times in the sequel the details of justifying differentiating under the integral;

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Therefore, multiplying (331) by $\left|\overline{Z}_{tt}(\alpha) - i\right|$ and splitting into two parts, we have

$$
\begin{aligned}
\left|\overline{Z}_{tt} - i\right| \partial_{\alpha'} A_1 &= \Im \frac{1}{2i} \int \frac{\pi}{2} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} (|Z_{tt}(\alpha') - i| - |Z_{tt}(\beta') - i|)\overline{Z}_{t,\beta'}(\beta')d\beta \\
&+ \Im \frac{1}{2i} \int \frac{\pi}{2} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} \left|\overline{Z}_{tt}(\beta') - i\right| \overline{Z}_{t,\beta'}(\beta')d\beta \\
&= I + II.
\end{aligned}
\tag{332}
$$

We need to control $\|I/A_1^2\|_{L^2}$ and $\|II/A_1^2\|_{L^2}$. Because $A_1 \geqslant 1$ (182), it suffices to control $\|I\|_{L^2}$ and $\|II\|_{L^2}$. By (241), we have

$$
\|I\|_{L^2} \lesssim \|Z_{t,\alpha}\|_{L^2} \left\|\overline{Z}_{tt,\alpha}\right\|_{L^2} \left\|\overline{Z}_{t,\alpha'}\right\|_{L^2} = \left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}^2 \left\|\overline{Z}_{tt,\alpha'}\right\|_{L^2}.
\tag{333}
$$

For, II, we use (328) to rewrite

$$
II = \Im \frac{1}{2i} \int \frac{\pi}{2} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} \left|\frac{(-iA_1(\beta'))}{Z_{,\beta'}}\right| \overline{Z}_{t,\beta'}(\beta')d\beta.
\tag{334}
$$

Now we use estimate (235):

$$
\|II\|_{L^2} \lesssim \|Z_{t,\alpha'}\|_{L^2} \left\|A_1 D_{\alpha'} \overline{Z}_t\right\|_{L^\infty} \leqslant \|Z_{t,\alpha'}\|_{L^2} \left\|D_{\alpha'} \overline{Z}_t\right\|_{L^\infty} \|A_1\|_{L^\infty}.
\tag{335}
$$

We conclude that

$$
\left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2} \lesssim \left\|\overline{Z}_{tt,\alpha'}\right\|_{L^2} (1 + \|Z_{t,\alpha'}\|_{L^2}^2) + \|Z_{t,\alpha'}\|_{L^2} \left\|D_{\alpha'} \overline{Z}_t\right\|_{L^\infty} \|A_1\|_{L^\infty}.
\tag{336}
$$

6.5 Controlling $\left\|\frac{h_{t\alpha}}{h_\alpha}\right\|_{L^\infty}$

We recall (56):

$$
h(\alpha, t) = \Phi(z(\alpha, t), t) = \Phi \circ z.
\tag{337}
$$

Therefore

$$
h_\alpha = (\Phi_z \circ z)z_\alpha
\tag{338}
$$

---

this may be made rigorous.

and
$$h_t = (\Phi_t \circ z) + (\Phi_z \circ z)z_t = (\Phi_t \circ z) + \frac{h_\alpha}{z_\alpha}z_t. \tag{339}$$

Therefore,
$$(h_t \circ h^{-1})(\alpha', t) = \Phi_t \circ Z + \frac{1}{Z_{,\alpha'}}Z_t. \tag{340}$$

Differentiating with respect to α' gives
$$(h_t \circ h^{-1})_{\alpha'} = \partial_{\alpha'}(\Phi_t \circ Z) + D_{\alpha'}Z_t + Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}. \tag{341}$$

It proves useful to replace the $D_{\alpha'}Z_t$ with its conjugate. We can do this by rewriting $D_{\alpha'}Z_t = 2\Re D_{\alpha'}Z_t - \frac{\overline{Z}_{t,\alpha'}}{\overline{Z}_{,\alpha'}}$, giving

$$(h_t \circ h^{-1})_{\alpha'} - 2\Re D_{\alpha'}Z_t = \partial_{\alpha'}(\Phi_t \circ Z) - \frac{\overline{Z}_{t,\alpha'}}{\overline{Z}_{,\alpha'}} + Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}. \tag{342}$$

Observe that the LHS is purely real. Therefore, if we apply $\Re(I - \mathbb{H})$ to both sides, the LHS remains unchanged. On the right-hand side, $(I - \mathbb{H})\partial_{\alpha'}(\Phi_t \circ Z)$ disappears by (158), and the remaining terms become commutators by (141) and (137):

$$
\begin{aligned}
(h_t \circ h^{-1})_{\alpha'} - 2\Re D_{\alpha'}Z_t &= \Re\left\{-(I - \mathbb{H})\left(\frac{1}{\overline{Z}_{,\alpha'}}\overline{Z}_{t,\alpha'}\right) + (I - \mathbb{H})\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\} \\
&= \Re\left\{-\left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{t,\alpha'} + [Z_t, \mathbb{H}]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\}.
\end{aligned}
\tag{343}
$$

We use (244) to control these:
$$\left\|\left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{t,\alpha'}\right\|_{L^\infty} \lesssim \left\|\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right\|_{L^2}\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2} \tag{344}$$

$$\left\|[Z_t, \mathbb{H}]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}. \tag{345}$$

Observe that
$$\frac{h_{t\alpha}}{h_\alpha} \circ h^{-1} = \partial_{\alpha'}(h_t \circ h^{-1}). \tag{346}$$

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We therefore conclude that

$$\left\|\frac{h_{t\alpha}}{h_\alpha}\right\|_{L^\infty} = \left\|\partial_{\alpha'}(h_t \circ h^{-1})\right\|_{L^\infty} \lesssim \|D_{\alpha'}Z_t\|_{L^\infty} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}. \tag{347}$$

6.6 Controlling $\left\|\frac{(A_1\circ h)_t}{(A_1\circ h)}\right\|_{L^\infty}$

Recall that

$$A_1 \circ h = \frac{\mathfrak{a}\,|z_\alpha|^2}{h_\alpha}. \tag{348}$$

Therefore,

$$\frac{\frac{d}{dt}(A_1 \circ h)}{A_1 \circ h} = \frac{\mathfrak{a}_t}{\mathfrak{a}} - \frac{h_{t\alpha}}{h_\alpha} + 2\Re D_\alpha z_t. \tag{349}$$

We have controlled each of the terms on the RHS in L^∞ in the previous sections. We conclude that

$$\left\|\frac{\frac{d}{dt}(A_1 \circ h)}{A_1 \circ h}\right\|_{L^\infty} \lesssim \left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty} + \left\|\frac{h_{t\alpha}}{h_\alpha}\right\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty}. \tag{350}$$

6.7 Controlling $\left\|(I + \mathbb{H})D_{\alpha'}Z_t\right\|_{L^\infty}$

By taking conjugates, using (141) to get a commutator, and using commutator estimate (244), we have

$$
\begin{aligned}
\|(I + \mathbb{H})D_{\alpha'}Z_t\|_{L^\infty} &= \left\|(I - \mathbb{H})\frac{1}{\overline{Z}_{,\alpha'}}\overline{Z}_{t,\alpha'}\right\|_{L^\infty} \\
&= \left\|\left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{t,\alpha'}\right\| \\
&\lesssim \left\|\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right\|_{L^2}\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}.
\end{aligned}
\tag{351}
$$

6.8 Controlling $\left\|(I - \mathbb{H})D_{\alpha'}\overline{Z}_{tt}\right\|_{L^\infty}$

We begin by expanding $D_{\alpha'}\overline{Z}_{tt}$ as in (175):

$$D_{\alpha'}\overline{Z}_{tt} = D_{\alpha'}(Z_t D_{\alpha'}\overline{Z}_t) + D_{\alpha'}(F_t \circ Z), \tag{352}$$

where $F(z(\alpha, t), t) := \overline{z}_t(\alpha, t)$. Under $(I - \mathbb{H})$, the second term disappears by (150), so we're left with

$$
\begin{aligned}
(I - \mathbb{H})D_{\alpha'}\overline{Z}_{tt} &= (I - \mathbb{H})\left\{D_{\alpha'}(Z_t D_{\alpha'}\overline{Z}_t)\right\} \\
&= (I - \mathbb{H})\left\{Z_t D_{\alpha'}^2 \overline{Z}_t\right\} + (I - \mathbb{H})\left\{\left(\frac{1}{Z_{,\alpha'}}D_{\alpha'}Z_t\right)\overline{Z}_{t,\alpha'}\right\} \\
&= [Z_t, \mathbb{H}]\, D_{\alpha'}^2 \overline{Z}_t + \left[\frac{1}{Z_{,\alpha'}}D_{\alpha'}Z_t, \mathbb{H}\right]\overline{Z}_{t,\alpha'}
\end{aligned}
\tag{353}
$$

by (139) and (141). We control both of these by (244),[44] to conclude that

$$
\begin{aligned}
\left\|(I - \mathbb{H})D_{\alpha'}\overline{Z}_{tt}\right\|_{L^\infty} &\lesssim \|Z_{t,\alpha'}\|_{L^2}\left\|D_{\alpha'}^2\overline{Z}_t\right\|_{L^2} + \left\|\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}}D_{\alpha'}Z_t\right)\right\|_{L^2}\|\overline{Z}_{t,\alpha'}\|_{L^2} \\
&\leqslant \|Z_{t,\alpha'}\|_{L^2}\left\|D_{\alpha'}^2\overline{Z}_t\right\|_{L^2} \\
&\quad + \left(\left\|D_{\alpha'}^2 Z_t\right\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|D_{\alpha'}Z_t\|_{L^\infty}\right)\|\overline{Z}_{t,\alpha'}\|_{L^2} \\
&\leqslant \|Z_{t,\alpha'}\|_{L^2}\left(\left\|D_{\alpha'}^2\overline{Z}_t\right\|_{L^2} + \left\|D_{\alpha'}^2 Z_t\right\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|D_{\alpha'}Z_t\|_{L^\infty}\right).
\end{aligned}
\tag{354}
$$

6.9 Controlling $\left\|D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}$

Recall from (183) that

$$
\frac{1}{Z_{,\alpha'}} = i\frac{\overline{Z}_{tt} - i}{A_1}.
\tag{355}
$$

Therefore,

$$
\begin{aligned}
\left\|D_\alpha \frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} &= \left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \\
&= \left\|\frac{\overline{Z}_{tt} - i}{A_1}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \\
&\leqslant \left\|(\overline{Z}_{tt} - i)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \\
&= \left\|(Z_{tt} + i)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty},
\end{aligned}
\tag{356}
$$

---

[44]Note that $\frac{1}{Z_{,\alpha'}}D_{\alpha'}Z_t\Big|_{\partial} = 0$ by (115).

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where we've used (182) $A_1 \geqslant 1$. We therefore will estimate this last term.[45] Because

$$(Z_{tt} + i)\partial_\alpha \frac{1}{Z_{,\alpha'}} = \partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}}(Z_{tt} + i)\right) - D_{\alpha'}Z_{tt}, \tag{357}$$

we can control

$$\left\|(Z_{tt} + i)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \leqslant \|D_{\alpha'}Z_{tt}\|_{L^\infty} + \left\|\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}}(Z_{tt} + i)\right)\right\|_{L^\infty}. \tag{358}$$

Because we can control $\|D_{\alpha'}Z_{tt}\|_{L^\infty}$, it suffices to focus on this second term. Observe that

$$\frac{A_1}{|Z_{,\alpha'}|^2} = i\frac{1}{\overline{Z}_{,\alpha'}}(\overline{Z}_{tt} - i) \tag{359}$$

is purely real. Therefore

$$\frac{1}{Z_{,\alpha'}}(Z_{tt} + i) \in i\mathbb{R}, \tag{360}$$

and so

$$\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}}(Z_{tt} + i)\right) \in i\mathbb{R}. \tag{361}$$

Therefore, we may apply $(I - \mathbb{H})$ and invert by taking imaginary parts:

$$\begin{aligned}
\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}}(Z_{tt} + i)\right) &= i\Im(I - \mathbb{H})\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}}(Z_{tt} + i)\right) \\
&= i\Im(I - \mathbb{H})\left\{(Z_{tt} + i)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\} + i\Im(I - \mathbb{H})D_{\alpha'}Z_{tt}.
\end{aligned} \tag{362}$$

It thus suffices to control the terms on the RHS of (362).

We rewrite the first term as a commutator by (137) and control it by (244)

$$\begin{aligned}
\left\|(I - \mathbb{H})(Z_{tt} + i)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} &\leqslant \left\|[Z_{tt} + i, \mathbb{H}]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \\
&\lesssim \|Z_{tt,\alpha'}\|_{L^2}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}.
\end{aligned} \tag{363}$$

Therefore, it suffices to control the second term on the RHS of (362), $\Im(I - \mathbb{H})D_{\alpha'}Z_{tt}$. If $\mathbb{H}$ were L^∞ to L^∞ bounded, then this would be fine, since $D_{\alpha'}Z_{tt} \in L^\infty$. But we don't have such a bound. We get around this by taking advantage of the fact that we know things

---

[45]We will also find it useful to estimate for its own sake, in a later section.

about holomorphicity.

Our technique relies crucially on the fact that we only care about the *imaginary* part of $(I - \mathbb{H})D_{\alpha'}Z_{tt}$. We write

$$\Im(I - \mathbb{H})D_{\alpha'}Z_{tt} = \Im(I - \mathbb{H})\left(\frac{1}{Z_{,\alpha'}}Z_{tt,\alpha'} - \frac{1}{\overline{Z}_{,\alpha'}}\overline{Z}_{tt,\alpha'}\right) + \Im(I - \mathbb{H})\left\{\frac{1}{\overline{Z}_{,\alpha'}}\overline{Z}_{tt,\alpha'}\right\}. \quad (364)$$

Observe that because what's inside the $(I - \mathbb{H})$ in the first term is purely imaginary, we can remove the $\mathbb{H}$ and so we no longer have the issue of L^∞ boundedness. This term, therefore, is controlled by $2\left\|D_{\alpha'}\overline{Z}_{tt}\right\|_{L^\infty}$, so it suffices to focus on the second term, $\Im(I - \mathbb{H})\left\{\frac{1}{\overline{Z}_{,\alpha'}}\overline{Z}_{tt,\alpha'}\right\}$. We now drop the $\Im$, and focus on $(I - \mathbb{H})\left\{\frac{1}{\overline{Z}_{,\alpha'}}\overline{Z}_{tt,\alpha'}\right\}$.

We begin by reducing this to controlling $(I - \mathbb{H})D_{\alpha'}\overline{Z}_{tt}$, by commuting out the $\frac{1}{\overline{Z}_{,\alpha'}}$, replacing it with $\frac{1}{Z_{,\alpha'}}$, and commuting it back in:

$$\left|(I - \mathbb{H})\left\{\frac{1}{\overline{Z}_{,\alpha'}}\overline{Z}_{tt,\alpha'}\right\}\right| = \left|\left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{tt,\alpha'} + \frac{1}{\overline{Z}_{,\alpha'}}(I - \mathbb{H})\overline{Z}_{tt,\alpha'}\right|$$

$$\leqslant \left|\left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{tt,\alpha'}\right| + \left|\frac{1}{\overline{Z}_{,\alpha'}}(I - \mathbb{H})\overline{Z}_{tt,\alpha'}\right|$$

$$= \left|\left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{tt,\alpha'}\right| + \left|\frac{1}{Z_{,\alpha'}}(I - \mathbb{H})\overline{Z}_{tt,\alpha'}\right|$$

$$= \left|\left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{tt,\alpha'}\right| + \left|(I - \mathbb{H})\frac{1}{Z_{,\alpha'}}\overline{Z}_{tt,\alpha'} - \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{tt,\alpha'}\right|$$

$$\leqslant \left|\left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{tt,\alpha'}\right| + \left|\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{tt,\alpha'}\right| + \left|(I - \mathbb{H})\frac{1}{Z_{,\alpha'}}\overline{Z}_{tt,\alpha'}\right|. \quad (365)$$

We control the first two terms by (244):

$$\left\|\left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{tt,\alpha'}\right\|_{L^\infty} + \left\|\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{tt,\alpha'}\right\|_{L^\infty} \lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left\|\overline{Z}_{tt,\alpha'}\right\|_{L^2}. \quad (366)$$

We've controlled the remaining term, $\left\|(I - \mathbb{H})D_{\alpha'}\overline{Z}_{tt}\right\|_{L^\infty}$, in (354) above.

We conclude that

$$
\left\| D_\alpha \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \leqslant \left\| (Z_{tt} + i)\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}
$$

$$
\lesssim \| D_{\alpha'} Z_{tt} \|_{L^\infty} + \| Z_{tt,\alpha'} \|_{L^2} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \| \overline{Z}_{tt,\alpha'} \|_{L^2} + \quad (354)
$$

$$
\lesssim \| D_{\alpha'} Z_{tt} \|_{L^\infty} + \| Z_{tt,\alpha'} \|_{L^2} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \| \overline{Z}_{tt,\alpha'} \|_{L^2}
$$

$$
+ \| Z_{t,\alpha'} \|_{L^2} \left(\| D_{\alpha'}^2 \overline{Z}_t \|_{L^2} + \| D_{\alpha'}^2 Z_t \|_{L^2} + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \| D_{\alpha'} Z_t \|_{L^\infty} \right).
$$

$$
(367)
$$

6.10 Controlling $\left\| \partial_{\alpha'} (I - \mathbb{H}) \frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty}$ and Related Terms

In this section, we estimate $\left\| \partial_{\alpha'} (I - \mathbb{H}) \frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty} = 2 \left\| \partial_{\alpha'} \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty}$, as well as some related quantities.

Because $\left. \frac{Z_t}{Z_{,\alpha'}} \right|_\partial = 0$ by (118), by (100) we may commute in the derivative, so

$$
\partial_{\alpha'} (I - \mathbb{H}) \frac{Z_t}{Z_{,\alpha'}} = (I - \mathbb{H}) \partial_{\alpha'} \frac{Z_t}{Z_{,\alpha'}}
$$

$$
= (I - \mathbb{H}) D_{\alpha'} Z_t + (I - \mathbb{H}) \left\{ Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\}.
$$

$$
(368)
$$

First, we control the first term. Rewriting $(I - \mathbb{H}) = 2I - (I + \mathbb{H})$, we have

$$
\| (I - \mathbb{H}) D_{\alpha'} Z_t \|_{L^\infty} \lesssim \| D_{\alpha'} Z_t \|_{L^\infty} + \| (I + \mathbb{H}) D_{\alpha'} Z_t \|_{L^\infty}.
$$

$$
(369)
$$

We estimated the second term at (351) above. We conclude that

$$
\| (I - \mathbb{H}) D_{\alpha'} Z_t \|_{L^\infty} \lesssim \| D_{\alpha'} Z_t \|_{L^\infty} + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \| \overline{Z}_{t,\alpha'} \|_{L^2}.
$$

$$
(370)
$$

We write the second term of (368) as a commutator by (137), and control this by (244):

$$
\left\| (I - \mathbb{H}) \left\{ Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\} \right\|_{L^\infty} = \left\| [Z_t, \mathbb{H}] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}
$$

$$
\lesssim \| Z_{t,\alpha'} \|_{L^2} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}.
$$

$$
(371)
$$

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We conclude from (370) and (371) that

$$\left\| \partial_{\alpha'}(I - \mathbb{H})\frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty} \lesssim \|D_{\alpha'}Z_t\|_{L^\infty} + \|Z_{t,\alpha'}\|_{L^2} \left\| \partial_{\alpha'}\frac{1}{Z_{,\alpha'}} \right\|_{L^2}. \tag{372}$$

Chapter 7

Controlling $\Re \int i \left(\frac{1}{\overline{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha$

In this chapter, we control from the RHS of (278) the term (282)

$$
\begin{aligned}
\Re \int i \left(\frac{1}{\overline{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha = & -\Re \int i \frac{\overline{z}_{t\alpha}}{\overline{z}_\alpha} \left(\frac{1}{\overline{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha \\
& + \Re \int i \left(\frac{1}{\overline{z}_\alpha} \partial_\alpha \partial_t \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha
\end{aligned}
\tag{373}
$$

for $\theta = D_\alpha^2 \overline{z}_t$.

We can control the first of these terms by

$$
\left| -\Re \int i \frac{\overline{z}_{t\alpha}}{\overline{z}_\alpha} \left(\frac{1}{\overline{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha \right| \lesssim \| D_\alpha \overline{z}_t \|_{L^\infty} \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \| D_{\alpha'}^2 \overline{Z}_t \|_{L^2}^2 .
\tag{374}
$$

Therefore, it suffices to focus on the second term on the RHS of (373). We expand it out:

$$
\begin{aligned}
\Re \int i \left(\frac{1}{\overline{z}_\alpha} \partial_\alpha \partial_t \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha = & \Re \int i \left(\frac{1}{\overline{z}_\alpha} \partial_\alpha \left(\frac{h_\alpha}{z_\alpha} \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right) \right) \right) |\theta|^2 h_\alpha d\alpha \\
= & \Re \int i \left(\frac{1}{\overline{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha \\
& + \Re \int i \left(\frac{1}{\overline{z}_\alpha} \frac{h_\alpha}{z_\alpha} \partial_\alpha \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right) \right) |\theta|^2 h_\alpha d\alpha.
\end{aligned}
\tag{375}
$$

We can estimate the first term on the RHS by

$$\left| \Re \int i\left(\frac{1}{\overline{z}_\alpha}\partial_\alpha \frac{h_\alpha}{z_\alpha} \right)\left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right)|\theta|^2 h_\alpha d\alpha \right|$$
$$\lesssim \left\| D_{\alpha'}\frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \left(\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty} \right)\|D^2_{\alpha'}\overline{Z}_t\|_{L^2}. \tag{376}$$

Therefore, it suffices to focus on the second term on the RHS of (375). Observe that because h is real-valued,

$$\Re \int i\left(\frac{1}{\overline{z}_\alpha}\frac{h_\alpha}{z_\alpha}\partial_\alpha\left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right) \right)|\theta|^2 h_\alpha d\alpha = -\Re \int i\left(\frac{1}{\overline{z}_\alpha}\frac{h_\alpha}{z_\alpha}\partial_\alpha\left(\frac{z_{t\alpha}}{z_\alpha} \right) \right)|\theta|^2 h_\alpha d\alpha. \tag{377}$$

For consistency with the quantities we've controlled elsewhere, we will conjugate, and focus on

$$\Re \int i\left(\frac{h_\alpha}{|z_\alpha|^2}\partial_\alpha\left(\frac{\overline{z}_{t\alpha}}{\overline{z}_\alpha} \right) \right)|\theta|^2 h_\alpha d\alpha. \tag{378}$$

We now drop $\Re$ and the i, write $D^2_\alpha\overline{z}_t = \theta$, and switch to Riemannian coordinates. We have

$$\int \left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\left(\frac{\overline{Z}_{t,\alpha'}}{\overline{Z}_{,\alpha'}} \right) \right)\left| D^2_{\alpha'}\overline{Z}_t \right|^2 d\alpha'. \tag{379}$$

We want to take advantage of the holomorphicity and antiholomorphicity of various of these factors. To do this, we first replace the $\frac{1}{\overline{Z}_{,\alpha'}}$ with a $\frac{1}{Z_{,\alpha'}}$ inside the inner part of the first factor to make it closer to holomorphic (since $\frac{\overline{Z}_{t,\alpha'}}{Z_{,\alpha'}} = D_{\alpha'}\overline{Z}_t$ is holomorphic):

$$\int \left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\left(\frac{\overline{Z}_{t,\alpha'}}{\overline{Z}_{,\alpha'}} \right) \right)\left| D^2_{\alpha'}\overline{Z}_t \right|^2 d\alpha' = \int \left(\frac{1}{|Z_{,\alpha'}|^2}\frac{Z_{,\alpha'}}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}\left(\frac{\overline{Z}_{t,\alpha'}}{Z_{,\alpha'}} \right) \right)\left| D^2_{\alpha'}\overline{Z}_t \right|^2 d\alpha'$$
$$+ \int \left(\frac{1}{|Z_{,\alpha'}|^2}\left(\partial_{\alpha'}\frac{Z_{,\alpha'}}{\overline{Z}_{,\alpha'}} \right)\left(\frac{\overline{Z}_{t,\alpha'}}{Z_{,\alpha'}} \right) \right)\left| D^2_{\alpha'}\overline{Z}_t \right|^2 d\alpha'$$
$$= \int \left(\frac{1}{\overline{Z}^2_{,\alpha'}}\partial_{\alpha'}D_{\alpha'}\overline{Z}_t \right)\left| D^2_{\alpha'}\overline{Z}_t \right|^2 d\alpha'$$
$$- \int \left(\frac{1}{|Z_{,\alpha'}|^2}\left(\partial_{\alpha'}\frac{Z_{tt} + i}{\overline{Z}_{tt} - i} \right)D_{\alpha'}\overline{Z}_t \right)\left| D^2_{\alpha'}\overline{Z}_t \right|^2 d\alpha', \tag{380}$$

where we have used (36) to replace the $\frac{Z_{,\alpha'}}{\overline{Z}_{,\alpha'}}$ with $-\frac{Z_{tt}+i}{\overline{Z}_{tt}-i}$. We can estimate the second term

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by

$$\left| \int \left(\frac{1}{|Z_{,\alpha'}|^2} \left(\partial_{\alpha'} \frac{Z_{tt}+i}{\overline{Z}_{tt}-i} \right) D_{\alpha'} \overline{Z}_t \right) \left| D^2_{\alpha'} \overline{Z}_t \right|^2 d\alpha' \right|$$

$$\lesssim \left\| \left(\frac{1}{|Z_{,\alpha'}|^2} \left(\partial_{\alpha'} \frac{Z_{tt}+i}{\overline{Z}_{tt}-i} \right) D_{\alpha'} \overline{Z}_t \right) \right\|_{L^\infty} \left\| D^2_{\alpha'} \overline{Z}_t \right\|^2_{L^2} . \tag{381}$$

$$\lesssim \left\| D_{\alpha'} Z_{tt} \right\|_{L^\infty} \left\| D_{\alpha'} \overline{Z}_t \right\|_{L^\infty} \left\| D^2_{\alpha'} \overline{Z}_t \right\|^2_{L^2} ,$$

where we have used $\frac{1}{A_1} = \left| \frac{1}{Z_{,\alpha'}(Z_{tt}+i)} \right|$ (183), $A_1 \geq 1$ (182), and (212).

