

Asymptotics of Equivariant Syzygies

by
Xin Zhou

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2014

Doctoral Committee:

Professor Robert K. Lazarsfeld, co-chair
Professor Mircea I. Mustață, co-chair
Professor Morgan V. Brown
Professor David E. Speyer
Professor James P. Tappenden

© Xin Zhou 2014
All Rights Reserved

ACKNOWLEDGEMENTS

I want to give my sincerest thanks to my adviser, Rob Lazarsfeld. His advisory is multidimensional, from building my mathematical sense of direction, to teaching me how to polish every facet of my papers, from sparking my problem solving ideas, to communicating results interestingly and effectively.

I also benefitted from the caring mentorship of many other professors in the department. Mircea Mustața has always brought me the clearest of concepts. William Fulton has always spiced my interest in topics. David Speyer has always kept momentum in my problem solving. Many others including Sergey Fomin, Mel Hochster, Thomas Lam, Gopal Prasad, Yongbin Ruan gave wonderful courses that shaped my mathematics.

Fellow grad-students and postdocs filled my graduate school years with math and friendship. I thank Bhargav Bhatt, Mihai Fulger, Linqun Ma, Claudiu Raicu, Matthew Satriano, Kevin Tucker, Liang Xiao, Xiaolei Zhao, Zhixian Zhu and those not mentioned.

I thank the administrative staff of the Department of Mathematics for their kind service.

Last, but not the least, I thank my family and friends for their support throughout the years.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS	ii
LIST OF FIGURES	v
CHAPTER	
I. Introduction	1
II. Syzygies as cohomologies	6
II.1 Koszul cohomology	6
II.2 Computing syzygies with vector bundles	8
III. Schur asymptotics of Veronese syzygies	13
III.1 Introduction	13
III.2 Some representation theory notations	17
III.3 Asymptotic plethysms	19
III.3.1 Integral points and $\bigotimes^p S^d$	20
III.3.2 $\mathbb{S}_\mu S^d$ via $\bigotimes^p S^d$	24
III.4 Asymptotic syzygy functors	28
III.4.1 $\mathbb{K}_{p,1}(d)$	28
III.4.2 $\mathbb{K}_{p,0}(b; d)$	29
III.5 $\mathbb{K}_{p,1}(b; d)$ when $b > 0$	32
III.6 Varying p	38
III.6.1 $\mathbb{K}_{p,0}(b; d)$	39
III.6.2 $\mathbb{K}_{p,1}(b; d)$	39
IV. Toric weights in asymptotic syzygies of toric varieties	43
IV.1 Introduction	43
IV.2 Idea of proof	46
IV.3 Surjectivity of map induced by a secant space	50
IV.3.1 Key lemma	50
IV.3.2 Proof of Lemma IV.3	53
IV.4 Enlarged secant space	60
IV.5 Asymptotics of normalized weights	63
IV.6 Boundary of $\Delta(a)$	72

BIBLIOGRAPHY	74
-------------------------------	-----------

LIST OF FIGURES

Figure

IV.1	Normalized torus weights for $K_{p,1}(\mathbb{P}^2; d)$ for $d = 2$ and $d = 4$	45
IV.2	Closure of normalized weights for $a = 0.33$, $b = 0.66$ and $a = 0.66$, $b = 1$ with $\mathbb{P} = \mathbb{P}^2$, $d = 4$	45
IV.3	$\Delta(\frac{1}{10})$ for the unit square.	46

CHAPTER I

Introduction

Classically, algebraic geometers were interested in studying the equations defining various projective embeddings of a given compact complex manifold X . Consider a line bundle L on X , whose sections realize X as an embedded subvariety of projective space:

$$X \hookrightarrow \mathbb{P}^r = \mathbb{P}\Gamma(X, L).$$

Castelnuovo [C1893] for curves, and Mumford [M66], [M70] in general, showed that if L is sufficiently positive, then X is cut out in \mathbb{P}^r by quadratic equations. For example, the map

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3 = \{[Z_0 : Z_1 : Z_2 : Z_3] \mid \text{not all } Z_i \text{ are } 0\}, \quad [s, t] \mapsto [s^3, s^2t, st^2, t^3]$$

embeds \mathbb{P}^1 as a cubic curve $C \subseteq \mathbb{P}^3$ cut out by the equations

$$Q_{12} = Z_0Z_2 - Z_1^2, \quad Q_{23} = Z_1Z_3 - Z_2^2, \quad Q_{13} = Z_0Z_3 - Z_1Z_2.$$

Around 1980, Green ([G84I], [G84II]) realized that one could see classical results of this sort as the first cases of more general statements involving higher syzygies. For example, not only is the twisted cubic $C \subseteq \mathbb{P}^3$ cut out by defining equations of

the lowest possible degree, but the relations among the quadrics Q_{ij} are spanned by syzygies whose coefficients all have degree one:

$$Z_0 \cdot Q_{23} - Z_1 \cdot Q_{13} + Z_2 \cdot Q_{12} = 0 \quad , \quad Z_1 \cdot Q_{23} - Z_2 \cdot Q_{13} + Z_3 \cdot Q_{12} = 0$$

To better state Green's result, let us introduce notations for syzygies. Let L be a very ample line bundle on a smooth projective variety X , and set

$$S = \text{Sym } H^0(X, L) \quad , \quad r = r(L) = h^0(L) - 1.$$

Let n be the dimension of X . Given a fixed divisor B on X , put

$$M = \bigoplus_m H^0(X, mL + B),$$

which we view as an S -module. We consider the graded minimal free resolution of M over S :

$$\mathbb{F} : \dots \longrightarrow F_p \longrightarrow \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where $F_p = \bigoplus_j S(-a_{p,j})$ is a free S -module. More canonically, write $K_{p,q}(X, B; L)$ for the finite dimensional vector space spanned by minimal generators of F_p in degree $(p+q)$, i.e.:

$$F_p \cong \bigoplus_q K_{p,q}(X, B; L) \otimes_{\mathbb{C}} S(-p-q)$$

When $B = 0$, in our asymptotic settings when L is sufficiently positive, \mathbb{F} amounts to the minimal resolution of the homogeneous coordinate ring of the image of X , we will write simply $K_{p,q}(X; L_d)$. For example, in our case of the twisted cubic above, the resolution looks like:

$$0 \rightarrow S(-3)^{\oplus 2} \rightarrow S(-2)^{\oplus 3} \rightarrow S \rightarrow M \rightarrow 0$$

and we have:

$$\dim K_{1,1}(\mathbb{P}^1; \mathcal{O}(3)) = 3 \quad , \quad \dim K_{2,1}(\mathbb{P}^1; \mathcal{O}(3)) = 2$$

and all the other $K_{p,q}$ vanish.

As noted above, higher syzygies arising from geometry were first studied by Green. In case of curves, Green ([G84I] Thm. 4.a.1) proved that:

$$K_{p,2}(X; L) = 0, \text{ if } \deg(L) \geq 2g + 1 + p$$

where g is the genus of X . Later for Veronese embeddings of projective spaces, he established that (corollary of [G84II] Thm. 2.2):

$$K_{p,q}(\mathbb{P}^n; \mathcal{O}(d)) = 0, \text{ if } q \geq 2, \text{ and } d \geq p.$$

In words, Green's result says that the first d modules of syzygies for the Veronese image are generated in the lowest possible degrees. Similar results were then obtained for other varieties ([EL93], [P05]) and some long standing open problems remain to give sharp bounds in special cases ([AF11], [AV03], [B03], [V02], [V05].)

A perspective changing discovery was made by Ein and Lazarsfeld in [EL12] where they make the observation that as we increase the embedding degree d , we can have on the order of d^n potential syzygies, while the old theory accounts for only on the order of d many syzygies. They give us a better asymptotic understanding of the full resolution in higher dimensions by showing that for any $1 \leq q \leq n$, $K_{p,q} \neq 0$ for almost all $p \leq r(L_d)$. This means that asymptotically, syzygies are as complicated as they can be in the sense that almost all allowed to be nonzero are actually nonzero.

The natural next set of questions, while also previously independently studied on its own before [EL12] (cf. [HSS06], [N11], [R13], [S13]), is to understand the possible

finer structures on syzygies. For example, when X is a projective variety in which a group G acts and L is a G -linear line bundle, then $K_{p,q}(X; L)$ carries induced G -actions. In this thesis, we focus on the two common equivariant behaviors studied in the literature, i.e. with respect to general linear groups and with respect to torus actions.

When X is the projective space \mathbb{P}^n , the vector spaces $K_{p,q}(\mathbb{P}^n; \mathcal{O}(d))$ are representations of GL_{n+1} . We can then ask what their asymptotic behaviors are as GL_{n+1} representations. In Chapter III, we address this problem and give a sharp order of growth for the number of distinct irreducible decompositions and the number of irreducible representations counting multiplicities in a fixed syzygy module. In our notation, we are studying the behavior of $K_{p,q}(\mathbb{P}^n; \mathcal{O}(d))$ for a fixed p as d becomes large. We study various other asymptotic behaviors in that chapter also. The work in that chapter is joint with Mihai Fulger.

The shortcoming of the GL -equivariant results is that we are not able to consider all $K_{p,q}$ at once as d increases. This flaw is overcome in the answer to the second question.

Specifically, when X is a toric variety with a very ample toric line bundle, $K_{p,q}$ are representations of the torus. Then the question arises what torus weights there are in these representations. More specifically, let Δ be the convex polytope associated to a toric very ample line bundle A and let $L_d = A^{\otimes d}$. Then the torus weights of $K_{p,q}(X; L_d)$ correspond to points in $(p+q)d \cdot \Delta$. Denote the collection of weights by:

$$\text{wts}(K_{p,q}(X; L_d)) := \{\text{Torus weights of } K_{p,q}(X; L_d)\} \subseteq (p+q)d \cdot \Delta$$

We normalize so that all points lie in Δ :

$$\text{wts}^{\text{nor}}(K_{p,q}(X; L_d)) := \frac{\text{wts}(K_{p,q}(X; L_d))}{(p+q)d} \subseteq \Delta$$

The question becomes in which area the normalized weights lie inside Δ .

If we look at all the $K_{p,q}$, it is shown in Chapter IV that they are asymptotically dense inside Δ . Their behavior is even more interesting if we look at a fixed segment of $K_{p,q}$ relative to $r(L_d)$. More precisely, in Chapter IV, we study the closure of the normalized weights inside Δ for all $K_{p,q}$ with $ar(L_d) \leq p \leq br(L_d)$. It turns out that this closure, denoted $\Delta(a, b)$ is not necessarily all of Δ . We give the exact characterization of this closure, from which we see that one of the key features of $\Delta(a, b)$ is that it depends only on a , so we can denote it by $\Delta(a)$. If Δ is the unit square (corresponding to $X = \mathbb{P}^1 \times \mathbb{P}^1, A = \mathcal{O}(1, 1)$), then $\Delta(a)$ consists of four segments of hyperbolas and eight line segments as we will see in an example in Chapter IV.

The thesis is organized as follows. In Chapter II, we introduce the basic tools we use to compute syzygies. In Chapter III, we present joint work with Mihai Fulger [FZ13] on the Schur asymptotics of Veronese syzygies. In Chapter IV, we present results on toric weights in asymptotic syzygies of toric varieties.

My previous work [Z12] is not included in this thesis. However, the material inspires section 2 and 3 of Chapter IV.

CHAPTER II

Syzygies as cohomologies

In this chapter, we introduce the two basic tools we use to compute syzygies in the next two chapters. In section 1, we introduce Koszul cohomology. In section 2, we introduce a vector bundle, cohomologies of whose wedge powers compute syzygies. In section 3, we introduce a surjective map of sheafs whose induced map in cohomologies we will study in Chapter IV. This chapter is an expository presentation of well-known material.

II.1 Koszul cohomology

Recall the set-up for the definition of $K_{p,q}(X, B; L)$. Let L be a very ample line bundle on a smooth projective variety X , and set

$$S = \operatorname{Sym} H^0(X, L) \quad , \quad r = r(L) = h^0(L) - 1.$$

Let n be the dimension of X . Given a fixed divisor B on X , put

$$M = \bigoplus_m H^0(X, mL + B),$$

which we view as an S -module. We consider the graded minimal free resolution of M over S :

$$\mathbb{F} : \dots \longrightarrow F_p \longrightarrow \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where $F_p = \bigoplus_j S(-a_{p,j})$ is a free S -module. We write $K_{p,q}(X, B; L)$ for the finite dimensional vector space spanned by minimal generators of F_p in degree $(p+q)$, i.e.:

$$(II.1.1) \quad F_p \cong \bigoplus_q K_{p,q}(X, B; L) \otimes_{\mathbb{C}} S(-p-q)$$

Below, we show that we have an alternative way to express $K_{p,q}$. Define the Koszul cohomology, temporarily denoted $K'_{p,q}(X, B; L)$ before we prove it to be equal to $K_{p,q}(X, B; L)$, to be cohomology in the middle of the following short complex:

$$(II.1.2) \quad \begin{aligned} \wedge^{p+1} H^0(X, L) \otimes H^0(B + (q-1)L) &\rightarrow \wedge^p H^0(X, L) \otimes H^0(B + qL) \\ &\rightarrow \wedge^{p-1} H^0(X, L) \otimes H^0(B + (q+1)L) \end{aligned}$$

and the maps are given by:

$$(II.1.3) \quad s_1 \wedge s_2 \dots \wedge s_p \otimes s \mapsto \sum_{i=1}^p (-1)^i s_1 \wedge \dots \hat{s}_i \dots \wedge s_p \otimes (s_i s)$$

where $s_i s$ is evaluated in $H^0(B + (q+1)L)$ with the evaluation map:

$$H^0(X, L) \otimes H^0(X, B + qL) \rightarrow H^0(X, B + (q+1)L).$$

and \hat{s}_i means omitting s_i . The key lemma of this section is that:

Lemma II.1. *There is a canonical isomorphism:*

$$K'_{p,q}(X, B; L) \cong K_{p,q}(X, B; L)$$

Proof. Let $S_+ \subset S$ be the irrelevant maximal ideal. We prove that:

$$K'_{p,q}(X, B; L) \cong K_{p,q}(X, B; L) \cong \mathrm{Tor}_p^S(M, S/S_+)_{p+q}.$$

\mathbb{F} is a minimal resolution which implies the maps are all 0 tensored with S/S_+ , hence the homology of $\mathbb{F} \otimes S/S_+$ is $K_{p,q}(X, B; L)$. \mathbb{F} is a free resolution of M , so by definition of Tor:

$$K_{p,q}(X, B; L) = \mathrm{Tor}_p^S(M, S/S_+)_{p+q}.$$

On the other hand, let

$$E_p = \wedge^p H^0(X, L)$$

We can form the Koszul complex:

$$\mathbb{E} : \dots \longrightarrow E_p \longrightarrow \dots \longrightarrow E_0 \longrightarrow S/S_+ \longrightarrow 0$$

and \mathbb{E} is the minimal free resolution of S/S_+ over S known as the Koszul resolution. So cohomologies of the complex $\mathbb{E} \otimes M$ also give us $\mathrm{Tor}_p^S(M, S/S_+)_{p+q}$. The degree q piece of E_p , tensored with the degree q piece of M is:

$$\wedge^p H^0(X, L) \otimes H^0(X, B + qL)$$

which is the mid-term of the short complex II.1.2. The other terms also match and the map in II.1.3 is the same as the map from the Koszul complex. Therefore, $K'_{p,q}(X, B; L) \cong K_{p,q}(X, B; L)$ and we denote both of them by $K_{p,q}(X, B; L)$. \square

II.2 Computing syzygies with vector bundles

In this section, we show that these Koszul cohomology groups are governed by the coherent cohomology of a vector bundle on X .

Let X be a smooth variety over \mathbb{C} . Let L be a very ample line bundle on X .

As in [GL85], [L89] and [EL12], for L in the evaluation map:

$$\nu_L : H^0(X, L) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow \mathcal{O}_X(L)$$

we put $M_L = \ker \nu_L$. Thus M_L is a vector bundle sitting in the basic exact sequence:

$$(II.2.1) \quad 0 \rightarrow M_L \rightarrow H^0(X, L) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow \mathcal{O}_X(L) \rightarrow 0$$

M_L is central in syzygy computations since:

Proposition II.2. *Assume that:*

$$(II.2.2) \quad H^1(\mathcal{O}_X(B + (q-1)L)) = 0,$$

Then:

$$K_{p,q}(X, B; L) = H^1(X, \wedge^{p+1} M_L \otimes \mathcal{O}_X(B + (q-1)L)).$$

Proof. From the defining sequence of M_L , II.2.1, we get

$$0 \rightarrow \wedge^p M_L \rightarrow \wedge^p V \otimes \mathcal{O}_X \rightarrow \wedge^{p-1} M_L \otimes L \rightarrow 0.$$

Twist the sequence by $\mathcal{O}_X(B + qL)$, we get:

$$0 \rightarrow \wedge^p M_L \otimes \mathcal{O}_X(B + qL) \rightarrow \wedge^p V \otimes \mathcal{O}_X(B + qL) \rightarrow \wedge^{p-1} M_L \otimes \mathcal{O}_X(B + (q+1)L) \rightarrow 0.$$

Then we get, as part of its cohomology long exact sequence:

$$\begin{aligned} 0 \rightarrow H^0(\wedge^p M_L \otimes \mathcal{O}_X(B + qL)) &\rightarrow H^0(\wedge^p V \otimes \mathcal{O}_X(B + qL)) \rightarrow \\ H^0(\wedge^{p-1} M_L \otimes \mathcal{O}_X(B + (q+1)L)) &\rightarrow H^1(\wedge^p M_L \otimes \mathcal{O}_X(B + qL)). \end{aligned}$$

Similarly, by taking the short exact sequences for other wedge powers, twisting and taking long exact sequence, we get:

$$\begin{aligned}
0 \rightarrow H^0(\wedge^{p+1} M_L \otimes \mathcal{O}_X(B + (q-1)L)) &\rightarrow H^0(\wedge^{p+1} V \otimes \mathcal{O}_X(B + (q-1)L)) \\
&\rightarrow H^0(\wedge^p M_L \otimes \mathcal{O}_X(B + qL)) \rightarrow H^1(\wedge^{p+1} M_L \otimes \mathcal{O}_X(B + (q-1)L)) \\
&\rightarrow H^1(\wedge^{p+1} V \otimes \mathcal{O}_X(B + (q-1)L)) = 0
\end{aligned}$$

where the last term vanishes because of assumption II.2.2. and another wedge power gives:

$$\begin{aligned}
0 \rightarrow H^0(\wedge^{p-1} M_L \otimes \mathcal{O}_X(B + (q+1)L)) &\rightarrow H^0(\wedge^{p-1} V \otimes \mathcal{O}_X(B + (q+1)L)) \rightarrow \\
H^0(\wedge^{p-2} M_L \otimes \mathcal{O}_X(B + (q+2)L)) &\rightarrow H^1(\wedge^{p-1} M_L \otimes \mathcal{O}_X(B + (q+1)L)).
\end{aligned}$$

