Rigidity in Complex Projective Space

by

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For my parents.

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CHAPTER I

Introduction

1.1 Background and motivation

The starting point of this thesis is the question:

Question I.1. *How much symmetry can a convex open set $\Omega \subset \mathbb{R}^d$ have before it is a symmetric space?*

Perhaps the most natural type of symmetry of a set Ω in $\mathbb{R}^d$ is an *affine symmetry*, that is an affine isomorphism of $\mathbb{R}^d$ that leaves Ω invariant. Despite being a natural type of symmetry, it is fairly clear that the affine symmetries of a bounded set will never act transitively. To obtain a larger group of symmetries, we can embed $\mathbb{R}^d$ into $\mathbb{P}(\mathbb{R}^{d+1})$ as an affine chart and then consider the *projective symmetries* of Ω, that is the projective isomorphsims of $\mathbb{P}(\mathbb{R}^{d+1})$ that leave Ω invariant. Notice that the projective symmetries will contain the affine symmetries. Moreover, it is possible for the projective symmetries of a bounded set to act transitively. Take for instance the ball

$$\mathcal{B} = \{[1 : x_1 : \cdots : x_d] \in \mathbb{P}(\mathbb{R}^{d+1}) : \sum x_i^2 < 1\}$$

then $\mathrm{PSO}(1, d)$ acts transitively on $\mathcal{B}$.

With this motivation we now state all the formal definitions. A set $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is *convex* if it does not contain any projective lines and its intersection with any projective line is connected. A convex set Ω is called *proper* if $\overline{L \cap \Omega} \neq L$ for all projective lines L. Properness is a natural condition because of the following observation:

Observation I.2. Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is an open convex set. Then there exists an affine chart $\mathbb{R}^d$ which contains Ω and in this chart Ω is of the form $\Omega' \times \mathbb{R}^k$ where $\Omega' \subset \mathbb{R}^{d-k}$ is an open bounded convex set. Moreover, if Ω is an open proper convex set then we can assume $k = 0$.

The symmetries or *automorphism group* of a convex set Ω is the group

$$\mathrm{Aut}(\Omega) = \{g \in \mathrm{PSL}(\mathbb{R}^{d+1}) : g\Omega = \Omega\}.$$

A proper convex open set Ω is called *symmetric* if $\mathrm{Aut}(\Omega)$ is a semi-simple Lie group and acts transitively on Ω. The symmetric convex sets were completely characterized by Koecher (see for instance [FK94, Koe99, Vin63]).

There are at least four natural notions of a "big" automorphism group:

1. Ω is *homogeneous*, that is $\mathrm{Aut}(\Omega)$ acts transitively on Ω. In this case Vinberg [Vin65] provided a complete classification and there are non-symmetric examples (see also [Rot66]).

2. Ω is *divisible*, that is there exists a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ which acts co-compactly on Ω. From the classification of homogeneous convex sets, every divisible homogeneous convex set is actually symmetric.

3. Ω is *quasi-divisible* which is obtained by relaxing the definition of divisible from co-compactness to finite volume (a proper convex set Ω has an intrinsic $\mathrm{Aut}(\Omega)$-invariant volume). See [Mar10, Mar12a, Mar12b] for examples and some properties of these sets.

4. Ω is *quasi-homogeneous* which means that there exists a compact set $K \subset \Omega$ such that $\mathrm{Aut}(\Omega)K = \Omega$. The automorphism group of every quasi-homogeneous strictly convex set is actually discrete and thus every quasi-homogeneous strictly convex set is actually divisible (see [Ben03, Corollary 4.3] or [Jo03, Proposition 5.15]). In fact, it appears that there are no known examples of quasi-homogeneous sets which are not homogenous or divisible.

From a geometric point of view the divisible case is the most interesting. In fact, if Ω is divisible then there exists a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ such that Γ acts properly discontinuously, freely, and co-compactly on Ω. Thus the quotient $\Gamma \backslash \Omega$ has a C^∞ manifold structure, moreover this manifold structure is compatible with a real projective structure. We will not pursue the geometric structures perspective in this thesis and refer the reader to Goldman's expository article on real projective manifolds [Gol09].

The divisible convex sets should also be the most accessible to study. In particular one could hope to apply techniques from discrete groups of Lie groups, Riemannian geometry (there is a natural Finsler metric), and geometric group theory.

For the two reasons above, the study of divisible convex sets have a rich history going back to Benzecri's [Ben55] work in the 1950's. Perhaps the most fundamental result is that there are many divisible sets, in particular:

Theorem I.3. *For any $d \geq 4$ there exists a proper convex set $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ which is divided by a discrete group Γ. Moreover,*

1. Ω is not projectively equivalent to the ball,

2. Γ is not quasi-isometric to any symmetric space,

3. Γ is word hyperbolic,

4. Ω is strictly convex,

5. $\partial\Omega$ is $C^{1+\alpha}$ for some $\alpha > 0$.

For $d = 2$ or $d = 3$ there exists a proper convex set $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ which is divided by a discrete group Γ satisfying (1), (3), (4), and (5).

This theorem as stated combines several results. First Gromov and Thurston constructed compact negatively curved Riemannian manifolds whose fundamental groups are not quasi-isometric to any symmetric space [GT87]. M. Kapovich later showed that some of these manifolds have a compatible convex projective structure [Kap07]. This implies (1), (2), and (3). Finally a result of Benoist [Ben04, Theorem 1.1] implies conditions (4) and (5).

There are many other examples of divisible sets. In low dimensions it is possible to give explicit constructions (see for instance [Ben06] or [VK67]). Further examples can be constructed by deformations. For instance, using the Klein model of hyperbolic geometry every compact hyperbolic manifold has a real projective structure. Goldman [Gol90] gave coordinates on the moduli space of projective structures on a compact surface genus $g \geq 2$ surface and showed its dimension is $16g - 16$. Thus we see that there are more projective structures than hyperbolic ones. In higher dimensions, results of Johnson and Millson [JM87], Koszul [Kos68], and Thurston (see [Gol88, Theorem 3.1]) imply that some compact hyperbolic manifolds have a nontrivial moduli space of real projective structures. All of these deformations can be realized as a quotient of the form $\Gamma\backslash\Omega$ where $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is an open proper convex set and $\Gamma \leq \mathrm{PSL}(\mathbb{R}^{d+1})$ is a discrete group, thus leading to additional examples of divisible convex sets.

Thus the answer to Question I.1 is that convex sets can have a lot of symmetry before they are symmetric spaces. This leads to a class of discrete groups of $\mathrm{PSL}(\mathbb{R}^{d+1})$ that can be studied by understanding the geometry of set they act on. For motivation we will state three theorems along these lines. The first is geometric, the second is algebraic, and the third is dynamical.

Theorem I.4. *[Ben04, Theorem 1.1] Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a proper convex open set. If Ω is divisible by $\Gamma \leq \mathrm{PSL}(\mathbb{R}^{d+1})$ then*

$$\Omega \text{ is strictly convex} \Leftrightarrow \partial\Omega \text{ is } C^1 \Leftrightarrow \Gamma \text{ is word hyperbolic.}$$

Theorem I.5. *[Ben03] Let $\Gamma \leq \mathrm{PSL}(\mathbb{R}^{d+1})$ be a discrete group which divides some proper convex open set $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$. If Ω is irreducible and is not symmetric then Γ is Zariski dense in $\mathrm{PSL}(\mathbb{R}^{d+1})$.*

For the next statement, given $\gamma \in \mathrm{PSL}(\mathbb{R}^{d+1})$ let $s_{d+1}(\gamma) \geq \cdots \geq s_1(\gamma)$ be the singular values of γ.

Theorem I.6. *[Cra09] Let $\Gamma \leq \mathrm{PSL}(\mathbb{R}^{d+1})$ be a discrete group which divides some proper convex open set $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$. If Ω is strictly convex then*

$$h_\Gamma = \lim_{R\to\infty} \frac{1}{R} \log \#\{\gamma \in \Gamma : \log(s_{d+1}(\gamma)/s_1(\gamma)) \leq R\}$$

exists and is at most $d-1$. If $h_\Gamma = d-1$ then Ω is projectively isomorphic to the ball.

There are many more results about these sets and the groups that divide them. The interested reader could look at the recent survey articles by Benoist [Ben08], Quint [Qui10], and Marquis [Mar13].

A final motivation for studying divisible sets is their connection with special representations. A theorem of Benoist [Ben04, Theorem 1.1] implies that when $\Gamma \leq \mathrm{PSL}(\mathbb{R}^{d+1})$ is word hyperbolic and divides a convex set $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ then the inclusion map $\rho : \Gamma \hookrightarrow \mathrm{PSL}(\mathbb{R}^{d+1})$ is a *convex Anosov representation*. These are part of "Higher Teichmüller theory" and were introduced by Labourie [Lab06] as a generalization of convex-cocompact groups. The theory of Anosov representations was further developed by Guichard and Wienhard [GW12] (among many others).

1.2 Results of this thesis

As mentioned above, the theory of real divisible sets has had remarkable success in finding interesting examples of discrete groups in Lie groups and providing a geometry to study them. Thus it is natural to try to extend these ideas to find more groups and more geometries.

If we return to the original question of looking at the symmetries of a set $\Omega \subset \mathbb{R}^d$ and assume that d is even, then there is one other type of symmetry coming from the complex projective isomorphisms. More precisely, we can identify $\mathbb{R}^d$ with $\mathbb{C}^{d/2}$ and embed $\mathbb{C}^{d/2}$ into $\mathbb{P}(\mathbb{C}^{d/2+1})$ as an affine chart. Then we can consider the complex projective transformation which leave Ω invariant. This leads to the following question:

Question I.7. *How "big" can the symmetry group be of an open "convex" set $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ have before it is symmetric?*

In real projective geometry every two distinct points are connected by a natural one-dimensional subspace, namely the projective line containing them. This unique one-dimensional subspace provides an obvious definition of convexity (its intersection with Ω should be connected). In complex projective geometry, this is no longer the case. Instead the natural subspace containing two distinct points is one complex dimensional. This makes the concept of convexity more ambiguous. Thankfully, there is a great deal of literature from the several complex variables community on different types of convexity in complex projective space and in this thesis we will use the definitions they developed. In particular, in the next chapter we will define three notions of convexity and state the basic properties of each.

As in the real case we will restrict our attention to the *divisible* sets, that is there exists a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ which acts co-compactly on Ω. In this thesis we will show that the complex case is much more rigid than the real counterpart. Delaying precise definitions to Chapter II, we will prove the following theorem:

Theorem I.8. *Suppose $d \geq 2$ and $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is a proper weakly linearly convex set with C^1 boundary. If Ω is divisible then Ω is a projective ball, that is Ω is projectively isomorphic to*

$$\{[1 : z_1 : \cdots : z_d] \in \mathbb{P}(\mathbb{C}^{d+1}) : \sum |z_i|^2 < 1\}.$$

A main tool in the proof is a complex analogue of the Hilbert metric discovered by Dubois [Dub09]. Dubois' main motivation for this construction came from developing contraction principles for linear maps on complex cones in complex Hilbert spaces. Later this metric was used to establish regularity properties of the entropy of certain families of random walks (see for instance [Led12]). In this thesis we will develop properties of the geometry on Ω induced by this metric. It turns out that there are many similarities to the real Hilbert metric, in particular,

1. the boundary behavior of the metric is closely related to the shape of the boundary,
2. the metric is invariant under projective automorphisms,
3. understanding translation distances of automorphisms is easy.

These properties allow us to understand a group Γ which acts co-compactly on a set Ω with C^1 boundary and eventually deduce that Ω must be projectively equivalent to the ball.

Given a group G acting by isometries on a metric space (X, d), the main way to relate (X, d) and G is the Švarc-Milnor lemma which (assuming some conditions) says that space (X, d) is quasi-isometric to G endowed with a word metric. In the common formulation of this lemma, one of these conditions is that the metric space is geodesic. Unfortunately, the complex Hilbert metric does not seem to have many geodesics. Instead, in the proof of Theorem I.8 we construct quasi-geodesics and then observe that the Švarc-Milnor Lemma holds for quasi-geodesic metric spaces.

This difficulty in finding geodesics motivates the next theorem in this thesis:

Theorem I.9. *Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is a proper strictly weakly linearly convex open set. If (Ω, d_Ω) is geodesic then Ω is a projective ball.*

It turns out that the complex Hilbert metric is related to the Apollonian metric introduced by Beardon [Bea98] (see Section 3.5 for details). For the Apollonian metric Gehring and Hag [GH00] proved a version of Theorem I.9 in the case when $d = 1$.

Both Theorem I.8 and Theorem I.9 will reduce to the following proposition:

Proposition I.10. *Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is an open set such that its intersection with any projective line is either empty or a projective disk. Then Ω is a projective ball.*

Remark I.11. If $\Omega \subset \mathbb{R}^d$ is an open set such that its intersection with any 2-plane is either empty or an ellipse then it is trivial to verify that Ω is an ellipse. The difficulty in the above proposition is that we are only assuming knowledge about the set intersected with certain 2-dimensional planes (namely the complex lines).

The main step in the proof of Proposition IV.1 is to show that $\mathrm{Aut}(\Omega)$ is very large. In particular, it acts transitively on Ω and $\partial\Omega$. Then we use the geometry of the complex Hilbert metric to deduce that Ω must be projectively equivalent to the ball.

In the final chapter of this thesis we will extend our results to "convex" sets in quaternionic projective space. In particular we will prove the following three results:

Theorem I.12. *Suppose $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is a proper weakly linearly convex set with C^1 boundary. If Ω is divisible then Ω is projectively equivalent to the ball.*

We will also construct a quaternionic Hilbert metric d_Ω and prove:

Theorem I.13. *Suppose $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is a proper strictly weakly linearly convex set. If (Ω, d_Ω) is geodesic then Ω is projectively equivalent to the ball.*

As in the complex case both theorems will eventually reduce down to the following proposition:

Proposition I.14. *Suppose $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is a open set such that its intersection with any projective line is either empty or projectively isomorphic to the disk. Then Ω is projectively equivalent to the ball.*

1.3 Prior work

Kobayashi and Ochiai's [KO80] classified compact complex surfaces with a projective structure. Using this classification Cano and Seade proved the following:

Theorem I.15. *[CS08] Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^3)$ is a divisible proper weakly linearly convex set. Then Ω is a projective ball.*

The above theorem is actually special case of the main result in [CS08]. In higher dimensions a complete classification is probably very difficult and thus this approach will not generalize.

There are also several related rigidity results coming from the complex analysis community. One remarkable theorem is the ball theorem of Rosay [Ros79] and Wong [Won77]:

Theorem I.16. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudo-convex domain. If the space of holomorphic automorphisms of Ω is non-compact then Ω is bi-holomorphic to a ball.*

By *strongly pseudo-convex* we mean that Ω has C^2 boundary and the Levi-form at each point in the boundary is positive definite. There are more general versions of the ball theorem requiring only that the boundary is strongly pseudo-convex at an orbit accumulation point. We refer the reader to the survey articles [IK99] and [Kra13] for more details.

A version of the ball theorem is also true for sets in real projective space:

Theorem I.17. *[SM02] Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is an open proper strongly convex set. If the space of projective automorphisms of Ω is non-compact then Ω is a projective ball.*

By *strongly convex set* we mean that Ω has C^2 boundary and the Hessian at each point in the boundary is positive definite. In the real projective world, Benoist showed that rigidity still holds if the boundary regularity is relaxed but at the cost of assuming the existence of a dividing group.

Theorem I.18. *[Ben04, Theorem 1.3] Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is an open, proper, divisible convex set. If $\partial\Omega$ is $C^{1+\alpha}$ for all $\alpha \in [0, 1)$ then Ω is a projective ball.*

The theorems of Benoist and Socié-Méthou and the examples of Benoist and M. Kapovich show that for real divisible sets there is a major difference between the case when the boundary has C^2 regularity and when it has C^1 regularity.

Another remarkable theorem in the complex case is due to Frankel.

Theorem I.19. *[Fra89] Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex (in the usual sense) domain and there exists a discrete group Γ of holomorphic automorphisms of Ω such that $\Gamma \backslash \Omega$ is compact. Then Ω is a symmetric domain.*

It is clear that if $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is convex (in the usual sense) in some affine chart then Ω will be weakly linearly convex. Moreover there exists bounded weakly linearly convex sets in $\mathbb{C}^d$ which are not bi-holomorphic to a convex set [NPZ08]. In particular the main theorem of this paper weakens the hypothesis of Frankel's result in one direction while strengthening the hypothesis in two directions: assuming additional boundary regularity and assuming the dividing group acts projectively instead of holomorphically. We should also mention that weak linear convexity is invariant under projective transformations while ordinary convexity is not. Thus for proving rigidity results about the group of projective automorphisms of a domain it seems more natural to look at weakly linearly convex domains.

CHAPTER II

Preliminaries

2.1 Convexity in complex projective space

In this section we will recall three natural definitions of convexity in complex projective space. Each comes from the several complex variables community and have been studied since the 1960's. Our presentation will closely follow the book by Andersson, Passare, and Sigurdsson [APS04] on convexity in complex projective geometry.

The three definitions we consider are motivated by the following proposition which characterizes convexity in real projective geometry:

Proposition II.1. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is an open connected set. Then the following are equivalent:*

1. *for every $p \in \partial\Omega$ there exists a hyperplane H containing p such that $H \cap \Omega = \emptyset$,*
2. *for every $p \in \mathbb{P}(\mathbb{R}^{d+1}) \setminus \Omega$ there exists a hyperplane H containing p such that $H \cap \Omega = \emptyset$,*
3. *for every projective line $L \subset \mathbb{P}(\mathbb{R}^{d+1})$ the intersection $L \cap \Omega$ is connected.*

The most difficult implication is that (3) implies (1), for a proof see [APS04, Lemma 1.3.9]. Using the proposition, one can then define convexity in real projective space as follows:

Definition II.2. A connected open set $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is *convex* if it satisfies one of the three equivalent conditions in Proposition II.1.

Unfortunately, in complex projective geometry the three corresponding conditions all yield a different class of sets, namely the *weakly linearly convex* sets, the *linear convex* sets, and the $\mathbb{C}$-*convex* sets.

Definition II.3.

1. An open set $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is called *weakly linearly convex* if for every $p \in \partial\Omega$ there exists a complex hyperplane H containing p such that $H \cap \Omega = \emptyset$. A compact set $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is called weakly linearly convex if there exists a basis of open weakly convex neighborhoods of Ω.

2. A set $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is called *linearly convex* if for every $p \in \mathbb{P}(\mathbb{C}^{d+1}) \setminus \Omega$ there exists a complex hyperplane H containing p such that $H \cap \Omega = \emptyset$.

3. Finally a set is called $\mathbb{C}$-*convex* if for every projective line $L \subset \mathbb{P}(\mathbb{C}^{d+1})$ the sets $L \cap \Omega$ and $L \setminus \Omega \cap L$ are both connected.

In $\mathbb{P}(\mathbb{C}^2)$ a complex hyperplane is just a point and hence any open set is linearly convex. One the other hand an open set in $\mathbb{P}(\mathbb{C}^2)$ is a $\mathbb{C}$-convex if and only if is simply connected. Thus there exists sets which are linearly convex but not $\mathbb{C}$-convex. The next theorem gives the general relationship between these three classes:

Theorem II.4. *[APS04, Theorem 2.3.9, Corollary 2.5.6] Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is an open or compact set. Then the following implications hold:*

$$\Omega \textit{ is } \mathbb{C}\textit{-convex} \Rightarrow \Omega \textit{ is linearly convex} \Rightarrow \Omega \textit{ is weakly linearly convex.}$$

Moreover, if $d > 1$, Ω is open, and $\partial\Omega$ is a C^1 hypersurface then

$$\Omega \textit{ is } \mathbb{C}\textit{-convex} \Leftrightarrow \Omega \textit{ is linearly convex} \Leftrightarrow \Omega \textit{ is weakly linearly convex.}$$

When $d > 2$ there are weakly linearly convex sets which are not linearly convex [APS04, Example 2.1.7] and we have already observed there are linearly convex sets which are not $\mathbb{C}$-convex.

2.1.1 The complex dual

An important concept in the study of convex sets is the dual:

Definition II.5. The *complex dual* of $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is the set

$$\Omega^* = \left\{ f \in \mathbb{P}(\mathbb{C}^{(d+1)*}) : f(x) \neq 0 \text{ for all } x \in \Omega \right\} \subset \mathbb{P}(\mathbb{C}^{(d+1)*}).$$

Remark II.6. Notice that Ω^* will be compact when Ω is open. Often in real projective geometry the dual of an open convex set $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is defined to be the set

$$\left\{ f \in \mathbb{P}(\mathbb{R}^{(d+1)*}) : f(x) \neq 0 \text{ for all } x \in \overline{\Omega} \right\} \subset \mathbb{P}(\mathbb{R}^{(d+1)*}).$$

This is, in some situations, a natural choice because then the dual of an open convex set will be an open convex set. In the complex case one has to be very careful with these definitions because $\mathbb{C}$-convexity is not preserved under taking closures or interiors. In particular, this choice of definition is made so that the complex dual of a $\mathbb{C}$-convex set is again $\mathbb{C}$-convex.

Since $\mathbb{P}(\mathbb{C}^{(d+1)*})$ can be identified with the space of complex hyperplanes in $\mathbb{P}(\mathbb{C}^{d+1})$ we have the following alternative definition of linear convexity:

Observation II.7. A set $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is linearly convex if and only if $\Omega^{**} = \Omega$.

It also turns out that the boundary of Ω and the boundary of Ω^* are closely related. We will call a complex hyperplane H *tangent* to a set Ω at $p \in \partial\Omega$ if H contains p but does not intersect Ω. With this language we have the following observation:

Observation II.8. Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is open then

$$f \in \partial\Omega^* \Leftrightarrow \text{the hyperplane } \ker f \text{ is tangent to } \Omega.$$

2.1.2 Properness

As in the real case, it is very natural to consider convex sets that are proper. In this thesis we will use the following definition of proper sets:

Definition II.9. A set $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is called *proper* if $\overline{L \cap \Omega} \neq L$ for every complex projective line L in $\mathbb{P}(\mathbb{C}^{d+1})$.

In real projective geometry, a convex set is proper if and only if its dual has non-empty interior. In the complex setting it is unclear if this is true, but the following is known:

Proposition II.10. *[APS04, Proposition 2.3.10] If $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is a proper $\mathbb{C}$-convex open set, then Ω^* is not contained in a complex hyperplane.*

2.1.3 Invariance

Since a set Ω is linearly convex if and only if $\Omega^{**} = \Omega$ we see that

Observation II.11. If Ω is linearly convex so is Ω^*.

Linear convexity is not invariant under projective maps, but $\mathbb{C}$-convexity is.

Theorem II.12. *[APS04, Theorem 2.3.6, Theorem 2.3.9] Suppose Ω is a open or compact $\mathbb{C}$-convex set. Then*

1. *any projective image or pre-image of Ω is $\mathbb{C}$-convex and*

2. *Ω^* is $\mathbb{C}$-convex.*

2.1.4 Topology

Despite only having a constraint on two-dimensional slices, it turns out that $\mathbb{C}$-convexity greatly restricts the topology of a set:

Theorem II.13. *Suppose $\Omega \subset \mathbb{C}^d$ is an open $\mathbb{C}$-convex set. Then Ω is homeomorphic to a ball.*

Since the idea behind the proof is simple we will sketch it: suppose that $0 \in \Omega$ then for $v \in \mathbb{C}^d$ let $\Omega_v = \{z \in \mathbb{C} : zv \in \Omega\}$. By the Riemann mapping theorem there exists a unique bi-holomorphic map $\varphi_v : \mathbb{D} \to \Omega_v$ such that $\varphi_v(0) = 0$ and $\varphi_v'(0) > 0$. Then one can show that the map

$$\Phi(v) = \frac{\varphi_{v/\|v\|}(\|v\|)v}{\|v\|}$$

is a homeomorphism $B \to \Omega$. For complete details see [APS04, Theorem 2.4.2].

2.1.5 Relationship to pseudo-convexity

It turns out that a weakly linearly convex set is also *pseudo-convex* (which we do not define here). See for instance [APS04, Proposition 2.1.8].

2.2 Quasi-geodesic metric spaces

As mentioned in the introduction, the results of this thesis makes use of a Hilbert metric defined for weakly linearly convex sets. This metric will have many of the nice properties of the classical Hilbert metric for convex sets in real projective space, but unfortunately is rarely geodesic. However we will show that (Ω, d_Ω) is a quasi-geodesic metric space when Ω satisfies the hypothesis of Theorem I.8. In this section we first recall the definitions of quasi-geodesics and quasi-geodesic metric spaces. Then we will state an important property of such spaces.

If (X, d_X) and (Y, d_Y) are metric spaces, a map $f : X \to Y$ is called a (A, B)-*quasi isometric embedding* if

$$\frac{1}{A} d_X(x_1, x_2) - B \leq d_Y(f(x_1), f(x_2)) \leq A d_X(x_1, x_2) + B$$

for all $x_1, x_2 \in X$. An embedding becomes an *isomorphism* when there exists some $R > 0$ such that Y is contained in the R-neighborhood of $f(X)$.

