Identities Relating Schur s-Functions and Q-Functions

by

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CHAPTER I

Introduction

Schur s-functions and Q-functions are two important families within the algebra Λ of symmetric functions with applications for other fields. The (non-skew) s-functions form a basis for Λ and are indexed by partitions (i.e. weakly decreasing sequences of non-negative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ converging to zero). They are fundamental objects in several fields: for example, they are the characters of irreducible polynomial representations of $GL_n(\mathbb{C})$, they encode the character tables of the symmetric groups, and they represent Schubert classes in the cohomology ring of the Grassmann manifold of k-planes in n-space. The skew s-functions are indexed by pairs of partitions λ/μ such that $\mu_i \leq \lambda_i$ for all i. They encode (not necessarily irreducible) representations that arise naturally, for example when branching from $GL(m+n)$ to $GL(m) \times GL(n)$. Both non-skew and skew s-functions have a combinatorial description as generating functions for semistandard tableaux.

Slightly less well known are the Q-functions. The (non-skew) Q-functions form a basis for a sub-algebra Ω of Λ and are indexed by strict partitions (i.e. partitions whose non-zero parts are strictly decreasing). They are also fundamental objects in several fields: for example, they are the characters of irreducible representations of a certain family of Lie superalgebras, they encode the character tables of the projective representations of the symmetric groups, and they represent Schubert classes in the cohomology ring of the isotropic Grassmannian in n-space. The skew Q-functions are indexed by pairs of strict partitions, and both non-skew and skew Q-functions are expressible as generating functions over shifted tableaux.

One of the stranger recent observations about skew s-functions is the discovery that there are unexpected collisions among them; that is, there are equalities $s_{\lambda/\mu} = s_{\alpha/\beta}$ with no combinatorially or algebraically obvious explanation , with the smallest such collision occurring in degree 8. The question of when two (possibly skew) s-functions are equal has been studied in a series of papers by van Willigenburg ([13]), Billera, Thomas and van Willigenburg ([3]), Reiner, Shaw, and van Willigenburg ([9]), and McNamara and van Willigenburg ([7]), but so far the results are not definitive and the complete answer appears to be complicated. McNamara and van Willigenburg ([7]) unify the work to date, providing a sufficiency condition explaining all currently known collisions and conjecturing a related necessary condition that may provide an algorithm for generating all s-function collisions.

Since Q-functions play a similar role in the sub-algebra Ω to the role of s-functions in Λ, and since both families have a combinatorial description in terms of tableux, it is natural to ask similar questions of Q-functions and the elements of Ω to those that are posed of s-functions within Λ. Whereas s-functions are monic, Q-functions are not; in determining equality up to constant multiple between possibly skew Q-functions, we begin in Chapter 2 by finding the leading coefficient of any skew Q-function (Theorem II.6). In Chapter 3 we turn to the question of equality up to constant multiple between an s-function and a Q-function, reducing the s-functions we need to consider by finding all the s-functions in Ω (Theorem III.5). In Chapter 4 we explore equality among Q-functions, and in particular, determine all the S-

functions that are a constant multiple of a single non-skew Q-function (Theorem IV.3). We return in Chapter 5 to relations between s- and Q-functions, providing a Q-expansion for all s-functions in Ω with non-negative coefficients (Theorem V.5) and determining all the s-functions that are a constant multiple of a single non-skew Q-function (Theorem V.3).

1.1 Partitions

Definition I.1. A *partition* λ is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots)$ of weakly decreasing non-negative integers containing finitely many non-zero terms. We will consider two partitions equivalent if they differ only by the number of zero terms.

Definition I.2. The *ith part of partition λ, λ_i,* is the ith largest term of λ.

Definition I.3. The *length of λ, $l(\lambda)$,* is the number of non-zero parts of λ.

Definition I.4. The *size of λ, $|\lambda|$,* is the sum of the parts of λ.

Definition I.5. A *strict partition* is a partition such that all non-zero parts are distinct, i.e. there does not exist an i such that $\lambda_i = \lambda_{i+1} \neq 0$.

Definition I.6. The *k-staircase partition, $\delta(k)$,* is the partition $(k, k-1, \ldots, 2, 1)$. For example, $\delta(5)$ is the partition 54321.

Definition I.7. On the set of partitions of size n, the *lexicographic order* is a total order such that, for partitions λ and μ, $\lambda \geq \mu$ if either $\lambda = \mu$ or the first non-zero difference $\lambda_i - \mu_i$ is positive.

Definition I.8. On the set of partitions of size n, the *dominance order* is a partial order such that, for partitions λ and μ, $\lambda \geq \mu$ if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for each $i \geq 1$.

1.2 Diagrams and Tableaux

Definition I.9. For any partition λ, we define the *(unshifted) diagram* $D(\lambda)$ to be $\{(i,j) \in \mathbb{Z}^2 : 1 \leq j \leq \lambda_i, i \geq 1\}$. As with matrices, we will use the convention that the (i,j) position of $D(\lambda)$ is the cell in the ith row from the top and the jth column from the left.

Notation I.10. We will use $\mu \subseteq \lambda$ to represent that the diagram of μ is contained in the diagram of λ, i.e. $\mu_i \leq \lambda_i$ for all $i \leq l(\mu)$.

Definition I.11. For any partitions λ and μ with $\mu \subseteq \lambda$, we define the *skew (unshifted) diagram* $D(\lambda/\mu)$ to be $D(\lambda) \backslash D(\mu)$ as sets, i.e. $\{(i,j) \in \mathbb{Z}^2 : \mu_i < j \leq \lambda_i, i \geq 1\}$.

Definition I.12. For a partition λ, we define *the conjugate of* λ, λ', to be the partition whose diagram $D(\lambda')$ interchanges the rows and columns of $D(\lambda)$; thus λ'_i is the number of cells in the ith column of $D(\lambda)$. In the skew case, we will use λ'/μ' to denote the conjugate of λ/μ, where $D(\lambda'/\mu') = D(\lambda') \setminus D(\mu')$.

Definition I.13. For any strict partition λ, we define the *shifted diagram* $D'(\lambda)$ to be $\{(i,j) \in \mathbb{Z}^2 : i \leq j \leq \lambda_i + i - 1, i \geq 1\}$, so that $D'(\lambda)$ may be viewed as the diagram formed when the ith row of the unshifted diagram $D(\lambda)$ is shifted $(i-1)$ positions to the right. As with matrices, we will use the convention that the (i,j) position of $D'(\lambda)$ is the cell in the ith row from the top and the jth column from the left. Since the rows are not left-justified and are instead a staircase-like shape, for all $D'(\lambda)$ if $i > j$ then $(i,j) \notin D'(\lambda)$.

Definition I.14. For any strict partitions λ and μ with $\mu \subseteq \lambda$, we define the *skew shifted diagram* $D'(\lambda/\mu)$ to be $D'(\lambda) \setminus D'(\mu)$ as sets, i.e. $\{(i,j) \in \mathbb{Z}^2 : \mu_i + i - 1 < j \leq \lambda_i + i - 1, i \geq 1\}$.

Definition I.15. The *kth diagonal* of $D'(\lambda/\mu)$, counting from left to right, is the set $\{(i, k{+}i{-}1) \in D'(\lambda/\mu)\}$. Viewing $D'(\lambda/\mu)$ as $D(\lambda/\mu)$ with the ith row shifted $(i-1)$ positions to the right, the kth diagonal of $D'(\lambda/\mu)$ is the image of the kth column of $D(\lambda/\mu)$, and so the length of (or number of cells in) the kth diagonal is $\lambda'_k - \mu'_k$. In particular, the *main* or *first diagonal* of $D'(\lambda/\mu)$ is the set $\{(i,i) \in D'(\lambda/\mu)\}$ and has length $l(\lambda) - l(\mu)$.

Definition I.16. We define any skew unshifted (or shifted) diagram $D(\lambda/\mu)$ (or $D'(\lambda/\mu)$) to be *connected* if one can move between any two cells in $D(\lambda/\mu)$ (or $D'(\lambda/\mu)$) through a series of horizontal and vertical steps while remaining within the diagram.

Definition I.17. For any partitions λ and μ with $\mu \subseteq \lambda$, an *(unshifted) tableau of shape* λ/μ is an assignment to the positions in $D(\lambda/\mu)$ of symbols from the ordered alphabet $\mathbf{P} = \{1 < 2 < 3 < \cdots\}$ such that the entries are weakly increasing left to right across each row right and are strictly increasing down each column.

Definition I.18. The *content* of (unshifted) tableau T is the sequence $(\alpha_1, \alpha_2, \dots)$ where α_a is the number of a entries in T for each $a \geq 1$.

Definition I.19. We define two unshifted shapes to be *combinatorially equivalent* if there exists a bijection between the positions such that each tableau of one shape corresponds to a valid tableau of the other. For example, two shapes are combinatorially equivalent if they differ by the insertion or deletion of a column or row of length zero, by the translation of connected components, by the reordering of connected components, or by some combination of these.

Notation I.20. We will let $\mathbf{P}'$ be the alphabet of ordered symbols $\{1' < 1 < 2' < 2 < \cdots\}$, where the entries $1, 2, \dots$ are said to be *unmarked* and the entries $1', 2', \dots$

are said to be *marked*. We will use a^* to represent a or a'.

Definition I.21. For any strict partitions λ and μ with $\mu \subseteq \lambda$, a *shifted tableau T of shape* λ/μ is an assignment to the positions in $D'(\lambda/\mu)$ of symbols from $\mathbf{P}'$ under the following conditions:

(a) The entries are weakly increasing left to right across each row and down each column;

(b) Each column contains at most one a for each $a \geq 1$;

(c) Each row contains at most one a' for each $a \geq 1$.

Definition I.22. The *content* of shifted tableaux T is the sequence $(\alpha_1, \alpha_2, \dots)$ where α_a is the number of entries a and a' in T for each $a \geq 1$.

Notation I.23. We will use $T(i,j)$ to represent the entry in the (i,j) position of unshifted or shifted tableau T. For example, using this notation the first condition in Definition I.21 is $T(i,j) \leq T(i,j+1)$ and $T(i,j) \leq T(i+1,j)$.

