Foliation Structure for Generalized Hénon Mappings

by

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Dedicated to my God

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TABLE OF CONTENTS

CHAPTER I

Introduction

1.1 History

Complex dynamics is the study of iteration of a holomorphic function. The history of complex dynamics goes back to E. Schröder's study of Newton's method in one complex variable (see [1]). It is an algorithm to approach a zero of $f(z) = 0$ for some holomorphic function f over $\mathbb{C}$ in terms of a sequence. Instead of finding such a root, we look at the sequence defined by $z_{n+1} = N(z_n)$ with an initial value, where

$$N(z) = z - \frac{f(z)}{f'(z)}$$

on a neighborhood of a root of the function $f(z)$. For a good choice of our initial point z_0, the iteratively defined sequence $\{z_n\}$ approximates the solution of $f(z) = 0$. Through this approximating sequence, even when we cannot algebraically solve the equation, we can still get information on the zeros of $f(z) = 0$. The study of iteration is not just limited to the study of the equation of the form $f(z) = 0$. For example, we can apply it to the study of differential equations. The Hénon mapping first appeared in that context.

The Hénon mapping was first considered in an attempt to describe the weather. In 1969, Hénon studied a polynomial mapping of $\mathbb{C}^2$ of the form $f : (x, y) \to$

$(x^2 + c - ay, x)$ as a simplified model of a Poincaré section of the Lorenz differential equation (see [19], [20]). Hénon mappings are simply defined but have a rich dynamical properties. For example, they show chaotic behavior and if we restrict our attention to the real case, a strange attractor is observed.

In addition to the perspective from physics, they are important in their own right. According to Friedland and Milnor (see [17]), every polynomial automorphism of $\mathbb{C}^2$ is conjugate by a polynomial automorphism to an affine map, an elementary map or a finite composition of generalized Hénon mappings. Affine maps and elementary maps are dynamically non-interesting in the sense that their dynamical degree is 1. So, it is the finite compositions of generalized Hénon mappings that are dynamically interesting.

For these reasons, Hénon mappings are the most studied mappings in complex dynamics. Since there has been an immense amount of work done for Hénon mappings, it is impossible to list all of them. We list some of them. In the real case, for example, see Benedicks and Carleson ([10]), Holmes ([21]), Holmes and Whitley ([22]), and Holmes and Williams ([23]). In the complex case, for example, see Bedford, Lyubich, and Smillie ([3], [4], [5], [2], [6], [7], [8], and [9]), Fornæss and Sibony ([16]), Friedland and Milnor ([17]), and Hubbard and Oberste-Vorth ([24] and [25]). We review some relevant results to this paper, focusing on foliation structure. In [3], they considered a hyperbolic case. They used the Stable Manifold Theorem to show that stable manifolds foliate the boundary of the set of forward non-escaping points. Also, they showed that the stable manifolds are biholomorphic to $\mathbb{C}$ using the subadditivity of modulus and that each leaf is dense. In [2], they removed the condition on hyperbolicity on the polynomial. Instead, they used the hyperbolicity of the Green current and ergodicity to draw a similar conclusion to [3]. In [16], the mappings of the form $(z, w) \to (z^2 + c - aw, az)$, which are conjugate to Hénon mappings of the form $(z, w) \to (z^2 + c - a^2 w, z)$ are considered. They are regarded as perturbations of a single variable map $P_c(z) = z^2 + c$. Under some conditions on the polynomial $P_c(z)$ and a, they showed that the set of forward non-escaping points has a foliation structure and that the set of backward non-escaping points with finite exceptional points removed has a foliation structure by biholomorphic images of $\mathbb{C}$. In both foliations, each leaf is dense. In [24], they focused on the set of forward escaping points. They used the scattering theory method to find a good coordinate chart in the sense that the Green function (see Chapter II) has a simple representation. Using this function, they showed that each level set of the Green function has a foliation structure. They verified that each leaf is a biholomorphic image of $\mathbb{C}$ using the subadditivity of modulus and that each leaf is dense.

1.2 Presentation of Results

In this paper, we study the generalized Hénon mapping. It is a holomorphic polynomial automorphism f of $\mathbb{C}^2$ defined by

$$f : (z, w) \to (p(z) - aw, z),$$

where p is a monic polynomial of degree $d \geq 2$ and $a \neq 0$. For the dynamical study of f, we also consider the Green function g associated to f, which will be precisely defined in Chapter II.

Our focus lies on the set of forward escaping points, that is, the set of points of forward unbounded orbit. We first look into their behavior as we approach the line at ∞. Since the forward escaping points are characterized by $g > 0$, we consider the set $\mathscr{C}_c := \{g = c\}$ for some $c > 0$ near infinity. For this purpose, it is natural to consider $\mathbb{C}^2$ as a subset of $\mathbb{P}^2$. Especially, we take the Fubini-Study metric of $\mathbb{P}^2$ instead of the standard $\mathbb{C}^2$ hermitian metric. Accordingly, we extend our map f to a meromorphic function F defined over $\mathbb{P}^2$ such that $F|_{\mathbb{C}^2} = f$. As a meromorphic map of $\mathbb{P}^2$, the map F has a point of indeterminancy, say I_+. Since f has an inverse over $\mathbb{C}^2$, we can define F^{-1} and I_-. We let $K_c := \{g \leq c\}$. The behavior of $\mathscr{C}_c$ can be stated in terms of K_c. Note that the set $\overline{K_c} = K_c \cup I_+$ in $\mathbb{P}^2$ for $c > 0$ (for example, see III). We prove the following:

Theorem I.1. *There is no non-trivial holomorphic curve, which passes through I_+, and is supported in $\overline{K_c} \subseteq \mathbb{P}^2$ for $c > 0$.*

Observe that $\overline{K_c} = K_c \cup I_+$ in $\mathbb{P}^2$ for $c > 0$ implies that the closure of $\mathscr{C}_c$ for $c > 0$ in $\mathbb{P}^2$ is in $\mathscr{C}_c \cup I_+$. Then, the theorem implies that all holomorphic curves in the closure of $\mathscr{C}_c$ in $\mathbb{P}^2$ are actually in $\mathscr{C}_c \subseteq \mathbb{C}^2$. The proof is based on the property that g is unbounded as we approach I_- and the property that I_- is a superattracting fixed point for f. Also, we use the covering space $\mathbb{C}^3 \setminus \{0\}$ of $\mathbb{P}^2$ for transitions among the local affine charts of $\mathbb{P}^2$.

Next, we move to the foliation structure of $\mathscr{C}_c$ for $c > 0$. According to [24], $\mathscr{C}_c$ is foliated by biholomorphic images of $\mathbb{C}$ and each leaf is dense. Our main result answers the following question: "What kind of Riemann surface is each leaf?" It is a further study of the foliation structure. In order to better understand the meaning of our main result, we give the definition of a Brody curve and some explanation about it.

Definition I.2 (Brody Curve)**.** Let M be a complex manifold with a smooth metric ds. Let $\psi : \mathbb{C} \to M$ be a non-constant holomorphic map of $\theta \in \mathbb{C}$ to M.
The map ψ is said to be *Brody* if $\sup_{\theta\in\mathbb{C}} ds(\psi(\theta), d\psi(\frac{d}{d\theta})) < C$ for some constant $C > 0$. We call the image $\psi(\mathbb{C})$ a *Brody curve* in M. The curve $\psi(\mathbb{C})$ is said to be *injective Brody* if the parametrization ψ is injective.

Especially, we consider an injective Brody curve. For injective Brody curves, the parametrization is unique in the sense that if $\phi_1, \phi_2 : \mathbb{C} \to M$ are two injective parametrizations of an injective Brody curve $\mathscr{B}$, then there exist $a, b \in \mathbb{C}$ with $a \neq 0$, $\phi_2(z) = \phi_1(az+b)$ (see Chapter VI). So, whenever we biholomorphically parametrize an injective Brody curve by $\mathbb{C}$, it has a uniformly bounded speed of expansion with respect to the smooth metric ds of M. In some sense, we can compare injective Brody curves to holomorphic curves parametrized by the unit disc in $\mathbb{C}$. The Kobayashi metric is a natural metric for hyperbolic spaces. Due to its distance decreasing property under a holomorphic mapping, any holomorphic curve over a unit disc shows some kind of tame behavior with respect to the Kobayashi metric. On the contrary, in general, $\mathbb{C}$ does not have such a metric with decreasing property, the behavior of holomorphic curve of $\mathbb{C}$ can be very wild. Indeed, the existence of the map $\varphi : \mathbb{C} \to \mathbb{C}$ defined by $\varphi(z) = 2z$ confirms that there is no such metric. However, as pointed out, injective Brody curves have a uniformly bounded expansion speed, which implies that they do not fluctuate too much. Thus, the following theorem implies that even though every leaf of $\mathscr{C}_c$ for $c > 0$ seems to be very complicated in the sense that each leaf is dense (see Chapter V and [24]), but actually, it is not too wild. Our main result is the following:

Theorem I.3. *Every leaf of $\mathscr{C}_c$ viewed as a subset of $\mathbb{P}^2$ is an injective Brody curve.*

As an consequence of the theorem, we find a short $\mathbb{C}^2$ domain with real analytic boundary and its boundary is foliated by injective Brody curves.

The main ingredient for the existence of a Brody curve in each leaf is to modify Brody's proof in such a way that the family of parametrizations has a fixed point. The classical Brody's proof does not provide us with the location of limit maps, and so, we need to modify it. We also use Theorem I.1 to use the compactness property of $\mathbb{P}^2$. The intuition behind this part is that over a small neighborhood of I_-, $\mathscr{C}_c$ is more like a strap stretching in the w-direction. Also, the action of f^{-1} preserves the structure. In result, the ratio of the maximum modulus of derivative to the minimum of a parametrization is uniformly bounded over the entire family of parametrizations. So, this fact makes it possible to reparametrize the family such that the family of parametrizations shares a fixed point and Brody's technique is still valid to the family. The major factor for the injectivity part is Hurwitz's theorem.

On the way to the proof of our main theorem, we consider a family of parametrizations. Using the family of parametrizations, we re-prove the result in [24] that each leaf of $\mathscr{C}_c$ for $c > 0$ is a biholomorphic image of $\mathbb{C}$ in a more analytic way. Namely, we compute the Kobayashi-Royden pseudometric of each leaf (see Chapter V).

We close this section by briefly explaining each chapter. In Chapter II, we introduce the basic dynamics terminology related to generalized Hénon mappings. In Chapter III, we investigate the tendency of the value of g near the line at ∞ and prove our first theorem. In Chapter IV, we discuss the work of [24]. Also, we introduce a compact exhaustion of a leaf of $\mathscr{C}_c$, a family of the parametrizations of the compact subsets and related properties. In Chapter V, we introduce the Kobayashi-

Royden pseudometric, compute it for a specific leaf, and re-prove that each leaf is biholomorphic to $\mathbb{C}$. In Chapter VI, we introduce the notion of a Brody curve, provide various examples of Brody curves and explore their properties. Then we prove our main theorem. In the last chapter, we discuss short $\mathbb{C}^2$ domains.

1.3 Notations

In this last section, we introduce the notations that we are going to use in this paper.

Let Δ denote the unit disc in $\mathbb{C}$ and Δ_α the disc centered at the origin and of radius $\alpha > 0$ in $\mathbb{C}$. We use $\overline{\Delta}, \overline{\Delta_\alpha}$ to mean their closures in $\mathbb{C}$ respectively. Throughout this paper, we express $p(z) = \sum_{i=0}^{d} a_i z^i$ with $a_d = 1$. We let $q(z) := p(z) - z^d = \sum_{i=0}^{d'} a_i z^i$ with $d' \leq d-1$. We may assume that $d' \geq 2$ because it is not hard to see that when $d' \leq 1$ or even $q(z) = 0$, we have better estimate and therefore, it does not affect the result at all. Given a polynomial $H(z) = \sum_{i=0}^{d_H} h_i z^i$, $|H|(x)$ means the real polynomial $|H|(x) := \sum_{i=0}^{d_H} |h_i| x^i$. The z-coordinate and the w-coordinate of $f^n(z,w)$ are denoted by $f_1^n(z,w)$ and $f_2^n(z,w)$. Up to Section 6.2, (z_i, w_i) refers to $f^i(z,w)$. When we want to emphasize that it is the image of (z_K, w_K) under f^i, we will rather use $((z_K)_i, (w_K)_i)$. The standard norms on $\mathbb{C}^2$ and $\mathbb{C}^3$ are denoted by $\|\cdot\|_{\mathbb{C}^2}$ and $\|\cdot\|_{\mathbb{C}^3}$, respectively.

CHAPTER II

Preliminaries

We are going to study the generalized Hénon mapping from a dynamical perspective in this paper. The generalized Hénon mapping is a holomorphic polynomial automorphism f of $\mathbb{C}^2$ defined by

$$f(z,w) = (p(z) - aw, z) \text{ for } (z,w) \in \mathbb{C}^2,$$

where $p(z)$ is a monic polynomial of z with degree $d \geq 2$ and $a \neq 0$. Indeed, f is a holomorphic polynomial automorphism; since it is a polynomial mapping, it is holomorphic and since the map $f^{-1} : (z,w) \to (w, \frac{p(w)-z}{a})$ is its inverse, it is an automorphism of $\mathbb{C}^2$.

We can also view $f : \mathbb{C}^2 \to \mathbb{C}^2$ as a restriction of a meromorphic endomorphism $F : \mathbb{P}^2 \to \mathbb{P}^2$ to $\mathbb{C}^2$. We projectivise f to recover the meromorphic endomorphism F. In the homogeneous coordinate system $[z : w : t] \in \mathbb{P}^2$, it can be written as $F([z : w : t]) = [t^d p(\frac{z}{t}) - awt^{d-1} : zt^{d-1} : t^d]$. Then, F has a point of indeterminancy $I_+ := [0 : 1 : 0]$. Since f is an automorphism over $\mathbb{C}^2$, f^{-1} exists and is defined over $\mathbb{C}^2$. Then, in the same way, we can find the meromorphic endomorphism $F^{-1} : \mathbb{P}^2 \to \mathbb{P}^2$ for $f^{-1} : \mathbb{C}^2 \to \mathbb{C}^2$. In the homogeneous coordinate system $[z : w : t] \in \mathbb{P}^2$, it can be written as $F^{-1}([z : w : t]) = [wt^{d-1} : \frac{1}{a}(t^d p(\frac{w}{t}) - zt^{d-1}) : t^d]$, and F^{-1} also has a point of indeterminancy $I_- := [1:0:0]$. Note that F and F^{-1} are not inverse to each other in $\mathbb{P}^2$.

In this chapter, we introduce terminology and properties related to f, f^{-1}, F, and F^{-1}. From now on, we are going to use (z, w) for the standard Euclidean coordinate system of $\mathbb{C}^2$ and $[z : w : t]$ for the standard homogeneous coordinate system of $\mathbb{P}^2$, unless otherwise stated.

2.1 Choice of a Large Number R

In our mapping $f(z, w) = (p(z) - aw, z)$, $p(z)$ is the only non-linear term. So, this can be thought of as the source of the complicated behaviors of the orbits for f. However, near $|z| = \infty$, $p(z)$ shows somewhat tame behavior. Also, if the effect of w is small in the action of f, for example, under the condition that $|w| \le |z|$, then f can be approximated by simpler maps of the form $(z, w) \to (az^d, z)$ for some constant a. In some sense, this is reflected in the property that f has a super-attracting fixed point at I_-. In other words, we want to find a neighborhood of I_- of the form $\{|z| > R, |z| \ge |w|\}$ where f shows somewhat regular behavior. At the same time, for later applications, we are expecting some more properties related to $p'(z)$ and so on. We want to find such an R with the above mentioned properties.

Recall our notations $p(z) = \sum_{i=0}^{d} a_i z^i$ with $a_d = 1$ and $q(z) = p(z) - z^d = \sum_{i=0}^{d'} a_i z^i$ with $a_{d'} \neq 0$ and $d' \le d-1$. Note that $q(z)$ may be 0 if no such $a_{d'}$ exists. Given a polynomial $H(z) = \sum_{i=0}^{d_H} h_i z^i$, we define a real polynomial $|H|(x) := \sum_{i=0}^{d_H} |h_i| x^i$.

We can choose $R > 2$ with the properties listed below:

1. R is $\geq$ the largest absolute value of the roots of the real polynomial equation $\frac{5}{4}x^d - |p|(x) - (|a|+2)x = 0$. Indeed, this condition implies that for any z with $|z| > R$, $\frac{3}{4}|z|^d \leq |p(z)| - (|a|+2)|z| \leq |p(z)| \leq |p(z)| + (|a|+2)|z| \leq \frac{5}{4}|z|^d$.

2. If $d' > 1$, R is $\geq$ the largest absolute value of the roots of the real polynomial equation $\frac{3}{2}|a_{d'}| x^{d'} - |q|(x) - |a| x = 0$. Indeed, this condition implies that $\frac{1}{2}|a_{d'}||z|^{d'} \leq \left|p(z) - z^d\right| - |az| \leq \left|p(z) - z^d\right| \leq \left|p(z) - z^d\right| + |az| \leq \frac{3}{2}|a_{d'}||z|^{d'}$. If $d' \leq 1$, we disregard this condition.

3. R is $\geq$ the largest absolute value of the roots of the real polynomial equation $\frac{5}{4}dx^{d-1} - |p'|(x) - 1 = 0$. Indeed, this condition implies that for any z with $|z| > R$, $\frac{3}{4}d|z|^{d-1} \leq |p'(z)| - 1 \leq |p'(z)| \leq |p'(z)| + 1 \leq \frac{5}{4}d|z|^{d-1}$.

4. For z with $|z| > R$, $\left|\left|\frac{p'(z)z}{p(z)}\right| - d\right| \leq \frac{1}{d}$.

5. $\frac{3(|a_{d'}|+|a|)}{2R} < \frac{1}{4}$.

6. For z with $|z| > R$, $\left|\dfrac{2a}{p'(z)}\right| \leq \dfrac{2|a|}{\frac{3}{4}d|z|^{d-1}} \leq \dfrac{8|a|}{3dR} \leq \dfrac{1}{2\sqrt{R}}$.

7. $\left|\dfrac{9}{320a}\right| R > 1$.

2.2 Sets of Escaping Points and Non-escaping Points

In this section, we introduce the concepts that we use to describe the dynamics of f and their relationships. First, we define the sets $K^{\pm} \subset \mathbb{C}^2$ of points of a bounded forward and backward orbit under f, respectively.

$$\begin{cases} K^+ := \{(z,w) \in \mathbb{C}^2 : \|f^n(z,w)\|_{\mathbb{C}^2} < C \text{ for some } C > 0 \text{ for all } n \in \mathbb{N}\} \\ K^- := \{(z,w) \in \mathbb{C}^2 : \|f^{-n}(z,w)\|_{\mathbb{C}^2} < C \text{ for some } C > 0 \text{ for all } n \in \mathbb{N}\}. \end{cases}$$

We define the set $U^{\pm}$ of points of unbounded forward orbit and backward orbit as $U^{\pm} = \mathbb{C}^2 \setminus K^{\pm}$, respectively.

In order to better study the behaviors of $K^{\pm}, U^{\pm}$ and f near line at infinity and to use the compactness property of $\mathbb{P}^2$, we projectivise the space $\mathbb{C}^2$ to $\mathbb{P}^2$. Recall the definitions of F, F^{-1}, I_+, and I_-.

Proposition II.1 (See [29]). *$K^{\pm}$, $U^{\pm}$, $I_{\pm}$, and F satisfy the following properties:*

1. *I_- and I_+ are the super-attracting fixed points of F and F^{-1}, respectively.*
2. *Any compact subset K of $U^{\pm}$ uniformly converges to $I_{\mp}$, respectively.*
3. *$F(\{t = 0\} \setminus I_+) = I_-$ and $F^{-1}(\{t = 0\} \setminus I_-) = I_+$.*
4. *$\overline{K^+} = K^+ \cup I_+$ and $\overline{K^-} = K^- \cup I_-$.*

For the proof of the first and second statements, see Proposition 2.2.10 in [29], for the third one, see Proposition 2.5.3 in [29], and for the last one, see Corollary 2.5.6 in [29].

Recall our choice of R. We define a filtration:

$$V^+ := \left\{(z, w) \in \mathbb{C}^2 : |z| \geq |w|, |z| \geq R\right\}$$

$$V^- := \left\{(z, w) \in \mathbb{C}^2 : |z| \leq |w|, |w| \geq R\right\}$$

$$W := \left\{(z, w) \in \mathbb{C}^2 : |z|, |w| < R\right\}.$$

The following are two properties related to this filtration that we are going to use later. For the proofs and details about this filtration, see [3], [24] and [29]. Here, we give short proofs.

Proposition II.2.

1. *$f(V^+) \subseteq V^+$ and $f^{-1}(V^-) \subseteq V^-$.*

2. *$U^+ = \cup_{i=0}^{\infty} f^{-i}(V^+)$ and $U^- = \cup_{i=0}^{\infty} f^{i}(V^-)$.*

Proof. In order to prove the first statement, it suffices to check $|z| \leq |p(z) - aw|$ for $(z, w) \in V^+$. It is clear by Condition 1 on R.

We consider the second statement. We let $U := \mathbb{C}^2 \setminus [\cup_{i=0}^{\infty} f^{-i}(V^+)]$. Then $U \subseteq W \cup V^-$ since $f(V^+) \subseteq V^+$. Let $U_1 = U \cap W$ and $U_2 = U \cap V^-$. Note that $f(W) \cap V^- = \emptyset$ from the definition of f. So, $f^i(U_1) \subseteq W$, for every $i = 0, 1, \ldots$. If $(z, w) \in U_2$, then the absolute value of the w coordinate decreases under f or $f(z, w) \in U_1$, which is easy to see from the definition of f. Since $|z| \leq |w|$ for $(z, w) \in U_2$, $\{\|f^i(z, w)\|_{\mathbb{C}^2}\}_{i=0}^{\infty}$ is bounded for $(z, w) \in U$. So, $U \subseteq K^+$. The other direction is obvious. So, $U^+ = \cup_{i=0}^{\infty} f^{-i}(V^+)$. The case of $U^- = \cup_{i=0}^{\infty} f^{i}(V^-)$ is similar. $\square$

2.3 Green Function

In this section, we define two Green functions G and g associated to F and f, respectively and look into their properties. Our focus lies on the relationship between G and g. Actually, it implies the behavior of g near the line at ∞.