They form a diagram like the following:

(II.2.3)

$$\begin{array}{ccccc}
0 & & & & 0 \\
\downarrow & & & & \downarrow \\
H^0(\wedge^{p+1} M_L \otimes \mathcal{O}_X(B + (q-1)L)) & & & & \\
\downarrow & & & & \\
H^0(\wedge^{p+1} V \otimes \mathcal{O}_X(B + (q-1)L)) & & & & \\
\downarrow \rho & \searrow \alpha & & & \\
H^0(\wedge^p M_L \otimes \mathcal{O}_X(B + qL)) & \xrightarrow{\tau} & H^0(\wedge^p V \otimes \mathcal{O}_X(B + qL)) & \xrightarrow{\lambda} & H^0(\wedge^{p-1} M_L \otimes \mathcal{O}_X(B + (q+1)L)) \\
\downarrow \chi & & \searrow \beta & & \downarrow \gamma \\
H^1(\wedge^{p+1} M_L \otimes \mathcal{O}_X(B + (q-1)L)) & & & & H^0(\wedge^{p-1} V \otimes \mathcal{O}_X(B + (q+1)L)) \\
\downarrow & & & & \downarrow \\
0 & & & & \dots
\end{array}$$

By Lemma II.1,

$$K_{p,q}(X, B; L) = \text{Ker } \beta / \text{Img } \alpha.$$

γ and τ are both injective, and the diagram above is commutative, so we have:

$$\text{Ker } \lambda \cong \text{Ker } \beta, \quad \text{Img } \alpha \cong \text{Img } \rho.$$

Since λ and τ lie on a horizontal long exact sequence above,

$$\text{Ker } \lambda \cong \text{Img } \tau \cong H^0(\wedge^p M_L \otimes \mathcal{O}_X(B + qL)).$$

Then

$$\begin{aligned} K_{p,q}(X, B; L) &\cong H^0(\wedge^p M_L \otimes \mathcal{O}_X(B + qL)) / \text{Img } \rho \cong \\ &H^0(\wedge^p M_L \otimes \mathcal{O}_X(B + qL)) / \text{Ker } \chi \cong H^1(\wedge^{p+1} M_L \otimes \mathcal{O}_X(B + (q-1)L)). \end{aligned}$$

□

Two useful corollaries of the above lemma used in [EL12], [Z12] and this thesis are:

Proposition II.3. *Assume that:*

$$H^i(\mathcal{O}_X(B + mL)) = 0 \text{ for } i > 0 \text{ and } m > 0,$$

Then for $q \geq 2$:

$$K_{p,q}(X, B; L) = H^{q-1}(X, \wedge^{p+q-1} M_L \otimes \mathcal{O}_X(B + L)).$$

If moreover, $H^1(X, \mathcal{O}_X(B)) = 0$, then the same statement also holds true when $q = 1$.

Proof. By II.2, we only need to prove

$$H^1(X, \wedge^{p+1} M_L \otimes \mathcal{O}_X(B + (q-1)L)) \cong H^{q-1}(X, \wedge^{p+q-1} M_L \otimes \mathcal{O}_X(B + L)).$$

Consider again the short exact sequence:

$$\begin{aligned} 0 \rightarrow \wedge^{p+2} M_L \otimes \mathcal{O}_X(B + (q-2)L) &\rightarrow \wedge^{p+2} V \otimes \mathcal{O}_X(B + (q-2)L) \\ &\rightarrow \wedge^{p+1} M_L \otimes \mathcal{O}_X(B + (q-1)L) \rightarrow 0. \end{aligned}$$

Taking long exact sequence, we have:

$$H^1(\wedge^{p+2}V \otimes \mathcal{O}_X(B + (q-2)L)) \rightarrow H^1(\wedge^{p+1}M_L \otimes \mathcal{O}_X(B + (q-1)L)) \rightarrow$$

$$H^2(\wedge^{p+2}M_L \otimes \mathcal{O}_X(B + (q-2)L)) \rightarrow H^2(\wedge^{p+2}V \otimes \mathcal{O}_X(B + (q-2)L))$$

Then if $q \geq 3$ ($q = 2$ being the assumption), we have $q - 2 > 0$, and by assumption:

$$H^1(\wedge^{p+2}V \otimes \mathcal{O}_X(B + (q-2)L)) = H^2(\wedge^{p+2}V \otimes \mathcal{O}_X(B + (q-2)L)) = 0.$$

Hence:

$$H^1(\wedge^{p+1}M_L \otimes \mathcal{O}_X(B + (q-1)L)) = H^2(\wedge^{p+2}M_L \otimes \mathcal{O}_X(B + (q-2)L)).$$

Then we can continue the same way to prove our conclusion. □

By the exact same argument, we have:

Proposition II.4. *Assume that:*

$$H^i(\mathcal{O}_X(B + mL)) = 0 \text{ for } i > 0 \text{ and } m \geq 0,$$

Then:

$$K_{p,q}(X, B; L) = H^q(X, \wedge^{p+q}M_L \otimes \mathcal{O}_X(B)).$$

CHAPTER III

Schur asymptotics of Veronese syzygies

This chapter is based on joint work with Mihai Fulger. It closely follows the paper [FZ13]. Throughout the thesis, we will be using standard limit notation as defined in the footnote.¹

III.1 Introduction

Projective spaces are the most fundamental among algebraic varieties. Hence, to understand equivariant structures on syzygies, we should first look at syzygies of Veronese embeddings. We work throughout the rest of the thesis over the complex numbers.

Recall that $K_{p,q}(X, B; L)$ denotes the vector space generated by the minial gen-

1

(i) Let g, f be functions in d . $g \in o(f)$ means that

$$\lim_{d \rightarrow \infty} \frac{g(d)}{f(d)} = 0.$$

(ii) Let g, f be functions in d . $g \in O(f)$ means that there exists constant C such that for all d :

$$g(d) \leq C \cdot f(d).$$

(iii) Let g, f be functions in d . $g \in \Theta(f)$ means that there exists constant c, C such that for all d :

$$c \cdot f(d) \leq g(d) \leq C \cdot f(d).$$

erators in degree $(p + q)$ of the p -th module of syzygies of the section ring of $\mathcal{O}_X(L)$ twisted by B as a $\text{Sym}^\bullet H^0(\mathcal{O}_X(L))$ -module. In this section, we focus on the case of X being a projective space, so $X = \mathbb{P}(V)$ for some vector space V . In this case, we use the following notations for simplicity:

$$K_{p,q}(V, b; d) := K_{p,q}(\mathbb{P}(V), \mathcal{O}(b); \mathcal{O}(d)), \quad K_{p,q}(V; d) := K_{p,q}(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}; \mathcal{O}(d)).$$

Concretely, $K_{p,q}(V; d)$ are representations of the group GL_V . We naturally ask how they decompose into irreducible representations.

In this chapter, we follow the notation and definitions of [FH91] and [F97] for the basic objects of the representation theory of the general linear group. In particular, we write $\lambda \vdash n$ when $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$, with $\sum_{i \geq 1} \lambda_i = n$, is a *partition* of n . We identify partitions with Young diagrams which are left-aligned rows of boxes with λ_i boxes in the i -th row for the partition λ . Recall also that irreducible representations of GL_n are in one to one correspondence with partitions with at most n rows. For example, \wedge^3 , Sym^5 corresponding to the partitions $(1, 1, 1)$, (5) with Young diagrams

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \end{array}.$$

Each partition λ corresponds to a functor denoted \mathbb{S}_λ which takes a vector space V to a GL_V representation $\mathbb{S}_\lambda(V)$.

The precise way to set up the problem of decomposing $K_{p,q}$ is the following. The association

$$V \rightarrow K_{p,q}(V; d)$$

is a functor from vector spaces to representations that we denote by $\mathbb{K}_{p,q}(d)$. As in

[EL12], [Ru04], [S13], we can then write

$$(\dagger) \quad \mathbb{K}_{p,q}(d) = \bigoplus_{\lambda \vdash (p+q)d} M_{\lambda}(p, q; d) \otimes_{\mathbb{C}} \mathbb{S}_{\lambda},$$

where $M_{\lambda}(p, q; d)$ is a complex vector space whose dimension measures the multiplicity in $\mathbb{K}_{p,q}(d)$ of the Schur functor \mathbb{S}_{λ} corresponding to the partition λ of $(p+q)d$.

The decompositions in general seem to be quite hard. For instance, for $d = 4$, a calculation by Claudiu Raicu (assuming maximal cancellation of irreducible representations in constructing the resolution) suggests that $\mathbb{K}_{4,1}(4)$ decomposes as follows:

$$\begin{aligned} & \mathbb{S}_{(13,5,2)} + \mathbb{S}_{(13,4,3)} + \mathbb{S}_{(12,7,1)} + 2 * \mathbb{S}_{(12,6,2)} + 2 * \mathbb{S}_{(12,5,3)} + \mathbb{S}_{(12,4,4)} + \mathbb{S}_{(11,8,1)} + 3 * \mathbb{S}_{(11,7,2)} + \\ & 4 * \mathbb{S}_{(11,6,3)} + 3 * \mathbb{S}_{(11,5,4)} + \mathbb{S}_{(10,9,1)} + 3 * \mathbb{S}_{(10,8,2)} + 5 * \mathbb{S}_{(10,7,3)} + 5 * \mathbb{S}_{(10,6,4)} + 2 * \mathbb{S}_{(10,5,5)} + \\ & \mathbb{S}_{(9,9,2)} + 3 * \mathbb{S}_{(9,8,3)} + 5 * \mathbb{S}_{(9,7,4)} + 4 * \mathbb{S}_{(9,6,5)} + 2 * \mathbb{S}_{(8,8,4)} + 3 * \mathbb{S}_{(8,7,5)} + \mathbb{S}_{(8,6,6)} + \mathbb{S}_{(7,7,6)}. \end{aligned}$$

Conceivably, the decomposition has more irreducible factors.

The Schur decomposition of $K_{p,q}$ has been studied by many authors ([N11], [R13], [S13]) and it seems that it is closely related to the well known intractible problem of plethysm. While it doesn't seem realistic to expect an exact decomposition of $\mathbb{K}_{p,q}(d)$, the view point of [EL12] suggests that an overall asymptotic picture may be possible. This is what we give here.

Because of Green's result (corollary of [G84II] Thm. 2.2)) metioned in the introduction, we have:

$$\mathbb{K}_{p,q}(d) = 0, \text{ if } q \geq 2, \text{ and } d \geq p.$$

Therefore, asymptotically the whole of the p -th syzygy,

$$\mathbb{K}_p(d) = \bigoplus_{q=1}^{\infty} \mathbb{K}_{p,q}(d)$$

is captured by $\mathbb{K}_{p,1}(d)$. The main theorem in this chapter describes the asymptotics of $\mathbb{K}_{p,1}(d)$:

Theorem (III.14). *Fix $p \geq 1$. Then as d grows, $\mathbb{K}_{p,1}(d)$ contains*

(i) $\Theta(d^p)$ *distinct Schur functors.*

(ii) $\Theta(d^{\binom{p+1}{2}})$ *Schur functors counting multiplicities.*

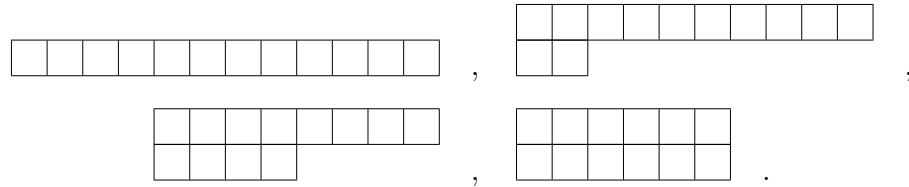
An explicit computation we can do to illustrate the statement of the theorem is the following. Let S, R be the homogeneous coordinate rings of the ambient project space and the Veronese image. The generators of the ideal of the Veronese image is represented by the degree 2 piece of the quotient map

$$R \leftarrow S.$$

By the definition of R and S , we want to know the kernel of the equivariant map:

$$\mathrm{Sym}^{2d} \leftarrow \mathrm{Sym}^2 \mathrm{Sym}^d.$$

$\mathrm{Sym}^2 \mathrm{Sym}^d$ is a known plethysm which decomposes to representations with Young diagrams of $2d$ boxes, with at most 2 rows where each row has an even number of boxes. For instance, if $d = 6$, we have the following Young diagrams in $\mathrm{Sym}^2 \mathrm{Sym}^4$:



The kernel consists of all these except the one rowed Young diagram representing Sym^{2d} . Since each Young diagram appears with multiplicity one, we have on the order of d many distinct irreps and also on the order of d many counting multiplicities.

After a brief section on some representation theory notations we adopt, sections 3 and 4 give the proof of Theorem III.14 as follows. We first describe the asymptotics of the Schur decomposition of $\bigotimes^{p+1} \text{Sym}^d$ with a convex-geometric approach. Next, we derive Theorem III.7, which allows us to compare the terms in the Koszul complex computing $K_{p,1}(d)$ to $\bigotimes^{p+1} \text{Sym}^d$. Then we arrive at the conclusion because we know two terms out of the three terms in the Koszul complex and the convex geometric description allows us to conclude that the third term is asymptotically insignificant. In sections 5 and 6, we prove results involving other ways to vary the parameters.

III.2 Some representation theory notations

- (i) We adopt the notation and definitions of [FH91] and [F97] for the basic objects of the Representation Theory of the general linear group. In particular, we write $\lambda \vdash n$ when $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$, with $\sum_{i \geq 1} \lambda_i = n$, is a *partition* of n . The *length* of λ , denoted $|\lambda|$, is the number of its nonzero parts. We write $(2^2, 1)$ for the partition $(2, 2, 1)$, we write $\lambda + \mu$ for the partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$, and 2λ for the partition $(2\lambda_1, 2\lambda_2, \dots)$, etc. Young diagrams are left-aligned rows of boxes with λ_i boxes in the i -th row for the partition λ . A Young tableau is obtained by filling in the boxes of the Young diagram with natural numbers. A tableau is called standard if the entries in each row and each column are increasing. A tableau is called semistandard if the entries are column strict but weakly increase in each row. Recording the number of times each number appears in a tableau gives a sequence known as the weight of the tableau. For

example $(d^p) = \underbrace{(d, \dots, d)}_p$ means we have d of $1, 2, \dots, p$ in the tableau.

- (ii) We write S^d for Sym^d . The symmetric group on p elements is denoted by Σ_p .
- (iii) In this paper we work with functors $\mathcal{F} : \text{Vect}_{\mathbb{C}} \rightarrow \text{Vect}_{\mathbb{C}}$ of finite dimensional complex vector spaces that have unique finite direct sum *decompositions*²

$$\mathcal{F} = \bigoplus_{\lambda} M_{\mathcal{F}, \lambda} \otimes_{\mathbb{C}} \mathbb{S}_{\lambda},$$

where $M_{\mathcal{F}, \lambda}$ are complex vector spaces, and \mathbb{S}_{λ} is the Schur functor corresponding to the partition λ . The dimension

$$(\mathcal{F}, \lambda) =_{\text{def}} \dim M_{\mathcal{F}, \lambda}$$

is the *multiplicity* of \mathbb{S}_{λ} in \mathcal{F} . The *total multiplicity* of \mathcal{F} is

$$N(\mathcal{F}) =_{\text{def}} \sum_{\lambda} (\mathcal{F}, \lambda).$$

The *complexity* of \mathcal{F} is the number of distinct types of Schur functors appearing in its decomposition, i.e.,

$$c(\mathcal{F}) =_{\text{def}} \#\{\lambda : (\mathcal{F}, \lambda) \neq 0\}.$$

If V is a complex vector space of finite dimension, then $\mathcal{F}(V)$ is naturally a $GL(V)$ -representation. We will often use this to reduce questions about functors to questions about representations of the general linear group.

- (iv) It is an elementary consequence of Pieri's rule that $(\bigotimes^p S^d, \lambda)$ is equal to the number of semistandard Young tableaux of shape λ and weight $\mu = (d^p)$.³ In

²Plethysm functors are polynomial, and syzygy functors have decompositions by [Ru04, §2]

³Such tableaux can be described by Gelfand–Tsetlin patterns (see [Sta99, p133] for a definition). The number of such Young tableaux is called the Kostka number $K_{\lambda\mu}$

particular, the Schur functor \mathbb{S}_λ appears in $\bigotimes^p S^d$ if and only if λ is a partition of pd with at most p parts.

- (v) Let V be a GL_n -representation. Denoting by U_n the unipotent group, by [F97, p144], the total multiplicity of V is equal to $\dim V^{U_n}$, the space of U_n -invariant vectors in V . This space of U_n -invariant vectors are the highest weight vectors in the representation, whose orbit span V . It is classical that $\mathbb{S}_\lambda(\mathbb{C}^p) \neq 0$ if and only if $|\lambda| \leq p$. If $\mathcal{F} : \text{Vect}_{\mathbb{C}} \rightarrow \text{Vect}_{\mathbb{C}}$ is a decomposable functor (as in Fact (iii)), such that any Schur subfunctor corresponds to a partition with at most p rows, then the Schur decomposition of \mathcal{F} can be read from the decomposition into irreducible GL_p -subrepresentations of $\mathcal{F}(\mathbb{C}^p)$. In particular, we can read the total multiplicity and the complexity of \mathcal{F} by applying the functor to \mathbb{C}^p , i.e., $N(\mathcal{F}) = \dim \mathcal{F}(\mathbb{C}^p)^{U_p}$.

III.3 Asymptotic plethysms

In this section we investigate the growth with d of the total multiplicity, and of the complexity of $\bigotimes^p \text{Sym}^d$, $\text{Sym}^p \text{Sym}^d$, and $\bigwedge^p \text{Sym}^d$ when p is fixed. As in several of the references listed in the remark below, the idea behind our study is convex-geometric. The contents of this section may be known to the experts, but we were unable to find precise references. We give a presentation here for the convenience of the reader.

Remark III.1. From the literature on the asymptotics of the decomposition of $\bigotimes^p S^d$ and other plethysms, we mention the following:

- The work of Kaveh and Khovanskii in [KK12] applies to the behavior of $\bigotimes^p S^d$ with fixed p and varying d . Their focus is on subtle properties such as Fujita-type approximation, the Brunn–Minkowski inequality, the Brion–Kazarnowskii formula, etc.
- In [K10], Kaveh computes the dimension of the moment body, varying p , for $\bigotimes^p V$, i.e., the growth with p of the number of its distinct subrepresentations, where V is a fixed GL_n -representation.
- Tate and Zelditch ([TZ04]) studied the asymptotics of Kostka numbers of $\bigotimes^p S_\lambda$ where p varies and S_λ is a fixed representation.
- Different asymptotic plethysms have been studied in [W90]. In [Man97], we again find the idea of the convex-geometric approach.