The real numbers $\mathbb{R}$ have a natural metric: $d_{\mathbb{R}}(x, y) = |x - y|$ and a map $f : [a, b] \to X$ is a called a (A, B)-*quasi geodesic segment* if f induces a (A, B)-quasi-isometric embedding $([a, b], d_{\mathbb{R}}) \to (X, d)$. A metric space (X, d) is called (A, B)-*quasi-geodesic* if for all $x, y \in X$ there exists a (A, B)-quasi-geodesic segment whose image contains x and y. If (X, d) is (A, B)-quasi geodesic for some A and B, then (X, d) is called a *quasi-geodesic metric space.*

We now observe that the Švarc-Milnor Lemma is true for quasi-geodesic metric spaces. More precisely: given a finitely generated group Γ and a set of generators $S = \{s_1, \ldots, s_k\}$ define the word metric d_S on Γ by

$$d_S(\gamma_1, \gamma_2) = \inf\{N : \gamma_2^{-1}\gamma_1 = s_{i_1} \ldots s_{i_N}\},$$

we then have the following:

Theorem II.14. *Suppose (X, d) is a proper quasi-geodesic metric space and Γ is a group acting on (X, d) by isometries. If the action is properly discontinuous and cocompact then Γ is finitely generated. Moreover, if $S \subset \Gamma$ is a finite generating set and $x_0 \in X$ then the map $\gamma \in \Gamma \to \gamma \cdot x_0 \in X$ is a quasi-isometry of (Γ, d_S) and (X, d).*

Proof. The proof of the theorem for geodesic metric spaces given in [dlH00, Chapter IV, Theorem 23]) can be extended to quasi-geodesic spaces essentially verbatim. □

CHAPTER III

A Hilbert type metric

In this section we will consider a projective metric defined on any proper weakly linearly convex set in complex projective space. This metric is an analogue the classical Hilbert metric. The metric was originally constructed by [Dub09] but many of the results in this chapter were established in [Zim13]. We will begin by recalling the definition of the real Hilbert metric which motivates the definition in the complex case.

3.1 The real Hilbert metric

Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a convex open set, that is a set satisfying any of the equivalent properties in Proposition II.1. Then Ω has a natural pseudo-metric d_Ω called the *Hilbert metric*. Given two distinct points $x, y \in \Omega$ let a and b be the intersection of the projective line $\overline{xy}$ with $\partial\Omega$ ordered a, x, y, b. Then the Hilbert metric is defined to be

$$d_\Omega(x, y) = \log \frac{|x-b|\,|y-a|}{|x-a|\,|y-b|}.$$

This expression is well defined and when Ω is proper the Hilbert metric is actually a metric on Ω generating the standard topology. Moreover d_Ω will be invariant under the projective automorphisms of Ω.

3.2 The complex Hilbert metric and basic properties

In this section we recall Dubois' construction of a complex Hilbert metric and prove some basic results. Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is a weakly linearly convex open set. Let $x, y \in \Omega$ be distinct points and let L_{xy} be the projective line containing x and y. Now L_{xy} has real dimension two and probably the most naive definition of a complex Hilbert metric would be the following:

$$d_\Omega(x,y) = \max_{a,b \in \partial(L_{xy} \cap \Omega)} \log \frac{|x-b|\,|y-a|}{|x-a|\,|y-b|}.$$

With $x, y \in \mathbb{P}(\mathbb{C}^{d+1})$ fixed and distinct, consider the map

$$F : L_{xy} \times L_{xy} \to \mathbb{R}$$

given by

$$F(a,b) = \log \frac{|x-b|\,|y-a|}{|x-a|\,|y-b|}.$$

Identifying L_{xy} with $\overline{\mathbb{C}}$ we see that the level sets of the maps

$$a \to \log\log \frac{|y-a|}{|x-a|} \text{ and } b \to \log \frac{|x-b|}{|y-b|}$$

are circles or lines. Thus F is an open map (that is F maps open sets in $L_{xy} \times L_{xy}$ to open sets in $\mathbb{R}$). This implies that

$$d_\Omega(x,y) = \max_{a,b \in L_{xy} \setminus L_{xy} \cap \Omega} \log \frac{|x-b|\,|y-a|}{|x-a|\,|y-b|}.$$

Dubois' key insight is that this choice is actually a metric when Ω is a proper weakly linearly convex open set. More precisely:

Theorem III.1. *[Dub09, Lemma 2.1, Lemma 2.2] If Ω is a proper weakly linearly convex open set then d_Ω is a complete metric on Ω such that the subspace topology* *on* $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ *and the topology on* Ω *induced by* d_Ω *coincide. Moreover, if* $W \subset \mathbb{P}(\mathbb{C}^{d+1})$ *is a complex projective subspace then the inclusion* $W \cap \Omega \hookrightarrow \Omega$ *induces an isometric embedding* $(W \cap \Omega, d_{W\cap\Omega}) \hookrightarrow (\Omega, d_\Omega)$*.*

Remark III.2. Dubois only considers linear convex sets (instead of weakly linearly convex sets) and does not explicitly state that the subspace topology and metric topology coincide, but both assertions follow from his arguments. For the reader's convenience we will provide a proof of Theorem III.1.

We start with an alternative definition of the complex Hilbert metric which makes the validity of the triangle inequality more transparent.

Proposition III.3. *If* $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ *is a proper weakly linearly convex open set then*

$$d_\Omega(x,y) = \max_{f,g\in\Omega^*} \log\left(\frac{|f(x)g(y)|}{|f(y)g(x)|}\right). \tag{3.1}$$

The proposition will follow from the next Lemma:

Lemma III.4. *Suppose* $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ *is a proper weakly linearly convex open set with* $x, y \in \Omega$ *and* $f, g \in \Omega^*$*. Let* L *be the projective line containing* x *and* y*. Let* $\{b_f\} = L \cap \ker f$ *and* $\{b_g\} = L \cap \ker g$*. If we identify* L *with* $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ *then*

$$\left|\frac{f(x)g(y)}{f(y)g(x)}\right| = \frac{|x-b_f|\,|y-b_g|}{|y-b_f|\,|x-b_g|}$$

for all $x, y \in L$*.*

Remark III.5. Notice that L and $\ker f$ do indeed intersect at a single point. Since L has complex dimension 1 and $\ker f$ has complex codimension 1, the subspaces must intersect. Moreover either $L \cap \ker f$ is a single point or $L \subset \ker f$. But $L \cap \Omega \neq \emptyset$ by assumption and $\Omega \cap \ker f = \emptyset$ because $f \in \Omega^*$. For the same reasons $\ker g \cap L$ will be a single point.

Proof. For clarity let $\Phi : L \rightarrow \overline{\mathbb{C}}$ be the identification. Pick $e_1, e_2 \in L$ such that $\Phi([e_1 + ze_2]) = z$ for all $z \in \mathbb{C}$. Then $x = [e_1 + \Phi(x)e_2]$ and $y = [e_1 + \Phi(y)e_2]$ where we define $[e_1 + (\infty)e_2] := [e_2]$.

For $h \in \Omega^*$ we will associate a special representative $\hat{h} \in (\mathbb{C}^{d+1})^*$. If $h(e_2) \neq 0$ then $\Phi(b_h) \in \mathbb{C}$ and we can pick $\hat{h}$ such that $\hat{h}(e_1 + ze_2) = z - \Phi(b_h)$ for all $z \in \mathbb{C}$. If $h(e_2) = 0$ then $\Phi(b_h) = \infty$ and we let $\hat{h}$ be a representative such that $\hat{h}(e_1 + ze_2) = 1$ for all $z \in \mathbb{C}$.

Then

$$\left| \frac{f(x)g(y)}{f(y)g(x)} \right| = \left| \frac{\hat{f}(e_1 + \Phi(x)e_2)\hat{g}(e_1 + \Phi(y)e_2)}{\hat{f}(e_1 + \Phi(y)e_2)\hat{g}(e_1 + \Phi(x)e_2)} \right| = \frac{|\Phi(x) - \Phi(b_f)|\,|\Phi(y) - \Phi(b_g)|}{|\Phi(y) - \Phi(b_f)|\,|\Phi(x) - \Phi(b_g)|}$$

where ∞ is handled in the obvious way. $\square$

Proof of Proposition III.3. By Lemma III.4 and the fact that for every $x \in \partial\Omega$ there exists $f \in \Omega^*$ such that $x \in [\ker f]$ we have

$$\begin{aligned} d_\Omega(x, y) &= \max_{a,b \in \partial(L_{xy} \cap \Omega)} \log \frac{|x - b|\,|y - a|}{|x - a|\,|y - b|} \\ &\leq \max_{f,g \in \Omega^*} \log \left(\frac{|f(x)g(y)|}{|f(y)g(x)|} \right). \end{aligned}$$

We have also observed that

$$d_\Omega(x, y) = \sup_{a,b \in L_{xy} \setminus (L_{xy} \cap \Omega)} \log \frac{|x - b|\,|y - a|}{|x - a|\,|y - b|}$$

and so using Lemma III.4 again we have that

$$d_\Omega(x, y) \geq \max_{f,g \in \Omega^*} \log \left(\frac{|f(x)g(y)|}{|f(y)g(x)|} \right).$$

$\square$

The next lemma will be used to show that the Hilbert metric is complete and generates the standard topology on Ω.

Lemma III.6. *Suppose* $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ *is a proper weakly linearly convex open set. Then* $d_\Omega : \Omega \times \Omega \to \mathbb{R}$ *is continuous with respect to the subspace topology and for any* $p \in \Omega$ *and* $R > 0$ *the closed ball*

$$B_R(p) = \{q \in \Omega : d_\Omega(p, q) \leq R\}$$

is a compact set in Ω *(with respect to the subspace topology).*

Proof. We will fist show that $d_\Omega : \Omega \times \Omega \to \mathbb{R}$ is continuous with respect to the subspace topology. The map $F : \Omega \times \Omega \times \Omega^* \times \Omega^* \to \mathbb{R}$ given by

$$F(v, w, f, g) = \log \left(\frac{|f(v)g(w)|}{|f(w)g(v)|} \right)$$

is clearly continuous in the subspace topology. Then since Ω^* is compact $d_\Omega : \Omega \times \Omega \to \mathbb{R}$ is continuous in the subspace topology.

We now show $B_R(p)$ is compact. Since d_Ω is continuous with respect to the subspace topology, $B_R(p)$ is closed in Ω with respect to the subspace topology. To see that $B_R(p)$ is compact it is enough to establish the following: if $\{q_n\}_{n \in \mathbb{N}} \subset \Omega$ is a sequence such that $q_n \to y \in \partial\Omega$ then $d_\Omega(p, q_n) \to \infty$. Let $f \in \Omega^*$ be such that $f(y) = 0$ (such a function exists since Ω is weakly linearly convex). Since Ω is proper there exists a function $g \in \Omega^*$ such that $g(y) \neq 0$. Then

$$d_\Omega(q_n, p) \geq \log \left(\frac{|f(p)g(q_n)|}{|f(q_n)g(p)|} \right)$$

and so $d_\Omega(q_n, p) \to \infty$ as $n \to \infty$. Thus $B_R(p)$ is compact in Ω. $\square$

Proof of Theorem III.1. We will first show that d_Ω is a metric. There are two things to check:

1. $d_\Omega(x, y) \geq 0$ with equality if and only if $x = y$,

2. $d_\Omega(x, y) \leq d_\Omega(x, z) + d_\Omega(z, y)$.

The first condition will follow from properness and the second will follow from weak linear convexity.

Since Ω is proper there exists $a \in L_{xy} \setminus \overline{L_{xy} \cap \Omega}$ thus

$$d_\Omega(x, y) \geq \log \frac{|x-a|\,|y-a|}{|x-a|\,|y-a|} = 0.$$

Since the map

$$(a, b) \in \mathbb{P}(\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^2) \to \log \frac{|x-b|\,|y-a|}{|x-a|\,|y-b|} \in [-\infty, \infty]$$

is open there exists $(a', b') \in L_{xy} \setminus \overline{L_{xy} \cap \Omega}$ near (a, a) such that

$$\log \frac{|x-b'|\,|y-a'|}{|x-a'|\,|y-b'|} > 0.$$

So $d_\Omega(x, y) > 0$. This establishes the first requirement.

Using Proposition III.3, we can establish the triangle inequality:

$$\begin{aligned}
d_\Omega(x, y) &= \max_{f,g \in \Omega^*} \log \left(\frac{|f(x)g(y)|}{|f(y)g(x)|} \right) = \max_{f,g \in \Omega^*} \log \left(\frac{|f(x)g(z)|}{|f(z)g(x)|} \frac{|f(z)g(y)|}{|f(y)g(z)|} \right) \\
&= \max_{f,g \in \Omega^*} \left(\log \left(\frac{|f(x)g(z)|}{|f(z)g(x)|} \right) + \log \left(\frac{|f(z)g(y)|}{|f(y)g(z)|} \right) \right) \\
&\leq \max_{f,g \in \Omega^*} \log \left(\frac{|f(x)g(z)|}{|f(z)g(x)|} \right) + \max_{f,g \in \Omega^*} \log \left(\frac{|f(z)g(y)|}{|f(y)g(z)|} \right) \\
&= d_\Omega(x, z) + d_\Omega(z, y).
\end{aligned}$$

The metric d_Ω is complete because of Lemma III.6. Moreover if $x_n \in \Omega \to x \in \Omega$ in the subspace topology, the the continuity of d_Ω implies that $x_n \to x$ in the metric topology.

Now suppose x_n converges to x in the metric topology, that is $d_\Omega(x_n, x) \to 0$. Suppose for a contradiction that x_n does not converge to x in the subspace topology. Then by passing to a subsequence we can suppose that x_n converges to $x^* \in \overline{\Omega}$ in the subspace topology of $\overline{\Omega}$ where $x^* \neq x$. Since $d_\Omega(x_n, x)$ is bounded, Lemma III.6

implies that $x^* \in \Omega$. Then x_n converges to x^* in the subspace topology of Ω. So by the argument above x_n converges to x^* in the topology induced by d_Ω. But d_Ω is a metric and so $x^* = x$ which contradicts our initial assumptions.

Finally the "moreover" part of the theorem follows directly from the definition of the metric. $\square$

3.3 Action of the automorphism group

Notice that if $\gamma \in \mathrm{Aut}(\Omega)$ then ${}^t\gamma$ will preserve Ω^* and hence we immediately see the following:

Lemma III.7. *Suppose Ω is a proper weakly linearly convex open set. If $\gamma \in \mathrm{Aut}(\Omega)$ then the action of γ on Ω is an isometry with respect to the Hilbert metric.*

Proposition III.8. *Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is a proper weakly linearly convex open set, then $\mathrm{Aut}(\Omega)$ is a closed subgroup of $\mathrm{PSL}(\mathbb{C}^{d+1})$ and acts properly on Ω.*

Remark III.9. There are two natural topologies on $\mathrm{Aut}(\Omega)$: the first comes from the inclusion $\mathrm{Aut}(\Omega) \hookrightarrow \mathrm{PSL}(\mathbb{C}^{d+1})$ and the second comes from the inclusion $\mathrm{Aut}(\Omega) \hookrightarrow \mathrm{Homeo}(\Omega)$ where $\mathrm{Homeo}(\Omega)$ has the compact-open topology. The proof of this proposition shows that the two topologies coincide.

Proof. We first establish that $\mathrm{Aut}(\Omega) \leq \mathrm{PSL}(\mathbb{C}^{d+1})$ is closed. Suppose $\varphi_n \in \mathrm{Aut}(\Omega)$ is a sequence and $\varphi_n \to \varphi \in \mathrm{PSL}(\mathbb{C}^{d+1})$. Now $\varphi(\Omega) \subset \overline{\Omega}$ and since $\varphi : \mathbb{P}(\mathbb{C}^{d+1}) \to \mathbb{P}(\mathbb{C}^{d+1})$ is a diffeomorphism the set $\varphi(\Omega)$ is open. Then $\varphi(\Omega) \cap \Omega \neq \emptyset$ since $\overline{\Omega} \setminus \Omega$ has empty interior. So suppose $p \in \Omega$ such that $\varphi(p) \in \Omega$.

Now for $q \in \Omega$ and $R > 0$ let $B_R(q) \subset \Omega$ be the ball of radius R about q with respect to the Hilbert metric. Then for n large, $\varphi_n(p) \in B_1(\varphi(p))$ and since φ_n acts by isometries with respect to the Hilbert metric we have $\varphi_n(B_R(p)) \subset B_{R+1}(\varphi(p))$

for all $R > 0$. This implies that $\varphi(B_R(p)) \subset \overline{B_{R+1}(\varphi(p))}$ for all $R > 0$ and hence that $\varphi(\Omega) \subset \Omega$. An identical argument shows that $\varphi^{-1}(\Omega) \subset \Omega$. So $\varphi(\Omega) = \Omega$ and $\varphi \in \mathrm{Aut}(\Omega)$. Thus $\mathrm{Aut}(\Omega)$ is closed.

Now we establish the properness of the action. It is enough to show that the set $\{\varphi \in \mathrm{Aut}(\Omega) : \varphi K \cap K \neq \emptyset\}$ is compact for any $K \subset \Omega$ compact. So assume $\{\varphi_n\}_{n\in\mathbb{N}} \subset \{\varphi \in \mathrm{Aut}(\Omega) : \varphi K \cap K \neq \emptyset\}$ for some compact K. We claim that φ_n has a convergent subsequence in $\mathrm{Aut}(\Omega)$. Let $f_n : \Omega \to \Omega$ be the homeomorphism induced by φ_n, that is $f_n(p) = \varphi_n(p)$. Since each f_n is an isometry with respect to d_Ω and $f_n(K) \cap K \neq \emptyset$, using the Arzelá-Ascoli theorem we may pass to a subsequence such that f_n converges uniformly on compact subsets of Ω to a continuous map $f : \Omega \to \Omega$. Moreover f is an isometry and hence injective.

Now we can pick $\hat{\varphi}_n \in \mathrm{GL}(\mathbb{C}^{d+1})$ representing $\varphi_n \in \mathrm{PSL}(\mathbb{C}^{d+1})$ such that $\|\hat{\varphi}_n\| = 1$. Then by passing to a subsequence we can suppose $\hat{\varphi}_n \to \hat{\varphi} \in \mathrm{End}(\mathbb{C}^{d+1})$. By construction, for $y \in \Omega \setminus (\Omega \cap \ker \hat{\varphi})$ we have that $\hat{\varphi}(y) = f(y)$. As f is injective this implies that $\hat{\varphi}$ has full rank. Thus $[\hat{\varphi}] \in \mathrm{PSL}(\mathbb{C}^{d+1})$ and $\varphi_n \to [\hat{\varphi}]$ in $\mathrm{PSL}(\mathbb{C}^{d+1})$. As $\mathrm{Aut}(\Omega)$ is closed in $\mathrm{PSL}(\mathbb{C}^{d+1})$ this implies that $\varphi_n \to [\hat{\varphi}]$ in $\mathrm{Aut}(\Omega)$.

Since φ_n was an arbitrary sequence in $\{\varphi \in \mathrm{Aut}(\Omega) : \varphi K \cap K \neq \emptyset\}$ this implies that $\{\varphi \in \mathrm{Aut}(\Omega) : \varphi K \cap K \neq \emptyset\}$ is compact. Since K was an arbitrary compact set of Ω the proposition follows. $\square$

3.4 Asymptotic properties

In this section we explore some asymptotic properties of the Hilbert metric. One nice feature of the Hilbert metric is that the behavior of the metric near the boundary is closely related to the geometry of the boundary.

Proposition III.10. *Suppose Ω is a proper weakly linearly convex open set. If*

$$\{p_n\}_{n\in\mathbb{N}}, \{q_n\}_{n\in\mathbb{N}} \subset \Omega$$

are sequences such that $p_n \to x \in \partial\Omega$, $q_n \to y \in \partial\Omega$, and $d_\Omega(p_n, q_n) < R$ for some $R > 0$ then every complex tangent hyperplane of Ω containing x also contains y.

Proof. Since Ω is proper there exists $g \in \Omega^*$ such that $g(x) \neq 0$ and $g(y) \neq 0$. If H is a complex tangent hyperplane containing x and $f \in \mathbb{P}(\mathbb{C}^{(d+1)*})$ is such that $[\ker f] = H$, then $[\ker f] \cap \Omega = H \cap \Omega = \emptyset$. Thus $f \in \Omega^*$ and

$$R \geq d_\Omega(p_n, q_n) \geq \log\left|\frac{f(q_n)}{f(p_n)}\right| + \log\left|\frac{g(p_n)}{g(q_n)}\right|.$$

Let $\hat{p}_n, \hat{q}_n, \hat{x}, \hat{y} \in \mathbb{C}^{d+1}$ and $\hat{f}, \hat{g} \in \mathbb{C}^{(d+1)*}$ be representatives of $p_n, q_n, x, y \in \mathbb{P}(\mathbb{C}^{d+1})$ and $f, g \in \mathbb{P}(\mathbb{C}^{(d+1)*})$ normalized such that

$$\left\|\hat{f}\right\| = \|\hat{g}\| = \|\hat{p}_n\| = \|\hat{q}_n\| = \|\hat{x}\| = \|\hat{y}\| = 1.$$

Then

$$R \geq \log\left|\frac{\hat{f}(\hat{q}_n)}{\hat{f}(\hat{p}_n)}\right| + \log\left|\frac{\hat{g}(\hat{p}_n)}{\hat{g}(\hat{q}_n)}\right|.$$

Since $f(x) = 0$, we see that $\hat{f}(\hat{p}_n) \to 0$. Since $g(x) \neq 0$ and $g(y) \neq 0$, we see that

$$\log\left|\frac{\hat{g}(\hat{p}_n)}{\hat{g}(\hat{q}_n)}\right|$$

is bounded from above and below (for n large). Thus we must have that $\hat{f}(\hat{q}_n) \to 0$ and then we see that $y \in [\ker f]$. $\square$

Another nice feature of the Hilbert metric is that it is possible to estimate the translation distance of elements of $\varphi \in \mathrm{Aut}(\Omega)$.

Proposition III.11. *Suppose Ω is a proper weakly linearly convex open set. If $x_0 \in \Omega$ then there exist $R > 0$ depending only on x_0 such that*

$$d_\Omega(\varphi x_0, x_0) \leq R + \log\left(\|\varphi\| \left\|\varphi^{-1}\right\|\right)$$

for all $\varphi \in \mathrm{Aut}(\Omega)$.

Proof. Let

$$\Lambda = \{f \in \mathbb{C}^{(d+1)*} : \|f\| = 1, [f] \in \Omega^*\}.$$

Since Ω^* is $\mathrm{Aut}(\Omega)$-invariant we see that

$${}^t\varphi f / \left\|{}^t\varphi f\right\| \in \Lambda$$

whenever $f \in \Lambda$ and $\varphi \in \mathrm{Aut}(\Omega)$. Let $\hat{x}_0 \in \mathbb{C}^{d+1}$ as a representative of $x_0 \in \mathbb{P}(\mathbb{C}^{d+1})$ with norm one. Since $f(\hat{x}_0) \neq 0$ for all $f \in \Lambda$ and Λ is compact, there exists $C > 0$ such that:

$$-C < \log|f(\hat{x}_0)| < C$$

for all $f \in \Lambda$. Now for $\varphi \in \mathrm{Aut}(\Omega)$

$$d_\Omega(\varphi x_0, x_0) = \sup_{f,g\in\Lambda} \log\left|\frac{f(\varphi\hat{x}_0)g(\hat{x}_0)}{f(\hat{x}_0)g(\varphi\hat{x}_0)}\right| \leq 2C + \sup_{f,g\in\Lambda} \log\left|\frac{f(\varphi\hat{x}_0)}{g(\varphi\hat{x}_0)}\right|$$

and for $f, g \in \Lambda$

$$\begin{aligned}
\log\left|\frac{f(\varphi\hat{x}_0)}{g(\varphi\hat{x}_0)}\right| &= \log\frac{\|{}^t\varphi f\|}{\|{}^t\varphi g\|} + \log\left|\left(\frac{{}^t\varphi f}{\|{}^t\varphi f\|}\right)(\hat{x}_0)\right| - \log\left|\left(\frac{{}^t\varphi g}{\|{}^t\varphi g\|}\right)(\hat{x}_0)\right| \\
&\leq \log\left(\|\varphi\| \left\|\varphi^{-1}\right\|\right) + \sup_{f',g'\in\Lambda} \log\left|\frac{f'(\hat{x}_0)}{g'(\hat{x}_0)}\right| \\
&\leq \log\left(\|\varphi\| \left\|\varphi^{-1}\right\|\right) + 2C.
\end{aligned}$$

Thus

$$d_\Omega(\varphi x_0, x_0) \leq 4C + \log\left(\|\varphi\| \left\|\varphi^{-1}\right\|\right)$$

and the proposition holds with $R := 4C$. $\square$

3.5 Connection to the Apollonian metric

Suppose $\Omega \subset \overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$ is an open set, then we can consider the function $A_\Omega : \Omega \times \Omega \to \mathbb{R}_{\geq 0}$ given by

$$A_\Omega(x_1, x_2) = \max_{b_1, b_2 \in \overline{\mathbb{R}^d} \setminus \Omega} \log \frac{|x_1 - b_1| \, |x_2 - b_2|}{|x_2 - b_1| \, |x_1 - b_2|}$$

where we handle ∞ in the obvious way. When Ω^c is not a proper subset of a sphere or hyperplane then A_Ω is a metric on Ω called the *Apollonian metric* [Bea98, Theorem 1.1]. This metric was apparently first considered by Barbilian [Bar35] and rediscovered by Beardon [Bea98]. For additional information about the Apollonian metric see Hästö [Häs06], Ibragimov [Ibr02], Rhodes [Rho97], and Seittenranta [Sei99].

By Proposition III.3 we have the following relationship between the complex Hilbert metric and the Apollonian metric:

Proposition III.12. *Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is a weakly linearly convex open set. If $x, y \in \Omega$ and L is the projective line containing x and y then*

$$d_\Omega(x, y) = A_{\Omega \cap L}(x, y).$$

One well known property of the Apollonian metric is the following:

Proposition III.13. *[Bea98, Lemma 3.1] If $B = \{x \in \mathbb{R}^d : \|x\| < 1\} \subset \overline{\mathbb{R}^d}$ then (B, d_B) is the Poincaré model of real hyperbolic space. In particular, (B, d_B) is a geodesic metric space.*

Unfortunately, as a result of Gehring and Hag demonstrates, the ball is essentially the only plane domain in which the Apollonian metric is geodesic.