It therefore remains only to control the first term on the RHS of (380). Now we take advantage of holomorphicity. We rewrite this as

$$\int \left(\frac{1}{\overline{Z}^2_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right) \left| D^2_{\alpha'} \overline{Z}_t \right|^2 d\alpha' = \int \left(\frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right) \left(\overline{\frac{1}{Z_{,\alpha'}} D^2_{\alpha'} \overline{Z}_t} \right) D^2_{\alpha'} \overline{Z}_t d\alpha'$$

$$= \int \left((\mathbb{P}_A + \mathbb{P}_H) \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right) \left(\overline{\frac{1}{Z_{,\alpha'}} D^2_{\alpha'} \overline{Z}_t} \right) \mathbb{P}_H D^2_{\alpha'} \overline{Z}_t d\alpha' , \tag{382}$$

where we have used (139) to insert $\mathbb{P}_H$ in front of the $D^2_{\alpha'} \overline{Z}_t$ and decomposed the first factor into the holomorphic and antiholomorphic projections. Now we use the adjoint property (98) to move the $\mathbb{P}_H$ into a $\mathbb{P}_A$ on the remaining factors, and control using Cauchy-Schwarz:

$$\left| \int \left((\mathbb{P}_A + \mathbb{P}_H) \left\{ \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\} \right) \left(\overline{\frac{1}{Z_{,\alpha'}} D^2_{\alpha'} \overline{Z}_t} \right) \mathbb{P}_H D^2_{\alpha'} \overline{Z}_t d\alpha' \right|$$

$$= \left| \int \left(\mathbb{P}_A \left\{ \left((\mathbb{P}_A + \mathbb{P}_H) \left\{ \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\} \right) \left(\overline{\frac{1}{Z_{,\alpha'}} D^2_{\alpha'} \overline{Z}_t} \right) \right\} \right) D^2_{\alpha'} \overline{Z}_t d\alpha' \right| \tag{383}$$

$$\lesssim \left\| \mathbb{P}_A \left\{ \left((\mathbb{P}_A + \mathbb{P}_H) \left\{ \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\} \left(\overline{\frac{1}{Z_{,\alpha'}} D^2_{\alpha'} \overline{Z}_t} \right) \right) \right\} \right\|_{L^2} \left\| D^2_{\alpha'} \overline{Z}_t \right\|_{L^2} .$$

It now remains only to control this first factor.

First we consider the term with the $\mathbb{P}_H$. In this case, we can rewrite this as a commutator:

$$\mathbb{P}_A \left\{ \left(\mathbb{P}_H \left\{ \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\} \right) \left(\overline{\frac{1}{Z_{,\alpha'}} D^2_{\alpha'} \overline{Z}_t} \right) \right\} = \frac{1}{2} \left[\overline{\frac{1}{Z_{,\alpha'}} D^2_{\alpha'} \overline{Z}_t}, \mathbb{H} \right] \mathbb{P}_H \left\{ \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\}$$

$$+ \frac{1}{4} \overline{\frac{1}{Z_{,\alpha'}} D^2_{\alpha'} \overline{Z}_t} \left(\fint \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right) . \tag{384}$$

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Here, the mean term appears because of (95). We now use commutator estimate (240) for the first term and Hölder for the second term, to conclude that

$$
\left\| \mathbb{P}_A \left\{ \left(\mathbb{P}_H \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right) \left(\overline{\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t} \right) \right\} \right\|_{L^2}
$$

$$
\lesssim \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\|_{\dot{H}^{1/2}} \left\| \mathbb{P}_H \left\{ \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\} \right\|_{L^2} + \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\|_{L^2} \left\| \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\|_{L^1}
$$

$$
\lesssim \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\|_{\dot{H}^{1/2}} \left\| \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\|_{L^2} + \left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \left\| D_{\alpha'}^2 \overline{Z}_t \right\|_{L^2} \left\| \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\|_{L^2}
$$

$$
\lesssim \left\| D_{\alpha'}^2 \overline{Z}_t \right\|_{L^2} \left(\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\|_{\dot{H}^{1/2}} + \left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \left\| D_{\alpha'}^2 \overline{Z}_t \right\|_{L^2} \right). \quad (385)
$$

Finally, we consider the $\mathbb{P}_A$ term in the first factor on the RHS of (383). By the L^2 boundedness of $\mathbb{P}_A$, it suffices to control

$$
\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t (I - \mathbb{H}) \left\{ \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\} \right\|_{L^2} \lesssim \left\| D_{\alpha}^2 \overline{Z}_t \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \left\{ \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\} \right\|_{L^2}
$$

$$
+ \left\| D_{\alpha'}^2 \overline{Z}_t (I - \mathbb{H}) \left\{ \frac{1}{\overline{Z}_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\} \right\|_{L^2}
$$

$$
\lesssim \left\| D_{\alpha}^2 \overline{Z}_t \right\|_{L^2} \left\| \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \left\{ \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\} \right\|_{L^\infty}
$$

$$
+ \left\| D_{\alpha}^2 \overline{Z}_t \right\|_{L^2} \left\| \left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H} \right] D_{\alpha'}^2 \overline{Z}_t \right\|_{L^\infty}
$$

$$
\lesssim \left\| D_{\alpha}^2 \overline{Z}_t \right\|_{L^2}^2 \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2},
$$

$$(386)$$

where we've used (139) to get the second commutator and used commutator estimate (244).

We now combine our estimates, concluding that

$$\Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha \lesssim (374) + (375)$$

$$\lesssim (374) + (376) + (381)$$
$$+ \left\| D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2} \cdot ((385) + (386))$$
$$\lesssim \left\| D_\alpha \bar{z}_t \right\|_{L^\infty} \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \left\| D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2}^2$$
$$+ \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \left(\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \left\| D_\alpha z_t \right\|_{L^\infty} \right) \left\| D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2}$$
$$+ \left\| D_{\alpha'} Z_{tt} \right\|_{L^\infty} \left\| D_{\alpha'} \bar{Z}_t \right\|_{L^\infty} \left\| D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2}^2$$
$$+ \left\| D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2}^2 \left(\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}} + \left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \left\| D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2} \right)$$
$$+ \left\| D_\alpha^2 \bar{Z}_t \right\|_{L^2}^3 \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}.$$

$$(387)$$

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Chapter 8

Controlling $\Re \int i \left(\dfrac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha$

We now show that we can control the term (289) from the RHS of (278):

$$\Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha = \Re \int i \left(2\frac{h_{t\alpha}}{h_\alpha} - 2\Re D_\alpha z_t \right) \frac{h_\alpha^2}{|z_\alpha|^2} \theta_\alpha \bar{\theta} d\alpha. \tag{388}$$

Here, all results will be expressed in terms of general energy $E_{a,\theta}$ for θ satisfying $\theta|_\partial = 0$ and $(I - \mathbb{H})(\theta \circ h^{-1}) = 0$, rather than specifying $\theta = D_\alpha^2 \bar{z}_t$.

8.1   Preliminary Estimate

We begin by rewriting this as

$$\begin{aligned}
\Re \int i \left(2\frac{h_{t\alpha}}{h_\alpha} - 2\Re D_\alpha z_t \right) \frac{h_\alpha^2}{|z_\alpha|^2} \theta_\alpha \bar{\theta} d\alpha &= \Re \int 2i \left(\frac{h_{t\alpha}}{h_\alpha} - \Re D_\alpha z_t \right) \left(\partial_\alpha \left(\theta \frac{h_\alpha}{z_\alpha} \right) \right) \bar{\theta} \frac{h_\alpha}{\bar{z}_\alpha} d\alpha \\
&\quad - \Re \int 2i \left(\frac{h_{t\alpha}}{h_\alpha} - \Re D_\alpha z_t \right) \left(\frac{h_\alpha}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) \theta \bar{\theta} d\alpha \\
&= I + II.
\end{aligned} \tag{389}$$

II is easy to control, via Hölder and a change of variables to Riemannian coordinates:

$$|II| \lesssim \left(\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty} \right) \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \|A_1\|_{L^\infty} E_{a,\theta}. \tag{390}$$

Therefore, we can focus on I from (389), which we will handle in the remaining sections of this chapter.

8.2   The Framework and Notation

For the following derivation, we will adopt some abbreviations for notational simplicity. We define:

$$\psi := \left(\frac{h_\alpha}{z_\alpha} \theta \right) \circ h^{-1}, \tag{391}$$

$$B := \left(\frac{h_{t\alpha}}{h_\alpha} - \Re D_\alpha z_t \right) \circ h^{-1}, \tag{392}$$

and

$$\Theta := \theta \circ h^{-1}. \tag{393}$$

We know ψ, B, and θ have the following properties:

Proposition 26. *Let ψ, B and Θ be as in (391), (392), and (393). Then*

$$B|_\partial = 0. \tag{394}$$

$$\psi|_\partial, \, \overline{\psi}\big|_\partial = 0 \tag{395}$$

$$B \in \mathbb{R} \tag{396}$$

$$(I - \mathbb{H})\Theta = 0 \tag{397}$$

$$(I - \mathbb{H})\psi = 0. \tag{398}$$

$$\|B\|_{L^\infty} \leqslant \left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \|D_{\alpha'} Z_t\|_{L^\infty} \tag{399}$$

$$\|\Theta\|_{L^2} \leqslant \|A_1\|_{L^\infty}^{1/2} E_{a,\theta}^{1/2} \tag{400}$$

$$\|\psi\|_{\dot{H}^{1/2}} \lesssim E_{a,\theta}^{1/2}. \tag{401}$$

Proof. For (394), this follows from (51) and (117). For (395), this follows from (114) and $\theta|_\partial = 0$; in the specific case $\theta = D_\alpha^2 \overline{z}_t$ this follows from (113) and (114). For (396), this follows because h is real-valued. For (397), this follows in the specific case from (139). For (398), this follows from (131), (397) and the second principle in §3.6; in the specific case from (144). The estimate (399) is immediate from the definition of B. Estimates (400) and

(401) follow immediately from the definition of $E_{a,\theta}$ and change of variables; in the specific case, they follow from (298) and (299). □

8.3 Green's Identity

We now show that we can control I from (389), using the properties listed in Proposition 26. A key insight is that $i\partial_{\alpha'}\psi = i\partial_{\alpha'}\mathbb{H}\psi$ by (398). The operator $i\partial_{\alpha'}\mathbb{H}$ corresponds to the Dirichlet-Neumann operator ∇_n.[46] We can therefore use Green's identity on the harmonic extensions of ψ and B.[47]

Since $i\partial_{\alpha'}\psi = i\partial_{\alpha'}\mathbb{H}\psi = \nabla_n\psi$, we can expand I from (389) as

$$
\begin{aligned}
I &= \Re \int 2i(B \circ h)(\partial_\alpha(\psi \circ h))(\overline{\psi} \circ h)d\alpha \\
&= \Re \int 2iB(\partial_{\alpha'}\psi)\overline{\psi}d\alpha' \\
&= \Re \int 2B(\nabla_n\psi)\overline{\psi}d\alpha' \\
&= \int B(\nabla_n\psi)\overline{\psi}d\alpha' + \int B\psi(\nabla_n\overline{\psi})d\alpha' \\
&= \int B\nabla_n(|\psi^\hbar|^2)d\alpha'.
\end{aligned}
\tag{402}
$$

Here $\psi^\hbar$ is the periodic harmonic extension of ψ to $P^- := [-1, 1] \times (-\infty, 0]$.

We now use Green's identity:

$$
\begin{aligned}
\int B\nabla_n(|\psi^\hbar|^2)d\alpha' &= \int (\nabla_n B)\,|\psi|^2\,d\alpha' + \int_{P^-} B^\hbar \Delta(|\psi^\hbar|^2)dv \\
&= I_1 + I_2.
\end{aligned}
\tag{403}
$$

---

[46] Recall that the Dirichlet-Neumann operator is defined by $\nabla_n f := \nabla_n f^\hbar$, where ∇_n is the outward-facing normal derivative and $f^\hbar$ is the extension of f that is harmonic and periodic in $P^- = [-1, 1] \times (-\infty, 0]$. For f real-valued, we can derive this by noting that $(I + \mathbb{H})f$ is holomorphic, so $i\partial_{\alpha'}(I + \mathbb{H})f = \nabla_n(I + \mathbb{H})f$. Taking real parts gives the identity.

[47] Here, to justify Green's identity, we can map the interior of P^- to the unit disc minus the slit, and then use the periodicity of all of the functions involved to consider the harmonic extensions of these functions to the whole unit disc.

We control the second term, I_2, by

$$
\begin{aligned}
|I_2| &= \left| \int_{P-} B^\hbar \Delta(|\psi^\hbar|^2) dv \right| \\
&= 2 \left| \int_{P-} B^\hbar \left| \nabla \psi^\hbar \right|^2 dv \right| \\
&\leqslant 2 \left\| B^\hbar \right\|_{L^\infty} \int_{P-} \left| \nabla \psi^\hbar \right|^2 dv \\
&= \left\| B^\hbar \right\|_{L^\infty} \int_{P-} \Delta(|\psi^\hbar|^2) dv \\
&= \|B\|_{L^\infty} \int \nabla_n(|\psi^\hbar|^2) d\alpha' \\
&= \|B\|_{L^\infty} \, 2\Re \int i(\partial_{\alpha'}\psi)\overline{\psi} d\alpha', \\
&= \|B\|_{L^\infty} \|\psi\|_{\dot{H}^{1/2}}
\end{aligned}
\tag{404}
$$

by the maximum principle and another application of Green's identity, where we can remove the absolute value because of the positivity of the integral in the third line.

We are left with the remaining term on the RHS of (403), I_1.

8.4   Controlling I_1

We are left from §8.3 with controlling

$$
\begin{aligned}
I_1 &= \int (\nabla_n B) |\psi|^2 d\alpha' \\
&= \Re \int (i\partial_{\alpha'} \mathbb{H} B) |\psi|^2 d\alpha' \\
&= \Re \int (i\mathbb{H} \partial_{\alpha'} B) |\psi|^2 d\alpha' \\
&= \Re \int \frac{1}{Z_{,\alpha'}} (i\mathbb{H}\partial_{\alpha'} B)\Theta \overline{\psi} d\alpha',
\end{aligned}
\tag{405}
$$

where we have commuted $\partial_{\alpha'}$ outside the $\mathbb{H}$ by (100) since $B|_\partial = 0$ (394).

We commute the $\frac{1}{\overline{Z}_{,\alpha'}}$ factor inside the $\mathbb{H}$, and then apply the adjoint property (91):

$$
\begin{aligned}
I_1 &= \Re \int i \left(\left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B \right) \Theta \overline{\psi} d\alpha' + \Re \int i \left(\mathbb{H} \left(\frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} B \right) \right) \Theta \overline{\psi} d\alpha' \\
&= \Re \int i \left(\left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B \right) \Theta \overline{\psi} d\alpha' - \Re \int i \left(\frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} B \right) \mathbb{H} \left(\Theta \overline{\psi} \right) d\alpha' \\
&= \Re \int i \left(\left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B \right) \Theta \overline{\psi} d\alpha' \\
&\quad - \Re \int i \left(\frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} B \right) \left(\mathbb{H} \left(\Theta \overline{\psi} \right) \overbrace{- \overline{\psi} \mathbb{H} \Theta + \overline{\psi} \mathbb{H} \Theta}^{0} \right) d\alpha' \\
&= \Re \int i \left(\left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B \right) \Theta \overline{\psi} d\alpha' \\
&\quad + \Re \int i \left(\frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} B \right) [\overline{\psi}, \mathbb{H}] \Theta d\alpha' - \Re \int i \left(\frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} B \right) \overline{\psi} \mathbb{H} \Theta d\alpha' \\
&= I_{11} + I_{12} + I_{13}.
\end{aligned}
\tag{406}
$$

Observe that because $\mathbb{H}\Theta = \Theta$ (397),

$$
\begin{aligned}
I_{13} &= -\Re \int i \left(\frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} B \right) \overline{\psi} \Theta d\alpha' \\
&= -\Re \int i \left(\partial_{\alpha'} B \right) |\psi|^2 d\alpha' \\
&= 0,
\end{aligned}
\tag{407}
$$

since $B \in \mathbb{R}$. It remains to control I_{11} and I_{12}.

We use Cauchy-Schwarz and then the $\dot{H}^{1/2} \times L^2$ commutator estimate (240) to control I_{12}:

$$
\begin{aligned}
|I_{12}| &\leqslant \|D_{\alpha'} B\|_{L^2} \left\| [\overline{\psi}, \mathbb{H}] \Theta \right\|_{L^2} \\
&\lesssim \|D_{\alpha'} B\|_{L^2} \left\| \overline{\psi} \right\|_{\dot{H}^{1/2}} \|\Theta\|_{L^2}.
\end{aligned}
\tag{408}
$$

We will control $\|D_{\alpha'} B\|_{L^2}$ by (430) in §8.5 below, we controlled $\left\| \overline{\psi} \right\|_{\dot{H}^{1/2}}$ at (401), and we controlled $\|\Theta\|_{L^2}$ at (400). We conclude that

$$
|I_{12}| \lesssim \|D_{\alpha'} B\|_{L^2} \|A_1\|_{L^\infty}^{1/2} E_{a,\theta}.
\tag{409}
$$

It remains to control I_{11} from (406). Here we use Proposition 10, identity (109). Because

$(I - \mathbb{H})\frac{1}{Z_{,\alpha'}} = \fint \frac{1}{Z_{,\alpha'}}$ (131) we can rewrite

$$\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B = \mathbb{H}\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B - \fint \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} B\right) + \left(\fint \frac{1}{Z_{,\alpha'}}\right) \underbrace{\left(\fint \partial_{\alpha'} B\right)}_{0 \text{ by } (394)} \qquad (410)$$

and so, by using this for one half of $\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B$,

$$\begin{aligned}
\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B &= \frac{(I + \mathbb{H})}{2}\left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B\right) - \frac{1}{2}\fint \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} B\right) \\
&= \mathbb{P}_H\left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B\right) - \frac{1}{2}\fint D_{\alpha'} B.
\end{aligned} \qquad (411)$$

We now use (411) to rewrite I_{11}, and then use adjoint property (98):

$$\begin{aligned}
I_{11} &= \Re \int i\left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B\right) \Theta \overline{\psi} d\alpha' \\
&= \Re \int i\left\{\mathbb{P}_H\left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B\right) - \frac{1}{2}\fint D_{\alpha'} B\right\} \Theta \overline{\psi} d\alpha' \\
&= \Re \int i\left\{\mathbb{P}_H\left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B\right)\right\} \Theta \overline{\psi} d\alpha' - \Re\left\{\left(\frac{1}{2}\fint D_{\alpha'} B\right) \int i\Theta \overline{\psi} d\alpha'\right\} \\
&= \Re \int i\left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B\right) \mathbb{P}_A\left(\Theta \overline{\psi}\right) d\alpha' - \Re\left\{\left(\frac{1}{2}\fint D_{\alpha'} B\right) \int i\Theta \overline{\psi} d\alpha'\right\} \\
&= I_{111} + I_{112}.
\end{aligned} \qquad (412)$$

To control I_{111}, we use Cauchy-Schwarz, and then control the first factor with the $L^2 \times L^\infty$ estimate (233) and control the second factor by rewriting it as a commutator by (397) and then using the $\dot{H}^{1/2} \times L^2$ estimate (240):

$$\begin{aligned}
|I_{111}| &\leqslant \left\|\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B\right\|_{L^2} \left\|\mathbb{P}_A\left(\Theta \overline{\psi}\right)\right\|_{L^2} \\
&= \left\|\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B\right\|_{L^2} \left\|\frac{1}{2}\left[\overline{\psi}, \mathbb{H}\right] \Theta\right\|_{L^2} \\
&\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \|B\|_{L^\infty} \|\overline{\psi}\|_{\dot{H}^{1/2}} \|\Theta\|_{L^2} \\
&\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \left(\left\|\frac{h_{t\alpha}}{h_\alpha}\right\|_{L^\infty} + \|D_{\alpha'} Z_t\|_{L^\infty}\right) \|A_1\|_{L^\infty}^{1/2} E_{a,\theta},
\end{aligned} \qquad (413)$$

100

where we have used (399), (400) and (401) in the last line.

It's easy to estimate I_{112} from (412); using (183) to rewrite $\frac{1}{\overline{Z}_{,\alpha'}} = -i\frac{Z_{tt}+i}{A_1}$, we estimate

$$
\begin{aligned}
|I_{112}| &= \left| \left(\frac{1}{2} \fint D_{\alpha'} B \right) \int i\Theta\overline{\psi} d\alpha' \right| \\
&\leqslant \|D_{\alpha'} B\|_{L^2} \int |\Theta\overline{\psi}| \, d\alpha' \\
&= \|D_{\alpha'} B\|_{L^2} \int \left| \frac{Z_{tt}+i}{A_1} |\Theta|^2 \right| d\alpha' \\
&\leqslant \|D_{\alpha'} B\|_{L^2} \|Z_{tt}+i\|_{L^\infty} E_{a,\theta}.
\end{aligned}
\tag{414}
$$

8.5 Controlling $\|D_{\alpha'} B\|_{L^2}$

We must control $\|D_{\alpha'} B\|_{L^2}$, where B is defined from (392) as

$$
B = \left(\frac{h_{t\alpha}}{h_\alpha} - \Re D_\alpha z_t \right) \circ h^{-1} = (h_t \circ h^{-1})_{\alpha'} - \Re D_{\alpha'} Z_t.
\tag{415}
$$

Recall from (343) that

$$
(h_t \circ h^{-1})_{\alpha'} - 2\Re D_{\alpha'} Z_t = \Re \left\{ -(I - \mathbb{H}) \left(\frac{1}{\overline{Z}_{,\alpha'}} \overline{Z}_{t,\alpha'} \right) + (I - \mathbb{H}) \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\}.
\tag{416}
$$

Therefore,

$$
\|D_{\alpha'} B\|_{L^2} \leqslant \|D_{\alpha'} \Re D_{\alpha'} Z_t\|_{L^2} + \left\| D_{\alpha'} \Re \left\{ -(I - \mathbb{H}) \left(\frac{1}{\overline{Z}_{,\alpha'}} \overline{Z}_{t,\alpha'} \right) + (I - \mathbb{H}) \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\} \right\|_{L^2}.
\tag{417}
$$

Note that $|\partial_{\alpha'} \Re f| \leqslant |\partial_{\alpha'} f|$ and so $|D_\alpha \Re f| \leqslant |D_\alpha f|$ for any f. Therefore,

$$
\|D_{\alpha'} B\|_{L^2} \leqslant \left\| D_{\alpha'}^2 Z_t \right\|_{L^2} + \left\| D_{\alpha'}(I - \mathbb{H}) \left(\frac{1}{\overline{Z}_{,\alpha'}} \overline{Z}_{t,\alpha'} \right) \right\|_{L^2} + \left\| D_{\alpha'}(I - \mathbb{H}) \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\|_{L^2}.
\tag{418}
$$

We've controlled $\|D_{\alpha'}^2 Z_t\|_{L^2}$, so it suffices to focus on the second and third terms.

For the second term, we begin by commuting the $D_{\alpha'}$ inside the $(I - \mathbb{H})$:

$$
D_{\alpha'}(I - \mathbb{H}) \left\{ \frac{1}{\overline{Z}_{,\alpha'}} \overline{Z}_{t,\alpha'} \right\} = \frac{1}{Z_{,\alpha'}} (I - \mathbb{H}) \partial_{\alpha'} \left(\frac{1}{\overline{Z}_{,\alpha'}} \overline{Z}_{t,\alpha'} \right) - \frac{1}{Z_{,\alpha'}} [\partial_{\alpha'}, \mathbb{H}] \left\{ \frac{1}{\overline{Z}_{,\alpha'}} \overline{Z}_{t,\alpha'} \right\}.
\tag{419}
$$

The $[\partial_{\alpha'}, \mathbb{H}]$ term is not necessarily zero, but it is easily controlled. By (101), it is:

$$\frac{1}{2i}\frac{1}{Z_{,\alpha'}}(\alpha')\cot(\frac{\pi}{2}(\alpha'-\beta'))\frac{1}{\overline{Z}_{,\beta'}}\overline{Z}_{t,\beta'}\bigg|_{\partial}. \tag{420}$$

There are now two cases. If we are in the trivial angle regime, this is zero, because $\frac{1}{\overline{Z}_{,\beta'}}\overline{Z}_{t,\beta'} = \frac{Z_{,\beta'}}{\overline{Z}_{,\beta'}}D_{\beta'}\overline{Z}_t$; the second factor always has periodic behavior, and the first does by (115). Otherwise, $\frac{1}{Z_{,\beta'}}(\beta') = 0$ at the corner, by the dichotomy (115), so we can subtract $\frac{1}{Z_{,\beta'}}(\beta')\cot(\cdots)\frac{1}{\overline{Z}_{,\beta'}}\overline{Z}_{t,\beta'}$ and control by Hölder and Hardy's inequality (218):

$$\left\|\frac{1}{2i}\frac{1}{Z_{,\alpha'}}(\alpha')\cot(\frac{\pi}{2}(\alpha'-\beta'))\frac{1}{\overline{Z}_{,\beta'}}\overline{Z}_{t,\beta'}\bigg|_{\partial}\right\|_{L^2_{\alpha'}}$$

$$= \left\|\frac{1}{2i}\left(\frac{1}{Z_{,\alpha'}}(\alpha') - \frac{1}{Z_{,\beta'}}(\beta')\right)\cot(\frac{\pi}{2}(\alpha'-\beta'))\frac{1}{\overline{Z}_{,\beta'}}\overline{Z}_{t,\beta'}\bigg|_{\partial}\right\|_{L^2_{\alpha'}}$$

$$\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty}. \tag{421}$$

Therefore, it suffices to consider the first term on the RHS of (419), which we write as

$$\left\|\frac{1}{Z_{,\alpha'}}(I-\mathbb{H})\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}\overline{Z}_{t,\alpha'}\right)\right\|_{L^2} = \left\|(I-\mathbb{H})D_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}\overline{Z}_{t,\alpha'}\right) - \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}\overline{Z}_{t,\alpha'}\right)\right\|_{L^2}$$

$$\lesssim \left\|D_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}\overline{Z}_{t,\alpha'}\right)\right\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left\|\frac{1}{\overline{Z}_{,\beta'}}\overline{Z}_{t,\beta'}\right\|_{L^\infty}$$

$$= \left\|D_{\alpha'}^2 Z_t\right\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left\|\frac{1}{\overline{Z}_{,\beta'}}\overline{Z}_{t,\beta'}\right\|_{L^\infty}, \tag{422}$$

where we used (233) in the second line (noting that this *doesn't* require control of $\frac{1}{\overline{Z}_{,\beta'}}\overline{Z}_{t,\beta'}\big|_{\partial}$).