III.3.1 Integral points and $\bigotimes^p S^d$

In this subsection, we show that the total multiplicity and the complexity of $\bigotimes^p S^d$ are counted by the number of lattice points inside slices over d of two rational convex cones. The growth with d of the number of such integral points is a polynomial of degree equal to the dimension of the cross section of the corresponding cone. We determine these two cones and compute the corresponding dimensions.⁴ These are captured by the following theorem:

Theorem III.2. *Fix $p \geq 1$. Then*

⁴In the language of [KK12], we are determining the dimension of the moment body and of the multiplicity body (which in our case is also the classical Gelfand–Tsetlin polytope) for the p -th product of a sufficiently large dimension projective space.

(i) $\lim_{d \rightarrow \infty} c(\bigotimes^p S^d)/d^{p-1}$ is a finite positive number.

(ii) $\lim_{d \rightarrow \infty} N(\bigotimes^p S^d)/d^{\binom{p}{2}}$ is a finite positive number.

Proof. We specify two graded sets Y_\bullet and \mathcal{Y}_\bullet such that the cardinalities of their d -th graded pieces count the complexity and the total multiplicity of $\bigotimes^p S^\bullet$, respectively.

Complexity. The set

$$Y_d =_{\text{def}} \left\{ (\lambda_2, \dots, \lambda_p, d) : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p \geq 0, \text{ with } \lambda_1 := pd - \sum_{k=2}^p \lambda_k \right\} \\ \subseteq \mathbb{N}^{p-1} \times \{d\}$$

is a parametrization of the partitions $\lambda \vdash pd$ with at most p parts, which is the set of distinct types of Schur functors appearing in the decomposition of $\bigotimes^p S^d$ by Fact (iv) from Sect.III.2. Therefore the complexity of $\bigotimes^p S^d$ is the number of integral points with the last coordinate d inside the cone contained in $\mathbb{R}_{\geq 0}^{p-1} \times \mathbb{R}_{\geq 0}$ over the cross section:

$$Y =_{\text{def}} \left\{ (\lambda_2, \dots, \lambda_p, 1) : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p \geq 0, \text{ with } \lambda_1 := p - \sum_{k=2}^p \lambda_k \right\} \\ \subset \mathbb{R}_{\geq 0}^{p-1} \times \{1\}.$$

This is $(p-1)$ -dimensional, which proves part i).

Total multiplicity. Guided by Fact (iv), we seek to parameterize the set of semistandard Young tableaux T with pd boxes, at most p rows, and weight (d^p) . A first parameterization is the $p \times p$ matrix

$$t_{ij} =_{\text{def}} \text{number of } j\text{'s in the } i\text{-th row of } T.$$

Strictness in columns for a semistandard tableau implies $t_{ij} = 0$ when $j < i$. The weight of T is (d^p) imposes

$$t_{jj} = d - \sum_{k=1}^{j-1} t_{kj} \text{ for all } j \in \{1, \dots, p\}.$$

Hence T is determined by only the parameters $(t_{ij})_{i < j}$ and d . Using only these parameters, the set of all such T is in bijection with the set \mathcal{Y}_d of points in $\mathbb{N}^{\binom{p}{2}} \times \{d\} = \{((t_{ij})_{i < j}, d)\}$ subject to the following conditions:

- Denoting $t_{ii} := d - \sum_{j=1}^{i-1} t_{ji}$, we ask that

$$(III.3.1) \quad t_{ii} \geq 0,$$

which we already discussed above. And the inequality holds since this represents the number of j 's in the j -th row in a tableau.

- We also ask that

$$(III.3.2) \quad \sum_{k=i}^{j-1} t_{ik} \geq \sum_{k=i+1}^j t_{i+1,k}, \text{ for all } 1 \leq i < p, 1 \leq j \leq p.$$

This inequality enforces the following condition. The LHS represents, for any label j , the total number of boxes with labels at most $j - 1$ in the i -th row. The RHS represents for the chosen j , the total number of boxes with labels at most j on the $i + 1$ -th row. It is straightforward to deduce from column strictness that this condition must be satisfied. Furthermore, since the rows are weakly increasing, the inequality ensures that we can fill out the $(i + 1)$ -th row with t_{i+1j} number of j 's given the i -th row.

Similar to the complexity problem, \mathcal{Y}_d is the set of integral points inside the cone with the last coordinate d in $\mathbb{R}_{\geq 0}^{\binom{p}{2}} \times \mathbb{R}_{\geq 0}$ over the cross section:

$$\mathcal{Y} := \left\{ ((t_{ij})_{i < j}, 1) : \begin{array}{l} 0 \leq t_{ii} := 1 - \sum_{k=1}^{i-1} t_{ki}, \text{ for all } 1 \leq i \leq p \\ \sum_{k=i}^{j-1} t_{ik} \geq \sum_{k=i+1}^j t_{i+1,k}, \text{ for all } 1 \leq i < p, 1 \leq j \leq p \end{array} \right\} \subset \mathbb{R}_{\geq 0}^{\binom{p}{2}} \times \{1\}.$$

This convex body is $\binom{p}{2}$ -dimensional,⁵ and part *ii*) follows. \square

Remark III.3. The limits in Theorem III.2 are at least algorithmically computable for each p , since they are the volumes of the convex bodies Y and \mathcal{Y} respectively.

Example III.4 (The plethysm of $\bigotimes^2 S^d$). The decomposition of $\bigotimes^2 S^d$ consists of all Schur functors of type $(2d - a, a)$ with $0 \leq a \leq d$, each with multiplicity 1. The total multiplicity and the complexity are both d in this case. \square

Remark III.5. Our constructions are similar to those of the Gelfand–Tsetlin patterns (see also [BZ89]). We use this particular form in order to compute the desired growth rates. \square

The following remark is a consequence of the proof of Theorem III.2. It allows us to deem certain collections of Schur functors asymptotically insignificant and will be used repeatedly:

Remark III.6. The coordinates corresponding to Young tableaux of weight (d^p) with at most $p - 1$ rows, and to Young diagrams with pd boxes and at most $p - 1$ rows lie in proper linear subspaces of the real vector spaces spanned by \mathcal{Y}_\bullet and Y_\bullet .

⁵To see this, it is enough to produce a point of \mathcal{Y} that satisfies strictly all the defining inequalities. Setting $t_{ij} = p^i \epsilon$ for all $i < j$, and for sufficiently small ϵ , defines such a point.

respectively. It follows that asymptotically all the Young tableaux and diagrams parameterized by \mathcal{Y}_\bullet and Y_\bullet respectively have p rows. \square

III.3.2 $\mathbb{S}_\mu S^d$ via $\bigotimes^p S^d$

We investigate the asymptotics of the Schur decomposition of $\mathbb{S}_\mu S^d$ when μ is a fixed partition of p . Denote by V_μ an irreducible complex Σ_p -representation of weight μ .

Theorem III.7. *Fix $p \geq 1$, and let μ be a partition of p . Then*

$$(i) \lim_{d \rightarrow \infty} N(\mathbb{S}_\mu S^d)/d^{\binom{p}{2}} = \frac{(\dim V_\mu)^2}{p!} \cdot \lim_{d \rightarrow \infty} N(\bigotimes^p S^d)/d^{\binom{p}{2}}.$$

$$(ii) \text{ As } d \text{ grows, } c(\mathbb{S}_\mu S^d) = \Theta(d^{p-1}).$$

Proof. We have inclusions of graded (by d) vector spaces A, B, C (whose graded pieces are denoted A_d, B_d, C_d respectively):

$$A := \bigoplus_{d \geq 0} (\mathbb{S}_\mu S^d \mathbb{C}^p)^{U_p} \hookrightarrow B := \bigoplus_{d \geq 0} (\bigotimes^p S^d \mathbb{C}^p)^{U_p} \hookrightarrow C := \bigoplus_{d \geq 0} \bigotimes^p S^d \mathbb{C}^p.$$

Observe that C is the section ring on the p -fold product of $\mathbb{P}(\mathbb{C}^p)$ of $\mathcal{O}_{\mathbb{P}(\mathbb{C}^p) \times p}(1, \dots, 1)$, and B is a subalgebra of C . The action of GL_p on \mathbb{C}^p induces a GL_p -representation structure on C_d for all d . The symmetric group Σ_p also acts on $C_d = \bigotimes^p S^d \mathbb{C}^p$ by permuting the factors of the tensor product. The two actions commute, hence the Σ_p -action restricts to B . We have the following:

(i) By [Gr97, Thm. 16.2], B is finitely generated.

(ii) No nontrivial σ in Σ_p acts as a scalar (depending only on σ and d) on B_d for each d . Otherwise, by the commutativity of the actions of Σ_p and GL_p , it also

acts as a scalar on the GL_p -span of B_d in C_d . As in Fact (v), the GL_p -span of B_d is C_d and we know a nontrivial σ does not act trivially on all of C_d .

By [P05] Thm. 1:

$$\dim A_d \sim \frac{(\dim V_\mu)^2}{|\Sigma_p|} \cdot \dim B_d.$$

By Theorem III.2 and Fact (v), we have part i). Part ii) is a consequence of i) and Lemma III.8. \square

Lemma III.8. *Fix $p \geq 1$, and let \mathcal{F}_d be a sequence of subfunctors of $\bigotimes^p S^d$ so that, as d grows, we have $N(\mathcal{F}_d) \in \Theta(d^{\binom{p}{2}})$. Then $c(\mathcal{F}_d) \in \Theta(d^{p-1})$.*

Proof. Consider the moment map

$$\begin{aligned} \mu : \mathbb{R}^{\binom{p}{2}} \times \mathbb{R} &\rightarrow \mathbb{R}^{p-1} \times \mathbb{R}, \quad \mu((t_{ij})_{i < j}, d) := (\lambda_2, \dots, \lambda_p, d), \text{ where} \\ \lambda_i &:= \sum_{j=i}^p t_{ij}, \text{ for all } 1 \leq i \leq p, \text{ and } t_{ii} := d - \sum_{k=1}^{i-1} t_{ki}, \text{ for all } 1 \leq i \leq p. \end{aligned}$$

By the constructions of the previous section, μ maps \mathcal{Y} onto Y , and \mathcal{Y}_d onto Y_d for all d . By assumption, there exists $C_1 > 0$ such that for large d ,

$$N(\mathcal{F}_d) \geq C_1 \cdot d^{\binom{p}{2}}.$$

On the other hand, denoting by $\overline{m}_{p,d}$ the maximal multiplicity of a Schur functor in $\bigotimes^p S^d$ which contains \mathcal{F}_d , we have

$$N(\mathcal{F}_d) \leq c(\mathcal{F}_d) \cdot \overline{m}_{p,d}.$$

By Theorem III.2 i) we obtain $c(\mathcal{F}_d) \leq c(\bigotimes^p S^d) \leq C_2 \cdot (d^{p-1})$, hence it is enough to show that

$$\overline{m}_{p,d} \leq C' \cdot d^{\binom{p-1}{2}}$$

for some $C' > 0$ independent of d . To see this, choose a basis for $\mathbb{Z}^{\binom{p}{2}} \subset \mathbb{R}^{\binom{p}{2}}$ so that μ is the projection onto the last p coordinates. Then choose an integer $l > 0$ such that

$$\mathcal{Y} \subset [-l, l]^{\binom{p-1}{2}} \times Y \subset \mathbb{R}^{\binom{p}{2}} \times \{1\}.$$

Then $\mathcal{Y}_d \subset [-dl, dl]^{\binom{p-1}{2}} \times Y_d$, so $\overline{m}_{p,d} \leq \#([-dl, dl] \cap \mathbb{Z})^{\binom{p-1}{2}}$. We can set $C' = (3l)^{\binom{p-1}{2}}$. \square

Corollary III.9. *Fix $p \geq 1$. Then*

$$\frac{1}{p!} N(\bigotimes^p S^d) \sim N(S^p S^d) \sim N(\bigwedge^p S^d).$$

Proof. This follows by applying Theorem III.7 for the trivial and alternating representations of Σ_p . \square

Remark III.10. [BCI11, Lem. 2.2] can be adjusted to show that if $\mathbb{S}_\lambda(\mathbb{C}^p)$ and $\mathbb{S}_{\lambda'}(\mathbb{C}^p)$ appear in the decompositions of $S^p S^d \mathbb{C}^p$ and $S^p S^{d'} \mathbb{C}^p$ respectively, then $\mathbb{S}_{\lambda+\lambda'}(\mathbb{C}^p)$ appears in the decomposition of $S^p S^{d+d'} \mathbb{C}^p$.⁶ In particular, the set of λ with $(S^p S^d, \lambda) > 0$, for some d , is a subsemigroup of Y_\bullet . Since $S^p S^d \neq 0$ for all $d \geq 0$, the semigroup is nonempty in all degrees. The existence and finiteness of

$$\lim_{d \rightarrow \infty} c(S^p S^d) / d^{p-1}$$

are easy applications of the semigroup techniques developed in [Kh93] and [LM09, §2.1]. We also have:

$$c(\bigwedge^p S^d) \sim c(S^p S^d).$$

⁶The product of two highest weight symmetric vectors is a highest weight symmetric vector

We leave it to the reader to deduce this from Remark III.6 and from the following result of Newell's (cf. [W90] (2.4a), (2.4b) with modern notation, [New51] for the original paper):

Lemma III.11 (Newell). *For any partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_p \geq 0)$, we have*

$$(i) \left(\bigwedge^p S^{d+1}, (\lambda_1 + 1, \dots, \lambda_p + 1) \right) = (S^p S^d, \lambda).$$

$$(ii) (S^p S^{d+1}, (\lambda_1 + 1, \dots, \lambda_p + 1)) = \left(\bigwedge^p S^d, \lambda \right).$$

Remark III.12 (Explicit constructions). Answering a question from [W90, Conj. 2.11], in [BCI11] it is shown that if $\lambda \vdash pd$ has $|\lambda| \leq p$, then

$$\mathbb{S}_{2\lambda} \text{ appears in the decomposition of } S^p S^{2d}, \text{ i.e., } (S^p S^{2d}, 2\lambda) > 0.$$

In [MM12] we find a constructive proof for this result. Because of the semigroup structure on types of partitions contributing to $S^p S^\bullet$ and using Newell's result, we also have:

$$\left(\bigwedge^p S^{2d+1}, 2\lambda + (1^p) \right) = (S^p S^{2d}, 2\lambda) > 0, \quad \text{and}$$

$$\left(\bigwedge^p S^{2d+2}, 2\lambda + (1^p) + (p) \right) = (S^p S^{2d+1}, 2\lambda + (p)) > 0.$$

Combining with Theorem III.2.ii), these provide a constructive proof for Theorem III.7.ii) in the particular cases when $\mu = (p)$ or $\mu = (1^p)$, but do not show that the corresponding limits exist, as it is the case in Remark III.10. The explicit constructions also show that $\lim_{d \rightarrow \infty} \frac{c(S^p S^d)}{d^{p-1}} \geq \frac{1}{2^{p-1}} \cdot \lim_{d \rightarrow \infty} \frac{c(\bigotimes^p S^d)}{d^{p-1}}$. Note that $\frac{1}{2^{p-1}} > \frac{1}{p!}$, when $p \geq 3$. In particular, it is not true in general that $c(S^p S^d) \sim \frac{1}{p!} c(\bigotimes^p S^d)$.

In the next section we will also use the following result:

Proposition III.13. *Fix $p \geq 1$. Then, as d grows, $N(\bigwedge^p S^d \otimes S^d) \sim \frac{1}{p!} \cdot N(\bigotimes^{p+1} S^d)$.*

Proof. This is the same argument as for Corollary III.9. Apply Theorem III.7.i) to the alternating representation of Σ_p with the action on the first p tensor factors of $\bigotimes^{p+1} S^d$. \square

III.4 Asymptotic syzygy functors

In this section we determine the precise asymptotic orders of growth for the total multiplicity and complexity of $\mathbb{K}_{p,1}(0; d)$ and $\mathbb{K}_{p,0}(b; d)$ in the nontrivial case $b \geq 1$.

Given $p, q, b, d \geq 0$, we defined the syzygy functor $\mathbb{K}_{p,q}(b; d)$ of the d -th Veronese embedding as the cohomology of the functorial Koszul-type complex

$$\bigwedge^{p+1} S^d \otimes S^{(q-1)d+b} \rightarrow \bigwedge^p S^d \otimes S^{qd+b} \rightarrow \bigwedge^{p-1} S^d \otimes S^{(q+1)d+b}.$$

In the introduction we explained that if we fix $p, q, b \geq 0$ and let d grow to infinity, we only have nontrivial behavior in $\mathbb{K}_{p,q}(b; d)$ when $p \geq 1$, and either $q = 1$, or $q = 0$ and $b \geq 1$. In this section we determine the precise asymptotic orders of growth for the total multiplicity and complexity of $\mathbb{K}_{p,1}(0; d)$ and $\mathbb{K}_{p,0}(b; d)$ in the nontrivial case $b \geq 1$. In the next section we will give a partial result for $\mathbb{K}_{p,1}(b; d)$ when $b \geq 1$.

III.4.1 $\mathbb{K}_{p,1}(d)$

As we did in the introduction, we write $\mathbb{K}_{p,1}(d)$ for $\mathbb{K}_{p,1}(0; d)$, i.e., the cohomology of the Koszul-type complex

$$(III.4.1) \quad \bigwedge^{p+1} S^d \rightarrow \bigwedge^p S^d \otimes S^d \rightarrow \bigwedge^{p-1} S^d \otimes S^{2d}.$$

The asymptotics of the decomposition of $\mathbb{K}_{p,1}(d)$ are described by:

Theorem III.14. *Fix $p \geq 1$. As d goes to infinity:*

$$(i) \ N(\mathbb{K}_{p,1}(d)) \sim \frac{p}{(p+1)!} \cdot N(\bigotimes^{p+1} S^d).$$

$$(ii) \ c(\mathbb{K}_{p,1}(d)) \in \Theta(d^p).$$

Proof. Since the first map in (III.4.1) is an inclusion, and by Remark III.6 the total multiplicity of the last term is asymptotically insignificant compared to that of $\bigotimes^{p+1} S^d$, we obtain that

$$N(\mathbb{K}_{p,1}(d)) \sim N(\bigwedge^p S^d \otimes S^d) - N(\bigwedge^{p+1} S^d)$$

once we have proved that the right hand side is in $\Theta(N(\bigotimes^{p+1} S^d))$. Using Corollary III.9 and Proposition III.13, we obtain:

$$N(\bigwedge^p S^d \otimes S^d) - N(\bigwedge^{p+1} S^d) = \frac{1}{p!} \cdot N(\bigotimes^{p+1} S^d) - \frac{1}{(p+1)!} \cdot N(\bigotimes^{p+1} S^d),$$

which is in $\Theta(N(\bigotimes^{p+1} S^d))$ and part *i*) follows. $\mathbb{K}_{p,1}(d)$ is a subquotient of $\bigwedge^p S^d \otimes S^d$ which is (noncanonically) a subfunctor of $\bigotimes^{p+1} S^d$. Hence, $\mathbb{K}_{p,1}(d)$ is a subfunctor of $\bigotimes^{p+1} S^d$. Part *ii*) is then a consequence of Lemma III.8. \square

Remark III.15. Unlike with $\bigotimes^{p+1} S^d$ and the plethysms of Remark III.12, we do not know how to construct explicit examples of λ with $(\mathbb{K}_{p,1}(d), \lambda) > 0$, nor do we know if the sequence $c(\mathbb{K}_{p,1}(d))/d^p$ has a limit.