With F viewed as a meromorphic mapping over $\mathbb{P}^2$, we can associate to F the Green function $G : \mathbb{C}^3 \to \mathbb{R} \cup \{-\infty\}$, which is a pluri-subharmonic function. For the details about pluri-subharmonic functions, see [27], or Appendix in [29]. Let $\tilde{F}$ be a lifting of F to $\mathbb{C}^3 \setminus \{0\}$ such that $\sup_{\|(z,w,t)\|_{\mathbb{C}^3}=1} \left\|\tilde{F}(z, w, t)\right\|_{\mathbb{C}^3} = 1$. In terms of the coordinates $(z,w,t) \in \mathbb{C}^3 \setminus \{0\}$, $\tilde{F}(z,w,t) = a_F(t^d p(\frac{z}{t}) - awt^{d-1}, zt^{d-1}, t^d)$ for a constant $a_F \neq 0$ satisfying $\sup_{\|(z,w,t)\|_{\mathbb{C}^3}=1} \left\|\tilde{F}(z,w,t)\right\|_{\mathbb{C}^3} = 1$. We can holomorphically extend $\tilde{F}$ over $\{0\}$. We replace $\tilde{F}$ by the extension. Then, G is defined by

$$G(z,w,t) = \lim_{n\to\infty} \frac{1}{d^n} \log \left\|\tilde{F}^n(z,w,t)\right\|_{\mathbb{C}^3}.$$

The existence of G is well-known. For example, see Theorem 1.6.1 in [29]. It is not hard to see from the definition that G satisfies

$$\begin{cases} G(\lambda z, \lambda w, \lambda t) = \log|\lambda| + G(z,w,t) \\ G(\tilde{F}(z,w,t)) = d \cdot G(z,w,t), \end{cases}$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ is a constant. By Theorem 1.6.5 in [29], G is continuous over $\mathbb{C}^3 \setminus \pi^{-1}(\{I_+\})$.

We turn our attention back to $\mathbb{C}^2$. We identify $\{(z,w,1)\} \in \mathbb{C}^3$ with $\mathbb{C}^2$. Thus, we can think of $G(z,w,1)$ as the restriction of G in $\mathbb{C}^2$. Since $\pi^{-1}(\{I_+\}) \cap (\mathbb{C}^2 \times \{1\}) = \emptyset$, $G(z,w,1)$ is well-defined as a continuous map over the entire $\mathbb{C}^2$.

We define $g(z,w) := G(z,w,1)$. We want to express g in terms of f. Before that, we need the following simple proposition.

Proposition II.3. *Let $\{a_n\}, \{b_n\} \subseteq \mathbb{R}$ be sequences such that $0 \leq b_n \leq a_n$ and let $\{c_n\} \subseteq \mathbb{R}$ be a sequence such that $\lim_{n\to\infty} c_n = \infty$. Assume that $\lim_{n\to\infty} a_n^{1/c_n}$ and $\lim_{n\to\infty}(a_n + b_n)^{1/c_n}$ exist. Then $\lim_{n\to\infty} a_n^{1/c_n} = \lim_{n\to\infty}(a_n + b_n)^{1/c_n}$*

Proof. The inequality $a_n \leq (a_n + b_n) \leq 2a_n$ implies the proposition. ∎

Then, from the definition of G in the above, we have

$$g(z,w) = G(z,w,1) = \lim_{n\to\infty} \frac{1}{d^n} \log \left\|\tilde{F}^n(z,w,1)\right\|_{\mathbb{C}^3}.$$

Next, due to Proposition II.3, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{d^n}\log\left\|\tilde{F}^n(z,w,1)\right\|_{\mathbb{C}^3} &= \lim_{n\to\infty}\frac{1}{d^n}\log\sqrt{\|f^n(z,w)\|^2_{\mathbb{C}^2}+1}\\ &= \lim_{n\to\infty}\frac{1}{d^n}\log^+\|f^n(z,w)\|_{\mathbb{C}^2}.\end{aligned}$$

In short, we can define the Green function g associated to f in $\mathbb{C}^2$ by

Definition II.4.

$$g(z,w) := G(z,w,1) = \lim_{n\to\infty}\frac{1}{d^n}\log^+\|f^n(z,w)\|_{\mathbb{C}^2},$$

where $\log^+ = \max\{0, \log\}$.

Then, g inherits the properties of G; g is continuous in the entire $\mathbb{C}^2$, g is pluri-subharmonic and g satisfies $g(f(z,w)) = d\cdot g(z,w)$. From the definition, it is clear that $g \geq 0$ on $\mathbb{C}^2$. We have $K^+ = \{g = 0\}$ (see Proposition 2.2.6 in [29]). The continuity of g implies that K^+ is closed in $\mathbb{C}^2$. In $U^+ = \mathbb{C}^2 \setminus K^+$, $g > 0$ and g is pluri-harmonic (see Proposition 2.2.10 in [29]).

We end this section with the following remark describing a useful relationship between g and G.

Remark II.5. Let $(\zeta,\omega,t) \in \mathbb{C}^3 \setminus \{0\}$. Then, for $t \neq 0$, by the homogeneity and regularity of $\tilde{F}$, we have

$$\begin{aligned}G(\zeta,\omega,t) &= \lim_{n\to\infty}\frac{1}{d^n}\log\left\|\tilde{F}^n(\zeta,\omega,t)\right\|_{\mathbb{C}^3}\\ &= \lim_{n\to\infty}\frac{1}{d^n}\Big(\log\left|t^{d^n}\right| + \log\left\|\tilde{F}^n(\frac{\zeta}{t},\frac{\omega}{t},1)\right\|_{\mathbb{C}^3}\Big)\\ &= \log|t| + G(\frac{\zeta}{t},\frac{\omega}{t},1) \qquad (2.1)\\ &= \log|t| + g(\frac{\zeta}{t},\frac{\omega}{t}) \qquad (2.2)\end{aligned}$$

CHAPTER III

The Behavior of the Level Sets at Infinity

3.1 The Closure of the Level Set $\mathscr{C}_c$

For the rest of this paper, we denote $\mathscr{C}_c := \{g = c\}$. For the study of $\mathscr{C}_c$ for $c > 0$, we first consider the following definition. Similarly to K^+, we define

Definition III.1. For $c > 0$, $K_c := \{g \leq c\}$.

The following lemma describes the behavior of the Green function g near I_-; g diverges to ∞ near I_-.

Lemma III.2. *For any $M > 0$, there exists a bidisc-shaped neighborhood $U_M \subseteq \mathbb{P}^2$ of I_- such that $g > M$ in $U_M \cap \mathbb{C}^2$. Indeed, $U_M = \{[1 : w : t] \in \mathbb{P}^2 : |w| < \epsilon_1, |t| < \epsilon_2\}$ for some sufficiently small $\epsilon_1, \epsilon_2 > 0$.*

Proof. Let $M > 0$ be arbitrarily given. Since the statement is local near I_-, we fix the affine coordinate chart centered at I_-, which is of the form $\{(1, w, t) : w, t \in \mathbb{C}\} = \{1\} \times \mathbb{C}^2 \subseteq \mathbb{C}^3 \setminus \{0\}$. We consider Equality 2.2 on this coordinate system. Recall that by Theorem 1.6.5 in [29], we know that G is continuous over $\mathbb{C}^3 \setminus \pi^{-1}(\{I_+\}) = \mathbb{C}^3 \setminus \{0\} \times \mathbb{C} \times \{0\}$. Since $(\{1\} \times \mathbb{C}^2) \cap (\{0\} \times \mathbb{C} \times \{0\}) = \emptyset$, G restricted to the coordinate system $\{1\} \times \mathbb{C}^2$ is continuous. Take a neighborhood $U \subseteq \{1\} \times \mathbb{C}^2$ of the origin $(1, 0, 0) \in \{1\} \times \mathbb{C}^2$ (which corresponds to I_-) with compact closure in the coordinate chart $\{(1,w,t) : w,t \in \mathbb{C}\}$. Then, G is bounded over U. Let m be the bound. We take ϵ_2 small enough so that $-\log|\epsilon_2| - m > M$. Take ϵ_1 small enough to satisfy $\{(1,w,t) : |w| < \epsilon_1 \text{ and } |t| < \epsilon_2\} \subseteq U$. Call this neighborhood $\tilde{U} = \{(1,w,t) : |w| < \epsilon_1 \text{ and } |t| < \epsilon_2\}$ and let $U_M = \{[1 : w : t] : |w| < \epsilon_1 \text{ and } |t| < \epsilon_2\}$ be the corresponding neighborhood of I_- to $\tilde{U}$ in $\mathbb{P}^2$. We have $\left|\log|t| + g(\frac{1}{t}, \frac{w}{t})\right| = |G(1,w,t)| < m$ over $U_M \cap \mathbb{C}^2$ from Equality 2.2. By the choice of $\epsilon_1, \epsilon_2 > 0$, we have $g(\frac{1}{t}, \frac{w}{t}) > -\log|t| - m > M$ over $U_M \cap \mathbb{C}^2$. So, the set $U_M \subseteq \mathbb{P}^2$ is the desired neighborhood of I_-. □

The following proposition implies that the level set $\mathscr{C}_c$ with $c > 0$ can only accumulate at I_+ near line at ∞.

Proposition III.3.

$$\overline{K_c} = K_c \cup I_+$$

Proof. Since $K^+ \subseteq K_c$, I_+ is in the closure of K_c by Proposition II.1. It suffices to show that for a sequence $\{x_n\} \subseteq K_c$ with $\|x_n\|_{\mathbb{C}^2} \to \infty$ as $n \to \infty$, $x_n \to I_+$ in $\mathbb{P}^2$. Since f is an automorphism over $\mathbb{C}^2$, we can consider the inverse images $\{f^{-1}(x_n)\}$ of $\{x_n\}$ under f. Consider the sequence $\{f^{-1}(x_n)\}$ in $\mathbb{P}^2$. Then, since $\mathbb{P}^2$ is compact, there exists a convergent subsequence in $\mathbb{P}^2$. By replacing by the convergent subsequence, we may assume that $f^{-1}(x_n) \to L$ for some $L \in \mathbb{P}^2$. $F|_{\mathbb{C}^2} = f$ in $\mathbb{C}^2$, so, if $L \in \mathbb{C}^2$, then $x_n \to f(L) \in \mathbb{C}^2$ and this contradicts $\|x_n\|_{\mathbb{C}^2} \to \infty$. Therefore, $L \in \{t = 0\}$. From Proposition II.1, we have $F(\{t = 0\} \setminus I_+) = I_-$. Thus, we conclude that $\{x_n\}$ should converge either to I_+ or I_-. However, by Lemma III.2, g is unbounded near I_-, so I_- cannot be a limit point of K_c. This proves the statement. □

3.2 Non-existence of Holomorphic Curves through I_+ in the Closure of $\mathscr{C}_c$

In this section, we prove Theorem I.1. We paraphrase the statement as follows:

Theorem III.4. *Let $\varphi : \Delta \to \mathbb{P}^2$ be a holomorphic mapping such that $\varphi(\Delta) \subseteq \overline{K_c}$ and $\varphi(0) = I_+$. Then φ is a constant map.*

Proof. We prove this proposition by contradiction. Suppose that there exists such a non-constant holomorphic mapping $\varphi : \Delta \to \mathbb{P}^2$ as in the statement. Then, $\varphi(\Delta) \subseteq \overline{K_c}$. We parametrize the mapping φ by $\varphi(\theta) = [z(\theta) : 1 : t(\theta)]$ where z, t are holomorphic functions of θ with $z(0) = t(0) = 0$. We will handle the cases where either of $z(\theta)$ or $t(\theta)$ is identically zero separately. So, we first assume that none of $z(\theta)$ and $t(\theta)$ are identically zero. Since z, t are non-zero holomorphic functions, by taking a sufficiently small neighborhood $V \subseteq \Delta$ of 0, we may assume that z, t have 0 only at $\theta = 0$. Since z, t are holomorphic functions of θ in a small neighborhood $V \subseteq \Delta$ of 0, we can write

$$z(\theta) = \theta^\alpha P(\theta)$$

$$t(\theta) = \theta^\beta Q(\theta)$$

where $\alpha, \beta \geq 1$, and P, Q are holomorphic functions of θ in V and $P(\theta), Q(\theta) \neq 0$ in V.

We denote by F the extension of f in $\mathbb{P}^2$. For $\theta \in V \setminus \{0\}$, $z(\theta), t(\theta)$ do not vanish, so the forward image of $[z(\theta) : 1 : t(\theta)]$ under F is well defined and is equal to $[t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1} : z(\theta)t(\theta)^{d-1} : t(\theta)^d]$. Since $t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}$ is a holomorphic function of θ, by the identity theorem in one complex variable, there are two cases:

I) $t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}$ has discrete zeros in V or

II) it is identically zero.

Recall the open neighborhood U_M of I_- in Lemma III.2. For the proof, we take $M = 2dc$ and denote the corresponding U_M by U_{2dc}. The same applies to U_{2d^2c}. We use the notations $\epsilon_1, \epsilon_2 > 0$ for the two positive numbers associated to U_{2dc} in Lemma III.2. We will keep the notations ϵ_1, ϵ_2 for U_{2d^2c} if there is no confusion.

I) By shrinking V if necessary, we may assume that $t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1} \neq 0$ for all $\theta \in V \setminus \{0\}$. We consider the forward image $[t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1} : z(\theta)t(\theta)^{d-1} : t(\theta)^d]$ of $[z : 1 : t]$ under f with respect to the affine coordinate chart of the form $(1, w, t) \in \{1\} \times \mathbb{C}^2 \subseteq \mathbb{C}^3 \setminus \{0\}$. Then, the coordinate of the forward image for $\theta \in V \setminus \{0\}$ is $(1, \frac{z(\theta)t(\theta)^{d-1}}{t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}}, \frac{t(\theta)^d}{t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}})$. Notice that $t(\theta)^d \neq 0$ for $\theta \in V \setminus \{0\}$ and therefore, the forward image under f is well-defined. Indeed, it should be well-defined since F restricted to $\mathbb{C}^2$ is f.

We want to find a sufficiently small non-zero θ such that $f(\varphi(\theta))$ lies inside U_{2dc} in this coordinate system, which implies $dg(\frac{z(\theta)}{t(\theta)}, \frac{1}{t(\theta)}) = g(f(\frac{z(\theta)}{t(\theta)}, \frac{1}{t(\theta)})) > 2dc$. This is a contradiction to $\varphi(\Delta) \subseteq \overline{K_c}$, which therefore proves the statement. If this strategy fails, then our plan B is to find θ such that $f^2(\varphi(\theta)) \in U_{2d^2c}$ for the same reasoning. In order to find such θ, we compute and compare the vanishing order of $t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}$, $z(\theta)t(\theta)^{d-1}$, and $t(\theta)^d$ at $\theta = 0$ in $\frac{z(\theta)t(\theta)^{d-1}}{t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}}$ and $\frac{t(\theta)^d}{t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}}$. Keep in mind that $a \neq 0$.

In this comparison, we have two cases: i) $\alpha \geq \beta$ or ii) $\alpha < \beta$.

i) $\alpha \geq \beta$. Let $\alpha = \beta + \gamma$ with $\gamma \geq 0$. We consider $\frac{z(\theta)t(\theta)^{d-1}}{t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}}$ first. The vanishing order of the numerator is simply $\alpha + \beta(d-1) = \beta d + \gamma$. The denominator becomes $t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1} = [\sum_{i=0}^{d} a_i P(\theta)^i Q(\theta)^{d-i} \theta^{\alpha i + \beta(d-i)}] - aQ(\theta)^{d-1}\theta^{\beta(d-1)}$. Since for every i, $\alpha i + \beta(d-i) = d\beta + \gamma i > \beta(d-1)$, the vanishing order of the denominator is $\beta(d-1)$. Comparing the vanishing orders, we have $\beta d + \gamma > \beta(d-1)$ and therefore, we can find a small positive real number δ_1 such that for every θ with $|\theta| < \delta_1$, $\left|\frac{z(\theta)t(\theta)^{d-1}}{t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}}\right| < \epsilon_1$ is true. In the same way, we can find another small positive real number δ_2 for $\left|\frac{t(\theta)^d}{t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}}\right| < \epsilon_2$. Then, we take θ with $|\theta| < \min\{\delta_1, \delta_2\}$.

ii) $\alpha < \beta$. Let $\beta = \alpha + \gamma$ with $\gamma > 0$. We have two subcases:

a) the case where there is no cancellation of the lowest term of $t(\theta)^d p(\frac{z(\theta)}{t(\theta)})$ by $at(\theta)^{d-1}$ in $t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}$, and

b) the case where there is cancellation of the lowest term of $t(\theta)^d p(\frac{z(\theta)}{t(\theta)})$ by $at(\theta)^{d-1}$ in $t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}$.

a) We do $\frac{z(\theta)t(\theta)^{d-1}}{t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}}$ first. Using the same argument for the denominator as in i), we obtain the vanishing order of the denominator $= \min\{\alpha d, \beta(d-1)\}$ since there is no cancellation. We compare it to $\beta d - \gamma$, the vanishing order of the numerator at $\theta = 0$. Since $\alpha d < \alpha d + \gamma(d-1) = \beta d - \gamma$ and $\beta(d-1) < \beta(d-1) + \alpha = \beta d - \gamma$, we can find a small positive real number δ_1 such that for every θ with $|\theta| < \delta_1$, $\left|\frac{z(\theta)t(\theta)^{d-1}}{t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}}\right| < \epsilon_1$ is true as in i). Again, in the same way, we can find another small positive real number δ_2 for $\left|\frac{t(\theta)^d}{t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}}\right| < \epsilon_2$. Then, we take θ with $|\theta| < \min\{\delta_1, \delta_2\}$.

b) In this case, we have $\alpha d = \beta(d-1)$.

1. *The case where the vanishing order of* $t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}$ *is at most* $\alpha d + \gamma(d-2)$. We do $\frac{z(\theta)t(\theta)^{d-1}}{t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}}$ first. Since $\alpha d + \gamma(d-2) < \alpha d + \gamma(d-1) = \beta d - \gamma$, we can pick a small positive real number δ_1 such that for every θ with $|\theta| < \delta_1$, $\left|\frac{z(\theta)t(\theta)^{d-1}}{t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}}\right| < \epsilon_1$ is true as in i). For $\frac{t(\theta)^d}{t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}}$, since $\alpha d + \gamma(d-2) < \alpha d + \gamma d = \beta d$, we can also choose another small positive real number δ_2 such that for every θ with $|\theta| < \delta_2$, $\left|\frac{t(\theta)^d}{t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}}\right| < \epsilon_2$ is true as in i). Then, we take θ with $|\theta| < \min\{\delta_1, \delta_2\}$.

2. *The case where the vanishing order of* $t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}$ *is* $\alpha d + \gamma(d-1)$. For $\theta \in V \setminus \{0\}$, f sends $[z : 1 : t]$ to $[\theta^{\beta d - \gamma} A_1(\theta) : \theta^{\beta d - \gamma} A_2(\theta) : \theta^{\beta d} A_3(\theta)] = [A_1(\theta) : A_2(\theta) : \theta^\gamma A_3(\theta)]$, where $A_1(\theta)$, $A_2(\theta)$, and $A_3(\theta)$ are holomorphic functions and do not vanish in a neighborhood of $\theta = 0$. By shrinking V if necessary, we may assume that A_1, A_2, and A_3 are bounded away from 0 in V. Therefore, the point should be in the usual $\mathbb{C}^2$ for $\theta \in V \setminus \{0\}$, and we can take one more iteration by f. Then, f sends $[A_1(\theta) : A_2(\theta) : \theta^\gamma A_3(\theta)] \to [(\theta^\gamma A_3(\theta))^d p(\frac{A_1(\theta)}{\theta^\gamma A_3(\theta)}) - aA_2(\theta)(\theta^\gamma A_3(\theta))^{d-1} : A_1(\theta)(\theta^\gamma A_3(\theta))^{d-1} : (\theta^\gamma A_3(\theta))^d]$. Observe that the first component has $A_1(\theta)^d$ as its leading term. Since $A_1(\theta)$ is bounded away from 0 in V, the first component is bounded away from 0 while the second and the third component can be shrunken as small as we want by letting $\theta \to 0$. Thus, we can find a θ such that $f^2([z(\theta) : 1 : t(\theta)]) \in U_{2d^2c}$. This is a contradiction to $\varphi(\Delta) \subseteq \overline{K_c}$ for $d^2 g(\frac{z(\theta)}{t(\theta)}, \frac{1}{t(\theta)}) = g(f^2(\frac{z(\theta)}{t(\theta)}, \frac{1}{t(\theta)})) > 2d^2c$.

3. *The case where the vanishing order of* $t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}$ *is* $\alpha d + \gamma d = \beta d$.

   In this case, $a_0 \neq 0$ since $t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}$ is not identically 0. This case is almost the same as the previous case. The only difference is that f sends $[z : 1 : t] \to [a_0 t(\theta)^d : z(\theta)t(\theta)^{d-1} : t(\theta)^d] = [a_0 t(\theta) : z(\theta) : t(\theta)] \to [t(\theta)^d p(a_0) - az(\theta)t(\theta)^{d-1} : a_0 t(\theta)^d : t(\theta)^d]$. Observe that $a \neq 0$.

II) $t(\theta)^d p(\frac{z(\theta)}{t(\theta)}) - at(\theta)^{d-1}$ is identically 0. Then, for all $\theta \in V \setminus \{0\}$, $p(\frac{z(\theta)}{t(\theta)}) = \frac{a}{t(\theta)}$. Notice that since $a \neq 0$, $\left|\frac{a}{t(\theta)}\right| \to \infty$ as $\theta \to 0$. Since $p(x)$ is a polynomial, the only way to make $\left|p(\frac{z(\theta)}{t(\theta)})\right| \to \infty$ is that $t(\theta)$ has a higher vanishing order than $z(\theta)$ does at $\theta = 0$. Since $t(\theta)^d$ is not 0 in $V \setminus \{0\}$, the forward image lies in the usual $\mathbb{C}^2$. We apply f one more time as previously. Then f sends $f([z : 1 : t]) = [0 : z(\theta) : t(\theta)] \to f^2([z : 1 : t]) = [a_0 t(\theta)^d - az(\theta)t(\theta)^{d-1} : 0 : t(\theta)^d]$ with a_0 is the constant term of $p(x)$. Observe that $a \neq 0$. Considering the vanishing order of $z(\theta), t(\theta)$, the vanishing order of $t(\theta)^d$ is greater than $a_0 t(\theta)^d - az(\theta)t(\theta)^{d-1}$. So, we can find a θ such that $f^2([z(\theta) : 1 : t(\theta)]) \in U_{2d^2c}$ and use the same argument used as in b).