Theorem III.14. *[GH00, Theorem 3.26] If $\Omega \subset \mathbb{C}$ is a bounded simply connected domain such that (Ω, d_Ω) is a geodesic metric space, then Ω is a disk.*

3.6 A model of complex hyperbolic space

Using Theorem III.1, Proposition III.13, and the projective ball model of complex hyperbolic d-space we can prove:

Proposition III.15. *Let $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ be a projective ball. Then (Ω, d_Ω) is isometric to complex hyperbolic d-space.*

Proof. We can pick coordinates such that

$$\Omega = \{[1 : z_1 : z_2 : \cdots : z_d] : \sum |z_i|^2 < 1\}.$$

Now let d be the complex hyperbolic metric on Ω described in Chapter 19 of [Mos73]. Then by Proposition III.13

$$d_\Omega(p, q) = d(p, q)$$

for all $p, q \in \Omega \cap L$ where L is the projective line $L = \{[1 : z : 0 : \cdots : 0] : |z| < 1\}$. Since $\mathrm{SU}(1, d)$ acts transitively on the set of projective lines intersecting Ω and both d and d_Ω are preserved by $\mathrm{SU}(1, d)$ we see that $d = d_\Omega$ on all of Ω. $\square$

3.7 Comparison with the Kobayashi metric

In this section we will compare the *Kobayashi metric* to the complex Hilbert metric. The results in this section are not used in the rest of thesis. For basic properties and applications of the Kobayashi metric see [Aba89].

Given a complex manifold Ω the *(infinitesimal) Kobayashi metric* is

$$K_\Omega(x; v) = \inf \{|\xi| : f \in \mathrm{Hol}(\Delta, \Omega),\ f(0) = x,\ df(\xi) = v\}$$

and the *Kobayashi pseudo-distance* is

$$d_\Omega^{\mathbb{K}}(x, y) = \inf \left\{ \int_0^1 K_\Omega(\gamma(t); \gamma'(t)) dt : \gamma \in C^\infty([0, 1], \Omega), \gamma(0) = x \text{ and } \gamma(1) = y \right\}.$$

Directly from the definitions one obtains the following result:

Proposition III.16. *Suppose $f : M_1 \to M_2$ is a holomorphic map between complex manifolds M_1 and M_2 then*

$$K_{M_2}\left(f(p); df(v)\right) \leq K_{M_1}\left(p; v\right)$$

and

$$d^{\mathbb{K}}_{M_2}\left(f(p_1), f(p_2)\right) \leq d^{\mathbb{K}}_{M_1}\left(p_1, p_2\right).$$

As a corollary we see that the Kobayashi metric is invariant under bi-holomorphisms:

Corollary III.17. *Suppose $f : M_1 \to M_2$ is a bi-holomorphic map between complex manifolds M_1 and M_2 then f induces an isometry $(M_1, d^{\mathbb{K}}_{M_1}) \to (M_2, d^{\mathbb{K}}_{M_2})$.*

In this section we will establish the following:

Proposition III.18. *Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is a proper open $\mathbb{C}$-convex set. Then*

$$\frac{1}{4} d_\Omega(x, y) \leq d^{\mathbb{K}}_\Omega(x, y)$$

for all $x, y \in \Omega$.

The proof of Proposition III.18 will use the next Lemma:

Lemma III.19. *Suppose $\Omega \subset \mathbb{C}$ is simply connected and $0 \notin \Omega$ then*

$$\frac{|v|}{4\,|p|} \leq K_\Omega(p; v) \text{ and } \frac{1}{4}\log\left(\frac{|p|}{|q|}\right) \leq d^{\mathbb{K}}_\Omega(p, q)$$

for all $p, q \in \Omega$ and $v \in \mathbb{C} \cong T_p\Omega$.

Proof. Since Ω is simply connected and $\Omega \neq \mathbb{C}$, there exists a bi-holomorphic map $f : \Delta \to \Omega$ with $f(0) = p$. Since f is bi-holomorphic, it is an isometry with respect to the Kobayashi metric. Since $0 \notin \Omega$ by Koebe's theorem, $|f'(0)| \leq 4\,|p|$. Since $K_\Delta(0; v) = |v|$ this establishes the first inequality.

To see the second inequality suppose $\gamma \in C^\infty([0,1],\Omega)$ is a curve with $\gamma(0) = p$ and $\gamma(1) = q$. Then by the first part of the lemma:

$$\int_0^1 K_\Omega(\gamma(t),\gamma'(t))dt \geq \frac{1}{4}\int_0^1 \frac{|\gamma'(t)|}{|\gamma(t)|}dt$$

Now $\frac{d}{dt}|\gamma(t)| \leq |\gamma'(t)|$ and so

$$\int_0^1 K_\Omega(\gamma(t),\gamma'(t))dt \geq \frac{1}{4}\int_0^1 \frac{\frac{d}{dt}|\gamma(t)|}{|\gamma(t)|}dt = \frac{1}{4}\int_{|p|}^{|q|}\frac{du}{u} = \frac{1}{4}\log\left(\frac{|p|}{|q|}\right).$$

$\square$

Proof of Proposition III.18. Suppose $p, q \in \Omega$ and L is the complex projective line containing p and q. Let $x, y \in \partial\Omega \cap L$ be such that

$$d_\Omega(p,q) = \log\frac{|p-x|\,|q-y|}{|p-y|\,|q-x|}.$$

Since Ω is weakly linearly convex, there exists a complex hyperplane H_x through x and a complex hyperplane H_y through y which do not intersect Ω. Then we can pick new coordinates such that

1. $x = [1:0:\dots:0]$,
2. $H_x = \{[z_1 : 0 : z_2 : \dots : z_d] : z_1, \dots, z_d \in \mathbb{C}\}$,
3. $y = [0:1:0:\dots:0]$, and
4. $H_y = \{[0 : z_1 : z_2 : \dots : z_d] : z_1, \dots, z_d \in \mathbb{C}\}$.

Then $L = \{[z_1 : z_2 : 0 : \dots : 0] : z_1, z_2 \in \mathbb{C}\}$ and

$$\Omega \subset \mathbb{P}(\mathbb{C}^{d+1}) \setminus H_y = \{[1 : z_1 : \dots : z_d] : z_1, \dots, z_d \in \mathbb{C}\}.$$

Now consider the projection $P(z_1, \dots, z_{d+1}) = (z_1, z_2)$. Since P is a holomorphic map

$$d^{\mathbb{K}}_{P(\Omega)}(p,q) \leq d^{\mathbb{K}}_\Omega(p,q).$$

By construction $P(\Omega) \subset \{[1 : z] : z \in \mathbb{C}\}$ which we can identify with $\mathbb{C}$. Since $H_x \cap \Omega = \emptyset$ we see that $0 \notin P(\Omega)$ and since $P(\Omega)$ is $\mathbb{C}$-convex we see that $P(\Omega)$ is simply connected. Thus

$$d_\Omega^{\mathbb{K}}(p,q) \geq d_{P(\Omega)}^{\mathbb{K}}(p,q) \geq \frac{1}{4} \log \frac{|p-x|}{|q-x|}.$$

Since $y = \infty$ in these coordinates

$$\log \frac{|p-x|}{|q-x|} = \log \frac{|p-x|\,|q-y|}{|p-y|\,|q-x|}$$

and thus

$$d_\Omega^{\mathbb{K}}(p,q) \geq \frac{1}{4} d_\Omega(p,q).$$

□

3.8 Why consider the complex Hilbert metric?

A useful approach to understanding a group is to understand the geometries it acts on. Under the hypothesis of Theorem I.8 we have a group $\Gamma \leq \mathrm{PSL}(\mathbb{C}^{d+1})$ acting co-compactly on a weakly linearly convex set $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$. Now Ω has (at least) two natural geometries: the complex Hilbert geometry and the Kobayashi geometry. In our proof of Theorem I.8 we will understand the properties of the group Γ by understanding the geometry of the complex Hilbert metric. We focus on this geometry because:

1. It is easy to estimate the translation distance.

2. If $W \subset \mathbb{P}(\mathbb{C}^{d+1})$ is a projective subspace then the inclusion $W \cap \Omega \hookrightarrow \Omega$ induces an isometric embedding $(W \cap \Omega, d_{W\cap\Omega}) \hookrightarrow (\Omega, d_\Omega)$. Thus making it easier to understand the geometry.

3. Although there are many parallels between the classical Hilbert metric and the Kobayashi metric (see for instance [Kob77] or [Lem87] or [Gol09]), the complex Hilbert metric is a direct analogue of the real Hilbert metric and thus many ideas from the well developed theory of real Hilbert geometry can be directly used in the complex setting.

CHAPTER IV

Rigidity from slices

In this chapter we will prove the following:

Proposition IV.1. *Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is an open set such that the intersection of Ω with any complex projective line is either empty or a projective disk. Then Ω is a projective ball.*

For the rest of chapter we will assume Ω is a open set satisfying the hypothesis of Proposition IV.1

4.1 Convexity

Notice that Ω is a proper $\mathbb{C}$-convex open set and hence is linearly convex by Theorem II.4. In particular, Ω is contained in lots of affine charts: if $f \in \Omega^*$ then $\mathbb{P}(\mathbb{C}^{d+1}) \setminus \ker f$ is an affine chart containing Ω. Now suppose $\mathbb{C}^d$ is an affine chart containing Ω. If $x, y \in \Omega$ and L is the complex line in $\mathbb{C}^d$ containing x and y then $\Omega \cap L$ is either a half space or a ball in L. In either case $\Omega \cap L \subset \mathbb{C}^d = \mathbb{R}^{2d}$ is convex in the usual sense. Since $x, y \in \Omega$ were arbitrary we see that Ω is convex in this affine chart. Summarizing:

Proposition IV.2. *Suppose $\mathbb{C}^d$ is an affine chart containing Ω. Then $\Omega \subset \mathbb{C}^d$ is convex.*

There are several corollaries of this observations:

Corollary IV.3. *Suppose $\mathbb{C}^d$ is an affine chart containing Ω. If $x_0 + \mathbb{R} \cdot v_0 \subset \overline{\Omega}$ for some $x_0 \in \overline{\Omega}$ and $v_0 \in \mathbb{C}^d$ then $x + \mathbb{R} \cdot v_0 \subset \Omega$ for any $x \in \Omega$.*

Proof. Suppose $x \in \Omega$. Since Ω is open there exists $\epsilon \in (0,1)$ such that

$$x + \frac{\epsilon}{(1-\epsilon)}(x - x_0) \in \Omega.$$

Then since Ω is convex for any $t \in \mathbb{R}$

$$x + tv_0 = (1-\epsilon)\left(x + \frac{\epsilon}{(1-\epsilon)}(x - x_0)\right) + \epsilon\left(x_0 + \frac{t}{\epsilon}v_0\right)$$

is in Ω. □

Corollary IV.4. *There exists an affine chart containing Ω as a bounded convex set.*

Proof. Since Ω is linearly convex, there exists an affine chart containing Ω. Since Ω is convex in this chart we can assume $0 \in \partial\Omega$ and $\Omega \subset \{(z_1, \dots, z_d) \in \mathbb{C}^d : \mathrm{Im}(z_1) > 0\}$. Then if $H = \{(-i, z_2 \dots, z_d) \in \mathbb{C}^d\}$ then Ω will be a bounded subset of the affine chart $\mathbb{P}(\mathbb{C}^{d+1}) \setminus H$. □

As a consequence of Corollary IV.4 we have:

Corollary IV.5. *Suppose L is a projective line then $L \cap \overline{\Omega} = \overline{L \cap \Omega}$.*

4.2 Constructing automorphisms

Let L be a projective line intersecting Ω and fix $p, q \in \partial\Omega \cap L$ distinct. Since Ω is linearly convex there exists complex hyperplanes $H_p, H_q \subset \mathbb{P}(\mathbb{C}^{d+1})$ such that $p \in H_p$, $q \in H_q$, $H_p \cap \Omega = \emptyset$, and $H_q \cap \Omega = \emptyset$. Since L intersects Ω it is transverse to H_p. Thus $q \notin H_p$ and so $H_p \neq H_q$.

Then by a projective transformation we may assume:

1. $p = [1 : 0 : \cdots : 0]$,

2. $q = [0 : 1 : \cdots : 0]$,

3. $H_p \cap H_q = \{[0 : 0 : z_1 : \cdots : z_{d-1}] : (z_1, \dots, z_{d-1}) \neq (0, \dots, 0)\}$.

Then $L = \{[z_1, z_2 : 0 : \cdots : 0] : (z_1, z_2) \neq (0, 0)\}$ and by another projective transformation we may assume:

4. $L \cap \Omega = \{[1 : z : 0 : \cdots : 0] : \mathrm{Im}(z) > 0\}$.

Lemma IV.6. *With the choice of coordinates above, if $g \in \mathrm{SL}(\mathbb{C}^2)$ and $[g] \in \mathrm{Aut}_0(L \cap \Omega)$ then $Aut_0(\Omega)$ contains the projective transformation*

$$\psi_g = \begin{pmatrix} g & 0 \\ 0 & Id \end{pmatrix}.$$

In particular,

1. if $\mathrm{Aut}_0(\Omega)$ acts transitively on $\partial\Omega \cap L$ and

2. if $x \in L \cap \partial\Omega$ then $P_x = \{\varphi \in \mathrm{Aut}_0(\Omega) : \varphi(x) = x\}$ acts transitively on $\Omega \cap L$.

Proof. The "in particular" assertion follows from the main assertion and well known facts about $\mathrm{Aut}(\mathcal{H}) = \mathrm{PSL}(\mathbb{R}^2)$ where $\mathcal{H} = \{[1 : z] : \mathrm{Im}(z) > 0\} \subset \mathbb{P}(\mathbb{C}^2)$.

Now $H_q = \{[0 : z_1 : z_2 : \cdots : z_d] : (z_1, \dots, z_d) \neq (0, \dots, 0)\}$ and as $\Omega \cap H_q = \emptyset$ we see that

$$\Omega \subset \{[1 : z_1 : \cdots : z_d] : z_1, \dots, z_d \in \mathbb{C}\}$$

which we can identify with $\mathbb{C}^d$. In this affine chart Ω is convex and contains the affine subspace

$$A = (i, 0, \dots, 0) + \mathbb{R}(1, 0, \dots, 0)$$

In particular by Corollary IV.3, $\mathrm{Aut}_0(\Omega)$ contains the projective transformation

$$\phi_t(\vec{z}) = \vec{z} + (t, 0, \dots, 0)$$

for all $t \in \mathbb{R}$. This implies that

$$\left\{ \psi_g : g = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in \mathbb{R} \right\} \subset \mathrm{Aut}_0(\Omega).$$

Now $H_p = \{[z_1 : 0 : z_2 : \cdots : z_d] : (z_1, \dots, z_d) \neq (0, \dots, 0)\}$ and as $\Omega \cap H_p = \emptyset$ we see that

$$\Omega \subset \{[z_1 : 1 : z_2 : \cdots : z_d] : z_1, \dots, z_d \in \mathbb{C}\}$$

which we can identify with $\mathbb{C}^d$. By our initial choice of coordinates

$$L \cap \Omega = \{[z : 1 : 0 : \cdots : 0] : \mathrm{Im}(z) < 0\}$$

and as Ω is convex in this new affine chart, $\mathrm{Aut}_0(\Omega)$ contains the transformation

$$\theta_s(\vec{z}) = \vec{z} + s(1, 0, \dots, 0)$$

for any $s \in \mathbb{R}$. Notice that θ_s is defined with respect to the new affine chart. This implies that for all $s \in \mathbb{R}$, $\mathrm{Aut}_0(\Omega)$ contains the subgroup

$$\left\{ \psi_g : g = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, s \in \mathbb{R} \right\}.$$

Finally it is well known that the one parameter groups

$$\left\{ \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \right\}_{s \in \mathbb{R}} \text{ and } \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\}_{t \in \mathbb{R}}$$

generate $\mathrm{SL}(\mathbb{R}^2)$ and thus the lemma follows. $\square$

4.3 Transitivity of the automorphism group

Lemma IV.7. *$Aut_0(\Omega)$ acts transitively on Ω.*

Proof. Since every two points in Ω are contained in a projective line intersecting Ω Lemma IV.6 immediately implies that $\mathrm{Aut}_0(\Omega)$ acts transitively on Ω. $\square$

Lemma IV.8. *$Aut_0(\Omega)$ acts transitively on $\partial\Omega$.*

This should be immediate from Lemma IV.6, except that given $x, y \in \partial\Omega$ it is not clear that the projective line containing x and y intersects Ω.

Proof. First observe that if L is a projective line intersecting Ω then by Lemma IV.6 for all $x, y \in L \cap \partial\Omega$ there exists $\phi \in \mathrm{Aut}_0(\Omega)$ such that $\phi(x) = y$.

We next observe that $\partial\Omega$ is connected. This follows since Ω is convex in any affine chart. Since $\partial\Omega$ is connected, it is enough to show that $\mathrm{Aut}_0(\Omega) \cdot x$ contains a neighborhood of x for all $x \in \partial\Omega$. Fix $x \in \partial\Omega$ and let L be a complex projective line such that $x \in L$ and L intersects Ω. Fix $z \in L \cap \partial\Omega$ distinct from x. By the first observation $z \in \mathrm{Aut}_0(\Omega) \cdot x$ and so $\mathrm{Aut}_0(\Omega) \cdot z = \mathrm{Aut}_0(\Omega) \cdot x$. Since Ω is open, there exists a neighborhood U of x in $\partial\Omega$ such that if $x' \in U$ then the complex line L' containing x' and z intersects Ω. Thus by the first observation $U \subset \mathrm{Aut}_0(\Omega) \cdot z = \mathrm{Aut}_0(\Omega) \cdot x$. $\square$

4.4 The boundary is smooth

To deduce the boundary is smooth we use the following well known fact.

Lemma IV.9. *Suppose G is a connected Lie group acting smoothly on a smooth manifold M. Then an orbit $G \cdot m$ is a smoothly embedded submanifold of M if and only if $G \cdot m$ is locally closed in M.*

Here smooth mean C^∞ and for a proof see [tD08, Theorem 15.3.7]. Since $\partial\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is closed Lemma IV.8 implies:

Proposition IV.10. *$\partial\Omega$ is a C^∞ embedded submanifold in $\mathbb{P}(\mathbb{C}^{d+1})$.*

4.5 Stabilizer subgroups

In this subsection we prove:

Proposition IV.11. *Suppose $p \in \Omega$ and $K_p = \{\varphi \in \mathrm{Aut}_0(\Omega) : \varphi p = p\}$. Then K_p acts transitively on $\partial\Omega$.*

We will need three lemmas.

Lemma IV.12. *Suppose $x \in \partial\Omega$ and L_1, L_2 are two projective lines which intersect Ω and contain x. Then*

$$0 = \inf\{d_\Omega(p_1, p_2) : p_1 \in L_1 \text{ and } p_2 \in L_2\}.$$

Proof. By Corollary IV.4 there exists an affine chart containing Ω as a bounded convex set. Since Ω is convex we may assume

1. $x = 0$,
2. $T_0\partial\Omega = \{(z_1, \dots, z_d) : \mathrm{Im}(z_1) = 0\}$, and
3. $\Omega \subset \{(z_1, \dots, z_d) : \mathrm{Im}(z_1) > 0\}$.

Now pick $p = (p_1, \dots, p_d) \in L_1 \cap \Omega$ and $q = (q_1, \dots, q_d) \in L_2 \cap \Omega$ such that $\mathrm{Im}(p_1) = \mathrm{Im}(q_1)$. Consider the lines $\ell_1, \ell_2 : \mathbb{R} \to \mathbb{K}^d$ given by $\ell_1(t) = tp$ and $\ell_2(t) = tq$. Since Ω is convex for $0 < t \leq 1$ we have $\ell_1(t), \ell_2(t) \in \Omega$. Moreover

$$\|\ell_1(t) - \ell_2(t)\| = \|p - q\|\, t.$$

Now let L_t be the complex line containing $\ell_1(t)$ and $\ell_2(t)$. Since $\mathrm{Im}(p_1) = \mathrm{Im}(q_1)$ the complex line L_t is parallel to $T_0\partial\Omega$. Then since $\partial\Omega$ is C^2 there exists $C_2 > 0$ such that

$$\|\ell_1(t) - w\| \geq C_2\sqrt{t} \text{ and } \|\ell_2(t) - w\| \geq C_2\sqrt{t}$$

for every $w \in L_t \cap \partial\Omega$. Then for $w \in L_t \cap \partial\Omega$ we have

$$\frac{|\ell_1(t) - w|}{|\ell_2(t) - w|} \leq \frac{|\ell_1(t) - \ell_2(t)| + |\ell_2(t) - w|}{|\ell_2(t) - w|} \leq 1 + (\|p - q\| / C_2)\sqrt{t}.$$

Similarly

$$\frac{|\ell_2(t) - w|}{|\ell_1(t) - w|} \leq 1 + (\|p - q\| / C_2)\sqrt{t}$$

for all $w \in L_t \cap \partial\Omega$. Thus

$$d_\Omega(\ell_1(t), \ell_2(t)) = d_{\Omega \cap L_t}(\ell_1(t), \ell_2(t)) \leq 2\log\left(1 + (\|p - q\| / C_2)\sqrt{t}\right).$$

As $\ell_1(t) \in L_1 \cap \Omega$ and $\ell_2(t) \in L_2 \cap \Omega$ for $0 < t \leq 1$ this proves the lemma. $\square$

Lemma IV.13. *For $x \in \partial\Omega$ let $P_x = \{\varphi \in \mathrm{Aut}_0(\Omega) : \varphi x = x\}$. Then P_x acts transitively on Ω.*

Proof. We first consider the special case in which $p, q \in \Omega$ and the complex line L containing them also contains x. Then, by Lemma IV.6, there exists $\varphi \in P_x$ such that $\varphi(p) = q$.

Now suppose $p, q \in \Omega$ are arbitrary. Let L_p (resp. L_q) be the complex projective line containing x and p (resp. q). By Lemma IV.12 there exists $p_n \in L_p$ and $q_n \in L_q$ such that $d_\Omega(p_n, q_n) \to 0$. By the special case there exists $\varphi_n, \psi_n \in P_x$ such that $\varphi_n(p) = p_n$ and $\psi_n(q) = q_n$. Then

$$d_\Omega(\psi_n^{-1}\varphi_n p, q) = d_\Omega(p_n, q_n) \to 0.$$

By Proposition III.8, $\mathrm{Aut}(\Omega)$ acts properly on Ω and so by passing to a subsequence we may suppose that $\psi_n^{-1}\varphi_n \to \varphi \in P_x$. Then $\varphi p = q$. As $p, q \in \Omega$ were arbitrary, this shows that P_x acts transitively on Ω. □

Proof of Proposition IV.11. Suppose $x, y \in \partial\Omega$. By Lemma IV.8, there exists $\varphi \in \mathrm{Aut}_0(\Omega)$ such that $\varphi x = y$. Let $q := \varphi p$. Then by the above lemma there exists $\psi \in P_y$ such that $\psi q = p$. Then $(\psi\varphi)(p) = p$ and $(\psi\varphi)x = y$. As $x, y \in \partial\Omega$ were arbitrary this shows that K_p acts transitively on $\partial\Omega$. □

4.6 Finishing the proof of Proposition IV.1

Fix $p \in \Omega$. Let $K_p = \{\varphi \in \mathrm{Aut}_0(\Omega) : \varphi p = p\}$. By Proposition III.8 K_p is compact. Let $\hat{K}_p$ be the pre-image of K_p under the map $\mathrm{SL}(\mathbb{C}^{d+1}) \to \mathrm{PSL}(\mathbb{C}^{d+1})$. By averaging the inner product

$$\langle w, z\rangle = \overline{z}_1 w_1 + \cdots + \overline{z}_{d+1} w_{d+1}$$

to obtain an $\hat{K}_p$-invariant inner product and then possibly changing coordinates we can assume $\hat{K}_p \leq \mathrm{SU}(d+1)$. Now $\hat{K}_p$ preserves the complex line p and by an orthogonal change of coordinates we may assume $p = \mathbb{C}\, e_1$. Then $\hat{K}_p$ also leaves invariant $\mathbb{C}\, e_2 + \cdots + \mathbb{C}\, e_{d+1}$ the orthogonal complement of $\mathbb{C}\, e_1$. So

$$\hat{K}_p \subset \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & A \end{pmatrix} : \lambda \in S^1, A \in \mathrm{U}(d) \right\}.$$

Now $\partial\Omega$ is not contained in a hyperplane and thus

$$\partial\Omega \cap \{[1 : z_1 : \cdots : z_d] : z_1, \dots, z_d \in \mathbb{C}\} \neq \emptyset.$$

So suppose that $x = [1 : x_1 : \cdots : x_d] \in \partial\Omega$ and let $R = \sum |x_i|^2$. Then K_p preserves the set

$$S = \{[1 : z_1 : \cdots : z_d] : \sum |z_i|^2 = R\}$$

and acts transitively on $\partial\Omega$. Thus $\partial\Omega$ is a subset of S. As $\partial\Omega$ is a $(2d-1)$-dimensional compact manifold this implies that $\partial\Omega = S$. Since $p = [1 : 0 : \cdots : 0] \in \Omega$ and Ω is convex in the affine chart $\{[1 : z_1 : \cdots : z_d] : z_1, \dots, z_d \in \mathbb{C}\}$ we have that

$$\Omega = \{[1 : z_1 : \cdots : z_d] : \sum |z_i|^2 < R^2\}.$$

CHAPTER V

Rigidity from symmetry

Recall that an open convex set $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is called *divisible* if there exists a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ such that $\Gamma \backslash \Omega$ is compact. In this section we will prove the main result of this thesis:

Theorem V.1. *Suppose Ω is a divisible proper weakly linearly convex open set with C^1 boundary. Then Ω is a projective ball.*

Remark V.2.

