Steady and Self-Similar Solutions to Two-Dimensional Hyperbolic Conservation Laws

by

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To my parents and sister

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CHAPTER I

Introduction

1.1 Preliminaries

A *system of conservation laws* is a system of nonlinear partial differential equations of the form

$$U_t + \sum_{i=1}^{n} f^i(U)_{x^i} = 0.$$

The unknown U is a function of $t \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$ and takes values in $\mathbb{R}^m$, and the flux functions $f^i, i = 1, ..., n$ also take values in $\mathbb{R}^m$. In the above and throughout, subscripts indicate partial differentiation with respect to the respective variables. We are particularly interested in the *two-dimensional compressible isentropic Euler equations*, which take the form

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \end{pmatrix}_y = 0,$$

where ρ is the density, ρu and ρv are the horizontal and vertical momentum densities, respectively, and p is the pressure. The system is closed with an equation of state $p = p(\rho)$. We are also interested in the *two-dimensional compressible full Euler* *equations*, which take the form

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(\rho E + p) \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(\rho E + p) \end{pmatrix}_y = 0,$$

where E is the total energy (kinetic plus internal) per unit mass. In this case the equation of state closing the system gives pressure as a function of density and internal energy. A common choice is the *polytropic* equation of state, in which

$$p = (\gamma - 1)\rho e,$$

where the constant γ is the *adiabatic exponent*, equal to 1.4 for air.

The Euler equations are believed to be one of the earliest partial differential equations to be written explicitly. The earliest was the one dimensional wave equation, developed by D'Alembert in 1749. The Euler equations were next, first published by Euler in 1757, though he presented a preliminary version of the incompressible equations in 1752. Since their discovery they have been extensively studied, yet our understanding is far from complete. See [10] for an overview of the history of the Euler equations that was published on the 250th anniversary of their formulation.

Existence and uniqueness of solutions to initial value problems are of primary interest in the study of conservation laws. Given smooth initial data $U(0, \vec{x})$, we would like to have a smooth solution at positive times. Using the classical theory of linear hyperbolic systems (see the discussion in [34]), it can be shown that a unique smooth solution exists locally in time. However, singularities can develop in *finite* time from smooth initial data. This is due to the fact that the propagation speed of information in the solution depends on the solution itself, and in the absence of diffusion smooth transitions can become steeper and steeper until discontinuities form. These discontinuities are known as *shock waves*. In gas dynamics experiments, these transitions are smooth and steep due to the diffusive effect of viscosity, but the first order conservation laws do not include these effects, and so the inviscid equations allow for these discontinuous solutions. Since these solutions are not differentiable, the differential equations must be interpreted in the integral or distributional sense.

Common experiments in gas dynamics study shock wave reflection phenomena — that is, how shock waves interact with solid boundaries — so as to better understand flow around an airfoil or inside a jet engine. The setting for many of these experiments is planar flow, so that at any given time the discontinuities lie on curves in the (x, y)-plane. Interesting behavior includes *regular reflection*, which is two shock waves meeting at a point on a solid wall (or four shocks meeting at a point in the plane), and *Mach reflection*, which is three shocks and a contact discontinuity meeting at a point. (A *contact discontinuity* is a curve of discontinuity through which the gas particles do not cross. The contact discontinuity could be a curve along which the gas slides past itself, or could separate regions with different temperatures and densities but at the same pressure so that the interface is not disturbed.) Regular reflection is studied in [8, 9, 14, 15, 16, 18, 25, 53], and Mach reflection is addressed in [2, 3, 27, 45, 47, 49]. See Figure 1.1 for an illustration of these types of shock reflections.

In either case, from the point of view of an observer moving at this interaction point, the flow is steady in time, and is to first order constant along rays emanating from this point. Whereas there are existence results for these and other certain cases, there is a famous non-existence result regarding *triple points* — that is, three shocks meeting at a point with smooth flow in between. This was originally investigated by

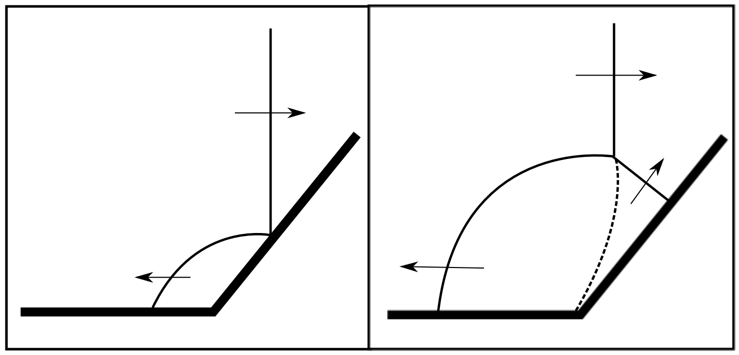

Figure 1.1: The initial configuration (not shown) is a vertical shock moving towards a ramp. In certain situations, a regular reflection (left) is produced. There are two shocks interacting at a point that moves up the ramp. For other parameters, a Mach reflection (right) is produced, and the interaction point is detached from the wall. It consists of three shocks and a contact discontinuity.

Von Neumann in 1943, and has been extended to more general equations of state (see [26, 50, 43]). It would be interesting to learn more about which configurations are possible or not possible. Therefore we will consider steady and self-similar solutions to two dimensional conservation laws — that is, solutions that satisfy

$$U(t,x,y) = U(\phi),$$

where ϕ is the standard polar angle in the plane.

This reduction is similar to the *Riemann problem* for a system of conservation laws in one dimension. A one-dimensional Riemann problem seeks solutions to

$$U_t + f(U)_x = 0, \tag{1.1}$$

with initial data

$$U(0,x) = \begin{cases} U_L, & x<0 \\ U_R, & x>0 \end{cases}. \tag{1.2}$$

Distributional solutions will not be unique unless an admissibility criterion is enforced, and requiring that an *entropy inequality* is satisfied is a typical choice. Since the Riemann problem is invariant under the transformation $(t, x) \mapsto (\alpha t, \alpha x)$ for $\alpha > 0$, the solutions that are sought are self-similar in the sense that they are functions of only x/t. The goal is then to connect U_L to U_R by a series of shocks, simple waves, or contact discontinuities with intermediate constant states in between. The following theorem was originally proved by Lax in 1957 for the *strictly hyperbolic* case (in which all eigenvalues are simple), and is also true for systems that are *non-strictly hyperbolic with constant multiplicity.*

Theorem I.1 (Lax [33, 42])**.** *Suppose*

- *that the* $m \times m$ *matrix* $f_U(U)$ *is diagonalizable for all* U *with* n *real eigenvalues of constant multiplicity,*
- *each eigenvalue is either genuinely nonlinear or linearly degenerate, and*
- $|U_R - U_L|$ *is sufficiently small.*

Then the Riemann problem consisting of (1.1) *and* (1.2) *has an entropy admissible solution for* $t > 0$ *that is self-similar and consists of up to* $n + 1$ *constant states, each close to the initial states, and they are successively connected to one another by a simple wave, a contact discontinuity, or a shock. Moreover, there is exactly one admissible solution with this preceding structure.*[1]

We will refer to this solution as Lax's solution.

Understanding the Riemann problem was fundamental in the development of the study of systems of conservation laws in one space dimension. In 1965, Glimm developed the random choice method [21], which proved global in time existence of admissible weak solutions, provided that the initial data has small total variation. The basic idea is to approximate the initial data by a piecewise constant function,

[1] We will define the notions of entropy admissibility, genuine nonlinearity, linear degeneracy, and simple waves in Chapter II.

use the Riemann solutions at each transition point for a small time (before any of the waves have a chance to interact), and then do the same procedure after a small time. Uniqueness of the random choice method was addressed in [7, 6, 37, 30]. In addition, numerical schemes such as the Gudonov scheme use exact Riemann solutions, and there is a wide collection of powerful approximate Riemann solvers as well (see for example [35]).

The natural extension of the Riemann problem to two dimensions would be to consider initial data constant along rays emanating from the origin — similar to the situation we consider, but instead considering unsteady solutions. To make this more tractable, usually the initial data is taken to be constant in several sectors instead of completely general self-similar data (see for example [29, 36, 52, 51]).

A more relevant motivation for this steady and self-similar reduction is found in [13], in which Elling found numerical evidence suggesting possible non-uniqueness for the initial value problem. Numerical simulations indicated an unsteady flow which took an analytical steady flow as initial data. Perhaps better understanding of the steady problem will lead to other similar examples of non-uniqueness or an analytical proof.

Another interesting question regards the appropriate function space for solutions of two-dimensional systems of conservation laws. Whereas the space of functions of bounded variation is ideal in one space dimension (it is the setting of Glimm's scheme mentioned above as well as in the vanishing viscosity results of Bressan and Bianchini in [4]), it is well known that BV is not appropriate for multidimensional conservation laws. In [40], Rauch showed that a necessary condition for BV estimates at positive times in terms of the variation of the initial data is that the matrices $f^x_U(U)$ and $f^y_U(U)$ commute. However, this is not satisfied for the Euler equations or most other systems of physical interest. Since the steady and self-similar form of two dimensional conservation laws is similar to the self-similar form for a one dimensional conservation law, perhaps BV is an appropriate function space for us to consider.

1.2 Summary of Results

In Chapter II, we consider general systems (possessing an entropy) for which the steady problem has a full basis of eigenvectors and eigenvalues of constant multiplicity that are either genuinely nonlinear or linearly degenerate. We show that any admissible steady and self-similar solution that is a sufficiently small L^∞ perturbation of a constant background solution is necessarily a special function of bounded variation. (It is well known that if U is a function of bounded variation, then U can be uniquely decomposed as $U_{ac} + U_S + U_c$, where U_{ac} is absolutely continuous, U_S is a saltus function of bounded varation, and U_c is a continuous singular function, for example the Cantor-Lebesgue function. $U \in BV$ is a *special function of bounded variation* if the continuous singular part vanishes.) Moreover, it must be constant outside of thin sectors centered at the characteristic directions corresponding to the background state. We demonstrate how to classify in which of these sectors the behavior is like that of a forward in time one-dimensional self-similar solution, and in which sectors the behavior is more like that of a backward in time solution. In these "forward sectors", there can be at most one wave — either a shock or a simple wave. "Backward sectors" can have infinitely many waves, but cannot have consecutive simple waves. There is no distinction between forward and backward sectors corresponding to linearly degenerate fields, and each such sector can have at most one contact discontinuity.

In Chapter III, we show that both the isentropic and full Euler equations satisfy the required assumptions for these results to hold, provided that the solutions we consider are small perturbations of a supersonic state. We also obtain as a corollary that a forward in time self-similar solution to a one-dimensional Riemann problem must correspond to Lax's solution in Theorem I.1 if it is a small L^∞ perturbation of a constant state, and that a backward in time solution must be a special function of bounded variation. We show an example with infinitely many shocks, so that this function space is sharp — any more restrictive commonly used space will not admit countably many discontinuities. We also present an example in which no Lax solution exists, in which the forward-in-time-like and backward-in-time-like sectors cannot be separated by a line on which we could prescribe Riemann data and obtain a Lax solution on the side containing forward-in-time-like sectors. See Figure 1.2 for a summary of the main results in the context of the Euler equations.

In Chapter IV, we consider the full Euler equations, and do not assume that the solution is a small perturbation of a constant solution. Assuming a polytropic equation of state, that the solution is bounded with density and internal energy bounded away from zero, and that the velocity does not vanish, we are able to show that such a solution is a special function of bounded variation. We are also able to obtain some results regarding the structure of possible solutions, but they are less specific than those for the small perturbation case, which is why it is interesting to confirm that full Euler fits into the perturbative framework (as done in Chapter III), while allowing for large variations and not restricting to supersonic flow (as done in Chapter IV). Whereas the notation and notions in Chapters II and III are shared, Chapter IV is largely self-contained.

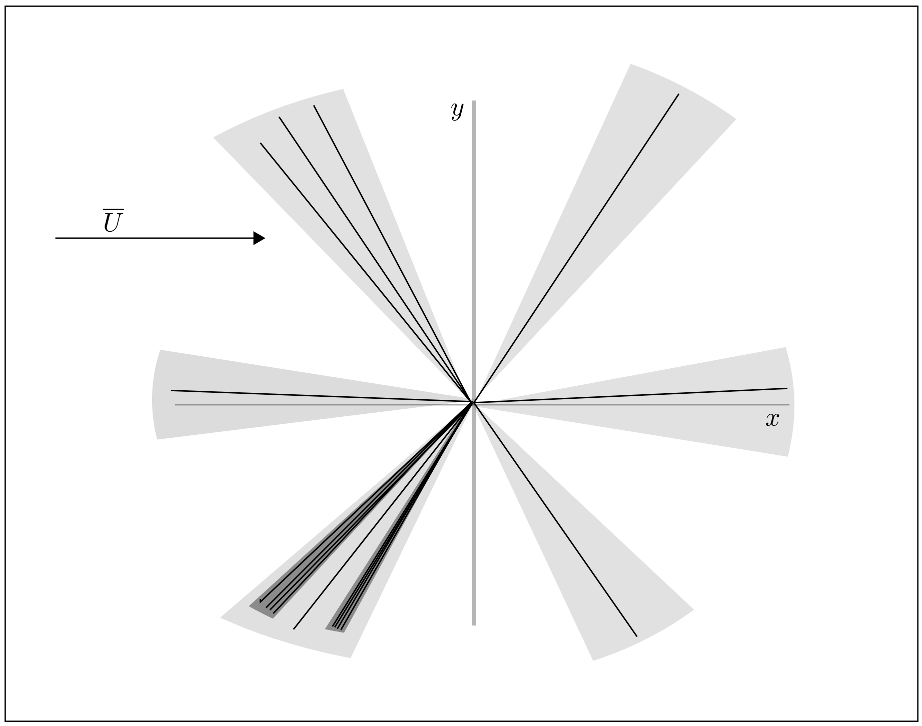

Figure 1.2: The background state $\overline{U}$ is supersonic horizontal velocity to the right, with Mach number $M_0 > 1$. $U \in L^\infty$, steady, self-similar and sufficiently close to $\overline{U} \implies U \in SBV$. U is constant outside of six narrow sectors (shaded light gray). In linearly degenerate sectors (centered on the positive and negative x-axis), there can be at most one contact discontinuity. In genuinely nonlinear forward sectors (centered at $\theta = \pm \arcsin(M_0^{-1})$ from the positive x-axis), there can be at most one shock or rarefaction wave (one shock pictured in each). In genuinely nonlinear backward sectors (centered at $\theta = \pm \arcsin(M_0^{-1})$ from the negative x-axis), there can be infinitely many waves, but no consecutive compression waves (three shocks pictured in the second quadrant, and a compression wave, shock, and compression wave pictured in the third quadrant). This picture applies to both isentropic and full Euler. If we imagine Riemann data prescribed on the y-axis, then the $x > 0$ part of the solution must be the "forward-in-time" Lax solution (with x functioning as time), while the $x < 0$ part is analogous to a backward-in-time solution to a one-dimensional problem and is therefore not uniquely determined from the data on the y-axis.

CHAPTER II

Small Perturbations for General Systems

2.1 Physical Systems and Entropy Solutions

We consider a *system of two-dimensional conservation laws* — that is, a system of nonlinear partial differential equations of the form

$$U_t + f^x(U)_x + f^y(U)_y = 0. \tag{2.1}$$

The unknown $U(t,x,y) = \big(U^1(t,x,y), U^2(t,x,y), ..., U^m(t,x,y)\big)$ is a function from $\mathbb{R}_+ \times \mathbb{R}^2$ to $\mathcal{P} \subset \mathbb{R}^m$, the individual components $U^\alpha, \alpha = 1, ..., m$ are called the *conserved quantities*, the set $\mathcal{P}$ is called the *phase space* of physically allowed values, and the smooth functions $f^x, f^y : \mathcal{P} \to \mathbb{R}^m$ are called the *horizontal* and *vertical flux functions*, respectively. (Like U, f^x and f^y are each column vectors with components $f^{x\alpha}, f^{y\alpha}, \alpha = 1, ..., m$.) We will throughout use subscripts to indicate differentiation.

Smooth functions $\eta, \psi^x, \psi^y : \mathcal{P} \to \mathbb{R}$ are an *entropy-entropy flux pair* for the system (2.1) if, for all $U \in \mathcal{P}$,

$$\eta_{UU} \text{ is positive definite }, \tag{2.2}$$

$$\psi^x_U = \eta_U f^x_U, \text{ and} \tag{2.3}$$

$$\psi^y_U = \eta_U f^y_U. \tag{2.4}$$

Notation II.1. Throughout, if a scalar function depends on a vector quantity, such as $\eta(U)$, then η_U is the gradient as a row vector and η_{UU} is the Hessian. If a vector valued function depends on a vector quantity, such as $f^x(U)$, then f^x_U is the Jacobian. When a symmetric bilinear form is evaluated, for example $r^T\eta_{UU}s$, we will write it as $\eta_{UU}rs$.

Definition II.2. A *physical* system is a choice of conserved quantities, phase space, flux functions, and entropy-entropy flux pair as described above.

As will be discussed in more detail later, some important examples of such systems are the *isentropic Euler* equations and the *full Euler* equations.

We follow the standard line of reasoning, as in for example [46] or [19], to motivate the notion of an entropy solution. Suppose that U is a differentiable solution to (2.1). Then (2.1) becomes

$$U_t + f^x_U U_x + f^y_U U_y = 0,$$

and left multiplying the row vector $\eta_U(U)$ with this result yields

$$\eta_U U_t + \eta_U f^x_U U_x + \eta_U f^y_U U_y = 0.$$

Then, using (2.3) and (2.4) and the chain rule we obtain

$$\eta(U)_t + \psi^x(U)_x + \psi^y(U)_y = 0.$$

We call a Lipschitz continuous solution to (2.1) a *classical solution* or *strong solution* — the differential equations hold in the usual sense pointwise almost everywhere. As shown above, classical solutions to physical systems satisfy an additional conservation law — or, in other words, the entropy η is an additional conserved quantity.

It is well known that singularities can develop in finite time from smooth initial data $U(0, x, y)$. Physically speaking, equations of the form (2.1) neglect the smoothing effects of viscosity, which causes steep yet smooth transitions to be realized as discontinuous shock waves. Therefore, we must relax what we mean by a solution to (2.1) beyond the notion of a classical solution.

Definition II.3. $U \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2; \mathcal{P})$ is a *weak solution* to an initial value problem for (2.1) if for any test function $\Phi = (\Phi^1, \Phi^2, ..., \Phi^m) \in C_c^\infty(R_+ \times \mathbb{R}^2; \mathbb{R}^m)$

$$-\int_{\mathbb{R}_+\times\mathbb{R}^2} \Phi_t \cdot U + \Phi_x \cdot f^x(U) + \Phi_y \cdot f^y(U) d(t,x,y) - \int_{\mathbb{R}^2} (\Phi \cdot U)\Big|_{t=0} d(x,y) = 0. \tag{2.5}$$

This definition is motivated by considering classical solutions, taking the dot product of (2.1) with a smooth test function, and integrating by parts. Therefore, it is clear (2.5) is satisfied by classical solutions for any Φ, but (2.5) makes sense if U is bounded and measurable.

Unfortunately, weak solutions are not unique for given initial data. Therefore, we need an *admissibility criterion* for weak solutions. Suppose we instead consider the associated *viscous* parabolic system to a physical system (2.1):

$$U_t^\epsilon + f^x(U^\epsilon)_x + f^y(U^\epsilon)_y = \epsilon \Delta U^\epsilon. \tag{2.6}$$

Here $\epsilon > 0$ is small, and the solutions for different values of ϵ are denoted by U^ϵ. Parabolic systems of this form tend to have unique smooth solutions global in time (given smooth initial data). Suppose this family of solutions $\{U^\epsilon\}_{\epsilon\geq 0}$ satisfies uniform L^∞ bounds and converges pointwise almost everywhere as $\epsilon \searrow 0$ to a weak solution of (2.1). Then, differentiating (2.6), left multiplying by $\eta_U(U^\epsilon)$, and continuing as before yields

$$\begin{aligned}\eta(U^\epsilon)_t + \psi^x(U^\epsilon)_x + \psi^y(U^\epsilon)_y &= \epsilon \eta_U(U^\epsilon)\Delta U^\epsilon \\ &= \epsilon \Delta\big(\eta(U^\epsilon)\big) - \epsilon \eta_{UU}(U^\epsilon) U_x^\epsilon U_x^\epsilon - \epsilon \eta_{UU}(U^\epsilon) U_y^\epsilon U_y^\epsilon.\end{aligned}$$

Multiply by a *non-negative* test function $\Theta \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2; \mathbb{R})$, integrate over $\mathbb{R}_+ \times \mathbb{R}^2$ and integrate by parts to obtain

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}^2} \Theta_t \eta(U^\epsilon) + \Theta_x \psi^x(U^\epsilon) + \Theta_y \psi^y(U^\epsilon)\ d(t,x,y) + \int_{\mathbb{R}^2} \big(\Theta\eta(U^\epsilon)\big)\Big|_{t=0} d(x,y) \\ = \int_{\mathbb{R}_+\times\mathbb{R}^2} -\epsilon\eta(U^\epsilon)\Delta\Theta + \epsilon\big(\eta_{UU}(U^\epsilon)U_x^\epsilon U_x^\epsilon + \eta_{UU}(U^\epsilon)U_y^\epsilon U_y^\epsilon\big)\Theta\ d(t,x,y) \\ \geq -\int_{\mathbb{R}_+\times\mathbb{R}^2} \epsilon\eta(U^\epsilon)\Delta\Theta\ d(t,x,y)\end{aligned}$$

by the convexity of η assumed in (2.2) and non-negativity of Θ. Taking $\epsilon \searrow 0$ and using dominated convergence we obtain

(2.7)

$$-\int_{\mathbb{R}_+\times\mathbb{R}^2} \Theta_t \eta(U) + \Theta_x \psi^x(U) + \Theta_y \psi^y(U)\ d(t,x,y) - \int_{\mathbb{R}^2} \big(\Theta\eta(U^\epsilon)\big)\Big|_{t=0} d(x,y) \leq 0.$$

This is what is usually referred to as the *integral form* of the differential inequality

$$\eta(U)_t + \psi^x(U)_x + \psi^y(U)_y \leq 0.$$

Thus, whereas classical solutions satisfy an additional differential *equation*, weak solutions satisfy an additional differential *inequality* in the *weak*, or *distributional* sense.

Though the most natural admissibility criterion for weak solutions to (2.1) is the *vanishing viscosity criterion* briefly described above, this is not feasible to use in practice — progress has been made for many one-dimensional systems and for initial data of small total variation (see for example [4]), but whether or not this can be expected in general two-dimensional problems is completely open. Therefore, the *entropy criterion* is often used instead.

Definition II.4. $U \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2; \mathbb{R}^m)$ is an *entropy solution* to a physical system (2.1) if it is a weak solution that also satisfies (2.7) for all non-negative test functions Θ.

2.2 Steady and Self-Similar Solutions

We are interested in entropy solutions that are *steady* in time. Therefore, there is a version of U that does not depend on time. Consider (2.7), and integrate the first term by parts in t. Then, use the compact support of Θ and integrate with respect to t to obtain

$$-\int_{\mathbb{R}^2} \Theta_x \psi^x(U) + \Theta_y \psi^y(U)\ d(x,y) \leq 0, \tag{2.8}$$

for all non-negative $\Theta \in C_c^\infty(\mathbb{R}^2;\mathbb{R})$, now taken to be time-independent. (We just showed how any non-negative $\Theta \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2;\mathbb{R})$ can define a non-negative $\Theta \in C_c^\infty(\mathbb{R}^2;\mathbb{R})$ — taking any non-negative $\Theta \in C_c^\infty(\mathbb{R}^2;\mathbb{R})$ and defining $\overline{\Theta}(t,x,y) := h(t)\Theta(x,y)$ for h non-negative, smooth and compactly supported gives rise to a non-negative $\overline{\Theta} \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$, so these unsteady and steady integral forms are equivalent for steady solutions.)

We are also only interested in entropy solutions that are constant on rays emanating from the origin. To derive the weak form, first consider nonnegative smooth compactly supported Θ with support in the right half plane. Change variables in (2.8) to (x,ξ) with $\xi = y/x$:

$$0 \geq -\int_0^\infty \int_{\mathbb{R}} \Big(\Theta_x(x,x\xi)\psi^x\big(U(\xi)\big) + \Theta_y(x,x\xi)\psi^y\big(U(\xi)\big)\Big) x\ d\xi\, dx. \tag{2.9}$$

Define

$$\theta(\xi) := \int_0^\infty \Theta(x,x\xi)\, dx = -\int_0^\infty x\big(\Theta_x(x,x\xi) + \xi\Theta_y(x,x\xi)\big)\, dx.$$

Then,

$$\theta_\xi(\xi) = \int_0^\infty x\Theta_y(x,x\xi)\, dx,$$

and

$$\int_0^\infty x\Theta_x(x, x\xi)\, dx = -\theta(\xi) - \xi\theta_\xi(\xi).$$

Then (2.9) is equivalent to

$$0 \geq \int_{\mathbb{R}} \theta(\xi)\psi^x\big(U(\xi)\big) - \theta_\xi(\xi)\Big(\psi^y\big(U(\xi)\big) - \xi\psi^x\big(U(\xi)\big)\Big)\, d\xi \tag{2.10}$$

for every smooth compactly supported nonnegative $\theta : \mathbb{R} \to \mathbb{R}$. Similar to before, these general and self-similar integral forms are equivalent for self-similar solutions. (2.10) is the integral form of

$$\big(\psi^y(U) - \xi\psi^x(U)\big)_\xi + \psi^x(U) \leq 0.$$

If instead Θ has support contained in the left half plane, there is an important difference — the change of variables in (2.9) introduces a factor of -1 from $d(x, y) = |x|d(x, \xi) = -xd(x, \xi)$. Therefore, in this case

$$\big(\psi^y(U) - \xi\psi^x(U)\big) + \psi^x(U) \geq 0.$$

Repeating this calculation for each of the components of (2.5) yields (the inequality is an equality in this case)

$$\big(f^y(U) - \xi f^x(U)\big)_\xi + f^x(U) = 0.$$

Therefore, we have the following definition.

Definition II.5. A *steady and self-similar entropy solution* $U \in L^\infty$ to a physical system (2.1) satisfies, in the sense of distributions,

$$\begin{cases} \big(f^y(U) - \xi f^x(U)\big)_\xi + f^x(U) = 0, \\ \big(\psi^y(U) - \xi\psi^x(U)\big)_\xi + \psi^x(U) \leq 0, \quad x > 0 \\ \big(\psi^y(U) - \xi\psi^x(U)\big)_\xi + \psi^x(U) \geq 0, \quad x < 0 \end{cases} \cdot \tag{2.11}$$

(Once either the right or left half plane is chosen, U is only a function of $\xi = y/x$.)

Remark II.6. We will later justify why we can ignore the case of $x = 0$ without loss of generality.

For the remainder of this chapter, we assume the following.

Assumption II.7. *The system of conservation laws under consideration,* (2.1)*, is a physical system, and all solutions considered are steady and self-similar entropy solutions.*

2.3 Smallness and Intuition

Many of our results require the implicit function theorem, and therefore a smallness assumption. Therefore we assume that our phase space $\mathcal{P}$ is a small neighborhood of a constant background state $\overline{U}$, and rename it $\mathcal{P}_\epsilon$.

Assumption II.8. *The phase space of allowed values for the conserved quantities is of the form*

$$\mathcal{P}_\epsilon := \Big\{ U \in \mathbb{R}^m \Big| |U - \overline{U}| \leq \epsilon \Big\}, \tag{2.12}$$

for some small $\epsilon > 0$. Thus the solutions we consider satisfy

$$||U(\cdot) - \overline{U}||_{L^\infty} \leq \epsilon.$$

We will reduce ϵ as necessary throughout this chapter, but only finitely many times. Note that this choice not only will this allow us to use any local result as global, but it also includes a compactness assumption — both of these will be important in the remainder of this chapter.

Recall that we are not assuming any regularity of the entropy solution $U(\cdot)$ — only that it is bounded. Therefore we cannot expect it a priori to be differentiable anywhere. We are not assuming it is of bounded variation either, and so we cannot even analyze its derivative in the sense of measures (or even talk about left or right limits). However, if we look at the differentiated form of the equations anyway, we obtain from the first line of (2.11)

$$\big(f_U^y - \xi f_U^x\big)U_\xi = 0.$$

Therefore, if on some interval ξ is not a generalized eigenvalue of the matrix pair $\Big(f_U^x\big(U(\xi)\big), f_U^y\big(U(\xi)\big)\Big)$, then $U_\xi = 0$ and therefore U is constant. If instead there is some interval of ξ on which ξ is an eigenvalue, then U_ξ must be the associated eigenvector, which is precisely the case of a *simple wave* for the steady problem. Either way, the smallness assumption on the phase space suggests that when ξ is not close to a generalized eigenvalue of the matrix pair $\big(f_U^x(\overline{U}), f_U^y(\overline{U})\big)$, U must be constant.

Shocks and contact discontinuities are also possible, but the smallness assumption and standard facts about conservation laws suggests that these waves also occur near the eigenvalues evaluated at the background state $\overline{U}$, since the phase space is a small neighborhood of $\overline{U}$.

All together, the differentiated form suggests that all interesting behavior must occur near the eigenvalues of the background state. So even if we were allowed to use the differentiated form it would be important to analyze the characteristic behavior of our system, which is the next step.

2.4 Pointwise Information

It is cumbersome to work with the integral form of a conservation law, so we instead investigate what pointwise information we can derive from it. Recall that we are assuming that $U \in L^\infty$. Rearranging the first line of (2.11) yields

$$\big(f^y(U) - \xi f^x(U)\big)_\xi = -f^x(U).$$

Therefore, the quantity

$$\big(f^y(U) - \xi f^x(U)\big)$$

has a distributional derivative that is L^∞ (since f^x is smooth on $\mathcal{P}$), and is therefore almost everywhere equal to a Lipschitz continuous function. Therefore, the fundamental theorem of calculus asserts that

$$\Big(f^y\big(U(\xi_2)\big) - \xi_2 f^x\big(U(\xi_2)\big)\Big) - \Big(f^y\big(U(\xi_1)\big) - \xi_1 f^x\big(U(\xi_1)\big)\Big)$$
$$= -\int_{\xi_1}^{\xi_2} f^x\big(U(\eta)\big)\, d\eta, \text{ for a.e. } \xi_1, \xi_2.$$

Similarly, the second line of (2.11) shows that the distributional derivative of

$$\Big(\psi^y\big(U(\xi)\big) - \xi\psi^x\big(U(\xi)\big)\Big) + \int_{\bar{\xi}}^{\xi} \psi^x\big(U(\eta)\big)\, d\eta \tag{2.13}$$

is a non-positive distribution, and therefore a non-positive measure. Therefore, there is a version of (2.13) that is a non-increasing function of bounded variation, and rearranging this statement we obtain

$$\Big(\psi^y\big(U(\xi_2)\big) - \xi_2\psi^x\big(U(\xi_2)\big)\Big) - \Big(\psi^y\big(U(\xi_1)\big) - \xi_1\psi^x\big(U(\xi_1)\big)\Big)$$
$$\leq -\int_{\xi_1}^{\xi_2} \psi^x\big(U(\eta)\big)\, d\eta, \text{ for } x > 0 \text{ and a.e. } \xi_1 < \xi_2.$$

Similarly, we can obtain

$$\Big(\psi^y\big(U(\xi_2)\big) - \xi_2\psi^x\big(U(\xi_2)\big)\Big) - \Big(\psi^y\big(U(\xi_1)\big) - \xi_1\psi^x\big(U(\xi_1)\big)\Big)$$
$$\geq -\int_{\xi_1}^{\xi_2} \psi^x\big(U(\eta)\big)\, d\eta, \text{ for } x < 0 \text{ and a.e. } \xi_1 < \xi_2.$$

We now fix a version of U so that these equations and inequalities hold *for all* $\xi_1 < \xi_2$.

Lemma II.9. *Suppose U is a steady and self-similar entropy solution to* (2.1)*, that the phase space is of the form* (2.15)*, and f^x, f^y, ψ^x, ψ^y are continuous on $\mathcal{P}_\epsilon$. Then there exists a version of U such that*

$$
(2.14) \qquad \begin{cases} \left(f^y(U) - \xi f^x(U)\right)\Big|_{\xi_1}^{\xi_2} = -\displaystyle\int_{\xi_1}^{\xi_2} f^x\big(U(\eta)\big)\ d\eta, \\ \left(\psi^y(U) - \xi\psi^x(U)\right)\Big|_{\xi_1}^{\xi_2} \leq -\displaystyle\int_{\xi_1}^{\xi_2} \psi^x\big(U(\eta)\big)\ d\eta, \quad x > 0, \\ \left(\psi^y(U) - \xi\psi^x(U)\right)\Big|_{\xi_1}^{\xi_2} \geq -\displaystyle\int_{\xi_1}^{\xi_2} \psi^x\big(U(\eta)\big)\ d\eta, \quad x < 0, \end{cases}
$$

for all $\xi_1 < \xi_2$.

This lemma immediately follows from Lemma A.1 in Appendix A.

Proof. Consider $x > 0$. Use Lemma A.1, with $\Omega = \mathbb{R}^2, z = (\xi_1, \xi_2), K = \mathcal{P}_\epsilon \times \mathcal{P}_\epsilon, W(z) = \big(U(\xi_1), U(\xi_2)\big)$. Clearly, the left hand side of the second line of (2.14) is a continuous function on $\mathbb{R}^2 \times \mathcal{P}_\epsilon \times \mathcal{P}_\epsilon$:

$$
g^1(z, w_1, w_2) := \big(\psi^y(w_2) - \xi_2\psi^x(w_2)\big) - \big(\psi^y(w_1) - \xi_1\psi^x(w_1)\big).
$$

The right side is a continuous function on $\mathbb{R}^2$:

$$
\tilde{g}^1(z) := -\int_{\xi_1}^{\xi_2} \psi^x\big(U(\eta)\big)\ d\eta,
$$

since U is L^∞. Splitting the first line of (2.14) into $2m$ inequalities defines the other components of g and $\tilde{g}$, and the lemma applies. The same argument can be used for $x < 0$. $\square$

2.5 Hyperbolicity

Consider the homogeneous polynomial

$$
P(x : y) := \det\left(\vec{x} \times \vec{f}_U(\overline{U})\right) = \det\left(x f^y_U(\overline{U}) - y f^x_U(\overline{U})\right),
$$

where $(x : y)$ are homogeneous coordinates on $\mathbb{RP}^1$, $\vec{x} = (x, y)$, and $\vec{f} = (f^x, f^y)$. We call the background state $\overline{U}$ *hyperbolic* if $P(x : y)$ has exactly m roots in $\mathbb{RP}^1$, counting multiplicity. If R is any rotation matrix, it is easy to see that $R\vec{x} \times R\vec{f}_U = \vec{x} \times \vec{f}_U$. Therefore, if $\vec{f}$ is replaced by $R\vec{f}$, the roots of P are rotated by the same amount. P has degree at most m, so there are at most m distinct roots and therefore we can rotate $\vec{f}$ to ensure that $(0 : 1)$ is not a root of P. Assume without loss of generality that this has been done.

The fact that $(0 : 1)$ is not a root of P immediately implies that

$$\det f_U^x(\overline{U}) \neq 0.$$

Now consider the polynomial

$$p(\xi) := \det\left(f_U^y(\overline{U}) - \xi f_U^x(\overline{U})\right).$$

From the above discussion, p has m real roots, counting multiplicity, since each of the m roots of P lead to a finite (since none lie on the y-axis) value of ξ that is a root of p. A *generalized eigenvalue* $\lambda(U)$ and an associated *generalized eigenvector* $r(U)$ of the matrix pair $\left(f_U^x(U), f_U^y(U)\right)$ satisfy

$$\left(f_U^y(U) - \lambda(U) f_U^x(U)\right) r(U) = 0.$$

(Note that the term *generalized eigenvectors* in this context refers to the elements of the kernel of the linear matrix pencil $(-\lambda f_U^x + f_U^y)$, not elements of the kernel of $(-\lambda f_U^x + f_U^y)^k$ for $k > 1$ in the context of defective geometric multiplicity.)