We conclude from (419), (421), and (422) that the second term on the RHS of (418) is controlled

$$\left\|D_{\alpha'}(I-\mathbb{H})\left\{\frac{1}{\overline{Z}_{,\alpha'}}\overline{Z}_{t,\alpha'}\right\}\right\|_{L^2} \lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty} + \left\|D_{\alpha'}^2 Z_t\right\|_{L^2}. \tag{423}$$

We now consider the last term in (418). We begin by writing this as a commutator using

(137), and then apply $D_{\alpha'}$:

$$
\begin{aligned}
D_{\alpha'}(I - \mathbb{H}) \left\{ Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\} &= D_{\alpha'} [Z_t, \mathbb{H}] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \\
&= \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \frac{1}{2i} \int (Z_t(\alpha') - Z_t(\beta')) \cot(\frac{\pi}{2}(\alpha' - \beta')) \partial_{\beta'} \frac{1}{Z_{,\beta'}} d\beta' \\
&= \frac{1}{Z_{,\alpha'}} Z_{t,\alpha'} \mathbb{H} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} - \frac{1}{Z_{,\alpha'}} \frac{1}{2i} \int \frac{\pi}{2} \frac{Z_t(\alpha') - Z_t(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} \partial_{\beta'} \frac{1}{Z_{,\beta'}} d\beta'.
\end{aligned}
$$
(424)

Via the boundedness of the Hilbert transform, we control the first of these terms by

$$
\|D_{\alpha'} Z_t\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}.
$$
(425)

Therefore, it suffices to focus on the second term. We commute the $\frac{1}{Z_{,\alpha'}}$ inside, getting

$$
\begin{aligned}
-\frac{1}{2i} \int \frac{\pi}{2} \frac{Z_t(\alpha') - Z_t(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} D_{\beta'} \frac{1}{Z_{,\beta'}} d\beta' \\
-\frac{1}{2i} \int \frac{\pi}{2} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \frac{\left(\frac{1}{Z_{,\alpha'}}(\alpha') - \frac{1}{Z_{,\beta'}}(\beta') \right)}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \partial_{\beta'} \frac{1}{Z_{,\beta'}} d\beta'.
\end{aligned}
$$
(426)

We control the first term by (235):

$$
\left\| \int \frac{\pi}{2} \frac{Z_t(\alpha') - Z_t(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} D_{\beta'} \frac{1}{Z_{,\beta'}} d\beta' \right\|_{L^2} \lesssim \|Z_{t,\alpha'}\|_{L^2} \left\| D_{\beta'} \frac{1}{Z_{,\beta'}} \right\|_{L^\infty}.
$$
(427)

We control the second term by (241):

$$
\left\| \int \frac{\pi}{2} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \frac{\left(\frac{1}{Z_{,\alpha'}}(\alpha') - \frac{1}{Z_{,\beta'}}(\beta') \right)}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \partial_{\beta'} \frac{1}{Z_{,\beta'}} d\beta' \right\|_{L^2} \lesssim \|Z_{t,\alpha'}\|_{L^2} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}^2.
$$
(428)

We conclude that the third term in (418) is controlled

$$
\begin{aligned}
\left\| D_{\alpha'}(I - \mathbb{H}) \left\{ Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\} \right\|_{L^2} &\lesssim \|D_{\alpha'} Z_t\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \\
&+ \|Z_{t,\alpha'}\|_{L^2} \left(\left\| D_{\beta'} \frac{1}{Z_{,\beta'}} \right\|_{L^\infty} + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}^2 \right).
\end{aligned}
$$
(429)

We conclude from (418) that

$$\|D_{\alpha'}B\|_{L^2} \leqslant \left\|D_{\alpha'}^2 Z_t\right\|_{L^2} + (423) + (429).$$
$$\lesssim \left\|D_{\alpha'}^2 Z_t\right\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty}$$
$$+ \|Z_{t,\alpha'}\|_{L^2}\left(\left\|D_{\beta'}\frac{1}{Z_{,\beta'}}\right\|_{L^\infty} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\right). \tag{430}$$

8.6 Conclusion

We now combine our various estimates. We have

$$|I| \lesssim |I_1| + |I_2|$$
$$\lesssim |I_1| + (404)$$
$$\lesssim |I_1| + \left(\left\|\frac{h_{t\alpha}}{h_\alpha}\right\|_{L^\infty} + \|D_{\alpha'}Z_t\|_{L^\infty}\right)E_{a,\theta}^{1/2}, \tag{431}$$

where

$$|I_1| \leqslant |I_{11}| + |I_{12}|$$
$$\leqslant |I_{111}| + |I_{112}| + |I_{12}|$$
$$\lesssim (413) + (414) + (409)$$
$$\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left(\left\|\frac{h_{t\alpha}}{h_\alpha}\right\|_{L^\infty} + \|D_{\alpha'}Z_t\|_{L^\infty}\right)\|A_1\|_{L^\infty}^{1/2}E_{a,\theta}$$
$$+ \|D_{\alpha'}B\|_{L^2}\|Z_{tt} + i\|_{L^\infty}E_{a,\theta} + \|D_{\alpha'}B\|_{L^2}\|A_1\|_{L^\infty}^{1/2}E_{a,\theta}. \tag{432}$$

From the original equation in this chapter (389), we conclude that

$$\left| \Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha \right| \leqslant |I| + |II|$$

$$\leqslant (431) + \left(\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty} \right) \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \|A_1\|_{L^\infty} E_{a,\theta}$$

$$\lesssim \left(\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \|D_{\alpha'} Z_t\|_{L^\infty} \right) E_{a,\theta}^{1/2}$$

$$+ \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \left(\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \|D_{\alpha'} Z_t\|_{L^\infty} \right) \|A_1\|_{L^\infty}^{1/2} E_{a,\theta}$$

$$+ \|D_{\alpha'} B\|_{L^2} \|Z_{tt} + i\|_{L^\infty} E_{a,\theta} + \|D_{\alpha'} B\|_{L^2} \|A_1\|_{L^\infty}^{1/2} E_{a,\theta}$$

$$+ \left(\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty} \right) \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \|A_1\|_{L^\infty} E_{a,\theta},$$

$$(433)$$

where we use (430) to control $\|D_{\alpha'} B\|_{L^2}$.

Chapter 9

Controlling G_θ of E_b

By (292), we must control

$$\left(\int \frac{1}{\mathfrak{a}} \left| D_\alpha(-i\mathfrak{a}_t \bar{z}_\alpha) + [(\partial_t^2 + i\mathfrak{a}\partial_\alpha), D_\alpha]\bar{z}_t \right|^2 d\alpha \right)^{1/2}. \tag{434}$$

We control the commutator in §9.1 at (435) and the first term in §9.2 at (472).

9.1 Controlling the Commutator for E_b

We use (261) to control

$$
\begin{aligned}
\left(\int \frac{1}{\mathfrak{a}} \left| [(\partial_t^2 + i\mathfrak{a}\partial_\alpha), D_\alpha]\bar{z}_t \right|^2 d\alpha \right)^{1/2} &\lesssim (\|D_\alpha z_{tt}\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty}^2) \|D_\alpha \bar{z}_t\|_{L^2(\frac{1}{\mathfrak{a}} d\alpha)} \\
&\quad + \|D_\alpha z_t\|_{L^\infty} \|D_\alpha \bar{z}_{tt}\|_{L^2(\frac{1}{\mathfrak{a}} d\alpha)} \\
&\lesssim (\|D_\alpha z_{tt}\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty}^2) E_b^{1/2} \\
&\quad + \|D_\alpha z_t\|_{L^\infty} \|D_\alpha \bar{z}_{tt}\|_{L^2(\frac{1}{\mathfrak{a}} d\alpha)},
\end{aligned}
\tag{435}
$$

where we have used $A_1 \geqslant 1$ (182) to insert the $A_1 \circ h$ weight corresponding to the lower-order term in energy E_b, $\|D_\alpha \bar{z}_t\|_{L^2(\frac{(A_1 \circ h)}{\mathfrak{a}})}^2$.

9.2 Controlling the Main Part for E_b

We show here that we can control $\left(\int \frac{1}{\mathfrak{a}}|D_\alpha(-i\mathfrak{a}_t\bar{z}_\alpha)|^2\,d\alpha\right)^{1/2}$. It turns out to be advantageous to think of $\mathfrak{a}_t\bar{z}_\alpha$ as $\frac{\mathfrak{a}_t}{\mathfrak{a}}\mathfrak{a}\bar{z}_\alpha$. We have

$$D_\alpha\left(-i\frac{\mathfrak{a}_t}{\mathfrak{a}}\mathfrak{a}\bar{z}_\alpha\right) = -i\mathfrak{a}\bar{z}_\alpha D_\alpha\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right) + \frac{\mathfrak{a}_t}{\mathfrak{a}}D_\alpha(-i\mathfrak{a}\bar{z}_\alpha) = -i\mathfrak{a}\bar{z}_\alpha D_\alpha\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right) + \frac{\mathfrak{a}_t}{\mathfrak{a}}D_\alpha\bar{z}_{tt}, \qquad (436)$$

since $-i\mathfrak{a}\bar{z}_\alpha = \bar{z}_{tt} - i$ (27). Therefore,

$$\left(\int \frac{1}{\mathfrak{a}}|D_\alpha(-i\mathfrak{a}_t\bar{z}_\alpha)|^2\,d\alpha\right)^{1/2} \leqslant \left(\int \frac{1}{\mathfrak{a}}\left|\mathfrak{a}\bar{z}_\alpha D_\alpha\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right)\right|^2\,d\alpha\right)^{1/2} + \left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty}\left(\int \frac{1}{\mathfrak{a}}|D_\alpha\bar{z}_{tt}|^2\,d\alpha\right)^{1/2}. \tag{437}$$

We controlled the factors in the second term on the RHS in (302) and (327). We can therefore concentrate on the first term. We recall that

$$\frac{1}{\mathfrak{a}} = \frac{|z_\alpha|^2}{A_1 \circ h}\frac{1}{h_\alpha}, \tag{438}$$

so

$$\begin{aligned}
\left(\int \frac{1}{\mathfrak{a}}\left|\mathfrak{a}\bar{z}_\alpha D_\alpha\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right)\right|^2\,d\alpha\right)^{1/2} &= \left(\int \frac{1}{\mathfrak{a}}\left|\mathfrak{a}\partial_\alpha\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right)\right|^2\,d\alpha\right)^{1/2} \\
&= \left(\int \frac{|z_\alpha|^2}{(A_1\circ h)h_\alpha}\left|\mathfrak{a}\partial_\alpha\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right)\right|^2\,d\alpha\right)^{1/2} \\
&\leqslant \left\|\frac{|z_\alpha|^2}{A_1\circ h}\right\|_{L^\infty}^{1/2}\left(\int \frac{1}{h_\alpha}\left|\mathfrak{a}\partial_\alpha\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right)\right|^2\,d\alpha\right)^{1/2} \\
&\leqslant \|z_\alpha\|_{L^\infty}\left(\int \left|\mathfrak{a}\frac{\partial_\alpha\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right)}{h_\alpha}\right|^2 h_\alpha\,d\alpha\right)^{1/2} \\
&= \|z_\alpha\|_{L^\infty}\left(\int \left|(\mathfrak{a}\circ h^{-1})\partial_{\alpha'}\left(\frac{A_t}{A}\right)\right|^2\,d\alpha'\right)^{1/2},
\end{aligned} \tag{439}$$

where in the penultimate step we have removed the factor of $1/A_1$ because $A_1 \geqslant 1$ (182).

We now seek a way of writing $(\mathfrak{a}\circ h^{-1})\partial_{\alpha'}\left(\frac{A_t}{A}\right)$ in terms of commutators. The derivation to do so will be slightly convoluted, since we have to take care that we can invert $(I - \mathbb{H})$ and we want to make sure that the advantageous weight $(\mathfrak{a}\circ h^{-1})$ is placed appropriately.

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Recall that $(\mathfrak{a}\bar{z}_\alpha) \circ h^{-1} = \mathcal{A}\overline{Z}_{,\alpha'}$. Because

$$-i\mathcal{A}_t\overline{Z}_{,\alpha'} = \frac{\mathcal{A}_t}{\mathcal{A}}(-i\mathcal{A}\overline{Z}_{,\alpha'}), \tag{440}$$

by the product rule

$$\partial_{\alpha'}\left(-i\mathcal{A}_t\overline{Z}_{,\alpha'}\right) = (-i\mathcal{A}\overline{Z}_{,\alpha'})\partial_{\alpha'}\left(\frac{\mathcal{A}_t}{\mathcal{A}}\right) + \frac{\mathcal{A}_t}{\mathcal{A}}\partial_{\alpha'}\left(-i\mathcal{A}\overline{Z}_{,\alpha'}\right). \tag{441}$$

The LHS is just the derivative of our standard quasilinear equation in Riemannian coordinates (186) and the second part of the second term in the RHS is simply $\partial_{\alpha'}(\overline{Z}_{tt}-i) = \partial_{\alpha'}\overline{Z}_{tt}$. Therefore, using the original equations (27) (in Riemannian coordinates) and (186), we have

$$(-i\mathcal{A}\overline{Z}_{,\alpha'})\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}} = \partial_{\alpha'}\left(\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'}\right) - \frac{\mathcal{A}_t}{\mathcal{A}}\partial_{\alpha'}\overline{Z}_{tt}. \tag{442}$$

We now apply $(I - \mathbb{H})$ to each side:

$$(I - \mathbb{H})\left\{(-i\mathcal{A}\overline{Z}_{,\alpha'})\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}\right\} = (I - \mathbb{H})\partial_{\alpha'}\left(\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'}\right) - (I - \mathbb{H})\left\{\frac{\mathcal{A}_t}{\mathcal{A}}\partial_{\alpha'}\overline{Z}_{tt}\right\}. \tag{443}$$

We want to move the factor $\mathcal{A}\overline{Z}_{,\alpha'}$ outside on the LHS, so we do this:

$$\begin{aligned}
(-i\mathcal{A}\overline{Z}_{,\alpha'})(I - \mathbb{H})\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}} &= (I - \mathbb{H})\partial_{\alpha'}\left(\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'}\right) \\
&\quad - (I - \mathbb{H})\left\{\frac{\mathcal{A}_t}{\mathcal{A}}\partial_{\alpha'}\overline{Z}_{tt}\right\} + [i\mathcal{A}\overline{Z}_{,\alpha'}, \mathbb{H}]\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}.
\end{aligned} \tag{444}$$

Now, by multiplying both sides by the modulus-one factor $\frac{|Z_{,\alpha'}|}{\overline{Z}_{,\alpha'}}$, we may remove the $(I - \mathbb{H})$ on the LHS by taking imaginary parts. We get that

$$\left|\mathcal{A}\overline{Z}_{,\alpha'}\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}\right| \leqslant \left|(I - \mathbb{H})\partial_{\alpha'}\left(\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'}\right) - (I - \mathbb{H})\left\{\frac{\mathcal{A}_t}{\mathcal{A}}\partial_{\alpha'}\overline{Z}_{tt}\right\} + [i\mathcal{A}\overline{Z}_{,\alpha'}, \mathbb{H}]\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}\right|. \tag{445}$$

Since $\mathcal{A}\overline{Z}_{,\alpha'} = (\mathfrak{a}\overline{z}_\alpha) \circ h^{-1}$, we may conclude that

$$\left\| (\mathfrak{a} \circ h^{-1})\partial_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2}$$

$$\leqslant \left\| \frac{1}{z_\alpha} \right\|_{L^\infty} \left\| (I - \mathbb{H})\partial_{\alpha'} \left(\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'} \right) - (I - \mathbb{H}) \left\{ \frac{\mathcal{A}_t}{\mathcal{A}} \partial_{\alpha'} \overline{Z}_{tt} \right\} - [i\mathcal{A}\overline{Z}_{,\alpha'}, \mathbb{H}]\partial_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2}.$$
(446)

We can easily control the second and third terms. Since $i\mathcal{A}\overline{Z}_{,\alpha'} = -(\overline{Z}_{tt} - i)$, by the L^2 boundedness of $\mathbb{H}$ and Hölder for the second term and estimate (233) for the third term,

$$\left\| -(I - \mathbb{H}) \left\{ \frac{\mathcal{A}_t}{\mathcal{A}} \overline{Z}_{tt,\alpha'} \right\} - [i\mathcal{A}\overline{Z}_{,\alpha'}, \mathbb{H}]\partial_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \lesssim \|Z_{tt,\alpha'}\|_{L^2} \left\| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty}.$$
(447)

We can therefore focus on controlling

$$\left\| (I - \mathbb{H})\partial_{\alpha'} \left(\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'} \right) \right\|_{L^2}.$$
(448)

9.2.1 Controlling $\left\| (I - \mathbb{H})\partial_{\alpha'} \left(\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'} \right) \right\|_{L^2}$

Recall from (116) that $\left. \left(\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'} \right) \right|_\partial = 0$. Therefore, we may commute $\partial_{\alpha'}$ outside $(I - \mathbb{H})$ for free by (100), so we are left with controlling

$$\left\| \partial_{\alpha'}(I - \mathbb{H}) \left(\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'} \right) \right\|_{L^2}.$$
(449)

Recall from (194) and $i\mathcal{A} = \frac{Z_{tt}+i}{Z_{,\alpha'}}$ that

$$\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'} = (D_{\alpha'}^2 \overline{Z}_t)Z_t^2 + 2Z_t D_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) + 2(D_{\alpha'}\overline{Z}_t)Z_{tt} + F_{tt} \circ Z + iD_{\alpha'}\overline{Z}_t, \quad (450)$$

where $F(z(\alpha,t),t) := \overline{z}_t(\alpha,t)$.

Under $(I - \mathbb{H})$, by (139), (154), (138) and (147) the last two terms disappear and the rest turn into commutators:

$$(I - \mathbb{H}) \left(\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'} \right) = [Z_t^2, \mathbb{H}]D_{\alpha'}^2 \overline{Z}_t + 2[Z_t, \mathbb{H}]D_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) + 2[Z_{tt}, \mathbb{H}]D_{\alpha'}\overline{Z}_t.$$
(451)

Therefore, by (449) and (451), we have to control

$$\left\| \partial_{\alpha'}[Z_t^2, \mathbb{H}]D_{\alpha'}^2\overline{Z}_t + 2\partial_{\alpha'}[Z_t, \mathbb{H}]D_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) + 2\partial_{\alpha'}[Z_{tt}, \mathbb{H}]D_{\alpha'}\overline{Z}_t \right\|_{L^2}. \tag{452}$$

We will treat the first term in (452) with part of the second term, and treat the remainder of the second term together with the third term. In doing so, we will have to be careful, because we will depend on the fact that from the second term $D_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t)$ is holomorphic when treated together (see (154)) but not when split apart.

We begin with the second and third terms of (452). Observe that, on expanding out, we would be able to control everything except

$$\partial_{\alpha'}2[Z_t, \mathbb{H}]\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\} \tag{453}$$

—which we will end up treating with the first term—if there were any inequality of the form

$$\|\partial_{\alpha'}[f, \mathbb{H}]g\|_{L^2} \overset{?}{\lesssim} \|f'\|_{L^2}\|g\|_{L^\infty}. \tag{454}$$

Unfortunately, this inequality doesn't hold. Indeed, we calculate that

$$\partial_{\alpha'}[f, \mathbb{H}]g = f'\mathbb{H}g - \frac{1}{2i}\int \frac{\pi}{2}\frac{f(\alpha') - f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}g(\beta')d\beta'. \tag{455}$$

The inequality does apply to the second term, by (235). The first term, though, we cannot control this way, since $\mathbb{H}$ is not $L^\infty \to L^\infty$ bounded. If, however, $\mathbb{H}g = g$, then we can rewrite $f'\mathbb{H}g = f'g$ and attain the desired inequality directly by Hölder. We will therefore follow this approach, relying on the fact that $(I - \mathbb{H})D_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) = 0$ (154) and $(I - \mathbb{H})D_{\alpha'}\overline{Z}_t = 0$ (138).[48] Because we are not using this method to control $\partial_{\alpha'}2[Z_t, \mathbb{H}]\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\}$, we will be left with a remainder term, which we will return to the form $\partial_{\alpha'}[Z_t, \mathbb{H}]\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\}$ at the expense of an error term. We will control this error term after this derivation, and then control $\partial_{\alpha'}[Z_t, \mathbb{H}]\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\}$ together with the first term of (452) at the end.

---

[48]An alternative approach would be to handle the terms like $Z_{t,\alpha'}\mathbb{H}D_{\alpha'}\overline{Z}_{tt}$ by commuting a $\frac{1}{Z_{,\alpha'}}$ outside

$$Z_{t,\alpha'}\mathbb{H}D_{\alpha'}\overline{Z}_{tt} = (D_{\alpha'}Z_t)\mathbb{H}\overline{Z}_{tt} - Z_{t,\alpha'}\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{tt,\alpha'} \tag{456}$$

and then using Hölder and (244).

We begin by expanding out the second and third terms of (452):

$$2\partial_{\alpha'}[Z_t, \mathbb{H}]D_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) + 2\partial_{\alpha'}[Z_{tt}, \mathbb{H}]D_{\alpha'}\overline{Z}_t$$

$$= 2Z_{t,\alpha'}\mathbb{H}D_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) + 2Z_{tt,\alpha}\mathbb{H}D_{\alpha'}\overline{Z}_t$$

$$- \frac{1}{i}\int\frac{\pi}{2}\frac{Z_t(\alpha') - Z_t(\beta')}{\sin^2(\frac{\pi}{2}(\alpha'-\beta'))}D_{\beta'}(\overline{Z}_{tt} - (D_{\beta'}\overline{Z}_t)Z_t)d\beta' \quad (457)$$

$$- \frac{1}{i}\int\frac{\pi}{2}\frac{Z_{tt}(\alpha') - Z_{tt}(\beta')}{\sin^2(\frac{\pi}{2}(\alpha'-\beta'))}D_{\beta'}\overline{Z}_t d\beta'.$$

Now we use the fact that $(I - \mathbb{H})D_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) = 0$ (154) and $(I - \mathbb{H})D_{\alpha'}\overline{Z}_t = 0$ (138) to remove the $\mathbb{H}$s from the RHS above. We get

$$2Z_{t,\alpha'}D_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) + 2Z_{tt,\alpha}D_{\alpha'}\overline{Z}_t - \frac{1}{i}\int\frac{\pi}{2}\frac{Z_t(\alpha') - Z_t(\beta')}{\sin^2(\frac{\pi}{2}(\alpha'-\beta'))}D_{\beta'}(\overline{Z}_{tt} - (D_{\beta'}\overline{Z}_t)Z_t)d\beta'$$

$$- \frac{1}{i}\int\frac{\pi}{2}\frac{Z_{tt}(\alpha') - Z_{tt}(\beta')}{\sin^2(\frac{\pi}{2}(\alpha'-\beta'))}D_{\beta'}\overline{Z}_t d\beta'. \quad (458)$$

We now want to put just the term with $(D_{\alpha'}^2\overline{Z}_t)Z_t$ back into commutator form. We can rewrite just that part as

$$-2Z_{t,\alpha'}(D_{\alpha'}^2\overline{Z}_t)Z_t + \frac{1}{i}\int\frac{\pi}{2}\frac{Z_t(\alpha') - Z_t(\beta')}{\sin^2(\frac{\pi}{2}(\alpha'-\beta'))}((D_{\beta'}^2\overline{Z}_t)Z_t(\beta'))d\beta'$$

$$= -2Z_{t,\alpha'}\mathbb{H}\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\}$$

$$- 2Z_{t,\alpha'}(I - \mathbb{H})\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\}$$

$$+ \frac{1}{i}\int\frac{\pi}{2}\frac{Z_t(\alpha') - Z_t(\beta')}{\sin^2(\frac{\pi}{2}(\alpha'-\beta'))}((D_{\beta'}^2\overline{Z}_t)Z_t(\beta'))d\beta'$$

$$= -2\partial_{\alpha'}[Z_t, \mathbb{H}]\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\}$$

$$- 2Z_{t,\alpha'}(I - \mathbb{H})\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\}.$$

$$(459)$$

We use (459) to rewrite (458) as

$$
\begin{aligned}
& 2Z_{t,\alpha'}(D_{\alpha'}\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)D_{\alpha'}Z_t) + 2Z_{tt,\alpha}D_{\alpha'}\overline{Z}_t \\
& \quad - \frac{1}{i}\int \frac{\pi}{2}\frac{Z_t(\alpha') - Z_t(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}(D_{\beta'}\overline{Z}_{tt} - (D_{\beta'}\overline{Z}_t)(D_{\beta'}Z_t))d\beta' \\
& \quad - \frac{1}{i}\int \frac{\pi}{2}\frac{Z_{tt}(\alpha') - Z_{tt}(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}D_{\beta'}\overline{Z}_t d\beta' \\
& \quad - 2\partial_{\alpha'}[Z_t, \mathbb{H}]\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\} - 2Z_{t,\alpha'}(I - \mathbb{H})\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\}.
\end{aligned}
\tag{460}
$$

We control all of this except for $-2\partial_{\alpha'}[Z_t, \mathbb{H}]\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\} - 2Z_{t,\alpha'}(I - \mathbb{H})\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\}$ using Hölder and (235):

$$
\begin{aligned}
\left\|(460) - \left(-2\partial_{\alpha'}[Z_t, \mathbb{H}]\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\} - 2Z_{t,\alpha'}(I - \mathbb{H})\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\}\right)\right\|_{L^2} \\
\lesssim \|Z_{t,\alpha'}\|_{L^2}\left(\left\|D_{\alpha'}\overline{Z}_{tt}\right\|_{L^\infty} + \left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty}^2\right) + \|Z_{tt,\alpha'}\|_{L^2}\left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty}. \quad (461)
\end{aligned}
$$

Now we must control the error term, $\left\|Z_{t,\alpha'}(I - \mathbb{H})\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\}\right\|_{L^2}$. By Hölder, we control

$$
\left\|Z_{t,\alpha'}(I - \mathbb{H})\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\}\right\|_{L^2} \leqslant \|Z_{t,\alpha'}\|_{L^2}\left\|(I - \mathbb{H})\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\}\right\|_{L^\infty}.
\tag{462}
$$

By (139), we write the last factor as a commutator and control it by (244):

$$
\begin{aligned}
\left\|(I - \mathbb{H})\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\}\right\|_{L^\infty} &= \left\|[Z_t, \mathbb{H}]D_{\alpha'}^2\overline{Z}_t\right\|_{L^\infty} \\
&\lesssim \|Z_{t,\alpha'}\|_{L^2}\left\|D_{\alpha'}^2\overline{Z}_t\right\|_{L^2}.
\end{aligned}
\tag{463}
$$

We conclude that

$$
\left\|Z_{t,\alpha'}(I - \mathbb{H})(D_{\alpha'}^2\overline{Z}_t)Z_t\right\|_{L^2} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2\left\|D_{\alpha'}^2\overline{Z}_t\right\|_{L^2}.
\tag{464}
$$

Thus we have reduced the problem to controlling

$$
\partial_{\alpha'}[Z_t^2, \mathbb{H}]D_{\alpha'}^2\overline{Z}_t - 2\partial_{\alpha'}[Z_t, \mathbb{H}]\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\},
\tag{465}
$$

where the first of these is the first term in (452) and the second of these is all we have left

from the second and third terms in (452). Observe the following identity:

$$-2[f, \mathbb{H}]fg + [f^2, \mathbb{H}]g$$
$$= \frac{1}{2i} \int (-2(f(\alpha') - f(\beta'))f(\beta') + f^2(\alpha') - f^2(\beta')) \cot(\frac{\pi}{2}(\alpha' - \beta'))g(\beta')d\beta'$$
$$= \frac{1}{2i} \int (f(\alpha') - f(\beta'))^2 \cot(\frac{\pi}{2}(\alpha' - \beta'))g(\beta')d\beta'$$
$$= [f, [f, \mathbb{H}]]g.$$

$$(466)$$

Letting $f = Z_t, g = D_\alpha^2 \overline{Z}_t$, we have

$$\partial_{\alpha'}[Z_t^2, \mathbb{H}]D_{\alpha'}^2\overline{Z}_t - 2\partial_{\alpha'}[Z_t, \mathbb{H}]\left\{(D_{\alpha'}^2\overline{Z}_t)Z_t\right\}$$
$$= \partial_{\alpha'}\frac{1}{2i} \int (Z_t(\alpha') - Z_t(\beta'))^2 \cot(\frac{\pi}{2}(\alpha' - \beta'))D_{\beta'}^2\overline{Z}_t d\beta' \qquad (467)$$
$$= \partial_{\alpha'}[Z_t, [Z_t, \mathbb{H}]]D_{\alpha'}^2\overline{Z}_t.$$

By (245), we have that

$$\left\|\partial_{\alpha'}[Z_t, [Z_t, \mathbb{H}]]D_{\alpha'}^2\overline{Z}_t\right\|_{L^2} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \left\|D_{\alpha'}^2\overline{Z}_t\right\|_{L^2}. \qquad (468)$$

We now bring together our various estimates from this subsection. By (461), (464), and (468),

$$\left\|(I - \mathbb{H})\partial_{\alpha'}\left(\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'}\right)\right\|_{L^2} \lesssim (461) + (464) + (468)$$
$$\lesssim \|Z_{t,\alpha'}\|_{L^2}(\left\|D_{\alpha'}\overline{Z}_{tt}\right\|_{L^\infty} + \left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty}^2) \qquad (469)$$
$$+ \|Z_{tt,\alpha'}\|_{L^2}\left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty} + \|Z_{t,\alpha'}\|_{L^2}^2\left\|D_{\alpha'}^2\overline{Z}_t\right\|_{L^2}.$$

9.2.2   Combining the Estimates

We now combine our various estimates.