Definition I.24. In a shifted tableau T, an entry a^* in position (i,j) is *free* if neither $T(i+1,j)$ nor $T(i,j-1)$ is a^* and thus the assignment conditions in Definition I.21 allow the entry to be a (unmarked) or a' (marked). In particular, for a shifted tableau of diagram $D'(\lambda/\mu)$, the $l(\lambda) - l(\mu)$ main diagonal entries are free since there are no entries immediately below or to the left of the main diagonal entries.

Remark I.25. As with unshifted shapes in Definition I.19, we define two shifted shapes to be *combinatorially equivalent* if there exists a bijection between the positions such that each shifted tableau of one shape corresponds to a valid shifted tableau of the other. For example, two shapes are combinatorially equivalent if they differ by the insertion or deletion of a column or row of length zero, by the translation of connected components, by the reordering of connected components, or by some combination of these. Note that only one connected component can have a leftmost diagonal of length ≥ 2, and if there is one it cannot be translated off the main diagonal.

1.3 Symmetric Functions

Let $x_1, x_2, \ldots$ be a countable set of independent variables.

Definition I.26. For each $r \geq 1$ the *rth power sum* in $x_1, x_2, \ldots$ is

$$p_r = \sum_i x_i^r.$$

Definition I.27. Let $\Lambda_{\mathbb{Q}}$ be the $\mathbb{Q}$-algebra generated by the power sums:

$$\Lambda_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, p_3, \ldots].$$

It is well known that $\Lambda_{\mathbb{Q}}$ consists of all formal linear combinations of monomials in the x_i's that are symmetric under the permutation of the variables (e.g. [6], 1.2).

Definition I.28. Let $\Omega_{\mathbb{Q}}$ be the sub-algebra of $\Lambda_{\mathbb{Q}}$ generated by the odd power sums:

$$\Omega_{\mathbb{Q}} = \mathbb{Q}[p_1, p_3, p_5, \ldots].$$

Definition I.29. Let ω be the ring involution $\omega : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ such that $\omega(p_r) = (-1)^{r-1} p_r$.

Notation I.30. With each (unshifted or shifted) tableau T, we may associate a monomial x^T where if the content of T is $\alpha = (\alpha_1, \alpha_2, \ldots)$ then $x^T = x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$.

We can now define combinatorially the four classes of symmetric functions we will be discussing.

Definition I.31. For any unshifted diagram $D(\lambda/\mu)$,

(a) The *s-function* $s_{\lambda/\mu}$ is

$$s_{\lambda/\mu}(x_1, x_2, \ldots) = \sum_T x^T$$

summed over all unshifted tableaux T with diagram $D(\lambda/\mu)$.

(b) The *S-function* $S_{\lambda/\mu}$ is

$$S_{\lambda/\mu}(x_1, x_2, \dots) = \sum_T x^T$$

summed over all shifted tableaux T with unshifted diagram $D(\lambda/\mu)$, i.e. the positions included satisfy Definition I.11 but the entries are from $\mathbf{P'}$ and satisfy the conditions of Definition I.21.

Definition I.32. For any shifted diagram $D'(\lambda/\mu)$,

(a) The *Q-function* $Q_{\lambda/\mu}$ is

$$Q_{\lambda/\mu}(x_1, x_2, \dots) = \sum_T x^T$$

summed over all shifted tableaux T with diagram $D'(\lambda/\mu)$.

(b) The *P-function* $P_{\lambda/\mu}$ is

$$P_{\lambda/\mu}(x_1, x_2, \dots) = 2^{-l(\lambda)+l(\mu)} Q_{\lambda/\mu}(x_1, x_2, \dots).$$

Remark I.33. (a) An unshifted shape $D(\lambda/\mu)$ includes the same positions as the shifted shape $D'(\lambda + \delta(k)/\mu + \delta(k))$ where $k = l(\lambda) - 1$. Since the entries of $S_{\lambda/\mu}$ satisfy the conditions of Definition I.21, we have $S_{\lambda/\mu} = Q_{\lambda+\delta(k)/\mu+\delta(k)}$; for example, $S_{532/1} = Q_{742/31}$. Thus the S-functions are the skew Q-functions for shifted shapes with a single position on the main diagonal.

(b) For any shifted diagram $D'(\lambda/\mu)$, the $l(\lambda) - l(\mu)$ entries on the main diagonal are free, so we may equivalently define $P_{\lambda/\mu}$ in terms of the sum of monomials for shifted tableaux:

$$P_{\lambda/\mu}(x_1, x_2, \dots) = {\sum_T}' x^T$$

summed over shifted tableaux T with diagram $D'(\lambda/\mu)$ where the notation $\sum'$ indicates that the sum is only over those tableaux with unmarked entries on the main diagonal.

(c) Although not obvious from the definition, s-functions are indeed symmetric (e.g. [6], 1.5). In fact, the non-skew s-functions s_λ form a basis for $\Lambda_{\mathbb{Q}}$ (e.g. [6], 1.3). Note also that the s-functions of combinatorially equivalent shapes are obviously equal.

(d) Similarly, although it is not obvious from the definitions, Q-functions (and hence S- and P-functions as well) are symmetric and are elements of $\Omega_{\mathbb{Q}}$; in fact, the non-skew Q-functions Q_λ and P-functions P_λ are each bases of $\Omega_{\mathbb{Q}}$ (e.g. [6], 3.8). Again it is clear that the Q-functions of combinatorially equivalent shapes are equal (and similarly for the S- and P-functions).

CHAPTER II

The Leading Term of $Q_{\lambda/\mu}$

Both for the purpose of finding equalities between symmetric functions and because of the combinatorial significance of their coefficients, it is natural to inquire after the coefficients of individual terms.

Definition II.1. The *leading term* of a symmetric function is the unique monomial $x_1^{\alpha_1}x_2^{\alpha_2}\cdots$ such that the content $\alpha = (\alpha_1, \alpha_2, \dots)$ is greatest in lexicographic order. Note that α is necessarily a partition.

Definition II.2. The *column set* of $D(\lambda/\mu)$ is the multi-set of the column lengths in the skew diagram $D(\lambda/\mu)$, listed in no particular order. The *column list* of $D(\lambda/\mu)$ is the list of column lengths in the skew diagram $D(\lambda/\mu)$, listed in order from left to right with the first entry the length of the leftmost column.

Definition II.3. The *diagonal set* of $D'(\lambda/\mu)$ is the multi-set of the diagonal lengths in the skew shifted diagram $D'(\lambda/\mu)$, listed in no particular order. The *diagonal list* of $D'(\lambda/\mu)$ is the list of diagonal lengths in the skew shifted diagram $D'(\lambda/\mu)$, listed in order from left to right with the first entry the length of the main diagonal $\lambda'_1 - \mu'_1 = l(\lambda) - l(\mu)$.

For s-functions, it is well known that the leading term comes from a single tableau and so has a coefficient of 1. This tableau has a 1 in every column, a 2 in every column of length at least two, and an i in every column of length at least i for all i, so that the content is the conjugate of the column set partition; for non-skew s_λ this content is λ itself. Further, the content of this leading term is greatest in dominance order.

The focus of this chapter is:

Question II.4. *What is the coefficient of the leading term of a (possibly skew) Q-function?*

Recall from Definition I.32 that for strict partitions λ and μ the Q-function is:

$$Q_{\lambda/\mu}(x_1, x_2, \dots) = \sum_T x^T \tag{2.1}$$

summed over all shifted tableaux T with diagram $D'(\lambda/\mu)$. It is not hard to determine the content of the leading term and the unique shifted tableau (up to the markings of the free entries) associated for any Q-function. This tableau is formed by including an i^* in every diagonal of length at least i for all i, so that the content is the conjugate of the diagonal set partition; we have included a proof in Lemma II.10 below. Thus, the leading coefficient is a power of two, where the exponent is the number of free entries in this leading term tableau. In the case of non-skew Q_λ, the content of the leading term is λ itself and the leading coefficient is $2^{l(\lambda)}$. In the case of skew $Q_{\lambda/\mu}$ the number of free entries has a lower bound of $l(\lambda) - l(\mu)$, which is the length of the first diagonal, and so $2^{-l(\lambda)+l(\mu)}Q_{\lambda/\mu}$, which is $P_{\lambda/\mu}$ of Definition I.32, has coefficients in $\mathbb{Z}$ ([6], 3.8). The primary result of this chapter is a closed formula for this leading coefficient for a general skew Q-function based only on its shape.

Notation II.5. For strict partitions μ and λ such that $\mu \subseteq \lambda$, we will use $b(\lambda, \mu)$ to represent the number of parts that appear in both λ and μ. For example, for $\lambda = 87521$ and $\mu = 5421$, $b(\lambda, \mu) = 3$.

Theorem II.6. *For any shifted diagram $D'(\lambda/\mu)$,*

(a) the leading coefficient of $Q_{\lambda/\mu}$ is $2^{l(\lambda)-b(\lambda,\mu)}$, and

(b) $2^{-l(\lambda)+b(\lambda,\mu)}Q_{\lambda/\mu}$ has coefficients in $\mathbb{Z}$.

Note that the second part of this theorem allows us to define another type of symmetric functions,

$$\overline{Q}_{\lambda/\mu} = 2^{-l(\lambda)+b(\lambda,\mu)}Q_{\lambda/\mu},$$

which are all monic. In the non-skew case, $b(\lambda, \emptyset) = 0$ so that $\overline{Q}_\lambda = P_\lambda$.

2.1 Unmarked Shifted Tableaux and the Leading Term of $Q_{\lambda/\mu}$

Definition II.7. A *border strip* of $D'(\lambda/\mu)$ (or unshifted $D(\lambda/\mu)$) is a connected subset of the diagram such that if position (i, j) is in the border strip then $(i+1, j+1))$ is not.