For the remainder of this chapter, we shall assume that the steady problem is hyperbolic in the following sense.

Definition II.10. The steady problem associated to the system (2.1) is called *hyperbolic* on the phase space $\mathcal{P}_\epsilon$ if the generalized eigenvalues of the matrix pair $\left(f_U^x(U), f_U^y(U)\right)$ are real, semisimple, and of constant multiplicity on $\mathcal{P}_\epsilon$ (thus the sum of the multiplicities equals m). It is called *strictly hyperbolic* if there are m distinct generalized eigenvalues, and *non-strictly hyperbolic with constant multiplicity* otherwise.

Assumption II.11. *The steady problem associated to* (2.1) *is hyperbolic on the phase space* $\mathcal{P}_\epsilon$ *(which implies* $f^x_U(U)$ *is non-degenerate for all* $U \in \mathcal{P}_\epsilon$*).*

2.6 Change of Dependent Variables

Since $f^x_U(\overline{U})$ is non-degenerate,

$$U \mapsto V := f^x(U)$$

is a local diffeomorphism. By Assumption II.8, our phase space is already a small neighborhood, so we can reduce ϵ and conclude that f^x is a *global diffeomorphism.* We then define

$$\overline{V} := f^x(\overline{U}), \qquad f(V) := f^y\big(U(V)\big),$$
$$e(V) := \psi^x\big(U(V)\big), \qquad q(V) := \psi^y\big(U(V)\big).$$

Then,

$$f_V = f^y_U U_V.$$

Also, we have

$$e_V = \psi^x_U U_V = \eta_U f^x_U U_V = \eta_U,$$

and

$$q_V = \psi^y_U U_V = \eta_U f^y_U U_V = e_V f_V.$$

Therefore, e and q form an “entropy-entropy flux pair” for the flux f. The term is applied loosely here because e is not necessarily convex. Properties of the entropy are only needed in two instances, and further properties of e will be discussed when they are needed.

Abusing notation, we shall continue to refer to our phase space as $\mathcal{P}_\epsilon$, but it will now refer to a small ball around V of permissible values:

$$\mathcal{P}_\epsilon := \Big\{ V \in \mathbb{R}^m \,\Big|\, |V - \overline{V}| \le \epsilon \Big\}. \tag{2.15}$$

The differential equations and entropy inequalities for V are then

$$\begin{cases} \big(f(V) - \xi V\big)_\xi + V = 0, \\ \big(q(V) - \xi e(V)\big)_\xi + e(V) \le 0, \quad x > 0, \\ \big(q(V) - \xi e(V)\big)_\xi + e(V) \ge 0, \quad x < 0. \end{cases} \tag{2.16}$$

The pointwise form is

$$\begin{cases} \big(f(V) - \xi V\big)\Big|_{\xi_1}^{\xi_2} = -\displaystyle\int_{\xi_1}^{\xi_2} V(\eta)\, d\eta, \\ \big(q(V) - \xi e(V)\big)\Big|_{\xi_1}^{\xi_2} \le -\displaystyle\int_{\xi_1}^{\xi_2} e\big(V(\eta)\big)\, d\eta, \quad x > 0, \\ \big(q(V) - \xi e(V)\big)\Big|_{\xi_1}^{\xi_2} \ge -\displaystyle\int_{\xi_1}^{\xi_2} e\big(V(\eta)\big)\, d\eta, \quad x < 0, \end{cases}$$

for all $\xi_1 < \xi_2$.

2.7 Eigenvalues and Eigenvectors

Since $f_V = f_U^y U_V = f_U^y (f_U^x)^{-1}$,

$$\det(f_U^y - \lambda f_U^x) = 0 \iff \big(\det(f_U^y - \lambda f_U^x)\big)\big(\det(f_U^x)^{-1}\big) = 0 \iff \det(f_V - \lambda I) = 0.$$

Therefore, the generalized eigenvalues of the matrix pair (f_U^x, f_U^y) are precisely the eigenvalues of the matrix f_V.

To that end, define

$$\lambda^1(V) < \lambda^2(V) < \cdots < \lambda^n(V)$$

to be the distinct values of λ solving

$$\det\big(f_V(V) - \lambda I\big) = 0.$$

If s is a generalized eigenvector with eigenvalue λ, then

$$0 = (f_U^y - \lambda f_U^x)s = (f_U^y - \lambda f_U^x)(f_U^x)^{-1} f_U^x s = (f_V - \lambda I) f_U^x s,$$

which means that $f_U^x s$ is an eigenvector of f_V. Since the generalized eigenvectors span $\mathbb{R}^m$, the eigenvectors of f_V do as well. Define

$$R^\alpha(V) := \ker\big(f_V(V) - \lambda^\alpha(V)I\big), \qquad p_\alpha(V) := \dim\, R^\alpha(V),$$

so that, under the hyperbolicity assumption

$$p_\alpha(V) \equiv: p_\alpha \text{ on } \mathcal{P}_\epsilon,$$

and

$$\mathbb{R}^m = \bigoplus_{\alpha=1}^{n} R^\alpha(V) \text{ for all } V \in \mathcal{P}_\epsilon.$$

In the strictly hyperbolic setting, it is relatively easy to prove that if the matrix f_V is a smooth function of V, then so are the eigenvalues and eigenvectors (see for example [19]). However, the situation is more delicate in the case of repeated eigenvalues — there are examples in which the eigenvalues and eigenvectors are not as smooth as the matrix. Fortunately, if the eigenvalues are semisimple and of constant multiplicity, they and the eigenvectors can be shown to be as smooth as the matrix, though the individual eigenvectors are only guaranteed to be locally defined as smooth functions (the *eigenspaces* are smooth when given a suitable topology). Since we are only considering small perturbations, we can simply reduce ϵ if necessary and have our right and left eigenvectors defined on all of $\mathcal{P}_\epsilon$.

The smoothness of the fluxes, the hyperbolicity assumption, and the discussion in Appendix B allow us to conclude that, for each $\alpha = 1, .., n$,

$$\lambda^\alpha : \mathcal{P}_\epsilon \to \mathbb{R}$$

is smooth. In addition, we have for all $V \in \mathcal{P}_\epsilon$ an orthonormal basis for $R^\alpha(V)$:

$$R^\alpha(V) = \text{Span}\left\{r^{\alpha,1}(V), ..., r^{\alpha,p_\alpha}(V)\right\}.$$

Reducing ϵ as necessary, we have that the right and left eigenvectors

$$r^{\alpha,i}(V), l^{\alpha,i}(V) : \mathcal{P}_\epsilon \to \mathbb{R}^m$$

are smooth, and satisfy the normalization

$$\begin{aligned} |r^{\alpha,i}(V)| &= 1, \\ l^{\alpha,i}(V) r^{\beta,j}(V) &= \delta_{\alpha\beta}\delta_{ij} \quad \forall \alpha, \beta = 1, .., n, i = 1, .., p_\alpha, j = 1, .., p_\beta. \end{aligned}$$

If for some α, $p_\alpha = 1$, then we omit the second index of the eigenvector and simply denote it by r^α.

2.8 Genuine Nonlinearity and Linear Degeneracy

The results in this chapter are valid under the assumption that the eigenvalues are either linearly degenerate on all of $\mathcal{P}_\epsilon$ or genuinely nonlinear on all of $\mathcal{P}_\epsilon$.

Definition II.12. An eigenvalue λ^α is *linearly degenerate* if

$$\lambda_V(V) r^{\alpha,i}(V) \equiv 0 \text{ on } \mathcal{P}_\epsilon, \quad i = 1, .., p_\alpha.$$

As it turns out, if $p_\alpha \geq 2$, λ^α must be linearly degenerate, and there is a nice geometrical structure created by the eigenspaces. This result is originally due to Boillat.

Theorem II.13 (Boillat as in [42]). *Suppose the hyperbolicity assumption, Assumption II.11, is satisfied. If an eigenvalue λ^α has multiplicity $p_\alpha \geq 2$, then it must be linearly degenerate. In addition, the affine subspaces $V + R^\alpha(V)$ are the tangent spaces to a family of sub-manifolds of dimension p_α. Each integral submanifold can be parameterized by $s \to W^\alpha(V^-, s)$, with s in $\mathbb{R}^{p_\alpha}$, so that*

$$W^\alpha(V^-, 0) = V^-$$
$$Span\Big\{W^\alpha_{s^i}(V^-, s)\Big\}_{i=1}^{p_\alpha} = R^\alpha(V^-).$$

They form a foliation of $\mathcal{P}_\epsilon$ called the characteristic foliation associated with λ^α.

Remark II.14. As we will see later, if V^- and V^+ are on the same leaf of this foliation, then there can be a *contact discontinuity* between them. Therefore, we will refer to $W^\alpha(V^-, s)$ as the *contact manifold* through V^-.

We now recall the definition of a genuinely nonlinear eigenvalue, which by the previous theorem must have multiplicity one.

Definition II.15. An eigenvalue λ^α is *genuinely nonlinear* if

$$\lambda_V(V) r^\alpha(V) \neq 0 \text{ on } \mathcal{P}_\epsilon.$$

We can orient $r^\alpha(V)$ so that, without loss of generality,

$$\lambda_V(V) r^\alpha(V) > 0 \text{ on } \mathcal{P}_\epsilon.$$

We make the following assumption.

Assumption II.16. *Each simple eigenvalue is either genuinely nonlinear or linearly degenerate.* (*Recall that an eigenvalue with multiplicity greater than one* must be *linearly degenerate, so no assumption is necessary in that case.*)

We now analyze properties of the "entropy" $e(V)$. Later we will need to know

$$\operatorname{sgn} e_{VV}(V) r^{\alpha}(V) r^{\alpha}(V)$$

(where e_{VV} is the Hessian of e with respect to the new variables V) for each genuinely nonlinear field and at all $V \in \mathcal{P}_{\epsilon}$. Of course if e_{VV} were positive definite we would immediately know that quantity is $+1$, but it is η_{UU} that we assumed to be positive definite. In addition, if this quantity vanishes anywhere on $\mathcal{P}_{\epsilon}$ then we will have trouble defining admissible discontinuities, so we must (and can) rule out this possibility.

Lemma II.17. *If λ^{α} is genuinely nonlinear, then $e_{VV} r^{\alpha} r^{\alpha} \neq 0$ on $\mathcal{P}_{\epsilon}$. If $f^{x}(U)$ has only positive (negative) eigenvalues, then e is strictly convex (concave).*

Proof. We shall use Proposition 6.1 from [44]. It states that if H is symmetric positive definite, and K is symmetric, then HK is diagonalizable with real eigenvalues. Moreover, the number of positive (negative) eigenvalues of K equals the number of positive (negative) eigenvalues of HK. First, we write

$$(f^{x}_{U})^{-1} = (\eta_{UU})^{-1} e_{VV}.$$

$(\eta_{UU})^{-1}$ is symmetric positive definite, and e_{VV} is symmetric. Then, applying the proposition, since $(f^{x}_{U})^{-1}$ is nondegenerate, all eigenvalues of e_{VV} are nonzero. Moreover, if all the eigenvalues of $(f^{x}_{U})^{-1}$ are positive (negative), then e_{VV} is positive (negative) definite, since a symmetric matrix is positive (negative) definite if and only if its eigenvalues are all positive (negative). All that is left is to show that $e_{VV} r^{\alpha} r^{\alpha} \neq 0$ in the case of eigenvalues of f^{x}_{U} having mixed signs.

As in Section 4.3 in [42], we consider

$$q_{V} = e_{V} f_{V}.$$

Then,

$$q_{VV} = e_{VV} f_V + e_V f_{VV}.$$

Therefore,

$$e_{VV} f_V = q_{VV} - e_V f_{VV}.$$

The first term on the right side is symmetric, and the second term on the right is a linear combination of symmetric matrices, and is thus symmetric. Therefore, the left side is also symmetric and thus defines a symmetric bilinear form. Then

$$\begin{aligned} e_{VV}(f_V r^\alpha) r^{\beta,i} &= e_{VV}(f_V r^{\beta,i}) r^\alpha \\ \lambda^\alpha e_{VV} r^\alpha r^{\beta,i} &= \lambda^\beta e_{VV} r^{\beta,i} r^\alpha \\ (\lambda^\alpha - \lambda^\beta) e_{VV} r^\alpha r^{\beta,i} &= 0. \end{aligned}$$

Therefore, for $\beta \neq \alpha$ and all $i = 1, .., p_\beta$, $e_{VV} r^\alpha r^{\beta,i} = 0$ since the other eigenvalues are distinct from λ^α (since genuinely nonlinear eigenvalues must be simple). Suppose that

$$e_{VV} r^\alpha r^\alpha = 0.$$

By bilinearity, this would imply that

$$e_{VV} r^\alpha s = 0$$

for all $s \in \mathbb{R}^m$. Therefore $e_{VV} r^\alpha$ must be the zero vector, but this contradicts the fact that e_{VV} has all eigenvalues nonzero. Therefore, for each α with λ^α genuinely nonlinear,

$$e_{VV} r^\alpha r^\alpha \neq 0.$$

$\square$

We finally note that changing variables to V does not affect linear degeneracy or genuine nonlinearity. As we showed before, if s is a generalized eigenvector of the matrix pair (f^x_U, f^y_U), then $r = f^x_U s$ is an eigenvector of f_V, and so

$$\lambda_V r = \lambda_V (f^x_U s) = \lambda_U U_V V_U s = \lambda_U s.$$

This does not matter much for this chapter, but when checking for linear degeneracy or genuine nonlinearity in a specific system it is more natural to check in terms of the original conserved quantities U, or other convenient variables.

2.9 Averaged Matrix

Though we are in some situations able to proceed if e is neither convex nor concave, in the most general setting we need to assume one more property about f^x and the background state.

To proceed with the analysis, we need to construct an averaged matrix $\hat{A}$ that is

- smooth and diagonalizable (with real eigenvalues),
- satisfies $\hat{A}(V^-, V^+)(V^+ - V^-) = f(V^+) - f(V^-)$, and
- satisfies $\hat{A}(V, V) = f_V(V)$.

The common choice in the literature is to define

$$\hat{A}(V^-, V^+) := \int_0^1 f_V\big(V^- + s(V^+ - V^-)\big)ds. \tag{2.17}$$

Clearly this satisfies most of the requirements — but in fact this definition only guarantees a real diagonalizable $\hat{A}$ in the *strictly hyperbolic* case. This is because the set of matrices with all simple real eigenvalues is open, and we are only interested in $V^\pm \in \mathcal{P}_\epsilon$, and so smoothness of the flux guarantees that this averaged matrix is a small perturbation of $f_V(\overline{V})$. However, the set of matrices with repeated real eigenvalues is *not open*, and so in the repeated eigenvalue case this $\hat{A}$ is not guaranteed to be diagonalizable.

An averaged matrix of this type is often used in numerical computations, and for some specific systems there is a *Roe averaged matrix* available. It has the special property that it can be defined as

$$\hat{A}(V^-, V^+) = f_V(\hat{V}),$$

where $\hat{V}$ is some appropriate averaging of the states V^- and V^+. In this case, diagonalizability is guaranteed from diagonalizability of f_V. (It also has the useful property that expressions for the eigenvalues and eigenvectors of f_V are available, which makes it especially well suited for numerical computations.) This is often accomplished by doing a line integral in phase space between the two states, but choosing a more sophisticated path than simply the line segment between the two states which allows one to analytically evaluate the integral.

Harten and Lax showed that physical systems always possess an averaged matrix with these properties. Therefore, we need either e_{VV} to be positive definite, or negative definite so that $(-e)$ can function as a convex entropy. In light of Lemma II.17, we will need to assume that the eigenvalues of $f_U^x(\overline{U})$ are all the same sign (from which it follows that they will all have the same sign on all of $\mathcal{P}_\epsilon$).

Theorem II.18 (Harten, Lax as in in [23])**.** *Suppose there exists an entropy/entropy-flux pair (e, q) for the flux function f with e_{VV} positive definite. Then we can define an averaged matrix $\hat{A}(V^{\pm})$, such that it is smooth in $V^{\pm}$, $\hat{A}(V, V) = f_V(V)$, and it is diagonalizable with real eigenvalues for all $V^{\pm} \in \mathcal{P}_\epsilon$. Most importantly,*

$$f(V^+) - f(V^-) = \hat{A}(V^-, V^+)(V^+ - V^-).$$

So that we can either define $\hat{A}$ as in (2.17) or obtain it using Theorem II.18, we make the following assumption.

Assumption II.19. *Assume that either*

- *the eigenvalues of f_V are all simple, or*
- *the eigenvalues of $f_U^x(\overline{U})$ are all positive or all negative.*

We denote the eigenvalues of $\hat{A}(V^{\pm})$ as $\hat{\lambda}^{\alpha,i}$. If for some α, $p_\alpha = 1$, then from the discussion in Appendix B it follows that $\hat{\lambda}^\alpha$ and $\hat{r}^\alpha$ are smooth functions of $V^{\pm}$. If instead $p_\alpha > 1$, we will not have in general that $\hat{\lambda}^{\alpha,1} = ... = \hat{\lambda}^{\alpha,p_\alpha}$. Since the multiplicity of these eigenvalues is not necessarily constant on $\mathcal{P}_\epsilon \times \mathcal{P}_\epsilon$, we cannot conclude that these eigenvalues and their associated eigenvectors are smooth functions of $V^{\pm}$.

However, as discussed in Appendix B, the eigenvalues will always be *continuous* functions of $V^{\pm}$, even if the multiplicity changes. We define, for $\alpha = 1,..,n$, the *α-group* to be

$$\left\{\hat{\lambda}^{\alpha,1}, \hat{\lambda}^{\alpha,2}, .., \hat{\lambda}^{\alpha,p_\alpha}\right\},$$

which can be continuously labeled so that

$$\hat{\lambda}^{\alpha,1}(V,V) = \hat{\lambda}^{\alpha,2}(V,V) = ... = \hat{\lambda}^{\alpha,p_\alpha}(V,V) = \lambda^\alpha(V)$$

for all $V \in \mathcal{P}_\epsilon$.

Many examples exist that demonstrate lack of continuity of eigenvectors when the multiplicity of an eigenvalue changes, even for symmetric matrices. Moreover, the projection operators onto the eigenspaces also can display this behavior — it is worse than just not being able to smoothly pick bases for these eigenspaces. However, if we instead consider, as in [28], the *total projection* for the $\alpha-$group, then this will be as smooth as $\hat{A}$. We have the formula

$$\hat{P}_\Gamma(V^\pm) = \frac{1}{2\pi i}\int_\Gamma (zI - \hat{A}(V^\pm))^{-1}dz,$$

where P_Γ is the sum of the projections onto the eigenspaces of all eigenvalues inside some counterclockwise contour Γ. By continuity, all eigenvalues in the α-group remain close to $\lambda^\alpha(\overline{V})$ for all $V^\pm$ in $\mathcal{P}_\epsilon$, and so we have for $\alpha = 1, .., n$ that

$$\hat{P}^\alpha(V^\pm) = \frac{1}{2\pi i}\int_{|z-\lambda^\alpha(\overline{V})|=\delta} (zI - \hat{A}(V^\pm))^{-1}dz$$

is the total projection for the α-group, where δ is small enough so these curves remain distinct for different choices of α, and large enough so as to include the entire α-group for all $V^\pm \in \mathcal{P}_\epsilon$. Clearly such a $\delta > 0$ can be found for ϵ sufficiently small. We will make use of the following properties of the projections.

$$\begin{aligned} \hat{P}^\alpha \hat{P}^\beta &= \delta_{\alpha\beta}\hat{P}^\alpha, \\ \hat{P}^\alpha \hat{A} = \hat{A}\hat{P}^\alpha &= \hat{P}^\alpha \hat{A}\hat{P}^\alpha, \\ \sum_{\alpha=1}^{n} \hat{P}^\alpha &= I. \end{aligned}$$

Note that this implies that $\hat{P}^\alpha\mathbb{R}^m$ is an invariant subspace for $\hat{A}$.

If for some α, $p_\alpha = 1$, then $\hat{\lambda}^\alpha$ and its associated right eigenvector $\hat{r}^\alpha$ are smooth functions of $V^\pm$. In this case we also normalize so that

$$\begin{aligned} &\hat{r}^\alpha(V, V) = r^\alpha(V), \\ &|\hat{r}^\alpha(V^\pm)| = 1, \text{ for all } \alpha \text{ with } p_\alpha = 1. \end{aligned}$$

Under Assumption II.19, we can rewrite the pointwise version as (after slight rearrangement)

$$
\begin{cases}
\Big(\hat{A}\big(V(\xi_1), V(\xi_2)\big) - \xi_1 I\Big)\big(V(\xi_2) - V(\xi_1)\big) & \\
\qquad\qquad\qquad\qquad = \displaystyle\int_{\xi_1}^{\xi_2} V(\xi_2) - V(\eta)\ d\eta, & \\
\big(q(V) - \xi e(V)\big)\Big|_{\xi_1}^{\xi_2} \leq - \displaystyle\int_{\xi_1}^{\xi_2} e\big(V(\eta)\big)\ d\eta, & x > 0, \\
\big(q(V) - \xi e(V)\big)\Big|_{\xi_1}^{\xi_2} \geq - \displaystyle\int_{\xi_1}^{\xi_2} e\big(V(\eta)\big)\ d\eta, & x < 0,
\end{cases}
\tag{2.18}
$$

for all $\xi_1 < \xi_2$.

2.10 Left and Right Sequences

Since $V(\cdot)$ is only assumed to be bounded, it is not necessarily piecewise smooth, and and at each point it is not necessarily either continuous, approximately continuous, or discontinuous with an approximate jump discontinuity (these are the hypotheses typically used in deriving the familiar Rankine-Hugoniot jump conditions). Arbitrary bounded functions do not even possess well defined limits from the left and right, and so we have to be more careful in this L^∞ setting.

We wish to use the pointwise form (2.18), so to that end we define pairs of sequences $(\tilde{\xi}_k^-), (\tilde{\xi}_k^+)$ such that $\tilde{\xi}_k^- < \tilde{\xi}_k^+$ for all k and $\tilde{\xi}_k^\pm \to \xi$.

Since $\mathcal{P}_\epsilon$ is compact, there exists a subsequence $\tilde{\xi}_{j(k)}^+$ such that $V(\tilde{\xi}_{j(k)}^+) \to V^+$. Similarly there exists a subsequence $\tilde{\xi}_{i(j(k))}^-$ such that $V(\tilde{\xi}_{i(j(k))}^-) \to V^-$. By construction, taking $\xi_k^\pm := \tilde{\xi}_{i(j(k))}^\pm$ yields a pair of sequences such that

$$
\xi_k^\pm \to \xi, \qquad V(\xi_k^\pm) \to V^\pm, \qquad \xi_k^- < \xi_k^+ \text{ for all } k.
$$

Understanding that this depends on the pair of sequences chosen, we define

$$
[g(V)] := g(V^+) - g(V^-)
$$

for any function g of V. Define

$$
J(g(V); \xi) := \sup |[g(V)]|,
$$

where the supremum is taken over all such sequences. Obviously $J(g(V);\xi) = 0 \iff g \circ V$ is continuous at ξ.

Applying (2.18) with $\xi_1 = \xi_k^-$ and $\xi_2 = \xi_k^+$ (this is why we insisted $\xi_k^- < \xi_k^+$ for all k) and taking the limit $k \to \infty$ yields

$$\begin{cases} \big(\hat{A}(V^-, V^+) - \xi I\big)[V] = 0, & \\ [q(V)] - \xi[e(V)] \leq 0, & x > 0, \\ [q(V)] - \xi[e(V)] \geq 0, & x < 0, \end{cases} \tag{2.19}$$

with the first line being equivalent to

$$[f(V)] - \xi[V] = 0,$$

which is the familiar *Rankine-Hugoniot condition.* Therefore, even in the absence of well defined left and right limits, we can make some sense of limits using these pairs of sequences, and the usual conditions apply to these sequence-dependent $V^\pm$.

The first line of (2.18) implies that, for a given pair of sequences

$$[V] = 0 \qquad \text{or} \qquad \xi = \hat{\lambda}^{\alpha,i} \text{ and } [V] \in \ker(\hat{A}(V^\pm) - \hat{\lambda}^{\alpha,i} I). \tag{2.20}$$

2.11 Sectors

We now start to make rigorous statements similar to the intuition developed in Section 2.3.

Theorem II.20. *Suppose V is continuous on an interval $I =]\xi_1, \xi_2[$ and that ξ is not an eigenvalue of $f_V\big(V(\xi)\big)$ for any $\xi \in I$. Then V is constant on I.*

Proof. Fix some $\xi \in I$. We first claim that V must be Lipschitz continuous at ξ. Suppose not, so that we can choose a sequence $\{h_n\} \to 0$ (with $h_n \neq 0$) such that

$$0 < \left| \frac{V(\xi + h_n) - V(\xi)}{h_n} \right| \nearrow \infty.$$

Divide both sides of the first line of (2.18) (taking $\xi_1 = \xi, \xi_2 = \xi + h_n$) by $|V(\xi + h_n) - V(\xi)|$ to obtain

$$\begin{aligned} \Big(\hat{A}\big(V(\xi), V(\xi + h_n)\big) - \xi I\Big) & \frac{V(\xi + h_n) - V(\xi)}{|V(\xi + h_n) - V(\xi)|} \\ &= \frac{1}{|V(\xi + h_n) - V(\xi)|} \int_{\xi}^{\xi + h_n} V(\xi + h_n) - V(\eta)\, d\eta \\ &= \frac{\mathcal{O}(h_n)}{|V(\xi + h_n) - V(\xi)|} = o(1). \end{aligned} \tag{2.21}$$

(the integral is $\mathcal{O}(h_n)$ since V is bounded). By assumption, $A(V(\xi)) - \xi I$ is non-degenerate, so for h sufficiently small $\hat{A}\big(V(\xi), V(\xi + h_n)\big) - \xi I$ will be uniformly non-degenerate. That is,

$$\exists \delta > 0\ \forall v \in \mathbb{R}^m : \Big|\Big(\hat{A}\big(V(\xi), V(\xi + h_n)\big) - \xi I\Big) v\Big| \geq \delta |v|$$

for some $\delta > 0$ (this follows from the continuity of the eigenvalues and diagonalizability of $\hat{A}$.) Taking $n \to \infty$, the left hand side of (2.21) stays bounded away from zero, while the right hand side goes to zero, leading to a contradiction.

Therefore, V must be Lipschitz on I, and therefore differentiable almost everywhere. Recall the differentiated form

$$\big(f_V(V(\xi)) - \xi V(\xi)\big) V_\xi = 0.$$

However, as we assumed the matrix was non-degenerate on I, it follows that $V_\xi = 0$ on I. A Lipschitz function is the integral of its derivative, so V is constant on I. $\square$

The continuity assumption is stronger than we need, so we proceed without assuming continuity but instead assuming ξ is bounded uniformly away from all eigenvalues of $f_V\big(V(\xi)\big)$.

Theorem II.21. *Consider an interval $I =]\xi_1, \xi_2[$. There is a $\delta_s = \delta_s(\epsilon) > 0$, with*

$$\delta_s \downarrow 0 \quad \textit{as} \quad \epsilon \downarrow 0,$$

so that

$$\forall \alpha = 1, ..n, \forall \xi \in I : |\lambda^\alpha(\overline{V}) - \xi| > \delta_s \tag{2.22}$$

implies V is constant on I.

Proof. Define

$$\delta_s := \sup |\lambda^\alpha(\overline{V}) - \hat{\lambda}^{\alpha,i}(V^\pm)|,$$

where the supremum is taken over all $V^\pm \in \mathcal{P}_\epsilon$, $i = 1, .., p_\alpha$, and $\alpha = 1, .., n$. We see that δ_s converges to zero as $\epsilon \searrow 0$ since $\hat{\lambda}^{\alpha,i}$ are continuous functions of $V^\pm$.

Assume V is discontinuous at $\xi \in I$. We may choose $(\xi_k^+), (\xi_k^-) \to \xi$ with $V(\xi_k^\pm) \to V^\pm$ with $[V] \neq 0$ and obtain, by (2.20), that

$$\xi = \hat{\lambda}^{\alpha,i}(V^\pm)$$

for some α and i. But then

$$|\lambda^\alpha(\overline{V}) - \xi| \leq \delta_s,$$

which contradicts (2.22).

Hence V is continuous on I.

Suppose that $\lambda^\alpha\big(V(\xi)\big) = \xi$ for some $\xi \in I$ and some α. Then, since

$$\lambda^\alpha\big(V(\xi)\big) = \hat{\lambda}^{\alpha,i}\big(V(\xi), V(\xi)\big),$$

we would have

$$|\lambda^\alpha(\overline{V}) - \xi| \leq \delta_s,$$

again a contradiction to (2.22). Therefore, Theorem II.20 applies and yields the conclusion. $\square$

Recall that under Assumption II.11,

$$P(0:1) = \det f_U^x(\overline{U}) \neq 0.$$

Find some ξ such that

$$P(1:\xi) = \det(f_U^y(\overline{U}) - \xi f_U^x(\overline{U})) \neq 0.$$

Rotate coordinates so that $(1:\xi)$ becomes $(0:1)$ and $(0:1)$ becomes $(-1,\xi)$. In these new coordinates, Assumption II.11 is satisfied, and so all results apply. Since P is invariant under rotation, in these new coordinates

$$P(-1,\xi) \neq 0,$$

which means that $-\xi \neq \lambda^\alpha(\overline{V})$ for all $\alpha = 1,..,n$. Reduce ϵ so that $|-\xi - \lambda^\alpha(\overline{V})| > \delta_s$ for all α. This will still be true on some small interval around $-\xi$, and so the previous theorem applies. Rotate back to the original coordinates to deduce that V must be constant in thin sectors containing the positive and negative y-axes. Therefore, we lost no generality in only considering test functions supported away from $x = 0$ and doing all the analysis in terms of ξ.

Now, construct n intervals of the form

$$I^\alpha := \quad]\lambda^\alpha(\overline{V}) - \delta_s, \lambda^\alpha(\overline{V}) + \delta_s[.$$

Considering both $x > 0$ and $x < 0$, we now have $2n$ thin sectors centered at $\dfrac{y}{x} = \lambda^\alpha(\overline{V})$ for some α. V is constant outside these sectors, and we have

$$\lim_{\substack{\xi \to \pm\infty \\ x>0}} V(\xi) = \lim_{\substack{\xi \to \mp\infty \\ x<0}} V(\xi).$$

All interesting behavior occurs within these sectors, so we will consider the behavior of V in these sectors individually.

2.12 Linearly Degenerate Sectors

2.12.1 Satisfying the Jump Conditions

We now analyze the possible behavior of V in I^α, where λ^α is linearly degenerate.

The first thing we recall is that if V is discontinuous at ξ, then we can find a pair of subsequences whose limits yield

$$[f(V)] - \xi[V] = 0, \tag{2.23}$$

with $[V] \neq 0$. We need to show that (2.23) is satisfied if and only if $V^\pm$ both lie on the same leaf of the foliation discussed earlier.

First, we notice that the eigenvalue λ^α is constant on each leaf of the foliation. By direct calculation, we see that

$$\frac{\partial}{\partial s^i}\Big(\lambda^\alpha(W^\alpha(V^-,s))\Big) = \lambda^\alpha_V(W^\alpha(V^-,s))W^\alpha_{s^i}(V^-,s) \equiv 0 \tag{2.24}$$

for all $i \in \{1,...,p_\alpha\}$ by linear degeneracy (since $W^\alpha_{s^i} \in R^\alpha$). Therefore,

$$s \mapsto \lambda^\alpha(W^\alpha(V^-,s)) \quad \text{is constant.}$$

We claim that for all s, if $V^+ = W^\alpha(V^-,s)$ and $\xi = \lambda^\alpha(V^-) = \lambda^\alpha(V^+)$ then (2.23) is satisfied. Make these choices for V^+ and ξ and consider

$$F(s) := f\big(W^\alpha(V^-,s)\big) - f(V^-) - \lambda^\alpha\big(W^\alpha(V^-,s)\big)\big(W^\alpha(V^-,s) - V^-\big).$$

Notice $F(0) = 0$, and (using (2.24))

$$F_{s^i}(s) = f_V\big(W^\alpha(V^-,s)\big)W^\alpha_{s^i}(W^\alpha(V^-,s)) - \lambda^\alpha\big(W^\alpha(V^-,s)\big)W^\alpha_{s_i}(V^-,s)\big) \equiv 0.$$

Now we check for entropy admissibility. Define

$$E(s) := q\big(W^\alpha(V^-,s)\big) - q(V^-) - \lambda^\alpha\big(W^\alpha(V^-,s)\big)\Big(e\big(W^\alpha(V^-,s)\big) - e(V^-)\Big).$$

Then $E(0) = 0$ and

$$E_{s^i}(s) = q_V\big(W^\alpha(V^-,s)\big)W^\alpha_{s^i}(V^-,s) - \lambda^\alpha\big(W^\alpha(V^-,s)\big)e_V\big(W^\alpha(V^-,s)\big)W^\alpha_{s^i}(V^-,s)$$
$$= e_V\big(W^\alpha(V^-,s)\big)\Big(f_V\big(W^\alpha(V^-,s)\big) - \lambda^\alpha\big(W^\alpha(V^-,s)\big)I\Big)W^\alpha_{s^i}(V^-,s) \equiv 0,$$

(using the property of entropy-entropy flux pairs). Therefore, this choice of $V^\pm$ and ξ satisfies (2.18) for either $x > 0$ or $x < 0$. This means that any two states on a leaf of the foliation can be the left and right sides of a *contact discontinuity* located at ξ, if ξ is the (constant) value of λ^α on that leaf. This is why we refer to $W^\alpha(V^-,s)$ as the contact manifold through V^-.

We now use a theorem due to Freistühler to prove that these are the only choices that satisfy (2.23) for ϵ sufficiently small.

Theorem II.22 (Freistühler [20])**.** *For any (V^-, V^+, ξ) in a sufficiently small neighborhood of $(\overline{V}, \overline{V}, \lambda^\alpha(\overline{V}))$, if (V^-, V^+, ξ) satisfy the Rankine-Hugoniot jump condition then V^- and V^+ must lie on the same leaf of the characteristic foliation, and $\xi = \lambda^\alpha(V^-) = \lambda^\alpha(V^+)$.*

We then have the following lemma.

Lemma II.23. *For all $\xi \in I^\alpha$, $\xi \mapsto \lambda^\alpha\big(V(\xi)\big)$ is continuous.*

In addition, if $\xi \neq \lambda^\alpha\big(V(\xi)\big)$ on some open interval in I^α, then V is constant on this interval.

Proof. Suppose V is discontinuous at $\xi \in I^\alpha$. Then we can find a pair of subsequences so that $[V] \neq 0$ and (2.23) is satisfied. However, Theorem II.22 applies and so $V^\pm$ lie on the same leaf of the foliation, and so $[\lambda^\alpha(V)] = 0$. This holds for any pair of subsequences, and so $\lambda\big(V(\xi)\big)$ is continuous at all $\xi \in I^\alpha$.

Similarly, if $\xi \neq \lambda^\alpha\big(V(\xi)\big)$, then by Theorem II.22, (2.23) cannot be satisfied for any pair of sequences unless $[V]=0$. Therefore, V itself must be continuous on such an open interval, and Theorem II.20 shows it is constant. $\square$

2.12.2 Intermediate State

Lemma II.24. *Let $W^\alpha(V^-,s)$ be as above. For every $V^-, V \in \mathcal{P}_\epsilon$, there exists a unique $s \in B_\delta(0) \subset \mathbb{R}^{p_\alpha}$ such that*

$$\hat{P}^\alpha\big(W^\alpha(V^-,s),V\big)\big(V - W^\alpha(V^-,s)\big) = 0,$$

(for some $\delta > 0$).