We begin by separately noting the estimate

$$\left\|(\mathfrak{a} \circ h^{-1})\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}\right\|_{L^2} = (446)$$
$$\leqslant \left\|\frac{1}{z_\alpha}\right\|_{L^\infty}((469) + (447)). \qquad (470)$$

From this, using (439), we have

$$\left(\int \frac{1}{\mathfrak{a}} \left|\mathfrak{a}\overline{z}_\alpha D_\alpha \left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right)\right|^2 d\alpha\right)^{1/2} \lesssim \|z_\alpha\|_{L^\infty} \left\|(\mathfrak{a}\circ h^{-1})\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}\right\|_{L^2}$$

$$\lesssim \|z_\alpha\|_{L^\infty} \left\|\frac{1}{z_\alpha}\right\|_{L^\infty} \left\{ \|Z_{t,\alpha'}\|_{L^2}\left(\left\|D_{\alpha'}\overline{Z}_{tt}\right\|_{L^\infty} + \left\|D_{\alpha'}\overline{Z}_t\right\|^2_{L^\infty}\right) \right.$$

$$+ \|Z_{tt,\alpha'}\|_{L^2}\left(\left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty} + \left\|\frac{\mathcal{A}_t}{\mathcal{A}}\right\|_{L^\infty}\right)$$

$$\left. + \|Z_{t,\alpha'}\|^2_{L^2}\left\|D^2_{\alpha'}\overline{Z}_t\right\|_{L^2} \right\},$$

(471)

which we use both here for E_b but also in our control of E_a.

Therefore,

$$\left(\int \frac{1}{\mathfrak{a}}|D_\alpha(-i\mathfrak{a}_t\overline{z}_\alpha)|^2 d\alpha\right)^{1/2} \leqslant (437)$$

$$\leqslant \underbrace{\left(\int \frac{1}{\mathfrak{a}}\left|\mathfrak{a}\overline{z}_\alpha D_\alpha\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right)\right|^2 d\alpha\right)^{1/2}}_{\lesssim(471)}$$

$$+ \left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty}\underbrace{\left(\int |D_\alpha\overline{z}_{tt}|^2 \frac{d\alpha}{\mathfrak{a}}\right)^{1/2}}_{=(302)\approx\|\overline{Z}_{tt,\alpha'}\|_{L^2}}.$$

(472)

By (292), combining estimates in §6 with (435) and (472), we conclude that $\frac{d}{dt}E_b$ is bounded by a universal polynomial of E, $\|z_\alpha\|_{L^\infty}$ and $\|1/z_\alpha\|_{L^\infty}$.

Chapter 10

Controlling G_θ of E_a

From (279), we must control

$$\left(\int \left| D_\alpha^2(-i\mathfrak{a}_t \bar{z}_\alpha) + [\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha^2]\bar{z}_t \right|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \right)^{1/2}. \tag{473}$$

We control the commutator in §10.1. We control the first, main part in §10.2.

10.1   Controlling the Commutator for E_a

We must control

$$\left\| [\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha^2]\bar{z}_t \right\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)}. \tag{474}$$

We use the commutator (262) and Hölder to control this by

$$\|D_\alpha z_{tt}\|_{L^\infty} \left\| D_\alpha^2 \bar{z}_t \right\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)} + \|D_\alpha z_t\|_{L^\infty}^2 \left\| D_\alpha^2 \bar{z}_t \right\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)}$$

$$+ \|D_\alpha z_t\|_{L^\infty} \left\| D_\alpha \partial_t D_\alpha \bar{z}_t \right\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)} + \left\| D_\alpha^2 z_{tt} \right\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)} \|D_\alpha \bar{z}_t\|_{L^\infty}$$

$$+ \|D_\alpha z_t\|_{L^\infty}^2 \left\| D_\alpha^2 z_t \right\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)} + \left\| D_\alpha^2 z_t \right\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)} \|D_\alpha \bar{z}_{tt}\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty} \left\| D_\alpha^2 \bar{z}_{tt} \right\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)}. \tag{475}$$

We have controlled all of these quantities in §6.1.[49]   Observe that we will have quite a complicated polynomial dependence on our original, due to the presence of factors like

---

[49]The only unfamiliar quantity, $\|D_\alpha \partial_t D_\alpha \bar{z}_t\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)}$, we controlled at (311).

$\|D_\alpha^2 z_{tt}\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)}$, which differ from our energy by the location of the ∂_t and/or by the lack of conjugation.

10.2 Controlling the Main Part for E_a

We need to control

$$\left(\int \left| D_\alpha^2 \left(\mathfrak{a}_t \overline{z}_\alpha \right) \right|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \right)^{1/2}. \tag{476}$$

Because $A_1 \geqslant 1$ (182), we will remove the $A_1 \circ h$ factor in the denominator and focus on estimating

$$\left(\int \left| D_\alpha^2 \left(\mathfrak{a}_t \overline{z}_\alpha \right) \right|^2 h_\alpha d\alpha \right)^{1/2}. \tag{477}$$

Our plan will be to rewrite this RHS into a series of commutators by inserting an $(I - \mathbb{H})$ in front, as in §3.7.2. Getting to this point will require a long and somewhat convoluted series of estimates; we finally reach our initial goal of controlling $(I - \mathbb{H})$ of the RHS (in Riemannian coordinates) at equation (503), below.

We begin by writing $\mathfrak{a}_t \overline{z}_\alpha = \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \mathfrak{a} \overline{z}_\alpha$. By the product rule

$$D_\alpha^2 \left(\mathfrak{a}_t \overline{z}_\alpha \right) = \left(D_\alpha^2 \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \right) \mathfrak{a} \overline{z}_\alpha + 2 \left(D_\alpha \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \right) D_\alpha(\mathfrak{a} \overline{z}_\alpha) + \frac{\mathfrak{a}_t}{\mathfrak{a}} D_\alpha^2(\mathfrak{a} \overline{z}_\alpha). \tag{478}$$

We can handle the second and third terms directly, using $\mathfrak{a} \overline{z}_\alpha = i(\overline{z}_{tt} - i)$ (27). Indeed, for the second term, we can reduce it to a term we estimated in §9.2:

$$\left\| \left(D_\alpha \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \right) D_\alpha(\mathfrak{a} \overline{z}_\alpha) \right\|_{L^2(h_\alpha)} \leqslant \left(\int \left| D_\alpha \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \right|^2 h_\alpha d\alpha \right)^{1/2} \|D_\alpha \overline{z}_{tt}\|_{L^\infty}$$

$$\leqslant \left\| \frac{1}{A_1} \right\|_{L^\infty}^{1/2} \left(\int \left| D_\alpha \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \right|^2 h_\alpha (A_1 \circ h) d\alpha \right)^{1/2} \|D_\alpha \overline{z}_{tt}\|_{L^\infty}$$

$$\leqslant \underbrace{\left(\int \left| \mathfrak{a} \overline{z}_\alpha D_\alpha \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \right|^2 \frac{d\alpha}{\mathfrak{a}} \right)^{1/2}}_{\lesssim (471) \text{ from §9.2.2}} \|D_\alpha \overline{z}_{tt}\|_{L^\infty}, \tag{479}$$

where we've used $A_1 \geqslant 1$ (182) to remove the $\|1/A_1\|_{L^\infty}^{1/2}$ factor. We control the third term

$$\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} D_\alpha^2(\mathfrak{a} \overline{z}_\alpha) \right\|_{L^2(h_\alpha)} \leqslant \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} \|D_\alpha^2 \overline{z}_{tt}\|_{L^2(h_\alpha)}. \tag{480}$$

We conclude from (478), (479) and (480) that

$$
\begin{aligned}
\left\| D_\alpha^2 \left(\mathfrak{a}_t \overline{z}_\alpha \right) \right\|_{L^2(h_\alpha)} &\leqslant \left\| \left(D_\alpha^2 \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \right) \mathfrak{a} \overline{z}_\alpha \right\|_{L^2(h_\alpha)} + 2 \left\| \left(D_\alpha \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \right) D_\alpha (\mathfrak{a} \overline{z}_\alpha) \right\|_{L^2(h_\alpha)} \\
&\quad + \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} D_\alpha^2 (\mathfrak{a} \overline{z}_\alpha) \right\|_{L^2(h_\alpha)} \\
&\lesssim \left\| \left(D_\alpha^2 \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \right) \mathfrak{a} \overline{z}_\alpha \right\|_{L^2(h_\alpha)} \\
&\quad + \underbrace{\left(\int \left| D_\alpha \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \right|^2 \mathfrak{a} \, |z_\alpha|^2 \, d\alpha \right)^{1/2}}_{\lesssim (471) \text{ from } \S 9.2.2} \left\| D_\alpha \overline{z}_{tt} \right\|_{L^\infty} + \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} \left\| D_\alpha^2 \overline{z}_{tt} \right\|_{L^2(h_\alpha)}.
\end{aligned}
$$

(481)

It therefore suffices to focus on the first term on the RHS of (481).

10.2.1 Controlling $\left\| \left(D_\alpha^2 \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \right) \mathfrak{a} \overline{z}_\alpha \right\|_{L^2(h_\alpha)}$

We now rearrange this term so that we can apply $(I - \mathbb{H})$ in a way so that we will be able to invert the operator by taking real parts. Note that $\frac{\mathfrak{a}_t}{\mathfrak{a}}$ is purely real. Unfortunately, our derivative $D_\alpha = \frac{1}{z_\alpha} \partial_a$ is not purely real. To get around this, we factor the derivative into a real-weighted derivative and a complex modulus-one weight. Recall our notation

$$
|D_\alpha| = \frac{1}{|z_\alpha|} \partial_\alpha, |D_{\alpha'}| = \frac{1}{|Z_{,\alpha'}|} \partial_{\alpha'}.
$$

(482)

Since $D_\alpha = \left(\frac{|z_\alpha|}{z_\alpha} \right) |D_\alpha|$, we rewrite

$$
D_\alpha^2 = \left(\frac{|z_\alpha|}{z_\alpha} \right)^2 |D_\alpha|^2 + \left(\frac{|z_\alpha|}{z_\alpha} \right) \left(|D_\alpha| \left(\frac{|z_\alpha|}{z_\alpha} \right) \right) |D_\alpha|.
$$

(483)

Therefore,

$$
\mathfrak{a} \overline{z}_\alpha D_\alpha^2 \frac{\mathfrak{a}_t}{\mathfrak{a}} = \mathfrak{a} \overline{z}_\alpha \left(\frac{|z_\alpha|}{z_\alpha} \right)^2 |D_\alpha|^2 \frac{\mathfrak{a}_t}{\mathfrak{a}} + \mathfrak{a} \overline{z}_\alpha \left(\frac{|z_\alpha|}{z_\alpha} \right) \left(|D_\alpha| \left(\frac{|z_\alpha|}{z_\alpha} \right) \right) |D_\alpha| \frac{\mathfrak{a}_t}{\mathfrak{a}}.
$$

(484)

We now switch to Riemannian coordinates, recalling that $\mathfrak{a}\overline{z}_\alpha = i(\overline{z}_{tt} - i)$. We get

$$
\begin{aligned}
i(\overline{Z}_{tt} - i)D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} &= i(\overline{Z}_{tt} - i)\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)^2 |D_{\alpha'}|^2 \frac{\mathcal{A}_t}{\mathcal{A}} \\
&+ i(\overline{Z}_{tt} - i)\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)\left(|D_{\alpha'}|\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)\right)|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}}.
\end{aligned}
\tag{485}
$$

It will turn out that we'll want to replace $|D_{\alpha'}|^2$ in the first term of the RHS of (485) with $\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|}|D_{\alpha'}|\right)$, since this will give us the proper commutator estimate,[50] and it further turns out that we want to make the switch now rather than later, when it would be in commutator form. Doing this, we get

$$
i(\overline{Z}_{tt} - i)D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} = i(\overline{Z}_{tt} - i)\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)^2 \partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|}|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}}\right)
$$

$$
\underbrace{-i(\overline{Z}_{tt} - i)\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)^2\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}} + i(\overline{Z}_{tt} - i)\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)\left(|D_{\alpha'}|\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)\right)|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}}}_{e},
$$

$$
\tag{486}
$$

where we will use

$$
\begin{aligned}
e :=\; &-i(\overline{Z}_{tt} - i)\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)^2\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}} \\
&+ i(\overline{Z}_{tt} - i)\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)\left(|D_{\alpha'}|\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)\right)|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}}
\end{aligned}
\tag{487}
$$

as an abbreviation to save space and de-emphasize the less central error terms, which we will control directly, below at (496). We now apply $(I - \mathbb{H})$ to both sides of (486):

$$
(I - \mathbb{H})\left\{i(\overline{Z}_{tt} - i)D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}}\right\} = (I - \mathbb{H})\left\{i(\overline{Z}_{tt} - i)\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)^2 \partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|}|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}}\right)\right\}
\tag{488}
$$

$$
+ (I - \mathbb{H})e.
$$

Observe that what's inside the $(I - \mathbb{H})$ in the first term on the RHS is purely real, except for the controllable factor $-i(\overline{Z}_{tt} - i)\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)^2$. We commute that part outside the $(I - \mathbb{H})$.

---

[50]See footnote 51 below for a further explanation.

We get

$$(I - \mathbb{H}) \left\{ i(\overline{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\} = i(\overline{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2 (I - \mathbb{H}) \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right)$$

$$+ \left[i(\overline{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2 , \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \qquad (489)$$

$$+ (I - \mathbb{H}) e.$$

By multiplying by a modulus-one constant and taking real parts, we can invert the $(I - \mathbb{H})$ on the first term of the RHS of (489), and therefore we can control

$$\left| i(\overline{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2 \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right|$$

$$\leqslant \left| (I - \mathbb{H}) \left\{ i(\overline{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\} \right|$$

$$+ \left| \left[i(\overline{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2 , \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right| \qquad (490)$$

$$+ |(I - \mathbb{H}) e| .$$

Now we may begin our estimates. Note that $L^2(h_\alpha)$ corresponds under the Riemann mapping to $L^2(d\alpha')$.

Recall that what we needed to control was (486). We can estimate this by

$$\left\| -i(\overline{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \lesssim \left\| i(\overline{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2 \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right\|_{L^2} + \|e\|_{L^2}$$

$$\lesssim \|(490)\|_{L^2} + \|e\|_{L^2}$$

$$\leqslant \left\| (I - \mathbb{H}) \left\{ i(\overline{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\} \right\|_{L^2}$$

$$+ \left\| \left[i(\overline{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right\|_{L^2}$$

$$+ \|(I - \mathbb{H})e\|_{L^2} + \|e\|_{L^2}$$

$$\lesssim \left\| (I - \mathbb{H}) \left\{ i(\overline{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\} \right\|_{L^2}$$

$$+ \left\| \left[i(\overline{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right\|_{L^2} + \|e\|_{L^2}.$$

(491)

Thus, it suffices to focus on these three terms. The first term will be the main term that will eventually give us the commutator structure that we want. We first control the remaining terms.

First we check the error term, e (487). We control the first term

$$\left\| -i(\overline{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2 \left(\partial_{\alpha'} \frac{1}{|Z_{,\alpha'}|} \right) |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \leqslant \left\| (\overline{Z}_{tt} - i) \partial_{\alpha'} \frac{1}{|Z_{,\alpha'}|} \right\|_{L^\infty} \left\| D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2}$$

(492)

and the second term

$$\left\| i(\overline{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right) \left(|D_{\alpha'}| \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right) \right) |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2}$$

$$\leqslant \left\| i(\overline{Z}_{tt} - i) \left(|D_{\alpha'}| \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right) \right) \right\|_{L^\infty} \left\| |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2}. \quad (493)$$

We control the $\left\|D_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}\right\|_{L^2}$ factors by using an estimate we made for E_b:

$$
\begin{aligned}
\left\|D_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}\right\|_{L^2} &= \left\||D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}}\right\|_{L^2} = \left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}\right\|_{L^2} \\
&= \left(\int \left|D_\alpha\frac{\mathfrak{a}_t}{\mathfrak{a}}\right|^2 h_\alpha d\alpha\right)^{1/2} \\
&= \left(\int \left|D_\alpha\frac{\mathfrak{a}_t}{\mathfrak{a}}\right|^2 \frac{\mathfrak{a}\,|z_\alpha|^2}{A_1\circ h}d\alpha\right)^{1/2} \\
&\leqslant \left(\int \left|\mathfrak{a}\bar{z}_\alpha D_\alpha\frac{\mathfrak{a}_t}{\mathfrak{a}}\right|^2 \frac{d\alpha}{\mathfrak{a}}\right)^{1/2} \\
&\lesssim (471),
\end{aligned}
\tag{494}
$$

where we've removed the $1/A_1$ factor by $A_1 \geqslant 1$ (182). Because $|\partial_{\alpha'}|f|| \leqslant |\partial_{\alpha'}f|$ (211), we have estimated the first factor in (492) directly at (367). For the first factor in (493), we use $\left|\partial_{\alpha'}\frac{f}{|f|}\right| \leqslant \left|\frac{f'}{|f|}\right|$ (212), and again

$$
\left\|i(\overline{Z}_{tt}-i)\left(|D_{\alpha'}|\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)\right)\right\|_{L^\infty} \leqslant \left\|(\overline{Z}_{tt}-i)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \lesssim (367).
\tag{495}
$$

We conclude from (492), (493), (494), and (495) that

$$
\|e\|_{L^2} \leqslant \left\|(\overline{Z}_{tt}-i)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \underbrace{\left\|D_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}\right\|_{L^2}}_{(471)},
\tag{496}
$$

where we have estimated $\left\|(Z_{tt}+i)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}$ as part of the estimate for $\left\|D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}$ at (367).

Now we estimate the second term on the RHS of (491). We use $L^2 \times L^\infty$ commutator estimate (233):

$$
\begin{aligned}
&\left\|\left[-i(\overline{Z}_{tt}-i)\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)^2,\mathbb{H}\right]\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|}|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}}\right)\right\|_{L^2} \\
&\qquad \lesssim \left\|\partial_{\alpha'}\left(-i(\overline{Z}_{tt}-i)\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)^2\right)\right\|_{L^2}\left\|\left(\frac{1}{|Z_{,\alpha'}|}|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}}\right)\right\|_{L^\infty}.
\end{aligned}
\tag{497}
$$

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Observe that because either (i) the water is flat, in which case $\left.\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right|_{\partial} = 0$, or else (ii) the angle is non-trivial and so $\overline{Z}_{tt} - i = 0$ at the corners (see (115)) this commutator does satisfy the boundary condition we need for (233).[51] To control the first factor, we replace $\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}$ with $\frac{\overline{Z}_{tt}-i}{|\overline{Z}_{tt}-i|}$ (36) and estimate:

$$\left\| \partial_{\alpha'}\left(-i(\overline{Z}_{tt}-i)\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)^2 \right) \right\|_{L^2} \leqslant \|Z_{tt,\alpha'}\|_{L^2} + \left\| (\overline{Z}_{tt}-i)\partial_{\alpha'}\frac{\overline{Z}_{tt}-i}{|\overline{Z}_{tt}-i|} \right\|_{L^2} \qquad (499)$$
$$\lesssim \|Z_{tt,\alpha'}\|_{L^2}.$$

What remains to be done is to show that we can control $\left\| \frac{1}{|Z_{,\alpha'}|}|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty}$. First we rewrite this by (183) as

$$\begin{aligned}
\left\| \frac{1}{|Z_{,\alpha'}|}|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty} &= \left\| \frac{\overline{Z}_{tt}-i}{A_1}|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty} \\
&\leqslant \left\| (\overline{Z}_{tt}-i)|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty} \\
&= \left\| (\mathfrak{a}\overline{z}_\alpha)D_\alpha\frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} \\
&\leqslant \left\| (D_\alpha(\mathfrak{a}\overline{z}_\alpha))\frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} + \|D_\alpha(\mathfrak{a}_t\overline{z}_\alpha)\|_{L^\infty} \\
&\leqslant \|D_\alpha\overline{z}_{tt}\|_{L^\infty}\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} + \|D_\alpha(\mathfrak{a}_t\overline{z}_\alpha)\|_{L^\infty},
\end{aligned} \qquad (500)$$

where we've used $A_1 \geqslant 1$ (182) in the second line and $-i\mathfrak{a}\overline{z}_\alpha = \overline{z}_{tt}-i$ in the third and last lines. Because we can control the first term on the RHS of the last line, it suffices to control the last term. We defer doing so until the end, in §10.2.9. For now, we combine (497), (499)

---

[51] Note that this estimate for $[f,\mathbb{H}]\partial_{\alpha'}g$, unlike the $L^\infty \times L^2$ estimate, does *not* require that $g|_\partial = 0$. This explains why we moved from $|D_{\alpha'}|^2 \frac{\mathcal{A}_t}{\mathcal{A}}$ to $\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|}|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}} \right)$; had we tried to estimate

$$\left\| \left[\frac{1}{|Z_{,\alpha'}|}\left(-i(\overline{Z}_{tt}-i)\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)^2 \right), \mathbb{H} \right]\partial_{\alpha'}|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \qquad (498)$$

by $L^\infty \times L^2$ commutator estimate (226), we would have been able to control each of the terms (they would be similar to the estimates for e in (496) above), but we don't know that $|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}}\big|_\partial$ is zero.

and (500) to estimate

$$\left\| \left[-i(\overline{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right\|_{L^2}$$

$$\lesssim \|Z_{tt,\alpha'}\|_{L^2} \left(\|D_\alpha \overline{z}_{tt}\|_{L^\infty} \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} + \|D_\alpha(\mathfrak{a}_t \overline{z}_\alpha)\|_{L^\infty} \right). \quad (501)$$

We're left with the first, main term of the end of (491). Observe that by (478) (changing between Riemannian and Lagrangian notation), this is

$$\left\| (I - \mathbb{H}) \left\{ i(\overline{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\} \right\|_{L^2} = \left\| (I - \mathbb{H}) \left\{ \left(\left(D_\alpha^2 \frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \mathfrak{a} \overline{z}_\alpha \right) \circ h^{-1} \right\} \right\|_{L^2}$$

$$\leqslant \left\| (I - \mathbb{H}) \left(D_\alpha^2(\mathfrak{a}_t \overline{z}_\alpha) \right) \circ h^{-1} \right\|_{L^2}$$

$$+ \|(I - \mathbb{H})\|_{L^2 \to L^2} \quad (502)$$

$$\cdot \left\| \left\{ 2 \left(D_\alpha \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \right) D_\alpha(\mathfrak{a} \overline{z}_\alpha) + \frac{\mathfrak{a}_t}{\mathfrak{a}} D_\alpha^2(\mathfrak{a} \overline{z}_\alpha) \right\} \circ h^{-1} \right\|_{L^2}$$

$$\lesssim \left\| (I - \mathbb{H}) \left(D_\alpha^2(\mathfrak{a}_t \overline{z}_\alpha) \right) \circ h^{-1} \right\|_{L^2} + (479) + (480).$$

We have therefore reduced things (except for the one term we're deferring to the end) to controlling

$$\left\| (I - \mathbb{H}) \left\{ \left(D_\alpha^2(\mathfrak{a}_t \overline{z}_\alpha) \right) \circ h^{-1} \right\} \right\|_{L^2} = \left\| (I - \mathbb{H}) D_{\alpha'}^2 (\overline{Z}_{ttt} + i\mathcal{A} \overline{Z}_{t,\alpha'}) \right\|_{L^2}, \quad (503)$$

that is, we have finally reduced things to controlling $(I - \mathbb{H})$ of the RHS of the equation, which will allow us to use desirable commutators.

10.2.2 Controlling $\left\| (I - \mathbb{H}) \left\{ \left(D_\alpha^2(\mathfrak{a}_t \overline{z}_\alpha) \right) \circ h^{-1} \right\} \right\|_{L^2}$

We now expand out the RHS of (503), $(I - \mathbb{H}) D_{\alpha'}^2 (\overline{Z}_{ttt} + i\mathcal{A} \overline{Z}_{t,\alpha'})$, as we did in the Riemann mapping derivation in §3.7.2. From (194), we have

$$\overline{Z}_{ttt} = (D_{\alpha'}^2 \overline{Z}_t) Z_t^2 + 2 Z_t D_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'} \overline{Z}_t) Z_t) + (D_{\alpha'} \overline{Z}_t) Z_{tt} + F_{tt} \circ Z, \quad (504)$$

where $F(z(\alpha, t), t) := \overline{z}_t(\alpha, t)$. By $Z_{tt} + i = i\mathcal{A} Z_{,\alpha'}$,

$$i\mathcal{A} \overline{Z}_{t,\alpha'} = (Z_{tt} + i) D_{\alpha'} \overline{Z}_t. \quad (505)$$

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Putting these together and applying $(I - \mathbb{H})D^2_{\alpha'}$, we have

$$(I - \mathbb{H})D^2_{\alpha'}(\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'})$$
$$= (I - \mathbb{H})D^2_{\alpha'}\left[(D^2_{\alpha'}\overline{Z}_t)Z^2_t + 2Z_tD_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) + (D_{\alpha'}\overline{Z}_t)(2Z_{tt} + i) + F_{tt} \circ Z\right].$$
$$(506)$$

Observe that each of these terms has holomorphic factors, so under $(I - \mathbb{H})$ we will get commutators.