III.4.2 $\mathbb{K}_{p,0}(b; d)$

We describe the asymptotic behavior of

$$(III.4.2) \quad \mathbb{K}_{p,0}(b; d) = \ker(\bigwedge^p S^d \otimes S^b \rightarrow \bigwedge^{p-1} S^d \otimes S^{b+d}),$$

in the the nontrivial cases when only d grows, in the following theorem:

Theorem III.16. *Fix $p \geq 1$ and $b \geq 1$. Then as d grows,*

$$(i) \ N(\mathbb{K}_{p,0}(b; d)) \in \Theta\left(d^{\binom{p}{2}}\right).$$

$$(ii) \ c(\mathbb{K}_{p,0}(b; d)) \in \Theta(d^{p-1}).$$

Proof. Note that Pieri's rule implies

$$\left(\bigwedge^{p-1} S^d \otimes S^{b+d}, \lambda\right) = 0, \text{ for any } \lambda \text{ with } |\lambda| = p+1.$$

As in Remark III.6, because b is finite, asymptotically all of the Schur functors appearing in $\bigwedge^p S^d$ correspond to λ with $\lambda_p \geq b$. For each occurrence of such \mathbb{S}_λ , by Pieri's rule, we have an occurrence of $\mathbb{S}_{(\lambda_1, \dots, \lambda_p, b)}$ in $\bigwedge^p S^d \otimes S^b$. Since $b > 0$, this corresponds to a partition with $p+1$ parts, hence it also appears in the decomposition of $\mathbb{K}_{p,0}(b; d)$ by Schur's lemma and (III.4.2). Therefore the total multiplicity and complexity of $\mathbb{K}_{p,0}(b; d)$ are bounded below by those of $\bigwedge^p S^d$, which are described in Corollary III.9 and Remark III.10 respectively.

The total multiplicity and complexity of $\mathbb{K}_{p,0}(b; d)$ are bounded above by those of $\bigotimes^p S^d \otimes S^b$. By Pieri's rule, there is at most a finite number $C(p, b)$ of ways of obtaining a Young tableau of weight (d^p, b) by adding b boxes labeled $p+1$ to a fixed one of weight (d^p) , or of ways of obtaining a partition $\lambda \vdash pd + b$ by adding b boxes to the corresponding Young diagram of a fixed partition μ with $|\mu| \leq p$. Therefore

$$(III.4.3) \quad N(\bigotimes^p S^d \otimes S^b) \in \Theta(N(\bigotimes^p S^d)) \quad \text{and} \quad c(\bigotimes^p S^d \otimes S^b) \in \Theta(c(\bigotimes^p S^d)).$$

□

Remark III.17. We have used the assumption $b \geq 1$ to say that adding b boxes on the $(p+1)$ -st row of a Young diagram with p non-empty rows produces a digram with $p+1$ rows. The assumption is also necessary for the theorem because, as in the introduction, $\mathbb{K}_{p,0}(0; d) = 0$.

Remark III.18. Except for the case of the total multiplicity when $b = 1$, which we will explain in the next section, we do not know if the sequences $N(\mathbb{K}_{p,0}(b; d))/d^{\binom{p}{2}}$ and $c(\mathbb{K}_{p,0}(b; d))/d^{p-1}$ have limits when $b \geq 1$.

Lemma III.19. *One can improve (III.4.3) to*

$$N(\bigotimes^p S^d \otimes S^b) \sim \binom{b+p}{p} \cdot N(\bigotimes^p S^d) \quad \text{and} \quad c(\bigotimes^p S^d \otimes S^b) \sim (b+1) \cdot c(\bigotimes^p S^d).$$

Proof. By Sect.III.2 Fact (iv), the total multiplicity of $\bigotimes^p S^d \otimes S^b$ counts Young tableaux of weight (d^p, b) . These are partitioned according to their truncation to tableaux of weight (d^p) , by forgetting the b boxes labeled $p+1$. Conversely, from any Young tableau T of weight (d^p) we obtain potential tableaux of weight (d^p, b) by arbitrarily placing a total number of b boxes labeled $p+1$ at the end of the first $p+1$ rows of T . The potential tableaux may fail to be actual tableaux only when the shape λ of T satisfies $\lambda_i < \lambda_{i+1} + b$ for some $i \leq p$. As in Remark III.6, this phenomenon is asymptotically insignificant. Consequently,

$$N(\bigotimes^p S^d \otimes S^b) \sim \binom{b+p}{p} \cdot N(\bigotimes^p S^d).$$

For the complexity problem, note that from a partition $\lambda \vdash pd$ with $|\lambda| \leq p$, one obtains $b+1$ potential partitions $\lambda[j] := (\lambda_1 + (b-j), \lambda_2, \dots, \lambda_p, j)$ with $j \in \{0, \dots, b\}$. By Pieri's rule, the set of all such $\lambda[j]$ that are true partitions (i.e., with $\lambda_p \geq j$)

is the set of partitions μ with $(\bigotimes^p S^d \otimes S^b, \mu) > 0$. Reasoning as in Remark III.6, asymptotically all $\lambda[j]$ are true partitions. \square

III.5 $\mathbb{K}_{p,1}(b; d)$ when $b > 0$

In this section we investigate the asymptotics in d of the decomposition of $\mathbb{K}_{p,1}(b; d)$, when $b \geq 1$. We explain why the strategy from the previous section fails and, using a restriction argument, we determine the asymptotic orders of the logarithms of the complexity and total multiplicity of $\mathbb{K}_{p,1}(b; d)$ as functions of $\log d$, when p and $b > 0$ are fixed, satisfying $p \geq b + 1$.

The functor $\mathbb{K}_{p,1}(b; d)$ is the cohomology of the functorial Koszul-type complex:

$$(III.5.1) \quad \bigwedge^{p+1} S^d \otimes S^b \rightarrow \bigwedge^p S^d \otimes S^{b+d} \rightarrow \bigwedge^{p-1} S^d \otimes S^{b+2d}.$$

Similar to Lemma III.19, one can show

$$N(\bigwedge^{p+1} S^d \otimes S^b) \sim \frac{\binom{b+p+1}{p+1}}{(p+1)!} \cdot N(\bigotimes^{p+1} S^d),$$

asymptotically $\frac{\binom{b+p}{p}}{(p+1)!} \cdot N(\bigotimes^{p+1} S^d)$ of which corresponds to partitions of length $p+1$,

$$N(\bigwedge^p S^d \otimes S^{b+d}) \sim \frac{1}{p!} \cdot N(\bigotimes^{p+1} S^d),$$

and because the last term contains no Schur functor corresponding to partitions of length $p+1$,

$$N(\bigwedge^{p-1} S^d \otimes S^{b+2d}) \in o(N(\bigotimes^{p+1} S^d)).$$

When $b > 0$, the sum of multiplicities corresponding to partitions of length $p+1$ of the leftmost term in equation III.5.1 is asymptotically at least as big as that of the

central term. Hence we cannot apply the strategy of the previous section. We do not know if in fact $N(\mathbb{K}_{p,1}(b; d)) \in o(N(\bigotimes^{p+1} S^d))$ for all $b \geq 1$, but with the help of a result of Raicu, we confirm this when $b = 1$:

Proposition III.20. *If $p \geq 1$, then as d grows, $N(\mathbb{K}_{p,1}(1; d)) \in o(N(\bigotimes^{p+1} S^d))$.*

Proof. Consider the sequence

$$\bigwedge^{p+1} S^d \otimes S^1 \rightarrow \bigwedge^p S^d \otimes S^{d+1} \rightarrow \bigwedge^{p-1} S^d \otimes S^{2d+1}.$$

The first two cohomologies are by definition $\mathbb{K}_{p+1,0}(1; d)$ and $\mathbb{K}_{p,1}(1; d)$. The total multiplicity of the third term is asymptotically insignificant compared to the first two. Hence

$$N(\mathbb{K}_{p+1,0}(1; d)) - N(\mathbb{K}_{p,1}(1; d)) \sim N(\bigwedge^{p+1} S^d \otimes S^1) - N(\bigwedge^p S^d \otimes S^{d+1}),$$

if we show that the latter difference is in $O(N(\bigotimes^{p+1} S^d))$. By the computations above,

$$N(\bigwedge^{p+1} S^d \otimes S^1) - N(\bigwedge^p S^d \otimes S^{d+1}) \sim \frac{1}{(p+1)!} \cdot N(\bigotimes^{p+1} S^d).$$

By [R13, Thm. 6.4], the decomposition of $\mathbb{K}_{p+1,0}(1; d)$ is obtained from the decomposition of $S^{p+1}S^{d-1}$ by replacing Schur subfunctors \mathbb{S}_λ of the latter with $\mathbb{S}_{\lambda+(1^{p+2})}$.⁷

In particular,

$$N(\mathbb{K}_{p+1,0}(1; d)) = N(S^{p+1}S^{d-1}) \sim \frac{1}{(p+1)!} \cdot N(\bigotimes^{p+1} S^d).$$

But this is the same approximation as for $N(\mathbb{K}_{p+1,0}(1; d)) - N(\mathbb{K}_{p,1}(1; d))$. \square

⁷With the contents of the thesis, we have another way to prove this result. By Pieri's rule and Lemma III.11, the referenced theorem states that $\mathbb{K}_{p+1,0}(1; d)$ contains exactly the representations with $p+2$ parts in $\bigwedge^{p+1} S^d \otimes S^1$. This is the same as stating that $(\mathbb{K}_{p+1,0}(1; d), \lambda) = 0$ if $|\lambda| < p+2$. For this, using Sect.III.2 Fact (v), it is enough to check that $K_{p+1,0}(\mathbb{C}^{p+1}, 1; d) = 0$, which follows from [EL12, Prop. 5.1].

We obtain asymptotic information about the Schur decomposition of $\mathbb{K}_{p,1}(b; d)$ via a restriction argument suggested by R. Lazarsfeld, inspired by ideas from [EL12]. The Schur subfunctors of $\mathbb{K}_{p,q}(b; d)$ corresponding to partitions of length at most n are captured by the GL_n -decomposition of $K_{p,q}(\mathbb{C}^n, b; d) := \mathbb{K}_{p,q}(b; d)(\mathbb{C}^n)$. By [EL12, Prop. 3.2],

$$(III.5.2) \quad \text{If } q > 0, \text{ then } K_{p,q}(\mathbb{C}^n, b; d) = H^1 \left(\bigwedge^{p+1} M_d \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(b + (q-1)d) \right),$$

where:

$$M_d := \ker(S^d(\mathbb{C}^n) \otimes \mathcal{O}_{\mathbb{P}^{n-1}} \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(d)).$$

Consider a splitting $\mathbb{C}^n = \mathbb{C}^{n-1} \oplus \mathbb{C}$, and the GL_{n-1} -equivariant short exact sequence:

$$\begin{aligned} \bigwedge^{p+1} M_d \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(b + (q-1)d) &\hookrightarrow \bigwedge^{p+1} M_d \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(b+1 + (q-1)d) \\ &\twoheadrightarrow \bigwedge^{p+1} M_d \otimes \mathcal{O}_{\mathbb{P}^{n-2}}(b+1 + (q-1)d). \end{aligned}$$

Then from the long sequence in cohomology we extract the GL_{n-1} -equivariant complex:

$$\begin{aligned} (III.5.3) \quad H^0 \left(\bigwedge^{p+1} M_d \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(b+1 + (q-1)d) \right) &\rightarrow H^0 \left(\bigwedge^{p+1} M_d \otimes \mathcal{O}_{\mathbb{P}^{n-2}}(b+1 + (q-1)d) \right) \\ &\rightarrow H^1 \left(\bigwedge^{p+1} M_d \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(b + (q-1)d) \right). \end{aligned}$$

Assume $n \geq 2$ is chosen such that

$$(III.5.4) \quad p+1 \geq h^0(\mathcal{O}_{\mathbb{P}^{n-1}}(b+1 + (q-1)d)).$$

By [G84II, Thm. 3.a.1], the leftmost term of (III.5.3) is zero, therefore the rightmost map is an inclusion of GL_{n-1} -representations. Next, we study the GL_{n-1} -representation

$$H^0 \left(\bigwedge^{p+1} M_d \otimes \mathcal{O}_{\mathbb{P}^{n-2}}(b+1 + (q-1)d) \right).$$

By restricting the map defining M_d , and by comparing with the corresponding map for \mathbb{P}^{n-2} ,

$$(III.5.5) \quad M_d \otimes \mathcal{O}_{\mathbb{P}^{n-2}} \simeq M'_d \oplus \left(\bigoplus_{j \leq d-1} S^j \mathbb{C}^{n-1} \otimes \mathcal{O}_{\mathbb{P}^{n-2}} \right),$$

where $M'_d = \ker(S^d(\mathbb{C}^{n-1}) \otimes \mathcal{O}_{\mathbb{P}^{n-2}} \rightarrow \mathcal{O}_{\mathbb{P}^{n-2}}(d))$. The isomorphism (III.5.5) is GL_{n-1} -equivariant. Then

$$\begin{aligned} & H^0\left(\bigwedge^{p+1} M_d \otimes \mathcal{O}_{\mathbb{P}^{n-2}}(b+1+(q-1)d)\right) = \\ &= H^0\left(\bigoplus_{i=0}^{p+1} \bigwedge^i M'_d \otimes \left(\bigwedge^{p+1-i} \left(\bigoplus_{j \leq d-1} S^j(\mathbb{C}^{n-1})\right)\right) \otimes \mathcal{O}_{\mathbb{P}^{n-2}}(b+1+(q-1)d)\right) \\ &= \bigoplus_{i=0}^{p+1} \left[H^0\left(\bigwedge^i M'_d \otimes \mathcal{O}_{\mathbb{P}^{n-2}}(b+1+(q-1)d)\right) \otimes \bigwedge^{p+1-i} \left(\bigoplus_{j \leq d-1} S^j(\mathbb{C}^{n-1})\right) \right]. \end{aligned}$$

Keeping only the term corresponding to $i = 0$, we have by (III.5.2), (III.5.3), (III.5.4), and [G84II, Thm. 3.a.1] an inclusion of GL_{n-1} -representations

$$(III.5.6) \quad S^{b+1+(q-1)d}(\mathbb{C}^{n-1}) \otimes \bigwedge^{p+1} \left(\bigoplus_{j \leq d-1} S^j(\mathbb{C}^{n-1}) \right) \hookrightarrow K_{p,q}(\mathbb{C}^n, b; d).$$

In particular, when $q = 1$, we obtain an inclusion of GL_{n-1} -representations

$$(III.5.7) \quad \bigoplus_{0 \leq e_p < e_{p-1} < \dots < e_0 \leq d-1} (S^{e_0}(\mathbb{C}^{n-1}) \otimes \dots \otimes S^{e_p}(\mathbb{C}^{n-1}) \otimes S^{b+1}(\mathbb{C}^{n-1})) \hookrightarrow K_{p,1}(\mathbb{C}^n, b; d).$$

The following lemma describes some of the partitions λ such that \mathbb{S}_λ appears in the decomposition of the LHS of (III.5.7).

Lemma III.21. *Let $n \geq 2$ and $b \geq 0$. Fix λ a partition with $|\lambda| = n - 1 \leq p + 2$, and $\lambda_{n-1} > b + 1$. We assume that λ satisfies*

$$\lambda \vdash L_0 + (b+1), \quad \text{with} \quad L_0 \geq \frac{p(p+1)}{2}.$$

For all $0 \leq i \leq p$, let

$$(III.5.8) \quad \begin{cases} e_i := \lceil \frac{L_i}{p+2-i} + \frac{p+1-i}{2} \rceil \\ L_{i+1} := L_i - e_i \end{cases}$$

Then $e_0 > e_1 > \dots > e_p \geq 0$ and $L_{p+1} = 0$. Assume furthermore that the following conditions hold:

$$(III.5.9) \quad \begin{cases} \lambda'_1 \geq e_0 \\ \lambda'_1 + \lambda'_2 \geq e_0 + e_1 \\ \vdots \\ \lambda'_1 + \lambda'_2 + \dots + \lambda'_{n-2} \geq e_0 + e_1 + \dots + e_{n-3}, \end{cases}$$

where $\lambda' := R(b+1, \lambda)$ is the removal of the last $b+1$ visible boxes in λ . The visible boxes of λ are by definition the λ_1 boxes of the Young diagram of λ that have no box directly below them. We order them increasingly from left to right, and then up to down. Then \mathbb{S}_λ appears in the decomposition of

$$S^{b+1} \otimes \bigotimes_{i=0}^p S^{e_i}.$$

If $e_0 \leq d-1$, then $\mathbb{S}_\lambda(\mathbb{C}^{n-1})$ appears in the decomposition of the left term of (III.5.7).

The statement of the lemma hints to its algorithmic proof that we leave as an exercise.

We are ready to show that the Schur decomposition of $\mathbb{K}_{p,1}(b; d)$ is asymptotically rich.

Theorem III.22. Fix $p \geq 1$ and $b \geq 1$. Assume that $p \geq b+1$. Then as d goes to infinity,

$$\log c(\mathbb{K}_{p,1}(b; d)) \in \Theta(\log d), \quad \log N(\mathbb{K}_{p,1}(b; d)) \in \Theta(\log d).$$

Proof. Choose n maximal such that (III.5.4) is satisfied. This verifies

$$n \geq \lceil \sqrt[b+1]{(p+1) \cdot (b+1)!} \rceil.$$

Assume $n \geq 5$. The plan of the proof of the proposition is as follows: The previous lemma constructs irreducible GL_{n-1} -subrepresentations of $K_{p,1}(\mathbb{C}^n, b; d)$. To prove the existence of many distinct types of irreducible GL_n -subrepresentations, by the branching rule [FH91, Ex. 6.12], it is enough to pick a large subset of the GL_{n-1} -representations that correspond to Young diagrams that differ pairwise in at least one column by at least two boxes. We force this by asking that the corresponding partitions satisfy $\lambda_{2i-1} = \lambda_{2i}$ for all $i \leq (n-1)/2$, and $\lambda_{n-1} = \lambda_{n-2}$. Geometrically, we ask that the Young diagram of λ sits in a tiling of the plane by 1×2 -sized boxes, instead of the standard 1×1 tiling, with the possible exception of using 1×3 boxes for the last three rows when $n-1$ is odd. We say that λ has a *twin pattern of length* $n-1$. The Young diagrams of distinct partitions with twin patterns of length $n-1$ (or of lengths of the same parity) automatically differ in at least one column by at least two 1×1 boxes.