1. By Theorem II.4 weak linear convexity, linear convexity, and $\mathbb{C}$-convexity are all equivalent when $\partial\Omega$ is C^1 the above theorem could be restated with any of the three types of convexity.

2. The examples of real divisible convex sets constructed by deforming a real hyperbolic lattice and the examples constructed by Kapovich have word hyperbolic dividing groups. Then a theorem of Benoist [Ben04, Theorem 1.1] implies that all these examples have C^1 boundary and so in real projective geometry there are many examples of divisible proper convex open sets with C^1 boundary.

5.1 Outline of proof

The proof is divided into roughly three parts:

1. Section 5.3: showing that Γ is quasi-isometric to (Ω, d_Ω). This will be accomplished by showing that (Ω, d_Ω) is a quasi-geodesic metric space.

2. Section 5.4 and 5.5: showing that Γ contains bi-proximal elements. This will be accomplished by understanding how the shape of the boundary of Ω constrains the spectrum of elements in Γ.

3. Section 5.6: using bi-proximal elements in Γ to construct additional elements in $\mathrm{Aut}(\Omega)$.

4. Section 5.7 and 5.8: using these additional automorphisms to show that Ω satisfies the slice condition of Proposition IV.1.

5.2 Tangent spaces versus supporting hyperplanes

Before starting the proof we begin with a simple observation about the relationship between the tangent spaces of $\partial\Omega$ and hyperplanes tangent to Ω (in the sense of Observation II.7). Suppose Ω is a proper weakly linearly convex open set in $\mathbb{P}(\mathbb{C}^{d+1})$ with C^1 boundary. Then if $x \in \partial\Omega$ we can find a affine chart $\mathbb{C}^d$ containing x. In this affine chart we can identify $T_x\partial\Omega$ with a real hyperplane. Then let $T_x^{\mathbb{C}}\partial\Omega$ be the maximal complex subspace contained in $T_x\partial\Omega$. This will be a complex hyperplane through x. Now since Ω is weakly linearly convex there exists at least one complex hyperplane H through x which does not intersect Ω. Since Ω is C^1 this hyperplane must actually coincide with $T_x^{\mathbb{C}}\partial\Omega$. Summarizing this discussion:

Observation V.3. Suppose that Ω is a proper weakly linearly convex open set with C^1 boundary. If $x \in \partial\Omega$ then $T_x^{\mathbb{C}}\partial\Omega$ is the unique complex hyperplane passing through x and not intersecting Ω.

5.3 The complex Hilbert metric is quasi-geodesic and consequences

In this section we will prove that (Ω, d_Ω) is quasi-geodesic when Ω satisfies the hypothesis of Theorem I.8. We will then give some applications using the Švarc-Milnor Lemma.

Theorem V.4. *Suppose Ω is a divisible proper weakly linearly convex open set with C^1 boundary. Then (Ω, d_Ω) is a quasi-geodesic metric space.*

Theorem V.4 will follow from the next Proposition which shows the Hilbert metric on a $\mathbb{C}$-convex planar domain with C^1 boundary are always quasi-geodesic. But first we need some notation: for a C^1 embedding $f : S^1 \to \mathbb{C}$ it is well known that $\operatorname{Im}(f)$ separates $\mathbb{C}$ into two components one of which is bounded. We will denote this bounded component by Ω_f.

Proposition V.5. *Suppose $f : S^1 \to \mathbb{C}$ is a C^1 embedding, then there exists $K > 0$ such that (Ω_f, d_{Ω_f}) is $(1, K)$-quasi-isometric to $(\mathbb{D}, d_{\mathbb{D}})$. Moreover for all $\epsilon > 0$ there exists $\delta > 0$ such that if $g : S^1 \to \mathbb{C}$ is a C^1 embedding with*

$$\max_{e^{i\theta} \in S^1} \left\{ \left| f(e^{i\theta}) - g(e^{i\theta}) \right| + \left| df(e^{i\theta}) - dg(e^{i\theta}) \right| \right\} < \delta$$

then (Ω_g, d_{Ω_g}) is $(1, K + \epsilon)$-quasi-isometric to $(\mathbb{D}, d_{\mathbb{D}})$.

We will delay the proof of Proposition V.5 until the end of the section.

Proof of Theorem V.4. We first claim that if L is a complex projective line intersecting Ω then $L \cap \partial\Omega \subset L$ is a C^1 embedded submanifold. It is enough to show that L intersects $\partial\Omega$ transversally at every point $x \in \partial\Omega \cap L$. Suppose not, then there exists $x \in \partial\Omega \cap L$ such that $L \subset T_x \partial\Omega$. Since L is a complex subspace, this implies that $L \subset T_x^{\mathbb{C}} \partial\Omega$. But then by weak linear convexity $L \cap \Omega = \emptyset$. Thus we have a contradiction and so $L \cap \partial\Omega \subset L$ is a C^1 embedded submanifold. Since Ω is weakly linearly convex and $\partial\Omega$ is C^1, Ω is $\mathbb{C}$-convex (see for instance [APS04, Corollary 2.5.6]) and thus $L \cap \partial\Omega$ is an embedded copy of S^1.

Then, by Proposition V.5, for each complex projective line L intersecting Ω there exists $k(L) > 0$ such that $(\Omega \cap L, d_{\Omega\cap L})$ is $(1, k(L))$-quasi-isometric to $(\mathbb{D}, d_{\mathbb{D}})$. Moreover, for L' sufficiently close to L Proposition V.5 implies that $(\Omega \cap L', d_{\Omega\cap L'})$ is $(1, k(L) + 1)$-quasi-isometric to $(\mathbb{D}, d_{\mathbb{D}})$.

Now let $\Gamma \leq \mathrm{PSL}(\mathbb{C}^{d+1})$ be a dividing group, then there exists $K \subset \Omega$ compact such that $\Omega = \cup_{\gamma\in\Gamma}\gamma K$. The set of complex projective lines intersecting K is compact and so by the remarks above there exists $k > 0$ such that $(L\cap\Omega, d_{L\cap\Omega})$ is $(1, k)$-quasi-isometric to $(\mathbb{D}, d_{\mathbb{D}})$ for any complex projective line L intersecting K. In particular, if $x \in K$ and $y \in \Omega$ there is a $(1, k)$-quasi geodesic joining x to y. As $\Omega = \cup_{\gamma\in\Gamma}\gamma K$ we then have that any two points in Ω are joined by a $(1, k)$-quasi geodesic. □

Now suppose $\Gamma \leq \mathrm{PSL}(V)$ is a discrete group dividing a proper weakly linearly convex open set Ω with C^1 boundary. By the above theorem (Ω, d_Ω) is a quasi-geodesic metric space and by Proposition III.8 Γ acts properly on (Ω, d_Ω). Then by Theorem II.14, Γ is finitely generated and so by applying Selberg's Lemma we obtain:

Corollary V.6. *Suppose Ω is a proper weakly linearly convex open set with C^1 boundary. If Ω is divisible, then there exists a torsion free discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ such that Γ acts co-compactly, freely, and properly discontinuously on Ω.*

For torsion free dividing groups we have the following:

Corollary V.7. *Suppose Ω is a proper weakly linearly convex open set and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a torsion free discrete group dividing Ω. Then there exists $\epsilon > 0$ such that $d_\Omega(\gamma p, p) > \epsilon$ for all $\gamma \in \Gamma \setminus \{1\}$ and for all $p \in \Omega$.*

Proof. By Proposition III.8 the action $\mathrm{Aut}(\Omega)$ on Ω is proper. Thus the stabilizer K_p of any point $p \in \Omega$ is compact. Since Γ is torsion free and discrete $\Gamma \cap K_p = \{1\}$ for any $p \in \Omega$ and hence

$$\inf_{\gamma \in \Gamma \setminus \{1\}} d_\Omega(\gamma p, p) > 0$$

for all $p \in \Omega$. Now since Γ divides Ω, there exists $K \subset \Omega$ compact such that $\Omega = \cup_{\gamma \in \Gamma} \gamma K$. Then

$$\inf_{p \in \Omega} \inf_{\gamma \in \Gamma \setminus \{1\}} d_\Omega(\gamma p, p) = \inf_{p \in K} \inf_{\gamma \in \Gamma \setminus \{1\}} d_\Omega(\gamma p, p) > 0.$$

□

5.3.1 Proof of Proposition V.5

Proposition V.5 will follow from the next three lemmas.

Lemma V.8. *Suppose $\Omega_1, \Omega_2 \subset \mathbb{C}$ are open bounded sets. If $F : \overline{\Omega}_1 \to \overline{\Omega}_2$ is a k-bi-Lipschitz homeomorphism with $F(\Omega_1) = \Omega_2$, then F induces a $(1, 4\log k)$-quasi-isometry $(\Omega_1, d_{\Omega_1}) \to (\Omega_2, d_{\Omega_2})$.*

Proof. Since

$$\frac{1}{k}|x-y| \leq |F(x) - F(y)| \leq k|x-y|$$

for all $x, y \in \overline{\Omega}_1$ we have that

$$d_{\Omega_1}(x,y) - 4\log k \leq d_{\Omega_2}(F(x), F(y)) \leq d_{\Omega_1}(x,y) + 4\log k$$

for all $x, y \in \Omega_1$. □

Lemma V.9. *Suppose $f : S^1 \to \mathbb{C}$ is a C^1 embedding, then for all $\epsilon > 0$ there exists $\delta > 0$ such that if $g : S^1 \to \mathbb{C}$ is a C^1 embedding with*

$$\max_{e^{i\theta} \in S^1} \left\{ \left| f(e^{i\theta}) - g(e^{i\theta}) \right| + \left| df(e^{i\theta}) - dg(e^{i\theta}) \right| \right\} < \delta$$

then there exists $F : \overline{\Omega}_f \to \overline{\Omega}_g$ *a* $(1+\epsilon)$*-bi-Lipschitz homeomorphism with* $F(\Omega_f) = \Omega_g$.

Proof. Since $f : S^1 \to \mathbb{C}$ is a C^1 embedding there exists a collar neighborhood extension $\Phi : \{1-\eta \leq |z| \leq 1+\eta\} \to \mathbb{C}$. Then if δ is small enough, $\mathrm{Im}(g)$ can be parameterized by $e^{i\theta} \to \Phi(r(e^{i\theta})e^{i\theta})$ for some C^1 function $r : S^1 \to (1-\eta, 1+\eta)$. By further shrinking δ, it is easy to construct a C^1 diffeomorphism $F : \overline{\Omega}_f \to \overline{\Omega}_g$ such that $|F'(z)|$ and $|(F^{-1})'(z)|$ are bounded by $(1+\epsilon)$. Thus $F : \overline{\Omega}_f \to \overline{\Omega}_g$ is $(1+\epsilon)$-bi-Lipschitz. □

Lemma V.10. *Suppose* $f : S^1 \to \mathbb{C}$ *is a* C^∞ *embedding, then there exists a* k*-bi-Lipschitz homeomorphism* $F : \overline{\mathbb{D}} \to \overline{\Omega}_f$ *with* $F(\mathbb{D}) = \Omega_f$.

Proof. This (and more) follows from the smooth version of the Riemann mapping theorem (see for instance [Tay11, Chapter 5, Theorem 4.1]). □

We can now prove Proposition V.5.

Proof of Proposition V.5. Suppose $f : S^1 \to \mathbb{C}$ is a C^1 embedding. Then since any C^1 embedding can be approximated by a C^∞ embedding, Lemma V.9 and Lemma V.10 implies the existence of a k-bi-Lipschitz map $F : \overline{\mathbb{D}} \to \overline{\Omega}_f$. By Lemma V.8 this induces a $(1, 4\log(k))$-quasi-isometry $(\mathbb{D}, d_{\mathbb{D}}) \to (\Omega_f, d_{\Omega_f})$. Finally the "moreover" part of the proposition is just Lemma V.8 and Lemma V.9. □

5.4 Every element is bi-proximal or almost unipotent

For V a complex $(d+1)$-dimensional vector space and $\varphi \in \mathrm{PSL}(V)$ let

$$\sigma_1(\varphi) \leq \sigma_2(\varphi) \leq \cdots \leq \sigma_{d+1}(\varphi)$$

be the absolute value of the eigenvalues (counted with multiplicity) of φ. Since we are considering absolute values this is well defined.

Definition V.11.

1. An element $\varphi \in \mathrm{PSL}(V)$ is called *proximal* if $\sigma_d(\varphi) < \sigma_{d+1}(\varphi)$ and is called *bi-proximal* if φ and φ^{-1} are proximal. When φ is bi-proximal let x_φ^+ and x_φ^- be the eigenlines in $\mathbb{P}(V)$ corresponding to $\sigma_{d+1}(\varphi)$ and $\sigma_1(\varphi)$.

2. An element $\varphi \in \mathrm{PSL}(V)$ is called *almost unipotent* if

$$\sigma_1(\varphi) = \sigma_2(\varphi) = \cdots = \sigma_{d+1}(\varphi) = 1.$$

The purpose of this section is to prove the following.

Theorem V.12. *Suppose Ω is a proper weakly linearly convex open set with C^1 boundary. If $\Gamma \leq \mathrm{PSL}(\mathbb{C}^{d+1})$ divides Ω then every $\gamma \in \Gamma \setminus \{1\}$ is bi-proximal or almost unipotent. Moreover if $\varphi \in \mathrm{Aut}(\Omega)$ is bi-proximal then*

1. $x_\varphi^+, x_\varphi^- \in \partial\Omega$,

2. $T^{\mathbb{C}}_{x_\varphi^+}\partial\Omega \cap \partial\Omega = \{x_\varphi^+\}$,

3. $T^{\mathbb{C}}_{x_\varphi^-}\partial\Omega \cap \partial\Omega = \{x_\varphi^-\}$, *and*

4. *if $U^+ \subset \overline{\Omega}$ is a neighborhood of x_φ^+ and $U^- \subset \overline{\Omega}$ is a neighborhood of x_φ^- then there exists $N > 0$ such that for all $m > N$ we have*

$$\varphi^m(\partial\Omega \setminus U^-) \subset U^+ \text{ and } \varphi^{-m}(\partial\Omega \setminus U^+) \subset U^-.$$

Remark V.13. Notice that in the second part of theorem we allow φ to be any bi-proximal element in $\mathrm{Aut}(\Omega)$.

Given an element $\varphi \in \mathrm{SL}(V)$ let $m^+(\varphi)$ be the size of the largest Jordan block of φ whose corresponding eigenvalue has absolute value $\sigma_{d+1}(\varphi)$. Next let $E^+(\varphi)$ be the span of the eigenvectors of φ whose eigenvalue have absolute value $\sigma_{d+1}(\varphi)$ and are part of a Jordan block with size $m^+(\varphi)$. Also define $E^-(\varphi) = E^+(\varphi^{-1})$.

Given $y \in \mathbb{P}(V)$ let $L(\varphi, y) \subset \mathbb{P}(V)$ denote the limit points of the sequence $\{\varphi^n y\}_{n\in\mathbb{N}}$. With this notation we have the following observations:

Proposition V.14. *Suppose $\varphi \in \mathrm{SL}(V)$ and $\{\varphi^n\}_{n\in\mathbb{N}} \subset \mathrm{SL}(V)$ is unbounded, then*

1. *there exists a proper projective subspace $H \subsetneq \mathbb{P}(V)$ such that $L(\varphi, y) \subset [E^+(\varphi)]$ for all $y \in \mathbb{P}(V) \setminus H$,*
2. *φ acts recurrently on $[E^+(\varphi)] \subset \mathbb{P}(V)$, that is for all $y \in [E^+(\varphi)]$ there exists $n_k \to \infty$ such that $\varphi^{n_k} y \to y$,*
3. *$E^-(\varphi) \subset \ker f$ for all $f \in E^+({}^t\varphi)$.*

Proof. All three statements follow easily once φ is written in Jordan normal form. □

Lemma V.15. *Suppose Ω is a proper weakly linearly convex open set with C^1 boundary and $\varphi \in \mathrm{Aut}(\Omega)$ such that $\{\varphi^n\}_{n\in\mathbb{N}} \subset \mathrm{PSL}(\mathbb{C}^{d+1})$ is unbounded. Then $E^{\pm}(\varphi) = x^{\pm}$ for some $x^{\pm} \in \partial\Omega$ and $E^{\pm}({}^t\varphi) = f^{\pm}$ for some $f^{\pm} \in \Omega^*$. Moreover $T^{\mathbb{C}}_{x^{\pm}}\partial\Omega = [\ker f^{\mp}]$.*

Proof. We will break the proof of the lemma into a series of claims.

Claim 1: *$[E^+({}^t\varphi)] \cap \Omega^*$ is non-empty.*

By part (1) of Proposition V.14 there exists a hyperplane $H \subset \mathbb{P}(V^*)$ such that $L({}^t\varphi, f) \subset [E^+({}^t\varphi)]$ for all $f \in \mathbb{P}(V^*) \setminus H$. By Proposition II.10, Ω^* is not contained in a hyperplane and so there exists $f \in \Omega^* \setminus \Omega^* \cap H$. Then as Ω^* is compact and ${}^t\varphi$-invariant, $L({}^t\varphi, f) \subset \Omega^*$ and thus $[E^+({}^t\varphi)] \cap \Omega^* \neq \emptyset$.

Claim 2: *$[E^-(\varphi)] \cap \Omega = \emptyset$ and $[E^-(\varphi)] \cap \partial\Omega \neq \emptyset$. In particular, since $\partial\Omega$ is C^1 if*

$x \in [E^-(\varphi)] \cap \partial\Omega$ then $[E^-(\varphi)] \subset T_x^{\mathbb{C}}\partial\Omega$.

By Proposition III.8, $\mathrm{Aut}(\Omega)$ acts properly on Ω and hence for any $y \in \Omega$ the set

$$\{n \in \mathbb{N} : d_\Omega(\varphi^{-n}y, y) \leq 1\}$$

is finite. So $[E^-(\varphi)] \cap \Omega = \emptyset$ by part (2) of Proposition V.14. Since Ω is open, part (1) of Proposition V.14 implies the existence of some $y \in \Omega$ such that $L(\varphi^{-1}, y) \subset [E^-(\varphi)]$. Since Ω in φ-invariant $L(\varphi^{-1}, y) \subset \overline{\Omega}$. Thus $[E^-(\varphi)] \cap \overline{\Omega} \neq \emptyset$.

Claim 3: *$\{f^+\} = [E^+({}^t\varphi)] \cap \Omega^*$ for some $f^+ \in \mathbb{P}(V^*)$ and $[\ker f^+] = T_x^{\mathbb{C}}\partial\Omega$ for any $x \in [E^-(\varphi)] \cap \partial\Omega$.*

Suppose $f \in [E^+({}^t\varphi)] \cap \Omega^*$ then by part (3) of Proposition V.14, $E^-(\varphi) \subset \ker f$ and by the definition of Ω^*, $[\ker f] \cap \Omega = \emptyset$. Thus if $x \in [E^-(\varphi)] \cap \partial\Omega$ then $[\ker f]$ is a complex tangent hyperplane of Ω at x. Since $\partial\Omega$ is C^1 this implies that $[\ker f] = T_x^{\mathbb{C}}\partial\Omega$. As $f \in [E^+({}^t\varphi)] \cap \Omega^*$ was arbitrary this implies the claim.

Claim 4: *$f^+ = E^+({}^t\varphi)$ for some $f^+ \in \Omega^*$.*

Pick representatives $\hat{\varphi}_n \in \mathrm{GL}(V^*)$ of ${}^t\varphi^n \in \mathrm{PSL}(V^*)$ such that $\|\hat{\varphi}_n\| = 1$. Then there exists $n_k \to \infty$ such that $\hat{\varphi}_{n_k}$ converges to a linear endomorphism $\hat{\varphi}_\infty \in \mathrm{End}(V^*)$. By construction $\hat{\varphi}_\infty(g) \in L({}^t\varphi, g)$ for any $g \in \mathbb{P}(V^*) \setminus [\ker \hat{\varphi}_\infty]$. Also by using the Jordan normal form one can check that $\hat{\varphi}_\infty(V^*) = E^+({}^t\varphi)$. Select $f \in \mathbb{P}(V^*)$ such that $\hat{\varphi}_\infty(f) = f^+$. Then viewing f and f^+ as complex one dimensional subspaces of V^* we see that

$$W := \{v \in V^* : \hat{\varphi}_\infty(v) \in f^+\} = f + \ker \hat{\varphi}_\infty.$$

Notice that

$$\dim_{\mathbb{C}} W = 1 + \dim_{\mathbb{C}} \ker \hat{\varphi}_\infty = d + 2 - \dim_{\mathbb{C}} E^+({}^t\varphi).$$

Finally assume for a contradiction that $\dim_{\mathbb{C}} E^+({}^t\varphi) > 1$. In this case $[W]$ is a proper projective subspace of $\mathbb{P}(V^*)$. By Proposition II.10, Ω^* is not contained in a hyperplane and thus there exists $g \in \Omega^* \setminus \Omega^* \cap [W]$. Since $\ker \hat{\varphi}_\infty \subset W$, $g \notin [\ker \hat{\varphi}_\infty]$ and so $\hat{\varphi}_\infty g$ is well defined in $\mathbb{P}(V^*)$. As $\hat{\varphi}_\infty(V^*) = E^+({}^t\varphi)$ we then have that $\hat{\varphi}_\infty g \in [E^+({}^t\varphi)]$. Since Ω^* is compact and ${}^t\varphi$-invariant, $\hat{\varphi}_\infty g \in L({}^t\varphi, g) \subset \Omega^*$. Thus by Claim 3 we must have that $\hat{\varphi}_\infty g = f^+$. But this contradicts the fact that $g \notin [W]$. So we have a contradiction and so $E^+({}^t\varphi)$ must be a one complex dimensional subspace.

Claim 5: *$x^+ = E^+(\varphi)$ for some $x^+ \in \partial\Omega$.*

The property of $E^+(\varphi)$ having dimension one depends only on the Jordan block structure of φ. As φ and ${}^t\varphi$ have the same Jordan block structure Claim 4 implies that $E^+(\varphi) = x^+$ for some $x^+ \in \mathbb{P}(\mathbb{C}^{d+1})$. By repeating the argument in the proof of Claim 2 we see that $x^+ \in \partial\Omega$.

Claim 6: *Lemma V.15 is true.*

Summarizing our conclusions so far: we have that $E^+(\varphi) = x^+$ for some $x^+ \in \partial\Omega$, $E^+({}^t\varphi) = f^+$ for some $f^+ \in \Omega^*$, and $[\ker f^+]$ is a complex tangent hyperplane of Ω containing $[E^-(\varphi)]$. Thus applying the above argument to φ^{-1} we see that $E^-(\varphi) = x^-$ for some $x^- \in \partial\Omega$, $E^-({}^t\varphi) = f^-$ for some $f^- \in \Omega^*$, and $[\ker f^-]$ is a complex tangent hyperplane of Ω containing $[E^+(\varphi)]$. Since $E^\pm(\varphi) = x^\pm$ we see that $[\ker f^\mp]$ is a complex tangent hyperplane containing $x^\pm$ and thus $T^{\mathbb{C}}_{x^\pm}\partial\Omega = [\ker f^\mp]$. □

Since each $\varphi \in \mathrm{SL}(V)$ is either almost unipotent or $\sigma_{d+1}(\varphi) > \sigma_1(\varphi)$, Theorem V.12 will follow from the next lemma.

Lemma V.16. *Suppose Ω is a proper weakly linearly convex open set with C^1 boundary and $\varphi \in \mathrm{Aut}(\Omega)$ is such that $\sigma_{d+1}(\varphi) > \sigma_1(\varphi)$. Then φ is bi-proximal and*

1. *$T^{\mathbb{C}}_{x^+_\varphi}\partial\Omega \cap \partial\Omega = \{x^+_\varphi\}$,*

2. *$T^{\mathbb{C}}_{x^-_\varphi}\partial\Omega \cap \partial\Omega = \{x^-_\varphi\}$,*

3. *if $U^+ \subset \overline{\Omega}$ is a neighborhood of x^+_φ and $U^- \subset \overline{\Omega}$ is a neighborhood of x^-_φ then there exists $N > 0$ such that for all $m > N$ we have*

$$\varphi^m(\partial\Omega \setminus U^-) \subset U^+ \text{ and } \varphi^{-m}(\partial\Omega \setminus U^+) \subset U^-.$$

Proof. Let $E^\pm({}^t\varphi) = f^\pm \in \Omega^*$ and $E^\pm(\varphi) = x^\pm \in \partial\Omega$. By the previous lemma $[\ker f^\pm] = T^{\mathbb{C}}_{x^\mp}\partial\Omega$. Since $\sigma_{d+1}(\varphi) > \sigma_1(\varphi)$ we see that $E^+({}^t\varphi) \neq E^-({}^t\varphi)$ and so $f^+ \neq f^-$. Since $\partial\Omega$ is C^1, x^+ is contained in unique complex tangent hyperplane and so $x^+ \notin [\ker f^+]$. For the same reason $x^- \notin [\ker f^-]$.