First we address the last two terms: $(I - \mathbb{H})D^2_{\alpha'}\left((D_{\alpha'}\overline{Z}_t)(2Z_{tt} + i) + F_{tt} \circ Z\right)$. We may rewrite

$$\begin{aligned}(I - \mathbb{H})D^2_{\alpha'}\left((D_{\alpha'}\overline{Z}_t)(2Z_{tt} + i) + F_{tt} \circ Z\right) &= (I - \mathbb{H})D^2_{\alpha'}\left\{(D_{\alpha'}\overline{Z}_t)(2Z_{tt} + i)\right\} \\ &= (I - \mathbb{H})D^2_{\alpha'}\left\{(D_{\alpha'}\overline{Z}_t)2(Z_{tt} + i)\right\} \\ &= 2(I - \mathbb{H})\left[(D^3_{\alpha'}\overline{Z}_t)(Z_{tt} + i) + 2(D^2_{\alpha'}\overline{Z}_t)(D_{\alpha'}Z_{tt}) + (D_{\alpha'}\overline{Z}_t)(D^2_{\alpha'}Z_{tt})\right],\end{aligned}$$
$$(507)$$

where we have relied on (152) and (140); note how we *added* $i(I - \mathbb{H})D^3_{\alpha'}\overline{Z}_t = 0$ in the second line, to go from $2Z_{tt} + i$ to $2(Z_{tt} + i)$.

By Hölder, we can control the second and third terms of the RHS of (507):

$$\begin{aligned}\left\|(I - \mathbb{H})\left[2(D^2_{\alpha'}\overline{Z}_t)(D_{\alpha'}Z_{tt}) + (D_{\alpha'}\overline{Z}_t)(D^2_{\alpha'}Z_{tt})\right]\right\|_{L^2} \\ \lesssim \left\|D^2_{\alpha'}\overline{Z}_t\right\|_{L^2}\left\|D_{\alpha'}Z_{tt}\right\|_{L^\infty} + \left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty}\left\|D^2_{\alpha'}Z_{tt}\right\|_{L^2}.\end{aligned}$$
$$(508)$$

To control the first term on the RHS of (507), we use (143) to write

$$\begin{aligned}\left\|(I - \mathbb{H})(D^3_{\alpha'}\overline{Z}_t)(Z_{tt} + i)\right\|_{L^2} &= \left\|\left[\frac{Z_{tt} + i}{Z_{,\alpha'}}, \mathbb{H}\right]\partial_{\alpha'}D^2_{\alpha'}\overline{Z}_t\right\|_{L^2} \\ &\lesssim \left\|\partial_{\alpha'}\frac{Z_{tt} + i}{Z_{,\alpha'}}\right\|_{L^\infty}\left\|D^2_{\alpha'}\overline{Z}_t\right\|_{L^2} \\ &\lesssim \left(\|D_{\alpha'}Z_{tt}\|_{L^\infty} + \left\|(Z_{tt} + i)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)\left\|D^2_{\alpha'}\overline{Z}_t\right\|_{L^2},\end{aligned}$$
$$(509)$$

where we have used the $L^\infty \times L^2$ commutator estimate (226).

We conclude from (507), (508) and (509) that we can control

$$
\left\| (I - \mathbb{H}) D_{\alpha'}^2 \left((D_{\alpha'} \overline{Z}_t)(2Z_{tt} + i) + F_{tt} \circ Z \right) \right\|_{L^2} \lesssim (508) + (509)
$$
$$
\lesssim \left\| D_{\alpha'}^2 \overline{Z}_t \right\|_{L^2} \left(\left\| D_{\alpha'} Z_{tt} \right\|_{L^\infty} + \left\| (Z_{tt} + i) \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \right) + \left\| D_{\alpha'} \overline{Z}_t \right\|_{L^\infty} \left\| D_{\alpha'}^2 Z_{tt} \right\|_{L^2}. \tag{510}
$$

We can now focus on what remains from (506),

$$
(I - \mathbb{H}) D_{\alpha'}^2 \left\{ (D_{\alpha'}^2 \overline{Z}_t) Z_t^2 + 2 Z_t D_{\alpha'} (\overline{Z}_{tt} - (D_{\alpha'} \overline{Z}_t) Z_t) \right\}
$$
$$
\begin{aligned}
&= (I - \mathbb{H}) \left((D_{\alpha'}^2 (Z_t^2))(D_{\alpha'}^2 \overline{Z}_t) + 2(D_{\alpha'}^2 Z_t) D_{\alpha'} (\overline{Z}_{tt} - (D_{\alpha'} \overline{Z}_t) Z_t) \right) \\
&\quad + (I - \mathbb{H}) \left(2(D_{\alpha'}(Z_t^2))(D_{\alpha'}^3 \overline{Z}_t) + 4(D_{\alpha'} Z_t) D_{\alpha'}^2 (\overline{Z}_{tt} - (D_{\alpha'} \overline{Z}_t) Z_t) \right) \\
&\quad + (I - \mathbb{H}) \left(Z_t^2 (D_{\alpha'}^4 \overline{Z}_t) + 2 Z_t D_{\alpha'}^3 (\overline{Z}_{tt} - (D_{\alpha'} \overline{Z}_t) Z_t) \right) \\
&= i + ii + iii.
\end{aligned} \tag{511}
$$

Terms i and ii are easy to control. The details are uninteresting and distract from our main derivation, so we defer them to §10.2.7, below. We may therefore focus on iii.

10.2.3   Controlling iii from (511)

We now focus on controlling

$$
iii = (I - \mathbb{H}) \left\{ Z_t^2 (D_{\alpha'}^4 \overline{Z}_t) \right\} + (I - \mathbb{H}) \left\{ 2 Z_t D_{\alpha'}^3 (\overline{Z}_{tt} - (D_{\alpha'} \overline{Z}_t) Z_t) \right\} = I + II. \tag{512}
$$

This quantity will be subdivided quite a few times, with various terms being controlled directly and certain terms canceling out with others, as we estimate iii in this subsection, §10.2.4 and §10.2.5. To help the reader confirm that we have covered every term, we give an exhaustive list of the various terms at (567) in §10.2.6, where we combine the various estimates.

These terms appear to require too much smoothness to close the estimate: we have up to four derivatives here, when our energy controls only two derivatives. The key observation is the following identity, obtained by integrating by parts in each of the commutators on the

LHS of the following,

$$
\begin{aligned}
[f^2, \mathbb{H}]\partial_{\alpha'} g - 2[f, \mathbb{H}]\partial_{\alpha'}(fg) - \partial &= \frac{1}{2i}\int (f^2(\alpha') - f^2(\beta'))\cot(\frac{\pi}{2}(\alpha' - \beta'))\partial_{\beta'} g(\beta')d\beta' \\
&\quad - \frac{1}{2i}\int 2(f(\alpha') - f(\beta'))\cot(\frac{\pi}{2}(\alpha' - \beta'))\partial_{\beta'}(f(\beta')g(\beta'))\,d\beta' - \partial \\
&= \frac{1}{2i}\int 2f(\beta)f'(\beta')\cot(\frac{\pi}{2}(\alpha' - \beta'))g(\beta')d\beta' - \frac{1}{2i}\int \frac{\pi}{2}\frac{f^2(\alpha') - f^2(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}g(\beta')d\beta' \\
&\quad - \frac{1}{2i}\int 2f'(\beta')\cot(\frac{\pi}{2}(\alpha' - \beta'))f(\beta')g(\beta')d\beta' \\
&\quad + \frac{1}{2i}\int \frac{\pi}{2}2\frac{f(\alpha') - f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}f(\beta')g(\beta')d\beta' \\
&= \frac{1}{2i}\int \frac{\pi}{2}\frac{-f^2(\alpha') + f^2(\beta') + 2(f(\alpha') - f(\beta'))f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}g(\beta')d\beta' \\
&= \frac{-1}{2i}\int \frac{\pi}{2}\frac{(f(\alpha') - f(\beta'))^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))}g(\beta')d\beta' \\
&=: -\frac{\pi}{2}[f, f; g],
\end{aligned}
$$

$$(513)$$

where ∂ is the boundary term from the two integrations by parts, which it is useful to keep on the LHS of the equation. Using the commutator estimate (247), we can control the RHS in L^2 by $\|f'\|_{L^\infty}^2 \|G\|_{L^2}$, where $G' = g$, if $G|_\partial = 0$ (so $\int g = 0$). Note this has the effect of smoothing two derivatives away from $\partial_{\alpha'} g$. Our goal will be to take the worst terms from (512) and put them in this form, using the fact that various of these factors are holomorphic. The natural choice would be to put $f = \frac{Z_t}{Z_{,\alpha'}}$ and $g = \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t$. Unfortunately, this poses a problem, since $\partial_{\alpha'}\frac{Z_t}{Z_{,\alpha'}}$ isn't controlled in L^∞. We can, however, control $\left\|\partial_{\alpha'}(I - \mathbb{H})\frac{Z_t}{Z_{,\alpha'}}\right\|_{L^\infty}$; we did this at (372). Therefore, we will try to attain an equation of the form (513) with $f = \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}$ (and still $g = \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t$), where $\mathbb{P}_A = \frac{(I - \mathbb{H})}{2}$ is the anti-holomorphic projection. To do this, we will have to do substantially more work, decomposing certain factors from (512) into their holomorphic and antiholomorphic projections.[52]

---

[52] An alternative approach is to observe that Z_t might be close to being as good of a weight as $Z_{tt} + i$ and therefore $\frac{1}{Z_{,\alpha'}}$, which would suggest that $\partial_{\alpha'}\frac{Z_t}{Z_{,\alpha'}}$ is close to $\partial_{\alpha'}\frac{1}{Z_{,\alpha'}^2}$, which *is* in L^∞. Suppose we replaced z_t with $z_t + c(t)$ for a $c(t)$ so that $c'(t) = i$ and $c(0) = -z_t(\pm 1, 0)$. Then $z_t + c(t)$ is zero at the corners at $t = 0$ and its time derivative is $z_{tt} + i$, which is zero at the corners for all time (assuming we're dealing with a non-trivial angle). Therefore $Z_t + c(t)$ is always zero. If we make the initial assumption that

$$
\left\|\frac{z_t + c(t)}{z_{tt} + i}\right\|_{L_\alpha^\infty}(0) < \infty, \tag{514}
$$

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Our goal, therefore, will be to isolate

$$\left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right)^2, \mathbb{H} \right] \partial_{\alpha'} (\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t) - 2 \left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right), \mathbb{H} \right] \partial_{\alpha'} \left(\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \left(\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t \right) \right) \quad (515)$$

from (512), and handle the remaining terms. Technically, in fact, we will want to isolate not this but the already *integrated-by-parts* version, since there's a problematic boundary term on integrating by parts. Luckily, this boundary term will cancel out with other problematic terms in our derivation. In other words, our goal will be to end up with $-\frac{\pi}{2} \left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha}} \right), \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right); \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t \right]$, which we will control via (247), where we rely on boundary condition (121) (and (113)):

$$\left\| \left[\mathbb{P}_A \frac{Z_t}{Z_{,\alpha}}, \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}; \partial_\beta D_{\beta'}^2 \overline{Z}_t \right] \right\|_{L^2} \leqslant \left\| \partial_{\alpha'} \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right\|_{L^\infty}^2 \left\| D_{\alpha'}^2 \overline{Z}_t \right\|_{L^2}, \quad (516)$$

where we have estimated the first factor at (372).

Thus we must expand I and II from (512). The basic approach is straightforward: we follow our nose to put things in the form (515). The only challenge will be that we have to take extra care about certain potentially singular holomorphic functions, which won't immediately disappear under $(I - \mathbb{H})$.[53] It turns out that these will be fine—everything cancels perfectly—but this imposes some extra bookkeeping, so the derivation below looks particularly complicated.

We begin by expanding I from (512):

$$I = (I - \mathbb{H}) \left\{ \frac{Z_t^2}{Z_{,\alpha'}^2} \partial_{\alpha'}^2 D_{\alpha'}^2 \overline{Z}_t \right\} + (I - \mathbb{H}) \left\{ Z_t^2 \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t \right\} = I_1 + I_2. \quad (517)$$

---

then we can derive a Gronwall inequality for $\frac{d}{dt} \frac{z_t + c(t)}{z_{tt} + i}$ in terms of our energy (the only issue is controlling $\left\| \frac{z_{ttt}}{z_{tt} + i} \right\|_{L^\infty}$, which by (37) can be reduced to controlling $\left\| \frac{a_t}{a} \right\|_{L^\infty}$), so we will continue to have the bound (514) for short periods of time. It's possible to finagle things to replace Z_t with $Z_t + c(t)$; it's not trivial—there's some accounting to make sure that the extraneous terms cancel or disappear—but it's less work than the derivation here of introducing $\mathbb{P}_A$. Then we're left with controlling $\left\| \partial_{\alpha'} \frac{Z_t + c(t)}{Z_{,\alpha'}} \right\|_{L^\infty}$, which we can do assuming (514). This technique can be used to simplify a few other derivations in this dissertation. In a regime where (514) held or was a benign assumption, this might be a useful trick, but it imposes restrictions on the initial data so we do not use it.

[53]More precisely, these functions are highly singular in the non-trivial angle regime. In general, these functions are not necessarily periodic (and their means are not necessarily zero), so there would be boundary terms remaining; in the non-trivial angle regime, these boundary terms would blow up.

We expand the identity operator into the sum of the projections $\mathbb{P}_A + \mathbb{P}_H$, allowing us to rewrite

$$
\begin{aligned}
I_1 &= (I - \mathbb{H}) \left\{ \left((\mathbb{P}_A + \mathbb{P}_H) \frac{Z_t}{Z_{,\alpha'}} \right)^2 \partial_{\alpha'}^2 D_{\alpha'}^2 \overline{Z}_t \right\} \\
&= (I - \mathbb{H}) \left\{ \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right)^2 \partial_{\alpha'}^2 D_{\alpha'}^2 \overline{Z}_t \right\} + (I - \mathbb{H}) \left\{ 2 \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'}^2 D_{\alpha'}^2 \overline{Z}_t \right\} \\
&\quad + (I - \mathbb{H}) \left\{ \left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \right)^2 \partial_{\alpha'}^2 D_{\alpha'}^2 \overline{Z}_t \right\} \\
&= I_{11} + I_{12} + I_{13}.
\end{aligned}
\tag{518}
$$

Sweeping the issues of too-singular holomorphic functions, mean conditions, etc., under the rug for a moment, we see that I_{11} would be one of the main terms that we want to end up with in (515) and I_{13} would disappear by holomorphicity, and we'd be left with I_{12}, which will cancel out with a term from the expansion of II (in (530)). From I, therefore, we would be left with controlling I_2, which we will handle at the end.

Now we handle those details that we had swept under the rug. To begin, we write I_{11} in commutator form. Note that $\partial_{\alpha'}^2 D_{\alpha'}^2 \overline{Z}_t$ is too singular to disappear under $(I - \mathbb{H})$, so we will have to be careful in writing things in commutator form. Luckily, the boundary remainder we will end up with to handle that singularity will cancel with the boundary terms we get when we integrate by parts for the main term, as was required in (515). We expand I_{11}:

$$
I_{11} = \left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right)^2, \mathbb{H} \right] \partial_{\alpha'}^2 D_\alpha^2 \overline{Z}_t + \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right)^2 (I - \mathbb{H}) \partial_{\alpha'}^2 D_{\alpha'}^2 \overline{Z}_t.
\tag{519}
$$

We now integrate by parts for the first term and commute a derivative outside of $(I - \mathbb{H})$ in

the second term:

$$I_{11} = -\frac{1}{2i} \int \left\{ \partial_{\beta'} \left(\left(\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right)^2 (\alpha') - \left(\mathbb{P}_A \frac{Z_t}{Z_{,\beta'}} \right)^2 (\beta') \right) \cot(\cdots) \right) \right\} \partial_{\beta'} D_{\beta'}^2 \overline{Z}_t d\beta'$$

$$+ \frac{1}{2i} \left(\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right)^2 (\alpha') - \left(\mathbb{P}_A \frac{Z_t}{Z_{,\beta'}} \right)^2 (\beta') \right) \cot(\cdots) \partial_{\beta'} D_{\beta'}^2 \overline{Z}_t \Big|_{\partial}$$

$$+ \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right)^2 \partial_{\alpha'} \underbrace{(I - \mathbb{H}) \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t}_{0 \text{ by (143)}} + \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right)^2 (\alpha') \underbrace{[\partial_{\alpha'}, \mathbb{H}] \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t}_{-\frac{1}{2i} \cot(\cdots) \partial_{\beta'} D_{\beta'}^2 |_{\partial} \text{ by (101)}} \tag{520}$$

$$= -\int \{\partial_{\beta'} \cdots\} \partial_{\beta'} D_{\beta'}^2 \overline{Z}_t (\beta') d\beta' \underbrace{- \frac{1}{2i} \left(\mathbb{P}_A \frac{Z_t}{Z_{,\beta'}} \right)^2 (\beta') \cot(\cdots) \partial_{\beta'} D_{\beta'}^2 \overline{Z}_t \Big|_{\partial}}_{R}.$$

Here, the first term is precisely what we want. The second term, R, we'll handle at (533), where it will cancel out with other terms.

From the remaining terms from I_1 (518), we will handle I_{12} below after (530) and we will handle I_{13} below in (532).

We defer I_2 until later, and now move on to II from (512). Once again expanding the identity as $\mathbb{P}_A + \mathbb{P}_H$, we have

$$II = 2(I - \mathbb{H}) \left\{ Z_t D_{\alpha'}^3 (\overline{Z}_{tt} - (D_{\alpha'} \overline{Z}_t) Z_t) \right\}$$

$$= 2(I - \mathbb{H}) \left\{ \left((\mathbb{P}_A + \mathbb{P}_H) \frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'} D_{\alpha'}^2 (\overline{Z}_{tt} - (D_{\alpha'} \overline{Z}_t) Z_t) \right\}$$

$$= 2 \left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right), \mathbb{H} \right] \partial_{\alpha'} D_{\alpha'}^2 (\overline{Z}_{tt} - (D_{\alpha'} \overline{Z}_t) Z_t) \tag{521}$$

$$+ \underbrace{(I - \mathbb{H}) \left\{ \left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'} D_{\alpha'}^2 (\overline{Z}_{tt} - (D_{\alpha'} \overline{Z}_t) Z_t) \right\}}_{0 \text{ by (157)}},$$

where we've used (156) to get the first commutator. We now expand what remains from (521):

$$II = 2 \left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right), \mathbb{H} \right] \partial_{\alpha'} (D_{\alpha'}^2 \overline{Z}_{tt} - (\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t) \frac{Z_t}{Z_{,\alpha'}} - 2(D_{\alpha'}^2 \overline{Z}_t) D_{\alpha'} Z_t - (D_{\alpha'} \overline{Z}_t) D_{\alpha'}^2 Z_t)$$

$$= II_1 + II_2 + II_3 + II_4. \tag{522}$$

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We control everything but II_2 directly by the $L^\infty \times L^2$ commutator estimate (226). To apply this, we need that both parts have periodic boundary behavior. We know $\left.\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right|_\partial = 0$ (121). The reason that

$$\left.(D_{\alpha'}^2 \overline{Z}_{tt} - 2(D_{\alpha'}^2 \overline{Z}_t)D_{\alpha'}Z_t - (D_{\alpha'}\overline{Z}_t)D_{\alpha'}^2 Z_t)\right|_\partial = 0 \tag{523}$$

is that it equals $\left.\left(D_{\alpha'}^2(F_{tz} \circ Z) + (\partial_{\alpha'}D_{\alpha'}^2 \overline{Z}_t)\frac{Z_t}{Z_{,\alpha'}}\right)\right|_\partial = 0$ by (112), (113), and (127). We get

$$
\begin{aligned}
\|II_1 + II_3 + II_4\|_{L^2} &\lesssim \left\|\partial_{\alpha'}\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right\|_{L^\infty} \\
&\quad \cdot \left(\left\|D_\alpha^2 \overline{Z}_{tt}\right\|_{L^2} + \left\|(D_{\alpha'}^2 \overline{Z}_t)D_{\alpha'}Z_t\right\|_{L^2} + \left\|(D_{\alpha'}\overline{Z}_t)D_{\alpha'}^2 Z_t\right\|_{L^2}\right) \\
&\lesssim \left\|\partial_{\alpha'}\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right\|_{L^\infty} \\
&\quad \cdot \left(\left\|D_\alpha^2 \overline{Z}_{tt}\right\|_{L^2} + \left\|D_{\alpha'}^2 \overline{Z}_t\right\|_{L^2}\|D_{\alpha'}Z_t\|_{L^\infty} + \left\|D_{\alpha'}^2 Z_t\right\|_{L^2}\left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty}\right).
\end{aligned}
\tag{524}
$$

We are now left with II_2 from (522), which we expand, rewriting the identity as $\mathbb{P}_A + \mathbb{P}_H$:

$$
\begin{aligned}
II_2 &= -2\left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right), \mathbb{H}\right]\partial_{\alpha'}\left\{(\partial_{\alpha'}D_{\alpha'}^2 \overline{Z}_t)(\mathbb{P}_A + \mathbb{P}_H)\frac{Z_t}{Z_{,\alpha'}}\right\} \\
&= -2\left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right), \mathbb{H}\right]\left\{\partial_{\alpha'}\left((\partial_{\alpha'}D_{\alpha'}^2 \overline{Z}_t)\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right) + \partial_{\alpha'}\left((\partial_{\alpha'}D_{\alpha'}^2 \overline{Z}_t)\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}}\right)\right\} \tag{525} \\
&= II_{21} + II_{22}.
\end{aligned}
$$

II_{21} will give us the main term that we want, after integrating by parts, which we do now:

$$
\begin{aligned}
II_{21} &= 2\frac{1}{2i}\int (\partial_{\beta'}\cdots)(\partial_{\beta'}D_{\beta'}^2 \overline{Z}_t)\mathbb{P}_A \frac{Z_t}{Z_{,\beta'}}(\beta')d\beta' \\
&\quad - 2\frac{1}{2i}\left(\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}(\alpha')\right) - \left(\mathbb{P}_A \frac{Z_t}{Z_{,\beta'}}(\beta')\right)\right)\cot(\cdots)\left(\mathbb{P}_A \frac{Z_t}{Z_{,\beta'}}(\beta')\right)\partial_{\beta'}D_{\beta'}^2 \overline{Z}_t\Big|_\partial \tag{526} \\
&= II_{211} + II_{212}.
\end{aligned}
$$

The first term, II_{211}, is what we need, combining with I_{11} to give the integrated-by-parts version of (515). We'll handle II_{212} together with II_{222} below in (529).

To handle II_{22} from (525), we will want to switch back from commutator form, so we do

that:

$$\begin{aligned}
II_{22} &= -2\left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right), \mathbb{H}\right]\partial_{\alpha'}\left((\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t)\mathbb{P}_H\frac{Z_t}{Z_{,\alpha'}}\right) \\
&= -2(I-\mathbb{H})\left\{\left(\mathbb{P}_A\frac{Z_t}{Z_{,\alpha'}}\right)\partial_{\alpha'}\left((\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t)\mathbb{P}_H\frac{Z_t}{Z_{,\alpha'}}\right)\right\} \\
&\quad + 2\left(\mathbb{P}_A\frac{Z_t}{Z_{,\alpha'}}\right)(I-\mathbb{H})\partial_{\alpha'}\left((\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t)\mathbb{P}_H\frac{Z_t}{Z_{,\alpha'}}\right) \\
&= II_{221} + II_{222}.
\end{aligned} \tag{527}$$

We begin by considering II_{222}. We expand carefully:

$$\begin{aligned}
II_{222} &= 2\left(\mathbb{P}_A\frac{Z_t}{Z_{,\alpha'}}\right)\partial_{\alpha'}\underbrace{(I-\mathbb{H})\left((\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t)\mathbb{P}_H\frac{Z_t}{Z_{,\alpha'}}\right)}_{0\text{ by }(159)} \\
&\quad + 2\left(\mathbb{P}_A\frac{Z_t}{Z_{,\alpha'}}\right)[\partial_{\alpha'},\mathbb{H}]\left((\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t)\mathbb{P}_H\frac{Z_t}{Z_{,\alpha'}}\right) \\
&= -2\frac{1}{2i}\left(\mathbb{P}_A\frac{Z_t}{Z_{,\alpha'}}(\alpha')\right)\cot(\cdots)\left((\partial_{\beta'}D_{\beta'}^2\overline{Z}_t)\mathbb{P}_H\frac{Z_t}{Z_{,\beta'}}\right)\bigg|_\partial,
\end{aligned} \tag{528}$$

where we used (101) for the commutator $[\partial_{\alpha'},\mathbb{H}]$. When we combine this with II_{212} from (526) we get

$$\begin{aligned}
II_{222} + II_{212} &= \underbrace{-2\frac{1}{2i}\left(\mathbb{P}_A\frac{Z_t}{Z_{,\alpha'}}(\alpha')\right)\cot(\cdots)\left((\partial_{\beta'}D_{\beta'}^2\overline{Z}_t)\underbrace{(\mathbb{P}_A+\mathbb{P}_H)}_{I}\frac{Z_t}{Z_{,\beta'}}\right)\bigg|_\partial}_{0} \\
&\quad + 2\frac{1}{2i}\left(\mathbb{P}_A\frac{Z_t}{Z_{,\beta'}}(\beta')\right)\cot(\cdots)\left((\partial_{\beta'}D_{\beta'}^2\overline{Z}_t)\mathbb{P}_A\frac{Z_t}{Z_{,\beta'}}\right)\bigg|_\partial.
\end{aligned} \tag{529}$$

The first term is zero because $D_{\beta'}^3\overline{Z}_t$ and Z_t have periodic boundary behavior by (112) and (113). The remaining term we'll control later at (533); it will cancel out with other terms.

We now expand II_{221} from (527):

$$
\begin{aligned}
II_{221} &= -2(I - \mathbb{H})\left\{\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right) \partial_{\alpha'}\left((\partial_{\alpha'}D_{\alpha'}^2 \overline{Z}_t)\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}}\right)\right\} \\
&= -2(I - \mathbb{H})\left\{\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right)\left((\partial_{\alpha'}^2 D_{\alpha'}^2 \overline{Z}_t)\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}}\right)\right\} \\
&\quad - 2(I - \mathbb{H})\left\{\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right)\left((\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t)\mathbb{P}_H \partial_{\alpha'} \frac{Z_t}{Z_{,\alpha'}}\right)\right\} \\
&= II_{2211} + II_{2212},
\end{aligned}
\tag{530}
$$

where we've used $\frac{Z_t}{Z_{,\alpha'}}\big|_{\partial} = 0$ (118) to commute the $\partial_{\alpha'}$ in the second term inside $\mathbb{P}_H$ by (100). The first of these terms, II_{2211}, cancels with I_{12} from (518).