Consider any two-by-two subtableau of a shifted tableau T of shape $D'(\lambda/\mu)$:

$$\begin{matrix} T(i-1, j-1) & T(i-1, j) \\ \circ & T(i, j). \end{matrix}$$

Note that if the entries $T(i-1, j-1)$ and $T(i, j)$ are on the main diagonal then the position marked $\circ$ is not in $D'(\lambda/\mu)$. The conditions of Definition I.21 require $T(i-1, j-1) \leq T(i-1, j) \leq T(i, j)$. Since there is at most one a in each column, if $T(i, j) = a^*$, then $T(i-1, j) \leq a'$. But since there is at most one a' in each row, $T(i-1, j-1) < a'$, and thus at most one entry in the diagonal is a^*.

Therefore, for a shifted tableau T the set of cells with entry a^* for each a is a disjoint union of border strips. For each border strip β with entries a^*, the assignment conditions of shifted tableaux uniquely determine whether each entry is marked or unmarked except for the cell closest to the main diagonal (the position (i, j) such that neither $T(i+1,j)$ nor $T(i,j-1)$ is a^*) which is free. Thus the number of border strips and the number of free entries in T are equal.

Notation II.8. We will use $fr(T)$ to represent the number of free entries in shifted tableau T.

Definition II.9. For any strict partitions λ and μ with $\mu \subseteq \lambda$, an *unmarked shifted tableau* T *of shape* λ/μ is an assignment to the positions in $D'(\lambda/\mu)$ of unmarked symbols from the set $P = (1 < 2 < 3 < \cdots)$ such that:

(a) The rows are weakly increasing, $T(i,j) \leq T(i,j+1)$;

(b) The columns are weakly increasing, $T(i,j) \leq T(i+1,j)$;

(c) The diagonals are strictly increasing, $T(i,j) < T(i+1,j+1)$.

As with marked shifted tableaux, we call the entry closest to the main diagonal in its border strip free. Every marked shifted tableau may be associated with an unmarked shifted tableau by simply removing the entry markings. Thus for each unmarked shifted tableau T there are $2^{fr(T)}$ marked shifted tableau with monomial x^T. Then for any $D'(\lambda/\mu)$, an alternative expression for the Q-function equivalent to (2.1) is:

$$Q_{\lambda/\mu} = \sum_T 2^{fr(T)} x^T \tag{2.2}$$

summed over all *unmarked* shifted tableaux T with diagram $D'(\lambda/\mu)$.

Lemma II.10. *There is exactly one unmarked shifted tableau of shape λ/μ whose content is that of the leading term of $Q_{\lambda/\mu}$. This content is greatest in dominance order and is the conjugate of the partition formed by the diagonal set of $D'(\lambda/\mu)$.*

Proof. For any shifted diagram $D'(\lambda/\mu)$, the unmarked tableau T with content α greatest in lexicographic order will need to maximize α_1. Since there can be at most one 1 in each diagonal, T must have exactly one 1 in every diagonal. Having maximized α_1, to then maximize α_2 there must be a 2 in every diagonal of length greater than or equal to two. Note that this also maximizes $\alpha_1 + \alpha_2$. To then maximize α_3, there must be a 3 in every diagonal of length greater than or equal to three, which then also maximizes $\alpha_1 + \alpha_2 + \alpha_3$, and so forth. Therefore, the unmarked tableau T such that x^T is the leading term of $Q_{\lambda/\mu}$ is unique, and the content of this unmarked tableau is greater in dominance order than the contents of all other unmarked tableau. Since the number of a entries in T is the number of diagonals of length at least a, the leading term of $Q_{\lambda/\mu}$ will have content equal to the conjugate of the partition that is the diagonal set of $D'(\lambda/\mu)$. □

2.2 Ascents and Descents of the Diagonal List of $D'(\lambda/\mu)$

For any strict partition λ, the length of the kth column in the unshifted diagram $D(\lambda)$ must be equal to or one more than the length of the $(k+1)$th column. Otherwise, if the kth column is shorter than the $(k+1)$th then λ is not a partition, and if the kth column has length at least two more than the length of $(k+1)$th column then λ is not strict. In particular, the rightmost (non-zero) column of λ must have length one, else $\lambda_1 = \lambda_2$. Therefore, there are four cases for the relative size and position of adjacent columns in the skew diagram $D(\lambda/\mu)$, and so, as $D'(\lambda/\mu)$ is the image of $D(\lambda/\mu)$ after shifting each row $(i-1)$ positions to the right, there are four cases for the relative length and position of adjacent diagonals in the shifted diagram $D'(\lambda/\mu)$. In the following sections, we denote the length of the kth diagonal of $D'(\lambda/\mu)$ by d_k. In Figure 2.1 below we provide an example of each case involving the first and second diagonal of $D'(\lambda/\mu)$.

Case 1: $\lambda'_k = \lambda'_{k+1}$ and $\mu'_k = \mu'_{k+1} + 1$. In this case $d_k = d_{k+1} - 1$, so that as we read from left to right, there is an *ascent* in the diagonal list of $D'(\lambda/\mu)$ from the kth element to the $(k+1)$th element.

Case 2: $\lambda'_k = \lambda'_{k+1}$ *and* $\mu'_k = \mu'_{k+1}$. In this case $d_k = d_{k+1}$. Since the diagonals are of equal length, there is neither a descent nor ascent in the diagonals list of $D'(\lambda/\mu)$ from the kth element to the $(k+1)$th element. We refer to this as a *horizontal step* between diagonals of equal length.

Case 3: $\lambda'_k = \lambda'_{k+1} + 1$ *and* $\mu'_k = \mu'_{k+1} + 1$. In this case $d_k = d_{k+1}$. Since the diagonals are of equal length, there is neither a descent nor ascent in the diagonals list of $D'(\lambda/\mu)$ from the kth element to the $(k+1)$th element. We refer to this as a *vertical step* between diagonals of equal length.

Case 4: $\lambda'_k = \lambda'_{k+1} + 1$ *and* $\mu'_k = \mu'_{k+1}$. In this case $d_k = d_{k+1} + 1$, so that as we read from left to right, there is a *descent* in the diagonal list of $D'(\lambda/\mu)$ from the kth element to the $(k+1)$th element.

Notation II.11. Let $desc(\lambda/\mu)$ be the number of descents in the diagonal list of $D'(\lambda/\mu)$ (i.e. the size of set $\{i|d_i > d_{i+1}\}$), read from left to right, including the descent from the rightmost diagonal of positive length to the right-adjacent diagonal of length 0.

Notation II.12. Let $asc(\lambda/\mu)$ be the number of ascents in the diagonal list of $D'(\lambda/\mu)$ (i.e. the size of set $\{i|d_i < d_{i+1}\}$), read from left to right.

Lemma II.13. *For any (possibly skew) shifted diagram $D'(\lambda/\mu)$,*

(a) $l(\lambda) - l(\mu) = desc(\lambda/\mu) - asc(\lambda/\mu)$*;*

(b) $asc(\lambda/\mu) = l(\mu) - b(\lambda, \mu)$*.*

Proof. We proceed by induction on the number of diagonals. When the number of diagonals is one, then the length of this single diagonal must be one and $\lambda/\mu = 1/\emptyset$.

Thus

$$l(\lambda) - l(\mu) = 1 = desc(\lambda/\mu) - asc(\lambda/\mu),$$

so hypothesis (a) holds. Since $\mu = \emptyset$ the length of μ and $b(\lambda, \mu)$ are both 0. Thus

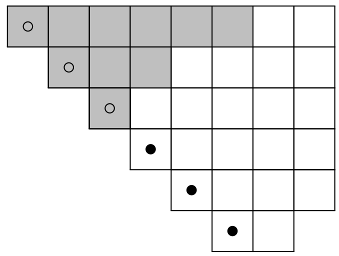
$$asc(\lambda/\mu) = 0 = l(\mu) - b(\lambda, \mu),$$

so hypothesis (b) holds.

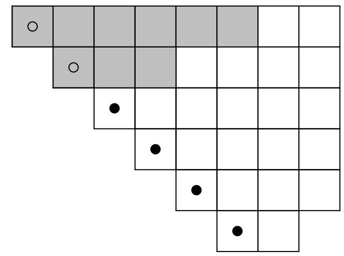
Now assume that the hypotheses hold for all (possibly skew) shifted diagrams with $n-1$ or fewer diagonals. Given any shifted diagram $D'(\lambda/\mu)$ with n diagonals, let $D'(\lambda/\mu)^{\geq 2}$ be the shifted diagram we get by deleting the first diagonal of $D'(\lambda/\mu)$. Since $D'(\lambda/\mu)^{\geq 2}$ has $n-1$ diagonals, by assumption the hypotheses hold. Let k be the length of the first diagonal of $D'(\lambda/\mu)^{\geq 2}$. $D'(\lambda/\mu)$ can be reformed by prepending a diagonal to $D'(\lambda/\mu)^{\geq 2}$, that is by shifting every row of $D'(\lambda/\mu)^{\geq 2}$ to the right one position and adding the first diagonal of $D'(\lambda/\mu)$ to the left of the first diagonal of $D'(\lambda/\mu)^{\geq 2}$. We will show that the hypotheses hold when the first and second diagonals of $D'(\lambda/\mu)$ satisfy each of the four possibilities for the relative size and position of adjacent diagonals in shifted diagrams. See Figure 2.1 for an example of each case where $D'(\lambda/\mu)^{\geq 2} = D'(765431/52)$ which has diagonal list $[4,3,4,3,2,2,1]$ with one ascent and five descents.