Proof. Recall that $\hat{P}^\alpha$ has rank p_α, so we can view the map $\mathbf{F}$ defined as

$$\mathbf{F}(V^-,s,V) := \hat{P}^\alpha\big(W^\alpha(V^-,s),V\big)\big(V - W^\alpha(V^-,s)\big)$$

mapping $\mathbb{R}^{p_\alpha+2m}$ to $\mathbb{R}^{p_\alpha}$. (More concretely, we may define $\tilde{\mathbf{F}}$ to simply be the p_α entries of $\mathbf{F}$ corresponding to the p_α linearly independent rows of $\hat{P}^\alpha$, which do not change on $\mathcal{P}_\epsilon$ since linear independence is an open condition and ϵ can be decreased.) Then,

$$\mathbf{F}_s(\overline{V},0,\overline{V}) = -\hat{P}^\alpha(\overline{V},\overline{V})\Big(W^\alpha_{s^1}(\overline{V},0)\big|...\big|W^\alpha_{s^{p_\alpha}}(\overline{V},0)\Big).$$

However, recall that each $W^\alpha_{s^i}(\overline{V},0)$ is an eigenvector of $f_V(\overline{V})$, and therefore lies in the total eigenspace (which at $(\overline{V},\overline{V})$ is just the eigenspace) that $\hat{P}^\alpha$ is projecting onto. Therefore,

$$\mathbf{F}_s(\overline{V},0,\overline{V}) = -\Big(W^\alpha_{s^1}(\overline{V},0)\big|...\big|W^\alpha_{s^{p_\alpha}}(\overline{V},0)\Big).$$

By construction of the contact manifold, this has rank p_α, since the span of the columns is precisely $R^\alpha(\overline{V})$. By the implicit function theorem we obtain $s(V^-,V)$ close to the origin satisfying $\mathbf{F}(V^-,s,V)=0$ for ϵ sufficiently small. $\square$

2.12.3 Regularity of β Components

The following lemma establishes the regularity of the total projections along the other groups $\beta \neq \alpha$.

Lemma II.25. *There exists a set E of full measure in I^α such that for all $\xi_0 \in E$, $\xi \in I^\alpha$, $\beta \neq \alpha$, and $i = 1, ..., p_\beta$ we have*

$$\begin{aligned}\hat{P}^\beta\Big(W^\alpha\big(V(\xi_0), s(\xi)\big), V(\xi)\Big)\Big(V(\xi) - W^\alpha\big(V(\xi_0), s(\xi)\big)\Big) \\ = o(|\xi - \xi_0|) + \mathcal{O}\big(|V(\xi) - V(\xi_0)| \cdot |\xi - \xi_0|\big),\end{aligned} \tag{2.25}$$

provided that $\xi_0 = \lambda^\alpha\big(V(\xi_0)\big)$, and defining $s(\xi) := s\big(V(\xi_0), V(\xi)\big)$ in the context of the previous lemma.

Proof. From (2.18), we have

$$\Big(f\big(V(\xi)\big) - f\big(V(\xi_0)\big)\Big) - \xi_0\big(V(\xi) - V(\xi_0)\big) = \int_{\xi_0}^{\xi} V(\xi) - V(\eta) d\eta.$$

We claim that the left hand side is equal to

$$\Big(f\big(V(\xi)\big) - f\big(W^\alpha\big(V(\xi_0), s(\xi)\big)\big)\Big) - \xi_0\Big(V(\xi) - W^\alpha\big(V(\xi_0), s(\xi)\big)\Big).$$

This is by design, since a contact at ξ_0 satisfies the Rankine-Hugoniot condition for the states $V(\xi_0)$ and $W^\alpha\big(V(\xi_0), s(\xi)\big)$. For ease of reading, abbreviate $V(\xi_0) = V_0$, $V(\xi) = V$, and $W^\alpha\big(V^-, s(\xi)\big)$ as W. Then

$$\begin{aligned}&\big(f(V) - f(V_0)\big) - \xi_0(V - V_0) \\ &\quad = \big(f(V) - f(W) + f(W) - f(V_0)\big) - \xi_0(V - W + W - V_0) \\ &\quad = \big(f(V) - f(W)\big) - \xi_0(V - W) + \big(f(W) - f(V_0)\big) - \xi_0(W - V_0) \\ &\quad = \big(f(V) - f(W)\big) - \xi_0(V - W) + 0,\end{aligned}$$

since $\xi_0 = \lambda^\alpha(V_0)$ by assumption. Then, using the averaged matrix we have

$$\Big(\hat{A}(W,V) - \xi_0\Big)\big(V - W\big) = \int_{\xi_0}^{\xi} V - V(\eta)d\eta.$$

Left multiply both sides by $\hat{P}^\beta(W,V)$ and add and subtract V_0 inside the integral on the right to obtain

$$\begin{aligned}\hat{P}^\beta(W,V)&\Big(\hat{A}(W,V) - \xi_0\Big)\big(V - W\big)\\ &= \hat{P}^\beta(W,V)\left(\int_{\xi_0}^{\xi} V_0 - V(\eta)d\eta + \big(V - V_0\big)(\xi - \xi_0)\right)\end{aligned}$$

Lebesgue's differentiation theorem asserts that the integral on the right side is $o(|\xi - \xi_0|)$ for $\xi_0 \in E$, a set of full measure. Using the properties of the total projection we obtain

$$\Big(\hat{A}(W,V) - \xi_0\Big)\hat{P}^\beta(W,V)(V - W) = o(|\xi - \xi_0|) + \mathcal{O}\big(|V(\xi) - V(\xi_0)| \cdot |\xi - \xi_0|\big)$$

However, since $\xi_0 \in I^\alpha$, and any eigenvalues in any β-group are thus uniformly bounded away from ξ_0, $(\hat{A}(W,V) - \xi_0)$ is uniformly non-degenerate on $\hat{P}^\beta\mathbb{R}^m$, and so

$$\left|\Big(\hat{A}(W,V) - \xi_0\Big)\hat{P}^\beta(W,V)(V - W)\right| \geq \delta\left|\hat{P}^\beta(W,V)(V - W)\right|,$$

for some $\delta > 0$, and the result follows. □

2.12.4 Regularity of V on Subsequences

We now use the main result in [17], which concerns Hölder continuity after restriction for arbitrary functions.

Theorem II.26 (Elling [17])**.** *Let $m \geq k$. Consider any $D \in \mathbb{R}^k$ and $f : D :\rightarrow \mathbb{R}^m$. For almost every $x \in D$ there is a sequence $(x_n) \in D \setminus \{x\}$ converging to x with*

$$\limsup_{n\to\infty} \frac{|f(x) - f(x_n)|}{|x - x_n|^{k/m}} < \infty.$$

There is are some differentiability after restriction results for arbitrary (though for our purposes bounded and measurable would suffice) real-valued functions, the most notable due to Saks (see Chapter 9, Section 4 in [41]), but the only result concerning arbitrary vector valued functions seems to be this Hölder result, though for our purposes continuity after restriction would be enough.

Lemma II.27. *Suppose E is as described in Lemma II.25. For almost every $\xi_0 \in E \subset I^\alpha$, there exists a subsequence $(\xi_n) \to \xi_0$ and $0 < C < \infty$ such that*

$$|V(\xi_n) - V(\xi_0)| \leq C|\xi_n - \xi_0|^{1/m}. \tag{2.26}$$

That is, for almost all $\xi_0 \in E \subset I^\alpha$ there exists a set $D := \{\xi_0\} \cup \{(\xi_n)\}$ such that $V|_D$ is Holder continuous with exponent $\frac{1}{m}$.

Proof. As $V(\xi)$ is a function from $\mathbb{R}$ to $\mathbb{R}^m$, the result follows from Theorem II.26. □

Lemma II.28. *For $(\xi_n) \to \xi_0$ as above, we have the following estimate for all $\beta \neq \alpha$:*

$$\begin{aligned} \hat{P}^\beta\Big(W^\alpha\big(V(\xi_0), s(\xi_n)\big), V(\xi_n)\Big)\Big(V(\xi_n) - W^\alpha\big(V(\xi_0), s(\xi_n)\big)\Big) \\ = o(|\xi_n - \xi_0|). \end{aligned} \tag{2.27}$$

Proof. The previous lemma proves that $V|_D$ is continuous, and so $|V(\xi_n) - V(\xi_0)|$ is $o(1)$ and the result follows immediately from (2.25).

□

Lemma II.29. *In the setting of the previous lemmas,*

$$\Big(V(\xi) - W^\alpha\big(V(\xi_0), s(\xi_n)\big)\Big) = o(|\xi_n - \xi_0|).$$

Proof. We have

$$
\begin{aligned}
&\Big(V(\xi_n) - W^\alpha\big(V(\xi_0), s(\xi_n)\big)\Big) \\
&= \sum_\beta \hat{P}^\beta\Big(W^\alpha\big(V(\xi_0), s(\xi_n)\big), V(\xi_n)\Big)\Big(V(\xi_n) - W^\alpha\big(V(\xi_0), s(\xi_n)\big)\Big) \\
&= \sum_{\beta\neq\alpha} \hat{P}^\beta\Big(W^\alpha\big(V(\xi_0), s(\xi_n)\big), V(\xi_n)\Big)\Big(V(\xi_n) - W^\alpha\big(V(\xi_0), s(\xi_n)\big)\Big) \\
&= o(|\xi_n - \xi_0|),
\end{aligned}
$$

due to the clever choice of $s(\xi_n)$. □

2.12.5 Existence of at Most One Contact

We now combine the results of the previous three sections to obtain the main result.

Theorem II.30. *On a linearly degenerate sector, V is either constant, or constant on each side of a single contact discontinuity.*

Proof. By Lemma II.23, $F := \{\xi \in \overline{I}^\alpha \mid \xi = \lambda^\alpha(V(\xi))\}$ is closed and V is constant on $I^\alpha \backslash F$.

Assume there are $\xi_1, \xi_2 \in F$ and $\eta \in I^\alpha \setminus F$ with $\xi_1 < \eta < \xi_2$. Then we can choose a maximal $]\eta^-, \eta^+[$ containing η but not meeting F. Necessarily $\eta^\pm \in F$, so $\eta^+ = \lambda^\alpha(V(\eta^+))$ and $\eta^- = \lambda^\alpha(V(\eta^-))$. But V is constant on $]\eta^-, \eta^+[$, so $\eta^+ = \eta^-$, which is a contradiction.

Hence F must be a closed interval.

Assume F has positive length, and pick $\xi_0 \in F$ with $F \supset (\xi_n) \to \xi_0$ such that

(2.26) implies (2.27). Then,

$$\begin{aligned}
\lambda^\alpha\big(V(\xi_n)\big) &= \lambda^\alpha\Big(W^\alpha\big(V(\xi_0), s(\xi_n)\big)\Big) + \mathcal{O}\Big(\Big|V(\xi_n) - W^\alpha\big(V(\xi_0), s(\xi_n)\big)\Big|\Big) \\
&= \lambda^\alpha\big(V(\xi_0)\big) + \mathcal{O}\Big(\Big|V(\xi_n) - W^\alpha\big(V(\xi_0), s(\xi_n)\big)\Big|\Big) \\
&= \xi_0 + o(|\xi_n - \xi_0|).
\end{aligned}$$

However, since $\xi_n \in F$, this implies

$$\xi_n - \xi_0 = o(|\xi_n - \xi_0|),$$

a contradiction.

Thus F must be a point or empty. However, since $\xi > \lambda^\alpha\big(V(\xi)\big)$ for $\xi > \lambda^\alpha(\overline{V}) + \delta_s$ and $\xi < \lambda^\alpha\big(V(\xi)\big)$ for $\xi < \lambda^\alpha(\overline{V}) - \delta_s$ (by construction of the sectors), and λ^α is continuous on I^α and away from the other sectors (which are well separated), $\lambda^\alpha\big(V(\xi)\big)$ must equal ξ for some $\xi \in I^\alpha$. Therefore there is exactly one point in F, and at most one contact discontinuity in $\overline{I}^\alpha$. □

2.13 Genuinely Nonlinear Sectors

This construction of various waves is carried out in many texts on conservation laws, for example [46, 19, 42].

Similar to the construction of the contact manifold, we will construct the α-*simple wave* curves, which are also sometimes referred to as *rarefaction curves*. Let $\tilde{W}^\alpha(V^-, s)$ be the integral curve of the vector field $r^\alpha(V)$ through V^-, parameterized by s, so that

$$\tilde{W}^\alpha(V^-, 0) = V^-, \qquad \tilde{W}^\alpha_s(V^-, s) = r^\alpha\big(\tilde{W}^\alpha(V^-, s)\big).$$

We may assume, for each $V^- \in \mathcal{P}_\epsilon$, that $\tilde{W}^\alpha(V^-, s)$ is defined on the interval $[0, s_{\max}(V^-)]$ so that $\tilde{W}^\alpha(V^-, s) \in \mathcal{P}_\epsilon$ for all $s \in [0, s_{\max}(V^-)]$ (since the eigenvector is unit length, this integral curve is parametrized by arc length and so $s_{\max}(V^-) \geq d(V^-, \partial\mathcal{P}_\epsilon))$. That is, extend each integral curve until it reaches the boundary of the compact domain $\mathcal{P}_\epsilon$. This lower bound of distance to $\partial\mathcal{P}_\epsilon$ for how for we can extend rarefaction curves will be utilized in Chapter III when we construct some examples for general systems.

Then, since we are in a genuinely nonlinear sector,

$$\lambda^\alpha\big(\tilde{W}^\alpha(V^-, s)\big)_s = \lambda^\alpha_V\big(\tilde{W}^\alpha(V^-, s)\big)r^\alpha\big(\tilde{W}^\alpha(V^-, s)\big) > 0,$$

and so

$$s \mapsto \lambda^\alpha\big(\tilde{W}^\alpha(V^-, s)\big)$$

is strictly increasing. If $\xi \mapsto s(\xi)$ is its inverse map, we can set

$$W(\xi) := \tilde{W}^\alpha(V^-, s(\xi)), \text{ for } \lambda^\alpha(V^-) \leq \xi \leq \lambda^\alpha(V^+),$$

and obtain a classical solution on the interval $[\lambda^\alpha(V^-), \lambda^\alpha(V^+)]$, since

$$\Big(f_V\big(W(\xi)\big) - \xi I\Big)W_\xi = \Big(f_V\big(W(\xi)\big) - \lambda^\alpha\big(W(\xi)\big)I\Big)r^\alpha\big(W(\xi)\big)s_\xi(\xi) \equiv 0.$$

Consider ξ in a genuinely nonlinear sector corresponding to the α field. Recalling (2.19) and (2.20), at a point of discontinuity and choice of a pair of left/right sequences we have

$$\xi = \hat{\lambda}^\alpha(V^\pm) \quad \text{and} \quad [V] \parallel \hat{r}^\alpha(V^\pm).$$

Define the function $\mathbf{G} : \mathbb{R}^{2m+1} \to \mathbb{R}^m$ as

$$\mathbf{G}(V^-, V^+, s) := V^+ - V^- - s\hat{r}^\alpha(V^\pm).$$

Notice that $\mathbf{G}(\overline{V}, \overline{V}, 0) = 0$, and

$$\mathbf{G}_{V^+}(\overline{V}, \overline{V}, 0) = I.$$

This has rank m, and so the implicit function guarantees (after reducing ϵ if necessary) that the *only* solutions to (2.20) with ξ in the α sector (since $\hat{r}^\alpha$ has unit length) are given by

$$V^+ = S^\alpha(V^-, s),$$

where

$$s \mapsto S^\alpha(V^-, s)$$

is a smooth curve through V^-, and is defined (for all $V^- \in \mathcal{P}_\epsilon$) for $s \in]-\epsilon, \epsilon[$. (We have parametrized the shock curve by the strength of the shock, that is $|s| = |[V]|$. Note that for $V^- \neq \overline{V}$ the entire shock curve will not necessarily lie in $\mathcal{P}_\epsilon$, but this is not important). We say that S^α is the $\alpha-$*shock curve* or $\alpha-$*Hugoniot locus* for V^-. Note that

$$S^\alpha_s(V^-, 0) = r^\alpha(V^-).$$

Since

$$\hat{\lambda}^\alpha(V^-, V^-) = \lambda^\alpha(V^-),$$

we have that

$$(\hat{\lambda}^\alpha_{V^-})(V^-, V^-) + (\hat{\lambda}^\alpha_{V^+})(V^-, V^-) = \lambda^\alpha_V(V^-).$$

The definition of $\hat{A}$ (in the repeated eigenvalue case the expression is not shown, but it is clear from the Harten-Lax proof) shows that $\hat{\lambda}^\alpha$ is symmetric in $V^\pm$, and so

$$(\hat{\lambda}^\alpha_{V^-})(V^-, V^-) = (\hat{\lambda}^\alpha_{V^+})(V^-, V^-).$$

Therefore,

$$\hat{\lambda}^\alpha_{V^+}(V^-, V^-) = \frac{1}{2}\lambda^\alpha_V(V^-).$$

We observe that

$$\frac{d}{ds}\hat{\lambda}^\alpha\big(V^-, S^\alpha(V^-, s)\big)\Big|_{s=0} = \frac{1}{2}\lambda^\alpha_V r^\alpha(V^-) > 0. \tag{2.28}$$

Therefore, by smoothness of the eigenvalues we can reduce ϵ if necessary so that

$$s \mapsto \hat{\lambda}^\alpha\big(V^-, S^\alpha(V^-, s)\big) \text{ is strictly increasing.} \tag{2.29}$$

We also have that

$$s \mapsto \lambda^\alpha\big(S^\alpha(V^-, s)\big) \text{ is strictly increasing,} \tag{2.30}$$

and

$$\Big|\lambda^\alpha\big(S^\alpha(V^-, s)\big) - \lambda^\alpha(V^-)\Big| > |\hat{\lambda}^\alpha\big(V^-, S^\alpha(V^-, s)\big) - \lambda^\alpha(V^-)\Big| \text{ for all } s \in]-\epsilon, \epsilon[. \tag{2.31}$$

(with possibly smaller ϵ). We now check these states $S^\alpha(V^-, s)$ for admissibility. Abbreviate $S^\alpha(V^-, s)$ as $S(s)$ and $\hat{\lambda}^\alpha\big(V^-, S(s)\big)$ as $\hat{\lambda}(s)$. Define

$$F(s) := f\big(S(s)\big) - f(V^-) - \hat{\lambda}(s)\big(S(s) - V^-\big) \equiv 0,$$

(identically zero since the jump conditions are satisfied for all s). Then

$$0 \equiv F'(s) = \Big(f_V\big(S(s)\big) - \hat{\lambda}(s)I\Big)S'(s) - \hat{\lambda}'(s)\big(S(s) - V^-\big). \tag{2.32}$$

Define

$$E(s) := q\big(S(s)\big) - q(V^-) - \hat{\lambda}(s)\Big(e\big(S(s)\big) - e(V^-)\Big).$$

By direct calculation, we see that $E(0) = 0$, and

$$\begin{aligned} E'(s) &= \Big(q_V\big(S(s)\big) - \hat{\lambda}(s)e_V\big(S(s)\big)\Big)S'(s) - \hat{\lambda}'(s)\Big(e\big(S(s)\big) - e(V^-)\Big) \\ &= e_V\big(S(s)\big)\Big(f_V\big(V(s)\big) - \hat{\lambda}(s)I\Big)S'(s) - \hat{\lambda}'(s)\Big(e\big(S(s)\big) - e(V^-)\Big) \\ &= e_V\big(S(s)\big)\hat{\lambda}'(s)\big(S(s) - V^-\big) - \hat{\lambda}'(s)\Big(e\big(S(s)\big) - e(V^-)\Big), \end{aligned}$$

using the entropy-entropy flux pair identity and (2.32). Therefore,

$$E'(0) = 0.$$

Then, we have

$$\begin{aligned}
E''(s) =& e_{VV}\big(S(s)\big)S'(s)\hat{\lambda}'(s)\big(S(s) - V^-\big) + e_V\big(S(s)\big)\hat{\lambda}''(s)\big(S(s) - V^-\big) \\
&+ e_V\big(S(s)\big)\hat{\lambda}'(s)S'(s) - \lambda''(s)\Big(e\big(S(s)\big) - e(V^-)\Big) - \hat{\lambda}'(s)e_V\big(S(s)\big)S'(s) \\
=& e_{VV}\big(S(s)\big)S'(s)\hat{\lambda}'(s)\big(S(s) - V^-\big) + e_V\big(S(s)\big)\hat{\lambda}''(s)\big(S(s) - V^-\big) \\
&- \lambda''(s)\Big(e\big(S(s)\big) - e(V^-)\Big).
\end{aligned}$$

Therefore,

$$E''(0) = 0.$$

Finally,

$$\begin{aligned}
E'''(0) &= \hat{\lambda}'(0)e_{VV}(V^-)r^\alpha(V^-)r^\alpha(V^-) \\
&= \left(\frac{1}{2}\lambda^\alpha_V(V^-)r^\alpha(V^-)\right)e_{VV}(V^-)r^\alpha(V^-)r^\alpha(V^-).
\end{aligned}$$

The first factor is positive, and so whether $E(s)$ is increasing or decreasing in a neighborhood of zero is dependent on the sign of the second factor. Therefore, recalling the sign of the second factor can never be zero due to Lemma II.17, we can check it at any point in phase space and determine that, for ϵ sufficiently small,

$$\operatorname{sgn} E(s) = (\operatorname{sgn} s)(\operatorname{sgn} e_{VV}r^\alpha r^\alpha) \tag{2.33}$$

To that end, we make the following definition.

Definition II.31. For a given α, if $e_{VV}(V^-)r^\alpha(V^-)r^\alpha(V^-) > 0$, we define the *forward α-sector* to be in the right half plane, and the *backward α-sector* to be in the left half plane. We reverse the definition if $e_{VV}(V^-)r^\alpha(V^-)r^\alpha(V^-) < 0$.

Lemma II.32. *For a given pair of left and right subsequences leading to* $[V] \neq 0$*,* (2.19) *implies the familiar Lax admissibility conditions*

$$\lambda^\alpha(V^-) > \xi > \lambda^\alpha(V^+) \text{ in a forward sector, and} \tag{2.34}$$
$$\lambda^\alpha(V^-) < \xi < \lambda^\alpha(V^+) \text{ in a backward sector.}$$

Moreover, we have the following "uniform" Lax conditions:

$$\lambda^\alpha(V^-) - \delta_L|[V]| > \xi > \lambda^\alpha(V^+) + \delta_L|[V]| \text{ in a forward sector, and} \tag{2.35}$$

$$\lambda^\alpha(V^-) + \delta_L|[V]| < \xi < \lambda^\alpha(V^+) - \delta_L|[V]| \text{ in a backward sector,} \tag{2.36}$$

for some $\delta_L > 0$*.*

Proof. Suppose $e_{VV}(V^-)r^\alpha(V^-)r^\alpha(V^-) > 0$ and $x > 0$, so we are in a forward sector. In order for the second line of (2.19) to be satisfied, from (2.33) we see we must have $s \leq 0$. However if $[V] \neq 0$, then $s < 0$. Then, using (2.29), (2.30), and (2.31) we obtain (2.34). Using (2.28) and the analogous computation for $\lambda^\alpha\big(S^\alpha(V^-, s)\big)$, we see we can obtain (2.35) by smoothness of the eigenvalue and taking ϵ sufficiently small. The other cases are obtained similarly. $\square$

We now decompose each sector depending on the behavior of V. Define

$$\mathcal{S} := \{\xi \in \overline{I^\alpha} \mid J(V;\xi) > 0\},$$
$$\mathcal{R} := \{\xi \in \overline{I^\alpha} \mid J(V;\xi) = 0,\ \xi = \lambda_k(V(\xi))\},$$
$$\mathcal{C} := \{\xi \in \overline{I^\alpha} \mid J(V;\xi) = 0,\ \xi \neq \lambda_k(V(\xi))\},$$

These labels stand for "shock", "resonant", and "constant", respectively. For the remainder of this section, complements are taken with respect to I^α.

2.13.1 Backward Sectors

We now analyze the behavior of V in a backward genuinely nonlinear sector. We first observe that each discontinuity must have neighborhoods on either side on which V is constant, and so left and right limits are well defined. Moreover, the size of these neighborhoods is lower bounded proportional to the strength of the shock.

Theorem II.33. *We have that $\mathcal{S} \subset I^\alpha$, that is, a shock cannot occur at an endpoint of I^α. If $\xi_0 \in \mathcal{S}$, then there are neighborhoods on either side of ξ_0 on each of which V is constant, and the size of these neighborhoods is lower bounded proportional to the strength of the shock. That is, for each $\xi_0 \in \mathcal{S}$ we have well defined*

$$\sigma^+(\xi_0) := \sup_{\overline{I^\alpha} \ni \eta > \xi_0} \{\eta \,\big|\, \lambda^\alpha\big(V(\xi)\big) > \xi \ \forall \xi \in]\xi_0, \eta[\,\} \ \textit{and} \tag{2.37}$$

$$\sigma^-(\xi_0) := \inf_{\overline{I^\alpha} \ni \eta < \xi_0} \{\eta \,\big|\, \lambda^\alpha\big(V(\xi)\big) < \xi \ \forall \xi \in]\eta, \xi_0[\,\}$$

that satisfy

$$V \textit{ is constant on } [\sigma^-(\xi_0), \xi_0[\,,\,]\xi_0, \sigma^+(\xi_0)] \subset \overline{I}^\alpha,$$

$$\sigma^+(\xi_0) \geq \xi_0 + \delta_L J(V; \xi_0), \tag{2.38}$$

$$\sigma^-(\xi_0) \leq \xi_0 - \delta_L J(V; \xi_0), \tag{2.39}$$

(where δ_L is as in (2.36)*). We also have that*

$$\sigma^\pm(\xi_0) \in \mathcal{R}$$

$$\lambda^\alpha(V(\xi_0+)) - \xi_0 \geq \delta_L J(V; \xi_0), \tag{2.40}$$

$$\lambda^\alpha(V(\xi_0-)) - \xi_0 \leq -\delta_L J(V; \xi_0). \tag{2.41}$$

Proof. If $\xi_0 \in \mathcal{S}$, we can find a pair of sequences $(\xi_k^\pm) \to \xi_0$ with $V(\xi_k^\pm) \to V^\pm$, with $\xi_k^+ \searrow \xi_0$ and apply to obtain

$$\lambda^\alpha(V^-) < \xi_0 < \lambda^\alpha(V^+). \tag{2.42}$$

Suppose there is no $\delta > 0$ so that $\lambda^\alpha\big(V(\xi)\big) - \xi > 0$ for $\xi \in]\xi_0, \xi_0 + \delta[$. Then we can find a sequence $\xi_k^{++} \searrow \xi_0$ such that $\lambda^\alpha\big(V(\xi_k^{++})\big) - \xi_k^{++} \leq 0$, and, taking subsequences if necessary, obtain $\xi_k^+ < \xi_k^{++}$ for all k and $V(\xi_k^{++}) \to V^{++}$. But this implies $\lambda^\alpha(V^{++}) > \xi_0$, a contradiction.

Therefore, $\sigma^+(\xi_0)$ as in (2.37) is well defined, since the supremum is over a non-empty set. Similar arguments show that $\sigma^-(\xi_0)$ is well defined. Also note that (2.42) implies $\mathcal{S} \subset I^\alpha$, since one of $\lambda^\alpha(V^\pm)$ must be farther from $\lambda^\alpha(\overline{V})$ than δ_s if $|\xi_0 - \lambda^\alpha(\overline{V})| = \delta_s$.

We now claim that V must be continuous on $]\xi_0, \sigma^+(\xi_0)]$ and $[\sigma^-(\xi_0), \xi_0[$. Suppose there is a $\xi_1 \in]\xi_0, \sigma^+(\xi_0)]$ with $\xi_1 \in \mathcal{S}$. Then, find $\eta \in]\xi_0, \sigma^+(\xi_0)[\cap]\sigma^-(\xi_1), \xi_1[$ so that

$$\lambda^\alpha\big(V(\eta)\big) < \eta < \lambda^\alpha\big(V(\eta)\big),$$

the first inequality coming from $\eta \in]\sigma^-(\xi_1), \xi_1[$, and the second coming from $\eta \in]\xi_0, \sigma^+(\xi_0)[$. This gives a contradiction, so V is continuous on $]\xi_0, \sigma^+(\xi_0)]$, and similar arguments show that it is continuous on $[\sigma^-(\xi_0), \xi_0[$.

Then, Theorem II.20 shows that V is constant on $]\xi_0, \sigma^+(\xi_0)[$ and $]\sigma^-(\xi_0), \xi_0[$. Therefore, the right and left limits are well defined, giving (2.40) and (2.41) from (2.36). Since we have also shown that V is continuous at $\sigma^\pm(\xi_0)$, it is constant on $]\xi_0, \sigma^+(\xi_0)]$ and $[\sigma^-(\xi_0), \xi_0[$.

Now, by definition of $\sigma^+(\xi_0)$, if $\sigma^+(\xi_0) \notin \partial\overline{I^\alpha}$ there exists $\eta_n \searrow \sigma^+(\xi_0)$ such that

$$\lambda^\alpha\big(V(\eta_n)\big) \leq \eta_n.$$

Then, continuity of V at $\sigma^+(\xi_0)$ requires that

$$\lambda^\alpha\Big(V\big(\sigma^+(\xi_0)\big)\Big) = \sigma^+(\xi_0),$$

implying $\sigma^+(\xi_0) \in \mathcal{R}$.

However, if $\sigma^+(\xi_0)$ is the right endpoint of I^α and not in $\mathcal{R}$ then continuity would require that $\lambda^\alpha\Big(V\big(\sigma^+(\xi_0)\big)\Big) > \sigma^+(\xi_0) = \lambda^\alpha(\overline{V}) + \delta_s$. But then

$$\begin{aligned}
\left|\lambda^\alpha\Big(V\big(\sigma^+(\xi_0)\big)\Big) - \lambda^\alpha(\overline{V})\right| &= \lambda^\alpha\Big(V\big(\sigma^+(\xi_0)\big)\Big) - \lambda^\alpha(\overline{V}) \\
&= \lambda^\alpha\Big(V\big(\sigma^+(\xi_0)\big)\Big) - \sigma^+(\xi_0) + \sigma^+(\xi_0) - \lambda^\alpha(\overline{V}) \\
&> \delta_s,
\end{aligned}$$

a contradiction. Therefore $\sigma^+(\xi_0) \in \mathcal{R}$. Therefore, we have that

$$\sigma^+(\xi_0) = \lambda^\alpha\Big(V\big(\sigma^+(\xi_0)\big)\Big) = \lambda^\alpha\big(V(\xi_0+)\big),$$

and so (2.39) follows from (2.40).

Similar arguments give (2.38). $\square$

Lemma II.34. *There is a constant $C_{\mathcal{S}}$, independent of V, so that for any $\xi_0 \in \mathcal{S}$,*

$$\xi \notin]\sigma^-(\xi_0), \sigma^+(\xi_0)[$$

implies

$$J(V;\xi_0), \left|\lambda^\alpha(V(\xi_0+)) - \xi_0\right|, \left|\lambda^\alpha(V(\xi_0-)) - \xi_0\right| \le C_{\mathcal{S}}|\xi - \xi_0|.$$

Proof. Consider $\xi \notin]\sigma^-(\xi_0), \sigma^+(\xi_0)[$. Then, (2.38) and (2.39) yield

$$\begin{aligned}
|\xi - \xi_0| &\ge \min\big(|\sigma^+(\xi_0) - \xi_0|, |\sigma^-(\xi_0) - \xi_0|\big) \\
&\ge \delta_L J(V;\xi_0), \\
\implies J(V;\xi_0) &\le \delta_L^{-1}|\xi - \xi_0|.
\end{aligned}$$

Next, we have that

$$\begin{aligned}
|\lambda^\alpha\big(V(\xi_0+)\big) - \xi_0| &= \left|\lambda^\alpha\big(V(\xi_0+)\big) - \hat{\lambda}^\alpha\big(V(\xi_0\pm)\big)\right| \\
&\le CJ(V;\xi_0) \quad \text{(by smoothness of } \hat{\lambda}^\alpha) \\
&\le C\delta_L^{-1}|\xi - \xi_0|.
\end{aligned}$$

The same estimate for $\lambda^\alpha\big(V(\xi_0-)\big)$ and taking $C_{\mathcal{S}} := \max(\delta_L^{-1}, C\delta_L^{-1})$ yield the result.

$\square$

Note that since the shock set is discrete, there can be at most countably many shocks, and all left and right limits are well defined. We can therefore modify V on a set of measure zero so that it is right continuous everywhere.

Lemma II.35. *If ξ_0 is a limit point of $\mathcal{S}$, then $\xi_0 \in \mathcal{R}$.*

Proof. Since ξ_0 is a limit point of $\mathcal{S}$, we can choose a monotonic sequence $(\xi_n) \in \mathcal{S}$ converging to ξ_0. Suppose it is a decreasing sequence. Choose some $\eta_n \in]\sigma^-(\xi_n), \xi_n[$ for each n. Then

$$\begin{aligned} |\lambda^\alpha(V(\eta_n)) - \eta_n| &= |\lambda^\alpha(V(\xi_n-)) - \eta_n| \\ &\leq |\lambda^\alpha(V(\xi_n-)) - \xi_n| + |\xi_n - \eta_n| \\ &\leq C_{\mathcal{S}}|\xi_n - \xi| + |\xi_n - \xi_0| \text{ by Lemma II.34,} \\ &\to 0. \end{aligned}$$

$(\eta_n) \to \xi_0$ and $\xi_0 \notin \mathcal{S}$, so $\lambda^\alpha \circ V$ is continuous at ξ_0 and therefore $\lambda^\alpha(V(\xi_0)) = \xi_0$. $\square$

Theorem II.36. *If $\xi_0 \in \mathcal{C}$, then V is constant on an interval $]\kappa^-(\xi_0), \kappa^+(\xi_0)[$ that contains ξ_0. Taking $\kappa^\pm$ so that we have the maximal such interval, $\kappa^\pm(\xi_0) \in \mathcal{R} \cup \mathcal{S} \cup \big(\mathcal{C} \cap \partial\overline{I^\alpha}\big)$.*

Proof. By Lemma II.35, ξ_0 is not a limit point of $\mathcal{S}$ and so V is continuous on a neighborhood of ξ_0. Then $\lambda^\alpha(V(\xi_0)) - \xi_0 \neq 0$ implies $\lambda^\alpha(V(\xi)) - \xi \neq 0$ on a neighborhood ξ_0. Since $\lambda^\beta(V(\xi)) - \xi \neq 0$ for $\beta \neq \alpha$ by definition of I^α, Theorem II.20 shows V is constant on this neighborhood. Therefore,

$$\kappa^+(\xi_0) := \sup_{\overline{I^\alpha} \ni \eta > \xi_0} \{\eta \,\big|\, V \text{ is constant on } [\xi_0, \eta[\,\}$$

is well defined. We define $\kappa^-(\xi_0)$ analogously.