Then there exists a basis $e_1, e_2, \dots, e_{d+1}$ of $\mathbb{C}^{d+1}$ such that $\mathbb{C}\cdot e_1 = x^+$, $\mathbb{C}\cdot e_2 = x^-$, and $\ker f^+ \cap \ker f^-$ is the span of $e_3, \dots, e_{d+1}$. Since x^+, x^-, and $\ker f^+ \cap \ker f^-$ are φ-invariant, with respect to this basis φ is represented by a matrix of the form

$$\begin{pmatrix} \lambda^+ & 0 & 0 \\ 0 & \lambda^- & 0 \\ 0 & 0 & A \end{pmatrix} \in \mathrm{SL}(\mathbb{C}^{d+1})$$

where A is some $(d-1)$-by-$(d-1)$ matrix. Finally since $E^+(\varphi) = x^+$ and $E^-(\varphi) = x^-$ we see that φ is bi-proximal.

We now show part (1) of the lemma, that is $[\ker f^-] \cap \partial\Omega = \{x^+\}$. If $x \in [\ker f^-] \cap \partial\Omega$ then with respect to the basis above $x = [w_1 : 0 : w_2 : \cdots : w_d]$ for some $w_1, \dots, w_d \in \mathbb{C}$. If $w_1 = 0$ then $x \in [\ker f^+]$. But then $[\ker f^+]$ and $[\ker f^-]$ are complex tangent hyperplanes to $\partial\Omega$ at x. Since $\partial\Omega$ is C^1 this implies that $[\ker f^+] = T^{\mathbb{C}}_x\partial\Omega = [\ker f^-]$ which contradicts the fact that f^+ and f^- are distinct points in $\mathbb{P}(V^*)$. So $w_1 \neq 0$, but then either $x = x^+$ or any limit point of $\{\varphi^{-n}x\}_{n\in\mathbb{N}}$

is in $[\ker f^+] \cap [\ker f^-] \cap \partial\Omega$ which we just showed is empty. So $[\ker f^-] \cap \partial\Omega = \{x^+\}$. A similar argument shows that $[\ker f^+] \cap \partial\Omega = \{x^-\}$.

By part (2) of the lemma, with respect to the basis above

$$\overline{\Omega} \setminus \{x^-\} \subset \{[1 : z_1 : \dots z_d] : z_1, \dots, z_d \in \mathbb{C}\}.$$

Thus for all $x \in \overline{\Omega}$ either $x = x^-$ or $\varphi^m x \to x^+$ as $m \to \infty$. In a similar fashion, for all $x \in \overline{\Omega}$ either $x = x^+$ or $\varphi^m x \to x^-$ as $m \to -\infty$. Thus part (3) holds. $\square$

5.5 There is a bi-proximal element

The purpose of this section is to prove the following.

Theorem V.17. *Suppose Ω is a proper weakly linearly convex open set with C^1 boundary. If $\Gamma \leq \mathrm{PSL}(\mathbb{C}^{d+1})$ divides Ω then some $\gamma \in \Gamma \setminus \{1\}$ is bi-proximal.*

Remark V.18. Using Proposition III.11, if $\varphi \in \mathrm{Aut}(\Omega)$ is almost unipotent and $x_0 \in \Omega$ then we have an estimate of the form:

$$d_\Omega(\varphi^N x_0, x_0) \leq R + \log\left(\left\|\varphi^N\right\| \left\|\varphi^{-N}\right\|\right) \leq A + B\log(N). \tag{5.1}$$

In particular, if we knew that every non-trivial element of Γ is an "axial isometry" then we would immediately deduce that every non-trivial element of Γ is bi-proximal. Unfortunately, we do not see a direct way of establishing that every non-trivial element of Γ is an "axial isometry."

We will start with a definition, but first let $\mathrm{SL}^*(\mathbb{C}^{d+1}) = \{\varphi \in \mathrm{GL}(\mathbb{C}^{d+1}) : |\det \varphi| = 1\}$.

Definition V.19. A connected closed Lie subgroup $G \leq \mathrm{SL}^*(\mathbb{C}^{d+1})$ is called *almost unipotent* if there exists a flag

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k \subsetneq V_{k+1} = \mathbb{C}^{d+1}$$

preserved by G such that if $G_{i+1} \leq \mathrm{GL}(V_{i+1}/V_i)$ is the projection of G then G_{i+1} is bounded.

Remark V.20. Notice that a group G is unipotent if and only if there exists a complete flag

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d \subsetneq V_{d+1} = \mathbb{C}^{d+1}$$

preserved by G such that if $G_{i+1} \leq \mathrm{GL}(V_{i+1}/V_i)$ is the projection of G then $G_{i+1} = \{1\}$.

The proof of Theorem V.17 will use the next two propositions.

Proposition V.21. *Suppose $\Gamma \leq \mathrm{SL}(\mathbb{C}^{d+1})$ is a subgroup such that every $\gamma \in \Gamma$ is almost unipotent. Let G be the Zariski closure of Γ in $\mathrm{SL}(\mathbb{R}^{2d+2})$. If G is connected, then G is almost unipotent.*

Proposition V.22. *Suppose $G \leq \mathrm{SL}(\mathbb{C}^{d+1})$ is a connected closed Lie subgroup. If G is almost unipotent and $g_1, \ldots, g_k$ are fixed elements in G then there exists a constant $C > 0$ such that for all $N > 0$ and $i_1, \ldots, i_N \in \{1, \ldots, k\}$*

$$\|g_{i_1} \cdots g_{i_N}\| \leq CN^d.$$

Delaying the proof of the propositions we will prove Theorem V.17.

Proof of Theorem V.17. Suppose for a contradiction that Γ contains no bi-proximal elements, then by Theorem V.12 every element of Γ is almost unipotent. If $\pi : \mathrm{SL}(\mathbb{C}^{d+1}) \to \mathrm{PSL}(\mathbb{C}^{d+1})$ is the natural projection, then there exists a finite index subgroup $\Gamma' \leq \pi^{-1}(\Gamma)$ such that the Zariski closure of Γ' is connected and Γ' is torsion free. Since Γ' is torsion free π induces an isomorphism $\Gamma' \to \pi(\Gamma')$ and by construction $\pi(\Gamma') \leq \Gamma$ will have finite index. Then $\pi(\Gamma')$ divides Ω and Γ' is finitely generated by Theorem V.4, Proposition III.8, and Theorem II.14. Now fix a finite generating set $S = \{s_1, \dots, s_k\} \subset \Gamma'$ and a point $x_0 \in \Omega$. By Proposition II.14 there exists $A, B > 0$ such that the map $\gamma \in \Gamma' \to \gamma \cdot x_0$ is an (A, B)-quasi-isometry between (Γ', d_S) and (Ω, d_Ω).

Since Γ' has infinite order, there exists a sequence $i_1, i_2, \dots \in \{1, \dots, k\}$ such that the map

$$N \in \mathbb{N} \to \gamma_{i_N} \cdots \gamma_{i_2} \gamma_{i_1}$$

is a geodesic with respect to the word metric, that is

$$d_S(\gamma_{i_N} \gamma_{i_{N-1}} \cdots \gamma_{i_1}, 1) = N$$

for all $N > 0$. Then

$$\frac{1}{A} N - B \leq d_\Omega(\gamma_{i_N} \gamma_{i_{N-1}} \cdots \gamma_{i_1} x_0, x_0) \leq AN + B \tag{5.2}$$

for all $N > 0$.

Since the Zariski closure of Γ' is connected, Proposition V.21 and Proposition V.22 implies the existence of $C > 0$ such that

$$\|\gamma_{i_1} \cdots \gamma_{i_N}\| \leq C N^d \text{ and } \left\|\gamma_{i_N}^{-1} \cdots \gamma_{i_1}^{-1}\right\| \leq C N^d$$

for all $N > 0$.

Then Proposition III.11 implies that:

$$d_\Omega(\gamma_{i_N} \cdots \gamma_{i_1} x_0, x_0) \leq R + \log\left(\|\gamma_{i_N} \cdots \gamma_{i_1}\| \left\|\gamma_{i_1}^{-1} \cdots \gamma_{i_N}^{-1}\right\|\right) \leq R + 2 \log C N^d$$

for some $R > 0$ depending only on x_0. This contradicts the estimate in equation (5.2) and hence Γ must contain a bi-proximal element. $\square$

We begin the proof of Proposition V.21 with a lemma that follows easily from the main result in [Pra94]:

Lemma V.23. *[Pra94] Suppose* $\Gamma \leq \mathrm{SL}(\mathbb{C}^{d+1})$ *is a subgroup such that every* $\gamma \in \Gamma$ *is almost unipotent. Let* G *be the Zariski closure of* Γ *in* $\mathrm{SL}(\mathbb{R}^{2d+2})$*. If* G *is connected and reductive, then* G *is compact.*

Proof of Proposition V.21. We will induct on d. When $d = 0$, the proposition is trivial so suppose $d > 0$.

If G is reductive then the above lemma implies that G is compact. Then G is an almost unipotent group with respect to the flag $\{0\} \subsetneq \mathbb{C}^{d+1}$.

If G is not reductive there exists a connected, non-trivial, normal unipotent group $U \leq G$. By Engel's theorem the vector subspace

$$V = \{v \in \mathbb{C}^{d+1} : uv = v \text{ for all } u \in U\}$$

is non-empty. Since U is non-trivial, V is a proper subspace. Since U is normal in G, G preserves the flag $\{0\} \subsetneq V \subsetneq \mathbb{C}^d$. Now let G_1 be the Zariski closure of $G|_V$ and let $\Gamma_1 = \Gamma|_V$ then Γ_1 is Zariski dense in G_1. Moreover each element of Γ_1 is almost unipotent and hence $\Gamma_1 \leq \mathrm{SL}^*(V)$. Since Γ_1 is Zariski dense in G_1 we see that $G_1 \leq \mathrm{SL}^*(V)$. Thus by induction G_1 preserves a flag of the form

$$\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k = V$$

where the image of G_1 into $\mathrm{GL}(V_{i+1}/V_i)$ is bounded.

In a similar fashion let G_2 be the Zariski closure of the image of G in $\mathrm{GL}(\mathbb{C}^d/V)$ and let Γ_2 be the image of Γ in $\mathrm{GL}(\mathbb{C}^d/V)$. Then Γ_2 will be Zariski dense in G_2. Moreover each element of Γ_2 is almost unipotent and hence $\Gamma_2 \leq \mathrm{SL}^*(\mathbb{C}^d/V)$. Since Γ_2 is Zariski dense in G_2 we see that $G_2 \leq \mathrm{SL}^*(\mathbb{C}^d/V)$. Thus by induction G_2 preserves a flag of the form

$$\{0\} = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_\ell = \mathbb{C}^d/V$$

where $W_i = V_{k+i}/V$ and the image of G_2 into $\mathrm{GL}(W_{i+1}/W_i)$ is bounded.

All this implies that G preserves the flag

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{k+\ell} = \mathbb{C}^d$$

and the image of G into $\mathrm{GL}(V_{i+1}/V_i)$ is bounded. Hence we see that G is almost unipotent. $\square$

Proof of Proposition V.22. After conjugating G, there exists a compact group $K \leq \mathrm{SU}(d+1)$ and a upper triangular group $U \leq \mathrm{SL}(\mathbb{C}^{d+1})$ with ones on the diagonal such that K normalizes U and $G \leq KU$. In particular, we can assume $G = KU$. Now for $\varphi \in \mathrm{End}(\mathbb{C}^{d+1})$ define $|\varphi| := \sup\{|u_{i,j}|\}$. Let $\|\cdot\|$ be the operator norm on $\mathrm{End}(\mathbb{C}^{d+1})$ associated to the standard inner product norm on $\mathbb{C}^{d+1}$. Then

$$\|k_1 \varphi k_2\| = \|\varphi\|$$

for $k_1, k_2 \in \mathrm{SU}(d+1)$ and $\varphi \in \mathrm{End}(\mathbb{C}^{d+1})$. Moreover, since $\|\cdot\|$ and $|\cdot|$ are norms on $\mathrm{End}(\mathbb{C}^{d+1})$ there exists $\alpha > 0$ such that

$$\frac{1}{\alpha}\,|\varphi| \leq \|\varphi\| \leq \alpha\,|\varphi|$$

for all $\varphi \in \mathrm{End}(\mathbb{C}^{d+1})$. In particular, for $k \in K$ and $u \in U$ we have

$$\left|kuk^{-1}\right| \leq \alpha \left\|kuk^{-1}\right\| = \alpha\,\|u\| \leq \alpha^2\,|u|\,.$$

Now let $g_1, \ldots, g_k$ be as in the statement of the proposition. Then $g_i = k_i u_i$ for some $k_i \in K$ and $u_i \in U$. Since K normalizes U we see that

$$g_{i_1} \cdots g_{i_N} = k u_1' \cdots u_N'$$

for some $k \in K$ and some $u_k' \in U$ with $|u_k'| \leq \alpha^2\,|u_{i_k}|$. Since

$$\|g_{i_1} \cdots g_{i_N}\| = \|u_1' \cdots u_N'\| \leq \alpha\,|u_1' \cdots u_N'|$$

the proposition will follow from the claim:

Claim: For any $R > 0$, there exists $C = C(R) > 0$ such that for any $u_1, \ldots, u_N \in U$ with $|u_i| < R$ we have $|u_1 \cdots u_N| \le CN^d$.

Now for $i < j$

$$(u_1 \cdots u_N)_{i,j} = \sum_{i=a_0 \le a_1 \le \cdots \le a_N = j} (u_1)_{ia_1}(u_2)_{a_1 a_2} \cdots (u_N)_{a_{N-1}j}$$

Since U is upper triangular with ones on the diagonal at most d terms in the product $(u_1)_{ia_1}(u_2)_{a_1a_2} \cdots (u_N)_{a_{N-1}j}$ are not equal to one and so

$$\left|(u_1)_{ia_1}(u_2)_{a_1a_2} \cdots (u_N)_{a_{N-1}j}\right| \le R^d.$$

Now we estimate the number of terms in the sum. Notice that $a_{k+1} - a_k \ge 0$ and

$$\sum_{i=0}^{N-1} a_{k+1} - a_k = j - i.$$

Thus we need to estimate the number of ways to write $j - i$ as the sum of N non-negative integers (where order matters). First let $C_n(j-i)$ be the number of ways to write $j - i$ as the sum of n positive integers. Next, at most $j - i$ of the $a_{k+1} - a_k$ are positive and hence the number of ways to write $j - i$ as the sum of N non-negative integers is at most

$$\sum_{n=1}^{j-i} \begin{pmatrix} N \\ n \end{pmatrix} C_n(j-i).$$

Then

$$|(u_1 \cdots u_N)_{i,j}| \le \sum_{i=a_0 \le a_1 \le \cdots \le a_N = j} \left|(u_1)_{ia_1}(u_2)_{a_1a_2} \cdots (u_N)_{a_{N-1}j}\right| \le R^d \sum_{n=1}^{j-i} \begin{pmatrix} N \\ n \end{pmatrix} C_n(j-i)$$

and since $j - i \leq d$ there exists $C = C(R) > 0$ such that

$$R^d \sum_{n=1}^{j-i} \begin{pmatrix} N \\ n \end{pmatrix} C_n(j-i) < CN^d$$

for any $i < j$. $\square$

5.6 Constructing additional automorphisms

Suppose Ω is a proper weakly linearly convex open set with C^1 boundary. If $\varphi \in \mathrm{Aut}(\Omega)$ is bi-proximal, then we have the following standard form. First let $H^{\pm}$ be the complex tangent hyperplane at $x^{\pm}_{\varphi}$. Then pick coordinates such that

1. $x^+_{\varphi} = [1 : 0 : \cdots : 0]$,
2. $x^-_{\varphi} = [0 : 1 : 0 : \cdots : 0]$,
3. $H^+ \cap H^- = \{[0 : 0 : z_2 : \cdots : z_d]\}$.

With respect to these coordinates, φ is represented by a matrix of the form

$$\begin{pmatrix} \lambda^+ & 0 & {}^t\vec{0} \\ 0 & \lambda^- & {}^t\vec{0} \\ \vec{0} & \vec{0} & A \end{pmatrix}$$

where A is a $(d-1)$-by-$(d-1)$ matrix. Since $H^- = \{[0 : z_1 : \cdots : z_d]\}$ and $\Omega \cap H^- = \emptyset$ we see that Ω is contained in the affine chart $\mathbb{C}^d = \{[1 : z_1 : \cdots : z_d] : z_1, \dots, z_d \in \mathbb{C}\}$. In this affine chart x^+_{φ} corresponds to 0 and $T^{\mathbb{C}}_0 \partial\Omega = \{0\} \times \mathbb{C}^{d-1}$. Then by a projective transformation we may assume that

4. $T_0 \partial\Omega = \mathbb{R} \times \mathbb{C}^{d-1}$.

Since $\partial\Omega$ is C^1 there exists open neighborhoods $V, W \subset \mathbb{R}$ of 0, an open neighborhood $U \subset \mathbb{C}^{d-1}$ of $\vec{0}$, and a C^1 function $F : V \times U \to W$ such that if $\mathcal{O} = (V + iW) \times U$ then

5. $\partial\Omega \cap \mathcal{O} = \mathrm{Graph}(F) = \{(x + iF(x, \vec{z}), \vec{z}) : x \in V, \vec{z} \in U\}$.

By another projective transformation we can assume

6. $\Omega \cap \mathcal{O} = \{(x + iy, \vec{z}) \in \mathcal{O} : y > F(x, \vec{z})\}$.

Theorem V.24. *With the choice of coordinates above,*

$$\Omega \cap \{[z_1 : z_2 : 0 : \cdots : 0]\} = \{[1 : z : 0, \cdots : 0] : \mathrm{Im}(z) > 0\}.$$

Moreover for $h \in \mathrm{SL}(\mathbb{R}^2)$ the projective transformation defined by

$$\psi_h = \begin{pmatrix} h & 0 \\ 0 & Id \end{pmatrix}$$

is in $\mathrm{Aut}_0(\Omega)$.

Proof. We can assume $\mathcal{O}$ is bounded. Then using part (4) of Theorem V.12 we can replace φ with a power of φ so that $\varphi(\mathcal{O}) \subset \mathcal{O}$.

We first claim that $F(x, \vec{z}) = F(0, \vec{z})$ for $(x, \vec{z}) \in V \times U$. Notice that with our choice of coordinates φ acts by

$$\varphi \cdot (z_1, \vec{z}) = \left(\frac{\lambda^- z_1}{\lambda^+}, \frac{A\vec{z}}{\lambda^+} \right)$$

where $\lambda^{\pm}$ and A are as above. Since φ is bi-proximal

$$\left(\frac{A}{\lambda^+} \right)^n \to 0.$$

Since φ preserves $T_0\partial\Omega = \mathbb{R} \times \mathbb{C}^{d-1}$ we see that $\lambda^-/\lambda^+ \in \mathbb{R}$. Since x_φ^+ is an attracting fixed point we have $\lambda^-/\lambda^+ \in (-1, 1)$. Finally since

$$\varphi \cdot (x + iF(x, \vec{z}), \vec{z}) = \left(\frac{\lambda^-}{\lambda^+} x + i \frac{\lambda^-}{\lambda^+} F(x, \vec{z}), \frac{A}{\lambda^+} \vec{z} \right)$$

and $\varphi(\mathcal{O}) \subset \mathcal{O}$ we see that

$$F\left(\frac{\lambda^-}{\lambda^+} x, \frac{A}{\lambda^+} \vec{z} \right) = \frac{\lambda^-}{\lambda^+} F(x, \vec{z}).$$

Differentiating with respect to x yields

$$(\partial_x F)(x, \vec{z}) = (\partial_x F)\left(\frac{\lambda^-}{\lambda^+}x, \frac{A}{\lambda^+}\vec{z}\right)$$

and repeated applications of the above formula shows

$$(\partial_x)F(x, \vec{z}) = (\partial_x F)\left(\left(\frac{\lambda^-}{\lambda^+}\right)^n x, \left(\frac{A}{\lambda^+}\right)^n \vec{z}\right)$$

for all $n > 0$. Taking the limit as n goes to infinity proves that $(\partial_x F)(x, \vec{z}) = (\partial_x F)(0, 0)$. Since $(\partial_x F)(0, 0) = 0$ we then see that $F(x, \vec{z}) = F(0, \vec{z})$.

Now for $t \in \mathbb{R}$ define the projective map u_t by $u_t \cdot (z_1, \dots, z_d) = (z_1 + t, z_2, \dots, z_d)$. Since $F(x, \vec{z}) = F(0, \vec{z})$, we see that there exists $\epsilon > 0$ and an open neighborhood U' of $0 \in \mathbb{C}^d$ such that $u_t(z_1, \dots, z_d) \in \Omega$ for all $(z_1, \dots, z_d) \in U' \cap \Omega$ and $|t| < \epsilon$. Now by construction

$$u_{(\lambda^-/\lambda^+)t} \circ \varphi = \varphi \circ u_t$$

and by part (4) of Theorem V.12 for any $x \in \Omega$ and $t \in \mathbb{R}$ there exist m such that $\varphi^m x \in U$ and $|(\lambda^-/\lambda^+)^m t| < \epsilon$. With this choice of m

$$\varphi^m u_t x = u_{(\lambda^-/\lambda^+)^m t} \varphi^m x$$

is in Ω. As Ω is φ-invariant this implies that $u_t x \in \Omega$. As $x \in \Omega$ and $t \in \mathbb{R}$ were arbitrary this implies that $u_t \in \mathrm{Aut}_0(\Omega)$ for all $t \in \mathbb{R}$. Also u_t corresponds to the matrix

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

in the action of $\mathrm{SL}(\mathbb{R}^2)$ defined in the statement of the theorem.

The same argument starting with φ^{-1} instead of φ (that is viewing Ω as a subset of the affine chart $\{[z_1 : 1 : z_2 : \dots : z_d]\}$) shows that $\mathrm{Aut}_0(\Omega)$ contains the one-parameter group of automorphisms corresponding to the matrices

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

in the action of $\mathrm{SL}(\mathbb{R}^2)$ defined in the statement of the theorem.

Finally it is well known that these two one-parameter subgroups generate all of $\mathrm{SL}(\mathbb{R}^2)$ and thus the second part of the theorem is proven.

The only assertion left to prove is that

$$\Omega \cap \{[z_1 : z_2 : 0 : \cdots : 0]\}$$

coincides with $\{[1 : z : 0 : \cdots : 0] : \mathrm{Im}(z) > 0\}$. But it well known that the action of $\mathrm{SL}(\mathbb{R}^2)$ restricted to $\{[1 : z : 0 : \cdots : 0] : \mathrm{Im}(z) > 0\}$ is transitive. As $\Omega \cap \{[z_1 : z_2 : 0 : \cdots : 0]\}$ is contained in this set, we see that the two sets are equal. $\square$

Notice that the projective map $\psi_t = \psi_{d_t}$ with

$$d_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

is bi-proximal, $x^+_{\psi_t} = x^+_\varphi$, and $x^-_{\psi_t} = x^-_\varphi$. We state this observation (and a little more) as a corollary.

Corollary V.25. *Suppose Ω is a proper weakly linearly convex open set with C^1 boundary. If $\varphi \in \mathrm{Aut}(\Omega)$ is bi-proximal, then there exists a one-parameter subgroup $\psi_t \in \mathrm{SL}(\mathbb{C}^{d+1})$ of bi-proximal elements such that $[\psi_t] \in \mathrm{Aut}_0(\Omega)$ and*

1. $(\psi_t)|_{x^+_\varphi} = e^t Id|_{x^+_\varphi}$,
2. $(\psi_t)|_{x^-_\varphi} = e^{-t} Id|_{x^-_\varphi}$,

3. $(\psi_t)|_{H^+ \cap H^-} = Id|_{H^+ \cap H^-}$ *where* $H^{\pm} = T^{\mathbb{C}}_{x^{\pm}_{\varphi}} \partial\Omega$.

Since $\mathrm{SL}(\mathbb{R}^2)$ acts transitively on $\{[1 : z] \in \mathbb{P}(\mathbb{C}^2) : \mathrm{Im}(z) > 0\}$, Theorem V.24 also implies the following:

Corollary V.26. *Suppose* Ω *is a proper weakly linearly convex open set with* C^1 *boundary,* $\varphi \in \mathrm{Aut}(\Omega)$ *is bi-proximal, and* L *is the complex projective line generated by* x^+_{φ} *and* x^-_{φ}*. Then for all* $p, q \in L \cap \Omega$ *distinct there exists* $\varphi_{pq} \in \mathrm{Aut}_0(\Omega)$ *such that* $\varphi_{pq}(p) = q$.

Since $\mathrm{SL}(\mathbb{R}^2)$ acts transitively on $\{[1 : z] \in \mathbb{P}(\mathbb{C}^2) : \mathrm{Im}(z) = 0\} \cup \{[0 : 1]\}$, Theorem V.24 almost implies the following:

Corollary V.27. *Suppose* Ω *is a proper weakly linearly convex open set with* C^1 *boundary,* $\varphi \in \mathrm{Aut}(\Omega)$ *is bi-proximal, and* L *is the complex projective line generated by* x^+_{φ} *and* x^-_{φ}*. Then for all* $x, y \in L \cap \partial\Omega$ *there exists* $\varphi_{xy} \in \mathrm{Aut}_0(\Omega)$ *such that* $\varphi_{xy}(x) = y$.

Proof. Theorem V.24 implies that for all $x, y \in \partial(L \cap \Omega)$ there exists $\varphi \in \mathrm{Aut}_0(\Omega)$ such that $\varphi(x) = y$. Thus we only have to show that $L \cap \partial\Omega = \partial(\Omega \cap L)$. To establish this, it is enough to show that L intersects $\partial\Omega$ transversally. Suppose this were not the case, then there exists $x \in L \cap \partial\Omega$ such that $L \subset T^{\mathbb{C}}_x \partial\Omega$. Then, since Ω is weakly linearly convex, $L \cap \Omega = \emptyset$ which is nonsense. Thus L intersects $\partial\Omega$ transversally and thus $L \cap \partial\Omega = \partial(\Omega \cap L)$. $\square$

5.7 Strict convexity

We call a weakly linearly convex open set Ω *strictly weakly linearly convex* if every complex tangent hyperplane of Ω intersects $\partial\Omega$ at exactly one point.