Thus we're left with the second term in (530), II_{2212}, which we combine with I_2 from (517):

$$
\begin{aligned}
I_2 + II_{2212} &= (I - \mathbb{H})\left\{Z_t^2\left(\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\partial_{\alpha'}D_{\alpha'}^2 \overline{Z}_t\right\} \\
&\quad - 2(I - \mathbb{H})\left\{\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right)(\partial_{\alpha'}D_{\alpha'}^2 \overline{Z}_t)\mathbb{P}_H \partial_{\alpha'}\frac{Z_t}{Z_{,\alpha'}}\right\} \\
&= (I - \mathbb{H})\left\{(\partial_{\alpha'}D_{\alpha'}^2 \overline{Z}_t)\left\{\frac{Z_t}{Z_{,\alpha'}}Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - 2\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right)\left(\mathbb{P}_H \partial_{\alpha'}\frac{Z_t}{Z_{,\alpha'}}\right)\right\}\right\} \\
&= (I - \mathbb{H})\left\{(\partial_{\alpha'}D_{\alpha'}^2 \overline{Z}_t)\left\{\frac{Z_t}{Z_{,\alpha'}}Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - 2\left(\underbrace{(\mathbb{P}_A + \mathbb{P}_H)}_{I}\frac{Z_t}{Z_{,\alpha'}}\right)\left(\mathbb{P}_H \partial_{\alpha'}\frac{Z_t}{Z_{,\alpha'}}\right)\right\}\right\} \\
&\quad + 2(I - \mathbb{H})\left\{(\partial_{\alpha'}D_{\alpha'}^2 \overline{Z}_t)\left\{\left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}}\right)\left(\mathbb{P}_H \partial_{\alpha'}\frac{Z_t}{Z_{,\alpha'}}\right)\right\}\right\} \\
&= A + B.
\end{aligned}
\tag{531}
$$

We begin by considering the second term, B, which will be part of a series of terms that

cancel. First, we combine B with I_{13} from (518):

$$
\begin{aligned}
I_{13} + B &= (I - \mathbb{H})\left\{\left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}}\right)^2 \partial_{\alpha'}^2 D_{\alpha'}^2 \overline{Z}_t\right\} \\
&\quad + 2(I - \mathbb{H})\left\{(\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t)\left\{\left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}}\right)\left(\mathbb{P}_H \partial_{\alpha'} \frac{Z_t}{Z_{,\alpha'}}\right)\right\}\right\} \\
&= (I - \mathbb{H})\left\{\left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}}\right)^2 \partial_{\alpha'}^2 D_{\alpha'}^2 \overline{Z}_t\right\} + (I - \mathbb{H})\left\{(\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t)\partial_{\alpha'}\left\{\left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}}\right)^2\right\}\right\} \\
&= (I - \mathbb{H})\partial_{\alpha'}\left\{(\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t)\left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}}\right)^2\right\} \\
&= \underbrace{\partial_{\alpha'}(I - \mathbb{H})\left\{(\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t)\left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}}\right)^2\right\}}_{0 \text{ by } (160)} + [\partial_{\alpha'}, \mathbb{H}]\left\{(\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t)\left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}}\right)^2\right\} \\
&= -\frac{1}{2i}\cot(\frac{\pi}{2}(\alpha' - \beta'))(\partial_{\beta'} D_{\beta'}^2 \overline{Z}_t)\left(\mathbb{P}_H \frac{Z_t}{Z_{,\beta'}}\right)^2\bigg|_{\partial}.
\end{aligned}
$$

$$(532)$$

Here we have in the second line used $\frac{Z_t}{Z_{,\alpha'}}\big|_{\partial} = 0$ (118) to commute the $\partial_{\alpha'}$ in the second term outside $\mathbb{P}_H$ by (100), and used (101) to expand the commutator $[\partial_{\alpha'}, \mathbb{H}]$ in the last line. We now combine this with R from (520) and with (529). Together, these are

$$
\begin{aligned}
R + (529) + (532) &= -\frac{1}{2i}\left(\mathbb{P}_A \frac{Z_t}{Z_{,\beta'}}\right)^2 (\beta')\cot(\cdots)\partial_{\beta'} D_{\beta'}^2 \overline{Z}_t\bigg|_{\partial} \\
&\quad + 2\frac{1}{2i}\left(\mathbb{P}_A \frac{Z_t}{Z_{,\beta'}}\right)^2 (\beta')\cot(\cdots)(\partial_{\beta'} D_{\beta'}^2 \overline{Z}_t)\bigg|_{\partial} \\
&\quad - \frac{1}{2i}\cot(\cdots)(\partial_{\beta'} D_{\beta'}^2 \overline{Z}_t)\left(\mathbb{P}_H \frac{Z_t}{Z_{,\beta'}}\right)^2 (\beta')\bigg|_{\partial} \\
&= \frac{1}{2i}\left(\underbrace{(\mathbb{P}_A - \mathbb{P}_H)}_{-\mathbb{H}}\frac{Z_t}{Z_{,\beta'}}\right)\left(\underbrace{(\mathbb{P}_A + \mathbb{P}_H)}_{I}\frac{Z_t}{Z_{,\beta'}}\right)\cot(\cdots)\partial_{\beta'} D_{\beta'}^2 \overline{Z}_t\bigg|_{\partial} \\
&= \frac{-1}{2i}\left(\mathbb{H}\frac{Z_t}{Z_{,\beta'}}\right)\left(\frac{Z_t}{Z_{,\beta'}}\right)\cot(\cdots)\partial_{\beta'} D_{\beta'}^2 \overline{Z}_t\bigg|_{\partial} \\
&= 0.
\end{aligned}
$$

$$(533)$$

This is zero because the first factor has periodic boundary behavior by (121), the Z_t has

periodic boundary behavior by (112), and the $\frac{1}{Z_{,\beta'}}$ joins with the $\partial_{\beta'} D_{\beta'}^2 \overline{Z}_t$ to ensure periodic boundary behavior by (113).

All that remains is the first term on the RHS of (531),

$$A = (I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t) \frac{Z_t}{Z_{,\alpha'}} \underbrace{\left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} - 2 \mathbb{P}_H \partial_{\alpha'} \frac{Z_t}{Z_{,\alpha'}} \right)}_{(*)} \right\}. \tag{534}$$

We expand $(*)$ as follows:

$$
\begin{aligned}
(*) &= Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} - (I + \mathbb{H}) D_{\alpha'} Z_t - (I + \mathbb{H}) \left\{ Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\} \\
&= -\mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) - (I + \mathbb{H}) D_{\alpha'} Z_t.
\end{aligned}
\tag{535}
$$

Therefore, we need to control

$$(I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t) \frac{Z_t}{Z_{,\alpha'}} \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\} \tag{536}$$

and

$$(I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t) \frac{Z_t}{Z_{,\alpha'}} (I + \mathbb{H}) D_{\alpha'} Z_t \right\}. \tag{537}$$

We will control (536) in §10.2.4 and (537) in §10.2.5.

10.2.4 Controlling (536)

We begin with (536). Our goal is to take advantage of the fact that we can control $\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\|_{\dot{H}^{1/2}}$. Therefore, we rewrite (536) so that we can isolate $\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t$ for a commutator estimate:

$$
\begin{aligned}
(536) &= (I - \mathbb{H}) \left\{ \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t \right) Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\} \\
&= (I - \mathbb{H}) \left\{ \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right) \right\} \\
&\quad - (I - \mathbb{H}) \left\{ \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \left(\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) D_{\alpha'}^2 \overline{Z}_t \right\} \\
&= (a) + (b).
\end{aligned}
\tag{538}
$$

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We expand (a) using $I = \mathbb{P}_A + \mathbb{P}_H$:

$$
\begin{aligned}
(a) &= (I - \mathbb{H}) \left\{ \left((\mathbb{P}_A + \mathbb{P}_H) \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right) \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right) \right\} \\
&= (I - \mathbb{H}) \left\{ \left(\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right) \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right) \right\} \\
&\quad + (I - \mathbb{H}) \left\{ \left(\mathbb{P}_H \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right) \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right) \right\} \\
&= \left[\left(\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right), \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right) \\
&\quad + \underbrace{(I - \mathbb{H}) \left\{ \left(\mathbb{P}_H \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right) \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right) \right\}}_{0 \text{ by (162)}}
\end{aligned}
\tag{539}
$$

by (161). We now use (236) to estimate the remaining, first term of (539):

$$
\| (a) \|_{L^2} \lesssim \left\| \partial_{\alpha'} \mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right\|_{L^2} \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\|_{\dot{H}^{1/2}}.
\tag{540}
$$

To apply (236), we require

$$
\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \bigg|_{\partial} = 0;
\tag{541}
$$

note that the second boundary condition,

$$
\left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right) \bigg|_{\partial} = 0,
\tag{542}
$$

holds by (113) and (114). It isn't obvious why this first boundary condition should hold. We will show below at (554) that it does, in fact, hold. Assuming that, because $\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\|_{\dot{H}^{1/2}}$ is part of our energy, we've reduced things to controlling $\left\| \partial_{\alpha'} \left(\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right) \right\|_{L^2}$.

We first find a useful way of rewriting $Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right)$. Observe that

$$
\begin{aligned}
[Z_t, [Z_t, \mathbb{H}]] &= Z_t^2 \mathbb{H} - Z_t \mathbb{H} Z_t - Z_t \mathbb{H} Z_t + \mathbb{H} Z_t^2 \\
&= Z_t^2 \mathbb{H} - 2 Z_t \mathbb{H} Z_t + \mathbb{H} Z_t^2.
\end{aligned}
\tag{543}
$$

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Therefore,

$$[Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} = Z_t^2 \underbrace{\mathbb{H} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}}_{\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \text{ by (137)}} - 2Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) + \mathbb{H} \left(Z_t^2 \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right)$$

$$= (I + \mathbb{H}) \left(Z_t^2 \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) - 2Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right). \tag{544}$$

Therefore,

$$2Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) = (I + \mathbb{H}) \left(Z_t^2 \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) - [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}. \tag{545}$$

Now we apply $\mathbb{P}_A$ to this, getting

$$2\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) = \mathbb{P}_A (I + \mathbb{H}) \left(Z_t^2 \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) - \mathbb{P}_A [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}. \tag{546}$$

By (95), the first term on the RHS is just a mean-term, because $\mathbb{P}_A \mathbb{P}_H = 0$ is zero on mean-zero functions. Because this mean will eventually disappear under the derivative, we denote it simply by c. We are left with

$$2\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) = c - \mathbb{P}_A [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}. \tag{547}$$

Now we will show, as promised, that $\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \Big|_\partial = 0$. We expand

$$[Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} = \frac{1}{2i} \int (Z_t(\alpha') - Z_t(\beta'))^2 \cot(\frac{\pi}{2}(\alpha' - \beta')) \partial_{\beta'} \frac{1}{Z_{,\beta'}} d\beta'. \tag{548}$$

Because $Z_t|_\partial = 0$, we see, by the fundamental theorem of calculus and Hölder,[54] that the kernel function $(Z_t(\alpha') - Z_t(\beta'))^2 \cot(\frac{\pi}{2}(\alpha' - \beta'))$ is continuous for $(\alpha', \beta') \in (I \times I) -$

---

[54]Here we use the fact that $\overline{Z}_t|_\partial = 0$ to assume by periodic extension that $|\alpha' - \beta'| \leqslant 1$, so we may replace the cotangent with $\frac{1}{\alpha' - \beta'}$. Note that

$$\left| \frac{f(x) - f(y)}{|x - y|^{1/2}} \right| \lesssim \frac{1}{|x - y|^{1/2}} \int_x^y |f'| \, dz \lesssim \frac{1}{|x - y|^{1/2}} \left(\int_x^y dz \right)^{1/2} \left(\int_x^y |f'|^2 \, dz \right)^{1/2} \lesssim \|f'\|_{L^2(I)}. \tag{549}$$

$\{d_{S^1}(\alpha', \beta') \neq 0\}$, is periodic in α', and satisfies

$$\left| (Z_t(\alpha') - Z_t(\beta'))^2 \cot(\frac{\pi}{2}(\alpha' - \beta')) \right| \lesssim \int |Z_{t,\alpha'}|^2 \, d\alpha'. \tag{550}$$

Therefore, by the Lebesgue dominated convergence theorem,

$$[Z_t, [Z_t, \mathbb{H}]]\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \bigg|_{\partial} = 0. \tag{551}$$

Furthermore we can estimate

$$\left\| [Z_t, [Z_t, \mathbb{H}]]\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \tag{552}$$

by Hölder and Hardy's inequality (217), and

$$\left\| \partial_{\alpha'}[Z_t, [Z_t, \mathbb{H}]]\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \tag{553}$$

by (245). We may then conclude by (547), (551), (552), (553), and (119) that

$$\mathbb{P}_A[Z_t, [Z_t, \mathbb{H}]]\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \bigg|_{\partial} = \mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \bigg|_{\partial} = 0. \tag{554}$$

Now we return to controlling $\left\| \partial_{\alpha'} \left(\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right) \right\|_{L^2}$. By (547),

$$\begin{aligned}
\left\| \partial_{\alpha'} \mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right\|_{L^2} &= \frac{1}{2} \left\| \partial_{\alpha'} \mathbb{P}_A[Z_t, [Z_t, \mathbb{H}]]\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}. \\
&= \frac{1}{2} \left\| \mathbb{P}_A \partial_{\alpha'}[Z_t, [Z_t, \mathbb{H}]]\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \\
&\lesssim \left\| \partial_{\alpha'}[Z_t, [Z_t, \mathbb{H}]]\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \\
&\lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2},
\end{aligned} \tag{555}$$

where we have used (100) and (551) to commute the $\partial_{\alpha'}$ inside the $\mathbb{P}_A$ in the second line; used the L^2 boundedness of $\mathbb{P}_A$ in the third line; and used estimate (553) in the final line.

Now we return to controlling part (b) from (538). We begin by decomposing two of the

factors into their holomorphic and antiholomorphic projections:

$$
\begin{aligned}
(b) &= -(I - \mathbb{H}) \left\{ \left(\mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) D^2_{\alpha'} \overline{Z}_t \right\} \\
&= -(I - \mathbb{H}) \left\{ \left(\mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \left((\mathbb{P}_A + \mathbb{P}_H) \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) D^2_{\alpha'} \overline{Z}_t \right\}.
\end{aligned}
\tag{556}
$$

Observe that $\mathbb{H} = -\mathbb{P}_A + \mathbb{P}_H$. The cross terms cancel, and we're left with

$$
(I - \mathbb{H}) \left\{ \left(\mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right)^2 D^2_{\alpha'} \overline{Z}_t \right\} - (I - \mathbb{H}) \left\{ \left(\mathbb{P}_H \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right)^2 D^2_{\alpha'} \overline{Z}_t \right\}.
\tag{557}
$$

The second of these terms disappears by (163). We control the first term of (557) by the boundedness of $\mathbb{H}$ and Hölder's inequality, and conclude

$$
\|(b)\|_{L^2} \lesssim \left\| D^2_{\alpha'} \overline{Z}_t \right\|_{L^2} \left\| \mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\|^2_{L^\infty},
\tag{558}
$$

whose second factor we controlled in (371).

We conclude from (538) that

$$
\begin{aligned}
\|(536)\|_{L^2} &\lesssim (540) + (558) \\
&\leqslant \|Z_{t,\alpha'}\|^2_{L^2} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \left\| \frac{1}{Z_{,\alpha'}} D^2_{\alpha'} \overline{Z}_t \right\|_{\dot{H}^{1/2}} + \left\| D^2_{\alpha'} \overline{Z}_t \right\|_{L^2} \left\| \mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\|^2_{L^\infty}.
\end{aligned}
\tag{559}
$$

10.2.5 Controlling (537)

Now we are left with controlling (537), which we expand using $I = \mathbb{P}_A + \mathbb{P}_H$:

$$
\begin{aligned}
(537) &= (I - \mathbb{H}) \left\{ (\partial_{\alpha'} D^2_{\alpha'} \overline{Z}_t) \frac{Z_t}{Z_{,\alpha'}} (I + \mathbb{H}) D_{\alpha'} Z_t \right\} \\
&= (I - \mathbb{H}) \left\{ (\partial_{\alpha'} D^2_{\alpha'} \overline{Z}_t) \left((\mathbb{P}_A + \mathbb{P}_H) \frac{Z_t}{Z_{,\alpha'}} \right) (I + \mathbb{H}) D_{\alpha'} Z_t \right\} \\
&= (I - \mathbb{H}) \left\{ (\partial_{\alpha'} D^2_{\alpha'} \overline{Z}_t) \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) (I + \mathbb{H}) D_{\alpha'} Z_t \right\},
\end{aligned}
\tag{560}
$$

where the $\mathbb{P}_H$ term disappears by (164). We expand again $I = \mathbb{P}_A + \mathbb{P}_H$ with what remains:

$$(537) = (I - \mathbb{H}) \left\{ [(I + \mathbb{H})D_{\alpha'}Z_t] \, (\mathbb{P}_A + \mathbb{P}_H) \left\{ (\partial_{\alpha'}D_{\alpha'}^2 \overline{Z}_t) \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right\} \right\}. \tag{561}$$

The $\mathbb{P}_H$ part would be zero by holomorphicity, except that there is a mean term; by (165), what we're left with from that part is, in absolute value,

$$\left| \frac{1}{2} \left(\fint D_{\alpha'}Z_t \right) \left(\fint (\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t) \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right) \right| = \left| \frac{1}{2} \left(\fint D_{\alpha'}Z_t \right) \left(\fint (D_{\alpha'}^2\overline{Z}_t) \left(\partial_{\alpha'}\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right) \right|$$

$$\lesssim \| D_{\alpha'}Z_t \|_{L^\infty} \| D_{\alpha'}^2\overline{Z}_t \|_{L^2} \left\| \partial_{\alpha'}\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty}. \tag{562}$$

By (121) and (113) there is no boundary term in the integration by parts.[55]

What's left from (561) to control is the $\mathbb{P}_A$ part,

$$(I - \mathbb{H}) \left\{ ((I + \mathbb{H})D_{\alpha'}Z_t) \, \mathbb{P}_A \left((\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t) \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right) \right\}. \tag{563}$$

Now we use the boundedness of $\mathbb{H}$ and Hölder to control this in L^2 by

$$\| (I + \mathbb{H})D_{\alpha'}Z_t \|_{L^\infty} \left\| \mathbb{P}_A \left\{ (\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t) \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right\} \right\|_{L^2}. \tag{564}$$

We rewrite the second factor as a commutator, using (143):

$$\left\| \mathbb{P}_A \left\{ (\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t) \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right\} \right\|_{L^2} = \left\| \frac{1}{2} \left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right), \mathbb{H} \right] \partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t \right\|_{L^2}$$

$$\lesssim \left\| \partial_{\alpha'}(I - \mathbb{H}) \frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty} \| D_{\alpha'}^2\overline{Z}_t \|_{L^2}, \tag{565}$$

where we've used the $L^\infty \times L^2$ commutator estimate (226).

We conclude from (561), (562), (564) and (565) that

$$\| (537) \|_{L^2} \lesssim (\| D_{\alpha'}Z_t \|_{L^\infty} + \| (I + \mathbb{H})D_{\alpha'}Z_t \|_{L^\infty}) \left\| \partial_{\alpha'}(I - \mathbb{H}) \frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty} \| D_{\alpha'}^2\overline{Z}_t \|_{L^2}. \tag{566}$$

---

[55]We have been inefficient, in using L^∞ estimates when easier estimates could apply, to keep things in terms of quantities we have controlled.

10.2.6 Concluding the Estimate for iii from (511)

We now outline our expansion for iii. We expanded

$$
iii = \underbrace{I}_{(517)} + \underbrace{II}_{(522)}
$$

$$
= (\underbrace{I_1}_{(518)} + I_2) + (II_1 + \underbrace{II_2}_{(525)} + II_3 + II_4)
$$

$$
= (I_{11} + I_{12} + I_{13}) + I_2 + (\underbrace{II_{21}}_{(526)} + \underbrace{II_{22}}_{(527)}) + (II_1 + II_3 + II_4)
$$

$$
= (I_{11} + I_{12} + I_{13}) + I_2 + (II_{211} + II_{212}) + (\underbrace{II_{221}}_{(530)} + II_{222}) + (II_1 + II_3 + II_4)
$$

$$
= (I_{11} + I_{12} + I_{13}) + I_2 + (II_{211} + II_{212}) + (II_{2211} + II_{2212}) + II_{222} + (II_1 + II_3 + II_4).
\tag{567}
$$

Of these terms, $I_{11} - R$ and II_{211} will give the main estimate we were seeking:

$$
\|I_{11} - R + II_{211}\| \lesssim (516).
\tag{568}
$$

A remaining estimate is

$$
\|II_1 + II_3 + II_4\| \lesssim (524).
\tag{569}
$$

I_{12} and II_{2211} cancel. If we add R and all the remaining terms, $I_{13}, I_2, II_{2212}, II_{222}, II_{212}$, after the cancellation from (533), we are left with

$$
(534) = (536) + (537).
\tag{570}
$$

We estimate the first at (559) and the second at (566). We conclude that

$$\|iii\| \lesssim (516) + (524) + (559) + (566)$$

$$\lesssim \left\| \partial_{\alpha'} \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right\|_{L^\infty}^2 \left\| D_{\alpha'}^2 \overline{Z}_t \right\|_{L^2}$$

$$+ \left(\left\| \partial_{\alpha'} \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty} \left(\left\| D_\alpha^2 \overline{Z}_{tt} \right\|_{L^2} + \left\| D_{\alpha'}^2 \overline{Z}_t \right\|_{L^2} \left\| D_{\alpha'} Z_t \right\|_{L^\infty} + \left\| D_{\alpha'}^2 Z_t \right\|_{L^2} \left\| D_{\alpha'} \overline{Z}_t \right\|_{L^\infty} \right) \right)$$

$$+ \|Z_{t,\alpha'}\|_{L^2}^2 \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\|_{\dot{H}^{1/2}} + \left\| D_{\alpha'}^2 \overline{Z}_t \right\|_{L^2} \left\| \mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\|_{L^\infty}^2$$

$$+ \left(\|D_{\alpha'} Z_t\|_{L^\infty} + \|(I + \mathbb{H}) D_{\alpha'} Z_t\|_{L^\infty} \right) \left\| \partial_{\alpha'} (I - \mathbb{H}) \frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty} \left\| D_{\alpha'}^2 \overline{Z}_t \right\|_{L^2}.$$

$$(571)$$

10.2.7 Estimating i and ii from (511)

We must estimate i and ii from (511). When we expand this out carefully, all the problematic terms cancel and what remains can be controlled easily using Hölder and the L^2 boundedness of $\mathbb{H}$.

We expand i:

$$i = (I - \mathbb{H}) \big\{ 2(D_{\alpha'}^2 Z_t) Z_t (D_{\alpha'}^2 \overline{Z}_t) + 2(D_{\alpha'} Z_t)^2 (D_{\alpha'}^2 \overline{Z}_t) + (2D_{\alpha'}^2 Z_t)(D_{\alpha'} \overline{Z}_{tt})$$

$$- 2(D_{\alpha'}^2 Z_t) Z_t D_{\alpha'}^2 \overline{Z}_t - 2(D_{\alpha'}^2 Z_t)(D_{\alpha'} Z_t)(D_{\alpha'} \overline{Z}_t) \big\} \tag{572}$$

$$= (I - \mathbb{H}) \big\{ 2(D_{\alpha'} Z_t)^2 (D_{\alpha'}^2 \overline{Z}_t) + (2D_{\alpha'}^2 Z_t)(D_{\alpha'} \overline{Z}_{tt}) - 2(D_{\alpha'}^2 Z_t)(D_{\alpha'} Z_t)(D_{\alpha'} \overline{Z}_t) \big\},$$

where the first and fourth terms of the first line of the RHS cancel. We can therefore estimate

$$\|i\|_{L^2} \lesssim \|D_{\alpha'} Z_t\|_{L^\infty}^2 \left\| D_{\alpha'}^2 \overline{Z}_t \right\|_{L^2} + \left\| D_{\alpha'}^2 Z_t \right\|_{L^2} \left(\left\| D_{\alpha'} \overline{Z}_{tt} \right\|_{L^\infty} + \|D_{\alpha'} Z_t\|_{L^\infty}^2 \right). \tag{573}$$

We expand ii:

$$ii = (I - \mathbb{H}) \big\{ 4(D_{\alpha'} Z_t) Z_t D_{\alpha'}^3 \overline{Z}_t + 4(D_{\alpha'} Z_t)(D_{\alpha'}^2 \overline{Z}_{tt}) - 4(D_{\alpha'} Z_t)(D_{\alpha'}^3 \overline{Z}_t) Z_t$$

$$- 8(D_{\alpha'} Z_t)^2 (D_{\alpha'}^2 \overline{Z}_t) - 4(D_{\alpha'} Z_t)(D_{\alpha'} \overline{Z}_t)(D_{\alpha'}^2 Z_t) \big\} \tag{574}$$

$$= (I - \mathbb{H}) \big\{ 4(D_{\alpha'} Z_t)(D_{\alpha'}^2 \overline{Z}_{tt}) - 8(D_{\alpha'} Z_t)^2 (D_{\alpha'}^2 \overline{Z}_t) - 4(D_{\alpha'} Z_t)(D_{\alpha'} \overline{Z}_t)(D_{\alpha'}^2 Z_t) \big\},$$

where the first and third terms of the first line of the RHS cancel out. We can therefore

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estimate

$$\|ii\|_{L^2} \lesssim \|D_{\alpha'}Z_t\|_{L^\infty}\left\|D^2_{\alpha'}\overline{Z}_{tt}\right\|_{L^2} + \|D_{\alpha'}Z_t\|^2_{L^\infty}\left\|D^2_{\alpha'}\overline{Z}_t\right\|_{L^2} + \|D_{\alpha'}Z_t\|^2_{L^\infty}\left\|D^2_{\alpha'}Z_t\right\|_{L^2}. \quad (575)$$

10.2.8   Concluding the Estimates of $(I - \mathbb{H})$ of the RHS

Combining our various estimates, we have, by (503) and (506), that

$$\left\|(I - \mathbb{H})\left(D^2_\alpha(\mathfrak{a}_t\overline{z}_\alpha)\right) \circ h^{-1}\right\|_{L^2}$$
$$\lesssim (510) + \|i\|_{L^2} + \|ii\|_{L^2} + \|iii\|_{L^2}$$
$$\lesssim (510) + (573) + (575) + (571)$$
$$\lesssim \left\|D^2_{\alpha'}\overline{Z}_t\right\|_{L^2}\left\{ \|D_{\alpha'}Z_{tt}\|_{L^\infty} + \left\|(Z_{tt}+i)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} + \|D_{\alpha'}Z_t\|^2_{L^\infty} + \left\|\mathbb{P}_A\left\{Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\}\right\|^2_{L^\infty} \right.$$
$$+ \left(\left\|\partial_{\alpha'}\left(\mathbb{P}_A\frac{Z_t}{Z_{,\alpha'}}\right)\right\|_{L^\infty} + \|D_{\alpha'}Z_t\|_{L^\infty} + \|(I+\mathbb{H})D_{\alpha'}Z_t\|_{L^\infty}\right)\left\|\partial_{\alpha'}\left(\mathbb{P}_A\frac{Z_t}{Z_{,\alpha'}}\right)\right\|_{L^\infty} \left.\right\}$$
$$+ \left\|D^2_{\alpha'}Z_{tt}\right\|_{L^2}\left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty}$$
$$+ \left\|D^2_{\alpha'}\overline{Z}_{tt}\right\|_{L^2}\left(\left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty} + \left\|\partial_{\alpha'}\left(\mathbb{P}_A\frac{Z_t}{Z_{,\alpha'}}\right)\right\|_{L^\infty}\right)$$
$$+ \left\|D^2_{\alpha'}Z_t\right\|_{L^2}\left(\left\|D_{\alpha'}\overline{Z}_{tt}\right\|_{L^\infty} + \|D_{\alpha'}Z_t\|^2_{L^\infty} + \|D_{\alpha'}Z_t\|_{L^\infty}\left\|\partial_{\alpha'}\left(\mathbb{P}_A\frac{Z_t}{Z_{,\alpha'}}\right)\right\|_{L^\infty}\right)$$
$$+ \|Z_{t,\alpha'}\|^2_{L^2}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left\|\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{1/2}}.$$
$$(576)$$

10.2.9   Using the Peter-Paul Trick, Combining the Estimates

We now combine the many terms we have estimated, and handle the remaining term we had deferred.