Finally, $\kappa^\pm(\xi_0) \notin \mathcal{C} \cap I^\alpha$ because it would violate their extremality. We need not define $\kappa^+(\xi_0)$ if ξ_0 is the right endpoint of I^α, for our purposes it is more than enough that we know V is constant on some open interval containing ξ_0. In that case V is constant on $]\kappa^-(\xi_0), \lambda^{\alpha+1}(\overline{V}) - \delta_s[$, or $]\kappa^-(\xi_0), \infty[$ in the case of $\alpha = n$. We also need not worry about defining $\kappa^-(\xi_0)$ if ξ_0 is the left endpoint of I^α. □

Lemma II.37. *For any $\xi_0 \in \mathcal{R}$, there is a neighborhood containing ξ_0 such that $\xi \mapsto \lambda^\alpha\big(V(\xi)\big)$ satisfies a Lipschitz condition based at ξ_0 for all ξ in this neighborhood. The Lipschitz constant is uniform for all such ξ_0 and is independent of V, and is given explicitly by $C_\mathcal{S} + 2$. That is,*

$$\Big|\lambda^\alpha\big(V(\xi)\big) - \lambda^\alpha\big(V(\xi_0)\big)\Big| \leq (C_\mathcal{S} + 2)|\xi - \xi_0|, \ \forall \xi \text{ sufficiently close to } \xi_0. \tag{2.43}$$

Proof. Fix $\xi_0 \in \mathcal{R}$, and consider $\xi > \xi_0$ sufficiently close to ξ_0 (If ξ_0 is the right endpoint of I^α then we are done, since V is continuous at ξ_0 and therefore constant on $[\xi_0, \xi_0 + \delta[$ for $\delta > 0$ small.) We will first estimate $\big|\lambda^\alpha\big(V(\xi)\big) - \xi\big|$, which is zero at ξ_0.

Suppose $\xi \in \mathcal{R}$. Then $\big|\lambda^\alpha\big(V(\xi)\big) - \xi\big| = 0$.

Suppose $\xi \in \mathcal{S}$. Necessarily $\xi_0 \leq \sigma^-(\xi)$ by definition of $\mathcal{R}$ and $\sigma^-(\xi)$. Then $\xi_0 \notin]\sigma^-(\xi), \xi[$ and so Lemma II.34 applies to the shock at ξ and we have

$$\big|\lambda^\alpha\big(V(\xi)\big) - \xi\big| = \big|\lambda^\alpha\big(V(\xi+)\big) - \xi\big| \leq C_\mathcal{S}|\xi - \xi_0|, \tag{2.44}$$

(recalling we have made V right continuous everywhere).

Finally, suppose $\xi \in \mathcal{C}$. Necessarily $\xi_0 \leq \kappa^-(\xi)$ by definition of $\mathcal{R}$ and $\kappa^-(\xi)$. If

$\kappa^-(\xi) \in \mathcal{R}$, then

$$\begin{aligned}\big|\lambda^\alpha\big(V(\xi)\big) - \xi\big| &= \Big|\lambda^\alpha\Big(V\big(\kappa^-(\xi)\big)\Big) - \xi\Big| \\ &= \Big|\lambda^\alpha\Big(V\big(\kappa^-(\xi)\big)\Big) - \kappa^-(\xi)\Big| + \Big|\kappa^-(\xi) - \xi\Big| \\ &\le 0 + |\xi - \xi_0|.\end{aligned}$$

If $\kappa^-(\xi) \in \mathcal{S}$, then

$$\begin{aligned}\big|\lambda^\alpha\big(V(\xi)\big) - \xi\big| &= \Big|\lambda^\alpha\Big(V\big(\kappa^-(\xi)\big)\Big) - \xi\Big| \\ &\le \Big|\lambda^\alpha\Big(V\big(\kappa^-(\xi)\big)\Big) - \kappa^-(\xi)\Big| + \big|\kappa^-(\xi) - \xi\big| \\ &\le C_{\mathcal{S}}\big|\kappa^-(\xi) - \xi_0\big| + \big|\kappa^-(\xi) - \xi\big| \\ &\le (C_{\mathcal{S}} + 1)|\xi - \xi_0|,\end{aligned}$$

by (2.44).

Similar arguments work for $\xi < \xi_0$. Since $\xi \mapsto \xi$ has Lipschitz constant 1, $\xi \mapsto \lambda^\alpha\big(V(\xi)\big)$ satisfies (2.43). $\square$

Theorem II.38. *For any $\xi_0 \in \mathcal{R}$, there is a neighborhood containing ξ_0 such that V satisfies a Lipschitz condition based at ξ_0 for all ξ in this neigborhood. The Lipschitz constant $C_{\mathcal{R}}$ is uniform for all such ξ_0 and is independent of V. That is,*

$$\big|V(\xi) - V(\xi_0)\big| \le C_{\mathcal{R}}|\xi - \xi_0|, \text{ for all } |\xi - \xi_0| < \delta,$$

for some $\delta > 0$. However, δ will depend on V.

Proof. Recall that

$$\Big(\hat{A}\big(V(\xi_0), V(\xi)\big) - \xi_0 I\Big)\big(V(\xi) - V(\xi_0)\big) = \mathcal{O}(|\xi - \xi_0|).$$

Left multiply by $\hat{P}^\beta\big(V(\xi_0), V(\xi)\big)$ and use the fact that the total projection commutes with the matrices on the left to obtain

$$\Big(\hat{A}\big(V(\xi_0), V(\xi)\big) - \xi_0 I\Big)\hat{P}^\beta\big(V(\xi_0), V(\xi)\big)\big(V(\xi) - V(\xi_0)\big) = \mathcal{O}(|\xi - \xi_0|).$$

Since $\xi_0 \in I^\alpha$, and any eigenvalues in any β-group are thus uniformly bounded away from ξ_0, $(\hat{A} - \xi_0)$ is uniformly non-degenerate on $\hat{P}^\beta \mathbb{R}^m$, and so

$$\hat{P}^\beta\big(V(\xi_0), V(\xi)\big)\big(V(\xi) - V(\xi_0)\big) = \mathcal{O}(|\xi - \xi_0|).$$

Summing over $\beta \neq \alpha$ yields

$$\sum_{\beta \neq \alpha} \hat{P}^\beta\big(V(\xi_0), V(\xi)\big)\big(V(\xi) - V(\xi_0)\big) = \mathcal{O}(|\xi - \xi_0|). \tag{2.45}$$

Define $g : \mathbb{R}^m \to \mathbb{R}^{m+1}$ as

$$\begin{aligned} &g^0(W) := \lambda^\alpha(W) \\ &g^i(W) = \Big(\sum_{\beta \neq \alpha} \hat{P}^\beta\big(V(\xi_0), W\big)\big(W - V(\xi_0)\big)\Big)^i, \; i = 1, .., m. \end{aligned}$$

Then,

$$g_W\big(V(\xi_0)\big) = \begin{pmatrix} \lambda_V\big(V(\xi_0)\big) \\ I - r^\alpha\big(V(\xi_0)\big) l^\alpha\big(V(\xi_0)\big) \end{pmatrix}_{(m+1)\times m}.$$

The map $z \mapsto g_W\big(V(\xi_0)\big)z$ is injective— since

$$\Big(I - r^\alpha\big(V(\xi_0)\big) l^\alpha\big(V(\xi_0)\big)\Big) z = 0 \implies z \parallel r^\alpha\big(V(\xi_0)\big),$$

and then if $z \parallel r^\alpha\big(V(\xi_0)\big)$,

$$\lambda_V\big(V(\xi_0)\big) z = 0 \implies z = 0,$$

since

$$\lambda_V\big(V(\xi_0)\big) r^\alpha\big(V(\xi_0)\big) > 0.$$

Therefore by the local immersion theorem there is a diffeomorphism G so that that

$$G \circ g\big(V(\xi)\big) = \big(V(\xi), 0\big) \in \mathbb{R}^{m+1}.$$

Lemma II.37 and (2.45) show that $\xi \mapsto g\big(V(\xi)\big)$ satisfies a Lipschitz condition based at ξ_0, and G being a local diffeomorphism (which is fine since V is continuous at ξ_0) show that $\xi \mapsto G \circ g\big(V(\xi)\big)$, and thus $\xi \mapsto V(\xi)$ also satisfy a Lipschitz condition based at ξ_0. It is clear that the Lipschitz constant depends on properties of the system such as $C_{\mathcal{S}}$, $\sup\limits_{V^\pm \in \mathcal{P}_\epsilon} |\lambda^\alpha(\overline{V}) - \hat{\lambda}^{\beta,i}(V^\pm)|^{-1}$, and $|\lambda_V^\alpha r^\alpha|^{-1}$, but not on V itself. □

Lemma II.39. *Define the jump part of V to be*

$$V_S(\xi) = \sum_{\eta \in \mathcal{S}, \eta < \xi} \big(V(\eta+) - V(\eta-)\big).$$

Then V_S is a right continuous saltus function (so, by definition, is of bounded variation), with total variation independent of V.

Proof. We have

$$\begin{aligned}\sum_{\eta \in \mathcal{S}} \big|V(\eta+) - V(\eta-)\big| &= \sum_{\eta \in \mathcal{S}} J(V;\eta) \\ &\le (2\delta_L)^{-1} \sum_{\eta \in \mathcal{S}} \big(\sigma^+(\eta) - \sigma^-(\eta)\big) \\ &\le (2\delta_L)^{-1} |I^\alpha| \le (\delta_L)^{-1}\delta_s < \infty,\end{aligned}$$

by Lemma II.33 and since the neighborhoods $]\sigma^-(\eta), \sigma^+(\eta)[$ are pairwise disjoint. Since $\mathcal{S}$ is countable and the jumps sum to a finite number, V_S as defined above is a right continuous saltus function, and it is clear that the total variation is independent of V. □

Lemma II.40. *For any $\xi_0 \in \mathcal{R}$, V_S satisfies a Lipschitz estimate based at ξ_0 for ξ sufficiently close to ξ_0. That is,*

$$\big|V_S(\xi) - V_S(\xi_0)\big| \le C_S |\xi - \xi_0|, \text{ for all } |\xi - \xi_0| < \delta,$$

for some $\delta > 0$. As in Theorem II.38, C_S is uniform in $\xi_0 \in \mathcal{R}$ and independent of V, but δ depends on V.

Proof. Consider $\xi > \xi_0$, and suppose $\xi \notin]\sigma^-(\eta), \sigma^+(\eta)[$ for any $\eta \in \mathcal{S}$. Then,

$$\begin{aligned}\left|V_S(\xi) - V_S(\xi_0)\right| &\leq \sum_{\xi_0 < \eta < \xi} J(V, \eta) \\ &\leq (2\delta_L)^{-1} \sum_{\xi_0 < \eta < \xi} \big(\sigma^+(\eta) - \sigma^-(\eta)\big) \\ &\leq (2\delta_L)^{-1} |\xi - \xi_0|,\end{aligned}$$

since the $]\sigma^-(\eta), \sigma^+(\eta)[$ are pairwise disjoint and contained in $[\xi_0, \xi]$ by assumption. If $\xi \in]\sigma^-(\eta), \eta[$ for some $\eta \in \mathcal{S}$, then the previous estimate holds for $\xi = \sigma^-(\eta)$, and V_S is constant on $]\sigma^-(\eta), \eta[$, and so the result follows. If $\xi \in [\eta, \sigma^+(\eta)[$ for some $\eta \in \mathcal{S}$, then apply the previous estimate for $\xi = \sigma^-(\eta)$, and then

$$\begin{aligned}\left|V_S(\xi) - V_S(\xi_0)\right| &\leq (2\delta_L)^{-1} \left|\sigma^-(\eta) - \xi_0\right| + J(V; \eta) \\ &\leq \big((2\delta_L)^{-1} + C_{\mathcal{S}}\big) |\xi - \xi_0|,\end{aligned}$$

from Lemma II.34. Similar arguments work for $\xi < \xi_0$, and then take $C_S := (2\delta_L)^{-1} + C_{\mathcal{S}}$. □

Theorem II.41. *On I^α, V must be of bounded variation. In fact,*

$$V = V_L + V_S,$$

where V_L is Lipschitz with constant independent of V, and V_S is a saltus function of bounded variation, with total variation independent of V. Note that this implies V is a special function of bounded variation, since the Cantor part vanishes. Moreover, the absolutely continuous part is in fact Lipschitz.

Proof. The statement about V_S has been covered in the previous lemmas. We claim that for any ξ_0, V_L satisfies a Lipschitz estimate based at ξ_0 for ξ sufficiently close to ξ_0.

If $\xi_0 \in \mathcal{C}$, then V is constant $]\kappa^-(\xi_0), \kappa^+(\xi_0)[$, and since there are no shocks on this interval it is clear that $V_L := V - V_S$ is constant and thus satisfies a local Lipschitz estimate based at ξ_0 with constant 0.

If $\xi_0 \in \mathcal{S}$, then the jump at ξ_0 is accounted for in V_S, and so V_L is constant $]\sigma^-(\xi_0), \sigma^+(\xi_0)[$, and so satisfies a local Lipschitz estimate based at ξ_0 with constant 0.

Finally, if $\xi_0 \in \mathcal{R}$, then for ξ sufficiently close to ξ_0, we have from Theorem II.38 and Lemma II.40 that

$$\begin{aligned}\big|V_S(\xi) - V_S(\xi_0)\big| &\le \big|V(\xi) - V(\xi_0)\big| + \big|V_S(\xi) - V_S(\xi_0)\big| \\ &\le C_{\mathcal{R}}|\xi - \xi_0| + C_S|\xi - \xi_0| := C_L|\xi - \xi_0|.\end{aligned}$$

Recall that the endpoints of I^α cannot be in $\mathcal{S}$, and so it is clear that these estimates can be obtained with Lipschitz constant 0 for $\xi > \lambda^\alpha(\overline{V}) + \delta_s$ sufficiently close to $\lambda^\alpha(\overline{V}) + \delta_s$, and similarly at the left endpoint.

Pick any ξ_1 and ξ_2. Consider the open cover

$$\bigcup_{\eta \in [\xi_1, \xi_2]} \Omega(\eta),$$

where for any η, $\Omega(\eta)$ is the neighborhood for which we have a Lipschitz estimate based at η with Lipschitz constant C_L (note we were able to use C_L uniformly in ξ_0). This has a finite subcover

$$\bigcup_{i=1}^{N} \Omega(\eta_i).$$

Then, adding in $\Omega(\xi_1)$ and $\Omega(\xi_2)$, we can express

$$\begin{aligned}\big|V(\xi_2) - V(\xi_1)\big| \le{}& \big|V(\zeta_1) - V(\xi_1)\big| + \big|V(\eta_1) - V(\zeta_1)\big| + \big|V(\zeta_2) - V(\eta_1)\big| \\ &+ \big|V(\eta_2) - V(\zeta_2)\big| + ... + \big|V(\xi_2) - V(\zeta_{N+1})\big| \\ \le{}& C_L|\xi_2 - \xi_1|,\end{aligned}$$

where $\eta_{i-1} \leq \zeta_i < \eta_i$, and $\zeta_i \in \Omega(\eta_{i-1}) \cap \Omega(\eta_i)$ for $i = 2, ..., N$. $\big($We take $\xi_1 < \zeta_1 < \eta_1, \zeta_1 \in \Omega(\xi_1) \cap \Omega(\eta_1)$, and $\eta_N < \zeta_{N+1} < \xi_2$, with $\zeta_{N+1} \in \Omega(\eta_N) \cap \Omega(\xi_2)$.$\big)$ Therefore, V_L is Lipschitz on all of $\overline{I^\alpha}$, and the rest follows. □

We now see that there cannot be consecutive simple waves.

Theorem II.42. *If V is continuous on an open interval $B \subset \overline{I^\alpha}$, then it is either constant or constant on either side of a single α-simple wave.*

Proof. Theorem II.38 and Lemma II.40 still apply on B, the only difference is that $B \cap \mathcal{S}$ must be empty. Since V is continuous, $\mathcal{C} \cap B$ is a countable union of disjoint open intervals. On each of these intervals, Theorem II.20 applies so V and therefore $\lambda^\alpha(V)$ is constant. Therefore, $\lambda^\alpha\big(V(\xi)\big) - \xi = 0$ can be satisfied at at most one endpoint of each open interval in $\mathcal{C} \cap B$. Therefore, there can be at most two open intervals in $\mathcal{C} \cap B$, making $\mathcal{R} \cap B$, on which

$$\lambda^\alpha\big(V(\xi)\big) = \xi,$$

a closed interval. Theorem II.41 shows that V is Lipschitz on B (since $B \cap \mathcal{S} = \emptyset$), and therefore is differentiable almost everywhere. Applying the strong form of the equations, we see that V_ξ must be parallel to $r^\alpha\big(V(\xi)\big)$, and since $\lambda^\alpha\big(V(\xi)\big) = \xi$, $\xi \mapsto V(\xi)$ must be the $\xi-$parametrization of R^α. Therefore, V is an $\alpha-$simple wave on $\mathcal{R} \cap B$. □

2.13.2 Forward Sectors

The behavior in the forward sectors is much simpler, as we see now.

Theorem II.43. *In a genuinely nonlinear forward sector, V is either constant, constant on either side of a single simple wave, or constant on either side of a single* *shock. In addition, V is a special function of bounded variation on I^α, with Lipschitz continuous part. The total variation and Lipschitz constant are independent of V.*

Proof. Suppose there is a shock at ξ_0. Choosing $(\xi_k^\pm) \to \xi_0$ as usual, with $[V] \neq 0$, we obtain the opposite comparisons

$$\lambda^\alpha(V^-) > \xi_0 > \lambda^\alpha(V^+).$$

Proceeding as before, but instead defining

$$\sigma^+(\xi_0) := \sup_{\overline{I^\alpha} \ni \eta > \xi_0} \{\eta \,\big|\, \lambda^\alpha\big(V(\xi)\big) < \xi \ \forall \xi \in]\xi_0, \eta[\,\},$$

once we argue as before that this supremum is over a non-empty set, we see that in fact

$$\sigma^+(\xi_0) = \lambda^\alpha(\overline{V}) + \delta_s.$$

(Before, we had $\lambda^\alpha(V^+) > \xi_0$, and so there can be $\xi > \xi_0$ satisfying $\xi > \lambda^\alpha(V^+)$, allowing V^+ to be the *left state* of a subsequent shock. This time, if $\lambda^\alpha(V^+) < \xi_0$, then $\lambda^\alpha(V^+) < \xi$ for all $\xi > \xi_0$.) Similarly, $\sigma^-(\xi_0)$ must be the left endpoint of I^α, and so if there is a shock anywhere in I^α, there is exactly one shock, and so its jump part is a single delta function, and its continuous part is constant. Hence in this case, V is a special function of bounded variation.

If V is continuous on I^α, then improve the result of Lemma II.37 and have that the constant for the local Lipschitz estimate for $\xi \to \lambda^\alpha\big(V(\xi)\big)$ is 2. Theorem II.38 follows as before, and so V is satisfies a local Lipschitz estimate with constant independent of ξ_0 and V based at any $\xi_0 \in \mathcal{R}$. This holds trivially for any $\xi_0 \in \mathcal{C}$, and so V is Lipschitz on I^α. Finally, Theorem II.42 can be applied to show V is constant on all of I^α or on either side of a single α-simple wave. $\square$

2.14 Global SBV Regularity

By Theorem II.30, there can be at most a single contact discontinuity in each linearly degenerate sector, and so the jump part is at most a single delta function, and the continuous part is constant. Therefore, V is a special function of bounded variation in degenerate sectors. Outside the union of all sectors, V is constant and so trivially Lipschitz. By choosing any ξ_1, ξ_2 and using a covering argument as in the proof of Theorem II.41, we see that V is indeed globally SBV with Lipschitz continuous part, with total variation and Lipschitz constant depending only on the system and background state.

CHAPTER III

Examples and Calculations

3.1 One-dimensional Conservation Laws

Consider a physical one-dimensional system of conservation laws

$$V_t + f(V)_x = 0$$

with entropy inequality (with e convex) given by

$$e(V)_t + q(V)_x \leq 0.$$

The same steps used in Section 2.2 show that a self-similar solution (in this case, this means a function of $\xi := \frac{x}{t}$) satisfies

$$\begin{cases} \big(f(V) - \xi V\big)_\xi + V = 0, \\ \big(q(V) - \xi e(V)\big)_\xi + e(V) \leq 0, \quad t > 0 \\ \big(q(V) - \xi e(V)\big)_\xi + e(V) \geq 0, \quad t < 0 \end{cases} \cdot$$

These are exactly the same equations as (2.16) but with t serving the role of x. Moreover, since e is convex, the *forward* sectors all lie in $t > 0$, and the *backward* sectors all lie in $t < 0$. Therefore, all the results in Chapter II apply, provided that the eigenvalues are of constant multiplicity (with full geometric multiplicity) and either genuinely nonlinear or linearly degenerate.

Corollary III.1. *Let V be a self-similar solution to a one-dimensional Riemann problem satisfying $||V(\cdot) - \overline{V}||_{L^\infty} \leq \epsilon$ for some $\overline{V}$ and $\epsilon > 0$ sufficiently small. Then, for $t > 0$, it must coincide with Lax's solution to the Riemann problem (from Theorem I.1). For $t < 0$, it will not be unique but must still be a special function of bounded variation.*

Examples of non-uniqueness can be easily constructed — it is easy to see how to construct examples where a combination of $\alpha-$waves interact at $(t, x) = (0, 0)$ to form a single outgoing α-wave. Then, extending this outgoing wave backward in time gives a different solution.

Example III.2. Consider Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0.$$

An entropy-entropy flux pair with convex entropy is $(u^2/2, u^3/3)$, leading to the Lax condition $u^- > \xi = \frac{u^+ + u^-}{2} > u^+$, where $\pm$ indicates which limit with respect to x (note that this is different from our notation in Chapter II — in that case u^+ would have been from the positive ξ-direction, which is from the negative x-direction for $x > 0$ and from the positive x direction for $x < 0$.)

With $\epsilon > 0$, consider the Riemann problem

$$u(0, x) = \begin{cases} \epsilon & x < 0 \\ -\epsilon & x > 0 \end{cases}.$$

The unique forward in time solution is

$$u(t, x) = \begin{cases} \epsilon & x < 0 \\ -\epsilon & x > 0 \end{cases},$$

but

$$u(t,x) = \begin{cases} \epsilon & x < \dfrac{\epsilon}{2}t \\ 0 & \dfrac{\epsilon}{2}t < x < -\dfrac{\epsilon}{2}t \\ -\epsilon & x > -\dfrac{\epsilon}{2}t \end{cases}$$

and

$$u(t,x) = \begin{cases} \epsilon & x < 0 \\ -\epsilon & x > 0 \end{cases}$$

are both admissible backward in time solutions that are small perturbations of $\overline{u} = 0$. There are in fact continuum many admissible backward in time solutions (see the example in Chapter 15 of [46]) to this Riemann problem that are small perturbations of the background solution $u \equiv 0$.

3.2 Isentropic Euler

For the *isentropic Euler equations*, we have $U = (\rho, \rho u, \rho v)$, where ρ is the density, and u and v are the horizontal and vertical velocity components, respectively. The fluxes are

$$f^x = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \end{pmatrix}, f^y = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \end{pmatrix},$$

where $p = p(\rho)$ is the pressure, and $c^2 := p'(\rho)$ is the square of the sound speed. We also assume that $c_\rho > -1$, which is satisfied by commonly used equations of state (polytropic gases with polytropic exponent $\gamma > 1$ satisfy $c_\rho > 0$ in fact — see Chapter IV for further discussion of polytropic gases). We will now use symmetry to simplify our background state without losing generality, classify the fields of the steady problem as linearly degenerate or genuinely nonlinear, confirm that we have a convex entropy η for the unsteady problem, and check the eigenvalues of f^x_U in order to determine the convexity or lack thereof for the steady "entropy" e.

3.2.1 Rotation Invariance

The Euler equations are invariant under rotation — if U is a weak, entropy, or classical solution, then for any 2×2 orthogonal matrix Q,

$$U' = (\rho', \vec{v}'), \quad \rho'(t, \vec{x}') = \rho(t, \vec{x}), \quad \vec{v}'(t, \vec{x}') = Q\vec{v}(t, \vec{x}), \quad \vec{x}' = Q\vec{x}$$

is another weak, entropy, or strong solution, respectively (here we define $\vec{v} := (u, v)$). Therefore, without loss of generality we can assume our background state has vertical velocity zero. In addition, we can choose units so that $\overline{\rho} = 1$, $\overline{c} = 1$. Therefore, we have

$$\overline{U} = (1, \overline{u}, 0),$$

3.2.2 Genuine Nonlinearity and Linear Degeneracy

Rewriting the fluxes in terms of the conserved quantities ρ, m, and n, where $(m, n) := (\rho u, \rho v)$ represents the momentum density vector, we have

$$f^x(U) = \begin{pmatrix} m \\ \rho m^2 \rho^{-1} + p \\ mn\rho^{-1} \end{pmatrix}, f^y(U) = \begin{pmatrix} n \\ mn\rho^{-1} \\ n^2\rho^{-1} + p \end{pmatrix}.$$

Then,

$$f^x_U = \begin{pmatrix} 0 & 1 & 0 \\ -m^2\rho^{-2} + c^2 & 2m\rho^{-1} & 0 \\ -mn\rho^{-2} & n\rho^{-1} & m\rho^{-1} \end{pmatrix}, f^y_U = \begin{pmatrix} 0 & 0 & 1 \\ -mn\rho^{-2} & n\rho^{-1} & m\rho^{-1} \\ -n^2\rho^{-2} + c^2 & 0 & 2n\rho^{-1} \end{pmatrix}.$$

Using Maple, we compute the generalized eigenvalues of the matrix pair (f_U^x, f_U^y) to be

$$\lambda^{\pm} = \frac{mn \pm \rho c\sqrt{m^2 + n^2 - (\rho c)^2}}{m^2 - (\rho c)^2}, \qquad \lambda^0 = \frac{n}{m}.$$

They are real, distinct, and smooth for $m^2 > \rho^2 c^2$, or equivalently, $|u| > c$. Therefore we choose our background state so that $|\overline{m}| > 1$ and $\epsilon > 0$ small enough so that $|u| > c$ for all $U \in \mathcal{P}_\epsilon$. Note that the $\lambda^{\pm}$ coincide for sonic flow ($|\vec{v}| = c$) and become complex for subsonic flow ($|\vec{v}| < c$). Therefore, steady Euler flow is hyperbolic if and only if the flow is supersonic.

The λ^0 eigenvalue corresponds to shear waves (where the density and pressure are constant but the gas slides past itself at different velocity — that is, the tangential velocity is discontinuous and no gas flows through the discontinuity). The corresponding eigenvector is $r^0 = (0, m, n)$ and it is linearly degenerate:

$$\lambda_U^0 r^0 = \begin{pmatrix} 0 & -\dfrac{n}{m^2} & \dfrac{1}{m} \end{pmatrix} \begin{pmatrix} 0 \\ m \\ n \end{pmatrix} \equiv 0.$$

Assume for the remainder of this section that the background state has horizontal velocity to the right, so that $\overline{u} > 1$. We now consider the $\pm$-fields, which correspond to acoustic waves. These fields are genuinely nonlinear, and since this is an open condition it suffices to check it at the background state $(1, M_0, 0)$, where M_0 is the *Mach number* of the background state and given by $\overline{u}/\overline{c} = \overline{u} = \overline{m}$, since $\overline{\rho} = \overline{c} = 1$ by assumption, and reduce ϵ if necessary.

At the background state, we have (from Maple)

$$r^{\pm}(\overline{U}) = \begin{pmatrix} \pm M_0 \\ \pm(M_0^2-1) \\ \sqrt{M_0^2-1} \end{pmatrix}.$$

Then,

$$\lambda_n^{\pm} = \frac{m \pm \rho c n\big(m^2+n^2-(\rho c)^2\big)^{-1/2}}{m^2-(\rho c)^2} \overset{\rho=c=1,n=0}{=} \frac{M_0}{M_0^2-1}.$$

For the ρ derivative, first set $m = M_0$ and $n = 0$ to obtain

$$\lambda^{\pm} \overset{m=M_0,n=0}{=} \pm\left(\frac{M_0^2}{(\rho c)^2}-1\right)^{-1/2},$$

so that

$$\begin{aligned}\lambda_\rho^{\pm} &= \mp\frac{1}{2}\left(\frac{M_0^2}{(\rho c)^2}-1\right)^{-3/2}\left(-2\frac{M_0^2}{(\rho c)^3}\right)(c+\rho c_\rho) \\ &\overset{\rho=c=1}{=} \pm\frac{1}{2}M_0^2(M_0^2-1)^{-3/2}(1+c_\rho).\end{aligned}$$

For the m derivative, set $n = 0$ and $\rho = c = 1$ to obtain

$$\lambda^{\pm} \overset{n=0,\rho=c=1}{=} \pm(m^2-1)^{-1/2},$$

so that

$$\lambda_m^{\pm} = \mp m(m^2-1)^{-3/2} \overset{m=M_0}{=} \mp M_0(M_0^2-1)^{-3/2}.$$

Finally, we have that

$$\begin{aligned}\lambda_U^{\pm} r^{\pm}\Big|_{\overline{U}} &= \frac{1}{2}\frac{M_0^3(1+c_\rho)}{(M_0^2-1)^{3/2}} - \frac{M_0}{\sqrt{M_0^2-1}} + \frac{M_0}{\sqrt{M_0^2-1}} \\ &= \frac{1}{2}\frac{M_0^3(1+c_\rho)}{(M_0^2-1)^{3/2}} > 0,\end{aligned}$$

and so $\lambda^{\pm}$ is uniformly genuinely nonlinear on the compact set $\mathcal{P}_\epsilon$.

3.2.3 Convexity of η, Analysis of e

For the isentropic Euler equations, it is the *physical* entropy that is assumed constant, and it turns out that the total energy can be used as a convex *mathematical* entropy. Define the *specific internal energy* as

$$e(\rho) := \int_0^\rho \frac{p}{\rho^2}\, d\rho,$$

and the *total energy* (per unit mass) as

$$E := e + \frac{u^2+v^2}{2}.$$

Defining

$$\eta(U) := \rho E, \qquad \vec{\psi}(U) := \big(\rho E + p\big)\vec{v}$$

gives an entropy-entropy flux pair. We can rewrite the fluxes as

$$f^x(U) = \frac{m}{\rho}U + \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix}, \quad f^y(U) = \frac{n}{\rho}U + \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix},$$

and the entropy and entropy fluxes as

$$\eta(U) = \rho e + \frac{m^2+n^2}{2\rho}, \quad \psi^x(U) = \frac{m}{\rho}\big(\eta(U)+p\big), \quad \psi^y(U) = \frac{n}{\rho}\big(\eta(U)+p\big).$$

Then, we need to verify that

$$\eta_U f^x_U = \psi^x_U, \qquad \eta_U f^y_U = \psi^y_U.$$

We have that

$$\eta_U f^x_U = \eta_U \left(\frac{m}{\rho} I + U \left(\frac{m}{\rho}\right)_U + \begin{pmatrix} 0 & 0 & 0 \\ c^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right),$$

and

$$\psi_U^x = \frac{m}{\rho}\left(\eta_U + \begin{pmatrix} c^2 & 0 & 0 \end{pmatrix}\right) + (\eta + p)\left(\frac{m}{\rho}\right)_U.$$

We see that $\eta_U f_U^x$ will equal ψ_U^x if

$$\eta_U U = \eta + p,$$

(since $\eta_m = \frac{m}{\rho}$). We have

$$\begin{aligned}
\eta_U U &= \begin{pmatrix} e + \frac{p}{\rho} - \frac{m^2+n^2}{2\rho^2} & \frac{m}{\rho} & \frac{n}{\rho} \end{pmatrix} \begin{pmatrix} \rho \\ m \\ n \end{pmatrix} \\
&= e\rho + p - \frac{m^2+n^2}{2\rho} + \frac{m^2}{\rho} + \frac{n^2}{\rho} \\
&= e\rho + p + \frac{m^2+n^2}{2\rho} \\
&= \eta + p,
\end{aligned}$$

and so $\psi_U^x = \eta_U f_U^x$. Similarly, $\psi_U^y = \eta_U f_U^y$.

Finally,

$$\eta_{UU} = \begin{pmatrix} \frac{c^2}{\rho} + \frac{m^2+n^2}{\rho^3} & -\frac{m}{\rho^2} & -\frac{n}{\rho^2} \\ -\frac{m}{\rho^2} & \frac{1}{\rho} & 0 \\ -\frac{n}{\rho^2} & 0 & \frac{1}{\rho} \end{pmatrix}.$$

The minors (starting from the lower right) are ρ^{-1}, ρ^{-2}, and $\det \eta_{UU} = c^2\rho^{-3}$, all of which are positive, and so η_{UU} is uniformly positive definite on the compact set $\mathcal{P}_\epsilon$, for ϵ sufficiently small.

Finally, recall

$$f_U^x = \begin{pmatrix} 0 & 1 & 0 \\ -\dfrac{m^2}{\rho^2} + c^2 & \dfrac{2m}{\rho} & 0 \\ -\dfrac{mn}{\rho^2} & \dfrac{n}{\rho} & \dfrac{m}{\rho} \end{pmatrix}.$$

The eigenvalues are $u, u \pm c$, and so we see (from Lemma II.17) that the *steady* entropy $e(V)$ (discussed in Section 2.8) is convex if $\overline{u} > 1$, and concave if $\overline{u} < -1$. Therefore, if the background state is horizontal supersonic velocity to the right, the forward sectors all lie in $x > 0$, and if the background state is horizontal supersonic velocity to the left, then the forward sectors all lie in $x < 0$. This is consistent with the fact that rightward supersonic velocity means the steady problem is hyperbolic in the positive x-direction, so that we can prescribe Riemann data on the y-axis and obtain a Lax solution of at most 3 waves separated constant states in the right half plane, and vice versa if the background state has supersonic horizontal velocity to the left.

3.2.4 Compression and Expansion Waves

For this section, assume that the background state is supersonic velocity to the right. Consider the +-sector in the upper right half plane. It is centered at $\frac{y}{x} = (M_0^2 - 1)^{-1/2}$ and so at an angle of $\arcsin(1/M_0)$ from the positive x-axis. Since the gas is moving towards the right, it is traveling in the direction of decreasing ξ here, and so U^+ is the state of the gas right before it passes through a shock, and U^- is the state of the gas after. The Lax condition in this forward sector is $\lambda^+(U^-) > \xi > \lambda^+(U^+)$, and since the +-Hugoniot curve through U^- has tangent r^+, we have from the above that $[U]$ is approximately a *negative* multiple of $r^+(\overline{U})$. Therefore, as the gas passes through the shock, the jump is approximately a positive multiple of $r^+(\overline{U})$, and so the density increases as the gas passes through the shock. In a simple wave, the derivative of U with respect to ξ is a positive multiple of r^+, and so as the gas travels through the wave, density decreases, and so this is a *rarefaction wave* or *Prandl-Meyer expansion*.

For the $-$-sector in the upper half left half plane, centered at a clockwise angle of $\arcsin(1/M_0)$ from the negative x-axis, gas is still moving in the decreasing ξ direction. This time, the Lax condition for a shock is $\lambda^-(U^-) < \xi < \lambda^+(U^+)$, and so $[U]$ is approximately a positive multiple of $r^-(\overline{U})$. As the gas passes through the shock, the jump is then approximately a negative multiple of $r^-(\overline{U})$, and so the density increases as the gas passes through the shock. In a simple wave, U_ξ is a positive multiple of r^-, and so as the gas travels through the wave, the density increases, and so this is a *compression wave* or *Prandl-Meyer compression*. The fact that the admissible shock curve and simple waves starting at V^- both cause an increase the density is where the proof of a Lax solution to a "backward in x" Riemann problem fails and uniqueness is lost.