We now construct an easy-to-count set of partitions λ with twin patterns of length $n-1$, also satisfying the conditions of Lemma III.21. We first ask that

$$(III.5.10) \quad \lambda_{n-1} \geq B := B(p, b, n) := \max \left\{ b+2, \frac{\frac{p(p+1)}{2} + b+1}{n-1} \right\}.$$

This implies $\lambda_{n-1} > b+1$, and $L_0 \geq \frac{p(p+1)}{2}$. We next impose the restriction

$$(III.5.11) \quad \lambda_1 \geq \max \left\{ \frac{n-3}{2}, 1 \right\} \cdot \left(\frac{L_0}{p+2} + \frac{p+3}{2} \right) \geq \max \left\{ \frac{n-3}{2}, 1 \right\} \cdot e_0.$$

This is a manageable condition that together with $\lambda_1 = \lambda_2$ implies the relations (III.5.9). The last requirement of Lemma III.21 is $e_0 \leq d-1$. For big enough d , this

is implied by

$$(III.5.12) \quad L_0 \leq pd.$$

A bound on λ_3 so that any further choice for $\lambda_3 \geq \lambda_5 \geq \dots \geq \lambda_{2[(n-1)/2]-1}$ satisfies via the twin pattern condition the previous restrictions is:

$$(III.5.13) \quad B \leq \lambda_3 \leq D(\lambda_1, d, p, n) := \min \left\{ \lambda_2 = \lambda_1, \frac{pd - 2\lambda_1}{n-3}, \frac{2(p-n+1)}{(n-2)(n-3)} \lambda_1 - \frac{(p+1)(p+2)}{2(n-3)} \right\}.$$

In the above set, λ_1 appears because λ is a Young diagram. The condition $\lambda_1 = \lambda_2$ is part of the tiling restriction. The middle term is explained by (III.5.12), and an algebraic manipulation shows that $\lambda_3 \leq \frac{2(p-n+1)}{(n-2)(n-3)} \lambda_1 - \frac{(p+1)(p+2)}{2(n-3)}$ implies (III.5.11).

We are reduced to counting partitions λ having twin patterns of length $n-1$, satisfying (III.5.10) and (III.5.13). For each fixed λ_1 in the interval $\left[B, \frac{pd-(n-3)B}{2} \right]$, these are counted by the binomial coefficients:

$$\binom{[D(\lambda_1, d, p, n)] - [B] + \left\lceil \frac{n-5}{2} \right\rceil}{\left\lceil \frac{n-5}{2} \right\rceil}.$$

When we sum over the range of λ_1 , the result is in $\Theta(d^{\lceil \frac{n-3}{2} \rceil})$. This conclusion also holds when $n < 5$. The cases $n \in \{2, 3, 4\}$ follow immediately from (III.5.7), and the case $n = 1$ is impossible, because (III.5.4) is satisfied for $n = 2$, when $p \geq b+1$. This gives the lower bound needed for the first statement, hence also for the second. The upper bounds follow from Lemma III.19 and from Theorem III.2. \square

III.6 Varying p

In this section, we study the growth of the complexities of $\mathbb{K}_{p,0}(b; d)$ and of $\mathbb{K}_{p,1}(b; d)$ when we vary p . A first result in the direction of varying p for syzygy functors is [EL12, Cor. 6.2]. It proves the nonvanishing of $\mathbb{K}_{p,q}(b; d)$ for fixed $q \geq 1$

and sufficiently large p . In this section, we study the growth of the complexities of $\mathbb{K}_{p,0}(b; d)$ and of $\mathbb{K}_{p,1}(b; d)$ when we vary p .

III.6.1 $\mathbb{K}_{p,0}(b; d)$

We evaluate the complexity of $\mathbb{K}_{p,0}(b; d)$ when we fix $b > 0$ and $d > 2$, and let p grow. It is elementary that $\mathbb{K}_{p,0}(0; d) = 0$, so we exclude this trivial case.

Theorem III.23. *Fix $b > 0$ and $d > 2$. Then as p grows to infinity,*

$$\log c(\mathbb{K}_{p,0}(b; d)) \in \Theta(p^{1/2}).$$

Proof. We argue that, on the one hand, $\log \mathbb{K}_{p,0}(b; d)$ is bounded below by a multiple of $(p^{1/2})$ by using Remark III.12. On the other, $\log c(\bigwedge^p S^d \otimes S^b)$ is bounded above by a multiple of $(p^{1/2})$, which provides the same upper bound for the cohomology functor.

As observed in the proof of Theorem III.16, any \mathbb{S}_λ with $|\lambda| = p + 1$ in $\bigwedge^p S^d \otimes S^b$ appears in $\mathbb{K}_{p,0}(b; d)$. Using Pieri's rule and Remark III.12, the complexity of $\mathbb{K}_{p,0}(b; d)$ is at least the number of ways one can partition $p \cdot [(d-1)/2]$ with at most p parts, the logarithm of which is bounded below by a multiple of $(p^{1/2})$ by [H99, p86]. The complexity of $\bigwedge^p S^d \otimes S^b$ is bounded above by the number of partitions of $pd + b$. The logarithm of these is $\Theta(p^{1/2})$ by [H99, p86]. \square

III.6.2 $\mathbb{K}_{p,1}(b; d)$

In this subsection we look for lower bounds on the complexity of $\mathbb{K}_{p,1}(b; d)$ when we increase p and d . We use the same restriction argument from the previous section,

but this time we count partitions λ of length $n - 1$ that have *almost triplet pattern*, i.e., the second and third parts of $\bar{\lambda} := \lambda - (1^{|\lambda|})$ are equal, the next three are equal, and so on. We ask that $|\bar{\lambda}|$ is a multiple of 3. The *mold* of a partition λ with almost triplet pattern is the partition obtained from $\bar{\lambda}$ by making its first part equal to the second (and third). One sees that irreducible GL_{n-1} -representations corresponding to partitions with different molds cannot branch out from the same irreducible GL_n -representation after restriction to GL_{n-1} .

Lemma III.24. *Fix $b \geq 0$. Let $n := n(p) \leq p + 1$ be such that $\lim_{p \rightarrow \infty} n(p) = \infty$. Then as p grows, for $d \geq 3 \cdot \left\lceil \frac{p+1}{n-3} \right\rceil + 3$, the number of molds of partitions λ of length $n - 1$ with almost triplet pattern, such that $\mathbb{S}_\lambda \mathbb{C}^{n-1}$ is a GL_{n-1} -subrepresentation of*

$$(III.6.1) \quad \bigoplus_{a_0 + \dots + a_{d-1} = p+1} \left(\bigotimes_{i=0}^{d-1} \bigwedge^{a_i} S^i \mathbb{C}^{n-1} \right) \otimes S^{b+1} \mathbb{C}^{n-1},$$

has its logarithm bounded below by a multiple of $(n^{1/2})$.

Proof. We prove a weaker form of the result by asking that $d \geq p + 2$. Fix $r \in \{1, 2, 3\}$ such that $d - r \equiv 1 \pmod{3}$. The representation (III.6.1) contains the nonzero subrepresentation:

$$(III.6.2) \quad \left(\bigwedge^{n-1} S^{d-r} \otimes (S^{d-r-1} \otimes S^{d-r-2} \otimes \dots \otimes S^{d-r-1-(p-n+1)}) \otimes S^{b+1} \right) (\mathbb{C}^{n-1}).$$

By Remark III.12, for any partition $\mu \vdash \left\lceil \frac{(n-1)(d-r-1)}{6} \right\rceil$, with $|\mu| \leq l := \left\lceil \frac{n-1}{3} \right\rceil$, the partition

$$\lambda'(\mu) := (1^{n-1}) + 2(\mu_1, \mu_1, \mu_1, \dots, \mu_l, \mu_l, \mu_l) + \epsilon(d) \cdot (n - 1)$$

is such that $\mathbb{S}_{\lambda'} \mathbb{C}^{n-1}$ is a subrepresentation of $\bigwedge^{n-1} S^{d-r} \mathbb{C}^{n-1}$, where $\epsilon(d)$ is 0 if

$d - r - 1$ is even, and 1 if $d - r - 1$ is odd. By Pieri's rule,

$$\lambda(\mu) := \lambda'(\mu) + ((\sum_{k=0}^{p-n+1} (d - r - 1 - k)) + b + 1)$$

produces a subrepresentation of (III.6.2). By construction, $\lambda(\mu)$ has almost triplet pattern, and length $n - 1$. If μ and μ' are different, then $\lambda(\mu)$ and $\lambda(\mu')$ have different molds. At least when p is large enough, the number of μ 's is bigger than the number of partitions of $[(n - 1)/3]$, the logarithm of which is in $O(n^{1/2})$ by [H99, p86].

When using the bound $d \geq 3 \cdot \lceil \frac{p+1}{n-4} \rceil + 3$, the result is proved similarly by considering a subrepresentation of (III.6.1) of form $(\bigwedge^{n_1} S^{d_1} \otimes \dots \otimes \bigwedge^{n_m} S^{d_m} \otimes S^{b+1})(\mathbb{C}^{n-1})$, with m as small as possible, such that $\sum_{i=1}^m n_i = p + 1$, with $n_i < n$ satisfying some conditions modulo 3, and with $d_1 > d_2 > \dots > d_m$ as large as possible, all congruent to 1 modulo 3 and smaller than $d - 1$. \square

As a corollary, we obtain:

Proposition III.25. *Fix $b \geq 0$. As p goes to infinity, for $n := \left\lceil {}^{b+1}\sqrt{(p+1) \cdot (b+1)!} \right\rceil$, and $d \geq 3 \cdot \lceil \frac{p+1}{n-3} \rceil + 3$, there is a positive constant C such that:*

$$\log(c(\mathbb{K}_{p,1}(b; d))) \geq C \cdot (p^{\frac{1}{2(b+1)}}).$$

Proof. The choice of n insures that (III.5.4) holds, hence by the discussion in the previous subsection, the GL_{n-1} -subrepresentations given by the previous lemma appear in the decomposition of $K_{p,1}(\mathbb{C}^n, b; d)$. We conclude by using the restriction argument of the previous section, the discussion in the preamble of this section, and noticing that $n \in \Theta(p^{1/(b+1)})$. \square

Remark III.26. Fixing d and increasing p alone is a more desirable problem to study. In attempting to use the restriction argument, our limitation in showing

nontrivial growth for fixed d stems from not knowing if there exist many Schur subfunctors of $\bigwedge^p S^d$ corresponding to partitions of relatively short length and with distinct twin patterns. (In Remark III.12, all Young diagrams are much too tall.)

CHAPTER IV

Toric weights in asymptotic syzygies of toric varieties

IV.1 Introduction

In this chapter, we consider weights in syzygies of toric varieties. We treat all of the syzygies at once. Let X be a smooth projective toric variety over \mathbb{C} . Let A be a very ample toric line bundle on X . Then following notation from the previous chapters, the embedding defined by $L_d = A^{\otimes d}$ is toric. Let Δ be the convex polytope associated to the very ample divisor A ([F93], Section 3.4, p66, P_A in notation of the book.)

Then by the Koszul cohomology description of $K_{p,q}$ in section II.1, the group $K_{p,q}(X; L_d)$ is the cohomology in the middle of the complex:

$$\begin{aligned} \text{(IV.1.1)} \quad \wedge^{p+1} H^0(X, L_d) \otimes H^0(X, (q-1)L_d) &\rightarrow \wedge^p H^0(X, L_d) \otimes H^0(X, qL_d) \\ &\rightarrow \wedge^{p-1} H^0(X, L_d) \otimes H^0(X, (q+1)L_d) \end{aligned}$$

By the definition of Δ , $H^0(X, L_d)$ is a representation of the torus with weights corresponding to points in $d\Delta$. Hence $\wedge^p H^0(X, L_d) \otimes H^0(X, qL_d)$ is a representation of the torus with weights corresponding to points in $(p+q)d\Delta$. Similarly, all terms in (IV.1.1) have weights in $(p+q)d\Delta$. The maps in (IV.1.1) are equivariant with

respect to the torus action, hence $K_{p,q}(X; L_d)$ is also a representation of the torus with weights in $(p+q)d\Delta$. In this chapter, we address the question which torus weights appear in the decompositions of $K_{p,q}(X; L_d)$.

Denote the collection of weights by:

$$\text{wts}(K_{p,q}(X; L_d)) = \{\text{Torus weights of } K_{p,q}(X; L_d)\} \subseteq (p+q)d \cdot \Delta.$$

We normalize so that all points lie in Δ :

$$\text{wts}^{\text{nor}}(K_{p,q}(X; L_d)) = \frac{\text{wts}(K_{p,q}(X; L_d))}{(p+q)d} \subseteq \Delta$$

We show first of all that as $d \rightarrow \infty$, the set of all normalized torus weights becomes dense in Δ :

Theorem (IV.19). *For every fixed q with $1 \leq q \leq n$,*

$$\bigcup_{\substack{d > 0 \\ 1 \leq p \leq r_d}} \text{wts}^{\text{nor}}(K_{p,q}(X; L_d))$$

is dense in Δ .

The Theorem is illustrated in Figure IV.1 below, which approximates $K_{p,1}(\mathbb{P}^2; d)$ for $d = 2$ and $d = 4$.

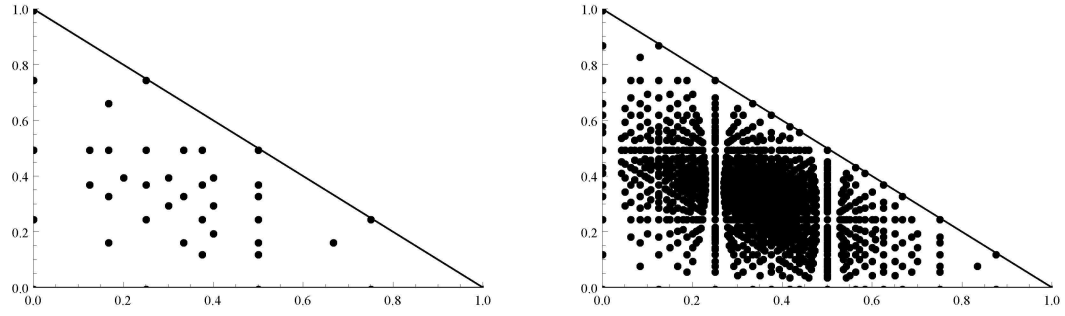


Figure IV.1: Normalized torus weights for $K_{p,1}(\mathbb{P}^2; d)$ for $d = 2$ and $d = 4$.

We can then ask for more precise information, i.e. what happens if we focus on only some of the syzygies appearing in the resolution. More specifically, restrict p to lie in a fixed interval relative to r_d ⁸, i.e. consider:

$$\Delta(a, b) = \overline{\bigcup_{\substack{d \gg 0 \\ a \cdot r_d \leq p \leq b \cdot r_d}} \text{wts}^{\text{nor}}(K_{p,q}(X; L_d))} \subseteq \Delta$$

where $0 \leq a < b \leq 1$. $K_{p,q}$ are no longer dense inside all of Δ . Figure IV.2 shows the normalized weights of $K_{p,1}(\mathbb{P}^2; 4)$ for $a = 0.33, b = 0.66$ and $a = 0.66, b = 1$:

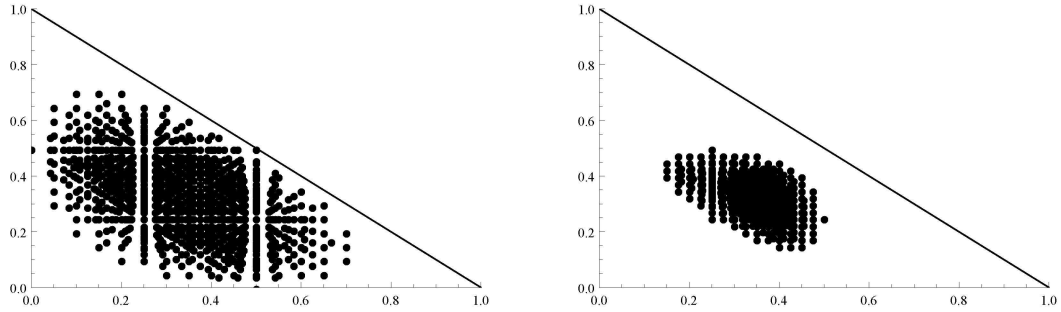


Figure IV.2: Closure of normalized weights for $a = 0.33, b = 0.66$ and $a = 0.66, b = 1$ with $\mathbb{P} = \mathbb{P}^2, d = 4$.

Quite surprisingly, we can explicitly describe $\Delta(a, b)$. The description involves the largest volume of a nice region supported at x . More precisely, define:

$$\tau_x = \sup \{ \text{vol}(S_x) \mid S_x = \text{finite union of cubes} \subset \Delta, \text{ with center of mass of } S_x = x \}$$

⁸Recall that $r_d = h^0(X, L_d) - 1$, the dimension of the ambient projective space.

Theorem (IV.21).

$$\Delta(a, b) = \left\{ x \in \Delta \mid \frac{\tau_x}{\text{vol}(\Delta)} \geq a \right\} =: \Delta(a)$$

Note that part of the statement of the theorem is that $\Delta(a, b)$ does not depend on b , so we write $\Delta(a)$ for it. The boundary of $\Delta(a)$ is also explicitly computable. For example, let Δ be the unit square. Then the boundary of $\Delta(\frac{1}{10})$ consists of 12 pieces, 4 segments of hyperbolas at the corners and 8 line segments in between as illustrated below.

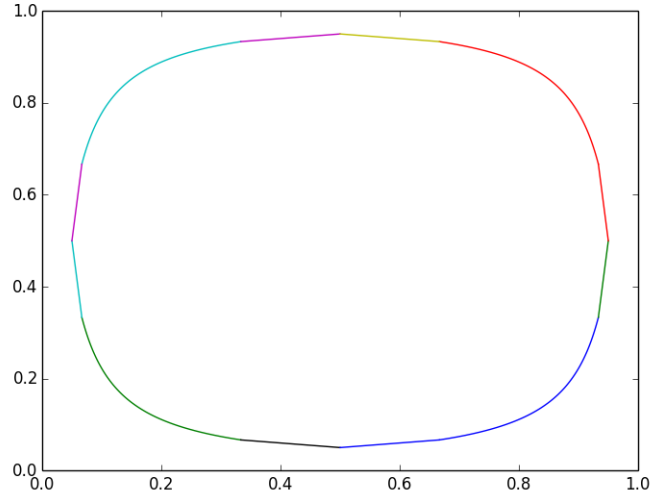


Figure IV.3: $\Delta(\frac{1}{10})$ for the unit square.