Starting in §10.2.1 we estimated $\left\|\left(D^2_\alpha\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right)\right)\mathfrak{a}\overline{z}_\alpha\right\|_{L^2(h_\alpha)}$. On changing to Riemannian

coordinates, this was equivalent to estimating

$$\left\|\left(D_\alpha^2\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right)\right)\mathfrak{a}\overline{z}_\alpha\right\|_{L^2(h_\alpha)} = \tag{491}$$

$$\lesssim \left\|(I-\mathbb{H})\left\{i(\overline{Z}_{tt}-i)D_{\alpha'}^2\frac{\mathcal{A}_t}{\mathcal{A}}\right\}\right\|_{L^2}$$

$$+\left\|\left[i(\overline{Z}_{tt}-i)\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)^2,\mathbb{H}\right]\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|}|D_{\alpha'}|\frac{\mathcal{A}_t}{\mathcal{A}}\right)\right\|_{L^2}+\|e\|_{L^2}$$

$$\lesssim (502)+(501)+(496)$$

$$\lesssim \left(\left\|(I-\mathbb{H})\left(D_\alpha^2(\mathfrak{a}_t\overline{z}_\alpha)\right)\circ h^{-1}\right\|_{L^2}+(479)+(480)\right)$$

$$+\left(\|Z_{tt,\alpha'}\|_{L^2}\left(\|D_\alpha\overline{z}_{tt}\|_{L^\infty}\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty}+\|D_\alpha(\mathfrak{a}_t\overline{z}_\alpha)\|_{L^\infty}\right)\right)$$

$$+(496)$$

$$\lesssim ((576)+(479)+(480))$$

$$+\left(\|Z_{tt,\alpha'}\|_{L^2}\left(\|D_\alpha\overline{z}_{tt}\|_{L^\infty}\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty}+\|D_\alpha(\mathfrak{a}_t\overline{z}_\alpha)\|_{L^\infty}\right)\right)$$

$$+(496). \tag{577}$$

By (481) and $A_1 \geqslant 1$ (182) we can conclude that

$$\left\|D_\alpha^2(\mathfrak{a}_t\overline{z}_\alpha)\right\|_{L^2\left(\frac{h_\alpha}{A_1\circ h}\right)} \leqslant \left\|D_\alpha^2(\mathfrak{a}_t\overline{z}_\alpha)\right\|_{L^2(h_\alpha)}$$

$$\leqslant \left\|\left(D_\alpha^2\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right)\right)\mathfrak{a}\overline{z}_\alpha\right\|_{L^2(h_\alpha)}+(479)+(480).$$

$$\leqslant (577)+(479)+(480) \tag{578}$$

$$\lesssim (576)+(479)+(480)+(496)$$

$$+\left(\|Z_{tt,\alpha'}\|_{L^2}\left(\|D_\alpha\overline{z}_{tt}\|_{L^\infty}\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty}+\|D_\alpha(\mathfrak{a}_t\overline{z}_\alpha)\|_{L^\infty}\right)\right).$$

Recall that we still needed to control $\|D_\alpha(\mathfrak{a}_t\overline{z}_\alpha)\|_{L^\infty}$. Here, we use the Peter-Paul trick Proposition 13. By (578), we now have an estimate

$$\left\|D_\alpha^2(\mathfrak{a}_t\overline{z}_\alpha)\right\|_{L^2\left(\frac{h_\alpha}{A_1\circ h}\right)} \leqslant c_1+c_2\|D_\alpha(\mathfrak{a}_t\overline{z}_\alpha)\|_{L^\infty}, \tag{579}$$

where

$$c_1 = C\left((576)+(479)+(480)+(496)+\|Z_{tt,\alpha'}\|_{L^2}\|D_\alpha\overline{z}_{tt}\|_{L^\infty}\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty}\right) \tag{580}$$

and

$$c_2 = C \, \|Z_{tt,\alpha'}\|_{L^2} \, . \tag{581}$$

We may now apply Proposition 13 with $\mu = \frac{h_\alpha}{A_1 \circ h}$.[56] We get that

$$
\begin{aligned}
\|D_\alpha(\mathfrak{a}_t \overline{z}_\alpha)\|_{L^\infty} &\lesssim c_2 \, \|D_\alpha(\mathfrak{a}_t \overline{z}_\alpha)\|_{L^2(\frac{|z_\alpha|^2 (A_1 \circ h)}{h_\alpha})} + c_1 + \|D_\alpha(\mathfrak{a}_t \overline{z}_\alpha)\|_{L^2} \\
&\leqslant C \, \|Z_{tt,\alpha'}\|_{L^2} \, \|D_\alpha(\mathfrak{a}_t \overline{z}_\alpha)\|_{L^2(\frac{(A_1 \circ h)^2}{\mathfrak{a}})} + (580) + \|\mathfrak{a}\|_{L^\infty}^{1/2} \, \|D_\alpha(\mathfrak{a}_t \overline{z}_\alpha)\|_{L^2(\frac{1}{\mathfrak{a}})} \\
&\leqslant C \, \|Z_{tt,\alpha'}\|_{L^2} \, \|A_1\|_{L^\infty} \, \|D_\alpha(\mathfrak{a}_t \overline{z}_\alpha)\|_{L^2(\frac{1}{\mathfrak{a}})} \\
&\quad + (580) + \|\overline{z}_{tt} - i\|_{L^\infty}^{1/2} \, \|1/z_\alpha\|_{L^\infty}^{1/2} \, \|D_\alpha(\mathfrak{a}_t \overline{z}_\alpha)\|_{L^2(\frac{1}{\mathfrak{a}})} \\
&\leqslant C \left(\left(\|Z_{tt,\alpha'}\|_{L^2} \, \|A_1\|_{L^\infty} \right) + \|\overline{z}_{tt} - i\|_{L^\infty}^{1/2} \, \|1/z_\alpha\|_{L^\infty} \right)^{1/2} (472) + (580),
\end{aligned}
\tag{582}
$$

where (472) was the main term to control for the RHS of E_b.

We now insert (582) into (578), and expand the various terms, to conclude that

$$
\begin{aligned}
\left\| D_\alpha^2(\mathfrak{a}_t \overline{z}_\alpha) \right\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)} &\lesssim (576) + (479) + (480) + (496) \\
&\quad + \left(\|Z_{tt,\alpha'}\|_{L^2} \left(\|D_\alpha \overline{z}_{tt}\|_{L^\infty} \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} + \|D_\alpha(\mathfrak{a}_t \overline{z}_\alpha)\|_{L^\infty} \right) \right) \\
&\leqslant (576) + (479) + (480) + (496) \\
&\quad + \|Z_{tt,\alpha'}\|_{L^2} \left(\|D_\alpha \overline{z}_{tt}\|_{L^\infty} \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} \right) \\
&\quad + \|Z_{tt,\alpha'}\|_{L^2} (582) \\
&\lesssim \left\| (I - \mathbb{H}) \left(D_\alpha^2(\mathfrak{a}_t \overline{z}_\alpha) \right) \circ h^{-1} \right\|_{L^2} \\
&\quad + \left(\|D_\alpha \overline{z}_{tt}\|_{L^\infty} + \left\| (\overline{Z}_{tt} - i) \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \right) \left(\int \frac{1}{\mathfrak{a}} \left| \mathfrak{a} \overline{z}_\alpha D_\alpha \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \right|^2 d\alpha \right)^{1/2} \\
&\quad + \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} \left\| D_\alpha^2 \overline{z}_{tt} \right\|_{L^2(h_\alpha)} \\
&\quad + \|Z_{tt,\alpha'}\|_{L^2} \, \|D_\alpha \overline{z}_{tt}\|_{L^\infty} \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} . \\
&\quad + \|Z_{tt,\alpha'}\|_{L^2} \left(\|Z_{tt,\alpha'}\|_{L^2} \, \|A_1\|_{L^\infty} + \|\overline{z}_{tt} - i\|_{L^\infty}^{1/2} \, \|1/z_\alpha\|_{L^\infty}^{1/2} \right) \\
&\quad \left\{ \left(\int \frac{1}{\mathfrak{a}} \left| \mathfrak{a} \overline{z}_\alpha D_\alpha \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \right|^2 d\alpha \right)^{1/2} + \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} \left(\int |D_\alpha \overline{z}_{tt}|^2 \frac{d\alpha}{\mathfrak{a}} \right)^{1/2} \right\},
\end{aligned}
\tag{583}
$$

---

[56]To do this, we need the a priori assumption that $\|D_\alpha(\mathfrak{a}_t \overline{z}_\alpha)\|_{L^\infty} < \infty$, since this is required for Peter-Paul. This holds so long as $D_\alpha(\overline{z}_{ttt} + i\mathfrak{a} \overline{z}_{t\alpha}) \in L^\infty$.

where we controlled

$$\left(\int \frac{1}{\mathfrak{a}} \left| \mathfrak{a} \bar{z}_\alpha D_\alpha \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \right|^2 d\alpha \right)^{1/2} \lesssim (471) \qquad (584)$$

and

$$\left\| (I - \mathbb{H}) \left(D_\alpha^2 (\mathfrak{a}_t \bar{z}_\alpha) \right) \circ h^{-1} \right\|_{L^2} \lesssim (576). \qquad (585)$$

By (290), combining estimates in §6, §7, and §8 with (475) and (583), we conclude that $\frac{d}{dt} E_a$ is bounded by a universal polynomial of E, $\|z_\alpha\|_{L^\infty}$ and $\|1/z_{\alpha L^\infty}\|$. This concludes the proof of Theorem 25. $\qquad \square$

Chapter 11

The Strength of the Energy

Our energy is expressed in terms of not only the free surface Z, the velocity Z_t and their spatial derivatives, but also time derivatives of these quantities. In this chapter, we give a characterization of our energy in terms of the free surface Z, the velocity Z_t, and their spatial derivatives. We do so in Riemannian variable, as it directly captures the geometry of the interface Z. We also, in §11.2, offer a heuristic discussion of the singularities inherent in the problem and the crest angles allowed by our energy, as indicated by the Riemann mapping.

11.1 A Characterization of the Energy

In this section, we translate the terms of our energy involving time derivatives[57] into terms depending only on the free surface Z, the velocity Z_t, and their spatial derivatives. We do this using the basic identity (183)

$$\frac{1}{Z_{,\alpha'}} = i \frac{\overline{Z}_{tt} - i}{A_1},\tag{586}$$

(180), and the holomorphicity of $\overline{Z}_t$ and various other quantities discussed in §3.6. These basic water wave equations allow us to show that quantities involving $\overline{Z}_{tt}$ can be controlled by analogous quantities involving $\frac{1}{Z_{,\alpha'}}$, along with various lower-order terms.[58]

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[57]We could call these the "acceleration" terms, although often the time derivative is outside the spatial derivatives, and the "acceleration" Z_{tt} is really the material derivative of the velocity, rather than the time derivative.

[58]We remark that for these estimates we *do not* ever rely on (high order) $\dot{H}^{1/2}$ parts of the energies.

The estimate we prove is

$$E(t) \leqslant C\bigg(\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}, \left\|D_{\alpha'}^2 \overline{Z}_t\right\|_{L^2}, \left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2},$$

$$\left\|D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\right\|_{L^2}, \left\|\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t\right\|_{\dot{H}^{1/2}}, \left\|D_{\alpha'} \overline{Z}_t\right\|_{\dot{H}^{1/2}}, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\bigg), \quad (587)$$

where the constant depends polynomially on its terms. We remark that this inequality can be reversed: each of the factors on the RHS of (587) is controlled by the energy. That is,

$$\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}, \left\|D_{\alpha'}^2 \overline{Z}_t\right\|_{L^2}, \left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2}, \left\|D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\right\|_{L^2}, \left\|\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t\right\|_{\dot{H}^{1/2}},$$

$$\left\|D_{\alpha'} \overline{Z}_t\right\|_{\dot{H}^{1/2}}, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \lesssim C(E(t)). \quad (588)$$

Therefore, these quantities fully characterize our energy. In the proof of our a priori estimate, we have shown (588) for every term except $\left\|D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\right\|_{L^2}$, which we never had a need to control. One can adapt the argument in §11.1.3 below to show that $\left\|D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\right\|_{L^2}$ can be controlled by the energy.[59]

We remark because both $\overline{Z}_t$ and $\frac{1}{Z_{,\alpha'}}$ are the boundary values of periodic holomorphic functions, the weighted derivative $D_{\alpha'}$ corresponds to the complex derivative ∂_z. We also note that $\frac{1}{Z_{,\alpha'}} = \frac{h_\alpha}{z_\alpha} \circ h^{-1}$ is a natural geometric quantity well-suited to this problem: it captures the free surface z_α *modulo* the singularities from the Riemann mapping h_α.

---

[59]To do this, it comes down once again to estimating $\left\|\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 A_1\right\|_{L^2}$, except this time we need to do this without the dependence on $\left\|D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\right\|_{L^2}$. That dependence comes from estimate (612). (It also comes from using $\left\|D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\right\|_{L^2}$ in the Sobolev inequality for $\left\|D_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}$; this is not a problem, since $\left\|D_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}$ is controlled by the energy.) To handle (612), we take advantage of the fact that $(I - \mathbb{H})\left\{\partial_{\alpha'} D_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\} = 0$ (this is due to (131), (137) and the second principle in §3.6) to rewrite the problematic term as a commutator and then use commutator estimate (233):

$$\left\|(I - \mathbb{H})\left\{\frac{A_1}{Z_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\}\right\|_{L^2} = \left\|\left[\overline{Z}_{tt}, \mathbb{H}\right] \partial_{\alpha'} D_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2}$$

$$\lesssim \left\|\overline{Z}_{tt,\alpha'}\right\|_{L^2} \left\|D_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}, \quad (589)$$

both of which are controlled by the energy. We remark that this, sharper argument shows that $\left\|\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 A_1\right\|_{L^2}$ is in some sense lower-order, since, by the Sobolev inequality with ε, $\left\|D_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}$ depends on the highest-order terms only with a weight of ε.

11.1.1 The Proof

Throughout the following proof we will rely on the fact that $A_1 \geqslant 1$ (182), the estimate (297)

$$\|A_1\|_{L^\infty} \lesssim 1 + \|\overline{Z}_{t,\alpha'}\|_{L^2}^2 , \tag{590}$$

the Sobolev estimate (300)

$$\|D_{\alpha'}\overline{Z}_t\|_{L^\infty} \lesssim \|\overline{Z}_{t,\alpha'}\|_{L^2} + \|D_{\alpha'}^2\overline{Z}_t\|_{L^2} , \tag{591}$$

and the estimate

$$\left\|D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \lesssim \left\|D_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} , \tag{592}$$

which holds by Sobolev inequality (203).[60]

We begin by noting that it suffices to control only the first term of E_a and E_b, since the remaining terms of the energy are (up to a factor of A_1) already on the RHS of (587).

For the first term of E_a, by the commutator identity (258)

$$
\begin{aligned}
\int |\partial_t D_\alpha^2 \overline{z}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha &\lesssim \int |D_\alpha^2 \overline{z}_{tt}|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha + \int |[\partial_t, D_\alpha^2]\overline{z}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \\
&\lesssim \|D_{\alpha'}^2\overline{Z}_{tt}\|_{L^2}^2 + \int |(D_\alpha z_t)D_\alpha^2\overline{z}_t + (D_\alpha^2 z_t)D_\alpha\overline{z}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \\
&\lesssim \|D_{\alpha'}^2\overline{Z}_{tt}\|_{L^2}^2 + \|D_\alpha z_t\|_{L^\infty}^2 (\|D_{\alpha'}^2\overline{Z}_t\|_{L^2}^2 + \|D_{\alpha'}^2 Z_t\|_{L^2}^2).
\end{aligned}
\tag{593}
$$

By (305) and (212)

$$
\begin{aligned}
\|D_{\alpha'}^2 Z_t\|_{L^2} &\lesssim \|D_{\alpha'}^2\overline{Z}_t\|_{L^2} + \|D_{\alpha'}\overline{Z}_t\|_{L^\infty} \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \\
&\lesssim C\left(\|D_{\alpha'}^2\overline{Z}_t\|_{L^2}, \|\overline{Z}_{t,\alpha'}\|_{L^2}, \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\right).
\end{aligned}
\tag{594}
$$

We conclude that

$$\int |\partial_t D_\alpha^2 \overline{z}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \lesssim C\left(\|D_{\alpha'}^2\overline{Z}_{tt}\|_{L^2}, \|\overline{Z}_{t,\alpha'}\|_{L^2}, \|D_{\alpha'}^2\overline{Z}_t\|_{L^2}, \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\right). \tag{595}$$

---

[60]Note that $\fint \left(D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)^2 = 0$ by the same argument that was used at (166) to show $\fint (D_{\alpha'}\overline{Z}_t)^2 = 0$.

For the first term of E_b, we use the commutator identity (257) to get

$$
\begin{aligned}
\int |\partial_t D_\alpha \bar{z}_t|^2 \frac{1}{\mathfrak{a}} d\alpha &\lesssim \int |D_\alpha \bar{z}_{tt}|^2 \frac{1}{\mathfrak{a}} d\alpha + \int |[\partial_t, D_\alpha]\bar{z}_t|^2 \frac{1}{\mathfrak{a}} d\alpha \\
&\lesssim \int |D_\alpha \bar{z}_{tt}|^2 \frac{(A_1 \circ h)}{\mathfrak{a}} d\alpha + \int |(D_\alpha z_t) D_\alpha \bar{z}_t|^2 \frac{1}{\mathfrak{a}} d\alpha \\
&\lesssim \left\| \bar{Z}_{tt,\alpha'} \right\|_{L^2}^2 + \|D_\alpha z_t\|_{L^\infty}^2 \int |D_\alpha \bar{z}_t| \frac{(A_1 \circ h)}{\mathfrak{a}} d\alpha \\
&\lesssim \left\| \bar{Z}_{tt,\alpha'} \right\|_{L^2}^2 + (\left\| \bar{Z}_{t,\alpha'} \right\|_{L^2} + \left\| D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2})^2 \left\| \bar{Z}_{t,\alpha'} \right\|_{L^2}^2 \\
&\leqslant C(\left\| \bar{Z}_{tt,\alpha'} \right\|_{L^2}, \left\| \bar{Z}_{t,\alpha'} \right\|_{L^2}, \left\| D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2}).
\end{aligned}
\tag{596}
$$

All that remains to do from (595) and (596) is to estimate $\left\| \bar{Z}_{tt,\alpha'} \right\|_{L^2}$ and $\left\| D_{\alpha'}^2 \bar{Z}_{tt} \right\|_{L^2}$ in terms of Z_t and $\frac{1}{Z_{,\alpha'}}$, which we now do, in §11.1.2 and §11.1.3.

11.1.2 Controlling $\left\| \bar{Z}_{tt,\alpha'} \right\|_{L^2}$

Using (586), we estimate

$$
\begin{aligned}
\left\| \partial_{\alpha'} \bar{Z}_{tt} \right\|_{L^2} &\lesssim \|A_1\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} + \|D_{\alpha'} A_1\|_{L^2} \\
&\lesssim C\left(\left\| \bar{Z}_{t,\alpha'} \right\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}, \|D_{\alpha'} A_1\|_{L^2} \right).
\end{aligned}
\tag{597}
$$

To control $\|D_{\alpha'} A_1\|_{L^2}$, we follow a similar procedure to what we did in (331)-(332), except instead of using $\bar{Z}_{tt} - i$, we use $\frac{1}{Z_{,\alpha'}}$ and estimate things in terms of $\frac{1}{Z_{,\alpha'}}$. We get

$$
\begin{aligned}
\|D_{\alpha'} A_1\|_{L^2} &\lesssim \left\| \bar{Z}_{t,\alpha'} \right\|_{L^2}^2 \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} + \|Z_{t,\alpha'}\|_{L^2} \left\| D_{\alpha'} \bar{Z}_t \right\|_{L^\infty} \\
&\leqslant C\left(\left\| \bar{Z}_{t,\alpha'} \right\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}, \left\| D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2} \right).
\end{aligned}
\tag{598}
$$

Combining (597) and (598) we conclude that

$$
\left\| \partial_{\alpha'} \bar{Z}_{tt} \right\|_{L^2} \leqslant C\left(\left\| \bar{Z}_{t,\alpha'} \right\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}, \left\| D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2} \right).
\tag{599}
$$

11.1.3   Controlling $\left\|D^2_{\alpha'}\overline{Z}_{tt}\right\|_{L^2}$

From (586), we have

$$iD^2_{\alpha'}\overline{Z}_{tt} = \underbrace{A_1 D^2_{\alpha'}\frac{1}{Z_{,\alpha'}} + 2(D_{\alpha'}A_1)D_{\alpha'}\frac{1}{Z_{,\alpha'}}}_{e_1} + \frac{1}{Z_{,\alpha'}}D^2_{\alpha'}A_1. \tag{600}$$

We estimate $\left\|D^2_{\alpha'}\overline{Z}_{tt}\right\|_{L^2}$ through the following procedure. First we note that the only challenging term to control on the RHS of (600) is the last one, $\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}A_1$. We observe that this is almost real, modulo factors of $\frac{1}{Z_{,\alpha'}}$ and its derivatives; therefore, we will be able to use the $\Re(I-\mathbb{H})$ trick and, through a series of commutators, reduce the estimate for $\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}A_1$ to an estimate of $(I-\mathbb{H})(D^2_{\alpha'}\frac{A_1}{Z_{,\alpha'}}) = (I-\mathbb{H})(iD^2_{\alpha'}\overline{Z}_{tt})$. Since $\overline{Z}_t$ is holomorphic, we will be able to rewrite $(I-\mathbb{H})(iD^2_{\alpha'}\overline{Z}_{tt})$ in terms of commutators and obtain favorable estimates. We now give the details.

We first estimate the error term e_1 in (600):

$$
\begin{aligned}
\|e_1\|_{L^2} &\lesssim \|A_1\|_{L^\infty}\left\|D^2_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \|D_{\alpha'}A_1\|_{L^2}\left\|D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \\
&\lesssim (1+\|\overline{Z}_{t,\alpha'}\|^2_{L^2})\left\|D^2_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + (598)\left(\left\|D^2_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\right) \\
&\lesssim C\left(\|\overline{Z}_{t,\alpha'}\|_{L^2},\left\|D^2_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2},\|D^2_{\alpha'}\overline{Z}_t\|_{L^2},\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\right).
\end{aligned} \tag{601}
$$

It remains to control $\left\|\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}A_1\right\|_{L^2}$. We want to use $(I-\mathbb{H})$ to turn our quantity into commutators, but to do so we need to factor $D_{\alpha'}$ into a real-weighted derivative $|D_{\alpha'}| := \frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}$ so that we may invert $(I-\mathbb{H})$.