Case 1: $d_1 = k-1 < k = d_2$ in $D'(\lambda/\mu)$. In this case the number of ascents in the diagonal list of $D'(\lambda/\mu)$ is one more than in the diagonal list of $D'(\lambda/\mu)^{\geq 2}$. Since each position in $D'(\lambda/\mu)^{\geq 2}$ is shifted to the right by one, to form λ each part of $\lambda^{\geq 2}$ is increased by one. To form μ each part of $\mu^{\geq 2}$ is increased by one and an additional part of length one is appended. Therefore,

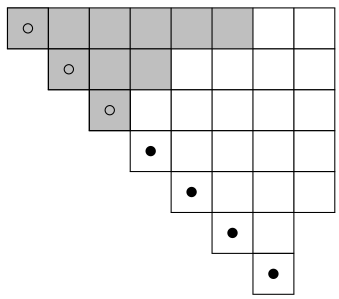
$$\begin{aligned} l(\lambda) - l(\mu) &= l(\lambda^{\geq 2}) - \Big(l(\mu^{\geq 2}) + 1\Big) = desc((\lambda/\mu)^{\geq 2}) - \Big(asc((\lambda/\mu)^{\geq 2}) + 1\Big) \\ &= desc(\lambda/\mu) - asc(\lambda/\mu), \end{aligned}$$



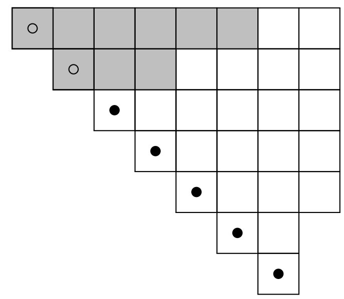
Case 1: $d_1 = d_2 - 1$, $\lambda/\mu = 876542/631$



Case 2: $d_1 = d_2$ with Horizontal Step, $\lambda/\mu = 876542/63$



Case 3: $d_1 = d_2$ with Vertical Step, $\lambda/\mu = 8765421/631$



Case 4: $d_1 = d_2 + 1$, $\lambda/\mu = 8765421/63$

Figure 2.1: The Four Shifted Diagrams $D'(\lambda/\mu)$ with $D'(\lambda/\mu)^{\geq 2} = D'(765431/52)$

and so hypothesis (a) holds. Since to form λ every existing part of $\lambda^{\geq 2}$ is increased by one and no additional part is appended, there are no parts of length one in λ, so that the part of length one appended to form μ does not contribute to $b(\lambda, \mu)$. Further, since every existing part of $\lambda^{\geq 2}$ and $\mu^{\geq 2}$ are increased by one to form λ and μ, every pair of equal parts remain equal, so that $b(\lambda, \mu) = b(\lambda^{\geq 2}, \mu^{\geq 2})$. Therefore,

$$\begin{aligned} asc(\lambda/\mu) &= 1 + asc((\lambda/\mu)^{\geq 2}) = 1 + l(\mu^{\geq 2}) - b(\lambda^{\geq 2}, \mu^{\geq 2}) \\ &= l(\mu) - b(\lambda, \mu), \end{aligned}$$

and hypothesis (b) holds.

Case 2: $d_1 = k = d_2$ in $D'(\lambda/\mu)$ with a horizontal step between the first and second diagonals. In this case the number of ascents and descents in the diagonal list of $D'(\lambda/\mu)$ are unchanged from those of $D'(\lambda/\mu)^{\geq 2}$. To form λ and μ, the shifting of each row of $D'(\lambda/\mu)^{\geq 2}$ to the right by one increases each part of $\lambda^{\geq 2}$ and $\mu^{\geq 2}$ by one; with a horizontal step no additional parts are appended to form either λ or μ.

Therefore,

$$l(\lambda) - l(\mu) = l(\lambda^{\geq 2}) - l(\mu^{\geq 2}) = desc((\lambda/\mu)^{\geq 2}) - asc((\lambda/\mu)^{\geq 2})$$
$$= desc(\lambda/\mu) - asc(\lambda/\mu),$$

so hypothesis (a) holds. Further, we have $b(\lambda, \mu) = b(\lambda^{\geq 2}, \mu^{\geq 2})$, so

$$asc(\lambda/\mu) = asc((\lambda/\mu)^{\geq 2}) = l(\mu^{\geq 2}) - b(\lambda^{\geq 2}, \mu^{\geq 2})$$
$$= l(\mu) - b(\lambda, \mu),$$

and hypothesis (b) holds.

Case 3: $d_1 = k = d_2$ in $D'(\lambda/\mu)$ with a vertical step between the first and second diagonals. In this case the number of ascents and descents in the diagonal list of $D'(\lambda/\mu)$ are unchanged from those in $D'(\lambda/\mu)^{\geq 2}$. To form λ and μ, the shifting of each row of $D'(\lambda/\mu)^{\geq 2}$ to the right by one increases each part of $\lambda^{\geq 2}$ and $\mu^{\geq 2}$ by one; in addition, we append a part of length one to each. Therefore,

$$l(\lambda) - l(\mu) = \Big(l(\lambda^{\geq 2}) + 1\Big) - \Big(l(\mu^{\geq 2}) + 1\Big) = desc((\lambda/\mu)^{\geq 2}) - asc((\lambda/\mu)^{\geq 2})$$
$$= desc(\lambda/\mu) - asc(\lambda/\mu),$$

and hypothesis (a) holds. With the addition of a part of length one to each, we have $b(\lambda, \mu) = b(\lambda^{\geq 2}, \mu^{\geq 2}) + 1$, so

$$asc(\lambda/\mu) = asc((\lambda/\mu)^{\geq 2}) = l(\mu^{\geq 2}) - b(\lambda^{\geq 2}, \mu^{\geq 2}) = \Big(l(\mu) - 1\Big) - \Big(b(\lambda, \mu) - 1\Big)$$
$$= l(\mu) - b(\lambda, \mu),$$

and hypothesis (b) holds.

Case 4: $d_1 = k + 1 > k = d_2$ in $D'(\lambda/\mu)$. In this case the number of descents in the diagonal list of $D'(\lambda/\mu)$ is one more than in the list of $D'(\lambda/\mu)^{\geq 2}$, but the number of ascents is unchanged. Since each row in $D'(\lambda/\mu)^{\geq 2}$ is shifted to the right by one to form λ and μ, each existing part of $\lambda^{\geq 2}$ and of $\mu^{\geq 2}$ is increased by one; the length of μ is equal to the length of $\mu^{\geq 2}$, but we append a part of length one to $\lambda^{\geq 2}$ to form λ. Therefore,

$$\begin{aligned} l(\lambda) - l(\mu) &= \left(1 + l(\lambda^{\geq 2})\right) - l(\mu^{\geq 2}) = \left(1 + desc((\lambda/\mu)^{\geq 2})\right) - asc((\lambda/\mu)^{\geq 2}) \\ &= desc(\lambda/\mu) - asc(\lambda/\mu), \end{aligned}$$

and so hypothesis (a) holds. Since the part appended to form λ has length one and μ has no parts of length one (all part of $\mu^{\geq 2}$ having been increase by one), we have $b(\lambda, \mu) = b(\lambda^{\geq 2}, \mu^{\geq 2})$. Therefore,

$$\begin{aligned} asc(\lambda/\mu) &= asc((\lambda/\mu)^{\geq 2}) = l(\mu^{\geq 2}) - b(\lambda^{\geq 2}, \mu^{\geq 2}) \\ &= l(\mu) - b(\lambda, \mu), \end{aligned}$$

and hypothesis (b) holds. □

Remark II.14. (a) To understand the significance of the equation in Lemma II.13.(a) consider that the lengths of consecutive diagonals can differ by at most one. Now if the first diagonal has length one, then every ascent in the diagonal list will have a matching descent since we must end at a rightmost diagonal of length one; we include the descent from this rightmost column of length one to a column of length zero, and so both sides of the equation equal one in this case. If the first diagonal has length greater than one, then every ascent still has a matching descent, but there will be a set of unmatched descents in order to get down to those final rightmost diagonals of length one and zero.

(b) To interpret the equation in Lemma II.13.(b) consider that for each part μ_i the leftmost cell in row i that is in $D'(\lambda/\mu)$ is the northwest cell of a diagonal that is either an ascent or a vertical step from the previous diagonal. In the case of a vertical step, the southeast cell of the previous diagonal is the rightmost cell of its row j, so that $\mu_i = \lambda_j$; see Case 3 in Figure 2.1 where the vertical step leads to $\mu_3 = \lambda_7 = 1$. In the case of an ascent, the southeast cells of the two diagonals are in the same row, so that $\mu_i \neq \lambda_j$ for all j; see Case 1 in Figure 2.1 where the vertical step leads to $\mu_3 = 1$ being unmatched in λ. Thus $b(\lambda, \mu)$ counts the number of vertical steps between diagonals of the same length in $D'(\lambda/\mu)$.

2.3 Proof of Theorem II.6

Recall that for unmarked shifted tableau T with diagram $D'(\lambda/\mu)$, we will use $fr(T)$ to represent the number of free entries in T, which is equal to the number of border strips in T. We will use T_L to represent the unique unmarked shifted tableau whose monomial x^{T_L} is the leading term of $Q_{\lambda/\mu}$ from Lemma II.10.

We begin by showing that the number of border strips in T is greater than or equal to the number of descents in the diagonal list of $D'(\lambda/\mu)$ with equality for T_L:

$$fr(T) \geq fr(T_L) = desc(\lambda/\mu). \tag{2.3}$$

We proceed by induction on the number of diagonals of $D'(\lambda/\mu)$. If $D'(\lambda/\mu)$ has one diagonal, then this diagonal has length one and $\lambda/\mu = 1/\emptyset$. Counting the descent from the rightmost diagonal of positive length to the right-adjacent diagonal of length zero, the diagonal list of $D'(1/\emptyset)$ has one descent. Since there is only one cell in $D'(1/\emptyset)$, all T contain exactly one border strip; for T_L the single entry is 1. Therefore, $fr(T) = desc(\lambda/\mu)$ and the hypothesis of (2.3) holds.