Now, we consider the case where either of $z(\theta), t(\theta)$ is identically 0. Since φ is not a constant mapping, exactly one of them should be identically 0. If $t(\theta)$ is identically 0, then since the only intersection between $\overline{K_c}$ and $\{t = 0\}$ is I_+, the mapping φ must be a constant mapping $\varphi(\theta) = I_+$, which is a contradiction. If $z(\theta)$ is identically 0, we have $[0 : 1 : t(\theta)] \to [a_0 t(\theta)^d - at(\theta)^{d-1} : 0 : t(\theta)^d]$. Use the same argument used in II) to draw the conclusion.

These cases prove the statement. $\square$

CHAPTER IV

Family of Parametrizations

In this chapter, we introduce a useful family of parametrizations.

4.1 Coordinate Functions

In this section, we will find a local coordinate system, on which g is represented in a simple way. Indeed, for our f, the z-coordinate determines the function g. So, we will find a coordinate chart where the action of f on the z-coordinate is simple. We follow Hubbard and Oberste-Vorth's work in [24].

Proposition IV.1 (See Proposition 5.2 in [24]). *There exist analytic functions $\varphi_\pm : V_\pm \to \mathbb{C} \setminus \overline{\Delta}$ such that*

$$\varphi_+(f(z,w)) = (\varphi_+(z,w))^d \text{ and } \varphi_-(f^{-1}(z,w)) = (\varphi_-(z,w))^d,$$
$$\lim_{\|(z,w)\|\to\infty} \left|\frac{\varphi_+(z,w)}{z}\right| = 1 \text{ in } V^+ \text{ and } \lim_{\|(z,w)\|\to\infty} \left|\frac{\varphi_-(z,w)}{Aw}\right| = 1 \text{ in } V^-,$$

where A is a non-zero constant only depending on a in f.

Proof. We only prove the statements for f and φ_+. Those for f^{-1} and φ_- are analogous. $\lim_{n\to\infty} z_n^{1/d^n}$ may be the first natural candidate for $\varphi_+(z,w)$, but it has a well-definedness problem. Alternatively, we define φ_+ to be the following telescoping infinite product, which is similar to $\lim_{n\to\infty} z_n^{1/d^n}$:

$$\varphi_+(z,w) = z \cdot \big(\frac{z_1}{z^d}\big)^{1/d} \cdot \dots \cdot \big(\frac{z_{n+1}}{z_n^d}\big)^{1/d^{n+1}} \cdot \dots$$

We show the existence and the analyticity of the limit function. Firstly, we check the definition of d^{n+1}-st root in each factor $\big(\frac{z_{n+1}}{z_n^d}\big)^{1/d^{n+1}}$ of the infinite product. We have

$$\left(\frac{z_{n+1}}{z_n^d}\right)^{1/d^{n+1}} = \left(\frac{p(z_n) - aw_n}{z_n^d}\right)^{1/d^{n+1}} = \left(1 + \frac{\sum_{i=0}^{d'} a_i z_n^i - aw_n}{z_n^d}\right)^{1/d^{n+1}}.$$

By Proposition II.2, we have for $(z_n, w_n) \in V^+$, $(z_{n+1}, w_{n+1}) \in V^+$. By the triangle inequality, Condition 1 on R, and the fact that $|z_n| \geq |w_n|$, we have $\left|\frac{\sum_{i=0}^{d'} a_i z_n^i - aw_n}{z_n^d}\right| \leq \frac{1}{4}$ for all $z \in V^+$. So, the angle of $\frac{z_{n+1}}{z_n^d} \in (-\arctan\frac{1}{4}, \arctan\frac{1}{4}) \subseteq (-\pi,\pi)$ for all $n > 0$ and all $(z,w) \in V^+$. Thus, we can take the principle branch of the d^{n+1}-st root in each factor so that each factor is well-defined and holomorphic in V^+.

Now we will check the convergence of the infinite product $z \cdot \big(\frac{z_1}{z^d}\big)^{1/d} \cdot \dots \cdot \big(\frac{z_{n+1}}{z_n^d}\big)^{1/d^{n+1}} \cdot \dots$ of holomorphic functions $\left\{\left(\frac{z_{n+1}}{z_n^d}\right)^{1/d^{n+1}}\right\}$ of V^+. We rely on the following theorem from the function theory of one complex variable.

Theorem IV.2 (See Theorem 8.1.9 in [18]). *Let $U \subseteq \mathbb{C}$ be open. Suppose $h_j : U \to \mathbb{C}$ are holomorphic and that $\sum_{j=1}^{\infty} |h_j|$ converges uniformly on compact sets. Then the sequence of partial products*

$$H_N(z) = \prod_{j=1}^{N} (1 + h_j(z))$$

converges uniformly on compact sets. In particular, the limit of these partial products defines a holomorphic function H on U.

We estimate $\left|\frac{z_{n+1}}{z_n^d}\right|^{1/d^{n+1}}$. From the argument about the validity of the d^{n+1}-st root, we have

$$\begin{aligned}
(4.1)\qquad \left|1+\frac{1}{4}\right|^{1/d^{n+1}} &\geq \left|1+\left|\frac{\sum_{i=0}^{d'} a_i z_n^i - a w_n}{z_n^d}\right|\right|^{1/d^{n+1}} \geq \left|\frac{z_{n+1}}{z_n^d}\right|^{1/d^{n+1}} \\
&\geq \left|1-\left|\frac{\sum_{i=0}^{d'} a_i z_n^i - a w_n}{z_n^d}\right|\right|^{1/d^{n+1}} \geq \left|1-\frac{1}{4}\right|^{1/d^{n+1}}.
\end{aligned}$$

We use this estimate to show the uniform convergence of the following series.

$$\sum_{i=0}^{\infty}\left|\left|\frac{z_{n+1}}{z_n^d}\right|^{1/d^{n+1}} - 1\right|.$$

We compare each term in the sum to that of a geometric series.

$$\left|\left|\frac{z_{n+1}}{z_n^d}\right|^{1/d^{n+1}} - 1\right| \leq \frac{\left|\left|\frac{z_{n+1}}{z_n^d}\right| - 1\right|}{\sum_{i=0}^{d^{n+1}-1}\left|\frac{z_{n+1}}{z_n^d}\right|^{i/d^{n+1}}} \leq \frac{\frac{1}{4}}{d^{n+1}\cdot\frac{3}{4}} = \frac{1}{3d^{n+1}}.$$

The second inequality comes from Inequality 4.1. This inequality proves that the series converges uniformly. Theorem IV.2 proves the existence and the analyticity of φ_+. If we apply the same argument to the reciprocal of the infinite product, we can prove that it converges to a non-zero constant.

The property $\varphi_+(f(z,w)) = (\varphi_+(z,w))^d$ is clear from the definition.

We prove the last property. Assume that $d' \geq 2$. From our choice of R, in a similar way to Inequality 4.1, we have

$$\begin{aligned}
&\left|1+\left|\frac{3a_{d'}}{2R^{d-d'}}\right|\right|^{1/d^{n+1}} \geq \left|1+\frac{\left|a_{d'}z^{d'}\right| + \left|\sum_{i=0}^{d'-1} a_i z_n^i - a w_n\right|}{\left|z_n^d\right|}\right|^{1/d^{n+1}} \\
&\geq \left|\frac{z_{n+1}}{z_n^d}\right|^{1/d^{n+1}} \\
&\geq \left|1-\frac{\left|a_{d'}z^{d'}\right| + \left|\sum_{i=0}^{d'-1} a_i z_n^i - a w_n\right|}{\left|z_n^d\right|}\right|^{1/d^{n+1}} \geq \left|1-\left|\frac{3a_{d'}}{2R^{d-d'}}\right|\right|^{1/d^{n+1}}.
\end{aligned}$$

So, we can estimate the difference between $\varphi_+(z,w)$ and z for $|z| > R$:

$$\left(1 - \left|\frac{3a_{d'}}{2 \cdot R^{d-d'}}\right|\right)^{1/(d-1)} \leq \left|\frac{\varphi_+(z,w)}{z}\right| \leq \left(1 + \left|\frac{3a_{d'}}{2 \cdot R^{d-d'}}\right|\right)^{1/(d-1)}. \tag{4.2}$$

We have just proved the convergence of φ_+ to z in magnitude. We need to show the angle of $\frac{\varphi_+(z,w)}{z}$ converges to 0 as $\|(z,w)\| \to \infty$.

The angle of $\left(\frac{z_{n+1}}{z_n^d}\right)^{1/d^{n+1}}$ is between $-\frac{\sin^{-1}(\left|\frac{3a_{d'}}{2R^{d-d'}}\right|)}{d^{n+1}}$ and $\frac{\sin^{-1}(\left|\frac{3a_{d'}}{2R^{d-d'}}\right|)}{d^{n+1}}$, where we take the branch of $\sin^{-1}$ to be $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Since

$$\sum_{i=0}^{\infty} \frac{1}{d^{n+1}} \sin^{-1}(\left|\frac{3a_{d'}}{2R^{d-d'}}\right|) = \frac{1}{d-1} \sin^{-1}(\left|\frac{3a_{d'}}{2R^{d-d'}}\right|) \sim \frac{1}{d-1}\left|\frac{3a_{d'}}{2R^{d-d'}}\right|,$$

the angle of $\frac{\varphi_+(z,w)}{z}$ is between $-\frac{2}{d-1}\left|\frac{3a_{d'}}{2R^{d-d'}}\right|$ and $\frac{2}{d-1}\left|\frac{3a_{d'}}{2R^{d-d'}}\right|$. Thus, as $R \to \infty$, this angle also shrinks to 0.

These two approximations prove the last property about the asymptotic behavior of φ_+. Note that from Inequality 4.2, we know that $\varphi_+(V^+) \subseteq \mathbb{C} \setminus \overline{\Delta}$.

In case of $d' \leq 1$, we can apply the same argument with a better bound. □

Remark IV.3. Additionally, we can estimate the error between $\varphi_+(z,w)$ and z from the magnitude and angle estimates in the proof.

We follow Hubbard and Oberste-Vorth's idea to find a useful local coordinate chart near I_- using the function φ_+. This appears in the proof of Proposition 6.2 in [24]. Because this is a crucial step in our work, we provide proofs in detail.

We consider two complex numbers z, y. Let $|y| \leq 1$. Define $\varphi_y(z) := \varphi_+(z, yz)$. Then, from the previous proposition, φ_y is well-defined and analytic for $|z| > R$. From the last property of Proposition IV.1, we know that φ_y has a simple pole at infinity. In order to get the coordinate function, we need the following lemma.

Lemma IV.4. *Let $r_1, r_2 > 0$. Let $\mathscr{F}$ = the space of analytic functions $f : \Delta_{r_1} \to \Delta_{r_2}$ such that $f(0) = 0$ and $f'(0) = 1$. Then $\mathscr{F}$ is compact with respect to the compact-open topology. In particular, there exist $r_3, r_4 > 0$ independent of $f \in \mathscr{F}$ such that every $f \in \mathscr{F}$ is injective in Δ_{r_3} and $\Delta_{r_4} \subseteq f(\Delta_{r_3})$.*

Proof. Montel's theorem and the analyticity of holomorphic functions prove the first part. We prove the second part. By replacing the domain of the definition by a smaller $0 < r_1' < r_1$ if necessary, we may assume that all $f \in \mathscr{F}$ are holomorphic on $\overline{\Delta}_{r_1}$. Let $f \in \mathscr{F}$. Since $f(0) = 0$, $f'(0) = 1$, and f is holomorphic near $\overline{D}$, we have $f(z) = z + z^2 Q_f(z)$ with $Q_f(z)$ holomorphic near $\overline{\Delta}_{r_1}$. Let $M_f := \max_{z \in \overline{\Delta}_{r_1}} \left\{ |Q_f(z)|, \left|Q_f'(z)\right| \right\}$. Take $r_f < 1$ small enough so that $3 r_f M_f < 1$. Then, for z_1, z_2 with $|z_1|, |z_2| < r_f$,

$$
\begin{aligned}
0 &= |f(z_1) - f(z_2)| = \left| z_1 - z_2 + z_1^2 Q_f(z_1) - z_2^2 Q_f(z_2) \right| \\
&\geq |z_1 - z_2| - \left| z_1^2 Q_j(z_1) - z_1^2 Q_j(z_2) \right| - \left| z_1^2 - z_2^2 \right| |Q_j(z_2)| \\
&\geq |z_1 - z_2| (1 - r_f^2 M_f - 2 r_f M_f) \leq (1 - 3 r_f M_f) |z_1 - z_2|
\end{aligned}
$$

By our choice of r_f, we have the injectivity of f over Δ_{r_f}.

We find a universal upper bound for M_f's to prove the second statement. We use the compactness property as follows:

$$
\sup_{f \in \mathscr{F}} \max_{z \in \overline{\Delta}_{r_1}} |Q_f| = \sup_{f \in \mathscr{F}} \max_{z \in \partial \Delta_{r_1}} \left| \frac{f - z}{z^2} \right| \leq \sup_{f \in \mathscr{F}} \max_{z \in \partial \Delta_{r_1}} \frac{|f| + |z|}{|z^2|}.
$$

The first equality comes from the maximum principle. The last term is finite since $\mathscr{F}$ is compact. Let $m_{\mathscr{F}}$ denote the last bound $\sup_{f \in \mathscr{F}} \max_{z \in \partial \Delta_{r_1}} \frac{|f|+|z|}{|z^2|}$.

Concerning $\sup_{f\in\mathscr{F}} \max_{z\in\overline{D}} |Q'|$, we consider the following:

$$\begin{aligned}\sup_{f\in\mathscr{F}} \max_{z\in\overline{\Delta}_{\frac{1}{3}r_1}} \left|Q'_f\right| &= \sup_{f\in\mathscr{F}} \max_{z\in\overline{\Delta}_{\frac{1}{3}r_1}} \left|\frac{1}{2\pi}\int_{\zeta\in\partial\Delta_{r_1}} \frac{f(\zeta)}{(\zeta-z)^2}d\zeta\right| \\ &\leq \frac{m_f}{2\pi} \max_{z\in\overline{\Delta}_{\frac{1}{3}r_1}} \int_{\zeta\in\partial\Delta_{r_1}} \frac{1}{|\zeta-z|^2}d|\zeta| \leq \frac{9m_{\mathscr{F}}}{4r_1}\end{aligned}$$

Thus, we take the universal bound M_f to be $\max\left\{m_{\mathscr{F}}, \frac{9m_{\mathscr{F}}}{4r_1}\right\}$. Let $r_{\mathscr{F}} > 0$ be such that $3r_{\mathscr{F}}M_{\mathscr{F}} < 1$. Then $r_3 = \min\left\{r_{\mathscr{F}}, \frac{1}{3}r_1\right\}$. This proves the universal injective radius statement. For the remaining part, simply apply Koebe's $\frac{1}{4}$ theorem. $\square$

We consider a function $h_y(Z) := \frac{1}{\varphi_+(1/Z, y/Z)}$ of Z with $|y| \leq 1$ and $0 < |Z| < 1/R$. Then h_y has a bounded singularity at $Z = 0$. By the Riemann removable singularity theorem, h_y extends to a map of $\Delta_{1/R}$. We redefine h_y to be the extension. Let $\mathscr{H} := \{h_y : |y| \leq 1\}$. Obviously, we have $h_y(0) = 0$ for all $h_y \in \mathscr{H}$. From Inequality 4.2, $h'_y(0) = 1$ for all $h_y \in \mathscr{H}$. Thus $\mathscr{H} \subseteq \mathscr{F}$ with $r_1 = \frac{1}{R}, r_2 = 1$. By Lemma IV.4, if $|x| \geq \frac{1}{r_4}$ and $|y| \leq 1$, then there exists a unique (z, w) with $|z| \geq \frac{1}{r_3}$ and $|w| \leq |z|$ such that

$$\begin{cases} \frac{1}{x} = \frac{1}{\varphi_+(z,w)} \\ y = \frac{w}{z} \end{cases}$$

From Inequality 4.2, if z is sufficiently large, then the corresponding x satisfies $|x| \geq \frac{1}{r_4}$. Let $R_\phi > 0$ such that $R_\phi > \frac{2}{r_4}$, $R_\phi > \frac{1}{r_3}$, and $R_\phi > R$. We can find a biholomorphism Ψ_+ of $D_{\Psi_+} := \{|z| \geq R_\phi, |w| \leq |z|\} \subseteq \mathbb{C}^2$ mapping $(z, w) \in D_{\Psi_+}$ to the corresponding $(x, y) \in \mathbb{C}^2$. In other words, $\{|z| \geq R_\phi, |w| \leq |z|\}$ has a nice coordinate chart for the action of f. For notational convenience, we will use the names, the ZW-coordinate chart and XY-coordinate chart, for (z, w) and (x, y) respectively. Notice that $\{|z| \geq R_\phi, |w| \leq |z|\} \subseteq \mathbb{C}^2$ is a neighborhood of I_-.

We compute our map f in the XY-coordinate system. To avoid a possible confusion, we use f_{XY} for the representation of f with respect to the XY-coordinate system. Let $(x,y) \in \Psi_+(D_{\Psi_+})$. Then

$$\begin{array}{ccc} (x,y) & \overset{\Psi_+^{-1}}{\to} & (z(x,y), yz(x,y)) \\ & & \downarrow f \\ (x^d, \frac{z(x,y)}{p(z(x,y))-ayz(x,y)}) & \overset{\Psi_+}{\leftarrow} & (p(z(x,y)) - ayz(x,y), z(x,y)) \end{array}$$

Thus, f_{XY} sends $(x,y) \in \Psi_+(D_{\Psi_+})$ to $(x^d, \frac{z(x,y)}{p(z(x,y))-ayz(x,y)})$.

We show that the XY coordinate chart is stable under the action of f. We want to check $f(D_{\Psi_+}) \subseteq D_{\Psi_+}$. By Condition 1 on R, we have $R_\phi \leq |z| \leq f_1(z,w)$ for $(z,w) \in D_{\Psi_+} \subseteq V^+$. This proves the z-coordinate part and the invariance of V^+ for f proves the w-coordinate part.

We also compute the Green function g in the XY-coordinate system. Let g_{XY} denote the representation of the Green function g with respect to the XY-coordinate chart, that is, $g_{XY}(x,y) = g(z,w)$. Since $(z,w) \in V^+$, we have

$$\begin{aligned} g_{XY}(x,y) = g(z,w) &= \lim_{n\to\infty} \frac{1}{d^n} \log^+ \sqrt{|z_n|^2 + |w_n|^2} = \lim_{n\to\infty} \frac{1}{d^n} \log \sqrt{|z_n|^2 + |w_n|^2} \\ &= \lim_{n\to\infty} \frac{1}{d^n} (\log|z_n| + O(1)) = \lim_{n\to\infty} \frac{1}{d^n} \log|z_n| \end{aligned}$$

Thanks to the last property of Proposition IV.1 and $|z_n| \to \infty$ as $n \to \infty$, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{d^n} \log|z_n| &= \lim_{n\to\infty} \frac{1}{d^n} \log|\varphi_+(z_n,w_n)| = \lim_{n\to\infty} \frac{1}{d^n} \log|\varphi_+(z,w)|^{d^n} \\ &= \log|\varphi_+(z,w)| = \log|x|. \end{aligned}$$

Summarizing the above, we have that for $(z, w) \in D_{\Psi_+}$,

$$g_{XY}(x, y) = g(z, w) = \log |\varphi_+(z, w)| = \log |x| \,. \tag{4.3}$$

4.2 Choice of Another Large Number c in $\mathscr{C}_c$

In V^+ and V^-, f and f^{-1} show tame behavior, respectively. So, we want to avoid the set W during our study of $\mathscr{C}_c$. Also, we want to use the XY-coordinate chart and require some more conditions listed below. Indeed, for a sufficiently large $c > 0$, all properties listed below are satisfied. Note that since $g(f(z, w)) = d \cdot g(z, w)$, we have $f^n(\mathscr{C}_c) = \mathscr{C}_{d^n c}$, and since $c > 0$, we have $d^n c \to \infty$ as $n \to \infty$. Also, the properties that we will consider for $\mathscr{C}_c$ are analytic and they are preserved by biholomorphisms f and f^{-1} of $\mathbb{C}^2$. Thus, we want to find such a sufficiently large $c > 0$ with good computational properties and to first verify the properties to be proved for the such large $c > 0$. Next, we generalize the result.

For notational convenience we will use r such that $r := e^c > 1$ instead of c in the following list of conditions.

1. $\frac{r}{2} \geq [\max_{|w| \leq R} |p(w)|] + |a|\, R$. Indeed, this condition implies that $\mathscr{C}_c \cap W = \emptyset$. Explanation will be given in Remark IV.8.

2. $r > 2R_\phi$, where R_ϕ is in the previous section. This is for the XY-coordinate chart. See Proposition IV.1 (or Inequality 6.1).

3. $R < \left|\frac{r}{5}\right|^{\frac{1}{d}}$

4. $48 \cdot 9\, |a|^4 \leq \dfrac{r^2}{128 \cdot 15^2}$

5. $r > 2R$

From now on, we assume that $r = e^c$ satisfies the above conditions unless stated otherwise. If we are referring to general $c > 0$, we will make it clear by stating the quantifier "for all $c > 0$". When we are refering to this list of conditions, we will alternatively use "Condition on r", "Condition on c", or "Condition on s". Here, the number s is a complex number such that $|s| = r$ and will appear in Section 4.4.

4.3 Smooth Manifold $\mathscr{C}_c$ and Dense Foliation Structure

Recall the notation $r := e^c > 1$. We denote $U^+(r) := g^{-1}(\log r)$ and $V^+(r) := \{(z,w) \in V^+ : g(z,w) = \log r\}$. From the property that $U^+ = \cup_{i=0}^{\infty} f^{-i}(V^+)$ in Proposition II.2, we have

$$U^+(r) = V^+(r) \cup f^{-1}(V^+(r^d)) \cup f^{-2}(V^+(r^{d^2})) \cup \cdots . \tag{4.4}$$

Here, the union in Expression 4.4 is increasing. Indeed, from $f(V^+) \subseteq V^+$ and $g(f(z,w)) = d \cdot g(z,w)$, it is clear that $f(V^+(r)) \subseteq V^+(r^d)$ and that $V^+(r) \subseteq f^{-1}(f(V^+(r))) \subseteq f^{-1}(V^+(r^d))$.