3.3 Full Euler

The full Euler equations are given by

$$f^x(U) = \begin{pmatrix} m \\ m^2\rho^{-1} + p \\ mn\rho^{-1} \\ \frac{1}{2}m(m^2+n^2)\rho^{-2} + me + mp\rho^{-1} \end{pmatrix},$$

$$f^y(U) = \begin{pmatrix} n \\ \\ mn\rho^{-1} \\ \\ n^2\rho^{-1} + p \\ \\ \frac{1}{2}n(m^2+n^2)\rho^{-2} + ne + np\rho^{-1} \end{pmatrix},$$

where ρ is the density, m and n are the horizontal and vertical momentum densities, p is the pressure, and e is the specific internal energy. The conserved quantities are ρ, m, n, and the total energy per unit volume $\rho E := \frac{1}{2\rho}(m^2+n^2) + \rho e$. Note that it is identical to the isentropic case, except for an additional conservation law for the conservation of energy. Moreover, the pressure is not just a function of density, and so the equation of state gives the pressure in terms of two thermodynamic variables —typically density and internal energy or density and entropy. We now check the properties of the full Euler equations.

3.3.1 Genuine Nonlinearity and Linear Degeneracy

For convenience, we can calculate the eigenvalues and check for genuine nonlinearity/linear degeneracy by considering the matrix pair (f^x_W, f^y_W) with convenient variables $W = (\rho, m, n, S)$ where S is the specific entropy (the invariance of eigenvalues and nonlinearity properties was discussed at the end of Section 2.8). It is well known (see [11]) that of the five thermodynamic quantities ρ, e, S, p and temperature T, only two are independent, so once two have been chosen the other three can be expressed in terms of those two. We have that

$$\begin{aligned} p_\rho(\rho, S) &=: c^2 > 0, \\ e_\rho(\rho, S) &= \frac{p}{\rho^2}, \\ e_S(\rho, S) &= T, \end{aligned}$$

where c is the *sound speed.* The derivatives of the fluxes in terms of the variables W are then

$$f_W^x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -m^2\rho^{-2} + c^2 & 2m\rho^{-1} & 0 & p_S \\ -mn\rho^{-2} & n\rho^{-1} & m\rho^{-1} & 0 \\ \dfrac{-m(m^2+n^2)}{\rho^3} + \dfrac{mc^2}{\rho} & \dfrac{3}{2}\dfrac{m^2}{\rho^2} + \dfrac{1}{2}\dfrac{n^2}{\rho^2} + e + \dfrac{p}{\rho} & \dfrac{mn}{\rho^2} & mT + \dfrac{mp_S}{\rho} \end{pmatrix},$$

$$f_W^y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -mn\rho^{-2} & n\rho^{-1} & m\rho^{-1} & 0 \\ -n^2\rho^{-2} + c^2 & 0 & 2n\rho^{-1} & p_S \\ \dfrac{-n(m^2+n^2)}{\rho^3} + \dfrac{nc^2}{\rho} & \dfrac{mn}{\rho^2} & \dfrac{3}{2}\dfrac{n^2}{\rho^2} + \dfrac{1}{2}\dfrac{m^2}{\rho^2} + e + \dfrac{p}{\rho} & nT + \dfrac{np_S}{\rho} \end{pmatrix}.$$

Using Maple, we can compute the generalized eigenvalues and eigenvectors. There are two simple eigenvalues corresponding to acoustic waves:

$$\lambda^{\pm}(U) = \frac{mn \pm \rho c\sqrt{m^2 + n^2 - (\rho c)^2}}{m^2 - (\rho c)^2},$$

and a double eigenvalue corresponding to shear waves and entropy waves:

$$\lambda^0(U) = \frac{n}{m}.$$

The background state we consider will be of the form $\overline{U} = (1, M_0, 0, E_0)$, where the units have been scaled (as in Section 3.2.2) so that the background density and sound speed are 1, and so the background *Mach Number* M_0 is the background horizontal momentum (we also assume $M_0 > 0$). So that these eigenvalues are real and distinct, we must again assume $M_0 > 1$, and it is clear that the eigenvalues and eigenvectors are smooth functions on a neighborhood of this $\overline{U}$. The eigenvectors considered below are also smooth on a neighborhood of $\overline{U}$.

Two linearly independent eigenvectors for λ^0 are

$$r^{0,1}(U) = \begin{pmatrix} 0 \\ m \\ n \\ 0 \end{pmatrix}, \qquad r^{0,2}(U) = \begin{pmatrix} -p_S \\ 0 \\ 0 \\ c^2 \end{pmatrix}.$$

As λ^0 is semisimple of constant multiplicity, Boillat's theorem (Theorem II.13) guarantees that λ^0 is linearly degenerate, though an easy calculation confirms this as well. For any contact discontinuity corresponding to λ^0, the flow will be tangential to the discontinuity. The $r^{0,1}$ field corresponds to a *shear wave*, which is a discontinuity in the tangential velocity, and the $r^{0,2}$ field corresponds to an *entropy wave*, in which the entropy and density are different but in a manner so that the pressure is constant on either side. A contact discontinuity for the full Euler equations will in general be a combination of these two effects, but it is instructive to separate them when choosing a basis for the eigenspace so as to compare with the isentropic case.

The acoustic characteristic fields $\lambda^\pm$ are again genuinely nonlinear — and as before it suffices to check at the background state. From Maple we have that

$$r^\pm(\overline{U}) = \begin{pmatrix} \pm M_0 \\ \pm(M_0^2 - 1) \\ \sqrt{M_0^2 - 1} \\ 0 \end{pmatrix}.$$

However, the first three entries are precisely the $r^\pm$ eigenvectors for the isentropic case investigated in Section 3.2.2, $\lambda^\pm$ are identical in the full and isentropic cases, and the first three W variables we consider in this case are precisely the conserved quantities for the isentropic case. Since $\lambda^\pm$ for full Euler are independent of S, it is clear that in our context of full Euler $\lambda^\pm_W r^\pm$ is identical to $\lambda^\pm_U r^\pm$ from the isentropic case in Section 3.2.2. Genuinely nonlinearity was demonstrated for $c_\rho > -1$, which is true for commonly used equations of state (in particular, any polytropic gas with $\gamma > 1$.)

3.3.2 Convexity of η, Analysis of e

As discussed in Chapter II, Section 1.1 of [22], taking $\eta(U) = -\rho S$, $\psi^x(U) = -mS$, $\psi^y(U) = -nS$ yields an entropy-entropy flux pair. The second law of thermodynamics implies that S is a strictly concave function of ρ^{-1} and e, which [22] proves is equivalent to $-\rho S$ being a strictly convex function of ρ, m, n, and ρE.

In Chapter II, Section 2 of [22], it is shown that the eigenvalues of f^x_U are

$$u \pm c, u, u.$$

(A change of coordinates was introduced that preserves the eigenvalues, since the actual expression for f^x_U is complicated due to the need of differentiating with respect to $\rho E = \frac{m^2+n^2}{2\rho} + \rho e$.) Therefore, if $M_0 > 1$, e_{VV} is positive definite (by Lemma II.17) and the system is hyperbolic in the positive x-direction and forward sectors lie in $x > 0$. If $M_0 < -1$, e_{VV} is negative definite and the system is hyperbolic in the negative x-direction.

3.4 Example with Infinitely Many Waves

Recall that we have parametrized the shock curves with respect to the size of the jump — that is,

$$|S^\alpha(V^-, s) - V^-| = s,$$

and that in a backward genuinely nonlinear sector, that taking $V^+ = S^\alpha(V^-, s)$ leads to an admissible shock at $\xi = \hat{\lambda}^\alpha\big(V^-, S^\alpha(V^-, s)\big)$.

Also, recall that since $|r^\alpha(V)| = 1$ for all $V \in \mathcal{P}_\epsilon$, that the simple wave curve is parametrized with respect to arc length and defined for a maximal interval so that $\tilde{W}(V^-, s)$ is in $\mathcal{P}_\epsilon$ for all $0 \le s \le s_{\max}(V^-)$. Then it follows that for any s in this interval,

$$|\tilde{W}^\alpha(V^-, s) - V^-| \le s.$$

Consider the backward α-sector for a genuinely nonlinear field. Starting with $V_0 = \overline{V}$, for all $k > 0$ define V_k to be one of the following:

$$V^k := \text{ either } S^\alpha\left(V^{k-1}, \frac{\epsilon}{2^{k+1}}\right) \text{ or } \tilde{W}^\alpha\left(V^{k-1}, \frac{\epsilon}{2^{k+1}}\right),$$

with the requirement that if V^k is on the simple wave curve of V^{k-1}, then V^{k+1} must be taken to be on the shock curve of V^k.

By construction

$$V^\infty := \lim_{k\to\infty} V^k$$

will be in $\mathcal{P}_\epsilon$, since

$$|V^\infty - \overline{V}| \le \sum_{k=1}^{\infty} |V^k - V^{k-1}| \le \frac{\epsilon}{2}.$$

Then, choose V^+ as

$$V^+ := \text{ either } \tilde{W}^\alpha\left(V^\infty, \frac{\epsilon}{4}\right) \text{ or } V^\infty.$$

V^+ is also guaranteed to be in $\mathcal{P}_\epsilon$ by construction. By definition of δ_s (in the proof of Theorem II.21), we also have that

$$\lambda^\alpha(V^k), \hat{\lambda}^\alpha(V^k, V^{k+1}) \in I^\alpha$$

for all $k \ge 0$, $k = \infty, +$. We now use these states to build a solution on I^α.

Define $V(\xi) = \overline{V}$ on $[\lambda^\alpha(\overline{V}) - \delta_s, \lambda^\alpha(\overline{V})]$, and depending on the choice made in the first step either put an α-simple wave on $[\lambda^\alpha(\overline{V}), \lambda^\alpha(V^1)]$, or a shock at $\hat{\lambda}(\overline{V}, V^1)$ between the states $\overline{V}$ and V^1.

Then, for $k \geq 1$:

- If $V^{k+1} = S^\alpha(V^k, \epsilon/2^{k+2})$ and $V^k = S^\alpha(V^{k-1}, \epsilon/2^{k+1})$, then $\hat{\lambda}^\alpha(V^{k-1}, V^k) < \lambda^\alpha(V^k) < \hat{\lambda}^\alpha(V^k, V^{k+1})$, and so we can place a shock counterclockwise from the previous shock.

- If $V^{k+1} = S^\alpha(V^k, \epsilon/2^{k+2})$ and $V^k = \tilde{W}^\alpha(V^{k-1}, \epsilon/2^{k+1})$, then $\lambda^\alpha(V^k) < \hat{\lambda}^\alpha(V^k, V^{k+1})$, so we can place a shock counterclockwise from the end of the previous simple wave.

- If $V^{k+1} = \tilde{W}^\alpha(V^k, \epsilon/2^{k+2})$, then necessarily $V^k = S^\alpha(V^{k-1}, \epsilon/2^{k+1})$, and so $\hat{\lambda}^\alpha(V^{k-1}, V^k) < \lambda^\alpha(V^k)$, and so we can start a simple wave counterclockwise from the previous shock, and we end it at $\lambda^\alpha(V^{k+1})$.

Then, either put a simple wave on $[\lambda^\alpha(V^\infty), \lambda^\alpha(V^+)]$ and set $V(\xi) = V^+$ on $[\lambda^\alpha(V^+), \lambda^\alpha(\overline{V}) + \delta_s]$, or set $V(\xi) = V^\infty$ on $[\lambda^\alpha(V^\infty), \lambda^\alpha(\overline{V}) + \delta_s]$.

All in all, we have constructed an example with infinitely many shocks, and possibly infinitely many simple waves (but with no consecutive simple waves). The limit point $\xi = \lambda^\alpha(V^\infty)$ of the shock set is necessarily in $\mathcal{R}$. In addition, if V^+ was chosen to be V^∞, then the solution is constant on $[\lambda^\alpha(V^\infty), \lambda^\alpha(\overline{V}) + \delta_s]$. However, if V^+ was chosen to be on the simple wave curve of V^∞, then we have a simple wave starting at the limit point of the shock set, and a constant solution on $[\lambda^\alpha(V^+), \lambda^\alpha(\overline{V}) + \delta_s]$.

To guarantee a solution exists for the entire $(x, y)-$plane, suppose that $f^x_U(\overline{U})$ has only positive eigenvalues, so that all the backward sectors lie in $x < 0$. Extend V to be V^+ from $\lambda^\alpha(\overline{V}) + \delta_s$ to the negative y-axis, and to be $\overline{V}$ from $\lambda^\alpha(\overline{V})$ to the positive y-axis. Then, since $\overline{V}$ and V^+ are sufficiently close, we can use Lax's solution to construct the solution for $x > 0$, since e is convex.

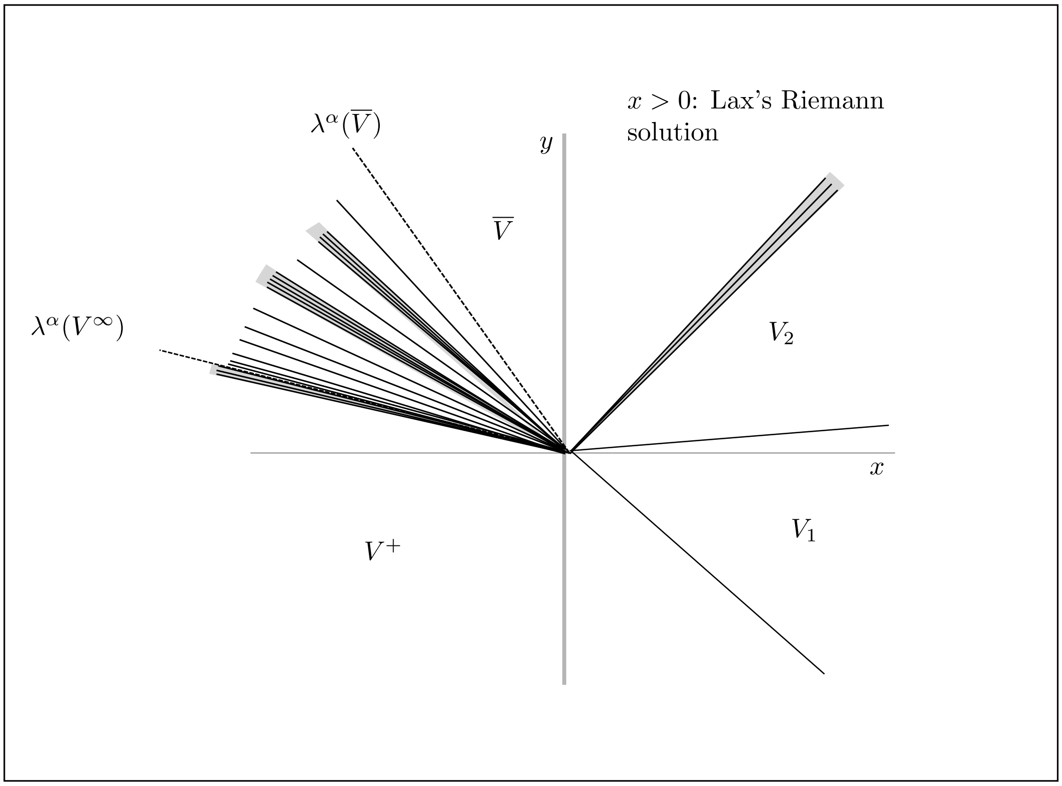

Figure 3.1: In a backward sector going counterclockwise, $\overline{V}$ is connected to successive states through a shock, a simple wave, a shock, a simple wave, and then infinitely many shocks which accumulate at $\xi = \lambda^\alpha(V^\infty)$. Then a simple wave between V^∞ and V^+ starts at $\lambda^\alpha(V^\infty)$. Riemann data of $\overline{V}$ and V^+ is prescribed on the y-axis, and Lax's solution (from Theorem I.1) is used to construct the remainder of the solution. In this case the Lax solution is two shocks and a simple wave, with intermediate constant states V_1 and V_2. (Simple waves are shaded gray.)

3.5 Example with Interspersed Forward and Backward Sectors

Most common examples and choices of background state can be rotated so that f_U^x is guaranteed to have eigenvalues of the same sign, so that the forward sectors all lie on one side of the y-axis, and the backward sectors all lie on the other. In this case, the steady system is hyperbolic in either the positive or negative x-direction, and our results show that on one side any admissible steady and self-similar solution must be Lax's solution. However, this need not always be the case.

It is easy to see that if we have a system of two conservation laws, that some rotation will be able to accomplish the above, and so the minimum example must be a system of three conservation laws. Consider

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix}_t + \begin{pmatrix} \frac{\tilde{u}^2}{4} \\ \frac{\tilde{v}^2}{4} \\ \frac{\tilde{w}^2}{4} \end{pmatrix}_x + \begin{pmatrix} \frac{\tilde{u}^4}{32} \\ \frac{\tilde{v}^4}{32} \\ \frac{\tilde{w}^4}{32} \end{pmatrix}_y = 0,$$

with

$$\eta(U) = \frac{1}{2}(\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2), \quad \psi^x(U) = \frac{1}{6}(\tilde{u}^3 + \tilde{v}^3 + \tilde{w}^3), \quad \psi^y(U) = \frac{1}{40}(\tilde{u}^5 + \tilde{v}^5 + \tilde{w}^5).$$

We easily see $(\eta, \vec{\psi})$ form an entropy-entropy flux pair with convex entropy. Consider the background state $\overline{U} = (2, -4, 8)$ so that

$$\begin{aligned} V &:= (u, v, w), & U(V) &= \big(2(u)^{1/2}, -2(v)^{1/2}, 2(w)^{1/2}\big), \\ f(V) = f^y\big(U(V)\big) &= \left(\frac{u^2}{2}, \frac{v^2}{2}, \frac{w^2}{2}\right), & \overline{V} &= (1, 4, 16). \end{aligned}$$

The entropy-entropy flux pair (e, q) is given by

$$\begin{aligned} e(V) &= \psi^x\big(U(V)\big) = \frac{4}{3}(u^{3/2} - v^{3/2} + w^{3/2}) \\ q(V) &= \psi^y\big(U(V)\big) = \frac{4}{5}(u^{5/2} - v^{5/2} + w^{5/2}). \end{aligned}$$

The eigenvectors of $f_U^x(\overline{U})$ are $1, -2, 4$, and so we know (from the discussion in the proof of Lemma II.17) that e_{VV} will have one negative and two positive eigenvalues.

By direct calculation, the eigenvalues of $f_V(V)$ are $\lambda^i(V) = u, v, w$ with eigenvectors $(1,0,0), (0,1,0)$, and $(0,0,1)$ for $i = 1, 2, 3$, respectively. We have that $\lambda_V^i r^i = 1$ for all i, and so each field is genuinely nonlinear. We have that

$$e_{VV}(V) = \begin{pmatrix} u^{-1/2} & 0 & 0 \\ 0 & -v^{-1/2} & 0 \\ 0 & 0 & w^{-1/2} \end{pmatrix}.$$

Then

$$e_{VV} r^1 r^1 > 0, \qquad e_{VV} r^2 r^2 < 0, \qquad e_{VV} r^3 r^3 > 0,$$

and so for $i = 1, 3$ the forward sectors have $x > 0$, and for $i = 2$ the forward sector has $x < 0$. It is clear that under no rotation of spatial coordinates can we achieve all forward sectors one one side of a line through the origin.

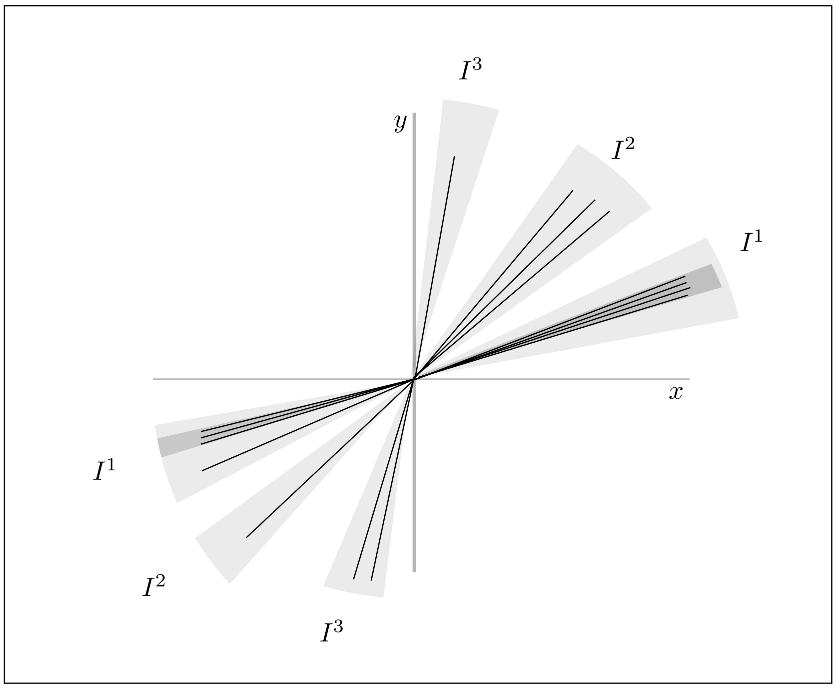

Figure 3.2: The forward 1- and 3-sectors and the backward 2-sector are in the first quadrant, while the backward 1- and 3-sectors and forward 2-sector are in the fourth quadrant. The forward 1-sector has a simple wave, the backward 2-sector has three shocks, and the forward 3-sector has a shock. The backward 1-sector has a simple wave and a shock, the forward 2-sector has a shock, and the backward 3-sector has two shocks. No line through the origin separates the three forward sectors from the three backward sectors, so this is not a Lax solution to any Riemann problem. (Sectors are shaded light gray, and simple waves are shaded dark gray.)

CHAPTER IV

More General Solutions of Full Euler

4.1 Preliminaries

We now consider steady and self-similar solutions to the *full Euler equations.* We will not restrict the phase space to a small ball around some background state. We will use explicit expressions for various quantities instead of the implicit function theorem.

The full Euler equations are given by

$$\begin{cases} U_t + f^x(U)_x + f^y(U)_y = 0, \\ \eta(U)_t + \psi^x(U)_x + \psi^y(U)_y \geq 0, \end{cases} \tag{4.1}$$

where

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}, f^x(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(\rho E + p) \end{pmatrix}, f^y(U) = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(\rho E + p) \end{pmatrix},$$

$$\eta(U) = \rho S, \qquad \psi^x(U) = \rho u S, \qquad \psi^y(U) = \rho v S.$$

As before, ρ is the density, u and v are the horizontal and vertical velocities, respectively, p is the pressure, and E is the total energy per unit mass. S is the entropy per unit mass.

We for now assume only that U is bounded, interpreting (4.1) in the sense of distributions. Proceeding as in Chapter II, we derive the weak form for steady, self-similar solutions U which only depend on θ, the polar angle $\angle(x, y)$.

The integral form of the entropy inequality in (4.1) is

$$-\int_{\mathbb{R}_+\times\mathbb{R}^2} \Phi_t\eta(U) + \Phi_x\psi^x(U) + \Phi_y\psi^y(U)\ d(t,x,y) - \int_{\mathbb{R}^2} \big(\Phi\eta(U)\big)\Big|_{t=0} \geq 0, \tag{4.2}$$

and a weak solution must satisfy (4.2) for any smooth, non-negative, compactly supported test function Φ.

Integrating by parts in t eliminates the first term in the integrand of (4.2), after which using compact-in-t support and integrating with respect to t yields the equivalent statement

$$-\int_{\mathbb{R}^2} \Phi_x\psi^x(U) + \Phi_y\psi^y(U) d(x,y) \geq 0$$

for all nonnegative smooth compactly supported t-independent functions $\Phi : \mathbb{R}^2 \to \mathbb{R}$. Now, change variables to polar coordinates to obtain

$$0 \leq -\int_0^\infty \int_0^{2\pi} \Big(\Phi_x\psi^x\big(U(\theta)\big) + \Phi_y\psi^y\big(U(\theta)\big)\Big) r d\theta dr. \tag{4.3}$$

Define a smooth 2π periodic function ϕ as

$$\phi(\theta) := \int_0^\infty \Phi(r\cos\theta, r\sin\theta)dr = -\cos\theta\int_0^\infty \Phi_x r dr - \sin\theta\int_0^\infty \Phi_y r dr.$$

Notice that

$$\phi'(\theta) = -\sin\theta\int_0^\infty \Phi_x r dr + \cos\theta\int_0^\infty \Phi_y r dr.$$

We then have that

$$-\int_0^\infty \Phi_x r dr = \cos\theta\phi(\theta) + \sin\theta\phi'(\theta),$$
$$-\int_0^\infty \Phi_y r dr = \sin\theta\phi(\theta) - \cos\theta\phi'(\theta).$$

Therefore (4.3) becomes

$$0 \leq \int_0^{2\pi} \Big(\cos\theta\phi + \sin\theta\phi'\Big)\psi^x\big(U(\theta)\big) + \Big(\sin\theta\phi - \cos\theta\phi'\Big)\psi^y\big(U(\theta)\big)d\theta.$$

This is the weak form of

$$\Big(-\sin\theta\psi^x\big(U(\theta)\big) + \cos\theta\psi^y\big(U(\theta)\big)\Big)_\theta + \cos\theta\psi^x\big(U(\theta)\big) + \sin\theta\psi^y\big(U(\theta)\big) \geq 0,$$

or equivalently,

$$\Big(\sin\theta\psi^x\big(U(\theta)\big) - \cos\theta\psi^y\big(U(\theta)\big)\Big)_\theta - \cos\theta\psi^x\big(U(\theta)\big) - \sin\theta\psi^y\big(U(\theta)\big) \leq 0. \tag{4.4}$$

As in Section 2.4, it follows that for almost every $\theta_1 < \theta_2$,

$$\big(\sin\theta\psi^x(U) - \cos\theta\psi^y(U)\big)\Big|_{\theta_1}^{\theta_2} \leq \int_{\theta_1}^{\theta_2} \cos\eta\psi^x(U(\eta)) + \sin\eta\psi^y(U(\eta))d\eta. \tag{4.5}$$

Similarly, we obtain that the first line of (4.1) for steady and self-similar solutions is

$$\Big(\sin\theta f^x\big(U(\theta)\big) - \cos\theta f^y\big(U(\theta)\big)\Big)_\theta = \cos\theta f^x\big(U(\theta)\big) + \sin\theta f^y\big(U(\theta)\big). \tag{4.6}$$

The right side is L^∞, and so the quantity being differentiated in the distributional sense on the left must have a version that is Lipschitz. Therefore, the fundamental theorem of calculus holds and we have, for almost every θ_1 and θ_2,

$$\Big(\sin\theta f^x(U) - \cos\theta f^y(U)\Big)\Big|_{\theta_1}^{\theta_2} = \int_{\theta_1}^{\theta_2} \cos\eta f^x(U(\eta)) + \sin\eta f^y(U(\eta))d\eta. \tag{4.7}$$

We now make some assumptions on our phase space and equation of state, so that we can utilize Lemma A.1 to obtain a version of U for which these pointwise conditions hold everywhere.

From thermodynamics, of the five quantities ρ, e, S, p, and temperature T, only two are independent — that is any of these can be written as a function of any two of the others. With this in mind, we make the following assumption.

Assumption IV.1. *We assume* $U = (\rho, \rho u, \rho v, \rho E) : S^1 \to \mathcal{P} \in \mathbb{R}^4$ *is* L^∞*, with* $\mathcal{P}$ *compact, and that any state in* $U \in \mathcal{P}$ *satisfies*

$$0 < C^{-1} \leq \rho \leq C < \infty, \tag{4.8}$$

$$0 < C^{-1} \leq e \leq C < \infty, \tag{4.9}$$

$$|U| \leq C < \infty,$$

where C *is some positive constant. We will also assume that*

$$T > 0,$$

$$p > 0.$$

We assume that any thermodynamic quantity can be expressed as a smooth function of any of the other two, as we are away from pathological cases such as vacuum, zero temperature, etc. Therefore the uniform lower bounds bounds (4.8) *and* (4.9) *show that* T, p*, and* S *also take values in a compact set, and* T *and* p *are bounded away form zero for any state in* $\mathcal{P}$*.*

Remark IV.2. Note we have not specified the exact form of $\mathcal{P}$ — later we will need convexity of the phase space in terms of different variables and so we will adjust $\mathcal{P}$ to ensure this.

As in Section 2.4, Lemma A.1 shows there is a version of U such that (4.5) and (4.7) are satisfied *for all* $\theta_1 < \theta_2$.

4.2 Jump Conditions

In Chapter II we used the eigenvalues and eigenvectors of the averaged matrix and the implicit function theorem to determine the properties of shock transitions, but now that we need to treat large jumps we need to investigate the algebraic properties of the Euler fluxes.

We have, for all θ_1 and θ_2,

$$\Big(-\cos\theta f^y(U(\theta)) + \sin\theta f^x(U(\theta)) \Big)\Big|_{\theta_1}^{\theta_2} = \mathcal{O}(|\theta_2 - \theta_1|). \tag{4.10}$$

On sequences $\{\theta_n^\pm\} \to \theta$ with $U(\theta^\pm) \to U_\pm$, we have

$$\sin\theta\Big[f^x(U)\Big] - \cos\theta\Big[f^y(U)\Big] = 0$$

(where $[g(U)] = g(U_+) - g(U_-)$ for any function g of U). Substituting in the Euler fluxes, we have

$$\sin\theta[\rho u] - \cos\theta[\rho v] = 0, \tag{4.11}$$

$$\sin\theta[\rho u^2 + p] - \cos\theta[\rho uv] = 0, \tag{4.12}$$

$$\sin\theta[\rho uv] - \cos\theta[\rho v^2 + p] = 0, \tag{4.13}$$

$$\sin\theta\,[u(\rho E + p)] - \cos\theta\,[v(\rho E + p)] = 0. \tag{4.14}$$

To separate the cases of shocks and contacts, we introduce the normal (angular) and tangential (radial) velocities at θ:

$$N := u\sin\theta - v\cos\theta, \qquad L := u\cos\theta + v\sin\theta,$$

$$u = N\sin\theta + L\cos\theta, \qquad v = -N\cos\theta + L\sin\theta.$$

We immediately observe that (4.11) is equivalent to

$$[\rho N] = 0. \tag{4.15}$$

(4.12) is equivalent to

$$\sin\theta[p] = [\cos\theta\rho uv - \sin\theta\rho u^2] = -[\rho u N]. \tag{4.16}$$

Similarly, (4.13) yields

$$\cos\theta[p] = [\sin\theta\rho uv - \cos\theta\rho v^2] = [\rho v N]. \tag{4.17}$$

Therefore,

$$0 = \sin\theta[\rho v N] + \cos\theta[\rho u N] = [\rho N L].$$

This means that $\rho_+ N_+ L_+ - \rho_- N_- L_- = 0$, or $\rho_+ N_+ (L_+ - L_-) = 0$ (from (4.15)). Therefore, if $\rho_+ N_+ = \rho_- N_- \neq 0$, $[L] = 0$. Hence, if there is mass flux through a shock, the tangential velocity is continuous. However, if $N_+ = N_- = 0$, the gas does not flow through the discontinuity and is we have a contact discontinuity. In this case there can be a jump in L.

Also, $\sin\theta\cdot$(4.16) + $\cos\theta\cdot$(4.17) yields

$$[p] = -\sin\theta[\rho u N] + \cos\theta[\rho v N] = -[\rho N^2] \implies [\rho N^2 + p] = 0. \tag{4.18}$$

Finally, (4.14) is equivalent to

$$0 = [N(\rho E + p)] = \left[\frac{1}{2}\rho N|\vec{u}|^2 + \rho N e + N p\right]. \tag{4.19}$$

For the case of a shock, divide by $\rho_+ N_+ = \rho_- N_-$ to obtain

$$\left[\frac{1}{2}|\vec{u}|^2 + e + p\tau\right] = 0, \tag{4.20}$$

where $\tau := \rho^{-1}$ is the specific volume (volume per unit mass). Denote $\rho_- N_- = \rho_+ N_+ =: \mathcal{M}$. Then (4.18) becomes

$$[p] = -\mathcal{M}[N]. \tag{4.21}$$

Therefore,

$$\begin{aligned} \frac{[p]}{[\tau]} &= -\mathcal{M}\frac{[N]}{[\tau]} = -\mathcal{M}\frac{N_+ - N_-}{\frac{1}{\rho_+} - \frac{1}{\rho_-}} \\ &= -\mathcal{M}\frac{N_+ - N_-}{\frac{\rho_- - \rho_+}{\rho_- \rho_+}} = -\mathcal{M}\frac{\rho_- \mathcal{M} - \rho_+ \mathcal{M}}{\rho_- - \rho_+} = -\mathcal{M}^2. \end{aligned}$$

Multiplying (4.21) by $(\tau_- + \tau_+)$ yields

$$\begin{aligned}
[p](\tau_- + \tau_+) &= -\mathcal{M}[N](\tau_- + \tau_+) \\
&= \mathcal{M}(N_- - N_+)\frac{\rho_- + \rho_+}{\rho_-\rho_+} \\
&= \frac{\mathcal{M}(\rho_- N_- - \rho_+ N_+ - \rho_- N_+ + \rho_+ N_-)}{\rho_-\rho_+} \\
&= \mathcal{M}(\tau_- N_- - \tau_+ N_+) = N_-^2 - N_+^2 = -[|\vec{u}|^2].
\end{aligned}$$

Substituting this into (4.20) yields

$$\begin{aligned}
[e + p\tau] &= \frac{1}{2}[p](\tau_- + \tau_+), \\
[e] &= -\frac{1}{2}(p_- + p_+)[\tau].
\end{aligned}$$

Therefore, for a shock wave with fixed (τ_+, p_+), the states (τ_-, p_-) that can be connected by a shock are defined by $H(\tau_-, p_-) = 0$, where $H(\tau, p)$ is the Hugoniot function defined below:

$$H(\tau, p) = e(\tau, p) - e(\tau_+, p_+) + \frac{1}{2}(\tau - \tau_+)(p + p_+).$$

4.3 Weak Form with Tangential and Normal Velocities

Analogous steps to those used to simplify the jump conditions to be in terms of radial and angular velocities can be done on the weak form of the equations. Multiplying a distribution (in this case an L^∞ function) by a smooth function ($\sin\theta$ or $\cos\theta$) results in another distribution, and the product rule applies for distributional derivatives of distributions multiplied with smooth functions.

(4.6) with the Euler fluxes becomes the following system:

$$\begin{aligned}
\Big(\sin\theta\rho u - \cos\theta\rho v\Big)_\theta &= \cos\theta\rho u + \sin\theta\rho v,\\
\Big(\sin\theta(\rho u^2+p) - \cos\theta(\rho uv)\Big)_\theta &= \cos\theta(\rho u^2+p) + \sin\theta(\rho uv),\\
\Big(\sin\theta(\rho uv) - \cos\theta(\rho v^2+p)\Big)_\theta &= \cos\theta(\rho uv) + \sin\theta(\rho v^2+p),\\
\Big(\sin\theta\big(u(\rho E+p)\big) - \cos\theta\big(v(\rho E+p)\big)\Big)_\theta &= \cos\theta\big(u(\rho E+p)\big) + \sin\theta\big(v(\rho E+p)\big).
\end{aligned}$$

Substituting the definitions of N and L we obtain:

$$(\rho N)_\theta = \rho L,$$

$$\big(u\rho N + p\sin\theta\big)_\theta = u\rho L + p\cos\theta, \tag{4.22}$$

$$\big(v\rho N - p\cos\theta)\big)_\theta = v\rho L + p\sin\theta, \tag{4.23}$$

$$\big(N(\rho E+p)\big)_\theta = L(\rho E+p).$$

Note that

$$\begin{aligned}
\sin\theta\big(u\rho N + p\sin\theta\big)_\theta &= \big(u\sin\theta(\rho N) + p\sin^2\theta\big)_\theta - \cos\theta\big(u\rho N + p\sin\theta\big),\\
-\cos\theta\big(v\rho N - p\cos\theta)\big)_\theta &= \big(-v\cos\theta(\rho N) + p\cos^2\theta\big)_\theta - \sin\theta\big(v\rho N - p\cos\theta)\big).
\end{aligned}$$

Thus, $\sin\theta\cdot(4.22) - \cos\theta\cdot(4.23)$ yields

$$\big(\rho N^2 + p\big)_\theta - \rho NL = \rho NL,$$

or

$$\big(\rho N^2 + p\big)_\theta = 2\rho NL.$$

Similarly,

$$\begin{aligned}
\cos\theta\big(u\rho N + p\sin\theta\big)_\theta &= \big(u\cos\theta(\rho N) + p\sin\theta\cos\theta\big)_\theta + \sin\theta\big(u\rho N + p\sin\theta\big),\\
\sin\theta\big(v\rho N - p\cos\theta)\big)_\theta &= \big(v\sin\theta(\rho N) - p\sin\theta\cos\theta\big)_\theta - \cos\theta\big(v\rho N - p\cos\theta)\big),
\end{aligned}$$

and so $\cos\theta \cdot (4.22) + \sin\theta \cdot (4.23)$ yields

$$\big(\rho LN\big)_\theta + \rho N^2 + p = \rho L^2 + p,$$

or

$$\big(\rho LN\big)_\theta = \rho L^2 - \rho N^2.$$

Therefore, the Euler equations are equivalent to

$$\begin{aligned}
(\rho N)_\theta &= \rho L, \\
(\rho N^2 + p)_\theta &= 2\rho N L, \\
(\rho L N)_\theta &= \rho L^2 - \rho N^2, \\
(N(\rho E + p))_\theta &= L(\rho E + p),
\end{aligned}$$

satisfied in the distributional sense.