Assume the hypothesis holds for all (possibly skew) shifted diagrams with $n-1$ or fewer diagonals. Given any shifted diagram $D'(\lambda/\mu)$ with n diagonals, let $D'(\lambda/\mu)^{\geq 2}$ be the shifted diagram we get by deleting the first diagonal of $D'(\lambda/\mu)$, as in the proof of Lemma II.13, and let $T^{\geq 2}$ be the resulting unmarked tableau with each entry unchanged, except that the entries of the first diagonal of $T^{\geq 2}$ are now all free. Since $D'(\lambda/\mu)^{\geq 2}$ has $n-1$ diagonals, by assumption the hypothesis holds. Let k be the

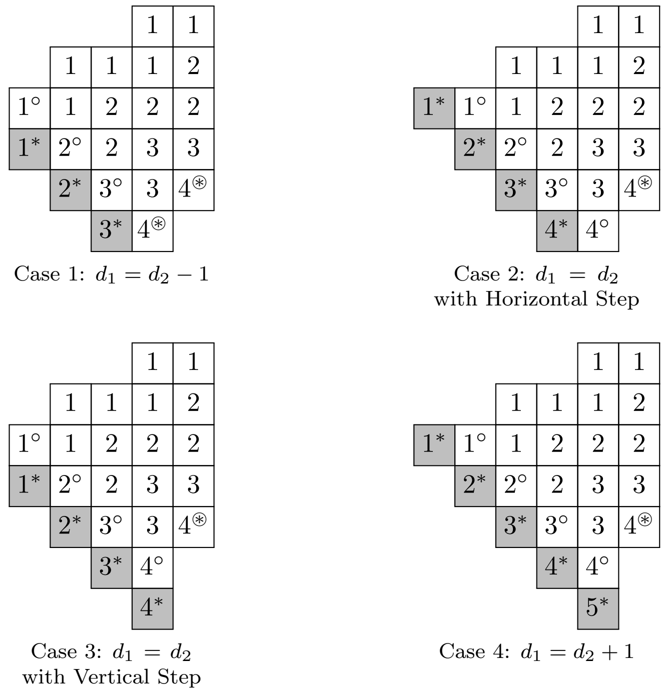


Figure 2.2: Free Entries in the Four Leading Term Tableau with $D'(\lambda/\mu)^{\geq 2} = D'(765431/52)$

length of the first diagonal of $D'(\lambda/\mu)^{\geq 2}$. We will show that the hypothesis holds when the first and second diagonals of $D'(\lambda/\mu)$ satisfy each of the four possibilities for the relative size and position of adjacent diagonals in shifted diagrams. See Figure 2.2 for an example of the leading term tableau for each case where $D'(\lambda/\mu)^{\geq 2} = D'(765431/52)$ in which the a° are the five free entries of $D'(765431/52)$ and the a^* are the free entries of each $D'(\lambda/\mu)$; recall from above that the diagonal list of $D'(765431/52)$ has five descents.

Case 1: $d_1 = k - 1 < k = d_2$ for $D'(\lambda/\mu)$. In this case we have added an ascent to the diagonal list, but the number of descents is unchanged from $D'(\lambda/\mu)^{\geq 2}$ to $D'(\lambda/\mu)$. For every unmarked tableau T with diagram $D'(\lambda/\mu)$, there are exactly k border strips that intersect the second diagonal and exactly $k - 1$ that intersect the first diagonal. If each entry $T(i, i)$ on the main diagonal equals the entry $T(i - 1, i)$ directly above, then all of the border strips that intersect the first diagonal also intersect the second, so that the total number of border strips is unchanged from $T^{\geq 2}$ to T. Certainly different first diagonal entries of T could increase the number of border strips, so that

$$fr(T) \geq fr(T^{\geq 2}) \geq desc((\lambda/\mu)^{\geq 2}) = desc(\lambda/\mu). \tag{2.4}$$

For T_L, the entries of the first diagonal are $1, 2, \ldots, (k-1)$ and the entries of the second diagonal are $1, 2, \ldots, (k-1), k$ with the a in the first diagonal directly below the a in the second diagonal for $1 \leq a \leq k-1$, so that equality holds in (2.4).

Case 2: $d_1 = k = d_2$ in $D'(\lambda/\mu)$ with a horizontal step between the first and second diagonals. In this case the number of descents in the diagonal list is unchanged from $D'(\lambda/\mu)^{\geq 2}$ to $D'(\lambda/\mu)$. For every unmarked tableau T with diagram $D'(\lambda/\mu)$, there are exactly k border strips that intersect the second diagonal and exactly k that intersect the first diagonal. If each entry $T(i,i)$ in the first diagonal equals the entry $T(i, i+1)$ directly to the right, then all of the border strips that intersect the first diagonal also intersect the second and the total number of border strips is unchanged from $T^{\geq 2}$ to T. Certainly different first diagonal entries could increase the number of border strips, so that

$$fr(T) \geq fr(T^{\geq 2}) \geq desc((\lambda/\mu)^{\geq 2}) = desc(\lambda/\mu). \tag{2.5}$$

For T_L, the entries of the first and second diagonals are $1, 2, \ldots, k$ with the a in the first diagonal directly to the left of the a in the second diagonal for $1 \leq a \leq k$, so that equality holds in (2.5).

Case 3: $d_1 = k = d_2$ in $D'(\lambda/\mu)$ with a vertical step between the first and second diagonals. Again in this case the number of descents in the diagonal list of $D'(\lambda/\mu)$ is equal to the number in the diagonal list of $D'(\lambda/\mu)^{\geq 2}$. For every unmarked tableau T with diagram $D'(\lambda/\mu)$, there are exactly k border strips that intersect the second diagonal and exactly k that intersect the first diagonal. If each entry $T(i,i)$ in the first diagonal equals the entry $T(i-1,i)$ directly above, then all of the border strips that intersect the first diagonal also intersect the second diagonal and the number of border strips is unchanged from $T^{\geq 2}$ to T. Certainly different first diagonal entries could increase the number of border strips, so that

$$fr(T) \geq fr(T^{\geq 2}) \geq desc((\lambda/\mu)^{\geq 2}) = desc(\lambda/\mu). \tag{2.6}$$

For T_L, the entries of the first and second diagonals are $1,2,\ldots,k$ with the a in the first diagonal directly below the a in the second diagonal for $1 \leq a \leq k$, so that equality holds in (2.6).

Case 4: $d_1 = k+1 > k = d_2$ in $D'(\lambda/\mu)$. In this case there is a descent from the first diagonal to the second diagonal of $D'(\lambda/\mu)$, so that the number of descents in the diagonal list of $D'(\lambda/\mu)$ is one more than the number in the diagonal list of $D'(\lambda/\mu)^{\geq 2}$. For every unmarked tableau T with diagram $D'(\lambda/\mu)$, there are exactly $k+1$ border strips that intersect the first diagonal. Since there are exactly k border strips that intersect the second diagonal, it must be that there is at least one border strip that intersects the first diagonal but does not intersect the second, and so does not intersect $T^{\geq 2}$. Thus the minimum number of border strips in T is one more than the number in $T^{\geq 2}$. Certainly different first diagonal entries could increase the number of border strips, so that

$$fr(T) \geq fr(T^{\geq 2}) + 1 \geq desc((\lambda/\mu)^{\geq 2}) + 1 = desc(\lambda/\mu). \tag{2.7}$$

For T_L, the entries of the first diagonal are $1,2,\ldots,k,(k+1)$ and the entries of the second diagonal are $1,2,\ldots,k$ with the a in the first diagonal directly left of the a in the second diagonal for $1 \leq a \leq k$. Thus, the number of border strips increases by exactly one from $T_L^{\geq 2}$ to T_L, and so equality holds in (2.7). Therefore, (2.3) holds in all four cases.

From (2.2), the leading coefficient of $Q_{\lambda/\mu}$ is $2^{fr(T_L)}$. Combining (2.3) and Lemma II.13, we have:

$$
\begin{aligned}
fr(T_L) &= desc(\lambda/\mu) \\
&= l(\lambda) - l(\mu) + asc(\lambda/\mu) \\
&= l(\lambda) - l(\mu) + l(\mu) - b(\lambda, \mu) \\
&= l(\lambda) - b(\lambda, \mu),
\end{aligned}
$$

proving part (a) of the theorem. Since $fr(T) \geq fr(T_L) = l(\lambda) - b(\lambda, \mu)$, the leading coefficient $2^{l(\lambda)-b(\lambda,\mu)}$ divides each term in (2.2), proving part (b). $\square$

Remark II.15. In the proof of the theorem, we see there are three ways to find the number of free entries or border strips in the leading term tableau. The method of the proof is to show the equivalence of counting the descents in the diagonal list and finding the tail or northeast cell of each border strip. The free entries are actually located at the $l(\lambda) - l(\mu)$ positions on the main diagonal and the southeast cell in each diagonal that is an ascent from the previous diagonal. And finally, the expression $l(\lambda) - b(\lambda, \mu)$ in the theorem statement has the advantage of needing only the shape and not the computation of the diagonal set.

CHAPTER III

Schur s-Functions in $\Omega_{\mathbb{Q}}$

Recall that an inspiration for our work is the exploration of answers to:

Question III.1. *When are two (possibly skew) Schur s-functions equal?*

Definition III.2. For any unshifted diagram $D(\lambda/\mu)$, the *rotation of* λ/μ, $Rot(\lambda/\mu)$, is the (possibly skew) shape whose diagram is the 180 degree rotation of $D(\lambda/\mu)$ viewed as a subset of $(\lambda_1)^{l(\lambda)}$, which is the smallest rectangle containing λ.

A basic partial answer to Question III.1 is:

Lemma III.3. *For any unshifted diagram $D(\lambda/\mu)$, $s_{Rot(\lambda/\mu)} = s_{\lambda/\mu}$.*

Proof. Given a tableau T with diagram $D(\lambda/\mu)$, let n be the minimum entry of T and N the maximum entry of T. We can form a tableau T' of shape $Rot(\lambda/\mu)$ by rotating the diagram $D(\lambda/\mu)$ by 180 degrees and replacing each entry a with $(n + N - a)$. The rows of T are weakly increasing and thus so are the rows of T'; the columns of T are strictly increasing and thus so are the columns of T'. Since s-functions are symmetric and the content of T' is a permutation of the the content of T, the terms of $s_{Rot(\lambda/\mu)}$ are simply a reordering of the terms of $s_{\lambda/\mu}$, and so the two s-functions are equal. $\square$

For the ring involution ω of Definition I.29, recall that $\omega(p_r) = p_r$ for r odd. Thus, since $\Omega_{\mathbb{Q}}$ is generated by the odd power sums,

$$\omega(f) = f \quad \text{for all symmetric functions } f \in \Omega_{\mathbb{Q}}.$$

For s-functions, $\omega(s_{\lambda/\mu}) = s_{\lambda'/\mu'}$ (e.g. [6], 1.3), so that

$$s_{\lambda'/\mu'} = \omega(s_{\lambda/\mu}) = s_{\lambda/\mu} \quad \text{for all } s_{\lambda/\mu} \in \Omega_{\mathbb{Q}}. \tag{3.1}$$

Question III.4. *Which s-functions are elements of $\Omega_{\mathbb{Q}}$?*

From (3.1) we see that answering this question provides a class of s-functions relevant for Question III.1, as well as being useful later when exploring identities between s- and Q-functions. The answer to Question III.4 is the main result of this chapter.