IV.2 Idea of proof

In this section, we discuss the basic strategy of the proof before giving full details.

Let L be a very ample toric line bundle on a smooth projective toric variety X . We wish to study which weights of the torus appear in $K_{p,q}(X; L)$. As in Chapter

II, let

$$M_L = \text{Ker} \left(H^0(X, L) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow L \right).$$

Thanks to Demazure vanishing and Prop. II.4, we have (cf. Prop. IV.2):

$$(IV.2.1) \quad K_{p,q}(X; L) = H^q(X, \wedge^{p+q} M_L).$$

So the issue is to identify the torus weights appearing in the right hand side of the equality.

To a first approximation, the idea is to find torus equivariant spaces

$$U, \quad W_1, \quad \text{with } \dim U = 1, \quad \dim W_1 \gg 0$$

together with a torus equivariant map:

$$(IV.2.2) \quad H^q(X, \wedge^{p+q} M_L) \longrightarrow U \otimes \wedge^{p+q} W_1.$$

Suppose one knew that (IV.2.2) is surjective. Then we can conclude that every weight appearing in $U \otimes \wedge^{p+q} W_1$ appears in:

$$K_{p,q}(X; L) = H^q(X, \wedge^{p+q} M_L).$$

On the other hand, one can compute combinatorially the weights of $U \otimes \wedge^{p+q} W_1$ from the weights of U and W_1 , and the results stated in the previous section would follow.

Strictly speaking, we do not achieve proving surjectivity of (IV.2.2). What we show is that we can find torus stable vector spaces W_0 of small dimension and W_1 of large dimension with W_0 a quotient of W_1 with the following property. Let W be any quotient of W_1 that factors the map to W_0 .

$$W_1 \twoheadrightarrow W \twoheadrightarrow W_0.$$

Then there is a surjective mapping:

$$H^q(X, \wedge^{p+q} M_L) \longrightarrow U \otimes \wedge^{p+q} W$$

with $\dim W = p + q$. As before, this allows us to produce many weights appearing in $K_{p,q}$ and the stated theorem follow.

The next point is to understand how to construct U , W_0 and W_1 . Take a w -dimensional torus stable quotient of $H^0(X, L)$ and denote it by W . W defines a toric stable linear subspace:

$$\mathbb{P}(W) \subset \mathbb{P}(H^0(L)).$$

Define by $Z \subset X$, the scheme theoretic intersection:

$$(IV.2.3) \quad Z = \mathbb{P}(W) \cap X.$$

Then there is a natural map:

$$W \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow L \otimes \mathcal{O}_Z.$$

Taking wedge powers, a local analysis (cf. (IV.3.2)) shows that there is a surjective homomorphism:

$$\wedge^w M_L \longrightarrow \mathcal{I}_{Z/X} \otimes \wedge^w W.$$

Hence, we have a map:

$$(IV.2.4) \quad H^q(X, \wedge^w M_L) \longrightarrow H^q(X, \mathcal{I}_{Z/X}) \otimes \wedge^w W$$

which is also toric equivariant. The goal is to choose W such that:

$$(IV.2.5) \quad U = H^q(X, \mathcal{I}_{Z/X}) = \mathbb{C} \neq 0.$$

In practice, (IV.2.4) is achieved by first choosing a toric stable subspace $Z \subset X$ such that (IV.2.5) holds, and then choosing W to satisfy (IV.2.3). This is carried out in Sect. 3. Furthermore, we will see in Sect. 4 that we can take as W quotients of a fixed very large W_1 (Prop. IV.17).

The main technical result of Sect. 3 is that when we follow this outline, the resulting map (IV.2.4) is surjective (Prop. IV.13). One key point here is that although the map (IV.2.4) is toric, to prove that it is surjective, we do not need to stay in the toric world. Hence, we can follow the inductive arguments in [EL12] and [Z12] with essentially no modification.

There is one further asymptotic ingredient. Namely, we are interested in the asymptotics of $K_{p,q}(X; L_d)$ where $L_d = A^{\otimes d}$ and A is very ample. When we go through the constructions just outlined for L_d , we arrive at the following situation.

We have torus stable subspaces $W_{0,d}, W_{1,d}, W_d$:

$$\text{weights}(\wedge^{p+q} W_d) + (\text{some fixed weight}) \subset \text{weights}(K_{p,q}(X; L_d)).$$

Moreover,

$$\dim W_d = p + q, \text{ and } W_{1,d} \rightarrow\rightarrow W_d \rightarrow\rightarrow W_{0,d}.$$

Furthermore,

$$\dim(W_{0,d}) \in o(d^n), \text{ and } h^0(X, L_d) - W_{1,d} \in o(d^n).$$

Thus, up to asymptotically insignificant contributions, all the weights of $\wedge^{p+q} W_{1,d}$ appear in $K_{p,q}$. It remains to prove a lemma on asymptotics of normalized weights for wedge powers. This is the content of Sect. 5. The asymptotic behavior we deduce applies to any sequence of toric quotient spaces W_d asymptotically equal

to $H^0(X, L_d)$ in dimension. Then specializing to $W_{1,d}$ gives us a lower bound on the weights that appear by the above discussion. Applying to $H^0(X, L_d)$ gives us an upper bound by the definition of Koszul cohomology. Hence, we get our sharp asymptotic description.

IV.3 Surjectivity of map induced by a secant space

In this section, we recall and adapt the computations in Chapter II to the toric case.

IV.3.1 Key lemma

We first recall the key vector bundle used to compute syzygies. Let X be a smooth projective toric variety over \mathbb{C} . Let A be a chosen toric very ample line bundle on X . We use L to denote any toric very ample line bundle on X (we will later replace L with $L_d = A^{\otimes d}$).

As in Chapter II, in the evaluation map:

$$\nu_L : H^0(X, L) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow L$$

we put $M_L = \ker \nu_L$. Thus M_L is a vector bundle sitting in the basic exact sequence:

$$0 \rightarrow M_L \rightarrow H^0(X, L) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow L \rightarrow 0.$$

We will need the following fact in this chapter:

Proposition IV.1. (*Demazure*) *For any projective toric variety X , and a very ample divisor A , then:*

$$H^m(X, \mathcal{O}_X(jA)) = 0 \text{ for } m \geq 1, j \geq 0.$$

Proof. This follows from Demazure vanishing (cf. p410 Thm 9.2.3 [CLS11]). \square

In our setting, we have:

Proposition IV.2. *For $1 \leq q \leq n$, $K_{p,q}(X; L_d) = H^q(X, \wedge^{p+q} M_{L_d})$.*

Proof. By Prop. II.4, the conclusion follows if we know:

$$H^i(X, \mathcal{O}_X(mL_d)) = 0 \text{ for } i > 0, m \geq 0.$$

This follows from the Proposition above. \square

Let W be a quotient of $H^0(X, L)$ of dimension w . Then we have

$$\mathbb{P}(W) \subset \mathbb{P}(H^0(X, L)).$$

Let Z :

$$Z = \mathbb{P}(W) \cap X$$

the scheme-theoretic intersection of $\mathbb{P}(W)$ with X . This gives rise to a surjective map of sheaves:

$$W_X = W \otimes \mathcal{O}_X \longrightarrow L \otimes \mathcal{O}_Z,$$

and we denote its kernel by Σ_W . So we get an exact diagram of sheaves:

$$(IV.3.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & M_L & \longrightarrow & V \otimes \mathcal{O}_X & \longrightarrow & L \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma_W & \longrightarrow & W \otimes \mathcal{O}_X & \longrightarrow & L \otimes \mathcal{O}_Z \longrightarrow 0 \end{array}$$

Through the local analysis of [EL12] (3.10), we get a diagram :

$$(IV.3.2) \quad \begin{array}{ccc} \bigwedge^w \Sigma_W & \longrightarrow & \bigwedge^w W_X \\ \downarrow & & \downarrow \\ \mathcal{I}_{Z/X} \otimes \bigwedge^w W_X & \longrightarrow & \mathcal{O}_X \otimes \bigwedge^w W_X \end{array}$$

and this induces a surjective map (cf. the map above [EL12] Def. 3.8):

$$(IV.3.3) \quad \sigma_\pi : \bigwedge^w M_L \rightarrow \mathcal{I}_{Z/X} \otimes \bigwedge^w W_X$$

Then σ_π induces a map:

$$(IV.3.4) \quad H^q(X, \bigwedge^w M_L) \rightarrow H^q(X, \mathcal{I}_{Z/X}) \otimes \bigwedge^w W$$

The above construction after Prop. IV.2 works in general. In our setting, when X, L, W are toric, all the above maps are toric equivariant. Following the notations above, the key conclusion of the section is the following lemma:

Lemma IV.3. *For $L = L_d$ with $d \gg 0$ and $1 \leq q \leq n$, there exists torus stable W with $Z = \mathbb{P}(W) \cap X$ and*

$$H^q(X, \mathcal{I}_{Z/X}) \neq 0,$$

such that the induced torus equivariant map:

$$H^q(X, \bigwedge^w M_L) \rightarrow H^q(X, \mathcal{I}_{Z/X}) \otimes \bigwedge^w W$$

where $w = \dim W$, is surjective. Moreover, combined with Prop. IV.2, this implies that any torus weight in $H^q(X, \mathcal{I}_{Z/X}) \otimes \bigwedge^w W$ also appears in $K_{w-q,q}(X; L)$.

IV.3.2 Proof of Lemma IV.3

Once torus equivariance has been established as in (IV.3.4), surjectivity has nothing to do with the torus action. So we will be able to prove the surjectivity of (IV.3.4) by proving the surjectivity of:

$$(IV.3.5) \quad H^q(X, \wedge^w M_L) \rightarrow H^q(X, \mathcal{I}_{Z/X}).$$

The rest of the proof is technical and follows the same lines of attack as in [Z12] and [EL12]. We will give the choice of Z and W later (Lemma IV.6, Prop. IV.13). For now, we introduce some terminology that will help in the induction.

For induction in the proof, we have to add in a twist of the map in IV.3.5. Let B be a line bundle and consider:

$$(IV.3.6) \quad H^q(X, \wedge^w M_L(B)) \rightarrow H^q(X, \mathcal{I}_{Z/X}(B))$$

Definition IV.4. Let W be a quotient of $H^0(X, L)$ as above. We say that W *carries weight q syzygies* for B if the map induced by σ_π in equation (IV.3.6) is surjective. (We also say the same for $q = 0$ for notational convenience even though it isn't necessarily directly related to syzygies.)

Let us set up some inductive notation. Take a general divisor $\overline{X} \in |A|$ so that \overline{X} is irreducible and diagram (IV.3.1) remains exact after tensoring with $\mathcal{O}_{\overline{X}}$. For $0 \leq i \leq q - 1$, let

$$X_0 = X, \quad Z_0 = Z, \quad A_0 = A.$$

Having made the definitions for $i - 1$, for i , choose a general $X_i \in |A_{i-1}|$ so that X_i is irreducible and the corresponding diagram (IV.3.1) for X_{i-1} remains exact after

tensoring with \mathcal{O}_{X_i} (and as previously defined, $\overline{X} = X_1$). Let

$$Z_i = Z_{i-1} \cap X_i, \quad A_i = A_{i-1}|_{X_{i-1}}.$$

Definition IV.5. We say that Z is *adapted* to the data X, B, A, n, q , if:

- (i) $H^q(X, \mathcal{I}_{Z/X}(B)) \neq 0$
- (ii) For all $i \geq 0$, $H^{q-i}(X_i, \mathcal{I}_{Z_i/X_i}(B + (i+1)A)) = 0$.
- (iii) For all $i \geq 0$, Z_i has dimension $q - 1 - i$.

Now we construct a toric Z in our smooth projective toric variety X and check the above conditions to conclude that Z is adapted to $X, \mathcal{O}_X, A, n, q$.

Let $-K_X = e_1 + \dots + e_m$ where $\{e_i\}$ are the prime toric invariant divisors. Let $c = n + 1 - q$ with $1 \leq q \leq n$.

Lemma IV.6. *We can order the e_i such that $Z = e_1 \cap \dots \cap e_{c-1} \cap (e_c + \dots + e_m)$ is a complete intersection.*

Proof. Choose e_1, \dots, e_n such that they generate an n -dimensional cone. Then $F = e_1 \cap \dots \cap e_{c-1}$ is a complete intersection. For any $i > c - 1$, F either does not meet e_i , or it does so transversely since adding a ray to a cone increase its dimension by at most 1. It meets at least one of them, e_c , since $c \leq n$. \square

Proposition IV.7. *With the above choice of Z :*

$$H^q(X, \mathcal{I}_{Z/X}) = \mathbb{C} \neq 0$$

Proof. If $q = 1$, then $c = n$, and in this case Z consists of two points. The short exact sequence

$$0 \rightarrow \mathcal{I}_{Z/X} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0$$

induces:

$$H^0(\mathcal{O}_X) \rightarrow H^0(\mathcal{O}_Z) \rightarrow H^1(\mathcal{I}_{Z/X}) \rightarrow H^1(\mathcal{O}_X)$$

where $h^0(\mathcal{O}_X) = 1$, $h^0(\mathcal{O}_Z) = 2$ and $h^1(\mathcal{O}_X) = 0$ (since structure sheaves of toric varieties do not have higher cohomology). Hence, $H^1(\mathcal{I}_{Z/X}) = \mathbb{C}$.

Assume $q \geq 2$. From

$$0 \rightarrow \mathcal{I}_{Z/X} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0$$

we get:

$$0 = H^{q-1}(\mathcal{O}_X) \rightarrow H^{q-1}(\mathcal{O}_Z) \rightarrow H^q(\mathcal{I}_{Z/X}) \rightarrow H^q(\mathcal{O}_X) = 0.$$

Then $H^{q-1}(\mathcal{O}_Z) = H^q(\mathcal{I}_{Z/X})$.

Let $F = e_1 \cap e_2 \dots \cap e_{c-1}$. From

$$0 \rightarrow \mathcal{I}_{Z/F} \rightarrow \mathcal{O}_F \rightarrow \mathcal{O}_Z \rightarrow 0.$$

we get:

$$0 = H^{q-1}(\mathcal{O}_F) \rightarrow H^{q-1}(\mathcal{O}_Z) \rightarrow H^q(\mathcal{I}_{Z/F}) \rightarrow H^q(\mathcal{O}_F) = 0.$$

we need to compute $H^q(\mathcal{I}_{Z/F})$. Now $Z = F \cap (e_c + \dots + e_m)$ and $e_c + \dots + e_m = -K_F$.

Since

$$\mathcal{I}_{Z/F} = \mathcal{O}_F(K_F), \quad \dim F = n - c - 1 = n - (n - q) = q,$$

and then Serre duality applies to give us:

$$H^q(\mathcal{I}_{Z/F}) = H^0(\mathcal{O}_F) = \mathbb{C}.$$

□

We next turn to verifying the conditions in Def. IV.5.

Proposition IV.8. *For all $m \geq 1, j \geq i \geq 0$:*

$$(i) \ H^m(\mathcal{O}_{X_i}(jA)) = 0.$$

$$(ii) \ H^m(\mathcal{O}_{Z_i}(jA)) = 0.$$

Proof. We prove the first assertion by induction on i . When $i = 0$, the conclusion follows from Demazure vanishing since X is toric. Suppose the conclusion is true for $i - 1$, then we have:

$$0 \rightarrow \mathcal{O}_{X_{i-1}}((j-1)A) \rightarrow \mathcal{O}_{X_{i-1}}(jA) \rightarrow \mathcal{O}_{X_i}(jA) \rightarrow 0$$

$H^m(\mathcal{O}_{X_{i-1}}(jA)) = H^{m+1}(\mathcal{O}_{X_{i-1}}((j-1)A)) = H^{m+1}(\mathcal{O}_{X_{i-1}}(jA)) = 0$ by inductive assumption, hence $H^m(\mathcal{O}_{X_i}(jA)) = 0$. The second assertion is analogous. \square

Proposition IV.9. *For all $i \geq 0$:*

$$H^{q-i}(X_i, \mathcal{I}_{Z_i/X_i}((i+1)A)) = 0$$

Proof. Consider on X_i the exact sequence:

$$0 \rightarrow \mathcal{I}_{Z_i/X_i}((i+1)A) \rightarrow \mathcal{O}_{X_i}((i+1)A) \rightarrow \mathcal{O}_{Z_i}((i+1)A) \rightarrow 0$$

then

$$H^{q-i-1}(\mathcal{O}_{X_i}((i+1)A)) \rightarrow H^{q-i-1}(\mathcal{O}_{Z_i}((i+1)A)) \rightarrow$$

$$H^{q-i}(\mathcal{I}_{Z_i/X_i}((i+1)A)) \rightarrow H^{q-i}(\mathcal{O}_{X_i}((i+1)A))$$

If $q - i = 1$, then Z_i has dimension $\dim Z - i = \dim Z - (q - 1) = n - (n + 1 - q) - (q - 1) = q - 1 - q + 1 = 0$. Then very ampleness and $H^{q-i}(\mathcal{O}_{X_i}((i+1)A)) = 0$ from Prop. IV.8 implies that $H^{q-i}(X_i, \mathcal{I}_{Z_i/X_i}((i+1)A)) = 0$.

Assume $q - i - 1 \geq 1$, then the two ends in the above sequence are 0 because of Prop. IV.8 and we get:

$$H^{q-i}(\mathcal{I}_{Z_i/X_{i-1}}((i+1)A)) = H^{q-i-1}(\mathcal{O}_{Z_i}((i+1)A)) = 0$$

□

Putting the computations together, we obtain:

Proposition IV.10. *For any $1 \leq q \leq n$, the scheme Z constructed above is adapted to $X, \mathcal{O}_X, A, n, q$*

Proof. Choosing the divisors as in Prop. IV.6, Def. IV.5 (iii) follows from the complete intersection condition. Def. IV.5 (i), (ii) are checked in Prop. IV.7, IV.9.

□

Having constructed Z , we next turn to the construction of quotients W as in Def. IV.4. The issue is to specify inductive conditions that will guarantee that Def. IV.4 holds. Recall that

$$V = H^0(X, L).$$

Let

$$(IV.3.7) \quad V' = V \cap H^0(X, I_{\overline{X}/X}(A)).$$

The intersection takes place inside V . Set $W' = \pi(V')$. Write

$$(IV.3.8) \quad \overline{V} = V/V', \quad \overline{W} = W/W', \quad \overline{L} = L|_{\overline{X}}, \quad \overline{B} = B|_{\overline{X}}, \quad \overline{Z} = Z \cap \overline{X}$$

As in [EL12, (3.14)], we get the analogue of (1.3) above for the barred objects and we have the surjection:

$$\overline{\sigma} : \wedge^{\overline{w}} M_{\overline{V}} \rightarrow I_{\overline{Z}/\overline{X}},$$

so we can study the behavior of \overline{W} with respect to carrying syzygies.