4.4 Shock Sides

We now argue that even in the L^∞ setting, in which left and right limits may not exist, there still exists a well defined notion of shocks and contacts, as well as front and back sides of shocks.

Lemma IV.3. *If U is discontinuous at θ_0, then it can be well defined as having either a forward facing shock, a backward facing shock, or a contact discontinuity at θ_0. If U has a contact discontinuity at θ_0, then N is continuous at θ_0 and $N(\theta_0) = 0$.*

Proof. Suppose $U(\theta)$ is discontinuous at θ_0. Then we can pick sequences $\{\theta_n\}, \{\theta_n'\} \to \theta_0$ with $U(\theta_n) \to U_0$, $U(\theta_n') \to U_0'$, with $U_0 \neq U_0'$. Then, from above, necessarily $\rho_0 N_0 = \rho_0' N_0'$. Based on our assumptions about ρ, it follows that N_0 and N_0' are either both positive, both negative, or both zero.

Case 1: $N_0 = N_0' = 0$. Let $\{\theta_n''\}$ be any other sequence converging to θ_0, and take any subsequence $\theta_{n(k)}''$. Since U is L^∞, there exists a subsequence $\left\{\theta_{n(k(j))}''\right\}$ such that $U(\theta_{n(k(j))}'') \to U_0''$. Applying the jump conditions to U_0 and U_0'' shows that $N_0'' = 0$, because $N_0 = 0$. Therefore, since any subsequence has a subsequence converging to zero, $N(\theta_n'') \to 0$, and as $\{\theta_n''\}$ was arbitrary, we see that in fact $N(\theta)$ is continuous at θ_0 and $N(\theta_0) = 0$. In this case we say that U has a contact discontinuity at θ_0.

Case 2: $N_0, N_0' > 0$. We claim that there exists some neighborhood containing θ_0 on which $N(\theta)$ is positive (except for at the point θ_0 itself, since U is discontinuous and not even necessarily well defined there). Suppose not. Then there exists $\{\theta_n''\} \to \theta_0$ such that $N(\theta_n'') \leq 0$ for all n. Again, there exists a subsequence $\left\{\theta_{n(k)}''\right\}$ such that $U(\theta_{n(k)}'') \to U_0''$, and by assumption $N_0'' \leq 0$. However, applying the jump conditions to N_0 and N_0'' gives a contradiction. Therefore, $N(\theta) > 0$ on some neighborhood of θ_0, and we call the shock forward facing. (The interpretation is that gas particles enter the front side of the shock, and leave the back side. Since the flow of mass through the shock is aligned with our choice of normal vector for the shock, we call this case forward facing.)

Case 3, for which $N_0, N_1 < 0$, is similar. We call this case backward facing. $\square$

4.5 Entropy

Recall the entropy inequality (4.4), with $\theta_n^- < \theta_n^+$,

$$\Big(-\cos\theta\psi^y(U(\theta)) + \sin\theta\psi^x(U(\theta))\Big)\Big|_{\theta_n^-}^{\theta_n^+} + \mathcal{O}(|\theta_n^+ - \theta_n^-|) \leq 0.$$

In the limit $\theta_n^\pm \to \theta$, if $U(\theta_n^\pm) \to U_\pm$, this becomes

$$\sin\theta[\psi^x(U)] - \cos\theta[\psi^y(U)] \leq 0.$$

For the case of the Euler equations, this reads

$$0 \geq \sin\theta[\rho u S] - \cos\theta[\rho v S] = [\rho N S],$$

which becomes

$$\mathcal{M}[S] \leq 0, \tag{4.24}$$

(where $\mathcal{M} = \rho_- N_- = \rho_+ N_+$). In the case of a forward facing shock, $S_+ \leq S_-$, and in the case of a backward facing shock, $S_+ \geq S_-$. In either case, the entropy cannot decrease when the gas passes through the shock from the front to the back.

We now need to derive Lax-type admissibility conditions from (4.24). Instead of using properties of eigenvalues and eigenvectors to obtain local results, we will need to use specific properties of the Euler system to derive global information. To do this, we will need some more assumptions regarding our equation of state.

Assumption IV.4. *We assume that the equation of state for pressure is given by $p = g(\tau, S)$ and that it is smooth for the phase space under consideration. We also assume*

$$g_\tau := -\rho^2 c^2 < 0, \tag{4.25}$$

$$g_{\tau\tau} > 0,$$

$$g_S > 0,$$

where $c > 0$ is the sound speed. Furthermore, from thermodynamics we have

$$\left(\frac{\partial e}{\partial \tau}\right)_{S=const} = -p$$
$$\left(\frac{\partial e}{\partial S}\right)_{\tau=const} = T.$$

Note these last relations are equivalent to

$$\left(\frac{\partial e}{\partial \rho}\right)_{S=\text{const}} = \frac{p}{\rho^2},$$
$$\left(\frac{\partial e}{\partial S}\right)_{\rho=\text{const}} = T.$$

The reader is warned of some possible confusion regarding the speed of sound defined in (4.25) as

$$c^2 := \left(\frac{\partial p}{\partial \rho}\right)_{S=\text{const}},$$

since we may also come across the definition

$$c^2 := \left(\frac{\partial p}{\partial \rho}\right)_{e=\text{const}} + \frac{p}{\rho^2}\left(\frac{\partial p}{\partial e}\right)_{\rho=\text{const}}.$$

However, they are the same. Notice

$$p(\rho, S) = p(\rho, e).$$

Differentiating with respect to ρ while holding S constant, we obtain

$$\begin{aligned}\left(\frac{\partial p}{\partial \rho}\right)_{S=\text{const}} &= \left(\frac{\partial p}{\partial \rho}\right)_{e=\text{const}}\left(\frac{\partial \rho}{\partial \rho}\right)_{S=\text{const}} + \left(\frac{\partial p}{\partial e}\right)_{\rho=\text{const}}\left(\frac{\partial e}{\partial \rho}\right)_{S=\text{const}} \\ &= \left(\frac{\partial p}{\partial \rho}\right)_{e=\text{const}} + \frac{p}{\rho^2}\left(\frac{\partial p}{\partial e}\right)_{\rho=\text{const}}.\end{aligned}$$

We now assume the case of a forward facing shock, and fix a pair of left and right sequences $\theta_n^{\pm}$ as usual, so that $\theta_n^- < \theta_n^+$ for all n, and $U(\theta_n^{\pm}) \to U_{\pm}$. We know that U_- must satisfy $H(\tau_-, p_-) = 0$ for the Hugoniot function for state (τ_+, p_+). We use the following result summarized by Courant and Friedrichs, which gives us our Lax-type conditions for a shock transition.

Lemma IV.5 (Bethe, Weyl as in Section 65 of [11])**.** *Assume that for fixed (τ_+, p_+), the set of states (τ, p) for which $H(\tau, p) = 0$ is a smooth curve in the (τ, p) plane* *that can be described by $p = G(\tau)$, and that the equation of state satisfies Assumption IV.4. Then, for the case of an entropy admissible forward facing shock, which requires $S_- > S_+$, we have*

1. $\tau_- < \tau_+$,
2. $N_+ > c_+ > 0, 0 < N_- < c_-$.

That is, the density increases as gas travels through the shock. The normal velocity for the gas before the shock is supersonic, and the normal velocity after passing through the shock is subsonic.

We have the analogous statement for an entropy admissible backward facing shock, which instead requires $S_- < S_+$:

1. $\tau_- > \tau_+$,
2. $-N_- > c_- > 0, 0 < -N_+ < c_+$.

Notation IV.6. For the remainder of this paper, a subscript *max* or *min* on a quantity refers to the maximum or minimum permissible value of that quantity for $U \in \mathcal{P}$.

Since we now know that one side of the shock must be supersonic, we can immediately conclude that one of N_+ or N_- must be greater in magnitude than $c_{\min}$, and using $\rho_+ N_+ = \rho_- N_-$ immediately gives us

$$|N_\pm| \geq \frac{c_{\min}\rho_{\min}}{\rho_{\max}}$$

for a shock. In the next section we will use this fact that our shock transitions cannot jump to arbitrary small normal velocities to determine how close a contact can be to a shock. Now that we have our global Lax conditions for shocks, we can start to analyze how close shocks and contact discontinuities can be to one another.

4.6 Properties of Discontinuities, Uniform Distances between Different Types of Discontinuities

We will again use the idea of an averaged matrix, which in this setting is a smooth matrix valued function of $U^{\pm}$ and θ that satisfies:

- $\sin\theta\big(f^x(U_+) - f^x(U_-)\big) - \cos\theta\big(f^y(U_+) - f^y(U_-)\big) = \hat{A}(U_-, U_+; \theta)(U_+ - U_-)$,
- $\hat{A}(U, U; \theta) = A(U; \theta) := \sin\theta f^x_U(U) - \cos\theta f^y_U(U)$, and
- $\hat{A}(U_-, U_+; \theta)$ is diagonalizable with real eigenvalues for all $U^{\pm} \in \mathcal{P}$, $\theta \in [0, 2\pi[$.

As discussed in Section 2.9, the existence of matrices with the important conservation and diagonalizability properties is guaranteed (see [23]) if our system possesses an entropy-entropy flux (η, ψ^x, ψ^y) with η strictly convex, but one that is easier for computations if the equation of state is *polytropic* (an assumption we will define and make later) is the *Roe averaged matrix*. It will be discussed in more detail as needed later, but for now we just need its existence and the properties listed above.

Rearranging (4.10), we obtain

$$\sin\theta_1\Big(f^x\big(U(\theta_2)\big) - f^x\big(U(\theta_1)\big)\Big) - \cos\theta_1\Big(f^y\big(U(\theta_2)\big) - f^y\big(U(\theta_1)\big)\Big) = \mathcal{O}(|\theta_2 - \theta_1|).$$

Using the averaged matrix, we have

$$\hat{A}\big(U(\theta_1), U(\theta_2); \theta_1\big)\big(U(\theta_2) - U(\theta_1)\big) = \mathcal{O}(|\theta_2 - \theta_1|). \tag{4.26}$$

Now, consider the matrix

$$A(U; \theta) = \sin\theta f^x_U(U) - \cos\theta f^y_U(U).$$

We can define

$$B(U; \theta) = \sin\theta f^x_W - \cos\theta f^y_W,$$

where the W variables are from Section 3.3.1.[1] Notice

$$B(U;\theta)W_U = A(U;\theta)$$

with $\det W_U \neq 0$. Using Maple, we have that

$$\det B(U;\theta) = 0 \iff |N| = c \text{ or } N = 0,$$

and therefore

$$\det A(U;\theta) = 0 \iff |N| = c \text{ or } N = 0.$$

We then have the following theorem which is analogous to Theorem II.20.

Theorem IV.7. *Suppose U is continuous on an interval $]\theta_1, \theta_2[$ and that $|N| \neq \pm c$ or 0 on this interval. Then U is constant on this interval.*

Proof. Fix some $\theta \in]\theta_1, \theta_2[$. We claim that U must be Lipschitz at θ. Suppose not. Then we can choose a sequence $\{\theta_n\} \to \theta$ (with $|\theta_n - \theta| \neq 0$) such that

$$0 < \left| \frac{U(\theta_n) - U(\theta)}{\theta_n - \theta} \right| \nearrow \infty.$$

Divide both sides of (4.26) by $|U(\theta_n) - U(\theta)|$ to obtain

$$\begin{aligned} \hat{A}\big(U(\theta), U(\theta_n); \theta\big) \left(\frac{U(\theta_n) - U(\theta)}{|U(\theta_n) - U(\theta)|} \right) &= \frac{1}{|U(\theta_n) - U(\theta)|} \mathcal{O}(|\theta_n - \theta|) \\ &= o(1) \quad \text{as } n \to \infty. \end{aligned}$$

By assumption, $A\big(U(\theta); \theta\big)$ is non-degenerate, so for $|\theta_n - \theta|$ sufficiently small, $\hat{A}\big(U(\theta), U(\theta_n); \theta\big)$ will be uniformly non-degenerate, because the eigenvalues of $\hat{A}$ are continuous functions of $U_\pm$, and U is continuous at θ by assumption. That is,

$$\exists \delta > 0 \; \forall z \in \mathbb{R}^m : \left| \Big(\hat{A}\big(U(\theta), U(\theta_n); \theta_1\big) \Big) z \right| \geq \delta |z|.$$

[1] We have $W = (\rho, \rho u, \rho v, S)$, which is advantageous because it is complicated to differentiate with respect to the total energy $\rho(\frac{u^2+v^2}{2} + e)$.

Taking $n \to \infty$, the left hand side stays bounded away from zero, while the right hand side goes to zero, leading to a contradiction.

Therefore, U must be Lipschitz on $]\theta_1, \theta_2[$. Assuming θ is a point of differentiability of U, we obtain

$$A\big(U(\theta); \theta\big) U_\theta = 0.$$

However, as we assumed the matrix was non-degenerate on $]\theta_1, \theta_2[$, it follows that $U_\theta = 0$ on this interval. A Lipschitz function is differentiable almost everywhere and is the integral of its derivative, so U is constant on $]\theta_1, \theta_2[$. $\square$

We now show that the shock set is discrete, analogous to Theorem II.33. However in this case we do not lower bound the size of the constant neighborhoods on either side of the shock yet — we will later when we pick a specific equation of state.

Theorem IV.8. *Suppose there is a shock at θ_0. Then there exist $\sigma^+(\theta_0) > \theta_0$ and $\sigma^-(\theta_0) < \theta_0$ so that U is constant on $]\sigma^-(\theta_0), \theta_0[,]\theta_0, \sigma^+(\theta_0)[$. In particular U has well defined left and right limits at shocks.*

Proof. We consider the case of a forward facing shock; backward facing shocks can be treated similarly. If there is a shock, choose left and right sequences $\theta_n^\pm \to \theta_0$ with $U(\theta_n^\pm) \to U_\pm$ with $\theta_n^- < \theta_n^+$ for all n. We have from above that

$$N_+ > c_+,$$
$$c_- > N_- > 0.$$

Suppose there is no η such that $N > c$ for all $\theta \in]\theta_0, \eta[$. Then pick a new sequence

$\theta_n^{++} \searrow \theta_0$ such that (passing to subseqences if necessary)

$$N(\theta_n^{++}) \leq c(\theta_n^{++}),$$

$$U(\theta_n^{++}) \to U_{++},$$

$$\theta_n^{+} < \theta_n^{++},$$

for all n. If $U_{++} \neq U_+$, then the Lax condition requires $N_{++} > c_{++}$, which contradicts our construction. If $U_{++} = U_+$, then $N_{++} = N_+ > c_+ = c_{++}$, also a contradiction. Therefore there exists $\eta > \theta_0$ such that $N > c$ for all $\theta \in]\theta_0, \eta[$.

Define

$$\sigma^+(\theta_0) := \sup \left\{ \eta > \theta_0 \,\middle|\, N(\theta) > c(\theta)\ \forall \theta \in]\theta_0, \eta[\right\}. \tag{4.27}$$

This supremum is being taken over a non-empty set, and is thus well defined.

The fact that there is some $\eta < \theta_0$ such that $N < c$ for all $\theta \in]\eta, \theta_0[$ is analogous. Combining this with our observation in the proof of Lemma IV.3 that there exists $\eta < \theta_0$ so that $N(\theta) > 0$ for all $\theta \in]\eta, \theta_0[$, we have that

$$\sigma^-(\theta_0) := \inf \left\{ \eta < \theta_0 \,\middle|\, 0 < N(\theta) < c(\theta)\ \forall \theta \in]\eta, \theta_0[\right\} \tag{4.28}$$

is well defined too.

All that is left to show is that U cannot possess any other discontinuities in these left and right open neighborhoods. Contacts are already ruled out — we know that $N = 0$ at a contact (note there could be a contact at $\sigma^-(\theta_0)$, but we are claiming there cannot be a contact on $]\sigma^-(\theta_0), \theta_0[$). Therefore we must show there cannot be shocks. Suppose there was a shock at $\theta_1 \in]\theta_0, \sigma^+(\theta_0)[$. It too must be forward facing, since N is positive for all such θ. Then, for some $\eta \in]\theta_0, \sigma^+(\theta_0)[\cap]\sigma^-(\theta_1), \theta_1[$ we would have

$$N(\eta) > c(\eta) < N(\eta),$$

where the first inequality is due to the shock at θ_0, and the second inequality is due to the shock at θ_1 — a contradiction. Therefore, U is continuous on $]\theta_0, \sigma^+(\theta_0)[$ and Theorem IV.7 shows it is constant. Similarly we can conclude U is constant on $]\sigma^-(\theta_0), \theta_0[$. $\square$

We now prove that if the velocity does not vanish then contact discontinuities are also isolated and must possess constant neighborhoods.

Theorem IV.9. *Suppose U has a contact discontinuity at θ_0. Then, either*

- *there exist $\sigma^+(\theta_0) > \theta_0$ and $\sigma^-(\theta_0) < \theta_0$ such that U is constant on $]\sigma^-(\theta_0), \theta_0[,]\theta_0, \sigma^+(\theta_0)[$, or*
- *θ_0 is contained in a closed interval on which $|\vec{u}| = 0$.*

Proof. From Lemma IV.3 we know that in fact N is continuous at θ_0 and that $N(\theta_0) = 0$. Therefore, choose a $\pi > \delta > 0$ such that

$$|\theta - \theta_0| < \delta \implies |N(\theta)| \leq \frac{c_{\min}\rho_{\min}}{2\rho_{\max}}.$$

From the discussion following Lemma IV.5, we have the lower bound for normal velocity at a shock:

$$|N_\pm| \geq \frac{c_{\min}\rho_{\min}}{\rho_{\max}}.$$

Therefore, for $|\theta - \theta_0| < \delta$ there can be no shocks. If there is another contact discontinuity, N must still be continuous, and so N is continuous for $|\theta - \theta_0| < \delta$. Therefore, the set

$$C := \left\{\theta \,\middle|\, |\theta - \theta_0| < \delta, N(\theta) = 0\right\}$$

is relatively closed in $]\theta_0 - \delta, \theta_0 + \delta[$ and U is constant on $]\theta_0 - \delta, \theta_0 + \delta[\setminus C$.

Suppose there are $\theta_1, \theta_2 \in C$ and $\eta \in]\theta_0 - \delta, \theta_0 + \delta[\setminus C$ with $\theta_1 < \eta < \theta_2$. Since this set is open we can find a maximal $]\eta^-, \eta^+[$ containing η but not meeting C. However, since N is continuous for $\eta \in]\theta_0 - \delta, \theta_0 + \delta[$ we can take limits and find (since U is constant on $]\eta^-, \eta^+[$)

$$N(\eta^-) = u(\eta)\sin\eta^- - v(\eta)\cos\eta^- = 0,$$

$$N(\eta^+) = u(\eta)\sin\eta^+ - v(\eta)\cos\eta^+ = 0.$$

Then, since $\eta^- \neq \eta^+$, the vectors $(\sin\eta^-, -\cos\eta^-)$ and $(\sin\eta^+, -\cos\eta^+)$ span $\mathbb{R}^2$ (since we took $\delta < \pi$), and so $\big(u(\eta), v(\eta)\big) = 0$. However this contradicts that $N \neq 0$ on $]\eta^-, \eta^+[$. Therefore, C is either a closed interval containing θ_0 or simply $\{\theta_0\}$.

Consider the conservation of mass equation,

$$(\rho N)_\theta = \rho L,$$

which is satisfied in the distributional sense. If $N \equiv 0$ on a closed interval, we can take a strong derivative at any θ in its interior to obtain that $L \equiv 0$ on the interior. Therefore the supposed contact at θ_0 could not have had a jump in tangential velocity, only density and entropy (continuity of pressure is required by the jump conditions). If we disregard this possibility, then it follows that this closed interval must be a single point, and so there can only be one contact for $|\theta - \theta_0| < \delta$.

This opens the possibility to very irregular solutions. If the velocity field is zero on some interval, then a completely arbitrary density distribution can be prescribed, as long as the entropy/internal energy/temperature is also prescribed to result in constant pressure. These very irregular solutions are not all that surprising, considering that when the velocity is identically zero the Euler equations become $p_\theta = 0$ in the sense of distributions, so $p \equiv$ constant. Note that for isentropic flow, this situation cannot occur, since constant pressure can only be attained if density is constant. $\square$

If we assume that the velocity does not vanish, we have shown that the set on which U has a discontinuity is countable and discrete. Therefore, right and left limits are well defined, and so from here on out we modify U to be right continuous at every point.

Lemma IV.10. *Assume there are no stagnation points (that is, $|\vec{u}| \neq 0$ everywhere). Then the set on which $N = 0$ is a finite set of points. Moreover, if $N(\theta_0) = 0$, and there is a shock at θ',*

$$|\theta' - \theta_0| \geq \delta > 0,$$

with δ independent of U.

Proof. Take any θ_0 with $N(\theta_0) = 0$. Define

$$\sigma^+(\theta_0) := \sup\left\{\eta > \theta_0 \,\middle|\, 0 < |N(\theta)| < \frac{c_{\min}\rho_{\min}}{2\rho_{\max}} \quad \forall \theta \in]\theta_0, \eta[\right\}.$$

This supremum is defined because it is taken over a nonempty set by Theorem IV.9. Moreover, since there cannot be any shocks on $]\theta_0, \sigma^+(\theta_0)]$ (since we have required $|N|$ be less than the minimum allowable value on either side of a shock), it follows that $|N(\sigma^+(\theta_0))| = 0$ or $\frac{c_{\min}\rho_{\min}}{2\rho_{\max}}$. We claim that if $\sigma^+(\theta_0) - \theta_0 < \pi$, then $|N(\sigma^+(\theta_0))| = \frac{c_{\min}\rho_{\min}}{2\rho_{\max}}$. If not, then since there could be no shocks or contacts between θ_0 and $\sigma^+(\theta_0)$, by Theorem IV.7, U would be constant. However, this would lead to a contradiction of $N(\theta_0) = N(\sigma^+(\theta_0)) = 0$ if they were separated by less than π (using the same argument as in the proof of Theorem IV.9). Therefore, $|N(\sigma^+(\theta_0))| = \frac{c_{\min}\rho_{\min}}{2\rho_{\max}}$, and U must be constant on $]\theta_0, \sigma^+(\theta_0)]$ (and continuous at $\sigma^+(\theta_0)$). When U is constant, $N_\theta = L$, and so

$$\begin{aligned}\frac{c_{\min}\rho_{\min}}{2\rho_{\max}} &= |N(\sigma^+(\theta_0)) - N(\theta_0)| \\ &= \left|\int_{\theta_0}^{\sigma^+(\theta_0)} L(\phi)d\phi\right| \leq |\vec{u}|_{\max}(\sigma^+(\theta_0) - \theta_0).\end{aligned}$$

Therefore, $(\sigma^+(\theta_0) - \theta_0)$ is bounded below independent of U, and another contact could only happen for $\theta > \sigma^+(\theta_0)$. A similar argument works for $\theta < \theta_0$, and so the total number of contacts is finite.

This same calculation shows that the distance between a contact and a shock is lower bounded independent of U. □

Lemma IV.11. *If there is a forward facing shock at θ, and a backward facing shock at θ', then*

$$|\theta - \theta'| \geq \delta > 0$$

for some δ independent of U.

Proof. For forward facing shocks, the normal velocity is positive on either side. Similarly, it is negative on either side of a backward facing shock. Therefore, between a forward facing and backward facing shock, N must transition through zero, not jump between positive and negative values. In the previous lemma, we showed that the distance between a point at which $N = 0$ and any kind of shock is uniformly bounded away from zero (independent of U), and so the claim follows. □

4.7 Shock Strengths and Neighborhood Sizes

To motivate the following calculations, we recall some key ideas to the regularity results in Chapter II. A necessary step was to prove that the jump part of the solution satisfied Lipschitz conditions based at points where there were no jumps (Lemma II.40), as well as establish that the jump part had finite variation (Lemma II.39). The key ingredient was that shocks have constant neighborhoods on each side *with size lower bounded proportional to the strength of the shock* (Theorem II.33). That was derived from the uniform Lax conditions (Lemma II.32), which followed from genuine nonlinearity and the fact that we were on a compact small neighborhood in phase space. This allowed the local condition of genuine nonlinearity to give estimates for the entire shock curves.

Another important part was to use that the eigenvalue itself was Lipschitz away from jumps — this, combined with genuine nonlinearity and the Lipschitz regularity of the other $m-1$ components corresponding to the other eigenspaces, was the key idea in Theorem II.38.

So, in our case, for a forward facing shock, $N - c$ is the quantity that we have a Lax-type condition for. Therefore, if we can establish lower bounds for $|N_\pm - c_\pm|$ proportional to the strength of the shock, as well as show that is is Lipschitz away from shocks, then we should be able to proceed very much like before. We need to find an appropriate way to measure the shock strength, and since we need global information about these shock curves, we now focus on an even more concrete example and pick our equation of state.

We now focus on the case of a polytropic gas, for which

$$e = \frac{p}{(\gamma - 1)\rho}.$$

We assume that $\gamma > 1$, recalling for air $\gamma = 1.4$. We now find expressions for various quantities at a shock, following [48]. Substituting the expression for e into (4.19) we obtain

$$\left[\frac{1}{2}\rho N(N^2 + L^2) + \frac{\gamma}{\gamma - 1}Np\right] = 0.$$

However, $[\rho N L^2] = 0$ for a shock, and so we have

$$\left[\frac{1}{2}\rho N^3 + \frac{\gamma}{\gamma - 1}Np\right] = 0.$$

This can be rewritten as (recalling $\rho_- N_- = \rho_+ N_+$)

$$0 = \frac{1}{2}(\rho_+ N_+)N_+^2 + \frac{\gamma}{\gamma - 1}N_+ p_+ - \frac{1}{2}(\rho_+ N_+)N_-^2 - \frac{\gamma}{\gamma - 1}N_- p_-.$$

However, since

$$N_- = N_+ + \frac{p_+ - p_-}{\rho_+ N_+},$$

we obtain

$$\begin{aligned} 0 &= \frac{1}{2}(\rho_+ N_+)N_+^2 + \frac{\gamma}{\gamma - 1}N_+ p_+ \\ &- \frac{1}{2}(\rho_+ N_+)\left(N_+ + \frac{p_+ - p_-}{\rho_+ N_+}\right)^2 - \frac{\gamma}{\gamma - 1}p_-\left(N_+ + \frac{p_+ - p_-}{\rho_+ N_+}\right). \end{aligned}$$

Then,

$$\begin{aligned} 0 &= \frac{\gamma}{\gamma - 1}N_+(p_+ - p_-) - N_+(p_+ - p_-) - \frac{1}{2}\frac{(p_+ - p_-)^2}{\rho_+ N_+} - \frac{\gamma}{\gamma - 1}p_- \frac{p_+ - p_-}{\rho_+ N_+} \\ &= \frac{p_+ - p_-}{\rho_+ N_+}\left(\frac{1}{\gamma - 1}\rho_+ N_+^2 - \frac{1}{2}(p_+ - p_-) - \frac{\gamma}{\gamma - 1}p_-\right) \\ &= \frac{p_+ - p_-}{(\gamma - 1)\rho_+ N_+}\left(\rho_+ N_+^2 - \frac{\gamma - 1}{2}(p_+ - p_-) - \gamma p_-\right) \\ &= \frac{p_+ - p_-}{(\gamma - 1)\rho_+ N_+}\left(\rho_+ N_+^2 - \frac{\gamma - 1}{2}p_+ - \frac{\gamma + 1}{2}p_-\right). \end{aligned}$$

Therefore,

$$\begin{aligned} N_+^2 &= \frac{1}{2\rho_+}\Big(p_+(\gamma - 1) + p_-(\gamma + 1)\Big) \\ &= \frac{p_+}{\rho_+}\left(\frac{\gamma - 1}{2} + \frac{p_-}{p_+}\frac{(\gamma + 1)}{2}\right). \end{aligned}$$

Introduce

$$z = \frac{p_- - p_+}{p_+} > 0$$

as the shock strength of a forward facing shock, so that

$$p_- = p_+(1 + z).$$

The remainder of these calculations are for forward facing shocks, and the same statements hold for backward facing shocks but with $\pm$ switched, and remembering all normal velocities will be negative.

$$\begin{aligned} N_+^2 &= \frac{p_+}{\rho_+}\left(\frac{\gamma-1}{2} + \frac{\gamma+1}{2}(1+z)\right) \\ &= \frac{\gamma p_+}{\rho_+}\left(1 + z\frac{\gamma+1}{2\gamma}\right) \\ &= c_+^2\left(1 + z\frac{\gamma+1}{2\gamma}\right). \end{aligned}$$

Therefore we have the relations (assuming a forward facing shock)

$$\begin{aligned} N_+ &= c_+\sqrt{1 + z\frac{\gamma+1}{2\gamma}} \\ N_+ - c_+ &= c_+\left(\sqrt{1 + z\frac{\gamma+1}{2\gamma}} - 1\right). \end{aligned} \tag{4.29}$$

We need similar relations for the state behind the shock.

$$\begin{aligned} N_- &= \frac{1}{\rho_+ N_+}\left(\rho_+ N_+^2 - p_- + p_+\right) \\ &= \frac{1}{\rho_+ N_+}\left(\rho_+ c_+^2\left(1 + z\frac{\gamma+1}{2\gamma}\right) - p_+(1+z) + p_+\right) \\ &= \frac{c_+^2}{N_+}\left(1 + z\frac{\gamma+1}{2\gamma} - \frac{z p_+}{\rho_+ c_+^2}\right) \\ &= \frac{c_+^2}{N_+}\left(1 + z\frac{\gamma+1}{2\gamma} - \frac{z}{\gamma}\right) \\ &= \frac{c_+^2}{N_+}\left(1 + z\frac{\gamma-1}{2\gamma}\right). \end{aligned}$$

Next, we obtain expressions for ρ_-, then c_-, $N_- - c_-$, and $\frac{N_-}{c_-}$.

$$\begin{aligned} \rho_- &= \frac{\rho_+ N_+}{N_-} = \frac{\rho_+ N_+^2}{c_+^2}\frac{1}{1 + z\frac{\gamma-1}{2\gamma}} \\ &= \rho_+\frac{1 + z\frac{\gamma+1}{2\gamma}}{1 + z\frac{\gamma-1}{2\gamma}}. \end{aligned} \tag{4.30}$$

$$c_-^2 = \frac{\gamma p_-}{\rho_-} = \frac{\gamma p_+(1+z)}{\rho_+}\frac{1 + z\frac{\gamma-1}{2\gamma}}{1 + z\frac{\gamma+1}{2\gamma}}$$

$$= c_+^2 (1+z) \frac{1 + z\frac{\gamma-1}{2\gamma}}{1 + z\frac{\gamma+1}{2\gamma}}.$$

$$N_- - c_- = \frac{c_+^2}{N_+}\left(1 + z\frac{\gamma-1}{2\gamma}\right) - c_+\sqrt{(1+z)\frac{1 + z\frac{\gamma-1}{2\gamma}}{1 + z\frac{\gamma+1}{2\gamma}}}$$

$$= c_+ \frac{1 + z\frac{\gamma-1}{2\gamma}}{\sqrt{1 + z\frac{\gamma+1}{2\gamma}}} - c_+\sqrt{(1+z)\frac{1 + z\frac{\gamma-1}{2\gamma}}{1 + z\frac{\gamma+1}{2\gamma}}}$$

$$= c_+ \sqrt{\frac{1 + z\frac{\gamma-1}{2\gamma}}{1 + z\frac{\gamma+1}{2\gamma}}}\left(\sqrt{1 + z\frac{\gamma-1}{2\gamma}} - \sqrt{1+z}\right). \tag{4.31}$$

$$\frac{N_-}{c_-} - 1 = \frac{\sqrt{1 + z\frac{\gamma-1}{2\gamma}}}{\sqrt{1+z}} - 1,$$

$$\frac{N_-}{c_-} = \frac{\sqrt{1 + z\frac{\gamma-1}{2\gamma}}}{\sqrt{1+z}} \tag{4.32}$$

We now argue that z is a suitable measure of shock strength.

Lemma IV.12. *At a shock, $[|U|]$ and z are equivalent measures of the strength of the shock, i.e. there exists $C > 0$ such that*

$$\frac{z}{C} \leq [|U|] \leq Cz.$$

Proof. First, recall that we assume pressure is bounded away from 0 and ∞, so we only need estimates valid for

$$0 \leq z \leq z_{\max} := \frac{p_{\max} - p_{\min}}{p_{\min}}.$$

(4.30) shows that

$$|[\rho]| = \left|\rho_+ \frac{1 + z\frac{\gamma+1}{2\gamma}}{1 + z\frac{\gamma-1}{2\gamma}} - \rho_+\right|$$

$$= \rho_+ \frac{z}{\gamma + z\frac{\gamma-1}{2}}.$$

This gives the estimate

$$\frac{\rho_{\min}}{\gamma + z_{\max}\frac{\gamma-1}{2}} z \leq |[\rho]| \leq \rho_{\max}\frac{1}{\gamma} z.$$

Clearly

$$p_{\min} z \leq |[p]| \leq p_{\max} z.$$

Finally, since $-\mathcal{M}[N] = [p]$, and, for a forward facing shock $\mathcal{M} = \rho_+ N_+ \geq \rho_{\min} c_{\min}$, and so

$$\rho_{\min} c_{\min} \leq |\mathcal{M}| \leq \rho_{\max} |\vec{u}|_{\max},$$

$$\frac{p_{\min}}{\rho_{\max} |\vec{u}|_{\max}} z \leq |[N]| \leq \frac{p_{\max}}{\rho_{\min} c_{\min}} z.$$

We have therefore shown that in terms of the primitive variables $V := (\rho, u, v, p)$ that for a shock

$$\frac{z}{C} \leq |[V]| \leq Cz$$

for some $C > 0$. $U = (\rho, \rho u, \rho v, \rho E)$ has derivative

$$U_V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u & \rho & 0 & 0 \\ v & 0 & \rho & 0 \\ \frac{u^2+v^2}{2} & \rho u & \rho v & \frac{1}{\gamma - 1} \end{pmatrix}.$$

Therefore it is C^1, and the operator norm of its derivative is bounded uniformly on the phase space under consideration. We now assume that $\mathcal{P}$ is chosen to be those states U which are obtained from a convex set of V of the form

$$0 < C^{-1} \leq \rho \leq C < \infty,$$

$$|\vec{u}| \leq C < \infty,$$

$$0 < C^{-1} \leq p \leq C < \infty,$$

for some $C > 0$. Then, since U is a C^1 function of V on a convex and compact set, it is easy to see that $|[U]| \leq C|[V]|$ for some $C > 0$. Furthermore $|[U]| > |[\rho]| > \frac{z}{C}$, and so putting it all together, there must exist $C > 0$ such that

$$\frac{z}{C} \leq |[U]| \leq Cz$$

for a shock. We will now denote $J(U;\theta) := |[U]|$ as the size of the jump in U at θ. $\square$

We now estimate the sizes of the neighborhoods on either side of a shock on which U must be constant.