Proposition IV.5 (See [24]). *For all $c > 0$, $\nabla g \neq 0$ over $\mathscr{C}_c$. In particular, $\mathscr{C}_c$ is a smooth manifold.*

Proof. We first consider sufficiently large c as in Section 4.2. For any $(z,w) \in \mathscr{C}_c$, take a small neighborhood $U_{(z,w)}$ of (z,w) with compact closure. Recall the coordinate change map Ψ_+ between the ZW- and XY- coordinate systems. Since every compact subset of U^+ converges uniformly to I_- under iteration of f, we have $f^N(U_{(z,w)}) \subseteq D_{\Psi_+}$ for sufficiently large $N \in \mathbb{N}$. The fibration of $\Psi_+(D_{\Psi_+})$ by g_{XY} in the XY-coordinate system is simply a trivial fibration. The coordinate change map Ψ_+ is a biholomorphism and f is a biholomorphic automorphism of $\mathbb{C}^2$. Therefore,

∇g does not vanish at $(z,w)\in\mathscr{C}_c$. Since (z,w) is arbitrary, the statement is proved. In particular, $\mathscr{C}_c$ is a smooth manifold.

We now consider the general case. This property is preserved by the biholomorphisms f and f^{-1}. Thus, the general case is obtained by considering the statement for $f^M(\mathscr{C}_c)$ for sufficiently large $M\in\mathbb{N}$. $\square$

From Proposition 2.2.10 in [29], we know that g is pluri-harmonic in U^+. Thus, $\mathscr{C}_c$ is Levi-flat for all $c>0$. This implies that $\mathscr{C}_c$ has a natural unique foliation structure by complex curves. This will be elaborated in the proof of Lemma VI.13. We will show that each leaf is a biholomorphic image of $\mathbb{C}$ later in Section 5.2. In the following, we prove that every leaf of $\mathscr{C}_c$ is dense.

Proposition IV.6 (See [24])**.** *For all $c>0$, each leaf is dense in $\mathscr{C}_c$.*

Proof. At the moment, we assume that c is sufficiently large as in Section 4.2 and the general case will be considered later. First, we look at the foliation structure of $V^+(r)$. Recall the relationship $g=\log|\varphi_+|$. Since ∇g does not vanish over $\mathscr{C}_c\cap V^+$, the same is true for $\nabla\varphi_+$ in $\mathscr{C}_c\cap V^+$. This implies that $\{\varphi_+=s\}\cap V^+$ is a smooth complex manifold in $\mathscr{C}_c$ for any $s\in\mathbb{C}$ with $|s|=r$. By the uniqueness of the foliation structure and Expression 4.4, a leaf of the fiber $\{g=\log r\}$ is of the following form:

$$\Phi_s=(\varphi_+^{-1}(s)\cap V^+)\cup f^{-1}(\varphi_+^{-1}(s^d)\cap V^+)\cup f^{-2}(\varphi_+^{-2}(s^{d^2})\cap V^+)\cup\cdots \tag{4.5}$$

for an $s\in\mathbb{C}$ with $|s|=r$. Since $f^{-1}(\varphi_+^{-1}(s^d))=\cup_{\{\omega^d=1\}}\varphi_+^{-1}(\omega s)$ in the neighborhood V^+ of I_-, we have

$$\{\varphi_+=s\}\cap V^+=\lim_{n\to\infty}\cup_{\{\omega^{d^n}=1\}}\varphi_+^{-1}(\omega s)\qquad\text{in the neighborhood } V^+ \text{ of } I_-.$$

Since the union of the d^n-th roots of unity is dense in S^1, $\overline{\{\varphi_+ = s\} \cap V^+} = V^+(r)$, which proves that $V^+(r)$ has a dense foliation structure.

Since f is a biholomorphic automorphism of $\mathbb{C}^2$, every $f^{-n}(V^+(r^{d^n}))$ also has a dense foliation structure. Expression 4.4 is increasing. Due to the uniqueness of the foliation structure, the increasing foliation structure is consistent. This proves the statement for large $c > 0$.

We now consider the general case. This property is preserved by the biholomorphisms f and f^{-1}. Thus, the general case is obtained by considering the statement for $f^M(\mathscr{C}_c)$ for sufficiently large $M \in \mathbb{N}$.

$\square$

Remark IV.7. For all $c > 0$, there is no algebraic leaf since each leaf is dense in $\mathscr{C}_c$.

4.4 Family of Parametrizations

From now on, we will consider a specific leaf. We consider $s \in \mathbb{C}$ such that $|s| = r$ and Φ_s, where r is as in Section 4.2 and Φ_s denotes the leaf corresponding to the parameter s. From the previous section,

$$\Phi_s = (\varphi_+^{-1}(s) \cap V^+) \cup f^{-1}(\varphi_+^{-1}(s^d) \cap V^+) \cup f^{-2}(\varphi_+^{-1}(s^{d^2}) \cap V^+) \cup \cdots$$

We define $\Phi_s^n := f^{-n}(\varphi_+^{-1}(s^{d^n}) \cap V^+)$ for $n = 0, 1, 2, \cdots$. Recall Condition 2 on r. We use the XY-coordinate system and the fact that f is a biholomorphic automorphism of $\mathbb{C}^2$ to find a parametrization $\Psi_s^n : \overline{\Delta} \to \Phi_s^n$ of the curve Φ_s^n for $n = 0, 1, 2, \cdots$. Indeed, for a given n, we can find a one-to-one correspondence between $\theta \in \overline{\Delta}$ and $(z, w) \in \Phi_s^n$ by

$$\begin{cases} \varphi_+(f^n(z,w)) = s^{d^n} \\ \frac{w_n}{z_n} = \theta \text{ for } |\theta| \leq 1. \end{cases}$$

We denote the n-th parametrization in the ZW-coordinate by $\Psi_s^n = ((\Psi_s^n)_1, (\Psi_s^n)_2)$.

Observe that Expression 4.5 is increasing. Since f is an automorphism of $\mathbb{C}^2$ and $\left|s^{d^i}\right| \geq |s|$, it suffices to prove the inclusion $\varphi_+^{-1}(s) \cap V^+ \subseteq f^{-1}(\varphi_+^{-1}(s^d) \cap V^+)$. From the parametrization, it suffices to prove that for every $(z, w) \in V^+$ such that $\varphi_+(z, w) = s$, $\left|\frac{w}{z}\right| \leq 1$, the following is true:

$$\begin{cases} \varphi_+(f(z,w)) = s^d \\ |z| \leq |p(z) - aw| \end{cases}$$

For our (z, w), the first condition is the same as $\varphi_+(z, w) = s$. The second condition is proved from the following inequality:

$$\left|\frac{z}{p(z) - aw}\right| \leq \frac{|z|}{|p(z)| - |a|\,|w|} \leq \frac{|z|}{|p(z)| - |a|\,|z|} \leq \frac{|z|}{3/4\,|z^d|} < 1$$

The last two inequalities are from Condition 1 on R. This shows that the union is increasing. In particular, $\Phi_s^n \subseteq \Phi_s^{n+1}$.

We introduce some more notations and some properties of the parametrizations.

For the parametrization Ψ_s^n, we define $\Psi_s^{n,i} := f^i(\Psi_s^n)$. Then, we have

1. $\Psi_s^{n,0} = \Psi_s^n$,

2. $f^i(\Phi_s) = \Phi_{s^{d^i}}$,

3. $f^i(\Psi_s^{n,j}(\theta)) = \Psi_s^{n,i+j}(\theta) \in \Phi_{s^{d^{i+j}}}$ for $\theta \in \overline{\Delta}$,

4. Φ_s^n is biholomorphic to $\overline{\Delta}$,

5. $\Phi_s^0 \subseteq \Phi_s \cap V^+$, and

6. $\{\Phi_s^n\}$ forms a compact exhaustion of Φ_s.

Note that in the 5-th statement, the equality may not hold.

We will denote the derivatives of Ψ_s^n, $(\Psi_s^n)_1$, $(\Psi_s^n)_2$, $\Psi_s^{n,i}$, $(\Psi_s^{n,i})_1$, and $(\Psi_s^{n,i})_2$ with respect to θ by $(\Psi_s^n)'$, $(\Psi_s^n)_1'$, $(\Psi_s^n)_2'$, $(\Psi_s^{n,i})'$, $(\Psi_s^{n,i})_1'$, and $(\Psi_s^{n,i})_2'$, where $\Psi_s^n = ((\Psi_s^n)_1, (\Psi_s^n)_2)$ and $\Psi_s^{n,i} = ((\Psi_s^{n,i})_1, (\Psi_s^{n,i})_2)$.

For the parametrization Ψ_s^n, we define the set of θ's $\Theta_s^{n,i} := \{|\theta| \leq 1 : (z_j, w_j) \in V^+$, for all j such that $i \leq j \leq n\}$ for $0 \leq i \leq n$. Clearly, $\Theta_s^{n,i} \subseteq \Theta_s^{n,i+1}$ and $\Theta_s^{n,n} = \{|\theta| \leq 1\}$.

Remark IV.8. We consider Condition 1 on r in Section 4.2. Let $(z,w) \in \varphi_+^{-1}(s) \cap V^+$ such that $|s| = r$ with r chosen as in Section 4.2. Then,

$$\left|\frac{p(w)-z}{a}\right| \geq \frac{|z| - |p(w)|}{|a|} > \frac{1}{|a|}(\frac{|s|}{2} - [\max_{|w|\leq R} |p(w)|]) \geq R,$$

which implies $f^{-1}(z,w) \in V^-$ or $f^{-1}(z,w) \in V^+$ and so, $f^{-1}(z,w) \notin W$. Using this, we verify $\Phi_s^n \cap W = \emptyset$ for our choice of s. Note that $\Psi_s^{n,i}(\overline{\Delta}) = f^i(\Phi_s^n) \subseteq \Phi_{s^{d^i}}$ and $\left|s^{d^i}\right| > |s|$ for $i = 1, ..., n$. So, we can apply the above inequality to all $(z,w) \in f^i(\Phi_s^n) \cap V^+$ to get $f^{-1}(z,w) \in V^-$ or $f^{-1}(z,w) \in V^+$ for such (z,w). Due to the invariance of V^- under f^{-1}, we conclude that $\Phi_s^n \cap W = \emptyset$. This argument is true for any n. Since $\Phi_s = \cup_n \Phi_s^n$, Φ_s does not intersect W. Also, the above argument is true for all s with $|s| = r$. This implies that $\mathscr{C}_c$ does not intersect W.

CHAPTER V

Non-hyperbolicity of Leaves of $\mathscr{C}_c$

In this chapter, we re-prove the theorem that Φ_s is biholomorphic to $\mathbb{C}$. In Hubbard and Oberste-Vorth, they used the subadditivity of the modulus and showed that the modulus of an annulus diverges. Here, however, we instead compute the Kobayashi-Royden pseudometric using the parametrizations found in Section 4.4. We start by introducing the Kobayashi-Royden pseudometric.

5.1 Kobayashi-Royden Pseudometric

For an analytic study, we introduce the notion of the Kobayashi-Royden pseudometric. It is the infinitesimal version of the Kobayashi pseudometric. The original definition can be found in [28]. Here, however, we adapt an equivalent definition found in [14]. Let M be a complex manifold. Let (p, ξ) be an element in a tangent bundle over M. We define the Kobayashi-Royden pseudometric as follows:

Definition V.1 (See [14], [28])**.** The Kobayashi-Royden pseudometric ds at (p, ξ) is defined by $ds(p, \xi) := \inf_f \left\{ \dfrac{1}{|c_f|} \right\}$, where the infimum is taken over all f's such that f is a holomorphic map of the unit disc Δ to M with $f(0) = p$ and $f'_*(\frac{\partial}{\partial z}) = c_f \xi$, and c_f is taken from the last condition.

Regarding this notion, we must point out an important property, which we are going to use.

Theorem V.2 (See [28])**.** *The Kobayashi-Royden pseudometric ds is a metric if and only if M is hyperbolic.*

The meaning of ds being a metric is that ds is non-degenerate. That is, for each $p \in M$, $ds(p, \xi) > 0$ for $\xi \neq 0$. This theorem together with the uniformization theorem for the Riemann surfaces will be used to distinguish the holomorphic images of Δ and the holomorphic images of $\mathbb{C}$. For notational convenience, we will call the Kobayashi-Royden pseudometric the KR-pseudometric.

5.2 Non-hyperbolicity of Leaf in $\mathscr{C}_c$

This section essentially proves that $\mathscr{C}_c$ is foliated by holomorphic images of $\mathbb{C}$. The precise statement is the following:

Theorem V.3 (See [24])**.** *For any complex number s with $|s| > 1$, the KR-pseudometric at every point of Φ_s is 0. Φ_s is simply connected. In particular, Φ_s is a biholomorphic image of $\mathbb{C}$.*

We start by discussing the simple connectedness of Φ_s.

Proposition V.4. *For any complex number s with $|s| > 1$, Φ_s is simply connected.*

Proof. We first assume that $|s|$ is sufficiently large as in Section 4.2. We focus on a topological property of Φ_s^n. f^n is a biholomorphic automorphism and the set $f^n(\Phi_s^n) = \varphi_+^{-1}(s^{d^n}) \cap V^+$ is a copy of the unit closed disc in the XY-coordinate system. So, Φ_s^n is simply connected. Since Φ_s is an increasing union of simply connected complex curves, Φ_s is simply connected. So, we have proved the simple connectedness for sufficiently large $|s|$. The general case is obtained by considering

$f^N(\Phi_s) = \Phi_{s^{d^N}}$ since the simple connectedness is preserved by the biholomorphisms f and f^{-1}. $\square$

We discuss the non-hyperbolicity of Φ_s. We will first prove that the Kobayashi-Royden pseudometric at $(z_K, w_K) \in \Phi_s^0 \subseteq \Phi_s \cap V^+$ is 0, and extend the proof over all Φ_s at the end of the proof. So, assume $(z_K, w_K) \in \Phi_s^0 \subseteq \Phi_s \cap V^+$. The sequence $\{\Phi_s^n\}$ is a compact exhaustion of Φ_s. Then, $(z_K, w_K) \in \Phi_s^n$ for all n. Recall that the sequence of parametrizations $\{\Psi_s^n\}$ of $\{\Phi_s^n\}$ is a sequence of mappings of $\overline{\Delta}$ into Φ_s. Note that since Φ_s is of complex dimension 1, all tangent vectors at (z_K, w_K) are parallel to one another. Thus, in order to verify that the Kobayashi-Royden pseudometric at (z_K, w_K) is 0, it suffices to show that $\left|\frac{d(\Psi_s^n)_2}{d\theta}\right| (= |(\Psi_s^n)_2'|)$ evaluated at (z_K, w_K) diverges to ∞ as $n \to \infty$.

Proposition V.5. *For $(z, w) \in V^+$ and all $n \geq 0$, $|z_n| \geq |z|^n$.*

Proof. The proof is by induction. The cases $n = 0$ and $n = 1$ are obvious. Indeed, $|z_0| = |z| > R > 2$ and $|z_1| = |p(z) - aw| \geq \frac{3}{4}|z|^d > |z|$ from Condition 1 on R. Similarly, when $n = 2$, we have $|z_2| \geq \frac{3}{4}|z_1|^d \geq \frac{3}{4}|z_1|^2 \geq \frac{3}{4}(\frac{3}{4}|z|^d)^2 \geq \frac{27}{64}|z|^4 \geq |z|^2$. Suppose that the inequality is true for $n = k \geq 2$. Then,

$$|z_{k+1}| \;\; \geq \;\; |p(z_k) - aw_k| \geq \frac{3}{4}|z_k|^d \geq \frac{3}{4}|z|^{dk} \geq \frac{3}{4}|z| \cdot |z|^{2k-1} \geq |z|^{k+1},$$

by $|z_k| > |z| > R$ and Condition 1 on R. So, it is proved. $\square$

We estimate the following infinite product.

Lemma V.6. *For some $r_0 > 0$ and sufficiently large $r > 1$ such that $\frac{r_0}{r} < 1$, $\sum_{n=1}^{\infty} \log(1 + \frac{r_0}{r^n}) \leq \frac{r_0}{r-1}$.*

Proof. We consider the function $h(x) = \log|1 + x|$ of real numbers over a closed interval $[-r_0/r, r_0/r]$. Elementary calculus tells us that for all $n > 0$, $\log(1+\frac{r_0}{r^n}) \leq \frac{r_0}{r^n}$.

This inequality proves the statement. □

We consider the sequence of matrices defined by:

$$\begin{pmatrix} \frac{\partial z_{n+1}}{\partial z} & \frac{\partial z_{n+1}}{\partial w} \\ \\ \frac{\partial w_{n+1}}{\partial z} & \frac{\partial w_{n+1}}{\partial w} \end{pmatrix} = \begin{pmatrix} p'(z_n) & -a \\ \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial z_n}{\partial z} & \frac{\partial z_n}{\partial w} \\ \\ \frac{\partial w_n}{\partial z} & \frac{\partial w_n}{\partial w} \end{pmatrix}$$

with the initial value

$$\begin{pmatrix} \frac{\partial z_1}{\partial z} & \frac{\partial z_1}{\partial w} \\ \\ \frac{\partial w_1}{\partial z} & \frac{\partial w_1}{\partial w} \end{pmatrix} = \begin{pmatrix} p'(z) & -a \\ \\ 1 & 0 \end{pmatrix}.$$

Lemma V.7. *For $(z,w) \in V^+$ and all $n \geq 0$,*

$$\left| \frac{\frac{\partial w_{n+1}}{\partial z}}{\frac{\partial z_{n+1}}{\partial z}} \right| \leq \frac{2}{d\,|z_n|}$$

Here z_0 means z.

Proof. The proof is by induction. We consider the initial case $n = 0$ first. If $n = 0$, then by Condition 3 on $R > 0$,

$$\left| \frac{\frac{\partial w_1}{\partial z}}{\frac{\partial z_1}{\partial z}} \right| = \left| \frac{1}{p'(z)} \right| \leq \frac{4}{3d\,|z|}.$$

Suppose that the statement is true for $n = k-1$. By Condition 3 on $R > 0$,

$$\begin{aligned} \left| \frac{\frac{\partial w_{k+1}}{\partial z}}{\frac{\partial z_{k+1}}{\partial z}} \right| &= \left| \frac{\frac{\partial z_k}{\partial z}}{\frac{\partial z_k}{\partial z} p'(z_k) - a \frac{\partial w_k}{\partial z}} \right| \leq \frac{1}{|p'(z_k)| - |a| \left| \frac{\frac{\partial w_k}{\partial z}}{\frac{\partial z_k}{\partial z}} \right|} \\ &\leq \frac{1}{\frac{3d}{4}\,|z_k| - \frac{2|a|}{d|z_{k-1}|}} \leq \frac{2}{d\,|z_k|}. \end{aligned}$$

The last inequality is from Condition 7 on R. □

Lemma V.8. *For $(z,w) \in V^+$ and all $n \geq 0$,*

$$\left| \frac{\frac{\partial w_{n+1}}{\partial w}}{\frac{\partial z_{n+1}}{\partial w}} \right| \leq \frac{2}{d\,|z_n|}$$

Proof. The proof is the same as for Lemma V.7 except that the induction step starts at $n = 1$ instead of $n = 0$. The cases $n = 0, 1$ are easy to check. Here we consider non-zero divided by 0 to be ∞. □

Lemma V.9.

$$\left|\frac{\frac{\partial z_n}{\partial w}}{\frac{\partial z_n}{\partial z}}\right| \le \prod_{i=1}^{n-1}\Big(1+\frac{8\,|a|}{d\,|z|^{2i}}\Big)\left|\frac{a}{p'(z)}\right| \textit{ for } (z,w)\in V^+ \textit{ and all } n>0$$

Proof.

$$\begin{aligned}
\left|\frac{\frac{\partial z_{n+1}}{\partial w}}{\frac{\partial z_{n+1}}{\partial z}}\right| &= \left|\frac{p'(z_n)\frac{\partial z_n}{\partial w}-a\frac{\partial w_n}{\partial w}}{p'(z_n)\frac{\partial z_n}{\partial z}-a\frac{\partial w_n}{\partial z}}\right| \le \frac{\left|p'(z_n)\frac{\partial z_n}{\partial w}\right|+|a|\left|\frac{\partial w_n}{\partial w}\right|}{\left|p'(z_n)\frac{\partial z_n}{\partial z}\right|-|a|\left|\frac{\partial w_n}{\partial z}\right|}\\
&\le \frac{1+\left|\frac{a}{p'(z_n)}\right|\cdot\left|\frac{\frac{\partial w_n}{\partial w}}{\frac{\partial z_n}{\partial w}}\right|}{1-\left|\frac{a}{p'(z_n)}\right|\cdot\left|\frac{\frac{\partial w_n}{\partial z}}{\frac{\partial z_n}{\partial z}}\right|}\left|\frac{\frac{\partial z_n}{\partial w}}{\frac{\partial z_n}{\partial z}}\right|\\
&= \frac{1+\left|\frac{a}{p'(z_n)}\right|\cdot\left|\frac{2}{d|z_n|}\right|}{1-\left|\frac{a}{p'(z_n)}\right|\cdot\left|\frac{2}{d|z_n|}\right|}\left|\frac{\frac{\partial z_n}{\partial w}}{\frac{\partial z_n}{\partial z}}\right|\\
&\le \left(1+\frac{8\,|a|}{d\,|p'(z_n)|\,|z_n|}\right)\left|\frac{\frac{\partial z_n}{\partial w}}{\frac{\partial z_n}{\partial z}}\right| \le \left(1+\frac{8\,|a|}{d\,|z_n|^2}\right)\left|\frac{\frac{\partial z_n}{\partial w}}{\frac{\partial z_n}{\partial z}}\right|.
\end{aligned}$$

In the third and last line, Lemma V.8, Lemma V.7, and Condition 3 and 7 on R are used. The initial value $\left|\frac{\frac{\partial z_1}{\partial w}}{\frac{\partial z_1}{\partial z}}\right|$ is easy to compute; we differentiate $p(z) - aw$ with respect to z and w and take the ratio. Proposition V.5 completes the proof of the lemma. □

We estimate $\frac{\frac{\partial \varphi_+}{\partial w}}{\frac{\partial \varphi_+}{\partial z}}$. Recall the definition of φ_+.

$$\varphi_+(z,w) = z\cdot\big(\frac{z_1}{z^d}\big)^{1/d}\cdot\ldots\cdot\big(\frac{z_{n+1}}{z_n^d}\big)^{1/d^{n+1}}\cdot\ldots$$

Note that our definition is not $\varphi_+ = \lim_{n\to\infty} z_n^{1/d^n}$ for $(z,w)\in V^+$. We define a sequence $\{(\varphi_+)_n\}_{n=1}^\infty$ of partial products of φ_+ by

$$(\varphi_+)_n(z,w) = z\cdot\prod_{i=0}^{n-1}\big(\frac{z_{i+1}}{z_i^d}\big)^{1/d^{i+1}},$$

where $i = 0$ corresponds to (z, w). We know that $(\varphi_+)_n \to \varphi_+$ locally uniformly. Observe that $[(\varphi_+)_n]^{d^n} = z_n$. Taking a partial derivative with respect to z, we have $d^n[(\varphi_+)_n]^{d^n-1}\frac{\partial(\varphi_+)_n}{\partial z} = \frac{\partial z_n}{\partial z}$. The same is true for differentiation with respect to w. Thus, we have

$$\frac{\frac{\partial\varphi_+}{\partial w}}{\frac{\partial\varphi_+}{\partial z}} = \lim_{n\to\infty}\frac{\frac{\partial(\varphi_+)_n}{\partial w}}{\frac{\partial(\varphi_+)_n}{\partial z}} = \lim_{n\to\infty}\frac{\frac{\partial z_n}{\partial w}}{\frac{\partial z_n}{\partial z}}.$$

The limit is bounded as follows:

$$\begin{aligned}\left|\frac{\frac{\partial\varphi_+}{\partial w}}{\frac{\partial\varphi_+}{\partial z}}\right| &\leq \lim_{n\to\infty}\prod_{i=1}^{n}\left(1+\frac{8\,|a|}{d\,|z|^{2i}}\right)\left|\frac{a}{p'(z)}\right| \leq \lim_{n\to\infty}\prod_{i=1}^{n}\left(1+\left|\frac{z^2}{10}\right|\cdot\frac{1}{|z|^{2i}}\right)\left|\frac{a}{p'(z)}\right| \\ &\leq e^{0.25}\left|\frac{a}{p'(z)}\right| \leq 2\left|\frac{a}{p'(z)}\right| \quad \text{for } (z,w)\in V^+\end{aligned}$$

The first inequality is from Lemma V.9, the second one from Condition 7 on R, and the third one from Lemma V.6. We have just proved the following lemma.