Lemma IV.11. *Fix $1 \leq q \leq n$. If \overline{W} carries weight $q - 1$ syzygies for $\overline{B} + \overline{A}$ on \overline{X} and if*

$$H^q(X, I_{Z/X}(B + A)) = 0,$$

then W carries weight q syzygies for B on X .

Proof. This follows from the same argument as [EL12] Thm 3.10 with $(q - 1)$ replaced by q and $B \otimes L$ with B in our case. \square

Proposition IV.12. *If $d \gg 0$, the following statements are true and so are their inductive counterparts after cutting down by hyperplanes as above:*

(i) *The map $H^0(X, L_d) \rightarrow H^0(Z, L_d)$ is surjective, equivalently,*

$$H^1(X, I_{Z/X}(L_d)) = 0.$$

(ii) *The map $H^0(Z, L_d) \rightarrow H^0(\overline{Z}, L_d)$ is surjective, equivalently,*

$$H^1(Z, L_d - A) = 0.$$

(iii) *$H^1(X, I_{Z/X}(L_d - A)) = 0$ (or equivalently, with W' chosen below, the map $V' \rightarrow W'$ is surjective.)*

(iv) *The map $H^0(X, L_d) \rightarrow H^0(\overline{X}, L_d)$ is surjective, or equivalently*

$$H^1(X, L_d - A) = 0.$$

(v) *$I_{Z/X} \otimes \mathcal{O}_X(L_d)$ is globally generated.*

Proof. These all follow from Serre vanishing. \square

Proposition IV.13. *Fix $1 \leq q \leq n$. Suppose there exists a subscheme Z of X adapted to X, B, A, n, q . Take $W_d = H^0(X, \mathcal{O}_Z(L_d))$. Then for $d \gg 0$, W_d carries weight q syzygies for B .*

Proof. Start with $W = W_d = H^0(Z, L_d)$. By definition in equation (IV.3.7), (IV.3.8) and surjectivity from Prop. IV.12, we have:

(IV.3.9)

$$\begin{array}{ccccccc}
 0 & \longrightarrow & V' = H^0(X, L_d - A) & \longrightarrow & V = H^0(X, L_d) & \longrightarrow & \bar{V} = H^0(\bar{X}, L_d) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & W' & \longrightarrow & W = H^0(Z, L_d) & \longrightarrow & \bar{W} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where

$$(IV.3.10) \quad W' = H^0(Z, L_d - A), \quad \bar{W} = H^0(\bar{Z}, L_d).$$

The sheaf $I_{Z/X} \otimes \mathcal{O}_X(L_d)$ is globally generated (Rmk. IV.12 (v)) so $Z = \mathbb{P}(W) \cap X$. Moreover, when we cut down by hyperplanes as in Lemma IV.11, we obtain the corresponding diagrams in lower dimensions.

We prove the Proposition by induction on q . Z is always of dimension $q-1$. When $q = 1$, Z consists of points. If X is of dimension 1, then surjectivity follows from the fact that sheaf surjective maps imply surjectivity in H^1 since there is no H^2 . If the dimension of X is at least 2, then we continue the induction with $\bar{Z} = \emptyset, \bar{W} = 0$. So the conclusion is trivially true for $q = 0$. Then the conclusion is true for $q = 1$ by Rmk. IV.12 (ii) and Lemma IV.11. Then apply Lemma IV.11 repeatedly. \square

IV.4 Enlarged secant space

In this section, our key conclusion (Prop. IV.17) is the following. Let $\text{wts}(U)$ denote the torus weights in a toric representation U . In the setting of syzygies, we prove that there exists torus stable quotient spaces $W_{0,d}, W_{1,d}$:

$$H^0(X, L_d) \twoheadrightarrow W_{1,d} \twoheadrightarrow W_{0,d},$$

with:

$$\dim(W_{0,d}) \in o(d^n), \text{ and } h^0(X, L_d) - W_{1,d} \in o(d^n),$$

such that for any W_d with:

$$\dim W_d = p + q, \text{ and } W_{1,d} \twoheadrightarrow W_d \twoheadrightarrow W_{0,d},$$

we have:

$$\text{weights}(\wedge^{p+q} W_d) + \text{some weight} \subset \text{weights}(K_{p,q}(X; L_d)).$$

Lemma IV.14. *Let X be a scheme with A a very ample divisor and $L_d = dA$. Let Z be a subscheme. $\{E_i\}$ a collection of divisors such that $Z = \cap_i E_i$. Then there exists a subspace $\mathcal{J}_d \subset H^0(X, L_d)$ such that \mathcal{J}_d generates $\mathcal{I}_{Z/X}(L_d)$ and $\dim \mathcal{J}_d \in O(1)$.*

Proof. If $\mathcal{J}_d^i \subset H^0(X, L_d)$ generates $\mathcal{I}_{E_i}(L_d)$, then $\sum_i \mathcal{J}_d^i$ generate $\mathcal{I}_{Z/X}(L_d)$, so we can assume $Z = E$ is a divisor. For some large N , $\mathcal{I}_E(L_N)$ is globally generated by sections F_1, \dots, F_{m_1} . Assume A is globally generated by sections s_1, \dots, s_{m_2} . Then for $d > N$, $\mathcal{I}_E(L_d) = \mathcal{O}_X(L_d - E)$ is globally generated by $\mathcal{J}_d = \langle F_i \cdot s_j^{d-N} \rangle$. It has finitely many vector space generators, so $\dim \mathcal{J}_d$ is finite. \square

Remark IV.15. It is straightforward to see that the above lemma is still true if we start with torus equivariant objects and want torus equivariant \mathcal{J}_d 's.

Proposition IV.16. *There exist quotients $W_{0,d}, W_{1,d}$ of $H^0(X, L_d)$, such that*

$$(IV.4.1) \quad \overline{W_{0,d}} = \overline{W_{1,d}} = \overline{W_d}, \quad Z = X \cap \mathbb{P}(W_{0,d}) = X \cap \mathbb{P}(W_{1,d})$$

and the dimensions satisfy

$$\dim W_{0,d} \in o(h^0(X, L_d))$$

and

$$\lim_{d \rightarrow \infty} \frac{\dim W_{1,d}}{h^0(X, L_d)} = 1$$

Moreover, $W_{0,d}$ and $W_{1,d}$ carry weight q syzygies for \mathcal{O}_X .

Proof. Pick $W_{0,d} = W_d$. This satisfies the conditions. To construct $W_{1,d}$, we will vary W_d while keeping Z and $\overline{W_d}$ the same, i.e. we look for large dimension quotients $W_{1,d}$ of $H^0(X, L_d)$ such that:

$$(IV.4.2) \quad Z = X \cap \mathbb{P}(W_{1,d})$$

and

$$(IV.4.3) \quad \overline{W_{1,d}} = \overline{W_{0,d}}.$$

By the argument of Prop. IV.13, if $W_{1,d}$ satisfy the above conditions, $W_{1,d}$ also carries weight q syzygies. We first construct $W_{1,d}$ such that they satisfy the conditions in (IV.4.2) and (IV.4.3) and then do a dimension count.

Let $W_{1,d}$ be the quotient of $V_d = H^0(X, L_d)$ by $\mathcal{J}_{1,d}$, i.e.

$$W_{1,d} = V_d / \mathcal{J}_{1,d}.$$

To satisfy (IV.4.2), $\mathcal{J}_{1,d}$ has to generate $\mathcal{I}_{Z/X}(L_d)$. By Lemma IV.14, we can pick $\mathcal{J}_{1,d}$ of bounded dimension. To satisfy (IV.4.3), for the consecutive quotient maps:

$$V_d \rightarrow W_{1,d} \rightarrow W_d,$$

we need:

$$(IV.4.4) \quad V'_d + J_d = V'_d + \mathcal{J}_{1,d}$$

where

$$V'_d = \ker(V_d \rightarrow \overline{V_d}), \quad J_d = \ker(V_d \rightarrow W_d).$$

Note that:

$$\begin{aligned} \dim(V'_d + J_d) - \dim V'_d &= \dim V_d - \dim \overline{W_d} - \dim V'_d \\ &= \dim \overline{V_d} - \dim \overline{W_d} \\ &\leq \dim \overline{V_d}. \end{aligned}$$

Hence (IV.4.4) requires $\mathcal{J}_{1,d}$ to be appropriate subspaces of $V'_d + J_d$ with the following range of dimensions:

$$\dim \overline{V_d} \leq \dim \mathcal{J}_{1,d} \leq \dim(V'_d + J_d).$$

Note that $\dim \overline{V_d} \in o(d^n)$. Therefore, to satisfy both (IV.4.2) and (IV.4.3), we can choose $\mathcal{J}_{1,d}$ such that $\dim \mathcal{J}_{1,d} \in o(d^n) + O(1)$. Since $W_{1,d} = V_d / \mathcal{J}_{1,d}$, we have $\dim W_{1,d} = \dim V_d - \dim \mathcal{J}_{1,d}$. Then $h^0(X, L_d) \in \Theta(d^n)$ imply that

$$\lim_{d \rightarrow \infty} \frac{\dim W_{1,d}}{h^0(X, L_d)} = 1.$$

□

Finally, we conclude:

Proposition IV.17. *For any torus equivariant W_d that fits in the following diagram of consecutive quotient maps:*

$$W_{1,d} \rightarrow W_d \rightarrow W_{0,d}$$

we have:

$$(IV.4.5) \quad \text{wts}\left(\bigwedge^{\dim W_d} W_d\right) + \text{wts}(H^q(X, I_{Z/X})) \subseteq \text{wts}(K_{\dim W_d - q, q}(X; L_d))$$

Proof. By the argument of Prop. IV.13, W_d carries weight q syzygies. Then the weight inclusions follow from Lemma IV.3. \square

IV.5 Asymptotics of normalized weights

Recall that we defined in the introduction:

$$\text{wts}(K_{p,q}(X; L_d)) = \{\text{Torus weights of } K_{p,q}(X; L_d)\} \subseteq (p+q)d \cdot \Delta.$$

where Δ is the convex polytope associated with the very ample divisor A . We are interested in the normalized weights:

$$\text{wts}^{\text{nor}}(K_{p,q}(X; L_d)) = \frac{\text{wts}(K_{p,q}(X; L_d))}{(p+q)d} \subseteq \Delta$$

In this section, we work asymptotically and we are interested in asymptotic closures:

$$\Delta(a, b) = \overline{\bigcup_{\substack{d \gg 0 \\ a \cdot r_d \leq p \leq b \cdot r_d}} \text{wts}^{\text{nor}}(K_{p,q}(X; L_d))} \subseteq \Delta$$

Then the contributions from weights in $H^q(X, I_{Z/X})$ and $W_{0,d}$ will be asymptotically insignificant, to normalized weights, in other words:

$$\overline{\bigcup_{\substack{d \gg 0 \\ a \cdot r_d \leq p \leq b \cdot r_d}} \text{wts}^{\text{nor}}\left(\bigwedge^{\dim W_d} W_d + H^q(X, I_{Z/X})\right)} = \overline{\bigcup_{\substack{d \gg 0 \\ a \cdot r_d \leq p \leq b \cdot r_d}} \text{wts}^{\text{nor}}\left(\bigwedge^p W_{1,d}\right)}$$

Hence, we can just work with the sequence $\{W_{1,d}\}$ in (IV.4.5) and for simplicity, we abuse notation and write W_d instead of $W_{1,d}$.

After this simplification, we arrive at the following setting. Let Δ be the convex polytope associated to a very ample divisor A in \mathbb{R}^n . For $d \in \mathbb{N}$, let

$$W_d \subset d\Delta \cap \mathbb{Z}^n.$$

Let us use $\bigwedge^{p_d} W_d$ to denote the collection of points in \mathbb{Z}^n expressible as nonrepetitive sums of p_d points in W_d . Assume that for $d \in \mathbb{N}$,

$$(IV.5.1) \quad \lim_{d \rightarrow \infty} \frac{|W_d|}{|d\Delta|} = 1$$

Take any point x inside the polytope Δ , suppose the largest sphere contained in Δ centered at x is S_x and has volume η_x .

The intuition for the two key lemmas (IV.18, IV.20) in this section are as follows. In Lemma IV.18, we prove that if there is a ball of volume η_x centered at x , we can pick any number between 1 and $(\eta_x - \epsilon)d^n$ such that the the average of these weights lie arbitrarily close to x asymptotically. In Lemma IV.20, we prove that too many points (exceeding the number of lattice points in dilations of maximal shapes centered at x) will not average close to x .

Lemma IV.18. *With the above notations, for any sequence*

$$1 \leq p_d \leq (\eta_x - \epsilon)d^n$$

and any open set $U_x \subset \mathbb{R}^n$ containing x , there exists d_0 such that for all $d > d_0$, U_x contains a point of

$$\frac{\text{wts}^{\text{nor}}(\bigwedge^{p_d} W_d)}{p_d \cdot d}$$

Proof. We will pick w_1, \dots, w_{p_d} in W_d and prove that they give rise to points in U_x . By the general theory of counting integral points via quasipolynomials [M], and approximating an arbitrary polytope with rational polytopes, given any full-dimensional convex polytope Γ ,

$$\lim_{d \rightarrow \infty} \frac{|d\Gamma \cap \mathbb{Z}^n|}{d^n} = \text{vol}(\Gamma)$$

For all d such that

$$p_d \leq \frac{1}{2}|(dU_x) \cap \mathbb{Z}^n|,$$

Notice that there are at least p_d points in $dU_x \cap W_d$ since otherwise,

$$\begin{aligned} |d\Delta - W_d| &\geq |(dU_x) \cap \mathbb{Z}^n - (dU_x) \cap W_d| \\ &\geq |dU_x \cap \mathbb{Z}^n| - p_d \\ &\geq \frac{1}{2}|(dU_x) \cap \mathbb{Z}^n|. \end{aligned}$$

The last term has order of growth d^n , that would be a contradiction to equation (IV.5.1) above. Therefore, we can choose p_d distinct points w_1, \dots, w_{p_d} from $(dU_x) \cap \mathbb{Z}^n$. Then, in $\frac{\Delta^{p_d} W_d}{p_d d}$, they correspond to the point

$$\frac{\sum_{i=1}^{p_d} w_i}{p_d d}$$

Each w_i is in dU_x , hence $\frac{w_i}{d}$ is in U_x . We can write the above sum as their average:

$$\frac{1}{p_d} \sum_{i=1}^{p_d} \frac{w_i}{d}$$

and it is evidently in U_x .

For all d such that:

$$p_d > \frac{1}{2}|(dU_x) \cap \mathbb{Z}^n|$$

Let $\lceil n \rceil$ denote the smallest integer greater than or equal to n . Consider the set of points

$$J_d = \left\{ \left(\lceil (p_d)^{1/n} \rceil \frac{S_x}{\text{vol}(S_x)^{1/n}} \right) \cap \mathbb{Z}^n \right\}$$

The intuition is that $\frac{S_x}{\text{vol}(S_x)^{1/n}}$ has volume 1, multiplied by $\lceil p_d^{1/n} \rceil$, it should have asymptotically on the growth of p_d many points, which we assert as (1) below and list some other estimates that are going to be useful. For large d , the following are true:

(i) We have:

$$\lim_{d \rightarrow \infty} \frac{|J_d|}{p_d} = 1, \quad |J_d| - p_d \in o(d^n)$$

In fact, given the lower bound on p_d and the upper bound $|d\Delta \cap \mathbb{Z}^n|$ for p_d , we know that $p_d \in \Theta(d)$. Hence, the first equality implies the second. By the existence of Ehrhart quasi-polynomials, approximation with rational polytopes and the fact that $\frac{S_x}{\text{vol}(S_x)^{1/n}}$ has volume 1, we have:

$$|J_d| = \lceil p_d^{1/n} \rceil^n + o(d^n).$$

Hence,

$$\lim_{d \rightarrow \infty} \frac{|J_d|}{p_d} = \lim_{d \rightarrow \infty} \frac{\lceil p_d^{1/n} \rceil^n}{p_d} = 1$$

(ii) Furthermore, it is evident that:

$$|J_d| - |J_d \cap W_d| \leq |d\Delta| - |d\Delta - W_d| \in o(d^n)$$

Hence:

$$\left| |J_d \cap W_d| - p_d \right| \leq \left| |J_d| - |J_d \cap W_d| \right| + \left| |J_d| - p_d \right| \in o(d^n).$$

Let $q_d = |J_d \cap W_d|$. Then the above inequality says

$$0 \leq p_d - q_d \in o(d^n).$$

(iii) The minimum distance between points in $\frac{J_d}{d}$ is $\frac{1}{d}$, which goes to 0. Hence, the sum:

$$\frac{1}{|J_d|} \sum_{w \in J_d} \frac{w}{d}$$

approaches the integral which defines the center of mass of S_x .

(iv) One has:

$$dS_x \cap W_d > p_d$$

because otherwise,

$$\begin{aligned} |W_d| &= |W_d \cap dS_x| + |W_d \cap \overline{dS_x}| \leq p_d + |\overline{dS_x}| \\ &\leq |dS_x| + |\overline{dS_x}| - (|dS_x| - p_d) \\ &\leq |d\Delta| - \epsilon d^n \end{aligned}$$

where $\overline{dS_x} = d\Delta - dS_x$. This is a contradiction to (IV.5.1).