Theorem III.5. *For any diagram $D(\lambda/\mu)$, $s_{\lambda/\mu} \in \Omega_{\mathbb{Q}}$ if and only if each connected component of $D(\lambda/\mu)$ is combinatorially equivalent to δ/η or $Rot(\delta/\eta)$, where δ is a staircase partition of any length and η is any partition such that $\eta \subseteq \delta$.*

That the s-functions for all δ/η and their rotations are in $\Omega_{\mathbb{Q}}$ was already known; what is new that we will establish here is that there are no additional shapes. Thus there are no other examples of equality between the s-functions of a shape and its conjugate that can be proved simply by showing inclusion in $\Omega_{\mathbb{Q}}$. Note that $Rot(\delta/\eta)$ is combinatorially equivalent to a shape of the form $\lambda/\delta(k)$, specifically with $\lambda_1 = l(\lambda)$ and $k = l(\lambda) - 1$ so that the first column and first row both have length one, but the converse does not hold for a general $\lambda/\delta(k)$. For example, $s_{444/21} \notin \Omega_{\mathbb{Q}}$.

It is well known that, for a disconnected skew diagram, the entries of each connected component are independent of the entries in the other connected components, and thus, following from the definition of s-functions in terms of tableaux (I.31.(a)), that the s-function for a disconnected skew diagram factors into the s-functions for the connected components. Since the power sum expansion of the product of s-functions includes only odd power sums if and only if the power sum expansion of the s-function for each connected component does, we may reduce the proof of the theorem to connected $D(\lambda/\mu)$, and so for the rest of the chapter we assume connectedness unless stated otherwise.

3.1 Domino Skimmings

The algebra $\Omega_{\mathbb{Q}}$ may alternatively be defined in terms of a cancellation law:

Lemma III.6 ([8], Theorem 2.11; [12])**.** *For a symmetric function f, we have $f \in \Omega_{\mathbb{Q}}$ if and only if*

$$f(t, -t, x_3, x_4, \dots) = f(0, 0, x_3, x_4, \dots).$$

Definition III.7. An *odd (even) run* in the (possibly skew) diagram $D(\lambda/\mu)$ is a set of exactly k consecutive columns from column i to column $(i+k-1)$ with k odd (even) such that $\mu'_i = \mu'_{i+1} = \cdots = \mu'_{i+k-1} > \mu'_{i+k}$ and, if column $(i-1)$ is non-empty, $\mu'_{i-1} > \mu'_i$. For example, $\lambda/\mu = 10, 7, 5, 5/5, 3, 3$ in Figure 3.1 below has runs, from left to right, of three, two, and five columns.

Definition III.8. An *odd (even) block of columns of length* c in the (possibly skew) diagram $D(\lambda/\mu)$ is a set of exactly k consecutive columns with k odd (even) such that the cells of these columns form a c by k rectangle, and if either the preceding or following column is included, the cells do not form a rectangle. Note that a block of length k is a subset of a run of at least k columns where for all columns i in the block not only are all the μ'_i equal but also all the λ'_i are equal. For example, $\lambda/\mu = 10, 7, 5, 5/5, 3, 3$ in Figure 3.1 below has, from left to right, blocks of three columns of length one, two columns of length three, two columns of length two, and three singleton columns.

Lemma III.9. *For any unshifted (possibly disconnected) diagram $D(\lambda/\mu)$*

$$s_{\lambda/\mu}(t, -t, x_3, x_4, \dots) = \sum_{\nu} (-1)^{N_2}\, t^{|\nu/\mu|}\, s_{\lambda/\nu}(x_3, x_4, \dots) \tag{3.2}$$

where the sum is over all ν such that $\mu \subseteq \nu \subseteq \lambda$, all columns of $D(\nu/\mu)$ have length $\nu_i' - \mu_i' \leq 2$, and all blocks of singleton columns in $D(\nu/\mu)$ are even, and where N_2 is the number of columns of length two in $D(\nu/\mu)$.

Proof. Recall that a skew s-function $s_{\lambda/\mu}$ is combinatorially defined as the sum of the monomials for all tableaux with diagram $D(\lambda/\mu)$. Any tableau T can be divided into two smaller tableaux, the possibly disconnected and skew tableau with diagram $D(\nu/\mu)$ formed by the 1 and 2 entries and the possibly disconnected and skew tableau with diagram $D(\lambda/\nu)$ formed by the entries of 3 and greater. Fixing ν, the entries within these two shapes are independent of each other, and so the sum of the contributions of all possible assignments to $D(\lambda/\nu)$ is the factor $s_{\lambda/\nu}(x_3, x_4, \dots)$ and the sum of the contributions of all possible assignments to $D(\nu/\mu)$ is the factor $s_{\nu/\mu}(t, -t))$.

Since the columns of any tableau are strictly increasing, the columns of $D(\nu/\mu)$ have length two or less. Each column of length two has entries $\substack{1\\2}$, so that each contributes a factor of $-t^2$ to the monomial for each T with diagram $D(\nu/\mu)$ in $s_{\nu/\mu}(t, -t)$.

For each block of k columns of length one in $D(\nu/\mu)$, if the block is connected to the preceding column then the preceding column must have length two forcing its top entry to be 1, and if the block is connected to the following column then the following column must have length two forcing its bottom entry to be 2. Thus the entries in the block of singleton columns are independent of all other entries in $D(\nu/\mu)$, and any weakly increasing filling with a 1's and b 2's, $a+b=k$, is valid, which contributes a factor of $(-1)^b t^k$ to the monomial for T. The sum of these contributions is 0 or t^k depending on whether k is odd or even respectively.

Therefore, in $s_{\nu/\mu}(t,-t)$, for each $D(\nu/\mu)$ which includes an odd block of singleton columns its set of tableaux contributes 0, and for all other ν the monomials for the set of tableaux sums to a single term that is the product of the independent factors just described. For example, for $\lambda/\mu = 10,7,5,5/5,3,3$ in Figure 3.1, if we take the gray shaded cells in each example to be ν/μ, then the product of independent factors is $t^2(-t^2)t^4 s_{10,7,5,5/9,4,4,2}$ and $t^2(-t^2)^3 t^4 s_{10,7,5,5/10,6,5,2}$ respectively. □

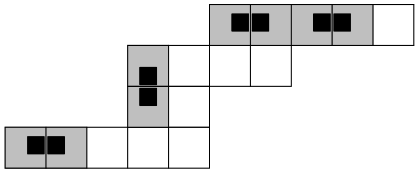
The grouping of terms in the expansion of $s_{\nu/\mu}(t,-t)$ above leads us to the following definition:

Definition III.10. For any diagram $D(\lambda/\mu)$, a partition ν such that $\mu \subseteq \nu \subseteq \lambda$ is a *domino skimming* if $D(\nu/\mu)$ only has columns of length ≤ 2 and can be tiled by horizontal and vertical dominoes so that every column of length two has exactly one vertical domino in it. See Figure 3.1 for an example of two domino skimmings, where $D(\nu/\mu)$ can be tiled by four dominoes in the first case and six in the second.

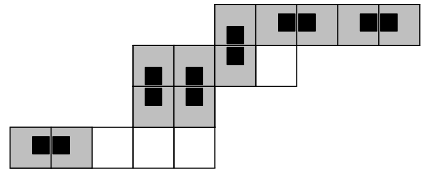
Definition III.11. A *maximum domino skimming* of $D(\lambda/\mu)$ is a domino skimming such that no other skimming of $D(\lambda/\mu)$ includes more dominoes. The figure on the right in Figure 3.1 shows a maximum skimming.

We next state a lemma but will save the proof for the end of the chapter.

Lemma III.12. *Among the maximum domino skimmings ν of any connected diagram $D(\lambda/\mu)$ with rightmost column of length at least two, there is exactly one*



Skimming $\nu = 9, 4, 4, 2$



Maximum Skimming $\nu = 10, 6, 5, 2$

Figure 3.1: Two Domino Skimmings of $\lambda/\mu = 10, 7, 5, 5/5, 3, 3$

skimming which maximizes the number of rows in the (possibly disconnected) diagram $D(\lambda/\nu)$.

3.2 Proof of Theorem III.5

First, we consider the case of the s-functions for shapes δ/η and $Rot(\delta/\eta)$. For $\lambda/\mu = Rot(\delta/\eta)$, since the northwest edge is a staircase-like shape with first column and first row each of length one, the only ν such that all blocks of singleton columns in $D(\nu/\mu)$ are even is μ itself, and so $s_{\lambda/\mu}(t, -t, x_3, x_4, \dots) = s_{\lambda/\mu}(x_3, x_4, \dots)$ by Lemma III.9. Thus $s_{\delta/\eta} = s_{Rot(\delta/\eta)} \in \Omega_{\mathbb{Q}}$ by Lemma III.6. Alternatively, it is easy to show using the Murnaghan-Nakayama Rule (e.g. [10], 7.17) that the expansion of $s_{\delta/\eta}(x_1, x_2, x_3, \dots)$ contains no even power sums.