Lemma V.10. $\left|\frac{\frac{\partial\varphi_+}{\partial w}}{\frac{\partial\varphi_+}{\partial z}}\right| \leq 2\left|\frac{a}{p'(z)}\right|$ *for* $(z, w) \in V^+$.

Now, we are ready to prove the Kobayashi-Royden pseudometric at some fixed point $(z_K, w_K) \in \Phi_s^0 \subseteq \Phi_s \cap V^+$ is 0.

The proof of Theorem V.3. We first assume that $|s|$ is large enough as in Section 4.2. We pick an arbitrary point (z_K, w_K) in the interior of Φ_s^0. Then, since $\Phi_s^n \subseteq \Phi_s^{n+1}$ for all n, $(z_K, w_K) \in \Phi_s^n$ for all n. Let $\theta_n \in \Delta$ denote the complex time such that $\Psi_s^n(\theta_n) = (z_K, w_K)$. We estimate the ratio $\frac{|(\Psi_s^{n+1})_2'(\theta_{n+1})|}{|(\Psi_s^n)_2'(\theta_n)|}$ and compare the ratio to a divergent geometric sequence so as to prove the divergence of the sequence $\{|(\Psi_s^n)_2'(\theta_n)|\}$ to ∞. We use the chain rule as follows:

$$
\begin{aligned}
(5.1)\qquad \frac{d\Psi_s^n}{d\theta}\Big|_{\theta=\theta_n} &= df^{-n}|_{f^n(z_K,w_K)} \cdot \frac{d}{d\theta}(f^n(\Psi_s^n))|_{\theta=\theta_n} \\
&= df^{-n}|_{f^n(z_K,w_K)} \cdot \frac{d}{d\theta}(\Psi^0_{s^{d^n}})|_{\theta=\theta_n} \\
(5.2)\qquad \frac{d\Psi_s^{n+1}}{d\theta}\Big|_{\theta=\theta_{n+1}} &= df^{-n}|_{f^n(z_K,w_K)} \cdot \frac{d}{d\theta}(f^n(\Psi_s^{n+1}))|_{\theta=\theta_{n+1}} \\
&= df^{-n}|_{f^n(z_K,w_K)} \cdot \frac{d}{d\theta}(\Psi^1_{s^{d^n}})|_{\theta=\theta_{n+1}}.
\end{aligned}
$$

The last equalities are valid since our parametrization satisfies $f^j(\Psi_s^i) = \Psi^{i-j}_{s^{d^j}}$ for all $0 \le j \le i$.

Our computation consists of 2 parts:

Part A Comparison of $\frac{d}{d\theta}(\Psi^0_{s^{d^n}})|_{\theta=\theta_n}$ and $\frac{d}{d\theta}(\Psi^1_{s^{d^n}})|_{\theta=\theta_{n+1}}$

Part B Computation of the matrix $df^{-n}|_{f^n(z_K,w_K)}$.

[**Part A**] We consider Ψ_s^i for $i = 0,1$ first. Since Ψ_s^i for $i = 0,1$ satisfy $\varphi_+(\Psi_s^i) = s$ near θ_i, the derivatives $\frac{d\Psi_s^i}{d\theta}|_{\theta=\theta_i}$ for $i = 0,1$ obviously satisfy $\frac{\partial\varphi_+}{\partial z}|_{(z_K,w_K)} \cdot (\Psi_s^i)_1'(\theta_i) + \frac{\partial\varphi_+}{\partial w}|_{(z_K,w_K)} \cdot (\Psi_s^i)_2'(\theta_i) = 0$ for $i = 0,1$. Since Φ_s is a smooth manifold in V^+, one of $\frac{\partial\varphi_+}{\partial z}|_{(z_K,w_K)}$ or $\frac{\partial\varphi_+}{\partial w}|_{(z_K,w_K)}$ must not equal to 0. From Lemma V.10, we know that $\frac{\partial\varphi_+}{\partial z}|_{(z_K,w_K)} \neq 0$. We denote by $\mu_{(z_K,w_K)} := -\frac{\frac{\partial\varphi_+}{\partial w}|_{(z_K,w_K)}}{\frac{\partial\varphi_+}{\partial z}|_{(z_K,w_K)}}$. Then, $(\Psi_s^i)_1'(\theta_i) = \mu_{(z_K,w_K)}(\Psi_s^i)_2'(\theta_i)$ for $i = 0,1$ and from Lemma V.10, $\left|\mu_{(z_K,w_K)}\right| \le \left|\frac{2a}{p'(z_K)}\right|$. For notational convenience, we will simply use μ for $\mu_{(z_K,w_K)}$. When we emphasize the point (z_K, w_K), we will use $\mu_{(z_K,w_K)}$.

From the other parametrizing equations of Ψ_s^0 and Ψ_s^1, we have $(\Psi_s^0)_2 = \theta(\Psi_s^0)_1$ near θ_0 for $i = 0$ and $(\Psi_s^1)_1 = \theta[p((\Psi_s^1)_1) - a(\Psi_s^1)_2]$ near θ_1 for $i = 1$, respectively.

For $i = 0$, differentiating with respect to θ at $\theta = \theta_0$, we have $(\Psi_s^0)_2'(\theta_0) = z_K + \theta_0(\Psi_s^0)_1'(\theta_0)$. Since $\theta_0 = \frac{w_K}{z_K}$ at (z_K, w_K), we have

$$\begin{cases} (\Psi_s^0)_1'(\theta_0) = \frac{\mu z_K^2}{z_K - \mu w_K} \\ (\Psi_s^0)_2'(\theta_0) = \frac{z_K^2}{z_K - \mu w_K}. \end{cases}$$

For $i = 1$, similarly, we differentiate with respect to θ at $\theta = \theta_1 = \frac{z_K}{p(z_K) - a w_K}$. Then,

$$\begin{cases} (\Psi_s^1)_1'(\theta_1) = \frac{\mu (p(z_K) - a w_K)^2}{a z_K + \mu(p(z_K) - a w_K) - \mu z_K p'(z_K)} \\ (\Psi_s^1)_2'(\theta_1) = \frac{(p(z_K) - a w_K)^2}{a z_K + \mu(p(z_K) - a w_K) - \mu z_K p'(z_K)} \end{cases}$$

The two vectors $\frac{d\Psi_s^0}{d\theta}|_{\theta=\theta_0}, \frac{d\Psi_s^1}{d\theta}|_{\theta=\theta_1}$ are not zero because Ψ_s^i is a biholomorphism of $\overline{\Delta}$ for all i. Since the Φ_s is of complex dimension 1, we know that $\frac{d\Psi_s^1}{d\theta}|_{\theta=\theta_1}$ is a complex scalar multiple of $\frac{d\Psi_s^0}{d\theta}|_{\theta=\theta_0}$. It suffices to find the lower bound just for $\left|\frac{(\Psi_s^1)_2'(\theta_1)}{(\Psi_s^0)_2'(\theta_0)}\right|$.

Since $z_K > R > 0$, we have $|\mu| \le \left|\frac{2a}{p'(z_K)}\right| \le \left|\frac{8a}{3dz_K^{d-1}}\right|$, $\left|\frac{3}{4}z_K^d\right| \le |p(z_K) - a w_K| \le \left|\frac{5}{4}z_K^d\right|$, and $\left|\frac{3}{4}dz_K^d\right| \le |p'(z_K)| \le \left|\frac{5}{4}dz_K^d\right|$ by our choice of R. So, we have the following:

1. $\left|\dfrac{(p(z_K) - a w_K)^2}{a z_K}\right| \ge \left|\dfrac{(\frac{3}{4}z_K^d)^2}{a z_K}\right| = \left|\dfrac{9}{16a} z_K^{2d-1}\right|$.

2. $\left|\dfrac{(p(z_K) - a w_K)^2}{\mu(p(z_K) - a w_K)}\right| = \left|\dfrac{p(z_K) - a w_K}{\mu}\right| \ge \left|\dfrac{9d}{32a} z_K^{2d-1}\right|$

3. $\left|\dfrac{(p(z_K) - a w_K)^2}{\mu z_K p'(z_K)}\right| \ge \left|\dfrac{(\frac{3}{4}z_K^d)^2}{\mu z_K \cdot \frac{5}{4}dz_K^{d-1}}\right| \ge \left|\dfrac{27}{160a} z_K^{2d-1}\right|$

We use the following inequality. The proof is quite straightforward.

Proposition V.11. *For $\{A_i\}_{i=1}^n$ with $A_i \neq 0$ for all i,*

$$\frac{1}{|\sum_{i=1}^n A_i|} \ge \frac{1}{\sum_{i=1}^n |A_i|} \ge \frac{1}{n} \cdot \min_{1\le i\le n} \left\{\frac{1}{|A_i|}\right\}$$

From Proposition V.11 and the above computations (1), (2), and (3), we have

$$\left|(\Psi_s^1)_2'(\theta_1)\right| \geq \left|\frac{9}{160a} z_K^{2d-1}\right| \tag{5.3}$$

On the other hand, from Lemma V.10 and Condition 7 on R, we have $|\mu| \leq \left|\frac{2a}{p'(z_K)}\right| \ll 1$. Then, $|z_K| - |\mu w_K| \geq 1/2\,|z_K|$, and therefore,

$$\left|(\Psi_s^0)_2'(\theta_0)\right| = \left|\frac{z_K^2}{z_K - \mu w_K}\right| \leq \frac{|z_K^2|}{1/2\,|z_K|} = 2\,|z_K|\,. \tag{5.4}$$

From the Inequalities 5.3 and 5.4, we have the following:

$$\left|\frac{(\Psi_s^1)_2'(\theta_1)}{(\Psi_s^0)_2'(\theta_0)}\right| \geq \left|\frac{9}{320a} z_K^{2d-2}\right|.$$

If $(z_K, w_K) \in \Phi_s \cap V^+$, then $((z_K)_n, (w_K)_n) \in \Phi_{s^{d^n}} \cap V^+$ and $\frac{(z_K)_n}{(w_K)_n} = \theta_n$ from our settings. Thus, we can use the above argument to $\Psi^i_{s^{d^n}}$ for $i = 0, 1$ in exactly the same way by replacing (z_K, w_K) by $((z_K)_n, (w_K)_n)$, θ_0, θ_1 by θ_n, θ_{n+1}, and s by s^{d^n}. Note that we use $\mu_{(z_K,w_K)} := -\frac{\frac{\partial\varphi_+}{\partial w}|_{(z_K,w_K)}}{\frac{\partial\varphi_+}{\partial z}|_{(z_K,w_K)}}$ for $\left|\frac{(\Psi^1_{s^{d^n}})_1'(\theta_{n+1})}{(\Psi^0_{s^{d^n}})_1'(\theta_n)}\right|$, which is the same as before. Hence, we obtain

Lemma V.12.

$$\left|\frac{(\Psi^1_{s^{d^n}})_2'(\theta_{n+1})}{(\Psi^0_{s^{d^n}})_2'(\theta_n)}\right| \geq \left|\frac{9}{320a}(z_K)_n^{2d-2}\right|.$$

[Part B] $df^{-n}|_{f^n} = (df^n)^{-1}$. So,

$$df^{-n}|_{f^n} = \frac{1}{a^n}\begin{pmatrix} \frac{\partial w_n}{\partial w} & -\frac{\partial z_n}{\partial w} \\ & \\ -\frac{\partial w_n}{\partial z} & \frac{\partial z_n}{\partial z} \end{pmatrix}$$

We combine the results from Part A) & B), Lemma V.7, Lemma V.10, and Lemma V.12 to compute the lower bound of $\frac{\left|(\Psi_s^{n+1})_2'(\theta_{n+1})\right|}{\left|(\Psi_s^n)_2'(\theta_n)\right|}$. By Chain Rule 5.1 and 5.2, we have the following: For $n \geq 2$,

$$\begin{aligned}
\frac{|(\Psi_s^{n+1})_2'(\theta_{n+1})|}{|(\Psi_s^n)_2'(\theta_n)|} &= \left| \frac{-\frac{\partial w_n}{\partial z}(\Psi^1_{s^{d^n}})_1'(\theta_{n+1}) + \frac{\partial z_n}{\partial z}(\Psi^1_{s^{d^n}})_2'(\theta_{n+1})}{-\frac{\partial w_n}{\partial z}(\Psi^0_{s^{d^n}})_1'(\theta_n) + \frac{\partial z_n}{\partial z}(\Psi^0_{s^{d^n}})_2'(\theta_n)} \right| \\
&\geq \frac{\left|\frac{\partial z_n}{\partial z}\right| \left|(\Psi^1_{s^{d^n}})_2'(\theta_{n+1})\right| - \left|\frac{\partial w_n}{\partial z}\right| \left|(\Psi^1_{s^{d^n}})_1'(\theta_{n+1})\right|}{\left|\frac{\partial z_n}{\partial z}\right| \left|(\Psi^0_{s^{d^n}})_2'(\theta_n)\right| + \left|\frac{\partial w_n}{\partial z}\right| \left|(\Psi^0_{s^{d^n}})_1'(\theta_n)\right|} \\
&\geq \frac{\left|(\Psi^1_{s^{d^n}})_2'(\theta_{n+1})\right| - \frac{2}{d|(z_K)_{n-1}|}\left|(\Psi^1_{s^{d^n}})_1'(\theta_{n+1})\right|}{\left|(\Psi^0_{s^{d^n}})_2'(\theta_n)\right| + \frac{2}{d|(z_K)_{n-1}|}\left|(\Psi^0_{s^{d^n}})_1'(\theta_n)\right|} \\
&\geq \left(\frac{1 - \frac{4|a|}{d|(z_K)_{n-1}||p'(z_K)|}}{1 + \frac{4|a|}{d|(z_K)_{n-1}||p'(z_K)|}} \right) \left| \frac{(\Psi^1_{s^{d^n}})_2'(\theta_{n+1})}{(\Psi^0_{s^{d^n}})_2'(\theta_n)} \right| \\
&\geq \left(\frac{1 - \frac{4|a|}{d|(z_K)_{n-1}||p'(z_K)|}}{1 + \frac{4|a|}{d|(z_K)_{n-1}||p'(z_K)|}} \right) \left| \frac{9}{320a}(z_K)_n^{2d-2} \right|
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1 - \frac{4|a|}{dR\frac{3}{4}dR^{d-1}}}{1 + \frac{4|a|}{dR\frac{3}{4}dR^{d-1}}} \left| \frac{9}{320a} \right| R^{n(2d-2)} \text{ due to Proposition V.5} \\
&= \frac{1 - \frac{16|a|}{3d^2R^d}}{1 + \frac{16|a|}{3d^2R^d}} \left| \frac{9}{320a} \right| R^{n(2d-2)} \\
&\geq \frac{19}{21} R^{2n-1} \text{ from our choice of } R.
\end{aligned}$$

The third line is due to Lemma V.7, the fourth line is due to Lemma V.10, and the fifth line is due to Lemma V.12. We know that for every n, $|(\Psi_s^n)_2'(\theta_n)| > 0$. As shown above, for $n \geq 2$, the sequence increases faster than a divergent geometric series with its common ratio $R > 1$. By the comparison test, we have proved that $|(\Psi_s^n)_2'(\theta_n)| \to \infty$ as $n \to \infty$ as desired, which means the KR-pseudometric of Φ_s at (z_K, w_K) is 0.

So far, we have proved the vanishing of the KR-pseudometric at $(z_K, w_K) \in \Phi_s^0 \subseteq \Phi_s \cap V^+$. We do the general case. For any $(z_K, w_K) \in \Phi_s$, we can find $N_K \in \mathbb{N}$ such that (z_K, w_K) is in the interior of $\Phi_s^{N_K}$. Then, we can apply the same argument to $f^{N_K}(z_K, w_K)$ with s replaced by $s^{d^{N_K}}$ since $f^{N_K}(\Phi_s^{N_K}) = \Phi^0_{s^{d^{N_K}}}$. Thus, the general case is proved.

Together with Proposition V.4, by the uniformization theorem, we just proved Theorem V.3. $\square$

CHAPTER VI

Brody Leaves of $\mathscr{C}_c$

In this chapter, we introduce the notion of the Brody curve, provide examples, discuss their properties, and finally prove the main theorem.

6.1 Brody Curves

Brody curves first appeared in Brody's proof that every compact non-Kobayashi hyperbolic manifold contains a non-trivial holomorphic image of $\mathbb{C}$ (see [12]). Recall the definition of the Brody curve defined in the Introduction.

Definition VI.1 (Brody Curve)**.** Let M be a complex manifold with a smooth metric ds. Let $\psi : \mathbb{C} \to M$ be a non-constant holomorphic map of $\theta \in \mathbb{C}$ to M. The map ψ is said to be *Brody* if $\sup_{\theta\in\mathbb{C}} ds(\psi(\theta), d\psi(\frac{d}{d\theta})) < C$ for some constant $C > 0$. We call the image $\psi(\mathbb{C})$ a *Brody curve* in M. The curve $\psi(\mathbb{C})$ is said to be *injective Brody* if the parametrization ψ is injective.

Note that the Brodyness heavily depends on the choice of the smooth metric. For example, $z \to (z, z^2)$ is a Brody curve with respect to the Fubini-Study metric of $\mathbb{P}^2$ while it is not with respect to the standard hermitian metric of $\mathbb{C}^2$. Thus, it is clear that the Brodyness is not biholomorphic invariant.

Example VI.2 (Brodyness in $\mathbb{C}^1$)**.** We use the standard hermitian metric for the space. Then, all injective holomorphic maps are linear maps, and therefore, Brody. However, if they are not injective, then they are not Brody.

Example VI.3 (Brodyness in $\mathbb{P}^1$)**.** We use the standard Fubini-Study metric for the space. Then, all injective maps are of the form $z \to [az+b : cz+d]$. Thus, they are all Brody. If we remove the injectivity condition, then we can find a non-Brody curve. Consider a map $z \to \sin z^2$. Then, over the real line, the derivative goes to ∞ while the point stays bounded. The function of the form $z \to p(z)e^z$ is Brody, where $p(z)$ is a polynomial of one complex variable.

Example VI.4 (Brodyness in $\mathbb{C}^2$)**.** Again, we use the standard hermitian metric for the space. Liouville's theorem tells us that a Brody curve in $\mathbb{C}^2$ is of the form $z \to (a_1z+b_1, a_2z+b_2)$ for some constants a_1, a_2, b_1, b_2. Indeed, the standard hermitian metric of $\mathbb{C}^2$ for the map $f(z) = (f_1(z), f_2(z))$ is of the from $\sqrt{|f_1'(z)|^2 + |f_2'(z)|^2}$. So, if a curve is Brody, that implies that f_1' and f_2' are bounded over $\mathbb{C}$. Liouville's theorem about the mapping of $\mathbb{C}$ into a compact set implies the constant derivative of f_1, f_2.

Now, we discuss the Brody curves in $\mathbb{P}^2$. We use the Fubini-Study metric. Then, it might be a natural question to ask how big the collection of Brody curves is. Indeed, there are plenty of Brody curves. In the following proposition, $ds(f, f')$ denotes the Fubini-Study metric of f' in $\mathbb{P}^2$.

Proposition VI.5. *Let α be a complex constant, p, q are polynomials of one complex variable z of degree d, d' respectively. Then, all curves of the form $(p(z)e^z, q(z)e^{\alpha z})$ are Brody.*

Proof. Let $\alpha = a+ib$, $z = x+iy$, and $f(z) = (p(z)e^z, q(z)e^{\alpha z})$. Without loss of generality, we may assume that $|z| = R > 0$. Then,

$$ds(f, f') = \frac{|(p+p')e^z|^2 + |(\alpha q + q')e^{\alpha z}|^2 + \left|(pq' + (\alpha - 1)pq - qp')e^{(\alpha+1)z}\right|^2}{(1 + |pe^z|^2 + |qe^{\alpha z}|^2)^2}.$$

Since there is no chance for the denominator being 0, the only way for the Fubini-Study metric of f' to blow off to ∞ is the growth rate of the numerator overwhelms that of the denominator as $R \to \infty$.