Theorem V.28. *Suppose Ω is a proper weakly linearly convex open set with C^1 boundary and $\Gamma \leq \mathrm{PSL}(\mathbb{C}^d)$ is a torsion-free group dividing Ω. Then Ω is strictly weakly linearly convex and for all $x, y \in \partial\Omega$ distinct there exists $\varphi \in \mathrm{Aut}_0(\Omega)$ bi-proximal such that $x = x_\varphi^+$ and $y = x_\varphi^-$.*

Remark V.29. Suppose, for a moment, that Γ is a torsion-free word hyperbolic group and $\partial\Gamma$ is the Gromov boundary. Then it is well known that the pairs of attracting and repelling fixed points of elements of Γ are dense in $\partial\Gamma \times \partial\Gamma$ (see for instance [Gro87, Corollary 8.2.G]). The proof of Theorem V.28 follows essentially the same argument, but with the additional technicalities coming from the possible existence of almost unipotent elements and the fact that we do not know that Γ is a word hyperbolic group (yet).

Theorem V.28 will follow from the next three propositions.

Proposition V.30. *If $x \in \partial\Omega$ then $T_x^{\mathbb{C}}\partial\Omega \cap \overline{\{x_\gamma^+ : \gamma \in \Gamma \text{ is bi-proximal}\}}$ is non-empty.*

Proposition V.31. *If $x, y \in \overline{\{x_\gamma^+ : \gamma \in \Gamma \text{ is bi-proximal}\}}$ and $T_x^{\mathbb{C}}\partial\Omega \neq T_y^{\mathbb{C}}\partial\Omega$ then there exists $\varphi \in \mathrm{Aut}_0(\Omega)$ bi-proximal such that $x = x_\varphi^+$ and $y = x_\varphi^-$.*

Proposition V.32. *If $x \in \overline{\{x_\gamma^+ : \gamma \in \Gamma \text{ is bi-proximal}\}}$ then $T_x^{\mathbb{C}}\partial\Omega \cap \partial\Omega = \{x\}$.*

Delaying the proof of the propositions, we prove Theorem V.28

Proof of Theorem V.28. First suppose that $x \in \partial\Omega$, then by Proposition V.30 there exists

$$z \in T_x^{\mathbb{C}}\partial\Omega \cap \overline{\{x_\gamma^+ : \gamma \in \Gamma \text{ is bi-proximal}\}}.$$

Since $T_x^{\mathbb{C}}\partial\Omega$ is a complex tangent hyperplane containing z and $\partial\Omega$ is C^1,

$$T_z^{\mathbb{C}}\partial\Omega = T_x^{\mathbb{C}}\partial\Omega.$$

But by Proposition V.32

$$\{z\} = T_z^{\mathbb{C}}\partial\Omega \cap \partial\Omega = T_x^{\mathbb{C}}\partial\Omega \cap \partial\Omega$$

and thus $x = z$. Since $x \in \partial\Omega$ was arbitrary $\partial\Omega = \overline{\{x_\gamma^+ : \gamma \in \Gamma \text{ is bi-proximal}\}}$. Then by Proposition V.32, Ω is strictly weakly linearly convex.

Now suppose $x, y \in \partial\Omega = \overline{\{x_\gamma^+ : \gamma \in \Gamma \text{ is bi-proximal}\}}$ then by Proposition V.32

$$T_x^{\mathbb{C}}\partial\Omega \cap \partial\Omega = \{x\} \text{ and } T_y^{\mathbb{C}}\partial\Omega \cap \partial\Omega = \{y\}.$$

In particular, if $x \neq y$ then $T_x^{\mathbb{C}}\partial\Omega \neq T_y^{\mathbb{C}}\partial\Omega$ and so by Proposition V.31 there exists $\varphi \in \mathrm{Aut}_0(\Omega)$ is bi-proximal such that $x = x_\varphi^+$ and $y = x_\varphi^-$. $\square$

5.7.1 Proof of Proposition V.30

Fix $x \in \partial\Omega$ and $p_n \in \Omega$ such that $p_n \to x$. Fix a base point $o \in \Omega$. Then since Γ acts co-compactly on Ω there exists $R < +\infty$ and $\varphi_n \in \Gamma$ such that $d_\Omega(\varphi_n o, p_n) < R$. By Proposition III.10: if $q \in \Omega$ then any limit point of $\{\varphi_n q\}_{n\in\mathbb{N}}$ is in $\partial\Omega \cap T_x^{\mathbb{C}}\partial\Omega$.

Now let $\hat{\varphi}_n \in \mathrm{GL}(\mathbb{C}^{d+1})$ be representatives of $\varphi_n \in \mathrm{PSL}(\mathbb{C}^{d+1})$ such that $\|\hat{\varphi}_n\| = 1$. By passing to a subsequence we may suppose $\hat{\varphi}_n \to \varphi \in \mathrm{End}(\mathbb{C}^{d+1})$. By construction, if $q \in \mathbb{P}(\mathbb{C}^{d+1}) \setminus [\ker\varphi]$ then $\varphi(q) = \lim_{n\to\infty}\varphi_n(q)$. In particular $\varphi(\Omega \setminus (\Omega \cap [\ker\varphi])) \subset T_x^{\mathbb{C}}\partial\Omega$. As Ω is an open set, this implies that $\varphi(\mathbb{C}^{d+1}) \subset T_x^{\mathbb{C}}\partial\Omega$.

We now claim that there exists $\gamma \in \Gamma$ bi-proximal such that $x_\gamma^+ \notin [\ker\varphi]$. To see this, let

$$W := \mathrm{Span}(x_\gamma^+ : \gamma \in \Gamma \text{ is bi-proximal}).$$

Since $\phi x_\gamma^+ = x_{\phi\gamma\phi^{-1}}^+$ we see that $[W]$ is Γ-invariant. Now suppose $\gamma \in \Gamma$ is bi-proximal, then $x_\gamma^+ \in [W]$ and so either $[W] \cap \Omega \neq \emptyset$ or $[W] \subset T_{x_\gamma^+}^{\mathbb{C}}\partial\Omega$. In the latter case

$$[W] \cap \partial\Omega \subset T_{x_\gamma^+}^{\mathbb{C}}\partial\Omega \cap \partial\Omega = \{x_\gamma^+\}$$

by Theorem V.12. Since $[W]$ contains $x_\gamma^- = x_{\gamma^{-1}}^+$ this case is impossible. Using Theorem II.4 we know that Ω is $\mathbb{C}$-convex. So by definition $\Omega' = [W] \cap \Omega$ is a $\mathbb{C}$-convex open set in $[W]$. Since $[W]$ is Γ-invariant, we see that Γ acts co-compactly and properly on Ω'. Now it is well known that a proper $\mathbb{C}$-convex open set is homeomorphic to an open ball (see for instance [APS04, Theorem 2.4.2]). Thus, by cohomological dimension considerations, we must have that $W = \mathbb{C}^{d+1}$.

Since $W = \mathbb{C}^{d+1}$ there exists $\gamma \in \Gamma$ bi-proximal such that $x_\gamma^+ \notin [\ker \varphi]$. Then $\varphi(x_\gamma^+) \in T_x^{\mathbb{C}} \partial\Omega$ and as

$$\varphi(x_\gamma^+) = \lim_{n \to \infty} \varphi_n x_\gamma^+ = \lim_{n \to \infty} x^+_{\varphi_n \gamma \varphi_n^{-1}}$$

we see that $T_x^{\mathbb{C}} \partial\Omega \cap \overline{\{x_\gamma^+ : \gamma \in \Gamma \text{ is bi-proximal}\}} \neq \emptyset$.

5.7.2 Proof of Proposition V.31 and Proposition V.32

We will need to know a little more about the action of almost unipotent elements on the boundary $\partial\Omega$.

Lemma V.33. *If $u \in \Gamma \setminus \{1\}$ is almost unipotent and $\psi \in \mathrm{Aut}(\Omega)$ is bi-proximal then x_ψ^+ is not a fixed point of u.*

Remark V.34. That Γ is a torsion free discrete group is critical here.

Proof. Suppose for a contradiction that there exists $\psi \in \mathrm{Aut}(\Omega)$ bi-proximal such that $u(x_\psi^+) = x_\psi^+$. Let $x^\pm := x_\psi^\pm$ and let L be the complex projective line containing x^+ and x^-. Let $H^\pm$ be the complex tangent hyperplane to Ω at $x^\pm$. By Theorem V.24 there exists coordinates such that

1. $x^+ = [1 : 0 : \cdots : 0]$,

2. $x^- = [0 : 1 : 0 : \cdots : 0]$,

3. $H^+ \cap H^- = \{[0:0:z_1:\dots:z_{d-1}] : z_1,\dots,z_{d-1} \in \mathbb{C}\}$,

4. $\Omega \cap L = \{[1:z:0:\dots:0] : \mathrm{Im}(z) > 0\}$,

and $\mathrm{Aut}_0(\Omega)$ contains the automorphisms

$$a_t \cdot [z_1 : z_2 : \dots : z_{d+1}] = [e^t z_1 : e^{-t} z_2 : z_3 : \dots : z_{d+1}].$$

Since u fixes x^+ it also fixes $H^+ = T^{\mathbb{C}}_{x^+}\partial\Omega$ and hence with respect to these coordinates u is represented by a matrix of the form:

$$\begin{pmatrix} 1 & b & \vec{x}^t \\ 0 & c & \vec{0}^t \\ \vec{0} & \vec{y} & A \end{pmatrix},$$

where $c \in \mathbb{C}$, $\vec{x}, \vec{y} \in \mathbb{C}^{d-1}$, and A is a $(d-1)$-by-$(d-1)$ matrix. Now a calculation shows that $u' = \lim_{t\to\infty} a_{-t} u a_t$ exists in $\mathrm{PSL}(\mathbb{C}^{d+1})$ and is represented by a matrix of the form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & \vec{0}^t \\ \vec{0} & \vec{0} & A \end{pmatrix}.$$

Since $\mathrm{Aut}(\Omega)$ is closed in $\mathrm{PSL}(\mathbb{C}^d)$ we have that $u' \in \mathrm{Aut}(\Omega)$. Since u' is the limit of almost unipotent elements, u' is almost unipotent and so $|c| = 1$. Since u leaves $\Omega \cap L$ invariant $c \in \mathbb{R}$. Then by possibly replacing u with u^2 we may assume that $c = 1$. Then $u'(z) = z$ for all $z \in L \cap \Omega$. Fix some $z \in \Omega \cap L$, then we have

$$\inf_{p\in\Omega} d_\Omega\left(u(p), p\right) \leq \lim_{t\to\infty} d_\Omega\left(u(a_t z), a_t z\right) = \lim_{t\to\infty} d_\Omega\left((a_{-t} u a_t)(z), z\right) = d_\Omega\left(u'(z), z\right) = 0.$$

This contradicts Corollary V.7. $\square$

We can use a standard argument to construct bi-proximal elements.

Lemma V.35. *Suppose $\phi, \gamma \in \Gamma$ are bi-proximal. If $x_\gamma^+, x_\gamma^-, x_\phi^+, x_\phi^-$ are all distinct and U^+, W^+ are neighborhoods of x_γ^+, x_ϕ^+ in $\overline{\Omega}$ then there exists $\psi \in \Gamma$ bi-proximal such that $x_\psi^+ \in U^+$ and $x_\psi^- \in W^+$.*

Proof. Pick neighborhoods U^-, W^- of x_γ^-, x_ϕ^- in $\overline{\Omega}$ and by possibly shrinking $U^\pm, W^\pm$ assume that $\overline{U^+}, \overline{U^-}, \overline{W^+}, \overline{W^-}$ are all disjoint. Let $\mathcal{O} = W^+ \cup W^- \cup U^+ \cup U^-$. Then by Theorem V.12, there exists m such that $\gamma^{\mp m}(\mathcal{O} \setminus U^\pm) \subset U^\mp$ and $\phi^{\mp m}(\mathcal{O} \setminus W^\pm) \subset W^\mp$. So if $\psi = \gamma^m \phi^{-m}$ then $\psi(U^+) \subset U^+$ and $\psi^{-1}(W^+) \subset W^+$. By Theorem V.12, ψ is either almost unipotent or bi-proximal. Moreover by Lemma V.15, $E^\pm(\psi) = x^\pm$ for some $x^\pm \in \partial\Omega$. Since $\psi(U^+) \subset U^+$ and $\psi^{-1}(W^+) \subset W^+$, part (1) of Proposition V.14 implies that $x^+ \in \overline{U^+}$ and $x^- \in \overline{W^+}$. Since $\overline{U^+}$ and $\overline{W^+}$ are disjoint this implies that $x^+ \neq x^-$. Thus ψ is not almost unipotent. $\square$

The next lemma constructs even more bi-proximal elements.

Lemma V.36. *Suppose $x_1, \dots, x_m \in \partial\Omega$ are distinct, then there exists $\gamma \in \Gamma$ bi-proximal such that $x_1, \dots, x_m, x_\gamma^+, x_\gamma^-$ are all distinct.*

Proof. First suppose that Γ contains a non-trivial almost unipotent element u. Let $\phi \in \Gamma$ be bi-proximal. Since x_ϕ^+ is a not fixed point of any power of u, $u^n(x_\phi^+) = u^m(x_\phi^+)$ if and only if $m = n$. Thus $x_1, \dots, x_m$ each appears at most once in the list

$$x_\phi^+,\ u(x_\phi^+),\ u^2(x_\phi^+),\ u^3(x_\phi^+), \dots$$

In a similar fashion, $x_1, \dots, x_m$ each appears at most once in the list

$$x_\phi^-,\ u(x_\phi^-),\ u^2(x_\phi^-),\ u^3(x_\phi^-), \dots$$

Thus for N sufficiently large $x_1, \dots, x_m, u^N(x_\phi^+), u^N(x_\phi^-)$ are all distinct and so $\gamma = u^N \phi u^{-N}$ satisfies the conclusion of the lemma.

Otherwise every element of $\Gamma \setminus \{1\}$ is bi-proximal. In this case, if there does not exist $\gamma \in \Gamma \setminus \{1\}$ bi-proximal such that $x_1, \dots, x_m, x_\gamma^+, x_\gamma^-$ are all distinct then

$$\Gamma = \cup_{i=1}^m \mathrm{Stab}_\Gamma(x_i)$$

and at least one of $\mathrm{Stab}_\Gamma(x_1), \dots, \mathrm{Stab}_\Gamma(x_m)$ has finite index in Γ (see for instance [Neu54, Lemma 4.1]). So by passing to a finite index subgroup and possibly relabeling we can assume that Γ fixes x_1.

Now let Γ' be the preimage of Γ under the map $\mathrm{SL}(\mathbb{C}^{d+1}) \to \mathrm{PSL}(\mathbb{C}^{d+1})$. Then we can conjugate Γ' to be a subset of the matrices of the form:

$$\begin{pmatrix} \lambda & {}^t\vec{v} \\ 0 & A \end{pmatrix}$$

where $\lambda \in \mathbb{C}^*$, $\vec{v} \in \mathbb{C}^d$ and A is a d-by-d matrix.

We claim that Γ' is commutative. Suppose, for a contradiction that $\gamma \in [\Gamma', \Gamma']$ is non-trivial. Then x_1 is an eigenline of γ with eigenvalue one. Since γ is non-trivial γ is bi-proximal (by assumption). Then by part (4) of Theorem V.12 the only fixed points of γ in $\partial\Omega$ are x_γ^+ and x_γ^-. Since $x_1 \in \partial\Omega$ is a fixed point with eigenvalue one and $\gamma \in \mathrm{SL}(\mathbb{C}^{d+1})$ we have a contradiction. So Γ' and hence Γ is commutative.

Now fix $\gamma_0 \in \Gamma \setminus \{1\}$ and let $x^\pm := x_{\gamma_0}^\pm$. Since Γ is commutative and the only fixed points of γ_0 in $\partial\Omega$ are $x^\pm$ we see that $\Gamma \cdot \{x^+, x^-\} = \{x^+, x^-\}$. Since the only fixed points of $\gamma \in \Gamma \setminus \{1\}$ are x_γ^+ and x_γ^- we have that

$$\{x^+, x^-\} = \{x_\gamma^+, x_\gamma^-\}.$$

Using Theorem V.24 we can pick coordinates such that $x^+ = [1 : 0 : \cdots : 0]$, $x^- = [0 : 1 : 0 : \cdots : 0]$, and if L is complex projective line containing x^+ and x^- then

$$\Omega \cap L = \{[1 : z : 0 : \cdots : 0] : \mathrm{Im}(z) > 0\}.$$

Since L is Γ-invariant and $\Gamma\backslash\Omega$ is compact, Γ acts co-compactly on $\Omega \cap L$. Also Γ acts by isometries on $(\Omega \cap L, d_{\Omega\cap L})$ which by Proposition III.13 is isometric to hyperbolic real 2-space. But this contradicts that Γ is commutative. $\square$

We can now prove Proposition V.31 and Proposition V.32.

Proof of Proposition V.31. Suppose $\{\gamma_n\}_{n\in\mathbb{N}}, \{\phi_n\}_{n\in\mathbb{N}} \subset \Gamma$ are sequences of bi-proximal elements such that $x^+_{\gamma_n} \to x$ and $x^+_{\phi_n} \to y$. By passing to subsequences we can assume $x^-_{\gamma_n} \to x'$ and $x^-_{\phi_n} \to y'$.

First suppose that x, x', y, y' are all distinct. Then for n large $x^+_{\gamma_n}, x^-_{\gamma_n}, x^+_{\phi_n}, x^-_{\phi_n}$ are all distinct and so using Lemma V.35 we can find a sequence $\{\psi_n\} \subset \Gamma$ of bi-proximal elements such that $x^+_{\psi_n} \to x$ and $x^-_{\psi_n} \to y$. Now let $H_x := T^{\mathbb{C}}_x \partial\Omega$, $H_y := T^{\mathbb{C}}_y \partial\Omega$, and $H^{\pm}_n := T^{\mathbb{C}}_{x^{\pm}_{\psi_n}} \partial\Omega$. Since $\partial\Omega$ is C^1 and $H_x \neq H_y$, for n large enough $H^+_n \neq H^-_n$ and in the of space complex codimension two subspaces $H^+_n \cap H^-_n \to H_x \cap H_y$. By Corollary V.25, there exist bi-proximal elements $\hat{\psi}_n \in \mathrm{SL}(\mathbb{C}^{d+1})$ such that

1. $[\hat{\psi}_n] \in \mathrm{Aut}_0(\Omega)$,
2. $\hat{\psi}_n|_{x^+_{\psi_n}} = 2Id|_{x^+_{\psi_n}}$,
3. $\hat{\psi}_n|_{x^-_{\psi_n}} = (1/2)Id|_{x^-_{\psi_n}}$, and
4. $\hat{\psi}_n|_{H^+_n \cap H^-_n} = Id|_{H^+_n \cap H^-_n}$.

Now since $x^+_{\psi_n} \to x$, $x^-_{\psi_n} \to y$, and $H^+_n \cap H^-_n \to H_x \cap H_y$, we see that $\hat{\psi}_n$ converges to $\hat{\psi} \in \mathrm{SL}(\mathbb{C}^{d+1})$ such that $\hat{\psi}|_x = 2Id|_x$, $\hat{\psi}|_y = (1/2)Id|_y$, and $\hat{\psi}|_{H_x\cap H_y} = Id|_{H_x \cap H_y}$. As $\mathrm{Aut}_0(\Omega) \subset \mathrm{PSL}(\mathbb{C}^{d+1})$ is closed, $[\hat{\psi}] \in \mathrm{Aut}_0(\Omega)$. This establishes the lemma in this special case.

Next consider case in which x, x', y, y' are not all distinct. By Lemma V.36 there exists $\varphi \in \Gamma$ bi-proximal such that x^+_φ and x^-_φ are not in the set $\{x, y, x', y'\}$. Then for n large $x^+_{\gamma_n}, x^-_{\gamma_n}, x^+_{\varphi}, x^-_{\varphi}$ are all distinct and we can use Lemma V.35 to find a sequence $\{\gamma'_n\} \subset \Gamma$ of bi-proximal elements such that $x^+_{\gamma'_n} \to x$ and $x^-_{\gamma'_n} \to x^+_{\varphi}$. So by replacing γ_n with γ'_n we may suppose $x^+_{\gamma_n} \to x$ and $x^-_{\gamma_n} \to x^+_{\varphi}$. A similar argument shows that we may assume $x^+_{\phi_n} \to y$ and $x^-_{\phi_n} \to x^-_{\varphi}$. Since $x, y, x^+_{\varphi}, x^-_{\varphi}$ are all distinct we can now apply the argument above. $\square$

Proof of Proposition V.32. Suppose $\{\gamma_n\}_{n\in\mathbb{N}} \subset \Gamma$ is a sequence of bi-proximal elements such that $x^+_{\gamma_n} \to x$. By passing to subsequences we can assume $x^-_{\gamma_n} \to x'$. Now pick $\gamma \in \Gamma$ bi-proximal such that

$$\{x, x'\} \cap \{x^+_{\gamma}, x^-_{\gamma}\} = \emptyset.$$

Then using Lemma V.35 we can find a sequence of bi-proximal elements $\{\phi_n\}_{n\in\mathbb{N}} \subset \Gamma$ such that $x^+_{\phi_n} \to x$ and $x^-_{\phi_n} \to x^+_{\gamma}$.

By Theorem V.12, $T^{\mathbb{C}}_{x^+_{\gamma}} \partial\Omega \cap \partial\Omega = \{x^+_{\gamma}\}$ and in particular $T^{\mathbb{C}}_{x^+_{\gamma}} \partial\Omega \neq T^{\mathbb{C}}_{x} \partial\Omega$. Thus by Proposition V.31, there exists $\varphi \in \mathrm{Aut}_0(\Omega)$ bi-proximal such that $x = x^+_{\varphi}$ and $x^+_{\gamma} = x^-_{\varphi}$. Then by Theorem V.12 we see that $T^{\mathbb{C}}_{x} \partial\Omega \cap \partial\Omega = \{x\}$. $\square$

5.8 Completing the proof of Theorem I.8

Suppose Ω is a divisible weakly linearly convex open set with C^1 boundary. Then it follows from Theorem V.24 and Theorem V.28 that $L \cap \Omega$ is either empty or a projective disk whenever L is a complex projective line. Thus by Proposition IV.1 Ω is a projective ball.

Remark V.37. The full argument in Proposition IV.1 is not needed, instead one can use the arguments in Sections 4.3, 4.4, and 4.5 to complete the proof.

CHAPTER VI

Rigidity from geodesics

In this chapter we will prove:

Theorem VI.1. *Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is a proper strictly weakly linear convex open set. Then the following are equivalent:*

1. (Ω, d_Ω) *is geodesic,*
2. Ω *is a projective ball,*
3. (Ω, d_Ω) *is isometric to* $\mathbb{C}\,\mathbb{H}^d$.

We call an open set Ω *strictly weakly linear convex* if for every complex tangent plane intersects Ω at exactly one point. Notice that in $\mathbb{P}(\mathbb{C}^2)$, complex hyperplanes are just points and hence when $d = 1$ being strictly weakly linear convex and weakly linear convex is the same.

In Section 3.5 we showed that for linearly convex domains in $\mathbb{P}(\mathbb{C}^2)$ the complex Hilbert metric and the Apollonian metric coincide and in particular Theorem I.9 can be seen as a generalization of the following result of Gehring and Hag:

Theorem VI.2. *[GH00] Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^2)$ is a proper weakly linear convex open set. If (Ω, A_Ω) is geodesic then Ω is projectively equivalent to the unit ball.*

Using Proposition IV.1 it is enough to prove the following:

Proposition VI.3. *Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is a proper strictly weakly linear convex open set and (Ω, d_Ω) is geodesic. If L is a projective line intersecting Ω then*

$$(\Omega \cap L, d_{\Omega \cap L})$$

is geodesic. In particular, the intersection of Ω with any projective line is either empty or a projective disk.

The proof of Proposition VI.3 will follow from the next two lemmas.

Lemma VI.4. *Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is a proper weakly linear convex open set. Fix $x, y \in \Omega$ and let $f, g \in \Omega^*$ be such that*

$$d_\Omega(x,y) = \log \left| \frac{f(x)g(y)}{f(y)g(x)} \right| .$$

Let L be the projective line containing x and y. If $\{b_f\} = \ker f \cap L$ and $\{b_g\} = \ker g \cap L$ then $b_f, b_g \in \partial\Omega$.

Proof. If we identify L with $\overline{\mathbb{C}}$ then Lemma III.4 and the results in Section 3.5 imply that:

$$\log \frac{|x-b_f|\,|y-b_g|}{|y-b_f|\,|x-b_g|} = d_\Omega(x,y) = A_{\Omega\cap L}(x,y) = \sup_{b_1,b_2 \in L \setminus (\Omega \cap L)} \log \frac{|x-b_1|\,|y-b_2|}{|y-b_1|\,|x-b_2|}.$$

Now the map

$$(b_1,b_2) \in (L \setminus (\Omega \cap L))^2 \to \log \frac{|x-b_1|\,|y-b_2|}{|y-b_1|\,|x-b_2|} \in \mathbb{R}$$

is an open map. Thus we must have that $b_f, b_g \in \partial(\Omega \cap L) \subset \partial\Omega$. □

Lemma VI.5. *Suppose $\Omega \subset \mathbb{P}(\mathbb{C}^{d+1})$ is a proper strictly weakly linear convex open set. If $x, y, z \in \Omega$ are all distinct and*

$$d_\Omega(x,z) + d_\Omega(z,y) = d_\Omega(x,y)$$

then z is contained in the complex projective line generated by x and y.