Theorem IV.13. *Suppose U has a forward facing shock at θ_0. Then the $\sigma^+(\theta_0) > \theta_0$ and $\sigma^-(\theta_0) < \theta_0$ (from Theorem IV.8) such that U is constant on $]\sigma^-(\theta_0), \theta_0[,]\theta_0, \sigma^+(\theta_0)[$ satisfy the following:*

$$\sigma^+(\theta_0) \geq \theta_0 + \delta_L J(U;\theta_0),$$

$$\sigma^-(\theta_0) \leq \theta_0 - \delta_L J(U;\theta_0),$$

where δ_L is a positive constant independent of U. Furthermore,

$$N_+ - c_+ \geq \delta_L J(U;\theta_0),$$

$$N_- - c_- \leq -\delta_L J(U;\theta_0).$$

The analogous statement holds for a backward facing shock.

Proof. Suppose the shock is forward facing. Recall, from (4.27),

$$\sigma^+(\theta_0) = \sup\left\{\eta > \theta_0 \,\middle|\, N(\theta) - c(\theta) > 0 \quad \forall \theta \in]\theta_0, \eta[\right\}.$$

It must be the case that $N(\sigma^+(\theta_0)) = c(\theta_0)$ — from Theorems IV.8 and IV.9 there could not be a shock or contact at $\sigma^+(\theta_0)$, and so U is continuous at $\sigma^+(\theta_0)$. Furthermore, it is constant on $]\theta_0, \sigma^+(\theta_0)[$ by Theorem IV.7. (Note however there could be another forward facing shock at $\theta' > \sigma^+(\theta_0)$ arbitrarily close to $\sigma^+(\theta_0)$). When U is constant, $N_\theta = L$, and so we have

$$N_+ - c_+ = N(\theta_0+) - N(\sigma^+(\theta_0)) = \int_{\sigma^+(\theta_0)}^{\theta_0} L(\phi)d\phi$$
$$\leq (\sigma^+(\theta_0) - \theta_0)|\vec{u}|_{\max}.$$

However, recall from (4.29)

$$N_+ - c_+ = c_+\left(\sqrt{1 + z\frac{\gamma+1}{2\gamma}} - 1\right) \geq c_{\min}\left(\sqrt{1 + z\frac{\gamma+1}{2\gamma}} - 1\right).$$

The function on the right satisfies the hypotheses of Lemma A.2 in Appendix A, thus

$$N_+ - c_+ \geq Cz$$

for some $C > 0$. All together, we obtain

$$(\sigma^+(\theta_0) - \theta_0) \geq \delta'_L J(U;\theta_0),$$

for some $\delta'_L > 0$. Similarly, recall from (4.28)

$$\sigma^-(\theta_0) = \inf\left\{\eta < \theta_0 \,\middle|\, 0 < N(\theta) < c(\theta) \quad \forall\theta \in]\eta, \theta_0[\right\}.$$

Similar to before, it must be the case that either $N(\sigma^-(\theta_0)) = c(\sigma^-(\theta_0))$, or $N(\sigma^-(\theta_0)+) = 0$, since there can be no shocks on $[\sigma^-(\theta_0), \theta_0[$, and no contacts on $]\sigma^-(\theta_0), \theta_0[$. However, if $N(\sigma^-(\theta_0)+) = 0$, the only possible discontinuity would be a contact, at which N is continuous, and so $N(\sigma^-(\theta_0)+) = N(\sigma^-(\theta_0)) = 0$ in that case. Either way, U is continuous on $]\sigma^-(\theta_0), \theta_0[$, hence constant by Theorem IV.7, and so we again can use $N_\theta = L$.

Suppose that $N(\sigma^-(\theta_0)) = c(\sigma^-(\theta_0)) = c_-$. Then

$$|N_- - c_-| = |N(\theta_0-) - N(\sigma^-(\theta_0))| = \left|\int_{\sigma^-(\theta_0)}^{\theta_0} L(\phi)d\phi\right|$$
$$\leq (\theta_0 - \sigma^-(\theta_0))|\vec{u}|_{\max}.$$

Recall from (4.31)

$$|N_- - c_-| = c_+\sqrt{\frac{1+z\frac{\gamma-1}{2\gamma}}{1+z\frac{\gamma+1}{2\gamma}}}\left(\sqrt{1+z}-\sqrt{1+z\frac{\gamma-1}{2\gamma}}\right)$$
$$\geq c_{\min}\sqrt{\frac{\gamma-1}{\gamma+1}}\left(\sqrt{1+z}-\sqrt{1+z\frac{\gamma-1}{2\gamma}}\right)$$
$$\geq Cz,$$

for some $C > 0$ by Lemma A.2. So,

$$(\theta_0 - \sigma^-(\theta_0)) \geq \delta_L'' J(U;\theta_0).$$

for some $\delta_L'' > 0$.

Suppose instead that $N(\sigma^-(\theta_0)) = 0$. Then,

$$N_- = N(\theta_0-) - N(\sigma^-(\theta_0)) = \int_{\sigma^-(\theta_0)}^{\theta_0} L(\phi)d\phi$$
$$\leq (\theta_0 - \sigma^-(\theta_0))|\vec{u}|_{\max}.$$

From (4.32),

$$N_- = c_-\frac{\sqrt{1+z\frac{\gamma-1}{2\gamma}}}{\sqrt{1+z}} \geq c_{\min}\sqrt{\frac{\gamma-1}{2\gamma}}.$$

Therefore,

$$(\theta_0 - \sigma^-(\theta_0)) \geq \delta > 0$$

in this case, for some $\delta > 0$. Then, taking

$$\delta_L = \min\left(\delta_L', \delta_L'', \frac{\delta}{J_{\max}}\right),$$

we obtain the desired result. (Recall that there is an upper limit to shock strength for the phase space under consideration). Analogous calculations yield the statement for backward-facing shocks. $\square$

4.8 Decomposition of the Domain and Regularity

We now divide $[0, 2\pi)$ into the sets at which different behavior occurs. We shall assume that there are no stagnation points.

$$\mathcal{C} := \left\{\theta \,\middle|\, U \text{ is continuous at } \theta, |N(\theta)| \neq c(\theta) \text{ and } \neq 0\right\}$$
$$\mathcal{S}_F := \left\{\theta \,\middle|\, U \text{ has a forward facing shock at } \theta\right\}$$
$$\mathcal{S}_B := \left\{\theta \,\middle|\, U \text{ has a backward facing shock at } \theta\right\}$$
$$\mathcal{S}_C := \left\{\theta \,\middle|\, N(\theta) = 0\right\}$$
$$\mathcal{R}_F := \left\{\theta \,\middle|\, U \text{ is continuous at } \theta, N(\theta) = c(\theta)\right\}$$
$$\mathcal{R}_B := \left\{\theta \,\middle|\, U \text{ is continuous at } \theta, N(\theta) = -c(\theta)\right\}$$

Lemma IV.14. *Recall Lemma IV.11. In a similar manner, $\mathcal{R}_F$ is uniformly separated from $\mathcal{R}_B \cup \mathcal{S}_B \cup \mathcal{S}_C$, and $\mathcal{R}_B$ is uniformly separated from $\mathcal{R}_F \cup \mathcal{S}_F \cup \mathcal{S}_C$.*

Proof. The proof is similar to the proof of Lemmas IV.10 and IV.11. Suppose $\theta_0 \in \mathcal{S}_C$, and recall from the proof of Lemma IV.10

$$\sigma^+(\theta_0) = \sup\left\{\eta > \theta_0 \,\middle|\, 0 < |N(\theta)| < \frac{c_{\min}\rho_{\min}}{2\rho_{\max}} \quad \forall \theta \in]\theta_0, \eta[\right\}.$$

It was shown that $\left(\sigma^+(\theta_0) - \theta_0\right)$ and the analogous $\left(\theta_0 - \sigma^-(\theta_0)\right)$ are bounded below independent of U. However, since

$$\frac{c_{\min}\rho_{\min}}{2\rho_{\max}} < c_{\min},$$

$\mathcal{R}_F$ and $\mathcal{R}_B$ are uniformly separated from $\mathcal{S}_C$. Between a point in $\mathcal{R}_F$ and a point in either $\mathcal{R}_B$ or $\mathcal{S}_B$, there must be a point in $\mathcal{S}_C$, since N cannot jump from a positive to a negative value — it must pass through zero. Therefore, $\mathcal{R}_F$ is uniformly separated from $\mathcal{R}_B \cup \mathcal{S}_B \cup \mathcal{S}_C$. The analogous statement holds for $\mathcal{R}_B$. $\square$

Lemma IV.15. *There exists a constant $C_{\mathcal{S}}$ such that if $\theta_0 \in \mathcal{S}_F$, and $\theta \notin]\sigma^-(\theta_0), \sigma^+(\theta_0)[$, then*

$$J(U;\theta_0), |N(\theta_0+) - c(\theta_0+)|, |N(\theta_0-) - c(\theta_0-)| \leq C_{\mathcal{S}}|\theta - \theta_0|.$$

Similarly, if $\theta_0 \in \mathcal{S}_B$, and $\theta \notin]\sigma^-(\theta_0), \sigma^+(\theta_0)[$, then

$$J(U;\theta_0), |N(\theta_0+) + c(\theta_0+)|, |N(\theta_0-) + c(\theta_0-)| \leq C_{\mathcal{S}}|\theta - \theta_0|.$$

Proof. Suppose $\theta_0 \in \mathcal{S}_F$. From Theorem IV.13, we have

$$\delta_L J(U;\theta_0) \leq \min\Big(|\sigma^+(\theta_0) - \theta_0|, |\theta_0 - \sigma^-(\theta_0)|\Big) \leq |\theta - \theta_0|.$$

Therefore,

$$J(U;\theta_0) \leq \delta_L^{-1}|\theta - \theta_0|.$$

From (4.29), we have that

$$N_+ - c_+ = c_+\left(\sqrt{1 + z\frac{\gamma+1}{2\gamma}} - 1\right) \leq c_{\max}\left(\sqrt{1 + z\frac{\gamma+1}{2\gamma}} - 1\right).$$

The function on the right is estimated using Lemma A.3 in Appendix A. Therefore, for some $C, C'_{\mathcal{S}} > 0$ we have

$$|N_+ - c_+| \leq Cz \leq C'_{\mathcal{S}} J(U;\theta_0) \leq C_{\mathcal{S}}\delta_L^{-1}|\theta - \theta_0|.$$

Also recall from (4.31) that

$$\begin{aligned} |N_- - c_-| &= c_+\sqrt{\frac{1 + z\frac{\gamma-1}{2\gamma}}{1 + z\frac{\gamma+1}{2\gamma}}}\left(\sqrt{1+z} - \sqrt{1 + z\frac{\gamma-1}{2\gamma}}\right) \\ &\leq c_{\max}\left(\sqrt{1+z} - \sqrt{1 + z\frac{\gamma-1}{2\gamma}}\right). \end{aligned}$$

Again using Lemma A.3,

$$|N_- - c_-| \leq Cz \leq C''_{\mathcal{S}} J(U;\theta_0) \leq C''_{\mathcal{S}}\delta_L^{-1}|\theta - \theta_0|,$$

for some $C, C''_{\mathcal{S}} > 0$. Taking

$$C_{\mathcal{S}} = \max(\delta_L^{-1}, C'_{\mathcal{S}}\delta_L^{-1}, C''_{\mathcal{S}}\delta_L^{-1})$$

gives the desired result. A similar argument works for $\theta_0 \in \mathcal{S}_B$. □

Lemma IV.16. *Recall that $\mathcal{S}_C$ is a finite set of points, and therefore can have no limit points. If θ is a limit point of $\mathcal{S}_F$, then $\theta \in \mathcal{R}_F$. If θ is a limit point of $\mathcal{S}_B$, then $\theta \in \mathcal{R}_B$.*

Proof. Consider $\{\theta_n\} \searrow \theta$ be a strictly decreasing sequence in $\mathcal{S}_F$ (the strictly increasing case is analagous). If θ is a limit point of $\mathcal{S}_F$, then $\theta \notin]\sigma^-(\theta_n), \sigma^+(\theta_n)[$ for all n. For all n, choose $\theta'_n \in]\sigma^-(\theta_n), \theta_n[$. Then,

$$\begin{aligned}
|N(\theta'_n) - c(\theta'_n)| &= |N(\theta'_n) - c(\theta_n-)|, \text{ since } U \text{ is constant on }]\sigma^-(\theta_n), \theta_n[, \\
&= |N(\theta'_n) - N(\theta_n-)| + |N(\theta_n-) - c(\theta_n-)| \\
&\leq |\vec{u}|_{\max}|\theta'_n - \theta_n| + C_{\mathcal{S}}|\theta - \theta_n|, \text{ from Lemma IV.15}, \\
&= \mathcal{O}(|\theta_n - \theta|).
\end{aligned}$$

Thus we have a sequence converging to θ such that

$$\lim_{n\to\infty} |N(\theta'_n) - c(\theta'_n)| = 0.$$

This eliminates the possibility of a shock or contact occurring at θ, and so U is continuous at θ. Therefore, $\theta \in \mathcal{R}_F$ by definition. A similar argument works for limit points of $\mathcal{S}_B$. □

Lemma IV.17. *If $\theta \in \mathcal{C}$, then U is constant on a neighborhood $]\kappa^-(\theta), \kappa^+(\theta)[$ containing θ. ($\kappa^\pm(\theta)$ are taken to be maximal so that each is in either $\mathcal{S}_C, \mathcal{S}_F, \mathcal{S}_B, \mathcal{R}_F$, or $\mathcal{R}_B$.)*

Proof. From Lemma IV.16, θ is not a limit point of $\mathcal{S}_C, \mathcal{S}_F$, or $\mathcal{S}_B$. Therefore, U is continuous on a neighborhood of θ. Therefore, if $|N(\theta)| \neq c(\theta)$ or 0, then this will also be true on a neighborhood of θ. Then, Theorem IV.7 applies and so U is constant on some neighborhood containing θ. Clearly $\kappa^{\pm}(\theta)$ can be taken to satisfy the requirement in the statement of the lemma. $\square$

Lemma IV.18. *Assume there are no stagnation points. Define, for $\theta \in [0, 2\pi[$,*

$$U_S(\theta) = \sum_{\phi \in [0,\theta[\cap (\mathcal{S}_C \cup \mathcal{S}_F \cup \mathcal{S}_B)} \big(U(\phi+) - U(\phi-)\big).$$

Then U_S is a right-continuous saltus function (so, by definition, is of bounded variation).

Proof. We have

$$\begin{aligned} \sum_{\phi \in (\mathcal{S}_C \cup \mathcal{S}_F \cup \mathcal{S}_B)} |U(\phi+) - U(\phi-)| &= \sum_{\phi \in (\mathcal{S}_F \cup \mathcal{S}_B)} J(U;\phi) + \sum_{\phi \in \mathcal{S}_C} |U(\phi+) - U(\phi-)| \\ &\leq (2\delta_L)^{-1} \sum_{\phi \in \mathcal{S}_F \cup \mathcal{S}_B} \big(\sigma^+(\phi) - \sigma^-(\phi)\big) + \sum_{\phi \in \mathcal{S}_C} C < \infty, \end{aligned}$$

since the number of contacts is finite, the neighborhoods $]\sigma^-(\phi), \sigma^+(\phi)[$ are pairwise disjoint, the phase space is compact, and the domain is compact. The BV norm only depends on the equation of state and the bounds for the phase space. $\square$

Lemma IV.19. *For any $\theta_0 \in \mathcal{R}_F \cup \mathcal{R}_B$, U_S satisfies a Lipschitz estimate based at θ_0 for θ sufficiently close to θ_0. That is, there exists $\delta > 0$ so that*

$$|U_S(\theta) - U_S(\theta_0)| \leq C_S|\theta - \theta_0|, \quad |\theta - \theta_0| < \delta.$$

Moreover, the Lipschitz constant C_S is uniform in θ and independent of U, though δ depends on U.

Proof. Consider $\theta_0 \in \mathcal{R}_F$, and $\theta > \theta_0$. We only need to consider other forward facing shocks occurring between θ_0 and θ, since $\mathcal{R}_F$ is uniformly separated from $\mathcal{S}_B$ and $\mathcal{S}_C$. Suppose $\theta \notin]\sigma^-(\phi), \sigma^+(\phi)[$ for any $\phi \in \mathcal{S}_F$. Then,

$$\begin{aligned}|U_S(\theta) - U_S(\theta_0)| &\leq \sum_{\theta_0<\phi<\theta} J(U;\phi) \\ &\leq (2\delta_L)^{-1} \sum_{\theta_0<\phi<\theta} (\sigma^+(\phi) - \sigma^-(\phi)) \leq (2\delta_L)^{-1}|\theta - \theta_0|,\end{aligned}$$

since the $]\sigma^-(\phi), \sigma^+(\phi)[$ are pairwise disjoint and contained in $[\theta_0, \theta]$ by assumption. If $\theta \in]\sigma^-(\phi), \phi[$ for some $\phi \in \mathcal{S}_F$, then the previous estimate holds for $\theta = \sigma^-(\phi)$, and U_S is constant on $]\sigma^-(\phi), \phi[$ and so the result follows. If $\theta \in [\phi, \sigma^+(\phi)[$ for some $\phi \in \mathcal{S}_F$, then apply the previous estimate for $\theta = \sigma^-(\phi)$, and then

$$\begin{aligned}|U_S(\theta) - U_S(\theta_0)| &\leq (2\delta_L)^{-1}\left|\sigma^-(\phi) - \theta_0\right| + J(U;\phi) \\ &\leq ((2\delta_L)^{-1} + C_{\mathcal{S}})|\theta - \theta_0|.\end{aligned}$$

from Lemma IV.15. Take $C_S = (2\delta_L)^{-1} + C_{\mathcal{S}}$. Similar arguments work for $\theta < \theta_0$, and $\theta_0 \in \mathcal{R}_B$. $\square$

Lemma IV.20. *For every $\theta_0 \in \mathcal{R}_F$, there exists a neighborhood containing θ_0 such that*

$$u(\theta)\sin\theta_0 - v(\theta)\cos\theta_0 - c(\theta)$$

satisfies a Lipschitz condition based at θ_0 for all θ in this neighborhood. The Lipschitz constant is uniform for all such θ_0 and is independent of U. That is,

$$\left|u(\theta)\sin\theta_0 - v(\theta)\cos\theta_0 - c(\theta)\right| \leq M|\theta - \theta_0|, \quad \text{for all } |\theta - \theta_0| < \delta$$

for some $\delta > 0$. (Recall that $(N(\theta_0) - c(\theta_0)) = 0$ for $\theta_0 \in \mathcal{R}_F$).

We have the similar estimate for

$$u(\theta)\sin\theta_0 - v(\theta)\cos\theta_0 + c(\theta)$$

with $\theta_0 \in \mathcal{R}_B$.

Proof. Suppose $\theta_0 \in \mathcal{R}_F$. We first prove the desired estimate for $|N(\theta) - c(\theta)|$. Take $\theta > \theta_0$ sufficiently close to θ_0. This means that either $\theta \in \mathcal{C}$, with $\kappa^-(\theta) \in \mathcal{R}_F \cup \mathcal{S}_F$, $\theta \in \mathcal{S}_F$, or $\theta \in \mathcal{R}_F$ (from Lemma IV.14).

Suppose $\theta \in \mathcal{R}_F$. Then $|N(\theta) - c(\theta)| = 0$.

Suppose $\theta \in \mathcal{S}_F$. Then, recalling

$$\sigma^-(\theta) = \inf\left\{\eta < \theta \middle| 0 < N(\phi) - c(\phi) < c(\phi), \quad \forall \phi \in]\eta, \theta[\right\},$$

it is clear that $\theta_0 \notin]\sigma^-(\theta), \theta[$. Therefore, Lemma IV.15 applies and

$$|N(\theta) - c(\theta)| = |N(\theta+) - c(\theta+)| \leq C_{\mathcal{S}}|\theta - \theta_0|,$$

(recalling we have made U right continuous everywhere).

Finally, suppose $\theta \in \mathcal{C}$. If $\kappa^-(\theta) \in \mathcal{R}_F$, then

$$\begin{aligned}
|N(\theta) - c(\theta)| &= |N(\theta) - c\big(\kappa^-(\theta)\big)| \\
&\leq |N(\theta) - N\big(\kappa^-(\theta)\big)| + |N\big(\kappa^-(\theta)\big) - c\big(\kappa^-(\theta)\big)| \\
&\leq |\vec{u}|_{\max}|\kappa^-(\theta) - \theta| + 0 \\
&\leq |\vec{u}|_{\max}|\theta - \theta_0|.
\end{aligned}$$

If $\kappa^-(\theta) \in \mathcal{S}_F$, then

$$\begin{aligned}
|N(\theta) - c(\theta)| &= |N(\theta) - c\big(\kappa^-(\theta)\big)| \\
&= |N(\theta) - N\big(\kappa^-(\theta)\big)| + |N\big(\kappa^-(\theta)\big) - c\big(\kappa^-(\theta)\big)| \\
&\leq |\vec{u}|_{\max}|\theta - \kappa^-(\theta)| + C_{\mathcal{S}}|\theta - \theta_0| \\
&\leq |\vec{u}|_{\max}|\theta - \theta_0| + C_{\mathcal{S}}|\theta - \theta_0|.
\end{aligned}$$

Taking $M' = \max(C_{\mathcal{S}}, |\vec{u}|_{\max})$ gives the desired estimate for $|N(\theta) - c(\theta)|$. Then,

$$\begin{aligned}|u(\theta)\sin\theta_0 - v(\theta)\cos\theta_0 - c(\theta)| =& |N(\theta) - c(\theta)| \\ &+ |u(\theta)\big(\sin\theta_0 - \sin\theta\big) - v(\theta)\big(\cos\theta_0 - \cos\theta\big)| \\ \leq& M'|\theta - \theta_0| + 2|\vec{u}|_{\max}|\theta - \theta_0|.\end{aligned}$$

Taking $M := M' + 2|\vec{u}|_{\max}$ gives the desired result for $\theta > \theta_0$, θ sufficiently close to $\theta_0 \in \mathcal{R}_F$.

Similar arguments work for $\theta < \theta_0$, and for $\theta_0 \in \mathcal{R}_B$. $\square$

We now recall the concept of genuine nonlinearity. For fixed θ, the quantities $N \pm c$ are genuinely nonlinear in the sense that

$$(N \pm c)_U r^{\pm}(U;\theta) \neq 0$$

for all U. For simplicity we can compute the derivative in terms of the primitive variables. We define

$$h := e + \frac{1}{2}(u^2 + v^2) + \frac{p}{\rho} = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}(u^2 + v^2).$$

to be the total enthalpy per unit mass, so that

$$\begin{aligned}c^2 &= \gamma\frac{p}{\rho} \\ &= (\gamma - 1)\left(h - \frac{u^2 + v^2}{2}\right).\end{aligned}$$

We then have

$$(N \pm c)_U r^{\pm}(U;\theta) = \left(N \pm \sqrt{\gamma \frac{p}{\rho}}\right)_V V_U r^{\pm}(U;\theta)$$

$$= \begin{pmatrix} \mp\frac{1}{2}\sqrt{\gamma\frac{p}{\rho^3}} \\ \sin\theta \\ -\cos\theta \\ \pm\frac{1}{2}\sqrt{\frac{\gamma}{p\rho}} \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ u & \rho & 0 & 0 \\ v & 0 & \rho & 0 \\ \frac{u^2+v^2}{2} & \rho u & \rho v & \frac{1}{\gamma-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ u \pm c\sin\theta \\ v \mp c\cos\theta \\ h \pm Nc \end{pmatrix}$$

$$= \begin{pmatrix} \mp\frac{1}{2}\frac{c}{\rho} \\ \sin\theta \\ -\cos\theta \\ \pm\frac{1}{2}\frac{c}{p} \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ (\gamma-1)\frac{u^2+v^2}{2} & -(\gamma-1)u & -(\gamma-1)v & \gamma-1 \end{pmatrix} \begin{pmatrix} 1 \\ u \pm c\sin\theta \\ v \mp c\cos\theta \\ h \pm Nc \end{pmatrix}$$

$$= \begin{pmatrix} \mp\frac{1}{2}\frac{c}{\rho} & \sin\theta & -\cos\theta & \pm\frac{1}{2}\frac{c}{p} \end{pmatrix} \begin{pmatrix} 1 \\ \pm\frac{c}{\rho}\sin\theta \\ \mp\frac{c}{\rho}\cos\theta \\ c^2 \end{pmatrix}$$

$$= \mp\frac{1}{2}\frac{c}{\rho} \pm \frac{c}{\rho} \pm \frac{1}{2}\frac{c^3}{p}$$

$$= \mp\frac{1}{2}\frac{c}{\rho} \pm \frac{c}{\rho} \pm \frac{\gamma}{2}\frac{c}{\rho}$$

$$= \pm\frac{1}{2}(\gamma+1)\frac{c}{\rho} \neq 0.$$

The Roe linearization for the full polytropic Euler equations has the advantage that it is simply the matrices f_U^x and f_U^y evaluated at some appropriately averaged state $\overline{U}$. It takes the form (see [48])

$$\hat{A}(U_-, U_+; \theta) := A(\overline{U}; \theta) =$$

$$\begin{pmatrix} 0 & \sin\theta & -\cos\theta & 0 \\ \frac{\Gamma}{2}\sin\theta|\overline{u}|^2 - \overline{u}\,\overline{N} & \overline{N} + (2-\gamma)\overline{u}\sin\theta & -\overline{u}\cos\theta - \Gamma\overline{v}\sin\theta & \Gamma\sin\theta \\ -\frac{\Gamma}{2}\cos\theta|\overline{u}|^2 - \overline{v}\,\overline{N} & \overline{v}\sin\theta + \Gamma\overline{u}\cos\theta & \overline{N} + (\gamma-2)\overline{v}\cos\theta & -\Gamma\cos\theta \\ \left(\frac{\Gamma}{2}|\overline{u}|^2 - \overline{h}\right)\overline{N} & \overline{h}\sin\theta - \Gamma\overline{N}\,\overline{u} & -\overline{h}\cos\theta - \Gamma\overline{N}\,\overline{v} & \gamma\overline{N} \end{pmatrix}.$$

Above we have abbreviated

$$\Gamma := \gamma - 1.$$

The averaged quantities are defined as

$$\begin{aligned} \overline{u} &= \frac{u_-\sqrt{\rho_-} + u_+\sqrt{\rho_+}}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \\ \overline{v} &= \frac{v_-\sqrt{\rho_-} + v_+\sqrt{\rho_+}}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \\ \overline{N} &= \overline{u}\sin\theta - \overline{v}\cos\theta, \\ |\overline{u}|^2 &= \overline{u}^2 + \overline{v}^2, \\ \overline{h} &= \frac{h_-\sqrt{\rho_-} + h_+\sqrt{\rho_+}}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \\ \overline{c}^2 &= (\gamma - 1)\left(\overline{h} - \frac{\overline{u}^2 + \overline{v}^2}{2}\right). \end{aligned}$$

The eigenvalues of $A(\overline{U}; \theta)$ are

$$\overline{N} \pm \overline{c}, \overline{N}, \overline{N},$$

and it has a full basis of eigenvectors. Moreover, it is clear that $A(\overline{U}; \theta)$ is a smooth function of $U_\pm$, the eigenvalues are smooth functions of $U_\pm$, (away from $\rho = 0$ of course), and by direct inspection of the eigenvectors (they are not needed here, but expressions for them are available) they too are smooth functions of $U_\pm$. Define the left and right eigenvectors so that

$$A(\overline{U};\theta)r^\pm(\overline{U};\theta) = (\overline{N} \pm \overline{c})r^\pm(\overline{U};\theta),$$

$$l^\pm(\overline{U};\theta)A(\overline{U};\theta) = (\overline{N} \pm \overline{c})l^\pm(\overline{U};\theta),$$

$$A(\overline{U};\theta)r^i(\overline{U};\theta) = \overline{N}\, r^i(\overline{U};\theta) \text{ for } i = 1,2,$$

$$l^i(\overline{U};\theta)A(\overline{U};\theta) = \overline{N}\, l^i(\overline{U};\theta), \text{ for } i = 1,2,$$

$$l^\alpha(\overline{U};\theta)r^\alpha(\overline{U};\theta) = \delta_{\alpha\beta}, \quad \alpha,\beta = +,-,1,2.$$

Theorem IV.21. *For any $\theta_0 \in \mathcal{R}_F \cup \mathcal{R}_B$, there is a neighborhood containing θ_0 such that U satisfies a Lipschitz condition based at θ_0 for all θ in this neighborhood. The Lipschitz constant is uniform for all such θ_0 and is independent of U. That is,*

$$\Big|U(\theta) - U(\theta_0)\Big| \leq M'|\theta - \theta_0|, \quad \textit{for all } |\theta - \theta_0| < \delta,$$

for some $\delta > 0$.

Proof. Suppose $\theta_0 \in \mathcal{R}_F$.

Recall that

$$\hat{A}\big(U(\theta_0), U(\theta); \theta_0\big)\big(U(\theta) - U(\theta_0)\big) = \mathcal{O}(|\theta - \theta_0|).$$

Denote $l^+\big(U(\theta_0), U(\theta); \theta_0\big) = l^+(\overline{U};\theta_0)$ where the average is taken between $U(\theta_0)$ and $U(\theta)$. Left multiply by $l^+\big(U(\theta_0), U(\theta); \theta_0\big)$ to obtain

$$\big(\overline{N} + \overline{c}\big)l^+\big(U(\theta_0), U(\theta); \theta_0\big)\big(U(\theta) - U(\theta_0)\big) = \mathcal{O}(|\theta - \theta_0|),$$

where the averages are taken between $U(\theta_0)$ and $U(\theta)$. Since $\theta_0 \in \mathcal{R}_F$, U is continuous at θ_0. Therefore for θ sufficiently close to θ_0, $\overline{N} + \overline{c}$ is uniformly bounded away from zero, since $\overline{N} - \overline{c}$ will be approaching zero. Therefore,

$$\theta \mapsto l^+\big(U(\theta_0), U(\theta_1); \theta_0\big)\big(U(\theta) - U(\theta_0)\big),$$

satisfies a Lipschitz estimate based at θ_0 for θ sufficiently close to θ_0, with Lipschitz constant uniformly bounded above by the bounds on the phase space, and proportional to $c_{\min}^{-1}$. Similarly,

$$\theta \mapsto l^i\big(U(\theta_0), U(\theta); \theta_0\big)\big(U(\theta) - U(\theta_0)\big),\ i = 1, 2,$$

also satisfiy a Lipschitz estimate based at θ_0 for θ sufficiently close to θ_0, with a similar upper bound on the Lipschitz constant. We claim that

$$W \mapsto g(W) = \begin{pmatrix} g^1(W) \\ g^2(W) \\ g^3(W) \\ g^4(W) \end{pmatrix} := \begin{pmatrix} \dfrac{W_2}{W_1}\sin\theta_0 - \dfrac{W_3}{W_1}\cos\theta_0 - c(W) \\ l^+\big(U(\theta_0), W; \theta_0\big)\big(W - U(\theta_0)\big) \\ l^1\big(U(\theta_0), W; \theta_0\big)\big(W - U(\theta_0)\big) \\ l^2\big(U(\theta_0), W; \theta_0\big)\big(W - U(\theta_0)\big) \end{pmatrix}$$

defines a diffeomorphism for W sufficiently close to $U(\theta_0)$, when combined with the previous lemma will prove the claim. Notice that

$$g_W\big(U(\theta_0)\big) = \begin{pmatrix} (u\sin\theta_0 - v\cos\theta_0 - c)_U\big|_{U(\theta_0)} \\ l^+\big(U(\theta_0); \theta_0\big) \\ l^1\big(U(\theta_0); \theta_0\big) \\ l^2\big(U(\theta_0); \theta_0\big) \end{pmatrix}.$$

Then, if

$$g_W^i\big(U(\theta_0)\big)z = 0, i = 2, 3, 4,$$

this implies $z \parallel r^-\big(U(\theta_0); \theta_0\big)$. But since

$$\Big((u\sin\theta_0 - v\cos\theta_0 - c)_U r^-(U; \theta_0)\Big)\Big|_{U(\theta_0)} \neq 0,$$

(by genuine nonlinearity) this implies $z = 0$. Therefore, for W sufficiently close to $U(\theta_0)$, g is a diffeomorphism. Since $U(\theta)$ approaches $U(\theta_0)$ as $\theta \to \theta_0$ by continuity, and $g\big(U(\theta)\big)$ satisfies the Lipschitz estimate based at θ_0 for θ sufficiently close to θ_0, U itself must satisfy a Lipschitz estimate based at θ_0 for θ sufficiently close to θ_0. The C^1 norm of g is bounded uniformly above by the phase space bounds and equation of state, as is the Lipschitz constant for $g\big(U(\theta)\big)$, and so the Lipschitz constant for U is as well. □

Theorem IV.22. *Assuming that there are no stagnation points, and that density and internal energy remain bounded away from zero, any L^∞ weak, steady, self similar solution to the 2-d full polytropic Euler equations must be of bounded variation. Moreover, U can be decomposed as*

$$U = U_L + U_S,$$

where U_L is Lipschitz with constant independent of U, and U_S is a saltus function of bounded variation, with total variation independent of U. (Note these constants will depend on the equation of state, the lower bound on density and internal energy, and the L^∞ norm of U.) Note that this implies U is a special function of bounded variation, since the Cantor part vanishes. Moreover, the absolutely continuous part is in fact Lipschitz.

Proof. The statement about U_S has been covered in previous lemmas. We claim that for any θ_0, U_L satisfies a Lipschitz estimate based at θ_0 for θ sufficiently close to θ_0.

If $\theta_0 \in \mathcal{C}$, then U is constant on a neighborhood containing θ_0, and since there are no shocks or contacts it is clear that $U_L := U - U_S$ is constant and thus satisfies a Lipschitz estimate based at θ_0 with constant 0.

If $\theta_0 \in \mathcal{S}_F \cup \mathcal{S}_B \cup \mathcal{S}_C$, then the jump at θ_0 is accounted for in U_S, and so U_L is constant on some neighborhood containing θ_0, and so satisfies a Lipschitz estimate based at θ_0 with constant 0.

If $\theta_0 \in \mathcal{R}_F \cup \mathcal{R}_B$, then, for θ sufficiently close to θ_0, we have from Lemma IV.19 and Theorem IV.21 that

$$\begin{aligned}|U_L(\theta) - U_L(\theta_0)| &\leq |U(\theta) - U(\theta_0)| + |U_S(\theta) - U_S(\theta_0)| \\ &\leq C_S|\theta - \theta_0| + M'|\theta - \theta_0| := C_L|\theta - \theta_0|.\end{aligned}$$

Global Lipschitz estimates with the same C_L can be obtained exactly as in the proof of Theorem II.41 in Chapter II, and we are done. $\square$

4.9 Structure of Flows

We now prove some results about the structure of possible solutions, and present several examples.

Assume that there are no stagnation points. We begin by decomposing the domain into a finite number of sectors[2]. Denote the points in $\mathcal{S}_C$ as $\theta_1, \theta_2, ..., \theta_N$ so that

$$0 \leq \theta_1 < \theta_2 < ... < \theta_N < 2\pi.$$

(Recall that $\theta \in \mathcal{S}_C$ means that $N(\theta) = 0$, and $\mathcal{S}_C$ is a finite set by Lemma IV.10.) Define the sectors I_i, for $i = 1, .., N$, as

$$I_i := [\theta_i, \theta_{i+1}],$$

(taking $\theta_{N+1} = \theta_1$ to unify the notation).