There are two cases: whether α is real or not. First, we consider the upper-bound of $ds(f, f')$ when α is not real. We consider $ds(f, f')$ over a line $L_\theta := \{R\theta \in \mathbb{C} : R \in \mathbb{R}\}$ in $\mathbb{C}$, where $\theta \in \mathbb{C}$ and $|\theta| = 1$. Notice that $\max\{|e^z|, |e^{\alpha z}|\}$ is not bounded above over L_θ for every θ. We fix a θ. Since exponential growth dominates any polynomial growth, we first compare the growth rate of $\left|e^{2(\alpha+1)z}\right|$ and $\max\{|e^{4z}|, |e^{4\alpha z}|\}$. The real part of $2(\alpha+1)z$ is an arithmetic average of the real parts of $4z$ and $4\alpha z$. So, $\left|e^{2(\alpha+1)z}\right|$ is bounded by $\max\{|e^{4z}|, |e^{4\alpha z}|\}$. If $\left|e^{2(\alpha+1)z}\right|$ is strictly smaller than $\max\{|e^{4z}|, |e^{4\alpha z}|\}$, $ds(f, f')$ is bounded above. If they are equal to each other, the fact that $|pq|^2$ is dominated by $|p|^4 + |q|^4$ proves that $ds(f, f')$ is bounded above over L_θ. This is true for every $\theta \in \mathbb{C}$ with $|\theta| = 1$. The statement is proved.

We consider the case where α is real. Again, we consider $ds(f, f')$ over a line $L_\theta := \{R\theta \in \mathbb{C} : R \in \mathbb{R}\}$ in $\mathbb{C}$, where $\theta \in \mathbb{C}$ and $|\theta| = 1$. We have two sub-cases: whether $\theta = \pm i$ or not. When θ is not a pure imaginary number, then, we can apply the same argument as the case where α is not real. So, we only consider the case where $\theta = \pm i$. Then, notice that $\max\{|e^z|, |e^{\alpha z}|\}$ is bounded above over L_θ, and automatically, the numerator is dominated by a polynomial of degree at most $2(d + d')$ while the degree of the dominating polynomial of the denominator is at least $4\max\{d,d'\}$. Thus, there is no chance for the metric to be unbounded.

Thus, the statement is proved. $\square$

However, not all holomorphic curves from $\mathbb{C}$ to $\mathbb{C}^2$ are Brody. The mapping $z \to (e^z, e^{iz^2})$ is not Brody. Take $z = bi$ for real b and let b to ∞. Even if we require them to be injective, not all injective curves from $\mathbb{C}$ to $\mathbb{C}^2$ are Brody. The following gives us some examples of injective non-Brody curves.

Proposition VI.6. *The map $f_n : z \to (z, e^{z^n})$ is not Brody in $\mathbb{C}^2 \subset \mathbb{P}^2$ for $n \geq 3$. In particular, not all holomorphic images of $\mathbb{C}$ in $\mathbb{P}^2$ are Brody.*

Proof. It is obvious that f_n is an holomorphic injective mapping to $\mathbb{C}^2 \subset \mathbb{P}^2$. In order to prove that it is not Brody, we consider a line $z = \alpha t$ where α is one of the n-th root of i and t is real. Then, we have

$$\begin{aligned} ds(f_n, f_n') &= \frac{1 + \left|nz^{n-1}e^{z^n}\right|^2 + \left|z \cdot nz^{n-1}e^{z^n} - 1 \cdot e^{z^n}\right|^2}{(1 + |z|^2 + |e^{z^n}|^2)^2} \geq \frac{\left|(nz^n - 1)e^{z^n}\right|^2}{(1 + |z|^2 + |e^{z^n}|^2)^2} \\ &\geq \frac{(n^2\,|t|^{2n} - 2n\,|t|^n)}{(2 + |t|^2)^2} \end{aligned}$$

Since $n \geq 3$, the Fubini-Study metric is unbounded as $|t| \to \infty$. $\square$

Thus, not all holomorphic images of $\mathbb{C}$ are Brody. As explained in the Introduction, in the sense that the derivative is bounded with respect to the Fubini-Study metric, Brodyness implies that the curve does not fluctuate too much.

We close this section by pointing out a property of the uniqueness of parametrization of injective Brody curves.

Proposition VI.7. *Let $\mathscr{C}$ be an biholomorphic image of $\mathbb{C}$. Let $\phi_1, \phi_2 : \mathbb{C} \to \mathscr{C}$ be two biholomorphic parametrizations of $\mathscr{C}$. Then $\phi_1(z) = \phi_2(az + b)$ for constants $a, b \in \mathbb{C}$ with $a \neq 0$.*

Proof. The composition $\phi_2^{-1} \circ \phi_1 : \mathbb{C} \to \mathbb{C}$ is a biholomorphism of $\mathbb{C}$ on $\mathbb{C}$. From the theorem of one complex variable, $\phi_2^{-1} \circ \phi_1(z) = az + b$ for constants $a, b \in \mathbb{C}$ with $a \neq 0$. □

Corollary VI.8. *Every Brody curve has a unique parametrization up to the inside composition with a linear map.*

6.2 Brody Leaves of $\mathscr{C}_c$

In this section, we prove our main theorem (Theorem I.3). Our space is $\mathbb{P}^2$. We use the standard Fubini-Study metric as a smooth Hermitian metric ds on $\mathbb{P}^2$. For notational convenience, we also use $\|\psi'(\theta_0)\|$ to mean $ds(\psi(\theta_0), d\psi(\frac{d}{d\theta}|_{\theta=\theta_0}))$, the Fubini-Study metric for a holomorphic mapping $\psi : \mathbb{C} \to \mathbb{P}^2$ of $\theta \in \mathbb{C}$ at $\theta = \theta_0$.

Let $c > 0$. We consider the level set $\mathscr{C}_c \subseteq \mathbb{C}^2$ as a subset of $\mathbb{P}^2$. From Chapter IV, we know that $\mathscr{C}_c$ has a foliation structure by complex curves and each leaf is dense. Every leaf is biholomorphic to $\mathbb{C}$ by Theorem V.3. We will show that the foliation of $\mathscr{C}_c$ enjoys more structural properties as stated below. Recall our notations for the parametrizations in Chapter IV.

Theorem VI.9. *For any $c > 0$, every leaf of $\mathscr{C}_c$ viewed as a subset of $\mathbb{P}^2$ is an injective Brody curve.*

Proof. First, we assume that our c is sufficiently large as in Section 4.2. We will prove the following two parts later:

Part 1 We will first show that each leaf Φ_s contains a Brody curve $\mathscr{B}_s \subseteq \mathbb{P}^2$.

Part 2 Next, we show that $\mathscr{B}_s$ is an injective Brody curve.

Once we prove the two parts, then, $\mathscr{B}_s$ in Φ_s is a biholomorphic image of $\mathbb{C}$. By Theorem V.3, each Φ_s is a biholomorphic image of $\mathbb{C}$. Suppose that $\mathscr{B}_s \neq \Phi_s$. Then, we can find a biholomoprhic mapping from $\mathbb{C}$ onto a proper subset of $\mathbb{C}$. This is a contradiction since there is no such mapping in one complex dimensional case. Hence, the Brody curve $\mathscr{B}_s$ should be all of the leaf Φ_s. The theorem is proved.

So far, we have proved the theorem for sufficiently large c. General case is obtained by making c in $\mathscr{C}_c$ large enough by applying f sufficiently many times so that we can apply the above argument. Then, Lemma VI.17 implies that the image of a Brody curve under f^{-N} is still a Brody curve for a finite N. This proves the general case. $\square$

For the rest of this paper, $|s|$ or equivalently c is assumed to be large enough as in Section 4.2.

[**Part 1**] We prove the following Proposition.

Proposition VI.10. *There exists a Brody curve $\mathscr{B}_s$ in Φ_s.*

We prove this theorem modifying the Brody reparametrization lemma. For the detail of the Brody reparametrization lemma, we refer the readers to [14], [26]. We modify the lemma so that the mappings in the family of the lemma have a fixed point and the mappings still form a normal family. Then, we apply the same technique as in the lemma. The limit mapping in the conclusion defines a Brody curve $\mathscr{B}_s$ in Φ_s as desired. We start by 3 lemmas. Lemma VI.12 will be proved later.

Lemma VI.11. *Consider a point $(z_K, 0) \in \Phi_s \cap V^+$. Let $\theta_n \in \Delta$ be such that $\Psi_s^n(\theta_n) = (z_K, 0)$ for each n. Then $ds((z_K, 0), d\Psi_s^n(\frac{d}{d\theta}|_{\theta=\theta_0})) \to \infty$ as $n \to \infty$. In a different notation, $\|(\Psi_s^n)'(\theta_n)\| \to \infty$ as $n \to \infty$.*

Proof. We are considering the Fubini-Study metric at a fixed point $(z_K, 0)$ and $(z_K, 0) \in V^+$. Thus, by Lemma VI.15, it suffices to show that $|(\Psi_s^n)_2'(\theta_n)| \to \infty$ as $n \to \infty$. In the proof of Theorem V.3, we proved the uniform divergence to ∞.

□

Lemma VI.12. *The sequence $\left\{ \dfrac{\sup_{|\theta|<1} ds(\Psi_s^n(\theta), d\Psi_s^n(\frac{d}{d\theta}))}{ds((z_K, 0), d\Psi_s^n(\frac{d}{d\theta}))} \right\}$ of the ratios as a sequence of n is bounded above. We call this bound M_s.*

Lemma VI.13. *Let $\xi : \mathbb{C} \to \mathscr{C}_c$ be a holomorphic mapping. Then, the image $\xi(\mathbb{C})$ should be inside a single leaf of the foliation of $\mathscr{C}_c$.*

Proof. Suppose, to the contrary, that the image does not lie inside a single leaf. Then, we can find an open subset $U \subseteq \mathbb{C}$ with compact closure such that $\xi(U)$ is not contained in a single leaf. Since $\xi(U)$ has compact closure and $\xi(U) \subseteq \mathscr{C}_c \subseteq U^+$, we can find a sufficiently large number n such that $f^n(\xi(U)) \subseteq V^+$. Note that $f^n(\xi(U))$ does not still sit inside a single leaf. We consider a holomorphic function $\varphi_+ \circ f^n \circ \xi : U \to \mathbb{C}$, where φ_+ is a holomorphic function defined over V^+ obtained in Proposition IV.1. Since $f^n(\xi(U))$ does not belong to a single leaf, $\varphi_+ \circ f^n \circ \xi : U \to \mathbb{C}$ is a non-constant holomorphic function over the open subset $U \subseteq \mathbb{C}$. So, the Open Mapping theorem implies that $\varphi_+ \circ f^n \circ \xi(U)$ should be open in $\mathbb{C}$. However, $\xi(U) \subseteq \mathscr{C}_c$ means that $\log|\varphi_+ \circ f^n \circ \xi(U)| = d^n c$, and therefore, $\varphi_+ \circ f^n \circ \xi(U)$ is a subset of a circle $|z| = e^{d^n c}$. This is a contradiction. Thus, Lemma VI.13 is proved.

□

The proof of Proposition VI.10. We modify the Brody reparametrization lemma. A brief description of the difference is that instead of moving a point of the maximum derivative modulus to $\theta = 0$ in the Brody reparametrization lemma, we will move a given fixed point to $\theta = 0$. However, by Lemma VI.12, we still have a compact family with respect to the compact-open topology, and moreover, we do have a fixed point. A precise argument comes below. We will follow the computation in [14].

Recall our parametrizations $\{\Psi_s^n\}_{n=0}^{\infty}$ of $\Phi_s^n \subseteq \Phi_s$ for $n = 0, 1, 2, \ldots$. By rescaling $\theta \to (1-\epsilon)\theta$ for a very small $\epsilon > 0$, we may assume that our parametrizations are holomorphic over $\overline{\Delta}$.

Let $H_n : \overline{\Delta} \to \mathbb{R}^+$ be defined by $H_n(\theta) = \|(\Psi_s^n)'(\theta)\|\,(1-|\theta|^2)$. As defined above, let $\theta_n \in \Delta$ be a point such that $\Psi_s^n(\theta_n) = (z_K, 0)$ for a prescribed fixed point $(z_K, 0) \in \Phi_s^0 \subseteq \Phi_s \cap V^+$. We define a sequence of Möbius transformation $\mu_n(\omega) = \frac{\omega+\theta_n}{1+\overline{\theta}_n\omega}$. Consider a sequence $\{g_n\}$ of mappings $g_n : \Delta \to \mathscr{C}_c$ defined by $g_n = \Psi_s^n \circ \mu_n$. Then,

$$\|g_n'(\omega)\|\,(1-|\omega|^2) = \|(\Psi_s^n)'(\theta)\|\,|\mu_n'(\omega)|\,(1-|\omega|^2) = \|(\Phi_s^n)'(\theta)\|\,(1-|\theta|^2).$$

By Lemma VI.12, we have

$$\|g_n'(\omega)\| \leq \frac{M_s\,\|g_n'(0)\|}{1-|\omega|^2},$$

where M_s denotes a bound for the sequence in Lemma VI.12.

Let $R_n = \|g_n'(0)\|$. By the definition of our parametrization Ψ_s^n in Section 4.4, $\theta_n \in \Delta$ such that $\Psi_s^n(\theta_n) = (z_K, 0)$ is the same as $\theta_n = \frac{f_2^n(z^+,0)}{f_1^n(z^+,0)}$. Since $f^n(z_+, 0)$ converges to I_-, θ_n converges to 0. Then, $|\mu_n'|$ converges to 1 and so, Lemma VI.11

implies $R_n \to \infty$. We define $k_n(\theta) := g_n(\frac{\theta}{R_n})$.

Then, over $\Delta_{R_n/2}$, we have

$$\|k_n'(\theta)\| = \frac{\|g_n'(\theta/R_n)\|}{R_n} \leq \frac{M_s \|g_n'(0)\|}{R_n} \cdot \frac{1}{1-|\theta/R_n|^2} \leq 2M_s$$

Thus, the set $\{k_n'(\theta)\}$ of derivatives is uniformly bounded and $k_n(\theta)$ has a fixed point, and therefore, $\{k_n(\theta)\}$ is a normal family. Moreover, $\|k_n'(0)\| = 1$. Thus, the limit maps of $\{k_n(\theta)\}$ are non-constant Brody maps.

We denote a limit map by Ψ_s and its image by $\mathscr{B}_s$. Notice that $\mathscr{B}_s$ is a holomorphic curve and passes through the fixed point $(z_K, 0)$. We have $\mathscr{B}_s \subseteq \overline{\Phi_s} \subseteq \mathscr{C}_c \cup I^+$ by Proposition III.3. By Theorem III.4, we have $\mathscr{B}_s \subseteq \mathscr{C}_c$. So, we can apply Lemma VI.13. Hence, the limit map should sit in one single leaf $\Phi_{s'}$ for some s'. However, the fixed point implies the image of the limit map should sit inside Φ_s. This proves the lemma. $\square$

[**Part 2**] We prove the injectivity of the limit map. Let $\Psi_s : \mathbb{C} \to \Phi_s$ be the limit map obtained through the Brody technique as in Part 1. Without loss of generality, we may assume that $k_n \to \Psi_s$, where $\{k_n\}$ is a convergent subsequence of the sequence in Part 1. It suffices to prove that for $\theta_a, \theta_b \in \mathbb{C}$, $\Psi_s(\theta_a) \neq \Psi_s(\theta_b)$.

Let $r > 0$ be such that $\theta_a, \theta_b \in \Delta_r \subseteq \mathbb{C}$. Let U_{Δ_r} denote an open subset of $\mathbb{C}^2$ with compact closure with respect to the standard $\mathbb{C}^2$ topology such that $\Psi_s(\Delta_r) \subseteq U_{\Delta_r}$. Since I_- is a super-attracting point, there exist a large number $N \in \mathbb{N}$ such that $f^N(\overline{U_{\Delta_r}}) \subseteq V^+$. $\theta = 0$ is the fixed point for the sequence of the functions $\{k_n(\theta)\}$ and therefore, we can find another large number $N' \in \mathbb{N}$ such that for all $n \geq N'$,

$k_n \to \Psi_s$ is uniform over Δ_r and $k_n(\Delta_{\frac{3}{2}r}) \subseteq U_{\Delta_r}$.

Note that $f^N(\Psi_s(\Delta_r)) \subseteq V^+ \cap \Phi_{s^{d^N}}$ from Proposition VI.10 and $f^N(k_n(\Delta_r)) \subseteq V^+ \cap \Phi_{s^{d^N}}$. Recall the biholomorphism Ψ_+ between the ZW-coordinate chart and the XY-coordinate chart in Chapter IV. Since $f^N(\Psi_s(\Delta_{2r}))$ lives in $V^+ \cap \left\{\varphi_+ = s^{d^N}\right\}$, it suffices to show the injectivity of $\frac{f_2^N(\Psi_s(\theta))}{f_1^N(\Psi_s(\theta))}$. From the locally uniform convergence of $\{k_n\}$ to Φ_s, we have that

$$\frac{f_2^N(k_n(\theta))}{f_1^N(k_n(\theta))} \to \frac{f_2^N(\Psi_s(\theta))}{f_1^N(\Psi_s(\theta))}$$

is uniform for $n \geq N'$. The functions $\left\{\frac{f_2^N(k_n(\theta))}{f_1^N(k_n(\theta))}\right\}$ in the sequence are a priori injective over Δ_r from our definition of parametrizations. The limit map is not constant. Hence, with Hurwitz's theorem, we have just proved that $\frac{f_2^N(\Psi_s(\theta))}{f_1^N(\Psi_s(\theta))}$ is injective over Δ_r. This proves the injectivity of Ψ_s.

6.3 Range of z in Terms of the Parameter s

We revise the proof of Inequality 4.2 to obtain a sharper range of $z \in V^+ \cap \Phi_s$ in terms of s. Recall that the leaf Φ_s satisfies $\{\varphi_+(z,w) = s\} \cap V^+ = \Phi_s \cap V^+$. Over $\Phi_s \cap V^+$, we have

$$\left(1 - \frac{c_1}{R^{c_2}}\right)^{1/(d-1)} \leq \left|\frac{s}{z}\right| \leq \left(1 + \frac{c_1}{R^{c_2}}\right)^{1/(d-1)}$$

where $c_1, c_2 > 0$ are the corresponding constants to Inequality 4.2. Note that c_1, c_2 are independent of R, s, and z. Then, we obtain the range of z as follows:

$$\frac{|s|}{(1 + \frac{c_1}{R^{c_2}})^{1/(d-1)}} \leq |z| \leq \frac{|s|}{(1 - \frac{c_1}{R^{c_2}})^{1/(d-1)}}.$$

By Condition 5 on R and Condition 5 on s, $\dfrac{|s|}{(1 + \frac{c_1}{R^{c_2}})^{1/(d-1)}} > R$. So, we can use

$\dfrac{|s|}{(1+\frac{\alpha}{R^\beta})^{1/(d-1)}}$ as our new R. Then,

$$\left(1-\frac{c_1(1+\frac{c_1}{R^{c_2}})^{c_2/(d-1)}}{|s|^{c_2}}\right)^{1/(d-1)} \le \left|\frac{s}{z}\right| \le \left(1+\frac{c_1(1+\frac{c_1}{R^{c_2}})^{c_2/(d-1)}}{|s|^{c_2}}\right)^{1/(d-1)}.$$

Here, c_1, c_2, R are independent of s, z but only dependent of $p(z), a$. Thus, we can redefine $c_1, c_2 > 0$ to write the above inequality in a simpler form:

$$(1-\frac{c_1}{|s|^{c_2}})^{1/(d-1)} \le \left|\frac{s}{z}\right| \le (1+\frac{c_1}{|s|^{c_2}})^{1/(d-1)}. \tag{6.1}$$

6.4 The Proof of Lemma VI.12

We will actually prove a more general statement as follows:

Lemma VI.14. $\frac{\sup_{|\theta|<1}\left\|(\Psi_s^{n,i})'(\theta)\right\|}{\inf_{\theta\in\Theta_s^{n,i}}\left\|(\Psi_s^{n,i})'(\theta)\right\|}$ *is uniformly bounded for any i, n with $0 \le i \le n$.*

For the definition of the set $\Theta_s^{n,i}$, see Section 4.4. Observe that the case where $i = 0$ implies Lemma VI.12. The main idea is the following. Since $f^n(\Phi_s^n) \subseteq V^+$, the behavior of the map $\Psi_s^{n,n}$ is quite regular and $\Theta_s^{n,n} = \{|\theta| < 1\}$. So, it is not difficult to find a bound of the ratio $\frac{\sup_{|\theta|<1}\left\|(\Psi_s^{n,n})'(\theta)\right\|}{\inf_{|\theta|<1}\left\|(\Psi_s^{n,n})'(\theta)\right\|}$. Then, we investigate the change of the quantity under the iteration of f^{-1}. The detail follows below.

By Condition 1 on s that $\Phi_s \cap W = \emptyset$, f^{-1} acts on $f^i(\Phi_s^n)$ for i with $1 \le i \le n$ in only three ways: i) sending a point in V^+ to a point in V^+, ii) sending a point in V^+ to a point in V^-, and iii) sending a point in V^- to a point in V^-. So, if a point escapes from V^+ during the n-times iteration of f^{-1}, the point moves from V^+ to V^- exactly only once during the n-times iteration and after that the point stays in V^- for the rest of the iteration. Thus, we want to show that

Statement 1 for the action of f^{-1} on $f^i(\Phi_s^n)$, the maximum increasing rate of the Fubini-Study metric in the case $V^- \to V^-$ is slower than the minimum increasing rate of the Fubini Study metric in the case $V^+ \to V^+$, and

Statement 2 the maximum increasing rate of the Fubini-Study metric of the case $V^+ \to V^-$ is bounded by a constant multiple of the case $V^+ \to V^+$, where the *increasing rate of the Fubini-Study metric* means the numerical quantity $\frac{\left\|(f^{-1}(\Psi_s^{n,i}))'(\theta)\right\|}{\left\|(\Psi_s^{n,i})'(\theta)\right\|}$.