Proof. Since Ω is open, Ω^* is compact. Thus there exists $f, g \in \Omega^*$ such that

$$d_\Omega(x,y) = \log \left| \frac{f(x)g(y)}{f(y)g(x)} \right|.$$

Let L_0 be the complex projective line containing x and y. If $\{b_f\} = \ker f \cap L_0$ and $\{b_g\} = \ker g \cap L_0$ then by the previous Lemma $b_f, b_g \in \partial\Omega$. Since $f \in \Omega^*$, $\ker f \cap \Omega = \emptyset$ and thus $\ker f$ is a complex tangent hyperplane to $b_f \in \partial\Omega$. Since Ω is strictly linearly convex this implies that $\ker f \cap \partial\Omega = \{b_f\}$. A similar argument shows that $\ker g \cap \partial\Omega = \{b_g\}$.

Also

$$\begin{aligned} d_\Omega(x,y) = \log \left| \frac{f(x)g(y)}{f(y)g(x)} \right| &= \log \left| \frac{f(x)g(z)}{f(z)g(x)} \right| + \log \left| \frac{f(z)g(y)}{f(y)g(z)} \right| \\ &\leq d_\Omega(x,z) + d_\Omega(z,y) = d_\Omega(x,y) \end{aligned}$$

thus we must have

$$\log \left| \frac{f(x)g(z)}{f(z)g(x)} \right| = d_\Omega(x,z) \text{ and } \log \left| \frac{f(z)g(y)}{f(y)g(z)} \right| = d_\Omega(z,y).$$

Now let L_1 be the projective line containing x and z. If $\{b_f'\} = \ker f \cap L_1$ and $\{b_g'\} = \ker g \cap L_1$ then by the previous Lemma $b_f', b_g' \in \partial\Omega$. But $\ker f \cap \partial\Omega = \{b_f\}$ and $\ker g \cap \partial\Omega = \{b_g\}$ so $b_f' = b_f$ and $b_g' = b_g$. Since x and y are distinct $d_\Omega(x,y) \neq 0$ and so b_g and b_f are distinct. Then since $b_f, b_g \in L_0 \cap L_1$ we must have that $L_1 = L_0$. So z is contained in the complex projective line generated by x and y. □

Proof of Proposition VI.3. Suppose L is a projective line intersecting Ω. If $x, y \in L$, then there exists a geodesic $\sigma : [0,T] \to \Omega$ joining x and y. Then for all $t \in [0,T]$

$$d_\Omega(x,y) = d_\Omega(x,\sigma(t)) + d_\Omega(\sigma(t),y).$$

Thus by Lemma VI.5, $\sigma \subset L \cap \Omega$. By Theorem III.1 the inclusion map $L \cap \Omega \hookrightarrow \Omega$ induces an isometric embedding $(L \cap \Omega, d_{L\cap\Omega}) \hookrightarrow (\Omega, d_\Omega)$ thus we see that σ is a geodesic in $(L \cap \Omega, d_{\Omega \cap L})$. Since $x, y \in L \cap \Omega$ were arbitrary, we see that $(L \cap \Omega, d_{L \cap \Omega})$ is a geodesic metric space. Then by Theorem VI.2, we see that $L \cap \Omega$ is a projective disk. $\square$

CHAPTER VII

Extensions to quaternionic projective space

In this section we will extend the results of Chapters IV, V, and VI to quaternionic projective space. In particular we will show:

Theorem VII.1. *Suppose $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is a proper weakly linearly convex set with C^1 boundary. If Ω is divisible then Ω is a projective ball.*

We will construct a quaternionic Hilbert metric and prove:

Theorem VII.2. *Suppose $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is a strictly weakly linearly convex set. If (Ω, d_Ω) is geodesic then Ω is a projective ball.*

As in the complex case, both results will reduce to the following proposition:

Proposition VII.3. *Suppose $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is an open set such that its intersection with any projective line is either empty or a projective disk. Then Ω is a projective ball.*

We will begin with some basic background on the quaternions.

7.1 The quaternions

The *quaternions* $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ form a complex two dimensional vector space with multiplication rules:

$$i^2 = j^2 = k^2 = ijk = -1.$$

The quaternions have a natural conjugation:

$$\overline{a + bi + cj + dk} = a - bi - ci - dk$$

and a corresponding absolute value:

$$|a + bi + cj + dk|^2 = (a + bi + cj + dk)\overline{(a + bi + cj + dk)} = a^2 + b^2 + c^2 + d^2.$$

One can also speak of the real part $\frac{1}{2}(x + \overline{x})$ and the imaginary part $\frac{1}{2}(x - \overline{x})$ of a quaternion.

Let $\mathbb{H}$ act on $\mathbb{H}^d$ as follows:

$$\alpha \cdot (z_1, \ldots, z_d) = (z_1\alpha, \ldots, z_d\alpha).$$

Now we can define $\mathrm{GL}(\mathbb{H}^d)$ to be the invertible linear transformations of $\mathbb{H}^d$ which commute with the action of $\mathbb{H}$. In this chapter we identify $\mathbb{H}^d$ with d-by-1 matrices with entries in $\mathbb{H}$. If $M_d(\mathbb{H})$ is the space of d-by-d matrices with entries in $\mathbb{H}$, then we can identify

$$\mathrm{GL}(\mathbb{H}^d) = \mathrm{GL}(\mathbb{C}^{2d}) \cap M_d(\mathbb{H}).$$

Since the quaternions are non-commutative this identification requires that the scalar multiplication acts on the right while $M_d(\mathbb{H})$ acts on the left.

Given $\varphi \in \mathrm{GL}(\mathbb{H}^d)$ we can define a determinant by viewing φ as an element of $\mathrm{GL}(\mathbb{C}^{2d})$:

$$D(\varphi) := \left|\det(\varphi : \mathbb{C}^{2d} \to \mathbb{C}^{2d})\right|.$$

There are more sophisticated ways to define determinants for matrices with quaternionic entries, but the simple definition given above is good enough for our purposes. Finally, define the *special linear group*

$$\mathrm{SL}(\mathbb{H}^{d+1}) = \{\varphi \in \mathrm{GL}(\mathbb{H}^{d+1}) : D(\varphi) = 1\}.$$

Now we can define the quaternionic projective space $\mathbb{P}(\mathbb{H}^{d+1})$ to be

$$\mathbb{P}(\mathbb{H}^{d+1}) = \{\vec{z} \in \mathbb{H}^{d+1}\}/\{\vec{z} \sim \alpha \cdot \vec{z}\}.$$

Then $\mathrm{GL}(\mathbb{H}^{d+1})$ acts on $\mathbb{P}(\mathbb{H}^{d+1})$ and an element $\varphi \in \mathrm{GL}(\mathbb{H}^{d+1})$ acts trivially if and only if

$$\varphi = \begin{pmatrix} \alpha & & \\ & \ddots & \\ & & \alpha \end{pmatrix}$$

for some $\alpha \in \mathbb{R}^*$. So the group

$$\mathrm{PSL}(\mathbb{H}^{d+1}) = \mathrm{SL}(\mathbb{H}^{d+1})/\{\pm Id\}$$

acts faithfully on $\mathbb{P}(\mathbb{H}^{d+1})$.

We now have the following observation, which motivates our choice of determinant:

Observation VII.4. If $\varphi \in \mathrm{GL}(\mathbb{H}^{d+1})$ then there exists $\varphi_1 \in \mathrm{SL}(\mathbb{H}^{d+1})$ so that the action of φ and φ_1 on $\mathbb{P}(\mathbb{H}^{d+1})$ coincide.

Proof. Given $\varphi \in \mathrm{GL}(\mathbb{H}^{d+1})$ there exists $t_\varphi \in \mathbb{R}$ such that $t_\varphi \varphi \in \mathrm{SL}(\mathbb{H}^{d+1})$. □

Given a set $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ we can define the projective automorphism group to be:

$$\mathrm{Aut}(\Omega) = \{\varphi \in \mathrm{PSL}(\mathbb{H}^{d+1}) : \varphi(\Omega) = \Omega\}.$$

7.2 Convexity in quaternionic projective space

We are unaware of a version of the Riemann mapping theorem in the quaternionic plane, thus it is unclear if $\mathbb{C}$-convexity has an analogue for sets in quaternionic projective space. However the other two types of projective convexity have obvious analogues.

Definition VII.5.

1. An open set $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is called *weakly linearly convex* if for every $p \in \partial\Omega$ there exists a quaternionic hyperplane H containing p such that $H \cap \Omega = \emptyset$.
2. A set $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is called *linearly convex* if for every $p \in \mathbb{P}(\mathbb{H}^{d+1}) \setminus \Omega$ there exists a quaternionic hyperplane H containing p such that $H \cap \Omega = \emptyset$.

We can also define a quaternionic dual. Let $\mathbb{H}^{(d+1)*}$ be the vector space of $\mathbb{H}$-linear functions $f : \mathbb{H}^{d+1} \to \mathbb{H}$.

Definition VII.6. The *quaternionic dual* of $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is the set

$$\Omega^* = \Big\{ f \in \mathbb{P}(\mathbb{H}^{(d+1)*}) : f(x) \neq 0 \text{ for all } x \in \Omega \Big\} \subset \mathbb{P}(\mathbb{H}^{(d+1)*}).$$

Since $\mathbb{P}(\mathbb{H}^{(d+1)*})$ can be identified with the space of hyperplanes in $\mathbb{P}(\mathbb{H}^{d+1})$ we have the following alternative definition of linear convexity:

Observation VII.7. A set $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is linearly convex if and only if $\Omega^{**} = \Omega$.

As in the complex case the boundary of Ω and the boundary of Ω^* are closely related. We will call a quaternionic hyperplane H *tangent* to a set Ω at $p \in \partial\Omega$ if H contains p but does not intersect Ω. With this language we have the following observation:

Observation VII.8. Suppose $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is open; then

$$f \in \partial\Omega^* \Leftrightarrow \text{the hyperplane } \ker f \text{ is tangent to } \Omega.$$

As in the real case, it is very natural to consider convex sets that are proper. In this thesis we will use the following definition of proper sets:

Definition VII.9. A set $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is called *proper* if $\overline{L \cap \Omega} \neq L$ for every quaternionic projective line L in $\mathbb{P}(\mathbb{H}^{d+1})$.

As the next proposition shows, when Ω is proper the dual is not too small.

Proposition VII.10. *If $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is a proper weakly linearly convex open set, then Ω^* is not contained in a quaternionic hyperplane.*

7.3 The quaternionic Hilbert metric

Let $x, y \in \Omega$ be distinct points and let L_{xy} be the projective line containing x and y. Now L_{xy} has real dimension four and we define the quaternionic Hilbert metric as:

$$d_\Omega(x,y) = \max_{a,b \in \partial(L_{xy} \cap \Omega)} \log \frac{|x-b|\,|y-a|}{|x-a|\,|y-b|}.$$

As in the complex case, this is actually a metric when Ω is a proper weakly linearly convex open set. More precisely:

Theorem VII.11. *If Ω is a proper weakly linearly convex open set then d_Ω is a complete metric on Ω such that the subspace topology on $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ and the topology on Ω induced by d_Ω coincide. Moreover, if $W \subset \mathbb{P}(\mathbb{H}^{d+1})$ is a quaternionic projective subspace then the inclusion $W \cap \Omega \hookrightarrow \Omega$ induces an isometric embedding $(W \cap \Omega, d_{W\cap\Omega}) \hookrightarrow (\Omega, d_\Omega)$.*

This theorem can be proven using exactly the same argument as in the complex case, moreover we can show the quaternionic Hilbert metric can be defined using the dual:

Proposition VII.12. *If* $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ *is a proper weakly linearly convex open set then*

$$d_\Omega(x,y) = \max_{f,g\in\Omega^*} \log\left(\frac{|f(x)g(y)|}{|f(y)g(x)|}\right). \tag{7.1}$$

Many of the properties of the complex Hilbert metric are also true for the quaternionic Hilbert metric. In particular the arguments in Section III can be used verbatim to establish the following three results:

Proposition VII.13. *Suppose* $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ *is a proper weakly linearly convex open set, then* $\mathrm{Aut}(\Omega)$ *is a closed subgroup of* $\mathrm{PSL}(\mathbb{H}^{d+1})$ *and acts properly on* Ω*.*

Proposition VII.14. *Suppose* $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ *is a proper weakly linearly convex open set. If* $\{p_n\}_{n\in\mathbb{N}}, \{q_n\}_{n\in\mathbb{N}} \subset \Omega$ *are sequences such that* $p_n \to x \in \partial\Omega$*,* $q_n \to y \in \partial\Omega$*, and* $d_\Omega(p_n,q_n) < R$ *for some* $R > 0$*, then every quaternionic tangent hyperplane of* Ω *containing* x *also contains* y*.*

Proposition VII.15. *Suppose* $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ *is a proper weakly linearly convex open set. If* $x_0 \in \Omega$*, then there exist* $R > 0$ *depending only on* x_0 *such that*

$$d_\Omega(\varphi x_0, x_0) \leq R + \log\left(\|\varphi\|\,\|\varphi^{-1}\|\right)$$

for all $\varphi \in \mathrm{Aut}(\Omega)$*.*

By the definition of the quaternionic Hilbert metric we have the following relationship with the Apollonian metric:

Proposition VII.16. *Suppose* $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ *is a weakly linearly convex open set. If* $x, y \in \Omega$ *and* L *is the quaternionic projective line containing* x *and* y*, then*

$$d_\Omega(x,y) = A_{\Omega\cap L}(x,y).$$

Using Beardon's calculation of the Apollonian metric on projective balls (see Proposition III.13) we see that the Hilbert metric yields a model of quaternionic hyperbolic space.

Proposition VII.17. *Let $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ be a projective ball. Then (Ω, d_Ω) is isometric to quaternionic hyperbolic d-space.*

Proof. We can pick coordinates such that

$$\Omega = \{[1 : z_1 : z_2 : \cdots : z_d] : \sum |z_i|^2 < 1\}.$$

Now let d be the quaternionic hyperbolic metric on Ω described in Chapter 19 of [Mos73]. Then by Proposition III.13

$$d_\Omega(p, q) = d(p, q)$$

for all $p, q \in \Omega \cap L$ where L is the projective line $L = \{[1 : z : 0 : \cdots : 0] : |z| < 1\}$. Since $\mathrm{Sp}(1, d)$ acts transitively on the set of projective lines intersecting Ω and both d and d_Ω are preserved by $\mathrm{Sp}(1, d)$ we see that $d = d_\Omega$ on all of Ω. $\square$

7.4 Möbius transformations

As in the complex case the proof of Proposition VII.3 and Theorem VII.1 require some knowledge about the symmetries of the upper half plane. In this section we will state and prove the necessary properties. We will always give elementary arguments, but everything follows from well known properties of rank one symmetric spaces.

We can identify $\mathbb{P}(\mathbb{H}^2)$ with $\overline{\mathbb{H}} = \mathbb{H} \cup \{\infty\}$ via the map

$$[z_1 : z_2] \to \begin{cases} z_1 (z_2)^{-1} & \text{if } z_2 \neq 0 \\ \infty & \text{otherwise.} \end{cases}$$

With this identification $\mathrm{PSL}(\mathbb{H}^2)$ acts on $\overline{\mathbb{H}}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = (az+b)(cz+d)^{-1}.$$

As in the complex case, Möbius transformations map spheres and hyperplanes to spheres and hyperplanes.

Observation VII.18. $\mathrm{PSL}(\mathbb{H}^2)$ maps spheres and hyperplanes to spheres and hyperplanes.

Proof. Every sphere and half plane can be described as a set of the form

$$\{z \in \mathbb{H} : |z-a| = R\,|z-b|\}$$

for some $a, b \in \mathbb{H}$ and $R > 0$. Moreover every set of this form is a sphere or half plane. A calculation shows that Möbius transformations map a set of this form to a set of this form. □

Let

$$\mathcal{H}_+ = \{z \in \mathbb{H} : \mathrm{Re}(z) > 0\}.$$

Now $\mathcal{H}_+$ is projectively equivalent to the unit ball by Möbius transformation

$$z \to (z-1)(z+1)^{-1}.$$

In particular, $\mathrm{Aut}(\mathcal{H}_+)$ is isomorphic with

$$\mathrm{Aut}(\{|z| < 1\}) = \mathrm{Sp}(1,1) \cong \mathrm{SO}(1,4).$$

The next proposition follows from basic properties of rank one symmetric spaces but we provide an elementary proof anyways.

Proposition VII.19. *1. If $x \in \partial\mathcal{H}_+ \subset \overline{\mathbb{H}}$ then the group*

$$P_x = \{\varphi \in \mathrm{Aut}_0(\mathcal{H}_+) : \varphi x = x\}$$

acts transitively on $\mathcal{H}_+$,

2. $\mathrm{Aut}_0(\mathcal{H}_+)$ *acts transitively on $\partial\mathcal{H}_+$,*

3. $\mathrm{Aut}_0(\mathcal{H}_+)$ *is generated by the two subgroups*

$$U = \left\{ \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} : \mathrm{Re}(w) = 0 \right\} \text{ and } V = \left\{ \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} : \mathrm{Re}(w) = 0 \right\}.$$

Proof. Since

$$P_\infty = \left\{ \begin{pmatrix} \lambda & w \\ 0 & \overline{\lambda}^{-1} \end{pmatrix} : \lambda, w \in \mathbb{H},\ \lambda \neq 0,\ \mathrm{Re}(w) = 0 \right\}$$

we see that P_∞ acts transitively on $\mathcal{H}_+$ and $\partial\mathcal{H}_+ \setminus \{\infty\}$. Since

$$P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} P_\infty \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we see that P_0 acts transitively on $\partial\mathcal{H}_+ \setminus \{0\}$. Thus $\mathrm{Aut}_0(\mathcal{H}_+)$ acts transitively on $\partial\mathcal{H}_+$. Since $\mathrm{Aut}_0(\mathcal{H}_+)$ acts transitively on the boundary, we see that every group P_x is conjugate to P_∞. Since P_∞ acts transitively on $\mathcal{H}_+$, we then have Part (1).

It remains to prove Part (3), let G be the group generated by U and V. Since

$$\left[\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \right] = \begin{pmatrix} wu & 0 \\ 0 & -uw \end{pmatrix}$$

and $\overline{w} = -w$ when $\mathrm{Re}(w) = 0$ we see that the Lie algebra of G contains

$$\left\{ \begin{pmatrix} \lambda & w \\ u & -\overline{\lambda} \end{pmatrix} : \lambda, w, u \in \mathbb{H}, \mathrm{Re}(w) = \mathrm{Re}(u) = 0 \right\}.$$

In particular G contains P_∞ and P_0. This implies that G acts transitively on the boundary. Now suppose $\varphi \in \mathrm{Aut}_0(\Omega)$. Since G acts transitively on $\partial\mathcal{H}_+$ there exists $\gamma \in G$ such that $(\gamma\varphi)(0) = 0$. Then $\gamma\varphi \in P_0 \subset G$ which implies that $\varphi \in G$. $\square$

7.5 Rigidity from slices

The proof of proposition VII.3 is identical to the proof in the complex case with one technicality. That Ω is linearly convex is used repeatedly in the proof of the proposition. In the complex case this follows from Ω being $\mathbb{C}$-convex and every $\mathbb{C}$-convex open set is linearly convex. In the quaternionic case we have to prove this. To make an induction argument work, we will relax the hypothesis a little bit.

Proposition VII.20. *Suppose $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is an open set such that its intersection with any projective line is either empty, the projective line minus a point, or projectively isomorphic to the disk. Then Ω is linearly convex.*

Remark VII.21.

1. This proposition is much easier to prove in the special case when there already exists an affine chart $\mathbb{H}^d$ containing Ω. In this situation, Ω is convex in this affine chart and then by the separating hyperplane theorem for real convex sets we not only have quaternionic hyperplanes through each point in the complement of Ω but real hyperplanes.

2. The proposition should hold in greater generality, but we do not pursue such matters here.

3. Our strategy for proving the proposition will closely follow the proof in [APS04] that an open $\mathbb{C}$-convex set is linear convex. In particular, we will pick a point $a \in \mathbb{P}(\mathbb{H}^{d+1}) \setminus \Omega$ then let $T : \mathbb{P}(\mathbb{H}^{d+1}) \to \mathbb{P}(\mathbb{H}^d)$ be the projection from a. We

then will show that $T(\Omega)$ satisfies the hypothesis of the proposition. Thus, by induction, there exists a hyperplane H in $\mathbb{P}(\mathbb{H}^d)$ which does not intersect $T(\Omega)$. Finally $T^{-1}(H)$ is a hyperplane in $\mathbb{P}(\mathbb{H}^{d+1})$ which contains a but does not intersect Ω. The most difficult part of the proof is showing that $T(\Omega) \neq \mathbb{P}(\mathbb{H}^d)$; this will require some cohomological arguments.

The proof will repeatedly use the following type of projective map:

Definition VII.22. We call a projective map $T : \mathbb{P}(\mathbb{H}^{d+1}) \to \mathbb{P}(\mathbb{H}^d)$ the *projection from* $a \in \mathbb{P}(\mathbb{C}^{d+1})$ if T is the map obtained by identifying $\mathbb{P}(\mathbb{H}^d)$ with the space of projective lines through a and $T(b)$ is the projective line containing a and b (notice T is not defined at a).

Lemma VII.23. *Suppose $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is an open set such that its intersection with any projective line is either empty, the projective line minus a point, or projectively isomorphic to the disk. If $a \notin \Omega$ and $T : \mathbb{P}(\mathbb{H}^{d+1}) \to \mathbb{P}(\mathbb{H}^d)$ is the projection from a, then $T(\Omega)$ is an open set such that its intersection with any projective line is either empty, the whole projective line, the projective line minus a point, or projectively isomorphic to the disk.*

Proof. Since T is an open map, $T(\Omega)$ is open. Now suppose L is a projective line in $\mathbb{P}(\mathbb{H}^d)$ intersecting $T(\Omega)$. If $a, b \in L \cap T(\Omega)$ are distinct then we claim there exists a projective disk in $T(\Omega) \cap L$ containing them. To see this, pick preimages $\widehat{a} \in T^{-1}(a)$ and $\widehat{b} \in T^{-1}(b)$ and consider the projective line $\widehat{L}$ containing them. By hypothesis there exists a projective disk in $\widehat{L} \cap \Omega$ containing $\widehat{a}, \widehat{b}$. Since T induces a projective isomorphism $\widehat{L} \to L$, the claim follows.

Now let $\widehat{\Omega} := L \cap T(\Omega)$. Now if $\widehat{\Omega} \neq L$ there exists an affine chart $\mathbb{H}$ in L containing $\widehat{\Omega}$. Since every two points $a, b \in \widehat{\Omega}$ are contained in a projective ball, we see that $\widehat{\Omega}$ is convex in this affine chart. Now, by convexity, if $\widehat{\Omega} \neq \mathbb{H}$ then $L \setminus \widehat{\Omega}$ has non-empty interior. So there exists a possibly different affine chart of L where $\widehat{\Omega}$ is bounded. Since $\widehat{\Omega}$ is convex we can translate so that $0 \in \partial\widehat{\Omega}$ and the real hyperplane $\{z \in \mathbb{H} : \mathrm{Re}(z) = 0\}$ is tangent to $\widehat{\Omega}$. We may also assume that $\widehat{\Omega} \subset \{z \in \mathbb{H} : \mathrm{Re}(z) > 0\}$. Since $\widehat{\Omega}$ is bounded in this affine chart, every projective disk in $\widehat{\Omega}$ has the form $\{z \in \mathbb{H} : |z - z_0| < r\}$ for some $z_0 \in \mathbb{H}$ and $r > 0$. Moreover by taking limits, for any $p \in \widehat{\Omega}$ there exists a closed disk $D_p \subset \overline{\widehat{\Omega}}$ which contains 0 and p. Then $0 \in \partial D_p$. Since $\{z \in \mathbb{H} : \mathrm{Re}(z) = 0\}$ is tangent to $\widehat{\Omega}$ we must have that $T_x\partial D_p = \{z \in \mathbb{H} : \mathrm{Re}(z) = 0\}$. Thus for any $p \in \widehat{\Omega}$ we have that

$$D_p = \{z \in \mathbb{H} : |z - r_p| \leq r_p\}.$$

But then

$$\widehat{\Omega} = \cup D_p = \{z \in \mathbb{H} : |z - r| < r\}$$

where $r = \sup r_p$. □

Lemma VII.24. *Suppose $E \subset \mathbb{P}(\mathbb{H}^{d+1})$ is a compact set and there exists $a \in E$ such that for all projective lines L through a the set $L \cap E$ has vanishing reduced cohomology. Then E has vanishing reduced cohomology, that is, $H^0(E) = \mathbb{Z}$ and $H^i(E) = 0$ for all $i > 0$.*

The proof of [APS04, Proposition 2.3.4] taken verbatim proves the lemma. The key step is to consider E_a the projective blow-up at a and then apply the Vietoris-Begle mapping theorem to the map $E_a \to \mathbb{P}(\mathbb{H}^d)$.

We are now ready to proof the Proposition VII.20]. This argument is taken from the proof of [APS04, Theorem 2.3.6].

Proof of Proposition VII.20. If $d = 1$, then every open set $\Omega \subset \mathbb{P}(\mathbb{H}^2)$ is linearly convex and so there is nothing to prove.