We are now able to proceed to the proof of the main result, that there are no other connected shapes whose s-functions are in $\Omega_{\mathbb{Q}}$. If $s_{\lambda/\mu} \in \Omega_{\mathbb{Q}}$, then from Lemma III.3 and (3.1) it follows that all four of $s_{\lambda/\mu}$, $s_{\lambda'/\mu'}$, $s_{Rot(\lambda/\mu)}$, and $s_{Rot(\lambda'/\mu')}$ are equal. Thus, in determining whether $s_{\lambda/\mu}$ is in or not in $\Omega_{\mathbb{Q}}$, it is sufficient to show the result for any one of the four. Any shape of size one is combinatorially equivalent to $1/\emptyset$ and $s_1 = p_1 \in \Omega_{\mathbb{Q}}$, so that we may assume that $|\lambda/\mu| > 1$. We may also assume that the rightmost column has length at least two, for if this condition does not hold for $D(\lambda/\mu)$ then, since $D(\lambda/\mu)$ is connected, the first row must have length $\lambda_1 - \mu_1 \geq 2$, so that $Rot(\lambda'/\mu')$ does satisfy the condition.

By Lemma III.6, we see that for $s_{\lambda/\mu}$ to be an element of $\Omega_{\mathbb{Q}}$, $s_{\lambda/\mu}(t,-t,x_3,x_4,...)$ must be independent of t. We may restate the expansion of $s_{\lambda/\mu}(t,-t,x_3,x_4,...)$ in (3.2) in terms of domino skimmings and group by the power of t:

$$s_{\lambda/\mu}(t,-t,x_3,x_4,\dots) = t^{max}\sum_{\nu}(-1)^{N_2}\ s_{\lambda/\nu}(x_3,x_4,\dots) \tag{3.3}$$
$$+ \text{(terms with lower degree in } t)$$

where max is twice the number of dominoes in each maximum skimming, N_2 is the number of columns of length two in $D(\nu/\mu)$, and the sum in the coefficient of t^{max} is over all partitions ν that are maximum skimmings of $D(\lambda/\mu)$.

For all diagrams $D(\lambda/\mu)$ (with rightmost column of length at least two) that are not of shape $Rot(\delta/\eta)$, there must be some column i such that either $\mu'_{i-1} \geq \mu'_i + 2$ (or the first column has length at least two) so that we may pack a vertical domino or $\mu'_{i-1} = \mu'_i$ so that we may pack a horizontal domino (since the northwest edge is not a staircase shape). Thus, domino skimmings ν with $|\nu/\mu| > 0$ do exist and we will show that the degree of $s_{\lambda/\mu}(t,-t,x_3,x_4,\dots)$ in t is positive. By Lemma III.12, we know that among the $s_{\lambda/\nu}$ in the coefficient of t^{max} there is exactly one λ/ν^p with strictly more non-empty rows than all others. We claim that the linear span of all skew s-functions for shapes with $< r$ non-empty rows does not contain the s-function for any shape with exactly r non-empty rows. Toward a contradiction, for any $D(\lambda/\nu)$ with exactly r non-empty rows we suppose:

$$s_{\lambda/\nu} = \sum_{\alpha/\beta} a(\alpha,\beta) s_{\alpha/\beta} \tag{3.4}$$

summed over shapes α/β with $< r$ rows of positive length and for some constants $a(\alpha,\beta)$. When we apply the involution ω to both sides of (3.4), we get the equivalent equation:

$$s_{\lambda'/\nu'} = \sum_{\alpha/\beta} a(\alpha,\beta) s_{\alpha'/\beta'}.$$

Since $D(\lambda/\nu)$ has exactly r rows of positive length, $D(\lambda'/\nu')$ has exactly r columns of positive length, and so, considering the leading monomial, the degree of $s_{\lambda'/\nu'}$ in x_1 is r. Since each $D(\alpha'/\beta')$ has strictly fewer than r columns of positive length, all $s_{\alpha'/\beta'}$ are of degree strictly less than r in x_1, and so $s_{\lambda'/\nu'}$ cannot be in their span.

Thus s_{λ/ν^p} is not in the span of the $s_{\lambda/\nu}$ for all other maximum skimmings ν, and therefore the coefficient of t^{max} in (3.3) is non-zero. $\square$

3.3 Proof of Lemma III.12

Definition III.13. Let m be the smallest column index i in a connected diagram $D(\lambda/\mu)$ such that either column $i-1$ is the rightmost column of an even run or $\mu'_i \leq \mu'_{i-1} - 2$. The *domino sparse part* of $D(\lambda/\mu)$, denoted $sparse(\lambda/\mu)$, is the diagram consisting of the first $m-1$ columns of $D(\lambda/\mu)$ and the *domino dense part* of $D(\lambda/\mu)$, denoted $dense(\lambda/\mu)$, is the diagram consisting of the columns to the right of and including the mth column. For the example in Figure 3.2, the first nine columns form $sparse(\lambda/\mu) = D(9999/743)$ and columns 10-23 form $dense(\lambda/\mu) = D(14, 14, 10, 6, 5, 5, 2, 2, 2, 1/11, 9, 3, 3, 3)$.

Lemma III.14. *Of the top two cells in each column for a connected diagram $D(\lambda/\mu)$, a domino skimming cannot include:*

(a) in $sparse(\lambda/\mu)$, the top cell of the rightmost column of each odd run as well as the second cell in each column;

(b) in $dense(\lambda/\mu)$, for each odd block of singleton columns, either the rightmost cell in the odd block or the second cell in the column of length at least two which precedes the odd block of singleton columns.

In the example in Figure 3.2, the top two cells in each column, which we would hope to include in a skimming, are colored gray. Of these cells, those that we claim

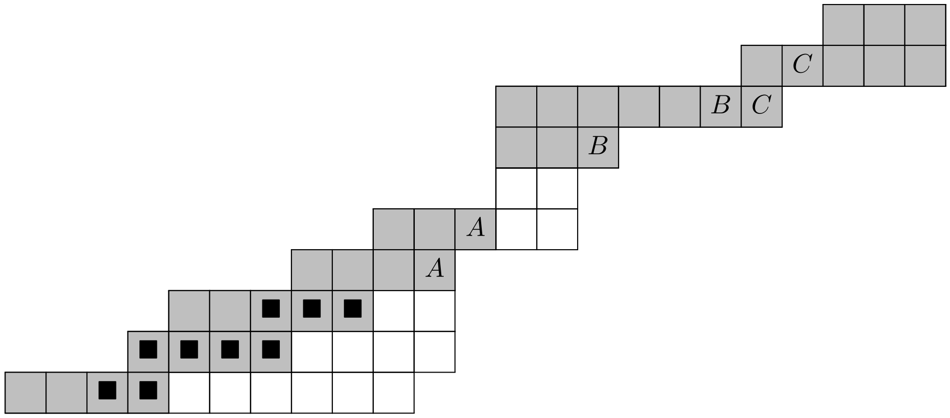


Figure 3.2: The Cells That Cannot Be Included in a Domino Skimming

cannot be included as described in part (a) of the lemma are marked with a $\blacksquare$. Three pairs of the cells described in part (b) are marked A, B, and C; we will show that at least one of each pair must also be excluded.

Proof. In $sparse(\lambda/\mu)$, by definition, the difference between μ'_{i-1} and μ'_i is at most one and every run is odd, except for possibly the rightmost run. Thus for any run of k columns only the top cell of the leftmost column is exposed so that we may not pack a vertical domino. When k is odd, we may cover the top cell of the first $k-1$ columns in the run with horizontal dominoes, forcing in any skimming the top cell of the rightmost column in each odd run to be uncovered. In the next run it remains that only the top cell of the leftmost column is exposed so that we still may not pack a vertical domino. In $dense(\lambda/\mu)$, the leftmost column must have length at least two since it begins a new run and $D(\lambda/\mu)$ is connected, so that every odd block of singleton columns in $dense(\lambda/\mu)$ must be preceded by a column of length at least two. Let us consider the intersection of the skimming and any group of a column of length at least two followed by an odd block of k singleton columns. We may pack a vertical domino in the column of length at least two and $\frac{k-1}{2}$ horizontal dominoes in the first $k-1$ singleton columns, leaving the rightmost singleton column uncovered, or we may pack $k/2$ horizontal dominoes in the group, which leaves the second cell in the leftmost column uncovered but covers the rightmost singleton column. So for each such group no skimming can include both the second cell in the leftmost column and the rightmost singleton column. □

For any connected diagram $D(\lambda/\mu)$ with rightmost column of length at least two, we propose a skimming ν^p which will include the top and second cell in each column except those cells which Lemma III.14 shows cannot be covered by a skimming, thereby proving that this skimming is indeed a maximum skimming. Within $sparse(\lambda/\mu)$, we pack horizontal dominoes in each odd run of k columns covering the top cell of the first $k-1$ columns in the run. If the final run of $sparse(\lambda/\mu)$ is an even run of k columns, then we pack $\frac{k}{2}$ horizontal dominoes in this run. Within $dense(\lambda/\mu)$, we pack dominoes according to the following rules:

(a) we cover an even block of k singleton columns with $\frac{k}{2}$ horizontal dominoes;

(b) we pack $\frac{k+1}{2}$ horizontal dominoes in any group of a column of length at least two followed by an odd block of k singleton columns;

(c) we pack a vertical domino in all other columns of length at least two.