The constant in Statement 2 is independent of the parametrization, n, and the number of actions of f^{-1}, i. We denote the constant in the Statement 2 by C_s. Then, once the above two statements are proved, we have

$$\frac{\sup_{|\theta|<1} \|(\Psi_s^{n,i})'(\theta)\|}{\inf_{\theta\in\Theta_s^{n,i}} \left\|(\Psi_s^{n,i})'(\theta)\right\|} \leq C_s \frac{\sup_{\theta\in\Theta_s^{n,i}} \|(\Psi_s^{n,i})'(\theta)\|}{\inf_{\theta\in\Theta_s^{n,i}} \left\|(\Psi_s^{n,i})'(\theta)\right\|}.$$

Then, it only remains to prove that

Statement 3 $\dfrac{\sup_{\theta\in\Theta_s^{n,i}} \|(\Psi_s^{n,i})'(\theta)\|}{\inf_{\theta\in\Theta_s^{n,i}} \left\|(\Psi_s^{n,i})'(\theta)\right\|}$ is uniformly bounded.

Hence, these three statements prove Lemma VI.12.

We move on to the proofs of the statements. Before proving the statements, we prepare some useful lemmas.

Lemma VI.15. *Let $|s|$ be a sufficiently large number as chosen previously, and i, n arbitrary numbers such that $1 \leq i \leq n$. Let $(z(\theta), w(\theta))$ denote $f^i(\Psi_s^n(\theta))$. Then, for any $\theta \in \Theta_s^{n,i}$, we have $|w'| \gg |z'|$ and*

$$\frac{|w'|^2}{2(1+|z|^2+|w|^2)} \leq \frac{|z'|^2+|w'|^2+|z'w-zw'|^2}{(1+|z|^2+|w|^2)^2} \leq \frac{3\,|w'|^2}{1+|z|^2+|w|^2} \tag{6.2}$$

Proof. Note that $(z(\theta), w(\theta)) \in V^+$, $|w| \leq |z|$ and from Lemma V.10 and Condition 6 on R, $|w'| \gg |z'|$ in V^+. Then,

$$\frac{|z'|^2+|w'|^2+|z'w-zw'|^2}{(1+|z|^2+|w|^2)^2} \geq \frac{|w'|^2+\frac{1}{2}\,|zw'|^2}{(1+|z|^2+|w|^2)^2} \geq \frac{|w'|^2}{2(1+|z|^2+|w|^2)}$$

On the other hand,

$$\frac{|z'|^2+|w'|^2+|z'w-zw'|^2}{(1+|z|^2+|w|^2)^2} \leq \frac{|z'|^2+|w'|^2+2(|z'w|^2+|zw'|^2)}{(1+|z|^2+|w|^2)^2}$$
$$\leq 2\frac{|z'|^2+|w'|^2}{1+|z|^2+|w|^2} \leq \frac{3\,|w'|^2}{1+|z|^2+|w|^2}$$

□

The following proposition estimates a lower bound of the increasing ratio of the Fubini-Study metric in the case $V^+ \to V^+$.

Proposition VI.16. *For $\theta \in \Theta_s^{n,i-1}$, the infimum of the increasing ratio $\inf_{\theta\in\Theta_s^{n,i-1}} \frac{\left\|(f^{-1}\Psi_s^{n,i})'(\theta)\right\|}{\left\|(\Psi_s^{n,i})'(\theta)\right\|}$ is bounded below by $\frac{d^2}{128\cdot 15^2|a|^2}\left|s^{d^i}\right|^{4-4/d}$.*

Proof. We compute $\inf_{\theta\in\Theta_s^{n,i-1}} \frac{\left\|(f^{-1}\Psi_s^{n,i})'(\theta)\right\|}{\left\|(\Psi_s^{n,i})'(\theta)\right\|}$ in terms of i. Let $\theta \in \Theta_s^{n,i-1}$. For a simpler computation, we use (z,w) for $\Psi_s^{n,i}(\theta)$, (z',w') for $(\Psi_s^{n,i})'(\theta)$, and (z_*,w_*) for $f^{-1}(\Psi_s^{n,i}(\theta))$. Then, $(z,w) \in f^i(\Phi_s) = \Phi_{s^{d^i}}$, $|w| \leq |z|$, $|z| > R$, $|z'| \ll |w'|$ and $(z,w),(z_*,w_*) \in V^+$.

Since $(z_*,w_*) \in V^+$, $|w| \geq \frac{1}{a}\,|p(w)-z|$. Then $|p(w)|+|aw| \geq |z| \geq |p(w)|-|aw|$, which gives the approximation of w in terms of s together with Inequality 6.1 and our choice of $R>0$ as follows:

$$\frac{3}{2}\left|s^{d^i}\right| \geq |z| \geq |p(w)|-|aw| \geq \frac{3}{4}\,|w|^d$$
$$\frac{5}{4}\,|w|^d \geq |p(w)|+|aw| \geq |z| \geq \frac{2}{3}\left|s^{d^i}\right|$$

Thus, we have $\left|2s^{d^i}\right|^{\frac{1}{d}} \geq |w| \geq \left|\frac{8s^{d^i}}{15}\right|^{\frac{1}{d}}$.

With this range of w, we compute a lower bound of the increasing rate of the

Fubini-Study metric in terms of the parameter s.

$$\begin{aligned}\left\|(\Psi_s^{n,i})'(\theta)\right\| &= \frac{|z|^2+|w|^2+|z'w-zw'|^2}{(1+|z|^2+|w|^2)^2} \leq \frac{|z'|^2+|w'|^2+2(|z'w|^2+|zw'|^2)}{(1+|z|^2+|w|^2)^2} \\ &\leq 2\frac{|z'|^2+|w'|^2}{1+|z|^2+|w|^2} \leq \frac{3\,|w'|^2}{|z|^2}.\end{aligned}$$

By Lemma VI.15, we have

$$\begin{aligned}\left\|(f^{-1}\Psi_s^{n,i})'(\theta)\right\| &= \frac{|w'|^2+\left|-\frac{1}{a}z'+\frac{p'(w)}{a}w'\right|^2+\left|\frac{1}{a}(p(w)-z)w'-w(-\frac{1}{a}z'+\frac{p'(w)}{a}w')\right|^2}{(1+|w|^2+\left|\frac{1}{a}(p(w)-z)\right|^2)^2} \\ &\geq \frac{\left|-\frac{1}{a}z'+\frac{p'(w)}{a}w'\right|^2}{6\,|w|^2} = \frac{|p'(w)w'|^2}{12\,|a|^2\,|w|^2}\end{aligned}$$

$\inf_{\theta\in\Theta_s^{n,i-1}}\frac{\left\|(f^{-1}\Psi_s^{n,i})'(\theta)\right\|}{\left\|(\Psi_s^{n,i})'(\theta)\right\|}$ is bounded by $\frac{|p'(w)|^2\,|z|^2}{36\,|a|^2\,|w|^2}$. So, by Inequality 6.1 and the range of w,

$$\begin{aligned}\frac{|p'(w)|^2\,|z|^2}{36\,|a|^2\,|w|^2} &\geq \frac{\left|\frac{3d}{4}\,|w|^{d-1}\right|^2|z|^2}{36\,|a|^2\,|w|^2} = \frac{d^2}{64\,|a|^2}\,|w|^{2d-4}\,|z|^2 \\ &\geq \frac{d^2}{128\,|a|^2}\left|\frac{8s^{d^i}}{15}\right|^{2-4/d}\left|s^{d^i}\right|^2 \geq \frac{d^2}{128\cdot 15^2\,|a|^2}\left|s^{d^i}\right|^{4-4/d}.\end{aligned}$$

□

[**Statement 1**] We will be proving the following:

Proposition VI.17. *For $1\leq i\leq n$, consider $\theta_-\in\Delta$ such that $\Psi_s^{n,i}(\theta_-)\in V^-$. The ratio $\frac{\|(f^{-1}\Psi_s^{n,i})'(\theta_-)\|}{\left\|(\Psi_s^{n,i})'(\theta_-)\right\|}$ is uniformly bounded by $2\,|a|^2\left(\frac{20}{9}\right)^2(1+d+\frac{1}{d})^2\frac{(1+2R^2)^2}{|R|^{2d}}$.*

Proof. To such θ_-, $\Psi_s^{n,i}$ assigns a point $\Psi_s^{n,i}(\theta_-)=f^i(\Psi_s^n(\theta_-))\in V^-$. We have $|(\Psi_s^{n,i})_1(\theta_-)|\leq|(\Psi_s^{n,i})_2(\theta_-)|$ and $|(\Psi_s^{n,i})_2(\theta_-)|>R$. For the simplicity of the calculation, we will use (z,w) for $\Psi_s^{n,i}(\theta_-)$ and (z',w') for $(\Psi_s^{n,i})'(\theta_-)$. Then, we have $|z|\leq|w|$ and $|w|\geq R$.

We want to find the ratio $\dfrac{\|(f^{-1}\Psi_s^{n,i})'(\theta_-)\|}{\left\|(\Psi_s^{n,i})'(\theta_-)\right\|}$.

$$\left\|(\Psi_s^{n,i})'(\theta_-)\right\| = \frac{|z'|^2 + |w'|^2 + |z'w - zw'|^2}{(1 + |z|^2 + |w|^2)^2} \geq \frac{|z'|^2 + |w'|^2}{(1 + 2|w|^2)^2}$$

On the other hand, by our choice of $R > 0$,

$$\begin{aligned}
\left\|(f^{-1}\Psi_s^{n,i})'(\theta_-)\right\| &= \frac{|w'|^2 + \left|\frac{z'}{a} - \frac{p'(w)w'}{a}\right|^2 + \left|w(-\frac{z'}{a} + \frac{p'(w)w'}{a}) - \frac{p(w)-z}{a}w'\right|^2}{(1 + |w|^2 + \left|\frac{1}{a}(p(w) - z)\right|^2)^2} \\
&\leq |a|^2 \frac{|aw'|^2 + (|z'| + |p'(w)w'|)^2 + [|wz'| + (|p'(w)w| + |p(w)| + |w|)|w'|]^2}{(|a|^2 + |aw|^2 + |p(w) - z|^2)^2} \\
&\leq |a|^2 \frac{|a|^2 + 2|p'(w)|^2 + (|w| + |p'(w)w| + |p(w)| + |w|)^2}{(|a|^2 + |aw|^2 + |p(w) - z|^2)^2} \max\{|z'|, |w'|\}^2 \\
&\leq 2|a|^2 \frac{[(1 + d + \frac{1}{d})|p(w)| + 2|w|]^2}{(\frac{3}{4}|w|^d)^4} \max\{|z'|, |w'|\}^2 \\
&\leq 2|a|^2 (1 + d + \frac{1}{d})^2 \frac{(\frac{5}{4}|w|^d)^2}{(\frac{3}{4}|w|^d)^4} \max\{|z'|, |w'|\}^2
\end{aligned}$$

The second last inequality is due to Condition 1, Condition 4, and Condition 7 on R, and the last inequality is due to the Condition 1 on R. Thus, the ratio is bounded by $2|a|^2 \left(\dfrac{20}{9}\right)^2 (1 + d + \dfrac{1}{d})^2 \dfrac{(1 + 2R^2)^2}{|R|^{2d}}$. □

By Condition 4 on s, this quantity is bounded by the increasing rate of the Fubini-Study metric of the case $V^+ \to V^+$.

[Statement 2] For $\theta \in \Theta_s^{n,i} \setminus \Theta_s^{n,i-1}$, we have $\Psi_s^{n,i}(\theta) \in V^+$ but $f^{i-1}(\Psi_s^n(\theta)) = f^{-1}(\Psi_s^{n,i}(\theta)) \in V^-$. Indeed, $\Theta_s^{n,i} \setminus \Theta_s^{n,i-1}$ is the set of points which just escaped out of V^+ to V^- at the i-th iteration. A more precise statement to prove is the following:

Proposition VI.18. *For all i, n such that $1 \leq i \leq n$, there exists a constant $C_s > 0$ independent of i, n such that*

$$\sup_{\theta \in \Theta_s^{n,i} \setminus \Theta_s^{n,i-1}} \frac{\|(f^{-1}\Psi_s^{n,i})'(\theta)\|}{\left\|(\Psi_s^{n,i})'(\theta)\right\|} \leq C_s \inf_{\theta \in \Theta_s^{n,i-1}} \frac{\|(f^{-1}\Psi_s^{n,i})'(\theta)\|}{\left\|(\Psi_s^{n,i})'(\theta)\right\|}.$$

Proof. Let $\theta \in \Theta_s^{n,i} \setminus \Theta_s^{n,i-1}$. Then, $\Psi_s^{n,i}(\theta) \in V^+$ but $f^{-1}(\Psi_s^{n,i})(\theta) \in V^-$. For a simpler computation, we substitute the notations (z,w) for $\Psi_s^{n,i}(\theta)$, (z',w') for $(\Psi_s^{n,i})'(\theta)$, (z_*,w_*) for $f^{-1}(\Psi_s^{n,i}(\theta))$, and (z'_*,w'_*) for $(f^{-1}(\Psi_s^{n,i})'(\theta))$. Since $\Psi_s^{n,i}(\theta) \in V^+$, we have $|w| \le |z|$, $|z'| \ll |w'|$, and $(z,w) \in \Phi_{s^{d^i}}$. Since $f^{-1}(\Psi_s^{n,i}) = \Psi_s^{n,i-1} \in V^-$, we have $|w| \le \left|\frac{p(z)-w}{a}\right|$.

Recall that

$$\begin{pmatrix} z'_* \\ w'_* \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{a} & \frac{p'(w)}{a} \end{pmatrix} \begin{pmatrix} z' \\ w' \end{pmatrix}.$$

$$\begin{aligned}
\left\|(f^{-1}\Psi_s^{n,i})'(\theta)\right\| &= \frac{|w'|^2 + \left|-\frac{1}{a}z' + \frac{p'(w)}{a}w'\right|^2 + \left|\frac{1}{a}(p(w)-z)w' - w(-\frac{1}{a}z' + \frac{p'(w)}{a}w')\right|^2}{(1+|w|^2+\left|\frac{1}{a}(p(w)-z)\right|^2)^2} \\
&\le \frac{|w'|^2 + (\left|-\frac{1}{a}z'\right| + \left|\frac{p'(w)}{a}w'\right|)^2 + (\left|\frac{1}{a}(p(w)-z)w' - \frac{p'(w)w}{a}w'\right| + \left|\frac{w}{a}z'\right|)^2}{(1+|w|^2+\left|\frac{1}{a}(p(w)-z)\right|^2)^2} \\
&\le 2\cdot\frac{|w'|^2 + \left|-\frac{1}{a}z'\right|^2 + \left|\frac{p'(w)}{a}w'\right|^2 + \left|\frac{1}{a}(p(w)-z)w' - \frac{p'(w)w}{a}w'\right|^2 + \left|\frac{w}{a}z'\right|^2}{(1+|w|^2+\left|\frac{1}{a}(p(w)-z)\right|^2)^2} \\
&\le 2\cdot\frac{(2+2\left|\frac{p'(w)}{a}\right|^2 + \left|\frac{1}{a}(p(w)-z) - \frac{p'(w)w}{a}\right|^2)\,|w'|^2}{(1+|w|^2+\left|\frac{1}{a}(p(w)-z)\right|^2)^2} \\
&\le 2\cdot\frac{(2+2\left|\frac{p'(w)}{a}\right|^2 + 2\left|\frac{1}{a}(p(w)-z)\right|^2 + 2\left|\frac{p'(w)}{a}w\right|^2)\,|w'|^2}{(1+|w|^2+\left|\frac{1}{a}(p(w)-z)\right|^2)^2} \\
&\le 4\cdot\frac{((1+|w|^2+\left|\frac{1}{a}(p(w)-z)\right|^2) + (-|w|^2 + \left|\frac{p'(w)}{a}\right|^2 + \left|\frac{p'(w)}{a}w\right|^2))\,|w'|^2}{(1+|w|^2+\left|\frac{1}{a}(p(w)-z)\right|^2)^2} \\
&\le \frac{4\,|w'|^2}{1+|w|^2+\left|\frac{1}{a}(p(w)-z)\right|^2} + 4\cdot\frac{-|w|^2 + \left|\frac{p'(w)}{a}\right|^2 + \left|\frac{p'(w)}{a}w\right|^2}{(1+|w|^2+\left|\frac{1}{a}(p(w)-z)\right|^2)^2}\,|w'|^2.
\end{aligned}$$

From Lemma VI.15, we have

$$\left\|(\Psi_s^{n,i})'(\theta)\right\| \ge \frac{|w'|^2}{2(1+|z|^2+|w|^2)} \ge \frac{|w'|^2}{6\,|z|^2}.$$

Thus, the ratio $\frac{\left\|(f^{-1}\Psi_s^{n,i})'(\theta)\right\|}{\left\|(\Psi_s^{n,i})'(\theta)\right\|}$ is bounded above by

$$\frac{24\left|z\right|^2}{1+\left|w\right|^2+\left|\frac{1}{a}(p(w)-z)\right|^2}+24\cdot\frac{(-\left|w\right|^2+\left|\frac{p'(w)}{a}\right|^2+\left|\frac{p'(w)}{a}w\right|^2)\left|z\right|^2}{(1+\left|w\right|^2+\left|\frac{1}{a}(p(w)-z)\right|^2)^2}. \tag{6.3}$$

Note that due to Condition 3 on r, we have $R<\left|\frac{s^{d^i}}{5}\right|^{\frac{1}{d}}$. We compute the upper bound of Quantity 6.3 in 4 cases:

Case 1 $|w|<R$,

Case 2 $R\le|w|<\left|\frac{s^{d^i}}{5}\right|^{\frac{1}{d}}$,

Case 3 $\left|\frac{s^{d^i}}{5}\right|^{\frac{1}{d}}\le|w|<\left|\frac{10s^{d^i}}{3}\right|^{\frac{1}{d}}$, and

Case 4 $\left|\frac{10s^{d^i}}{3}\right|^{\frac{1}{d}}\le|w|$.

[**Case 1**] Let $M_R=\max_{|w|\le R}\left\{-\left|w\right|^2+\left|\frac{p'(w)}{a}\right|^2+\left|\frac{p'(w)w}{a}\right|^2\right\}$. From Condition 1 on s and Inequality 6.1, $|z|-|p(w)|\ge\left|\frac{s^{d^i}}{4}\right|$. Then, Quantity 6.3 is bounded above by

$$[\ 6.3]\le\frac{24\left|z\right|^2}{\left|\frac{1}{a}(|z|-|p(w)|)\right|^2}+\frac{24\left|z\right|^2M_R}{\left|\frac{1}{a}(|z|-|p(w)|)\right|^4}\le\frac{48\left|s^{d^i}\right|^2}{\left|\frac{s^{d^i}}{4a}\right|^2}+\frac{48\left|s^{d^i}\right|^2M_R}{\left|\frac{s^{d^i}}{4a}\right|^4},$$

due to Inequality 6.1 and $|z|-|p(w)|\ge\left|\frac{s^{d^i}}{4}\right|$. The last quantity is bounded for all i.

[**Case 2**] From $R\le|w|<\left|\frac{s^{d^i}}{5}\right|^{\frac{1}{d}}$ and Inequality 6.1, we have $|p(w)|+2\left|aw\right|\le\frac{5}{4}\left|w\right|^d\le\left|\frac{s^{d^i}}{4}\right|\le\frac{1}{2}\left|z\right|$, which means $|z|-|p(w)|\ge\left|\frac{s^{d^i}}{4}\right|$.

Then, the bound 6.3 satisfies

$$\begin{aligned}
[\,6.3] \quad &\leq \frac{24\,|z|^2}{\left|\frac{1}{a}(|z|-|p(w)|)\right|^2} + \frac{24\,|z|^2\,(-\,|w|^2 + \left|\frac{p'(w)}{a}\right|^2 + \left|\frac{p'(w)}{a}w\right|^2)}{\left|\frac{1}{a}(|z|-|p(w)|)\right|^4} \\
&\leq \frac{24\,|a|^2\,|z|^2}{(|z|-|p(w)|)^2} + \frac{24\,|a|^2\,|z|^2\,(\frac{5}{4}d\,|w|^{d-1})^2(1+|w|^2)}{(|z|-|p(w)|)^4} \\
&\leq \frac{48\,|a|^2\left|s^{d^i}\right|^2}{\left|\frac{s^{d^i}}{4}\right|^2} + \frac{6d^2\,|a|^2\left|s^{d^i}\right|^4}{\left|\frac{s^{d^i}}{4}\right|^4} = 4^2\cdot 48\,|a|^2 + 4^4\cdot 6d^2\,|a|^2\,,
\end{aligned}$$

due to Condition 3 on R, Inequality 6.1, $R \leq |w| < \left|\frac{s^{d^i}}{5}\right|^{\frac{1}{d}}$, and $|z| - |p(w)| \geq \left|\frac{s^{d^i}}{4}\right|$.

[Case 4] From $\left|\frac{10s^{d^i}}{3}\right|^{\frac{1}{d}} \leq |w|$ and Inequality 6.1, we have $|p(w)| - 2\,|aw| \geq \frac{3}{4}\,|w|^d \geq \frac{5}{2}\left|s^{d^i}\right| \geq 2\,|z|$, which means $|p(w)| - |z| \geq \left|\frac{s^{d^i}}{2}\right|$

Then, the bound 6.3 satisfies

$$\begin{aligned}
[\,6.3] \quad &\leq \frac{24\,|z|^2}{\left|\frac{1}{a}(|p(w)|-|z|)\right|^2} + \frac{24\,|z|^2\,(-\,|w|^2 + \left|\frac{p'(w)}{a}\right|^2 + \left|\frac{p'(w)}{a}w\right|^2)}{\left|\frac{1}{a}(|p(w)|-|z|)\right|^4} \\
&\leq \frac{24\,|a|^2\,|z|^2}{(|p(w)|-|z|)^2} + \frac{24\,|a|^2\,|z|^2\,(\frac{5}{4}d\,|w|^{d-1})^2(1+|w|^2)}{(|p(w)|-|z|)^4} \\
&\leq \frac{48\,|a|^2\left|s^{d^i}\right|^2}{\left|\frac{s^{d^i}}{2}\right|^2} + \frac{12\,|a|^2\,(\frac{3}{4}\,|w|^d)^2(\frac{5}{4}d\,|w|^d)^2}{(\frac{1}{2}\,|p(w)|)^4}.
\end{aligned}$$

The denominator and the numerator of the very last term have the same order of w and the denominator is not 0 for $|w| \geq \left|\frac{10s^{d^i}}{3}\right|^{\frac{1}{d}}$. This means that Quantity 6.3 is bounded by a constant.