Now assume $d > 1$ and $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$. By induction we may assume that any set $\widehat{\Omega} \subset \mathbb{P}(\mathbb{H}^d)$ satisfying the hypothesis of the proposition is linearly convex.

Now pick $a \in \mathbb{P}(\mathbb{H}^{d+1}) \setminus \Omega$ and consider the projection $T : \mathbb{P}(\mathbb{H}^{d+1}) \to \mathbb{P}(\mathbb{H}^d)$ from a. If for every projective line L in $\mathbb{P}(\mathbb{H}^d)$ we have $T(\Omega) \cap L \neq L$ then by Lemma VII.23 there exists a hyperplane $H \subset \mathbb{P}(\mathbb{H}^d) \setminus T(\Omega)$. Then $T^{-1}(H)$ is a hyperplane in $\mathbb{P}(\mathbb{H}^{d+1})$ which contains a but does not intersect Ω.

So assume for a contradiction that $T(\Omega) \cap L = L$ for some projective line L. Let $\widehat{L}$ be the preimage of L, this set is projectively isomorphic to $\mathbb{P}(\mathbb{H}^3)$. So there exists a set $\widehat{\Omega} \subset \mathbb{P}(\mathbb{H}^3)$ satisfying the hypothesis of the proposition and $\widehat{a} \in \mathbb{P}(\mathbb{H}^3) \setminus \widehat{\Omega}$ such that if $\widehat{T} : \mathbb{P}(\mathbb{H}^3) \to \mathbb{P}(\mathbb{H}^2)$ is the projection from $\widehat{a}$ then $\widehat{T}(\widehat{\Omega}) = \mathbb{P}(\mathbb{H}^2)$.

Then for any projective line L through $\widehat{a}$ the intersection $\widehat{\Omega} \cap L$ is non-empty. By the hypothesis of the proposition $L \setminus \widehat{\Omega} \cap L$ has vanishing reduced cohomology. So by Lemma VII.24 the set $\mathbb{P}(\mathbb{H}^3) \setminus \widehat{\Omega}$ has vanishing reduced cohomology.

Now fix $b \in \widehat{\Omega}$ and consider the map $\pi : \mathbb{P}(\mathbb{H}^3) \setminus \widehat{\Omega} \to \mathbb{P}(\mathbb{H}^2)$ induced by the projection $\mathbb{P}(\mathbb{H}^3) \to \mathbb{P}(\mathbb{H}^2)$ from b. Now by the hypothesis of the proposition, the fibers of π have trivial reduced cohomology, thus the Vietoris-Begle mapping theorem implies that $\pi(\mathbb{P}(\mathbb{H}^3) \setminus \widehat{\Omega})$ has trivial reduced cohomology. Thus π cannot be onto, but this is only possible if $\widehat{\Omega}$ contains an entire projective line through b which is a contradiction. $\square$

7.6 Rigidity from symmetry

The majority of the proof Theorem VII.1 is nearly identical to the proof in the complex case. But there are two places in the proof where the argument needs to be modified.

First, the proof that (Ω, d_Ω) is a quasi-geodesic metric space used the Riemann mapping theorem and as this theorem is not available in the quaternionic case we will need a different argument to establish that (Ω, d_Ω) is quasi-geodesic. Second, in the proof of Proposition V.30 we needed to show that there are no proper Γ-invariant projective subspace that intersects Ω. The argument we gave used that every linear convex set with C^1 boundary is $\mathbb{C}$-convex and every open proper $\mathbb{C}$-convex is homeomorphic to a ball. Neither of these facts are available and thus we will need different arguments to show that there are no proper Γ-invariant projective subspace that intersect Ω.

Beyond these two technicalities, the rest of the proof in the complex case can be taken verbatim.

We begin with the proof that (Ω, d_Ω) is quasi-geodesic.

Proposition VII.25. *Suppose $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is a proper weakly linearly convex set with C^1 boundary. If Ω is divisible then (Ω, d_Ω) is quasi-geodesic.*

We will need several lemmas.

Lemma VII.26. *Suppose $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is a proper weakly linearly convex set with C^1 boundary. If $\epsilon > 0$ and L is a quaternionic projective line that intersects Ω then there exists a neighborhood $\mathcal{O}$ of L (in the space of projective lines) such that for every $L' \in \mathcal{O}$ the metric spaces $(\Omega \cap L, d_{\Omega\cap L})$ and $(\Omega \cap L', d_{\Omega \cap L'})$ are $(1, \epsilon)$-quasi-isometric.*

Proof. We first observe that any projective line L that intersects Ω must intersect $\partial\Omega$ transversally. If not, then there exists a point $x \in \partial\Omega \cap L$ such that $L \subset T_x\partial\Omega$ but then $L \subset T_x^{\mathbb{H}}\partial\Omega$ and by weak linear convexity $T_x^{\mathbb{H}}\partial\Omega$ does not intersect Ω.

Since $\partial\Omega$ is C^1, for any $\delta > 0$ there exists a neighborhood $\mathcal{O}_\delta$ of L such that for any

$L' \in \mathcal{O}_\delta$ there exists a $(1+\delta)$-bi-Lipschitz homeomorphism $f : \overline{\Omega \cap L} \to \overline{\Omega \cap L'}$ which maps $\Omega \cap L$ to $\Omega \cap L'$. This induces, as in the proof of Lemma V.8, a $(1, 4\log(1+\delta))$-quasi-isometry. Since $\delta > 0$ was arbitrary, the lemma follows. $\square$

We now turn our attention to planar domains. Suppose $\Omega \subset \mathbb{H}$ is open and $p \in \Omega$ let

$$\delta_\Omega(p) := \inf \{ |x - p| : x \in \partial\Omega \}.$$

If $\partial\Omega$ is C^1 and $x \in \partial\Omega$ let n_x be the inward pointing unit normal vector at x.

Lemma VII.27. *Suppose $\Omega \subset \mathbb{H}$ is a bounded open set with C^1 boundary. Then there exists $\epsilon > 0$, $A \geq 1$, and $B \geq 0$ such that for any $x \in \partial\Omega$ the line segment $(x, x + \epsilon n_x]$ parametrized as*

$$\gamma(t) = (1 - e^{-t})x + e^{-t}(x + \epsilon n_x) = x + e^{-t}\epsilon n_x$$

is a (A, B)-quasi-geodesic in (Ω, d_Ω).

Proof. Since $\partial\Omega$ is C^1 and compact there exists $\epsilon > 0$ such that for any $x \in \partial\Omega$ and $t \in (0, \epsilon)$ we have

$$\delta_\Omega(x + tn_x) \geq t/2.$$

Now consider the Riemannian metric

$$g_p(v, w) = \frac{\langle v, w \rangle}{\delta_\Omega(p)}$$

on Ω. If d_δ is the corresponding distance on Ω then Beardon [Bea98, Theorem 3.2] showed that

$$d_\Omega = A_\Omega \leq 2 d_\delta.$$

In particular if $t_1 < t_2$ then

$$\begin{aligned} d_\Omega(\gamma(t_1),\gamma(t_2)) &\leq 2\int_{t_1}^{t_2} \sqrt{g_{\gamma(s)}(\gamma'(s),\gamma'(s))}ds \\ &= 2\int_{t_1}^{t_2} \sqrt{\frac{\left|e^{-s}\epsilon n_x\right|}{\delta_\Omega(x+e^{-s}\epsilon n_x)}}ds \\ &\leq 2\int_{t_1}^{t_2} 2ds = 4\left|t_2-t_1\right|. \end{aligned}$$

To see the lower bound assume $t_1 < t_2$ then

$$d_\Omega(\gamma(t_1),\gamma(t_2)) = \sup_{a,b\in\overline{\mathbb{H}}\setminus\Omega} \log\frac{|\gamma(t_1)-a|\,|\gamma(t_2)-b|}{|\gamma(t_1)-b|\,|\gamma(t_2)-a|} \geq \log\frac{|\gamma(t_1)-x|}{|\gamma(t_2)-x|} = |t_2-t_1|$$

where we used $a = x$ and $b = \infty$. $\square$

Lemma VII.28. *Suppose $\Omega \subset \mathbb{H}$ is a bounded open set with C^1 boundary. If $K \subset \Omega$ is compact then there exists $A \geq 1$ and $B \geq 0$ such that for any $p \in K$ and any $q \in \Omega$ there exists an (A,B)-quasi-geodesic in the Hilbert metric joining them.*

Proof. By the previous lemma, there exists $\epsilon > 0$, $A \geq 1$, and $B' \geq 0$ such that for any $x \in \partial\Omega$, the line segment $(x, x+\epsilon n_x]$ can be parametrized by $\gamma_x(t)$ to be a (A',B')-quasi-geodesic.

By possibly enlarging K we can assume that

$$\Omega = K \cup \{q \in \Omega : \delta_\Omega(q) < \epsilon\}.$$

Notice that the set $\{\gamma_x(0) : x \in \partial\Omega\}$ is a compact subset of Ω and hence there exists $R > 0$ such that

$$d_\Omega(p,\gamma_x(0)) \leq R$$

for any $p \in K$ and $x \in \partial\Omega$. By enlarging R we may also assume that

$$d_\Omega(p,q) \leq R$$

for all $p, q \in K$.

Now let $p \in K$ and $q \in \Omega$, we claim that there exists an (A, B)-quasi-geodesic joining them where

$$B = B' + R + \frac{1}{A}.$$

If $q \in K$ then the map $\gamma : [0, 1] \to \Omega$ given by

$$\gamma(t) = \begin{cases} p & \text{if } t < 1, \\ q & \text{if } t = 1 \end{cases}$$

is a $(1, R)$-quasi-geodesic. So we can assume $q \notin K$ and thus $\delta_\Omega(q) < \epsilon$. Then there exists $x \in \partial\Omega$ such that $q = \gamma_x(t)$ for some $t \geq 0$. Thus it is enough to show that the map $\gamma : [0, \infty) \to \Omega$ given by

$$\gamma(t) = \begin{cases} p & \text{if } t \leq 1 \\ \gamma_x(t-1) & \text{otherwise} \end{cases}$$

is an (A, B)-quasi-geodesic.

If $1 \leq t_1, t_2 < \infty$ then

$$\frac{1}{A} |t_2 - t_1| - B \leq d_\Omega(\gamma(t_1), \gamma(t_2)) \leq A |t_2 - t_1| + B$$

since γ_x is an (A, B')-quasi-geodesic and $B \geq B'$.

If $0 \leq t_1, t_2 \leq 1$ then $d_\Omega(\gamma(t_1), \gamma(t_2)) = 0$ and hence

$$|t_2 - t_1| - 1 \leq d_\Omega(\gamma(t_1), \gamma(t_2)) \leq |t_2 - t_1|.$$

So, it remains to consider the case when $0 \leq t_1 \leq 1 < t_2 < \infty$. Then

$$\begin{aligned} d_\Omega(\gamma(t_1), \gamma(t_2)) &= d_\Omega(p, \gamma_x(t_2 - 1)) \leq d_\Omega(p, \gamma_x(0)) + d_\Omega(\gamma_x(0), \gamma_x(t_2 - 1)) \\ &\leq R + A |t_2 - 1| + B' \leq A |t_2 - t_1| + (R + B') \end{aligned}$$

and

$$
\begin{aligned}
d_\Omega(\gamma(t_1),\gamma(t_2)) &= d_\Omega(p,\gamma_x(t_2-1)) \geq d_\Omega(\gamma_x(0),\gamma_x(t_2-1)) - d_\Omega(p,\gamma_x(0)) \\
&\geq \frac{1}{A}\left|t_2-1\right| - B' - R \geq \frac{1}{A}\left|t_2-t_1\right| - (R+B'+\frac{1}{A})
\end{aligned}
$$

Thus, γ is an (A,B)-quasi-geodesic and the lemma follows. $\square$

Proof of Proposition VII.25. Since Ω is divisible we can find a compact set $K \subset \Omega$ such that $\Gamma K = \Omega$. Now let Λ be the set of quaternionic projective lines that intersect K. This set is compact and by the previous lemma, for every $L \in \Lambda$ there exists $A_L \geq 1$ and $B_L \geq 0$ such that for any $p \in K \cap L$ and $q \in \Omega \cap L$ there exists an (A_L, B_L)-quasi-geodesic joining them. Moreover, if L' is sufficiently close to L then for any $p \in K \cap L'$ and $q \in \Omega \cap L'$ there exists a (A_L, B_L+1)-quasi-geodesic joining them. So, by the compactness of Λ there exists $A \geq 1$ and $B \geq 0$ such that for any $p \in K$ and $q \in \Omega$ there is an (A,B)-quasi-geodesic joining them. Then since $\Gamma K = \Omega$ for any $p, q \in \Omega$, there is an (A,B)-quasi-geodesic joining them. $\square$

Now we turn our attention to the second and final necessary modification:

Proposition VII.29. *Suppose $\Omega \subset \mathbb{P}(\mathbb{H}^{d+1})$ is a proper weakly linearly convex set with C^1 boundary. If Ω is divisible by Γ and W is a Γ-invariant projective subspace that intersects Ω then $W = \mathbb{P}(\mathbb{H}^{d+1})$.*

Proof. First suppose that there exists an affine chart $\mathbb{H}^d$ which contains Ω as a bounded subset. And assume for a contradiction $W \neq \mathbb{P}(\mathbb{H}^{d+1})$. Now by compactness there exists $x \in \partial\Omega$ such that

$$d(x, W) = \inf\{\|x - w\| : w \in W\}$$

is as large as possible. Then since $\partial\Omega$ is C^1 we must have that $x + W \subset T_x\partial\Omega$. So $W \cap T_x\partial\Omega = \emptyset$. Now let $p_n \in \Omega$ be a sequence such that $p_n \to x$. Fixing a point $o \in W \cap \Omega$ there exists $\varphi_n \in \Gamma$ such that $d_\Omega(\varphi_n o, p_n) \leq R$. By passing to a subsequence we can suppose that $\varphi_n o \to y \in \partial\Omega$. But then by Proposition VII.14 we must have $y \in T_x\partial\Omega$ but by construction $y \in W$. So we have a contradiction.

We now show that there exists an affine chart $\mathbb{H}^d$ which contains Ω as a bounded subset. Using the proof of Theorem V.24 we can find coordinates such that

$$\Omega \subset \{[1 : z_1 : \cdots : z_d] : \mathrm{Re}(z_1) > 0\}.$$

Then for $t < 0$ the quaternion hyperplane

$$H = \{[z_1 : tz_1 : z_2 : \cdots : z_d] : z_1, \ldots, z_d \in \mathbb{H}\}$$

is a hyperplane in the interior of $\mathbb{P}(\mathbb{H}^{d+1}) \setminus \Omega$. Then Ω is a bounded subset of $\mathbb{P}(\mathbb{H}^{d+1}) \setminus H$. $\square$

7.7 Rigidity from geodesics

The proof in the complex case can be taken verbatim to prove the corresponding result in the quaternion case.

BIBLIOGRAPHY

BIBLIOGRAPHY

[Aba89] Marco Abate, *Iteration theory of holomorphic maps on taut manifolds*, Research and Lecture Notes in Mathematics. Complex Analysis and Geometry, Mediterranean Press, Rende, 1989. MR 1098711 (92i:32032)

[APS04] Mats Andersson, Mikael Passare, and Ragnar Sigurdsson, *Complex convexity and analytic functionals*, Progress in Mathematics, vol. 225, Birkhäuser Verlag, Basel, 2004. MR 2060426 (2005a:32011)

[Bar35] Dan Barbilian, *Einordnung von Lobatschewsky's Maßbestimmung in gewisse allgemeine Metrik der Jordanschen Bereiche*, Časopis pro pěstování matematiky a fysiky **064** (1935), no. 6, 182–183 (ger).

[Bea98] A. F. Beardon, *The Apollonian metric of a domain in* $\mathbf{R}^n$, Quasiconformal mappings and analysis (Ann Arbor, MI, 1995), Springer, New York, 1998, pp. 91–108. MR 1488447 (99k:30075)

[Ben55] Jean Paul Benzecri, *Varietes localement plates*, ProQuest LLC, Ann Arbor, MI, 1955, Thesis (Ph.D.)–Princeton University. MR 2612352

[Ben03] Yves Benoist, *Convexes divisibles. II*, Duke Math. J. **120** (2003), no. 1, 97–120. MR 2010735 (2004m:22018)

[Ben04] ———, *Convexes divisibles. I*, Algebraic groups and arithmetic, Tata Inst. Fund. Res., Mumbai, 2004, pp. 339–374. MR 2094116 (2005h:37073)

[Ben06] ———, *Convexes divisibles. IV. Structure du bord en dimension 3*, Invent. Math. **164** (2006), no. 2, 249–278. MR 2218481 (2007g:22007)

[Ben08] ———, *A survey on divisible convex sets*, Geometry, analysis and topology of discrete groups, Adv. Lect. Math. (ALM), vol. 6, Int. Press, Somerville, MA, 2008, pp. 1–18. MR 2464391 (2010h:52013)

[Cra09] Mickaël Crampon, *Entropies of strictly convex projective manifolds*, J. Mod. Dyn. **3** (2009), no. 4, 511–547. MR 2587084 (2011g:37079)

[CS08] A. Cano and J. Seade, *On discrete subgroups of automorphism of* $\mathbf{P}^n_{\mathbf{C}}$, ArXiv e-prints (2008).

[dlH00] Pierre de la Harpe, *Topics in geometric group theory*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2000. MR 1786869 (2001i:20081)

[Dub09] Loïc Dubois, *Projective metrics and contraction principles for complex cones*, J. Lond. Math. Soc. (2) **79** (2009), no. 3, 719–737. MR 2506695 (2010g:47160)

[FK94] Jacques Faraut and Adam Korányi, *Analysis on symmetric cones*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1994, Oxford Science Publications. MR 1446489 (98g:17031)

[Fra89] Sidney Frankel, *Complex geometry of convex domains that cover varieties*, Acta Math. **163** (1989), no. 1-2, 109–149. MR 1007621 (90i:32037)

[GH00] F. W. Gehring and K. Hag, *The Apollonian metric and quasiconformal mappings*, In the tradition of Ahlfors and Bers (Stony Brook, NY, 1998), Contemp. Math., vol. 256, Amer. Math. Soc., Providence, RI, 2000, pp. 143–163. MR 1759676 (2001c:30043)

[Gol88] William M. Goldman, *Geometric structures on manifolds and varieties of representations*, Geometry of group representations (Boulder, CO, 1987), Contemp. Math., vol. 74, Amer. Math. Soc., Providence, RI, 1988, pp. 169–198. MR 957518 (90i:57024)

[Gol90] ______, *Convex real projective structures on compact surfaces*, J. Differential Geom. **31** (1990), no. 3, 791–845. MR 1053346 (91b:57001)

[Gol09] ______, *Projective geometry on manifolds*, 2009.

[Gro87] M. Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263. MR 919829 (89e:20070)

[GT87] M. Gromov and W. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math. **89** (1987), no. 1, 1–12. MR 892185 (88e:53058)

[GW12] Olivier Guichard and Anna Wienhard, *Anosov representations: domains of discontinuity and applications*, Invent. Math. **190** (2012), no. 2, 357–438. MR 2981818

[Häs06] Peter A. Hästö, *Gromov hyperbolicity of the j_G and $\tilde{j}_G$ metrics*, Proc. Amer. Math. Soc. **134** (2006), no. 4, 1137–1142 (electronic). MR 2196049 (2007g:30062)

[Ibr02] Zair Ibragimov, *The apollonian metric, sets of constant width and möbius modulus of ring domains*, Ph.D. thesis, University of Michigan, 2002.

[IK99] A. V. Isaev and S. G. Krantz, *Domains with non-compact automorphism group: a survey*, Adv. Math. **146** (1999), no. 1, 1–38. MR 1706680 (2000i:32053)

[JM87] Dennis Johnson and John J. Millson, *Deformation spaces associated to compact hyperbolic manifolds*, Discrete groups in geometry and analysis (New Haven, Conn., 1984), Progr. Math., vol. 67, Birkhäuser Boston, Boston, MA, 1987, pp. 48–106. MR 900823 (88j:22010)

[Jo03] Kyeonghee Jo, *Quasi-homogeneous domains and convex affine manifolds*, Topology Appl. **134** (2003), no. 2, 123–146. MR 2009094 (2004i:57020)

[Kap07] Michael Kapovich, *Convex projective structures on Gromov-Thurston manifolds*, Geom. Topol. **11** (2007), 1777–1830. MR 2350468 (2008h:53045)

[KO80] Shoshichi Kobayashi and Takushiro Ochiai, *Holomorphic projective structures on compact complex surfaces*, Math. Ann. **249** (1980), no. 1, 75–94. MR 575449 (81g:32021)

[Kob77] Shoshichi Kobayashi, *Intrinsic distances associated with flat affine or projective structures*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), no. 1, 129–135. MR 0445016 (56 #3361)

[Koe99] Max Koecher, *The Minnesota notes on Jordan algebras and their applications*, Lecture Notes in Mathematics, vol. 1710, Springer-Verlag, Berlin, 1999, Edited, annotated and with a preface by Aloys Krieg and Sebastian Walcher. MR 1718170 (2001e:17040)

[Kos68] J.-L. Koszul, *Déformations de connexions localement plates*, Ann. Inst. Fourier (Grenoble) **18** (1968), no. fasc. 1, 103–114. MR 0239529 (39 #886)

[Kra13] Steven G. Krantz, *The impact of the theorem of bun wong and rosay*, Complex Variables and Elliptic Equations (to appear in 2013).

[Lab06] François Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165** (2006), no. 1, 51–114. MR 2221137 (2007c:20101)

[Led12] François Ledrappier, *Analyticity of the entropy for some random walks*, Groups Geom. Dyn. **6** (2012), no. 2, 317–333. MR 2914862

[Lem87] László Lempert, *Complex geometry in convex domains*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986) (Providence, RI), Amer. Math. Soc., 1987, pp. 759–765. MR 934278 (89f:32045)

[Mar10] Ludovic Marquis, *Espace des modules marqués des surfaces projectives convexes de volume fini*, Geom. Topol. **14** (2010), no. 4, 2103–2149. MR 2740643 (2012b:57038)

[Mar12a] ———, *Exemples de variétés projectives strictement convexes de volume fini en dimension quelconque*, Enseign. Math. (2) **58** (2012), no. 1-2, 3–47. MR 2985008

[Mar12b] ———, *Surface projective convexe de volume fini*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 1, 325–392. MR 2986273

[Mar13] ———, *Around groups in hilbert geometry*, Handbook of Hilbert geometry, European Mathematical Society Publishing, to appear in 2013.

[Mos73] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Princeton University Press, Princeton, N.J., 1973, Annals of Mathematics Studies, No. 78. MR 0385004 (52 #5874)

[Neu54] B. H. Neumann, *Groups covered by permutable subsets*, J. London Math. Soc. **29** (1954), 236–248. MR 0062122 (15,931b)

[NPZ08] Nikolai Nikolov, Peter Pflug, and Włodzimierz Zwonek, *An example of a bounded* $\mathbf{C}$*-convex domain which is not biholomorphic to a convex domain*, Math. Scand. **102** (2008), no. 1, 149–155. MR 2420684 (2009b:32014)

[Pra94] Gopal Prasad, $\mathbf{R}$*-regular elements in Zariski-dense subgroups*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 541–545. MR 1315463 (96a:22022)

[Qui10] Jean-François Quint, *Convexes divisibles (d'après Yves Benoist)*, Astérisque (2010), no. 332, Exp. No. 999, vii, 45–73, Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011. MR 2648674 (2011k:22023)

[Rho97] A. D. Rhodes, *An upper bound for the hyperbolic metric of a convex domain*, Bull. London Math. Soc. **29** (1997), no. 5, 592–594. MR 1458720 (98e:30005)

[Ros79] Jean-Pierre Rosay, *Sur une caractérisation de la boule parmi les domaines de* $\mathbf{C}^n$ *par son groupe d'automorphismes*, Ann. Inst. Fourier (Grenoble) **29** (1979), no. 4, ix, 91–97. MR 558590 (81a:32016)

[Rot66] O. S. Rothaus, *The construction of homogeneous convex cones*, Ann. of Math. (2) **83** (1966), 358–376. MR 0202156 (34 #2029)

[Sei99] Pasi Seittenranta, *Möbius-invariant metrics*, Math. Proc. Cambridge Philos. Soc. **125** (1999), no. 3, 511–533. MR 1656825 (2000g:30017)

[SM02] Edith Socié-Méthou, *Caractérisation des ellipsoïdes par leurs groupes d'automorphismes*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 4, 537–548. MR 1981171 (2004j:53055)

[Tay11] Michael E. Taylor, *Partial differential equations I. Basic theory*, second ed., Applied Mathematical Sciences, vol. 115, Springer, New York, 2011. MR 2744150 (2011m:35001)

[tD08] Tammo tom Dieck, *Algebraic topology*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2008. MR 2456045 (2009f:55001)

[Vin63] È. B. Vinberg, *The theory of homogeneous convex cones*, Trudy Moskov. Mat. Obšč. **12** (1963), 303–358. MR 0158414 (28 #1637)

[Vin65] ______, *Structure of the group of automorphisms of a homogeneous convex cone*, Trudy Moskov. Mat. Obšč. **13** (1965), 56–83. MR 0201575 (34 #1457)

[VK67] È. B. Vinberg and V. G. Kac, *Quasi-homogeneous cones*, Mat. Zametki **1** (1967), 347–354. MR 0208470 (34 #8280)

[Won77] B. Wong, *Characterization of the unit ball in* $\mathbf{C}^n$ *by its automorphism group*, Invent. Math. **41** (1977), no. 3, 253–257. MR 0492401 (58 #11521)

[Zim13] Andrew M. Zimmer, *Rigidity of complex convex divisible sets*, Submitted, 2013.