To finish showing existence of a point in U_x in the case of big p_d 's, we take all the points in $J_d \cap W_d$ to be w_1, \dots, w_{q_d} . Pick any of the rest $p_d - q_d$ points from $dS_x \cap W_d - J_d \cap W_d$ (in fact (c) above, we showed that we have enough points to do

this). Then:

$$\begin{aligned}
& \left| \frac{\sum_{i=1}^{p_d} w_i}{p_d d} - x \right| = \left| \frac{1}{dp_d} \left(\sum_{i=1}^{p_d} w_i \right) - x \right| \\
& \leq \left| \frac{1}{dp_d} \left(\sum_{i \in J_d \cap W_d} w_i \right) - x \right| + \left| \frac{1}{dp_d} \left(\sum_{i \in W_d - J} w_i \right) \right| \\
& \leq \left| \frac{1}{dp_d} \left(\sum_{i \in J_d} w_i \right) - x \right| + \left| \frac{1}{dp_d} \left(\sum_{i \in W_d - J_d} w_i \right) \right| + \left| \frac{1}{dp_d} \left(\sum_{i \in W_d - J_d} w_i \right) \right| \\
& \leq \left| \frac{|J_d|}{dp_d} \frac{1}{|J_d|} \left(\sum_{i \in J_d} w_i \right) - \frac{|J_d|}{dp_d} x \right| + \left| \frac{|J_d|}{dp_d} x - x \right| + 2 \left| \frac{1}{dp_d} \left(\sum_{i \in W_d - J_d} w_i \right) \right| \\
& \leq \frac{|J_d|}{dp_d} \left| \frac{1}{|J_d|} \left(\sum_{i \in J_d} w_i \right) - x \right| + \left| \frac{|J_d|}{dp_d} - 1 \right| |x| + 2 \frac{1}{p_d} \left(\sum_{i \in W_d - J_d} \frac{|w_i|}{d} \right)
\end{aligned}$$

$|x|$ and $\frac{|w_i|}{d}$ are both bounded as they lie in the polytope Δ . Suppose they are bounded by $\Gamma > 0$, i.e.

$$|x| \leq \Gamma, \quad \frac{|w_i|}{d} \leq \Gamma$$

Then the above sum is bounde by:

$$\begin{aligned}
\left| \frac{\sum_{i=1}^{p_d} w_i}{p_d d} - x \right| & \leq \frac{|J_d|}{dp_d} \left| \frac{1}{|J_d|} \left(\sum_{i \in J_d} w_i \right) - x \right| + \left| \frac{|J_d|}{dp_d} - 1 \right| \Gamma + 2 \frac{|W_d - J_d|}{p_d} \Gamma \\
& \leq \frac{|J_d|}{dp_d} \left| \frac{1}{|J_d|} \left(\sum_{i \in J_d} w_i \right) - x \right| + \left| \frac{|J_d|}{dp_d} - 1 \right| \Gamma + 2 \frac{p_d - q_d}{p_d} \Gamma
\end{aligned}$$

As previously discussed, when d is large, $\frac{|J_d|}{dp_d}$ tends to 1, $\left| \frac{1}{|J_d|} \left(\sum_{i \in J_d} w_i \right) - x \right|$ tends to 0. $p_d - q_d \in o(d^n)$, p_d is bounded below by $\frac{1}{2} |(p_d d U_x) \cap \mathbb{Z}^n|$ which has order of growth d^n , hence the last quotient tends to 0. Therefore, the whole sum tends to 0 and we get arbitrarily close to x . \square

Theorem IV.19. Fix $1 \leq q \leq n$. Then

$$\bigcup_{\substack{d > 0 \\ 1 \leq p \leq r_d}} \text{wts}^{\text{nor}}(K_{p,q}(X; L_d))$$

is dense in Δ .

Proof. By Prop. IV.16 and IV.17, we know that $\text{wts}^{\text{nor}}(K_{p,q}(\mathbb{P}; L_d))$ contains weights corresponding to nonrepetitive sums of weights in $W_{1,d}$. By Prop. IV.16 and Lemma IV.18, they are dense in Δ . \square

Take any point x inside the polytope Δ . Recall we denote:

$$\tau_x = \sup \{ \text{vol}(\mathbb{S}_x) \mid \mathbb{S}_x \text{ is a finite union of cubes } \subset \Delta, \text{ center of mass of } \mathbb{S}_x = x \}$$

Lemma IV.20. Take $y \in \Delta$. Assume

$$(IV.5.2) \quad y = \lim_{d \rightarrow \infty} y_d$$

where

$$y_d = \frac{1}{i_d} \sum_{1 \leq j \leq i_d} \frac{w_{d,i}}{d}, \quad w_{d,i} \in (d \cdot \Delta) \cap \mathbb{Z}^n, \quad w_{d,i} \text{ distinct.}$$

Then for any subsequence $\{i_{d_j}\}$:

$$\limsup_{j \rightarrow \infty} \frac{i_{d_j}}{d^n} \leq \tau_y$$

Proof. For simplicity, we call the finite union of cubes $\subset \Delta$ a shape. Suppose there is a subsequence $i_{d'_j}$ such that

$$\liminf_{j \rightarrow \infty} \frac{i_{d'_j}}{d^n} \geq \tau,$$

we claim that for any constant $\epsilon > 0$, there is a shape centered at y with volume at least $\tau - \epsilon$. Assuming the claim, the lemma follows since if we had

$$\limsup_{j \rightarrow \infty} \frac{i_{d_j}}{d^n} = \tau > \tau_y,$$

there is a subsequence $i_{d'_j}$ such that

$$\liminf_{j \rightarrow \infty} \frac{i_{d'_j}}{d^n} = \frac{\tau + \tau_y}{2} > \tau_y$$

By the claim, this is a shape with center of mass at y and volume $\frac{\tau + \tau_y}{2} > \tau_y$.

Contradiction to τ_y being the biggest volume supported at y .

Let's turn to the claim. For convenience, we write i_j for i_{d_j} . We construct a shape centered at y with volume at least $\tau - \epsilon$ by taking small cubes centered at $w_{d_j,i}$, then make two adjustments: boundary adjustment and center adjustment.

More specifically, for each $w_{d_j,i}$ with $1 \leq i \leq d_j$, take the $1 \times \dots \times 1$ cubes of dimension n centered at $w_{d_j,i}$. Denote each cube by $C_{d_j,i}$. Suppose the $2 \times \dots \times 2$ cube centered at $w_{d_j,i}$ does not intersect the boundary of $d\Delta$ for $1 \leq i \leq i'_d$. The boundary of $d\Delta$ is bounded by $d(1 - \epsilon)\Delta$ and $d(1 + \epsilon)\Delta$ for large d and any $\epsilon > 0$. Hence, $i_d - i'_d \in o(d^n)$.

Take the union:

$$\Sigma'_d = \bigcup_{i=1}^{i'_d} C_{d_j,i}$$

and denote by y' the center of mass for Σ'_d . Δ is bounded, so by the previous paragraph and assumption:

$$|y' - y| \leq |y' - y_d| + |y_d - y| \in o(d^n) + O(1).$$

Move Σ'_d by:

$$\frac{y - y'}{i'_d}$$

Then it is straightforward to see that for large d , the shape above is contained in Δ and has volume arbitrarily close to τ . \square

Recall that we define:

$$\Delta(a, b) = \overline{\bigcup_{\substack{d \gg 0 \\ a \cdot r_d \leq p \leq b \cdot r_d}} \text{wts}^{\text{nor}}(K_{p,q}(\mathbb{P}; L_d))} \subseteq \Delta$$

Theorem IV.21. *One has:*

$$\Delta(a, b) = \left\{ x \in \Delta \mid \frac{\tau_x}{\text{vol}(\Delta)} \geq a \right\} =: \Delta(a)$$

Proof. The closure $\Delta(a, b) \supseteq \left\{ x \in \Delta \mid \frac{\tau_x}{\text{vol}(\Delta)} \geq a \right\}$ follows from a similar argument as in Lemma IV.19 replacing a sphere with shapes supported at x with volumes arbitrarily close to τ_x . For sequences asymptotically small, we can pick points in a sphere around x . For sequences asymptotically close to τ_x , we can pick points from finite cube unions (possibly further scaled down) approximating τ_x .

Recall Lemma II.1, $K_{p,q}(X; L_d)$ can be computed as cohomology in the middle of the following short complex:

$$\begin{aligned} \wedge^{p+1} H^0(X, L_d) \otimes H^0(X, (q-1)L_d) &\rightarrow \wedge^p H^0(X, L_d) \otimes H^0(X, qL_d) \\ &\rightarrow \wedge^{p-1} H^0(X, L_d) \otimes H^0(X, (q+1)L_d) \end{aligned}$$

Hence all weights of $K_{p,q}(X, L_d)$ correspond to nonrepetitive sums of points in $(p+q)\Delta$. The length of the above Koszul complex is $h^0(X, L_d) = \text{vol}(\Delta) \cdot d^n + O(d^{n-1})$. q is bounded, so asymptotically insignificant. By Lemma IV.20, if a normalized weight is in

$$\overline{\text{wts}^{\text{nor}}(\wedge^{p_d} H^0(X, L_d) \otimes H^0(X, qL_d))},$$

then it satisfies

$$\tau_x \geq \limsup_{d \rightarrow \infty} \frac{p_d}{d^n}$$

We are computing for the range $[a\text{vol}(\Delta)d^n, b\text{vol}(\Delta)d^n]$, hence

$$\limsup_{d \rightarrow \infty} \frac{p_d}{d^n} \geq b\text{vol}(\Delta).$$

so

$$x \geq a\text{vol}(\Delta).$$

Then by the definition of $K_{p,q}(X, L_d)$ above:

$$\begin{aligned} & \overline{\text{wts}^{\text{nor}}(K_{p,q}(X, L_d))} \\ & \subseteq \overline{\text{wts}^{\text{nor}}(\wedge^p H^0(X, L_d) \otimes H^0(X, qL_d))} \subseteq \left\{ x \in \Delta \mid \frac{\tau_x}{\text{vol}(\Delta)} \geq a \right\} \end{aligned}$$

which gives us the other inclusion. \square

IV.6 Boundary of $\Delta(a)$

In this section, we describe the boundary of $\Delta(a)$.

Fix a convex body $\Delta \subset \mathbb{R}^n$, and a constant a in $[0, 1]$. Let v be a unit vector in \mathbb{R}^n . Then $H(v, c) := \{x \in \mathbb{R}^n \mid v \cdot x \leq c\}$ forms a family of parallel half spaces. There is a unique constant c so that:

$$\text{vol}(\Delta \cap H(v, c)) = a\text{vol}(\Delta).$$

Call this constant c_v . Let x_v be the center of gravity of $\Delta \cap H(v, c_v)$.

Proposition IV.22. *The points x_v , as v ranges over all unit vectors, form the boundary of $\Delta(a)$.*

Remark IV.23. We give the intuition but skip the proof of the proposition. Think of Δ as a container holding water of volume a . When one leans it to the direction with v pointing downward, water flows to lower its center of mass (potential energy). In this position, the center of mass, call it x_v , is the extreme of Δ in direction v , hence a boundary point of $\Delta(a)$. The water level in this setting corresponds to the boundary of $H(v, c_v)$ and we can actually see that the tangent of the boundary is the hyperplane perpendicular to v through x_v .

Example IV.24. In the example given in the introduction, when Δ is the unit square and $a = \frac{1}{10}$. The interested reader can work out that when the water surface divides the square into a triangle and a pentagon, the center of mass of the water is parametrized by

$$(1 - \frac{1}{15k}, \frac{1}{3}k) \text{ for } \frac{1}{5} \leq k \leq 1$$

and its symmetric images, these account for the 4 hyperbola segments. When the water surface divides the unit square into two trapezoids, the center of mass lies on:

$$(\frac{2}{3} - \frac{5}{3}k, \frac{29}{30} - \frac{1}{6}k) \text{ for } \frac{1}{10} \leq k \leq \frac{1}{5}$$

or its symmetric images. These account for the 8 line segments.

BIBLIOGRAPHY

BIBLIOGRAPHY

- [AF11] M. Aprodu and G. Farkas, *The Green Conjecture for smooth curves lying on arbitrary $K3$ surfaces*, Compositio Mathematica 147 (2011), 839–851.
- [AV03] M. Aprodu and C. Voisin. *Green-Lazarsfeld’s conjecture for generic curves of large gonality*, C. R. Acad. Sci. Paris 336 No. 4 (2003) 335–339.
- [B03] A. Beauville, *La conjecture de Green gnrrique*, Sminaire Bourbaki 46 (2003-2004): 1–14.
- [BCI11] P. Bürgisser, M. Christandl, C. Ikenmeyer, *Even partitions in plethysms*, Journal of Algebra 328 (2011), 322–329.
- [BCR09] W. Bruns, A. Conca, T. Römer, *Koszul cycles, in combinatorial aspects of Commutative Algebra and Algebraic Geometry*. Proceedings of the Abel Symposium (2009), 17–32.
- [BCR11] W. Bruns, A. Conca, T. Römer, *Koszul homology and syzygies of Veronese subalgebras*, Math. Ann. **351** (2011), 761–779.
- [BZ89] A. Berenstein, A. Zelevinski, *Tensor product multiplicities and convex polytopes in partition space*, J. Geom. Phys. **5**, no.3 (1989), 453–472.
- [C1893] G. Castelnuovo, *Sui multipli di uni serie lineare di gruppi di punti apparetmenente as una curva algebrica*, Rend. Circ. Mat. Palermo 7 (1893), 99–119.
- [CLS11] D. Cox, J. Little, H. Schenck, *Toric Varieties, Graduate Studies in Mathematics*, Amer. Math. Soc., Providence, RI, (2011).
- [EEL12] L. Ein, D. Erman and R. Lazarsfeld, *Asymptotics of random Betti tables*, to appear in Crelle.
- [EL12] L. Ein, R. Lazarsfeld, *Asymptotic syzygies of algebraic varieties*, , Invent. Math. 190 (2012), pp. 603-646.
- [EL93] L. Ein and R. Lazarsfeld, *Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension*, Invent. Math. **111** (1993), 51-67.
- [F93] W. Fulton, *Introduction to Toric Varieties, Princeton University Press*, Princeton, NJ., 1993.
- [F97] W. Fulton, *Young tableaux, With applications to Representation Theory and Geometry*, Cambridge Univ. Press (1997).
- [FH91] W. Fulton, J. Harris, *Representation theory, a first course*, Springer–Verlag, New York (1991).
- [FZ13] M. Fulger and X. Zhou, *Schur asymptotics of Veronese syzygies*, arXiv:1209.4007 [math.AG].

- [G84I] M. Green, *Koszul cohomology and the geometry of projective varieties*, J. Diff. Geom. 19 (1984), 125–171.
- [G84II] M. Green, *Koszul cohomology and the geometry of projective varieties, II*, J. Diff. Geom. 20 (1984), 279–289.
- [GL85] M. Green and R. Lazarsfeld, *A simple proof of Petri’s theorem on canonical curves*, in Geometry Today, Progress in Math. 60, (1985), pp. 129–142.
- [Gr97] F. Grosshans, *Algebraic homogeneous spaces and invariant theory*, Springer Berlin / Heidelberg (1997).
- [H99] G. H. Hardy, *Ramanujan: Twelve lectures on subjects suggested by his life and work*, 3rd ed., Chelsea Publ. Co., New York (1999), Ch.6 83–100, and Ch.8 113–131.
- [Ho89] R. Howe, *Asymptotics of dimensions of invariants for finite groups*, J. Algebra 122 (1989), 374–379.
- [HSS06] M. Hering, H. Schenck and G. Smith, *Syzygies, multigraded regularity and toric varieties*, Compos. Math. 142 (2006), 1499–1506.
- [K10] K. Kaveh, *A remark on asymptotic of highest weights in tensor powers of a representation*, arXiv:1002.1984v1 [math.RT] (2010).
- [KK12] K. Kaveh, A. G. Khovanskii, *Convex bodies associated to actions of reductive groups*, arXiv:1001.4830v2 [math.AG] (2012).
- [Kh93] A. G. Khovanskii, *Newton polyhedron, Hilbert polynomial and sums of finite sets*, Funct. Anal. Appl. 26 (1993), 276–281.
- [L89] R. Lazarsfeld *A sampling of vector bundle techniques in the study of linear series*, in M. Cornalba et. al. (eds.), Lectures on Riemann Surfaces, (1989) pp. 500–560.
- [LM09] R. Lazarsfeld, M. Mustață, *Convex bodies associated to linear series*, Ann. Sci. Éc. Norm. Supér. (4) 42, no.5 (2009), 783–835.
- [M] M. Mustata, *Lecture notes on toric varieties*, Chapter 13. Ehrhart polynomials.
- [M66] D. Mumford, *On the equations dening abelian varieties*, Invent. Math. 1 (1966), 287–354.
- [M70] D. Mumford, *Varieties dened by quadratic equations*, Corso CIME 1969, in Questions on algebraic varieties, Rome (1970), 30 –100.
- [Man97] L. Manivel, *Applications de Gauss et pléthysme*, Ann. de l’institut Fourier, 47, no.3 (1997), 715–773.
- [MM12] L. Manivel, M. Michalek, *Effective constructions in plethysms and Weintraub’s conjecture*, arXiv:1207.5748v1 [math.RT] (2012).
- [New51] M. J. Newell, *A theorem on the plethysm of S-functions*, Quart. J. Math. (1951), 161–166.
- [N11] I. Netay, *Syzygy algebras for the Segre embedding*, arXiv:1108.3733 [math.AG].
- [O96] A. Okounkov, *Brunn–Minkowski inequality for multiplicities*, Invent. Math. 125, no.3 (1996), 405–411.
- [OP01] G. Ottaviani, R. Paoletti, *Syzygies of Veronese embeddings*, Compos. Math. (2001), 31–37.

- [OR05] G. Ottaviani, E. Rubei, *Resolutions of homogeneous bundles on \mathbb{P}^2* , Ann. Inst. Fourier (Grenoble) 55, no.3 (2005), 973–1015.
- [Pa00] G. Pareschi, *Syzygies of abelian varieties*, Journal of the AMS 13 (2000), 651–664.
- [P05] R. Paoletti, *The asymptotic growth of equivariant sections of positive and big line bundles*, Rocky Mountain J. Math. 35, 6 (2005), 2089–2105.
- [R13] C. Raicu, *Representation stability for syzygies of line bundles on Segre–Veronese varieties*, arXiv:1209.1183 [math.AG] (2012).
- [Ru04] E. Rubei, *A result on resolutions of Veronese embeddings*, Ann. Univ. Ferrar Sez. VII 50 (2004), 151–165.
- [S13] A. Snowden, *Syzygies of Segre embeddings and Delta-modules*, Duke Math Journal 162 (2013), no. 2, 225–277.
- [Sta99] R. P. Stanley, *Enumerative combinatorics, Vol.2.*, Cambridge Univ. Press, Cambridge (1999).
- [Sch86] F. Schreyer, *Syzygies of canonical curves and special linear series*, Math. Ann. 275 (1986), 105–137.
- [TZ04] T. Tate, S. Zelditch, *Lattice path combinatorics and asymptotics of multiplicities of weights in tensor powers*, J. Funct. Anal. 217, no.2 (2004), 402–447.
- [V02] C. Voison, *Green’s conjecture for curves of even genus lying on a K3 surface*, J.Eur. Math. Soc. 4, (2002) 363–404 .
- [V05] C. Voison, *Green’s canonical syzygy conjecture for generic curves of odd genus*, Compos. Math. 141 no. 5, (2005) 1163–1190 .
- [W90] S. H. Weintraub, *Some observations on plethysms*, J. Algebra 129 (1) (1990), 103–114.
- [Z12] X. Zhou, *Effective non-vanishing of asymptotic adjoint syzygies*, to appear in Proc. of AMS.