[Case 3] If $\left|\frac{s^{d^i}}{5}\right|^{\frac{1}{d}} \leq |w| < \left|\frac{10s^{d^i}}{3}\right|^{\frac{1}{d}}$, then

$$\begin{aligned}
[\,6.3] \quad \leq \quad & \frac{24\,|z|^2}{|w|^2} + \frac{24\,|z|^2\,(-\,|w|^2 + \left|\frac{p'(w)}{a}\right|^2 + \left|\frac{p'(w)}{a}w\right|^2)}{|w|^4} \\
\leq \quad & \frac{24\,|z|^2}{|w|^2} + \frac{24\,|z|^2\,(\frac{5}{4}d\,|w|^{d-1})^2(1+|w|^2)}{|a|^2\,|w|^4} \\
\leq \quad & 48 \cdot 5^{\frac{2}{d}}\left|s^{d^i}\right|^{2-\frac{2}{d}} + \frac{15000d^2 \cdot 5^{\frac{4}{d}}\left|s^{d^i}\right|^{4-\frac{4}{d}}}{9\,|a|^2}
\end{aligned}$$

due to Inequality 6.1.

We compare the bounds of Quantity 6.3 obtained from the 4 cases to the infimum of the increasing rate from Lemma VI.16. Since the bounds of Quantity 6.3 obtained from the 4 cases have the same or less order of s^{d^i} than the infimum of the increasing rate from Lemma VI.16. The number from Lemma VI.16 is not 0. Thus, we can find the maximum ratio of the bounds of Quantity 6.3 to the infimum of the increasing rate from Lemma VI.16. This maximum ratio is the desired constant. So, the Statement is proved. Independence is clear from the proof. $\square$

[Statement 3] We will prove the following:

Proposition VI.19. $\frac{\sup_{\theta\in\Theta_s^{n,i}}\left\|(\Psi_s^{n,i})'(\theta)\right\|}{\inf_{\theta\in\Theta_s^{n,i}}\left\|(\Psi_s^{n,i})'(\theta)\right\|}$ *is uniformly bounded for* $0 \leq i \leq n$.

We first prove that $\frac{\sup_{\theta\in\Theta_s^{n,i}}\left|(\Psi_s^{n,i})'_2(\theta)\right|}{\inf_{\theta\in\Theta_s^{n,i}}\left|(\Psi_s^{n,i})'_2(\theta)\right|}$ is uniformly bounded for $0 \leq i \leq n$ and prove Proposition VI.19 using Inequality 6.1 and Lemma VI.15.

Proposition VI.20. *For all* $n \in \mathbb{N}$,

$$\frac{\sup_{\theta\in\Theta_s^{n,n}}|(\Psi_s^{n,n})'_2(\theta)|}{\inf_{\theta\in\Theta_s^{n,n}}|(\Psi_s^{n,n})'_2(\theta)|} \leq \left(\frac{1+\frac{c_1}{|s|^{c_2}}}{1-\frac{c_1}{|s|^{c_2}}}\right)^{1/(d-1)}\frac{R+1}{R-1}.$$

Proof. Recall the definition of our parametrization. Note that $\Psi_s^{n,n}(\theta) \in \Phi_{s^{d^n}} \cap V^+$ for $\theta \in \Delta$. For simplicity, we denote $\Psi_s^{n,n}(\theta)$ by (z,w) or $(z(\theta),w(\theta))$ for $\theta \in \Delta$. We have

$$\begin{cases} (\varphi_+)_z \cdot z' + (\varphi_+)_w \cdot w' = 0 \\ w = zt \end{cases}$$

Differentiating and solving this system for w' in z, w, we have $w' = \dfrac{z^2}{z - \mu w}$, where μ is very small as in Lemma V.10.

$$\frac{|z|}{1+|\mu|} = \frac{|z|^2}{|z| + |\mu z|} \le |w'| \le \frac{|z|^2}{|z| - |\mu z|} = \frac{|z|}{1-|\mu|}$$

From Inequality 6.1, we have

$$\frac{\sup_{|\theta|<1} |w'|}{\inf_{|\theta|<1} |w'|} \le \left(\frac{1 + \frac{c_1}{\left|s^{d^n}\right|^{c_2}}}{1 - \frac{c_1}{\left|s^{d^n}\right|^{c_2}}} \right)^{1/(d-1)} \frac{1+|\mu|}{1-|\mu|} \le \left(\frac{1 + \frac{c_1}{|s|^{c_2}}}{1 - \frac{c_1}{|s|^{c_2}}} \right)^{1/(d-1)} \frac{R+1}{R-1},$$

and therefore it is bounded independently of n. □

Lemma VI.21. *There exists a fixed constant $C > 0$ such that* $\dfrac{\sup_{\Theta_s^{n,i}} |(\Psi_s^{n,i})_2'|}{\inf_{\Theta_s^{n,i}} \left|(\Psi_s^{n,i})_2'\right|} \le C \dfrac{\sup_{\Theta_s^{n,n}} |(\Psi_s^{n,n})_2'|}{\inf_{\Theta_s^{n,n}} \left|(\Psi_s^{n,i})_2'\right|}$ *for i, n with $0 \le i \le n-1$. C is independent of i, n.*

For the proof of Lemma VI.21, we need the following lemma.

Lemma VI.22. *For all $\theta \in \Theta_s^{n,i}$ and $i \le j \le n-1$,*

$$\frac{|p'((\Psi_s^{n,j})_1)| - 1}{|a|} \left|(\Psi_s^{n,j+1})_2'\right| \le \left|(\Psi_s^{n,j})_2'\right| \le \frac{|p'((\Psi_s^{n,j})_1)| + 1}{|a|} \left|(\Psi_s^{n,j+1})_2'\right|.$$

Proof. It is a direct result from our choice of $R > 0$ and

$$df^{-1}|_{\Psi_s^{n,j+1}(\theta)} = \begin{pmatrix} p'((\Psi_s^{n,j})_1) & -a \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{a} & \frac{p'((\Psi_s^{n,j})_1)}{a} \end{pmatrix}.$$

Note that $|(\Psi_s^{n,j+1})_1'| \ll |(\Psi_s^{n,j+1})_2'|$ for $\theta \in \Theta_s^{n,i}$ by Lemma V.10. □

The proof of Lemma VI.21. For the simplicity of the proof, we use (z, w) for $\Psi_s^n(\theta)$, (z_i, w_i) for $\Psi_s^{n,i}(\theta)$, (z', w') for $\Psi_s^n(\theta)'$, and (z_i', w_i') for $\Psi_s^{n,i}(\theta)'$. Then $(z, w) \in \Phi_s$ and

$(z_i, w_i) \in \Phi_{s^{d^i}}$. From the previous lemma, we have

$$\frac{\max_{\Theta_s^{n,j}} |w_j'|}{\min_{\Theta_s^{n,j}} |w_j'|} \leq \frac{\max_{\Theta_s^{n,j}} |p'(z_j)| + 1}{\min_{\Theta_s^{n,j}} |p'(z_j)| - 1} \cdot \frac{\max_{\Theta_s^{n,j+1}} |w_{j+1}'|}{\min_{\Theta_s^{n,j+1}} |w_{j+1}'|}.$$

Multiple use of this inequality gives us

$$\frac{\max_{\Theta_s^{n,i}} |w_i'|}{\min_{\Theta_s^{n,i}} |w_i'|} \leq \frac{\max_{\Theta_s^{n,n}} |w_n'|}{\min_{\Theta_s^{n,n}} |w_n'|} \prod_{j=i+1}^{n-1} \frac{\max_{\Theta_s^{n,j}} |p'(z_j)| + 1}{\min_{\Theta_s^{n,j}} |p'(z_j)| - 1}. \tag{6.4}$$

Consider i, n such that $0 \leq i \leq n$. For any $\theta_a, \theta_b \in \Theta_s^{n,i}$ and z_a, z_b with $z_a = \Psi_s^{n,i}(\theta_a) \in \Phi_{s^{d^i}}, z_b = \Psi_s^{n,i}(\theta_b) \in \Phi_{s^{d^i}}$ respectively, by Inequality 6.1, we have

$$\frac{1 - \frac{c_1}{|s^{d^i}|^{c_2}}}{1 + \frac{c_1}{|s^{d^i}|^{c_2}}} \leq \frac{(1 - \frac{c_1}{|s^{d^i}|^{c_2}})^{1/(d-1)}}{(1 + \frac{c_1}{|s^{d^i}|^{c_2}})^{1/(d-1)}} \leq \frac{|z_a|}{|z_b|} \leq \frac{(1 + \frac{c_1}{|s^{d^i}|^{c_2}})^{1/(d-1)}}{(1 - \frac{c_1}{|s^{d^i}|^{c_2}})^{1/(d-1)}} \leq \frac{1 + \frac{c_1}{|s^{d^i}|^{c_2}}}{1 - \frac{c_1}{|s^{d^i}|^{c_2}}}.$$

Note that this inequality is independent of n, i.

We want to express the $\varepsilon > 0$ such that $\left|\frac{z_a - z_b}{z_b}\right|^2 < \varepsilon$.

$$\begin{aligned} |z_a - z_b|^2 &= |z_a|^2 + |z_b|^2 - 2 z_a \cdot z_b \\ &\leq \left| |z_a|^2 + |z_b|^2 - 2|z_a||z_b| \sqrt{1 - \left(\frac{2c_1}{(d-1)|s^{d^i}|^{c_2}}\right)^2} \right| \\ &\leq (|z_a| - |z_b|)^2 + 2|z_a||z_b| \left(\frac{2c_1}{(d-1)|s^{d^i}|^{c_2}}\right). \end{aligned}$$

Note that in the computation of $z_a \cdot z_b$, we used the computation of the bound of the angle in Proposition IV.1 and the fact that $\sin x \leq x$ for small positive x.

Thus,

$$\begin{aligned} \left|\frac{z_a - z_b}{z_b}\right|^2 &\leq \left(\left|\frac{z_a}{z_b}\right| - 1\right)^2 + 4\left(\frac{c_1}{(d-1)|s^{d^i}|^{c_2}}\right)\left|\frac{z_a}{z_b}\right| \\ &\leq \left(\frac{1 + \frac{c_1}{|s^{d^i}|^{c_2}}}{1 - \frac{c_1}{|s^{d^i}|^{c_2}}} - 1\right)^2 + 8\left(\frac{c_1}{(d-1)|s^{d^i}|^{c_2}}\right) \\ &\leq \frac{C_{ab}}{|s^{d^i}|^{c_2}} = \varepsilon, \end{aligned}$$

where $C_{ab} \geq 0$ is a constant independent of z_a, z_b, i, n.

Let $p'(z) = dz^{d-1} + q'(z)$ according to our notation. Let $M_{q'}$ be defined by $M_{q'} = \max_{1/2 \leq x \leq 3/2} |q'|(x) < \infty$. Then,

$$\frac{|p'(z_a)|+1}{|p'(z_b)|-1} = \frac{|(p'(z_a)-p'(z_b))+p'(z_b)|+1}{|p'(z_b)|-1} \leq \frac{|p'(z_a)-p'(z_b)|+2}{|p'(z_b)|-1}+1$$

By Condition 3 on R and $|z_b| > R$, we have $|p'(z_b)| - 1 \geq \frac{3}{4} d |z_b|^{d-1}$. So,

$$\begin{aligned}
\frac{|p'(z_a)-p'(z_b)|+2}{|p'(z_b)|-1}+1 &\leq \frac{4}{3}\frac{\left|d(z_a^{d-1}-z_b^{d-1})+(q'(z_a)-q'(z_b))\right|+2}{d\,|z_b|^{d-1}}+1 \\
&\leq \frac{4}{3}\left|\left(\frac{z_a}{z_b}\right)^{d-1}-1\right| + \frac{4}{3}\frac{M_{q'}+2}{d\,|z_b|}+1 \\
&\leq \frac{4}{3}((\sqrt{\varepsilon}+1)^{d-1}-1) + \frac{8}{3}\frac{M_{q'}+2}{d\,\left|s^{d^i}\right|}+1 \\
&\leq \frac{C}{\left|s^{d^i}\right|^{\max\left\{\frac{1}{2}c_2,1\right\}}}+1,
\end{aligned}$$

where C is a constant independent of z_a, z_b, i, n. Considering the same argument for the reciprocal, we know that $\left|\left|\frac{|p'(z_a)|+1}{|p'(z_b)|-1}\right|-1\right|$ is $o(\left|s^{d^i}\right|^{-\max\left\{\frac{1}{2}c_2,1\right\}})$. Since $\sum_{i=1}^{\infty}\frac{1}{(s^{d^i})^{\max\left\{\frac{1}{2}c_2,1\right\}}}$ converges, we have the convergence of Infinite Product 6.4 to a finite number by Theorem IV.2. From the setting, we know that this finite number ≥ 1.

This completes the proof of Lemma VI.21. $\square$

The proof of Proposition VI.19. So far, in Lemmas VI.20 and VI.21, we have proved that $\frac{\sup_{\theta\in\Theta_s^{n,i}}\left|(\Psi_s^{n,i})_2'(\theta)\right|}{\inf_{\theta\in\Theta_s^{n,i}}\left|(\Psi_s^{n,i})_2'(\theta)\right|}$ is uniformly bounded for $0 \leq i \leq n$.

By Lemma VI.15,

$$\frac{\sup_{\theta\in\Theta_s^{n,i}}\left\|(\Psi_s^{n,i})'(\theta)\right\|}{\inf_{\theta\in\Theta_s^{n,i}}\left\|(\Psi_s^{n,i})'(\theta)\right\|} \leq 9\frac{\sup_{\theta\in\Theta_s^{n,i}}\left|(\Psi_s^{n,i})_1\right|}{\inf_{\theta\in\Theta_s^{n,i}}\left|(\Psi_s^{n,i})_1\right|}\cdot\frac{\sup_{\theta\in\Theta_s^{n,i}}\left|(\Psi_s^{n,i})_2'(\theta)\right|}{\inf_{\theta\in\Theta_s^{n,i}}\left|(\Psi_s^{n,i})_2'(\theta)\right|}$$

Finally, Inequality 6.1 implies the desired boundedness. $\square$

CHAPTER VII

Short $\mathbb{C}^2$

Due to the Riemann Mapping Theorem, we know that there is no proper biholomoprhic image of $\mathbb{C}$ in $\mathbb{C}$. However, this is no longer true in higher dimension. Fatou and Bieberbach found such example in [11] and [13]. One of the methods to construct such a domain is using complex dynamics; for example, Fatou-Bieberbach domains can be obtained as basins of attraction for Hénon mappings. In these cases, the boundary behavior can be very nasty. In [3], Beford and Smillie proved that the boundary is not a topological manifold anywhere if a Hénon mapping has at least three attracting fixed points. However, if we are not using Hénon mappings, then we can obtain much better boundary regularity. in [30], Stensønes used a sequence of shear maps to construct a Fatou-Bieberbach domain with smooth boundary. A natural question to ask is the existence of a Fatou-Bieberbach domain with real analytic boundary. Also, in Stensønes's example, the boundary is foliated by Riemann surfaces. What kind of Riemann surface would they be?

In Stensønes's construction and the construction using a basin of attraction for Hénon mappings, the resulting Fatou-Bieberbach domains are unions of holomorphic balls and their Kobayashi pseudometric is identically 0. It was an interesting question what an increasing union of holomorphic balls with Kobayashi metric identically 0 would be. In [15], it turned out that it may not be biholomorphic to $\mathbb{C}^k$. Such domains are called short $\mathbb{C}^k$.

It is interesting to think of the same boundary regularity questions for short $\mathbb{C}^k$'s as for Fatou-Bieberbach domains. We consider the boundary behavior of short $\mathbb{C}^2$ domains. Can they have real analytic boundary? Can they be foliated by complex curves? If so, what kind of Riemann surfaces can the leaves be? Indeed, according to Theorem 1.12 in [15], the set $K_c = \{g < c\}$ for $c > 0$ is a short $\mathbb{C}^2$. Since g is pluri-harmonic in U^+, the boundary $\partial K_c = \mathscr{C}_c$ is real analytic. Moreover, due to our main result, we know that the boundary is foliated by injective Brody curves. Summarizing the results, we have proved that

Corollary VII.1. *K_c with $c > 0$ is a short $\mathbb{C}^2$ with real analytic boundary. The boundary is foliated by injective Brody curves.*

CHAPTER VIII

Conclusion and Further Directions

Recall the definitions and notations for our Hénon mapping f, the Green function g associated to f, and the level set $\mathscr{C}_c = \{g = c\}$. In previous chapters, we first studied the shape of the level set $\mathscr{C}_c$ for $c > 0$ of the Green function g in $\mathbb{P}^2$, looking at its behavior at the line at ∞. We recalled that the closure of $\mathscr{C}_c$ in $\mathbb{P}^2$ is $\mathscr{C}_c \cup I_+$, where I_+ is the set of indeterminancy when f is projectivised. Then, we showed that there is no holomorphic curve in $\mathbb{P}^2$ which passes through I_+ at the line at ∞ and which is supported in $\overline{\mathscr{C}_c}$. Roughly speaking, in some sense, it implies that near I_+, $\mathscr{C}_c$ is so narrow that $\overline{\mathscr{C}_c}$ in $\mathbb{P}^2$ cannot contain any holomorphic curves of the form $z^\alpha = w^\beta$ for $\alpha, \beta \in \mathbb{Z}$ with respect to the local affine coordinate chart centered at I_+.

Next, we reviewed the foliation structure of the level set $\mathscr{C}_c$ in $\mathbb{C}^2$ for $c > 0$ of the Green function g; it known that $\mathscr{C}_c$ is foliated by Riemann surfaces biholomorphic to $\mathbb{C}$ and each leaf is dense in the level set. Then, our second theorem answers the question what kind of Riemann surface the leaves would be: injective Brody curves. The meaning of our theorem can be understood as follows. In the sense that the leaf is not algebraic and is dense in the level set, one might expect that the shape of the curve is very complicated. Actually, however, it is not too complicated in the following respect. Whenever we biholomorphically parametrize an injective Brody curve, it has a uniformly bounded speed of expansion with respect to the Fubini-Study metric. For better understanding, one may consider a holomorphic curve parametrized by the unit disc in $\mathbb{C}$ and its natural metric, the Kobayashi metric. Then, due to the distance decreasing property, we see that every such curve parametrized by the unit disc in $\mathbb{C}$ shows tame behavior with respect to the Kobayashi metric. On the contrary, we do not have such metric with the distance decreasing property for Riemann surfaces biholomorphic to $\mathbb{C}$. In this aspect, injective Brody curves can be understood as Riemann surfaces biholomorphic to $\mathbb{C}$ with tame behavior with respect to a given metric. Also, as a consequence of our theorem, we found an example of the short $\mathbb{C}^2$ domains with real analytic boundary and with its boundary foliated by injective Brody curves.

For further study, we look at the intuition behind the computation. Recall our filtration and note that in this paragraph, when we are talking about the Fubini-Study metric at a point, we consider a local holomorphic curve through that point and its parametrization and mean the Fubini-Study metric of the derivative of the parametrization at that point. The essential parts in the computation are the following three: the vertical shape of each leaf of $\mathscr{C}_c$ in V^+ for sufficiently large c (see Statement 3 in Chapter VI), the property that when f moves a point from V^+ to V^-, the increase of the Fubini-Study metric at that point by f is not too much (see Statement 2 in Chapter VI), and the property that the Fubini-Study metric increases by a very small amount or sometimes even decreases in V^- under f (see Statement 1 in Chapter VI). These three properties contribute to generating a fixed point for a family of parametrizations, keeping the normal family property. Note that f decreases the Fubini-Study metric near I_+. The reason why we can use the above three properties to reparametrize a given family of holomorphic curves so that the modified family shares a fixed point seems to be that we do not have recurrence in U^+; once a point has escaped V^+, then, after a finite time, the point reaches V^-, escapes to I_+, and never comes back to W nor V^+.

As a further direction, we can ask the same question about Brody foliation or injective Brody foliation in another situation. It is known that the boundary $J^+ = \partial K^+$ of non-escaping points also has foliation structure and its leaf is biholomorphic to $\mathbb{C}$. In [3], [2], and [16], they approached it in two different ways; in [3] and [2], Bedford, Lyubich, and Smillie approached using hyperbolicity and in [16], Fornæss and Sibony considered a certain type of Hénon mapping of the form $(z, w) \to (z^2 + c - a^2 w, z)$ as a perturbation of a logistic map $(z, w) \to (z^2 + c, 0)$. The set $J = \partial K^+ \cap \partial K^- \subseteq J^+$ is invariant under f and f^{-1}, and therefore, we have recurrence. So, it is an interesting question whether such foliation in J^+ is also Brody folation or even injective Brody foliation. Namely, it is a question whether recurrence affects the foliation structure.

Concerning Brody foliation, we know that there is a Brody curve in J^+ and that it intersects with J. Indeed, by the Stable Manifold theorem, we know that there exists a biholomorphic image of $\mathbb{C}$ in J^+. Due to the non-Kobayashi hyperbolicity, one can find a sequence of holomorphic curves parametrized by the unit disc in $\mathbb{C}$ and a sequence of its parametrizations, the sequence of whose derivative is not bounded with respect to the Fubini-Study metric. Apply the Brody reparametrization lemma. Then, a limit curve exists in the closure of the stable manifold which is $J^+ \cup I_+$. Due to Theorem III.4, it should stay inside the J^+. Now, we prove that the limit curve should intersect J. Since the limit curve does not lie in a level set of g^-, where g^- is the Green function associated to f^{-1}, the limit curve always intersect the set K^-. Indeed, let $\varphi : \mathbb{C} \to J^+$ be the parametrization of the limit curve, not necessarily injective. Then, $g^- \circ \varphi$ is a non-constant harmonic function. Then, it is a non-constant linear function, and therefore, $g^- \circ \varphi$ should have zero. This is a contradiction. In this perspective, recurrence may not be an obstacle to the existence of a Brody curve. However, at the moment, we still do not have any information about injectivity, its location, or whether it is the entire stable manifold. Recurrence seems to be an obstacle to getting the information about the location of a Brody curve.

As another further step, one can think of Fatou-Bieberbach domains. Since it is biholomorphic to $\mathbb{C}^2$, it contains a copy of $\mathbb{C}$. Then, we can ask the question whether such a Riemann surface is also Brody or even injective Brody.

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