We say that I_i is a *clockwise sector*[3] if $N|_{I_i} \geq 0$, and that I_i is a *counterclockwise sector* if $N|_{I_i} \leq 0$. By construction, each sector will either be one or the other, and N will be positive on the interior of a clockwise sector, and negative on the interior of a counterclockwise sector. Moreover, L is continuous on the interior of each sector, since L is continuous at shocks and there are no contacts in the interior of a sector.

[2] This notion of sectors is completely different from that in Chapter II.

[3] We use clockwise since we have chosen to measure normal velocity at θ by using a vector that always points clockwise.

For all the figures in the remainder of this chapter, the flow direction is what is indicated. The length of the arrows is not meant to suggest anything about the length of the velocity vectors.

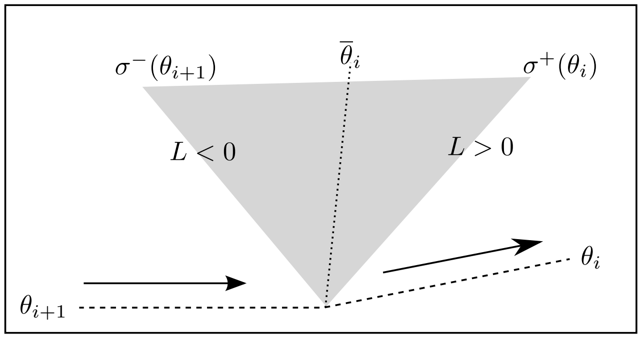

Figure 4.1: In a clockwise sector $I_i = [\theta_i, \theta_{i+1}]$, $L(\theta_i+) > 0$ and $L(\theta_{i+1}-) < 0$. L is strictly decreasing on $]\theta_i, \theta_{i+1}[$, and equal to zero at a unique $\overline{\theta}_i$. The flow is constant on $]\theta_i, \sigma^+(\theta_i)[$ and $]\sigma^-(\theta_{i+1}), \theta_{i+1}[$. If $\theta_{i+1} \neq \theta_i + \pi$, then there must be some wave structure in the grey shaded region.

Lemma IV.23. (*See Figure 4.1.*) *Suppose I_i is a clockwise sector. Then, $L(\theta_i+) > 0$, $L(\theta_{i+1}-) < 0$, and L is strictly decreasing on $]\theta_i, \theta_{i+1}[$. Similarly, if I_i is a counterclockwise sector then $L(\theta_i+) < 0$, $L(\theta_{i+1}-) > 0$, and L is strictly increasing on $]\theta_i, \theta_{i+1}[$. Therefore, there exists a unique $\overline{\theta}_i \in]\theta_i, \theta_{i+1}[$ such that $L(\overline{\theta}_i) = 0$.*

Proof. Consider the strong form of the conservation of mass and tangential momentum equations,

$$(\rho N)_\theta = \rho L,$$

$$(\rho L N)_\theta = \rho L^2 - \rho N^2.$$

Manipulating these, we obtain for any point of differentiability on the interior of I_i that

$$L_\theta \rho N + L(\rho N)_\theta = \rho L^2 - \rho N^2,$$

$$L_\theta \rho N + \rho L^2 = \rho L^2 - \rho N^2,$$

$$L_\theta = -N, \tag{4.33}$$

since ρ is bounded away from zero by assumption, and $N \neq 0$ on the interior of I_i.

Theorem IV.22 shows that U is Lipschitz almost everywhere (since U_S is constant except on at most a countable, discrete set), and since jumps in U_S on the interior of I_i must be shocks (not contacts since the boundaries of the sectors are where contacts may occur), L_S is constant on the interior of I_i. Therefore, L is Lipschitz (hence differentiable almost everywhere) on the interior of I_i, and so the fundamental theorem of calculus can be applied to L. Therefore (4.33) shows that L is strictly decreasing (increasing) on $]\theta_i, \theta_{i+1}[$ if I_i is a clockwise (counterclockwise) sector.

Recall that when U is constant, $N_\theta = L$. Also recall that by Theorem IV.9 there exist $\sigma^+(\theta_i) > \theta_i$ and $\sigma^-(\theta_{i+1}) < \theta_{i+1}$ such that U is constant on $]\theta_i, \sigma^+(\theta_i)[$ and on $]\sigma^-(\theta_{i+1}), \theta_{i+1}[$. Therefore, the following right limits are defined and we have for small $\delta > 0$ that

$$N(\theta_i + \delta) - N(\theta_i +) = N(\theta_i + \delta) = \int_{\theta_i +}^{\theta_i + \delta} L(\eta) d\eta.$$

Since U is constant on $]\theta_i, \theta_i + \delta[$ and there are no stagnation points $0 \neq |\vec{u}(\theta_i +)|^2 = |N(\theta_i +)|^2 + |L(\theta_i +)|^2 = |L(\theta_i +)|^2$. Therefore, by continuity, $\operatorname{sgn}(L)$ is constant on $]\theta_i, \theta_i + \delta[$, and so $\operatorname{sgn}\big(N(\theta_i + \delta)\big) = \operatorname{sgn}\big(L(\theta_i +)\big)$. Therefore, for a clockwise sector, $L(\theta_i +) > 0$. Similar arguments work for $L(\theta_{i+1} -)$ and for counterclockwise sectors.

Since $L(\theta_i +)$ and $L(\theta_{i+1} -)$ must have opposite signs, and L is monotone on $]\theta_i, \theta_{i+1}[$, there is a unique $\overline{\theta}_i$ such that $L(\overline{\theta}_i) = 0$. $\square$

Now we define in this context a *Prandtl-Meyer wave.* A *forward (facing) Prandtl-Meyer wave* is a closed interval $[\alpha, \beta]$ such that $N(\theta) = c(\theta)$ for all $\theta \in [\alpha, \beta]$.

A *backward (facing) Prandtl-Meyer wave* is the same except that $N(\theta) = -c(\theta)$. Moreover, U is differentiable almost everywhere on $]\alpha, \beta[$, and U_θ is in the kernel of $A(U;\theta)$ when it is defined. This follows from the the strong form of the Euler equations

$$A(U;\theta)U_\theta = 0.$$

It is well known that $p = A(S)\rho^\gamma$ for a polytropic gas, where $A(S) = C\exp(S)$ for some constant C. Therefore

$$c^2 = p_\rho = A(S)\gamma\rho^{\gamma-1},$$

and so

$$c = \sqrt{A(s)\gamma}\rho^{\frac{\gamma-1}{2}}.$$

S is constant away from discontinuities, and so, for θ in the interior of a forward Prandtl-Meyer wave we have that

$$\begin{aligned}(\rho N)_\theta = (\rho c)_\theta &= \sqrt{A(S)\gamma}\big(\rho^{\frac{\gamma+1}{2}}\big)_\theta \\ &= \sqrt{A(s)\gamma}\frac{\gamma+1}{2}\rho^{\frac{\gamma-1}{2}}\rho_\theta = \rho L,\end{aligned}$$

and so

$$\mathrm{sgn}(\rho_\theta) = \mathrm{sgn}(L).$$

Therefore, for a forward Prandtl-Meyer wave, as the gas particles pass through it (corresponding to the decreasing θ direction by our choice of coordinates), ρ increases if L is negative (compression wave), and decreases if L is positive (expansion wave). Since the flow is isentropic inside the wave,

$$\mathrm{sgn}(p_\theta) = \mathrm{sgn}(\rho_\theta) = \mathrm{sgn}(L).$$

Similar calculations for backward waves can be done. Therefore, in light of Lemma IV.23, we have the classifications:

- forward expansion wave: $\alpha < \beta \leq \overline{\theta_i}$, $N(\theta) = c(\theta), L(\theta) \geq 0$ for all $\theta \in [\alpha, \beta]$,
- forward compression wave: $\overline{\theta_i} \leq \alpha < \beta$, $N(\theta) = c(\theta), L(\theta) \leq 0$ for all $\theta \in [\alpha, \beta]$,
- backward expansion wave: $\overline{\theta}_i \leq \alpha < \beta$, $N(\theta) = -c(\theta), L(\theta) \geq 0$ for all $\theta \in [\alpha, \beta]$,
- backward compression wave: $\alpha < \beta \leq \overline{\theta}_i$, $N(\theta) = -c(\theta), L(\theta) \leq 0$ for all $\theta \in [\alpha, \beta]$.

It is possible to join a forward compression wave between $[\alpha_1, \beta_1]$ to a forward expansion wave between $[\alpha_0, \beta_0]$ if $\beta_0 = \overline{\theta}_i = \alpha_1$. In that case the compression wave ends when the flow is precisely sonic at $\overline{\theta}_i$ (since $N = c$ and $L = 0$, so $\vec{u} = c$), and an expansion wave immediately starts at $\overline{\theta}_i$. We have the following theorem.

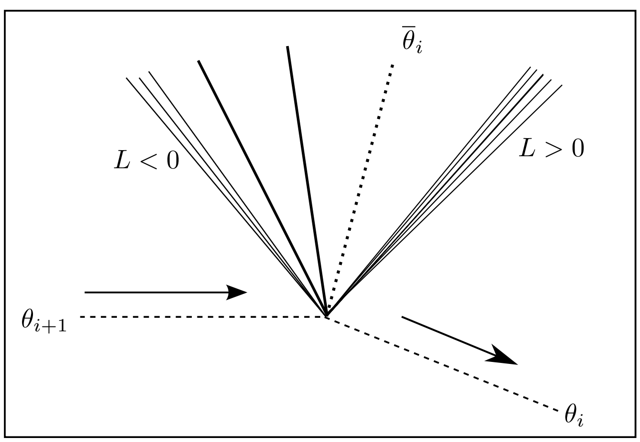

Figure 4.2: In a clockwise sector $I_i = [\theta_i, \theta_{i+1}]$, $L > 0$ on $]\theta_i, \overline{\theta}_i[$, and $L < 0$ on $]\overline{\theta}_i, \theta_{i+1}[$. The flow is constant on $]\theta_i, \sigma^+(\theta_i)[$ and $]\sigma^-(\theta_{i+1}), \theta_{i+1}[$. There is at most one shock or rarefaction in $[\sigma^+(\theta_i), \overline{\theta}_i]$, and possibly infinitely many shocks and compression waves in $]\overline{\theta_i}, \sigma^-(\theta_{i+1})]$. However, there cannot be consecutive compression waves. In this particular example, the flow consists of a compression wave and two shocks in the $L < 0$ part, and a rarefaction wave in the $L > 0$ part.

Theorem IV.24. (*See Figure 4.2.*) *Suppose I_i is a clockwise sector, and that U is continuous on an open interval $B \subset]\overline{\theta}_i, \theta_{i+1}[$. (In this case $L < 0$ on B.) Then,* *either U is constant on this open interval or constant on either side of a single forward compression wave.*

Similarly, in a counterclockwise sector, on an open interval $B \in]\theta_i, \overline{\theta}_i[$ on which U is continuous, U must be constant or constant on either side of a single backward compression wave.

Proof. Suppose I_i is a clockwise sector. Since U is continuous on B, the set on which $N(\theta) = c(\theta)$ is closed in B. Therefore, its complement in B is a countable union of open intervals, on which U is constant by Theorem IV.7 (since $N(\theta)$ cannot be 0 or $-c(\theta)$ in the interior of I_i). Since $L(\theta)$ is negative on B, and $N_\theta = L$ on this complement, $N(\theta) = c(\theta)$ can be satisfied at at most one endpoint of an open interval in the complement. Therefore at least one endpoint must be an endpoint of B, making $\mathcal{R}_F \cap B$ a closed interval in B.

Theroem IV.22 shows that U is Lipschitz on B, since $B \subset \mathcal{C} \cup \mathcal{R}_F$ and thus U_S is constant on B. Therefore, U is differentiable almost everywhere in $\mathcal{R}_F \cap B$ and the strong form of the equations implies U_θ is in the kernel of $A(U;\theta)$ everywhere it is defined, and so $\mathcal{R}_F \cap B$ defines a forward Prandtl-Meyer wave. The fact that $B \subset]\overline{\theta}_i, \theta_{i+1}[$ shows it must be a forward compression wave. The argument for a counterclockwise sector is similar. □

Note that there may be multiple forward compression waves in a clockwise sector — this theorem requires only that there is at least one forward facing shock in between. Since L is negative on $]\overline{\theta}_i, \theta_{i+1}[$, on any interval on which U is constant N is decreasing. In a forward sector, this corresponds to N increasing along particle paths of the gas particles. Therefore, upon exiting a compression wave, the normal velocity is sonic, but as the gas particles continue traveling in the clockwise (negative θ direction), the normal velocity increases and becomes supersonic, leading to the possibility of a forward facing shock, which upon exit the normal velocity will be subsonic. Normal velocity can then increase along particle paths back to the sound speed, and the gas can enter another compression wave.

Theorem IV.25. *(See Figure 4.2.) Suppose I_i is a clockwise sector. Then, on $]\theta_i, \overline{\theta}_i]$, exactly one of the following is true:*

- *U is constant on either side of a forward facing shock,*
- *U is constant on either side of a forward expansion wave,*
- *U has an expansion wave on $[\alpha, \overline{\theta_i}]$ and is constant on $]\theta_i, \alpha[$,*
- *U has a normal shock (that is, $L = 0$) at $\overline{\theta}_i$ and is constant on $]\theta_i, \overline{\theta}_i[$,*
- *U is constant on $]\theta_i, \overline{\theta}_i]$.*

We have the similar statement if I_i is a counterclockwise sector, for the interval $[\overline{\theta}_i, \theta_{i+1}[$.

Proof. Suppose U has a shock at $\theta_0 \in]\theta_i, \overline{\theta}_i]$. Then, we know that $N(\theta_0-) < c(\theta_0-)$. Recall that there exists $\sigma^-(\theta_0) < \theta_0$ such that U is constant on $]\sigma^-(\theta_0), \theta_0[$, and that $N\big(\sigma^-(\theta_0)+\big) = 0$ or $c\big(\sigma^-(\theta_0)+\big)$. However, $N_\theta = L$ on $]\sigma^-(\theta_0), \theta[$, and on this interval $L > 0$, and so N decreases as θ decreases, and so $N\big(\sigma^-(\theta_0)+\big) = 0$, making $\sigma^-(\theta_0) = \theta_i$. Therefore, there can be no shocks in $]\theta_i, \theta_0[$, and U is constant on $]\theta_i, \theta_0[$ by Theorem IV.7. Similar arguments show that $\sigma^+(\theta_0) \geq \overline{\theta}_i$ (since for N to be sonic it must decrease from $N(\theta_0) > c(\theta_0)$, which is impossible since $N_\theta = L > 0$ on $]\theta_0, \sigma^+(\theta_0)[$) and so either the first or fourth statement is true.

If there is not a shock, then U is continuous on this interval, and similar arguments as in the proof of Theorem IV.24 show that there can be at most one expansion wave, and we are done. Similar arguments work in counterclockwise sectors. $\square$

Examples with infinitely many shocks can be constructed (these theorems show that they must occur in the parts of the sectors where $L < 0$), or with infinitely many shocks interspersed with compression waves (with the restriction that compression waves cannot occur consecutively, by Theorem IV.24). Therefore, since infinitely many discontinuities may occur BV is the sharpest commonly used function space we may use.

We notice that even though we don't have hyperbolicity in general in any direction for the steady Euler equations, we can still see some parallels to the small perturbation case treated in Chapter II. The $L < 0$ parts of clockwise and counterclockwise sectors resemble the "backward sectors" in Chapter II — in that multiple waves of a given family are possible, just like backward in time solutions to one dimensional problems. The $L > 0$ parts of clockwise and counterclockwise sectors can contain at most one wave, resembling the "forward sectors" from earlier, just like forward-in-time solutions to one-dimensional Riemann problems can contain at most one wave of each family.

4.10 Maximum Number of Contacts

We note for a both rarefaction and compression waves that the velocity turns towards the origin as the gas particles travel through the wave. This can be seen by manipulating

$$(\rho N)_\theta = \rho L,$$

$$(\rho N^2 + p)_\theta = 2\rho N L,$$

to obtain the following.

$$(\rho N)_\theta N + (\rho N) N_\theta + p_\theta = 2\rho N L,$$

$$\rho LN + (\rho N)N_\theta + p_\theta = 2\rho NL,$$

$$LN - NN_\theta = \frac{p_\theta}{\rho}.$$

We consider the angle of the flow, $\phi := \angle(u, v)$, as in Figure 4.3 , as in [32].

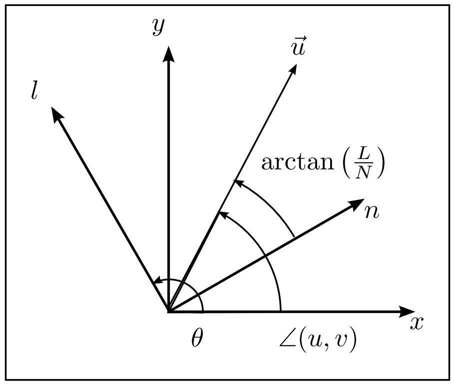

Figure 4.3: Computing the angle of the flow in terms of N, L, and θ. n is the angular coordinate vector, and l is the radial coordinate vector.

Since we are considering $N > 0$, $\angle(N, L) \in (-\pi/2, \pi/2)$, and so $\angle(N, L) = \arctan\left(\frac{L}{N}\right)$. Then,

$$\begin{aligned}\angle(u, v) &= \theta - \frac{\pi}{2} + \arctan\left(\frac{L}{N}\right) \\ &= \theta - \arctan\left(\frac{N}{L}\right).\end{aligned}$$

Then,

$$\begin{aligned}\phi_\theta &= 1 - \frac{1}{1 + \left(\frac{N}{L}\right)^2} \frac{LN_\theta - NL_\theta}{L^2} \\ &= 1 - \frac{LN_\theta + N^2}{L^2 + N^2} \\ &= \frac{L^2 - LN_\theta}{L^2 + N^2} \\ &= \frac{Lp_\theta}{\rho N(N^2 + L^2)}.\end{aligned}$$

However, recall that in a forward wave

$$\operatorname{sgn}(p_\theta) = \operatorname{sgn}(L),$$

and so ϕ_θ is positive. Therefore, as the gas particles travel through the shock, θ *decreases*, and so $\angle(u,v)$ *decreases* as well.

For backward waves,

$$\phi = \theta - \arctan\left(\frac{N}{L}\right) + \pi,$$

giving the same expression for ϕ_θ. But in this case

$$\operatorname{sgn}(p_\theta) = -\operatorname{sgn}(L),$$

but $N = -c$ and so ϕ_θ is still positive. However, for backward waves the gas particles move in the increasing θ direction, and so the flow still turns towards the origin as the gas particles travel through the wave. See Figure 4.4 for some examples of waves.

For shocks, since L is continuous and $|N|$ decreases as the gas particles pass through the shocks, the flow is turned away from the origin if L is positive, and toward the origin if L is negative.

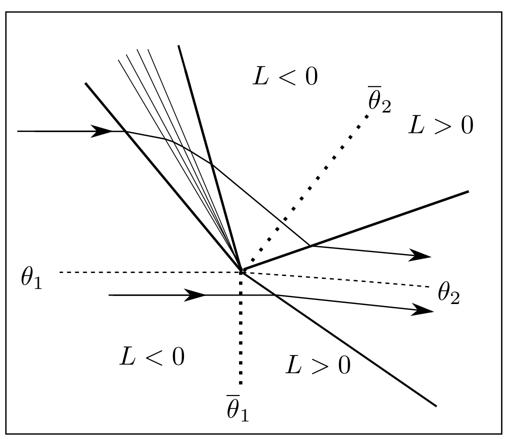

Figure 4.4: An example with two sectors. $I_1 = [\theta_1, \theta_2]$ is on the bottom and is a counterclockwise sector, $I_2 = [\theta_2, \theta_1]$ is on the top and is a clockwise sector. In I_1, there is a single shock in the $L > 0$ part. In I_2, the gas passes through a shock, a compression wave, then a shock in the $L < 0$ part, and a shock in the $L > 0$ part. As the gas particles travel through shocks, the flow is turned toward the shock line, and compression waves turn the flow toward the origin. Rarefaction waves would turn the flow toward the origin as well.

Theorem IV.26. (*See Figure 4.5*). *For* $\gamma = 1.4$*, there can be a maximum of two sectors. For other values of* $\gamma > 1$*, there can be up to three sectors, but there are no values of* $\gamma > 1$ *that lead to flows with more than three sectors.*

Proof. Choose coordinates so that there is a clockwise sector $I = [\alpha, \pi]$ where $0 < \alpha < \pi$, so that $N(\pi) = N(\alpha) = 0$. Then, by Lemma IV.23 $L(\alpha+) > 0$, $L(\pi-) < 0$, and so the flow needs to be turned away from the origin by an angle of α. Following IV.23, denote $\overline{\theta}$ the unique value between α and π such that $L(\overline{\theta}) = 0$, and recall that L is positive on $]\alpha, \overline{\theta}[$, and negative on $]\overline{\theta}, \pi[$.

By Theorem IV.24, the discussion after it, and the discussion preceding this theorem, any shocks or compression waves on $]\overline{\theta}, \pi[$ turn the flow towards the origin. Therefore, $\angle\big(u(\overline{\theta}), v(\overline{\theta})\big) \leq 0$. Since we are interested in finding the maximum possible α that the flow can be turned upwards, the best possible situation is for there to be no compression waves or shocks on $]\overline{\theta}, \pi[$, which yields $\overline{\theta} = \frac{\pi}{2}, \angle\big(u(\pi/2), v(\pi/2)\big) = 0$.

If there is a shock at $\frac{\pi}{2}$, then it is a normal shock and so the flow is absolutely subsonic, and thus constant, for $]\alpha, \frac{\pi}{2}[$. Therefore $\alpha = 0$ since the flow can never be turned away from the origin.

Therefore, to accomplish the maximum upwards turning, the flow should be constant on $[\frac{\pi}{2}, \pi[$. By Theorem IV.25, the flow is either constant (again resulting in $\alpha = 0$), has exactly one rarefaction, or exactly one shock on $]\alpha, \frac{\pi}{2}[$. A rarefaction wave turns the flow towards the origin, resulting in $\alpha < 0$, and so there must be a single shock to accomplish $\alpha > 0$.

Using the well known $\theta - \beta - M$ equation (see [31], Chapter 4) to relate the incident Mach number $M := \frac{|\vec{u}|_+}{c_+}$, the turning angle α, and the shock angle θ, we have that

$$\alpha = \arctan\left(\frac{2\cot\theta(M^2\sin^2\theta - 1)}{M^2(\gamma + \cos(2\theta)) + 2}\right).$$

It is well known (see Section 122 in [11]) that the curves $\alpha(\theta)$ for fixed values of M all lie below the limiting case $M \to \infty$, and solving for the maximum α yields

$$\alpha_{\max} = \arcsin\left(\frac{1}{\gamma}\right).$$

The flow can only be turned upward when $L > 0$, so there can never be more than three sectors since $\alpha_{\max} = \frac{\pi}{2}$ is only attained in the limit as $\gamma \searrow 1$. For $\gamma = 1.4$, $\alpha_{\max} \approx 45.5°$, and so the flow cannot turn the required $60°$ needed to have more than two sectors. For $1 < \gamma < 1.15$, $\alpha_{\max} > 60°$, and so there will exist finite incoming Mach numbers for which the flow can turn $60°$, allowing for the existence flows with three sectors for some values of $\gamma > 1$. $\square$

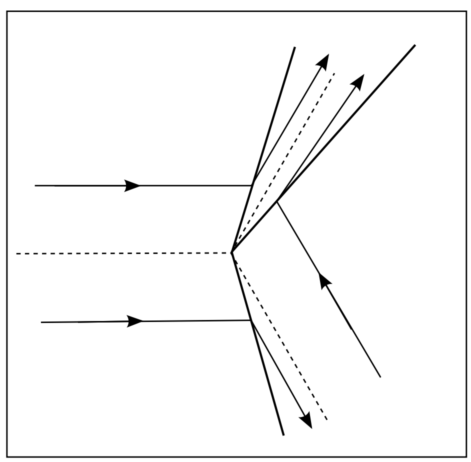

Figure 4.5: For $\gamma < 1.15$, the maximum turning angle is greater than $60°$. Therefore there exist flows with three contact discontinuities, such as the one above. In this example each sector has one shock in the region where $L > 0$, causing the flow to turn away from the origin.

CHAPTER V

Conclusions and Discussion

We have thus shown, in a wide variety of cases, that steady and self-similar admissible solutions that are only assumed to be bounded and satisfy a smallness condition are special functions of variation. In addition, in the situations in which a Lax solution exists, we have shown that the "forward in time" part of the solution must coincide with the Lax solution.

There are some related results in the literature. In [24], Heibig considered admissible self-similar L^∞ solutions to one-dimensional Riemann problems. Under the assumption that all fields are genuinely nonlinear (and thus all eigenvalues are simple), he was able to show that if the Riemann states are sufficiently close that the solution must coincide with the Lax solution. The fact that our analysis can treat linearly degenerate fields and non-strictly hyperbolic systems of constant multiplicity improves this result. In addition, [24] did not consider backward-in-time solutions, which correspond to the backward genuinely nonlinear sectors analyzed in Section 2.13.1. That case was more difficult than the case of forward sectors in Section 2.13.2, since for forward sectors it was guaranteed that $\sigma^\pm(\xi_0)$ are the endpoints of I^α, so at most one shock could occur (recall that for a shock occurring at ξ_0, we had that V must be constant on $]\xi_0, \sigma^+(\xi_0)[$ and on $]\sigma^-(\xi_0), \xi_0[$ — these are the intervals on which $\lambda^\alpha(V) - \xi$ had the wrong sign to be a limit state for a subsequent shock). All the delicate estimates in 2.13.1 were needed since the entropy inequality there allows for multiple waves of the same family, so our results are stronger than what was available before. In addition, we are able to get SBV regularity for systems in which there are no Lax solutions due to the forward in time and backward in time like sectors being interspersed with one another (see the example in Section 3.5).

There has also been some work regarding the SBV regularity of solutions to one-dimensional problems. In [1], Ambrosio and De Lellis showed that (not self-similar) L^∞ entropy solutions to *scalar* one-dimensional conservation laws with convex flux are special functions of bounded variation (as a function of x, for all but countably many values of t). Scalar conservation laws are known to be much better behaved than systems in general — one of the most important results is due to Oleĭnik [39] and shows that L^∞ data is immediately smoothed to be of locally bounded variation at positive times, and this was how [1] proceeded to improve the regularity to SBV. Therefore it is more appropriate to compare our results to those for systems. In [12], Dafermos showed that BV self-similar solutions to one-dimensional systems with genuinely nonlinear fields are SBV (without reference to entropy). In [5], Bianchini and Caravenna showed that BV entropy solutions to one-dimensional systems with genuinely nonlinear fields are SBV for all but countably many values of t. In each of these, the assumed BV regularity of the solution was used extensively, and so the fact that we can treat merely L^∞ solutions is interesting, although we have made smallness and self-similarity assumptions. It is the entropy inequality that prevents oscillatory solutions with unbounded variation from occurring — and so assuming BV and ignoring entropy to get SBV (as in [12]) is interesting to compare to our approach — using entropy to get from L^∞ to BV, and having SBV follow without additional effort.

The treatment of sonic and subsonic full Euler flows in Chapter IV is quite different from these one-dimensional results, since a notion of hyperbolicity is not satisfied. Even though the SBV regularity of the full Euler equations is treated in Chapter IV, it is still worthwhile to confirm that the structural results of Chapter II apply if we restrict to small perturbations of a constant state, as was done in Section 3.3. It is also interesting to see the similarities between the portions of the counterclockwise and clockwise sectors with negative tangential velocity (discussed in Section 4.9) and the backward sectors in Section 2.13.1. In each of those cases, there was the possibility of having another admissible shock or simple wave following a shock of the same type. Similarly, the portions of the counterclockwise and clockwise sectors with *positive* tangential velocity was comparable to the forward sectors in Section 2.13.2. In that case, there were only two choices of possible waves, and at most one of them could occur. Therefore, since we were not restricting to small perturbations in Chapter IV, we could not say exactly where certain kinds of waves could occur, but we could make broader statements such as "If the tangential velocity is positive and the normal velocity does not change sign on some interval, then there can be at most one shock or rarefaction." We could not determine where this interval was as determined by a background state, since we had no background state to speak of.

There does not seem to be much in the literature with regard to examining regularity of steady and self-similar solutions to two-dimensional problems, or investigating the structure by tracking how the convex entropy η for the unsteady problem translates into entropy inequalities for the steady problem.

APPENDICES

APPENDIX A

Analysis Lemmas

Lemma A.1. *Suppose* $\Omega \subset \mathbb{R}^n$ *is measurable and nonempty,* $K \subset \mathbb{R}^l$ *is compact,* $W \in L^\infty(\Omega)$ *so that* $W(z) \in K$ *for a.e.* $z \in \Omega$*, and that* $g : \Omega \times K \to \mathbb{R}^k$*,* $\tilde{g} : \Omega \to \mathbb{R}^k$ *are continuous. If*

$$g(z, W(z)) \leq \tilde{g}(z) \quad \textit{for a.e. } z \in \Omega,$$

(meaning $g_i(z, W(z)) \leq \tilde{g}_i(z)$ *for all* i*, where* $g = (g_1, ..., g_k)$*,* $\tilde{g} = (\tilde{g}_1, ..., \tilde{g}_k)$*), then we can find a version* $\tilde{W}$ *of* W*, with values in* K everywhere*, so that*

$$g\big(z, \tilde{W}(z)\big) \leq \tilde{g}(z) \quad \textit{for } \text{all } z \in \Omega. \tag{A.1}$$

Proof. We immediately modify W, on a set of measure 0, to have values in K everywhere. Let $E = \{z \mid g(z, W(z)) \leq \tilde{g}(z)\}$. Then $\complement E$ has measure zero, so every $z \in \complement E$ is the limit of a sequence (z_n) in E.

Pick any $z \in \complement E$, and choose a sequence of points $(z_n) \in E$ such that $z_n \to z$. Since $W(z) \in K$ for all $z \in \Omega$, there is a subsequence (z_n') of (z_n) such that $\big(W(z_n')\big)$ converges to an element of K. For this z then define $\tilde{W}(z) := \lim \big(W(z_n')\big)$. Then,

$$g(z, \tilde{W}(z)) \leftarrow g(z_n', W(z_n')) \leq \tilde{g}(z_n') \to \tilde{g}(z).$$

Repeat this choice of sequence and subsequence for all $z \in \complement E$, and (A.1) will be satisfied. (Note that this choice of version $\tilde{W}$ is in no way unique, but that is not important for our purposes.) □

Lemma A.2. *Suppose $g : [0, z_{max}] \to \mathbb{R}$ is smooth and satisfies $g(0) = 0$, $g'(0) > 0$, and $g(z) > 0$ for all $z \in]0, z_{max}]$. Then, there exists $C > 0$ such that*

$$g(z) \geq Cz$$

for all $z \in [0, z_{max}]$.

Proof. Taylor theorem and the fact that $g'(0) > 0$ yield that $g(z) \geq C'z$ for all $z \in [0, \delta[$ for some $\delta > 0$ and $C' > 0$. Then, $\dfrac{g(z)}{z}$ is continuous on the compact set $[\delta, z_{\max}]$ and thus attains a minimum positive value C''. Taking $C := \min\{C', C''\}$ yields the result. □

Lemma A.3. *Suppose $g : [0, z_{max}] \to \mathbb{R}$ is smooth and satisfies $g(0) = 0$. Then there exists $C > 0$ such that*

$$g(z) \leq Cz$$

for all $z \in [0, z_{max}]$.

Proof. We immediately observe that

$$g(z) = g(z) - g(0) = \int_0^z g'(y)dy \leq Cz,$$

since g' is bounded above on the compact interval $[0, z_{\max}]$. □

APPENDIX B

Regularity of Eigenvalues and Eigenvectors

The standard implicit function theorem argument for smoothness of simple eigenvalues does not work if there are repeated eigenvalues. (We will use the convention that when the eigenvalues are indexed with a subscript they are repeated according to their multiplicity, where superscript indices only label the distinct eigenvalues.) It is well known (see [44]) that the unordered set of eigenvalues (repeated according to multiplicity) of an $m \times m$ matrix is a continuous function of the matrix entries, with the spectrum being an element of $\mathbb{C}^m \setminus \sim$, where $\{\lambda_\alpha\}_{\alpha=1}^m \sim \{\mu_\alpha\}_{\alpha=1}^m$ if $\lambda_\alpha = \mu_{\sigma(\alpha)}$ for all $\alpha = 1..m$ and some $\sigma \in S_n$. The metric on this quotient space is given by

$$d\left(\{\lambda_\alpha\}, \{\mu_\alpha\}\right) = \min_{\sigma \in S_n} \max_{1 \leq \alpha \leq m} |\mu_\alpha - \lambda_{\sigma(\alpha)}|.$$

However, if the matrices in question are continuous functions of say $z \in D \subset \mathbb{R}^k$ such that the eigenvalues are real for all $z \in D$, then it is clear we can label

$$\lambda_1(z) \leq \lambda_2(z) \leq \ldots \leq \lambda_m(z)$$

such that $\lambda_\alpha(z)$ is a continuous function of z for $\alpha = 1, .., m$.

The smoothness of the eigenvalues and eigenvectors is more delicate when the eigenvalues are not simple — in fact there are many counterexamples. However, if the matrices $A(z)$ have the property that each distinct $\lambda^\alpha(z)$ has constant algebraic multiplicity p_α and p_α linearly independent eigenvectors for all $z \in D$, then around each $z_0 \in D$ there exists a neighborhood $D_{z_0} \ni z_0$ such that, for $\alpha, \beta = 1, .., n, i = 1, .., p_\alpha, j = 1, .., p_\beta$,

$$\lambda^\alpha(z) : D \to \mathbb{R}$$
$$r^{\alpha,i}(z) : D_{z_0} \to \mathbb{R}^m$$
$$l^{\alpha,i}(z) : D_{z_0} \to \mathbb{R}^m$$

are smooth functions satisfying for all $z \in D_{z_0}$

$$\begin{aligned} A(z)r^{\alpha,i}(z) &= \lambda^\alpha(z)r^{\alpha,i}(z), \\ l^{\alpha,i}(z)A(z) &= l^{\alpha,i}(z)\lambda^\alpha(z), \\ l^{\alpha,i}(z)r^{\beta,j}(z) &= \delta_{\alpha\beta}\delta_{ij}, \\ |r^{\alpha,i}(z)| &= 1. \end{aligned} \tag{B.1}$$

Moreover, the set of right (and left) eigenvectors is linearly independent for all z, and for each given family the right eigenvectors can be taken to be orthonormal. A proof of these statements for a single semisimple eigenvalue of constant multiplicity can be found in [38].

Theorem B.1 (Nomizu [38])**.** *Let $D \subset \mathbb{R}^k$ be open and $A : D \to \mathbb{M}_m(\mathbb{R})$ be a smooth mapping such that $A(z)$ is diagonalizable for all z. If λ is a continuous function on D such that for every $z \in D$ the value $\lambda(z)$ is an eigenvalue of $A(z)$ with the common multiplicity p, then λ is smooth. Furthermore, for each z_0, there exist smooth eigenvectors $r^1, ..., r^p$ of a neighborhood D_{z_0} of z_0 into $\mathbb{R}^m$ such that, for each $z \in D_{z_0}$, $r^1(z), ..., r^p(z)$ form an orthonormal basis of the eigenspace of $A(z)$ for $\lambda(z)$.*

Apply this theorem to each eigenvalue, taking the left eigenvectors to be the rows of the inverse matrix of the matrix of right eigenvectors to obtain the normalization (B.1).

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