From (483), we have

$$\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}A_1 = \frac{1}{Z_{,\alpha'}}\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)^2 |D_{\alpha'}|^2 A_1 + \underbrace{\frac{1}{Z_{,\alpha'}}\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\left(|D_{\alpha'}|\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)|D_{\alpha'}|A_1}_{e_2}. \tag{602}$$

We multiply both sides by $\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\right)^3$ so that the first term on the RHS is purely real:

$$\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\right)^3\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}A_1 = \left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\right)^3\frac{1}{Z_{,\alpha'}}\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)^2|D_{\alpha'}|^2 A_1 + \left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\right)^3 e_2. \tag{603}$$

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Now we apply $\Re(I - \mathbb{H})$ to each side, and conclude from the fact that $A_1 \in \mathbb{R}$ that

$$\left| \frac{1}{Z_{,\alpha'}} |D_{\alpha'}|^2 A_1 \right| \lesssim \left| (I - \mathbb{H}) \left\{ \left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right)^3 \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 A_1 \right\} \right| + \left| (I - \mathbb{H}) \left\{ \left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right)^3 e_2 \right\} \right|. \quad (604)$$

We conclude from (602) and (604) that

$$\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 A_1 \right\|_{L^2} \lesssim \|e_2\|_{L^2} + \left\| (I - \mathbb{H}) \left\{ \left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right)^3 \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 A_1 \right\} \right\|_{L^2}. \quad (605)$$

By (212) and (598) we estimate

$$\|e_2\|_{L^2} \lesssim \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \|D_{\alpha'} A_1\|_{L^2} \lesssim (592)(598). \quad (606)$$

It remains to estimate $\left\| (I - \mathbb{H}) \left\{ \left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right)^3 \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 A_1 \right\} \right\|_{L^2}$. To get the right commutator estimate, we first rewrite this as

$$\left\| (I - \mathbb{H}) \left\{ \left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right)^3 \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 A_1 \right\} \right\|_{L^2} \lesssim \left\| (I - \mathbb{H}) \left\{ \left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right)^3 \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right) \right\} \right\|_{L^2}$$
$$+ \left\| (I - \mathbb{H}) \left\{ \left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right)^3 \frac{1}{Z_{,\alpha'}} \left(\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) D_{\alpha'} A_1 \right\} \right\|_{L^2}. \quad (607)$$

We estimate the second term on the RHS of (607) directly:

$$\left\| (I - \mathbb{H}) \left\{ \left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right)^3 \frac{1}{Z_{,\alpha'}} \left(\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) D_{\alpha'} A_1 \right\} \right\|_{L^2} \lesssim \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \|D_{\alpha'} A_1\|_{L^2}$$
$$\lesssim (592)(598). \quad (608)$$

For the first term on the RHS of (607), we commute the non-holomorphic factor $\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right)^3$ outside the $(I - \mathbb{H})$, bringing along $\frac{1}{Z_{,\alpha'}}$ to ensure that the commutator is controllable, and

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then bringing the $\frac{1}{Z_{,\alpha'}}$ back inside:

$$\left\| (I - \mathbb{H}) \left\{ \left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right)^3 \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right) \right\} \right\|_{L^2}$$

$$\lesssim \left\| \left[\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right)^3 \frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right) \right\|_{L^2} \tag{609}$$

$$+ \left\| \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right) \right\|_{L^2} + \left\| (I - \mathbb{H}) \left\{ \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right) \right\} \right\|_{L^2}.$$

We estimate the first two terms on the RHS of (609) using commutator estimate $(233)^{61}$:

$$\left\| \left[\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right)^3 \frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right) \right\|_{L^2} + \left\| \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right) \right\|_{L^2}$$

$$\lesssim \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right\|_{L^\infty}. \tag{610}$$

We will postpone estimating $\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right\|_{L^\infty}$ until the end of this long series of calculations. For the moment, we take the last term from the RHS of (609):

$$\left\| (I - \mathbb{H}) \left\{ \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right) \right\} \right\|_{L^2} \lesssim \left\| (I - \mathbb{H}) D_{\alpha'}^2 \left(\frac{1}{Z_{,\alpha'}} A_1 \right) \right\|_{L^2}$$

$$+ \left\| (I - \mathbb{H}) \left\{ D_{\alpha'} \left(A_1 D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\} \right\|_{L^2}. \tag{611}$$

We estimate the second term by

$$\left\| (I - \mathbb{H}) \left\{ D_{\alpha'} \left(A_1 D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\} \right\|_{L^2} \lesssim \left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \| A_1 \|_{L^\infty} + \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \| D_{\alpha'} A_1 \|_{L^2}$$

$$\lesssim \left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \left(1 + \left\| \overline{Z}_{t,\alpha'} \right\|_{L^2}^2 \right) + (592)(598). \tag{612}$$

Finally, for the first term on the RHS of (611), we recall from (175) that

$$\frac{1}{Z_{,\alpha'}} A_1 = i(\overline{Z}_{tt} - i) = i(Z_t D_{\alpha'} \overline{Z}_t) + i F_t \circ Z + 1 \tag{613}$$

---

[61] Note that this does not require that $\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1$ has periodic boundary behavior; this explains why we chose this commutator and moved $\frac{1}{Z_{,\alpha'}}$ around instead of using the $L^\infty \times L^2$ estimate. Cf. footnote 51.

for $F(z(\alpha, t), t) := \overline{z}_t(\alpha, t)$. We apply $(I - \mathbb{H})D_{\alpha'}^2$ to this, and the last two terms disappear, the first by (151). We get

$$\left\|(I - \mathbb{H})D_{\alpha'}^2 \left(\frac{1}{Z_{,\alpha'}} A_1\right)\right\|_{L^2} = \left\|(I - \mathbb{H})D_{\alpha'}^2 (Z_t D_{\alpha'} \overline{Z}_t)\right\|_{L^2}$$
$$\lesssim \left\|(I - \mathbb{H})\left\{(D_{\alpha'}^2 Z_t) D_{\alpha'} \overline{Z}_t\right\}\right\|_{L^2}$$
$$+ \left\|(I - \mathbb{H})\left\{(D_{\alpha'} Z_t)(D_{\alpha'}^2 \overline{Z}_t)\right\}\right\|_{L^2} \qquad (614)$$
$$+ \left\|(I - \mathbb{H})\left\{\frac{Z_t}{Z_{,\alpha'}} \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right\}\right\|_{L^2}.$$

We estimate the first two terms directly by

$$\left\|(I - \mathbb{H})\left\{(D_{\alpha'}^2 Z_t) D_{\alpha'} \overline{Z}_t\right\}\right\|_{L^2} + \left\|(I - \mathbb{H})\left\{(D_{\alpha'} Z_t)(D_{\alpha'}^2 \overline{Z}_t)\right\}\right\|_{L^2}$$
$$\lesssim \left\|D_{\alpha'} \overline{Z}_t\right\|_{L^\infty} \left(\left\|D_{\alpha'}^2 Z_t\right\|_{L^2} + \left\|D_{\alpha'}^2 \overline{Z}_t\right\|_{L^2}\right) \qquad (615)$$
$$\lesssim (591)(\left\|D_{\alpha'}^2 \overline{Z}_t\right\|_{L^2} + (594)).$$

We are left with the last term on the RHS of (614). We first decompose $\frac{Z_t}{Z_{,\alpha'}}$ into its holomorphic and antiholomorphic projections. The term with the holomorphic projection disappears by (159); with what remains, we use (143) to get a commutator, which we control by commutator estimate (226):

$$\left\|(I - \mathbb{H})\left\{\frac{Z_t}{Z_{,\alpha'}} \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right\}\right\|_{L^2} = \left\|(I - \mathbb{H})\left\{\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right) \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right\}\right\|_{L^2}$$
$$= \left\|\left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right), \mathbb{H}\right] \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right\|_{L^2}$$
$$\lesssim \left\|\partial_{\alpha'} \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}\right\|_{L^\infty} \left\|D_{\alpha'}^2 \overline{Z}_t\right\|_{L^2}$$
$$\lesssim \left(\left\|D_{\alpha'} Z_t\right\|_{L^\infty} + \left\|Z_{t,\alpha'}\right\|_{L^2} \left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2}\right) \left\|D_{\alpha'}^2 \overline{Z}_t\right\|_{L^2}$$
$$\lesssim \left(\left\|D_{\alpha'}^2 \overline{Z}_t\right\|_{L^2} + \left\|Z_{t,\alpha'}\right\|_{L^2} \left(1 + \left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2}\right)\right) \left\|D_{\alpha'}^2 \overline{Z}_t\right\|_{L^2}$$
$$\qquad (616)$$

by (372).

We now give the estimate for $\left\|\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1\right\|_{L^\infty} = \left\|\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} A_1\right\|_{L^\infty}$ in (610). We do so

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using (331). We have

$$
\begin{aligned}
\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 &= \Im\frac{1}{2i}\int\frac{\pi}{2}\frac{(Z_t(\alpha')-Z_t(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha'-\beta'))}\left(\frac{1}{|Z_{,\alpha'}|^2}-\frac{1}{|Z_{,\beta'}|^2}\right)\overline{Z}_{t,\beta'}(\beta')d\beta \\
&+ \Im\frac{1}{2i}\int\frac{\pi}{2}\frac{(Z_t(\alpha')-Z_t(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha'-\beta'))}\frac{1}{\overline{Z}_{,\beta'}}D_{\beta'}\overline{Z}_t(\beta')d\beta \\
&= I + II.
\end{aligned}
\tag{617}
$$

From Hölder's inequality, Hardy's inequality (217) and the mean value theorem,[62] we have

$$
\|I\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \left\|D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \ (592).
\tag{618}
$$

We rewrite II using integration-by-parts identity (227):

$$
\begin{aligned}
\frac{1}{2i}\int\frac{\pi}{2}&\frac{(Z_t(\alpha')-Z_t(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha'-\beta'))}\frac{1}{\overline{Z}_{,\beta'}}D_{\beta'}\overline{Z}_t(\beta')d\beta \\
&= -[Z_t,\mathbb{H}]\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}D_{\alpha'}\overline{Z}_t\right) + \mathbb{H}\left(Z_{t,\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}D_{\alpha'}\overline{Z}_t\right) \\
&= -[Z_t,\mathbb{H}]\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}D_{\alpha'}\overline{Z}_t\right) - \left[\frac{1}{\overline{Z}_{,\alpha'}}\overline{D_{\alpha'}\overline{Z}_t},\mathbb{H}\right]\overline{Z}_{t,\alpha'} + \left|D_{\alpha'}\overline{Z}_t\right|^2.
\end{aligned}
\tag{619}
$$

Using (244) on the first two terms on the RHS of (619), we get

$$
\begin{aligned}
\|II\|_{L^\infty} &\lesssim \|Z_{t,\alpha'}\|_{L^2}\left\|\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}D_{\alpha'}\overline{Z}_t\right)\right\|_{L^2} + \left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty}^2 \\
&\lesssim \|Z_{t,\alpha'}\|_{L^2}\left(\left\|D_{\alpha'}^2\overline{Z}_t\right\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty}\right) + \left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty}^2.
\end{aligned}
\tag{620}
$$

Combining (617), (618), (620), (592), and (591), we have

$$
\begin{aligned}
\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_{L^\infty} &\leqslant \|I\|_{L^\infty} + \|II\|_{L^\infty} \\
&\leqslant C\left(\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}, \left\|D_{\alpha'}^2\overline{Z}_t\right\|_{L^2}, \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}, \left\|D_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\right).
\end{aligned}
\tag{621}
$$

---

[62]Note that we may use the periodicity of $\frac{1}{|Z_{,\alpha'}|^2}$ to assume that $|\alpha'-\beta'|\leqslant 1$ and therefore replace the $\frac{1}{\sin}$ with $\frac{1}{\alpha'-\beta'}$ to apply the mean value theorem.

We now sum up these estimates, from (605):

$$\left\|\frac{1}{Z_{,\alpha'}}D_{\alpha'}^2 A_1\right\|_{L^2} \lesssim (606) + (608) + (610) + (612) + (615) + (616)$$

$$\lesssim (592)(598) + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left\|\frac{1}{Z_{,\alpha'}}D_{\alpha'}A_1\right\|_{L^\infty}$$

$$+ \left\|D_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2}(1 + \|\overline{Z}_{t,\alpha'}\|_{L^2}^2) \tag{622}$$

$$+ (591)(\|D_{\alpha'}^2\overline{Z}_t\|_{L^2} + (594))$$

$$+ \left(\|D_{\alpha'}^2\overline{Z}_t\|_{L^2} + \|Z_{t,\alpha'}\|_{L^2}\left(1 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\right)\right)\|D_{\alpha'}^2\overline{Z}_t\|_{L^2}$$

$$\lesssim C\left(\left\|D_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2}, \|\overline{Z}_{t,\alpha'}\|_{L^2}, \|D_{\alpha'}^2\overline{Z}_t\|_{L^2}, \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\right),$$

where we used (621) to estimate $\left\|\frac{1}{Z_{,\alpha'}}D_{\alpha'}A_1\right\|_{L^\infty}$.

Combining (601) and (622) we conclude that

$$\|D_{\alpha'}^2\overline{Z}_{tt}\|_{L^2} \lesssim C\left(\left\|D_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2}, \|\overline{Z}_{t,\alpha'}\|_{L^2}, \|D_{\alpha'}^2\overline{Z}_t\|_{L^2}, \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\right). \tag{623}$$

11.2 Singularities and the Angle of the Crest

In §11.1 at (587) and (588) we characterized our energy in terms of various L^2 and $\dot{H}^{1/2}$ norms of quantities in Riemannian coordinates (as well as a single quantity, $\frac{1}{Z_{,\alpha'}}$, in L^∞). When there is a non-trivial angle ν at the corner, or when there is a singularity in the middle of the free surface, the Riemann mapping will have a singularity. In this section, we discuss this singularity and what it suggests about the angle ν, as well as what the singularity suggests about what types of quantities we can expect to control. This section is a *heuristic* discussion relying on crude power-law asymptotics; we emphasize that all other parts of the dissertation are fully rigorous, and do not depend on this discussion.

We will phrase our discussion in terms of a singularity at the corner—i.e., what we have called a non-trivial angle—but it applies more broadly to singularities in the middle of the free surface, where ν is half of the interior angle of the crest. We thus henceforth focus on non-trivial angles ν at the corner. Of course, our energy is finite in the trivial regime when $\nu = \frac{\pi}{2}$, so we can focus on the case where $\nu < \frac{\pi}{2}$.[63]

---

[63]Recall from the discussion at (183) that we cannot have $\nu > \frac{\pi}{2}$.

Throughout this section, we will abuse notation and say, e.g., $\Phi(z) \approx z^r$ or $h(\alpha) \approx \alpha^r$ at the corner, when in fact the corners are at $z, \alpha = \pm 1$, not 0.

If ν is the angle of the water at the corner, the Riemann mapping $\Phi(z)$ should behave like z^r at the corner, where $r\nu = \frac{\pi}{2}$. For $\nu < \frac{\pi}{2}$, we have $r > 1$. (See Figure 3 from §3.1.)

Recall from (587) and (588) that $\left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2}$ and $\left\|D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\right\|_{L^2}$ help characterize the energy, and so in particular if the energy is finite they must be finite. This gives us a more precise characterization of what angles ν our energy allows. Note that

$$Z(\alpha') = \Phi^{-1}(\alpha') \approx (\alpha')^{1/r} \tag{624}$$

so

$$Z_{,\alpha'} = \partial_{\alpha'}(\Phi^{-1}) \approx (\alpha')^{1/r-1}, \tag{625}$$

$$\frac{1}{Z_{,\alpha'}} \approx (\alpha')^{1-1/r}, \tag{626}$$

and

$$\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \approx (\alpha')^{-1/r} \quad (r \neq 1). \tag{627}$$

Therefore, assuming $r > 1$, $\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \in L^2$ if and only if $r > 2$ if and only if $\nu < \frac{\pi}{4}$. Similarly, $D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \in L^2$ so long as $r > 2$.[64]

We conclude from this discussion that our energy will be finite only when $\nu < \frac{\pi}{4}$ (or there is flat water $\nu = \frac{\pi}{2}$). This coincides precisely with the angles in the self-similar construction of [Wu12].[65] For singularities in the middle of the free surface, this suggests that the interior angle 2ν must be less than $\frac{\pi}{2}$.[66]

The preceding discussion depends only $\frac{1}{Z_{,\alpha'}}$, which is defined by the Riemann mapping. Now we consider other quantities, comparing their behavior in Lagrangian and Riemann coordinates. For this discussion, we work under the assumption that $0 < c_1 \leqslant |z_\alpha| \leqslant c_2 < \infty$ for all t in our timeframe, so the function $z(\alpha)$ doesn't affect the singularity (to first order). We can work under this assumption by the control of $\|z_\alpha\|_{L^\infty}$ and $\|1/z_\alpha\|_{L^\infty}$ from §6.2.

Since $z \approx \alpha$, we have

$$h(\alpha) = \Phi(z(\alpha)) \approx \alpha^r \tag{628}$$

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[64]We remark that, even though E_a (which roughly includes $\left\|D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\right\|_{L^2}$) is higher-order than E_b (which roughly includes $\left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2}$) in terms of the number of derivatives, the two energies are comparable in the sense that they allow precisely the same angles.

[65]We recall that our energy is finite for these solutions.

[66]We note that our energy does not apply to Stokes waves (interior angle $2\nu = \frac{\pi}{3}$).

and

$$h^{-1}(\alpha') \approx (\alpha')^{1/r}. \tag{629}$$

On differentiating (628), we get that

$$h_\alpha(\alpha) \approx \alpha^{r-1} \tag{630}$$

and so

$$h_\alpha \circ h^{-1}(\alpha') \approx (\alpha')^{\frac{1}{r}(r-1)} = (\alpha')^{1-\frac{1}{r}}. \tag{631}$$

Now we can see the impact of the Riemann mapping change of coordinates on derivatives. For the derivatives in Riemannian coordinates,

$$\partial_{\alpha'}(f \circ h^{-1}) = \frac{f_\alpha}{h_\alpha} \circ h^{-1}. \tag{632}$$

The singularity introduced by the denominator will be of the order of $(\alpha')^{\frac{1}{r}-1}$. This means that unless $f_\alpha \to 0$ at the corner, we cannot expect $\partial_{\alpha'}(f \circ h^{-1})$ to be in L^∞ for non-trivial angles.

On the other hand, the weighted spatial derivative $D_{\alpha'}$ does not introduce any such singularity:

$$D_{\alpha'}(f \circ h^{-1}) = \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}(f \circ h^{-1}) = \frac{f_\alpha}{z_\alpha} \circ h^{-1}, \tag{633}$$

which should be in L^∞ as long as $f_\alpha \in L^\infty$.

We remark that the behavior of the angle ν over time is of significant interest. This angle is determined by z_α at the corner. Therefore, the behavior of $z_{t\alpha}$ and $z_{tt\alpha}$ at the corner should determine how the angle changes. Since $Z_{t,\alpha'} = \frac{z_{t\alpha}}{h_\alpha} \circ h^{-1}$ and $\frac{1}{h_\alpha} \circ h^{-1} \approx (\alpha')^{1/r-1}$, we must have $z_{t\alpha} \to 0$ at the corner if $Z_{t,\alpha'} \in L^2$, as our energy assumes. A similar argument holds for $Z_{tt,\alpha'}$. Careful analysis of such behavior at the corner could be an avenue for future research.

Appendix A

Summary of Notation

We list here the various notations we've introduced in the dissertation.

- $f|_\partial := f(1,t) - f(-1,t) = \lim_{x_1 \to -1^+} f(x_1, t) - \lim_{x_2 \to 1^-} f(x_2, t)$ (38). Note that this is always with respect to space.

- We will use ν for the angle the water wave makes with the wall. We will say the water is *flat* or the angle is *trivial* at the corner if $\nu = 90° = \frac{\pi}{2}$; otherwise, we say the angle at the corner is *non-trivial*. See §1.3 for further details, and §11.2 for a discussion of what our energy being finite implies about ν.

- $\Omega_0(t) \subset [0,1] \times (-\infty, c)$ is the initial fluid domain; on Schwarz reflection, we get fluid domain $\Omega(t) \subset [-1,1] \times (-\infty, c)$. (See §1.3.) This is mapped by the Riemann mapping to
$$P^- := \{(x,y) \;:\; x \in [-1,1], y \leqslant 0\}. \tag{634}$$

- $I := [-1,1]$ (except when it's used for the identity or as an abbreviation for a quantity to be controlled).

- For a complex number $z = x + iy$, $\Re z = x, \Im z = y$.

- Details about our function spaces $C^k(-1,1), C^k[-1,1], C^k(S^1)$ and L^p are given in §1.6. We define $\|f\|_{\dot{H}^{1/2}} := \left(\frac{\pi}{8} \iint \frac{|f(\alpha') - f(\beta')|^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} d\alpha' d\beta' \right)^{1/2}$ at (18).

- $\fint f = \fint_I f := \frac{1}{2} \int_I f(\beta') d\beta'$.

- I_ε and $(I \times I)_\varepsilon$ are defined at (20) and (21). We use this for our definition of the principal value at (22).

- $z(\alpha, t)$ is the Lagrangian parametrization, and $z_t(\alpha, t) = \mathbf{v}(z(\alpha, t), t)$ is the velocity. $z_{tt} = \partial_t z_t$, etc.

- $\mathfrak{a} = \frac{|\bar{z}_{tt} - i|}{|z_\alpha|} = \frac{-\partial P}{\partial \mathbf{n}} \frac{1}{|z_\alpha|}$ is the *Taylor coefficient*; $\mathbf{n}$ is the outward-facing normal to $\Omega(t)$. We will sometimes also refer to $\frac{-\partial P}{\partial \mathbf{n}}$ as the Taylor coefficient.

- $h : \alpha \mapsto \alpha'$ is defined by $h(\alpha) = \Phi(z(\alpha, t), t)$ and gives the Riemannian coordinates, where Φ is the Riemann mapping defined in §3.1. Full details are in §3.2.

- α and β are our variables in Lagrangian coordinates; α' and β' are our variables in Riemannian coordinates.

- $\mathbb{H}$ is the Hilbert transform in Riemannian coordinates, defined by

$$\mathbb{H}f(\alpha') := \frac{1}{2i} \operatorname{pv} \int_I \cot(\frac{\pi}{2}(\alpha' - \beta'))f(\beta')d\beta'. \tag{635}$$

We will occasionally refer to variants of the Hilbert transform (e.g., using $\mathfrak{H}$ to refer to a Hilbert transform for the curved domain in Lagrangian coordinates), and will provide details in the text.

- We define $\mathbb{P}_A := \frac{(I - \mathbb{H})}{2}$ and $\mathbb{P}_H := \frac{(I + \mathbb{H})}{2}$ as the antiholomorphic and holomorphic projections.

- $[f, g; h](\alpha') := \frac{1}{2i} \int \frac{(f(\alpha') - f(\beta'))(g(\alpha') - g(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} h(\beta')d\beta'$. We use this for the higher-order Calderon commutator.

- We use $F(z(\alpha, t), t) := \bar{z}_t(\alpha, t)$ at several points (and take care not to use F for any other purpose).

- $Z := z \circ h^{-1}, Z_t := z_t \circ h^{-1}, Z_{tt} := z_{tt} \circ h^{-1}, Z_{,\alpha'} = \partial_{\alpha'}(z \circ h^{-1}), Z_{t,\alpha'} = \partial_{\alpha'}(z_t \circ h^{-1})$, etc., where composition and inverses are with respect to the spatial variable.

- $\mathcal{A} := (\mathfrak{a} h_\alpha) \circ h^{-1}$.

- $\mathcal{A}_t := (\mathfrak{a}_t h_\alpha) \circ h^{-1}$.

- $D_\alpha := \frac{1}{z_\alpha}\partial_\alpha, |D_\alpha| := \frac{1}{|z_\alpha|}\partial_\alpha$.

- $D_{\alpha'} := \frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}, |D_{\alpha'}| := \frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}$.

- For generic function G on $\Omega(t)$, $G \circ Z := G(Z(\alpha', t), t)$.

- $A_1 = \mathcal{A}\, |Z_{,\alpha'}|^2 = iZ_{,\alpha'}(\overline{Z}_{tt} - i) \in \mathbb{R}$ (180). On changing variables, we have

$$A_1 \circ h = \frac{\mathfrak{a}\, |z_\alpha|^2}{h_\alpha}, \tag{636}$$

  originally derived at (184); we will use this repeatedly without citation. We will often use $A_1 \geqslant 1$ (182), and we also use $\frac{1}{Z_{,\alpha'}} = i\frac{\overline{Z}_{tt} - i}{A_1}$ (183).

- We define our energies in §5.1. We define generic energies $E_{a,\theta}$ and $E_{b,\theta}$, and then specialize to $E_a := E_{a, D_\alpha^2 \overline{z}_t}$ and $E_b := E_{b, D_\alpha \overline{z}_t}$. We use G_θ to describe the RHS of the equation $(\partial_t^2 + i\mathfrak{a}\partial_\alpha)\theta = G_\theta$. For $\theta = D_\alpha^k \overline{z}_t$, $G_\theta = D_\alpha^k(-i\mathfrak{a}_t \overline{z}_\alpha) + [\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha^k]\overline{z}_t$.

- $\psi := \left(\frac{h_\alpha}{z_\alpha}\theta\right) \circ h^{-1}$ (391).

- $B := \left(\frac{h_{t\alpha}}{h_\alpha} - \Re D_{\alpha'} z_t\right) \circ h^{-1}$ (392).

- $\Theta := \theta \circ h^{-1}$ (393).

- See §1.5 for a discussion of how broadly I, II, I_1, I_{12}, etc., are defined: in short, they are unambiguous within each chapter, but ambiguous between chapters. We list here the distinct places where they are defined: (332) in §6.4; (389) in §8.1; (512) in §10.2.3; and (617) in §11.1.3. Of these, the penultimate is most extensively subdivided, and is fully expanded in (567).

- In §10.2.2, we define a subdivision i, ii, iii at (511).

Appendix B

Main Quantities Controlled

We list here the various quantities that are controlled by our energy, for ease of reference. We don't list every single quantity we have controlled, but we do include any quantities that we give at the end of a concluding inequality without further explanation.

- In §6.1, we control $\|A_1\|_{L^\infty}$ by the lower-order term in E_b at (297). We know $A_1 \geqslant 1$ by (182).

- In §6.1, we use E_a to control $\left\|D_{\alpha'}^2 \overline{Z}_t\right\|_{L^2\left(\frac{d\alpha'}{A_1}\right)}$, $\left\|D_{\alpha'}^2 \overline{Z}_{tt}\right\|_{L^2\left(\frac{d\alpha'}{A_1}\right)}$, $\left\|D_{\alpha'}^2 Z_t\right\|_{L^2\left(\frac{d\alpha'}{A_1}\right)}$, and $\left\|D_{\alpha'}^2 Z_{tt}\right\|_{L^2\left(\frac{d\alpha'}{A_1}\right)}$, although except in the first case there is also dependence on E_b, which comes from commuting ∂_t and handling conjugations correctly.

- From this, with an extra factor of $\|A_1\|_{L^\infty}^{1/2}$, we can control $\left\|D_{\alpha'}^2 \overline{Z}_t\right\|_{L^2}$, $\left\|D_{\alpha'}^2 \overline{Z}_{tt}\right\|_{L^2}$, $\left\|D_{\alpha'}^2 Z_t\right\|_{L^2}$, and $\left\|D_{\alpha'}^2 Z_{tt}\right\|_{L^2}$.

- In §6.1, we control $\left\|\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t\right\|_{\dot{H}^{1/2}}$ directly by E_a at (299).

- In §6.1, we control $\|D_\alpha \overline{z}_t\|_{L^2\left(\frac{(A_1 \circ h)}{\mathfrak{a}}\right)} = \|D_\alpha z_t\|_{L^2\left(\frac{(A_1 \circ h)}{\mathfrak{a}}\right)} = \left\|\overline{Z}_{t,\alpha'}\right\|_{L^2} = \|Z_{t,\alpha'}\|_{L^2}$ directly by the lower-order term in E_b.

- In §6.1, we control $\|D_\alpha \overline{z}_{tt}\|_{L^2\left(\frac{1}{\mathfrak{a}}\right)} = \|D_\alpha z_{tt}\|_{L^2\left(\frac{1}{\mathfrak{a}}\right)}$, primarily by E_b, although there are other terms (which come from commuting ∂_t). By change of variables, we can control $\|Z_{tt,\alpha'}\|_{L^2} = \left\|\overline{Z}_{tt,\alpha'}\right\|_{L^2}$ in terms of this and $\|A_1\|_{L^\infty}$.

- In §6.1, we control

$$\|D_\alpha z_t\|_{L^\infty} = \|D_{\alpha'} Z_t\|_{L^\infty} = \|D_\alpha \overline{z}_t\|_{L^\infty} = \left\|D_{\alpha'} \overline{Z}_t\right\|_{L^\infty} \leqslant (301)$$

$$\|D_\alpha z_{tt}\|_{L^\infty} = \|D_{\alpha'} Z_{tt}\|_{L^\infty} = \|D_\alpha \bar{z}_{tt}\|_{L^\infty} = \left\|D_{\alpha'} \overline{Z}_{tt}\right\|_{L^\infty} \leqslant (315)$$

$$\|z_{tt} + i\|_{L^\infty} = \|Z_{tt} + i\|_{L^\infty} = \|\bar{z}_{tt} - i\|_{L^\infty} = \left\|\overline{Z}_{tt} - i\right\|_{L^\infty} \leqslant (313)$$

$$\left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \lesssim (314). \tag{637}$$

- $\|z_\alpha\|_{L^\infty}$ and $\left\|\frac{1}{z_\alpha}\right\|_{L^\infty}$ are controlled by initial assumption in (25) and then we use Gronwall to get absolute bound on these so long as $E < \infty$ in §6.2.

- $\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty} = \left\|\frac{A_t}{\mathcal{A}}\right\|_{L^\infty}$ is controlled at (327) in §6.3.

- $\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}$ is controlled at (336) in §6.4.

- $\left\|\frac{h_{t\alpha}}{h_\alpha}\right\|_{L^\infty}$ is controlled at (347) in §6.5.

- $\left\|\frac{(A_1 \circ h)_t}{A_1 \circ h}\right\|_{L^\infty}$ is controlled at (350) in §6.6.

- $\|(I + \mathbb{H})D_{\alpha'}Z_t\|_{L^\infty}$ is controlled at (351) in §6.7.

- $\left\|(I - \mathbb{H})D_{\alpha'}\overline{Z}_{tt}\right\|_{L^\infty}$ is controlled at (354) in §6.8.

- $\left\|D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}$ is controlled at (367) in §6.9. The related term $\left\|(Z_{tt} + i)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}$ is also estimated there.

- $\left\|\partial_{\alpha'}\mathbb{P}_A\frac{Z_t}{Z_{,\alpha'}}\right\|_{L^\infty}$ is controlled at (372) in §6.10. The related term $\left\|\mathbb{P}_A\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_{L^\infty}$ is estimated at (371).

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