The leftmost column of $dense(\lambda/\mu)$ (column m of $D(\lambda/\mu)$) must have length at least two and by definition either $m = 1$, $\mu'_{m-1} - 2 \geq \mu'_m$, or the final run of $sparse(\lambda/\mu)$ is even with the top cell in each column included in ν^p, so that we may always pack a vertical domino in column m. Thus the rules force every instance in $dense(\lambda/\mu)$ of the three types of column groups described to be preceded by either a column of length one which we have covered by a horizontal domino or a column of length at least two in which we have packed a vertical domino, so that the top two cells in

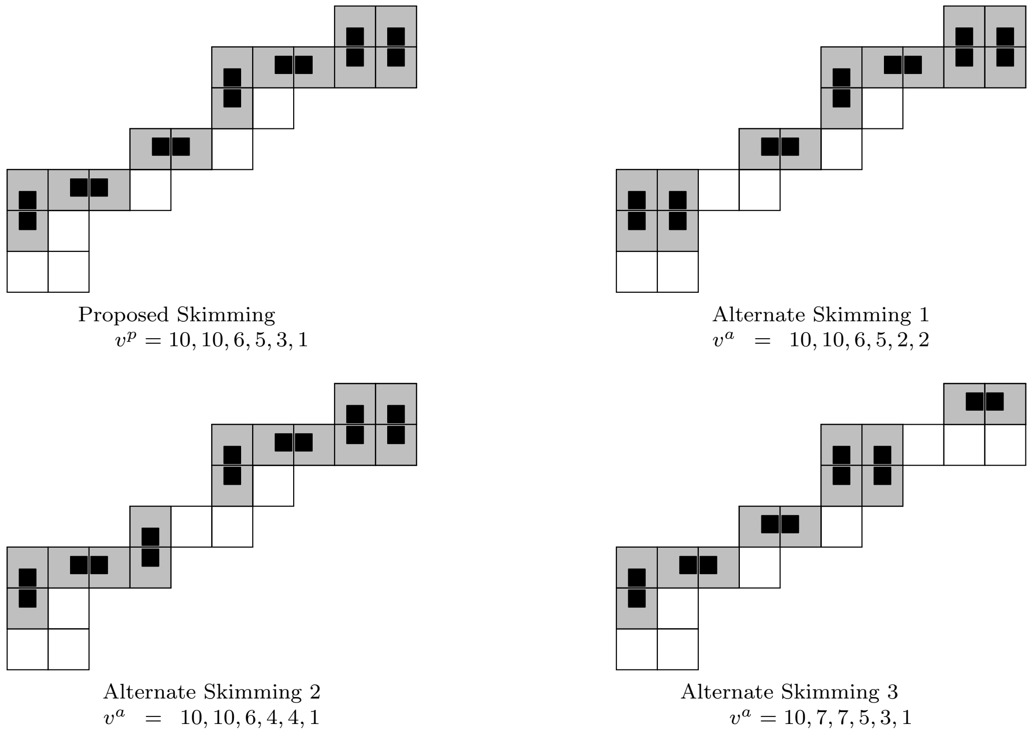


Figure 3.3: Domino Skimmings of $\lambda/\mu = 10, 10, 7, 6, 4, 2, 2/8, 5, 5, 3$

each column can be included in ν^p. See the Proposed Skimming in Figure 3.3 for an example (with $m = 1$).

Proof of Lemma III.12 : We let $D(\lambda/\nu^p)$ be the possibly disconnected diagram we get by deleting from connected $D(\lambda/\mu)$ the cells in the proposed skimming ν^p just described. We will show that any alternate skimming ν^a is either not a maximum skimming or that $D(\lambda/\nu^a)$ has fewer (non-empty) rows than $D(\lambda/\nu^p)$.

In Lemma III.14, we showed that within $sparse(\lambda/\mu)$ we are forced to pack only horizontal dominoes with the rightmost cell in each odd run left uncovered, and that within $dense(\lambda/\mu)$ we are forced to leave uncovered one cell from each group of a column of length at least two followed by an odd run of singleton columns. Since the proposed skimming covers every top and second cell in each column except these, so must every other maximum skimming. Thus the only differences between maximum skimmings occur within $dense(\lambda/\mu)$ in the groups of a column of length at least two followed by an odd block of k singleton columns. Recall that for such a group we can cover $k+1$ cells either with $\frac{k+1}{2}$ horizontal dominoes leaving the second cell in the leftmost column uncovered, as in the proposed skimming ν^p, or with a vertical domino and $\frac{k-1}{2}$ horizontal dominoes leaving the rightmost singleton column uncovered. Any other difference would mean that there is a cell uncovered in the alternate skimming that is covered in the proposed skimming, but there does not exist a column where the alternate skimming can make up for the missing cell.

Since the theorem assumes that the rightmost column of $D(\lambda/\mu)$ must have length at least two, every instance in $dense(\lambda/\mu)$ of a column of length at least two followed by an odd block of singleton columns must be followed by a right-adjacent column of length at least two. We will focus on such a group where a difference occurs.

If with the proposed skimming the bottom cell of the right-adjacent column is uncovered, which occurs when the right-adjacent column has length at least three or has length two and is followed by an odd block of singleton columns, then packing $\frac{k+1}{2}$ horizontal dominoes leaves two (non-empty) rows that intersect $D(\lambda/\nu^p)$ in these $k+2$ columns, a singleton row formed by the second cell in the leftmost column and a row whose leftmost cell is the bottom cell in the right-adjacent column. If instead, the alternate skimming packs a vertical domino and $\frac{k-1}{2}$ horizontal dominoes, then the uncovered cell from the group of a column of length at least two and the odd block of singleton columns is in the same row as the bottom cell of the right-adjacent column, and so $D(\lambda/\nu^a)$ has at least one fewer (non-empty) row than $D(\lambda/\nu^p)$. In Figure 3.3, compare the Proposed Skimming to columns 2-4 of Alternate Skimming 1 and columns 4-6 of Alternate Skimming 2. In both cases $D(\lambda/\nu^a)$ has only four (non-empty) rows compared to five for $D(\lambda/\nu^p)$.

Now consider the case when the bottom cell of the right-adjacent column is covered (i.e. it has length exactly two and is not followed by an odd block of singleton columns) with the proposed skimming. If the alternate skimming packs a vertical domino and $\frac{k-1}{2}$ horizontal dominoes, then the uncovered rightmost singleton column will prevent the packing of a vertical domino in the right-adjacent column, so that the second cell in this column will be left uncovered. There is no column where the alternate skimming can make up for this missing cell and so the alternate skimming is not a maximum skimming. In Figure 3.3, compare the Proposed Skimming to columns 7-10 of Alternate Skimming 3 and notice the alternate skimming only includes six dominoes instead of seven. $\square$

CHAPTER IV

Equality between S_λ and Q_ν

In this chapter, we discuss partial answers to the following questions:

Question IV.1. *When are two (possibly skew) Q-functions equal up to a constant multiple?*

and, as a particular case of this:

Question IV.2. *When is a skew Q-function equal to a constant multiple of a non-skew Q-function?*

The primary focus of this chapter is the special case where the shifted skew shape has exactly one position on the main diagonal. In this case, the diagram is also a valid unshifted shape and so these skew Q-functions are the S-functions of Definition I.31:

$$S_{\lambda/\mu}(x_1, x_2, \dots) = \sum_T x^T$$

summed over all shifted tableaux T for the unshifted diagram $D(\lambda/\mu)$. From Remark I.33, the Q_ν form a basis of $\Omega_{\mathbb{Q}}$ and so every element of the sub-algebra can be written as a linear combination of the non-skew Q-functions. Thus our second question reduces to determining whether for a skew Q-function there is more than one term with non-zero coefficient in its Q-expansion. The main result of this chapter answers this question for all S-functions.

Theorem IV.3. *If $D(\lambda/\mu)$ is an unshifted diagram such that*

$$S_{\lambda/\mu} = c\, Q_\nu$$

for some strict partition ν and scalar c, then λ/μ is combinatorially equivalent either to $21/1$, in which case $S_{21/1} = 2Q_2$, or to $m^k/\emptyset$, for which

$$S_{(m^k)} = Q_{(m+k-1, m+k-3, \ldots, |m-k|+1)}.$$

4.1 The Shifted Littlewood-Richardson Rule

Since the Schur s-functions form a basis of $\Lambda_{\mathbb{Q}}$, we may define coefficients $c^\lambda_{\mu\nu}$ by:

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu.$$

These coefficients are known to be non-negative integers and have a combinatorial interpretation known as the Littlewood-Richardson rule (e.g. [6], 1.9).

Definition IV.4. For a (possibly skew) unshifted tableau T, the *word* $w(T) = w_1 w_2 \cdots$ is the sequence formed by reading the rows of T from right to left beginning with the top row, so that w_1 is the rightmost entry in the top row.

Definition IV.5. A word $w = w_1 w_2 \cdots w_n$ over the alphabet $\mathbf{P}$ is said to satisfy the *lattice property* if the number of instances of $i-1$ in $w_1 w_2 \cdots w_r$ is greater than or equal to the number of instances of i for $1 \le r \le n$ and $i > 1$.

We can now state the Littlewood-Richardson rule:

Theorem IV.6. *(LR rule) For partitions λ, μ, and ν, $c^\lambda_{\mu\nu}$ is the number of tableaux T with diagram $D(\lambda/\mu)$ and content ν such that $w(T)$ satisfies the lattice property.*

In Chapter 2, we discussed the well known fact that the leading term of $s_{\lambda/\mu}$ has content equal to the conjugate of the column set partition and is greatest in dominance order. Since there is a single tableau with this content, the LR coefficient of the s-function for this leading term partition is 1. Note also that unless $\mu, \nu \subseteq \lambda$, $|\nu| = |\lambda/\mu|$, and the leading term partition dominates ν, we have $c^{\lambda}_{\mu\nu} = 0$.

For the proof of Theorem IV.3, we will use the shifted analogue of the Littlewood-Richardson rule developed by Stembridge ([11], Theorem 8.3).

Definition IV.7. For a (possibly skew) shifted tableau T, the *word* $w(T) = w_1 w_2 \cdots$ is the sequence formed by reading the rows of T from left to right, beginning with the last row, so that w_1 is the leftmost entry in the bottom row.

Definition IV.8. For any word $w = w_1 w_2 \cdots w_n$ over alphabet $\mathbf{P}'$, we define $m_i(j)$, the *multiplicity of i at the j-th step of w*, for $i \geq 1$ and $0 \leq j \leq 2n$, as:

$$m_i(j) = \text{the multiplicity of } i \text{ in } w_{n-j+1}, \ldots, w_n \qquad 0 \leq j \leq n$$

$$m_i(j) = m_i(n) + \text{ the multiplicity of } i' \text{ in } w_1, \ldots, w_{j-n} \qquad n < j \leq 2n.$$

The multiplicities can be computed by scanning the word first from right-to-left counting the instances of i and then from left-to-right adding the instances of i'. So in particular, $m_i(0) = 0$ for all i and $(m_1(2n), m_2(2n), \ldots)$ is the content of T.

Definition IV.9. A word w over alphabet $\mathbf{P}'$ is said to satisfy the *lattice property* if, whenever $m_i(j) = m_{i-1}(j)$:

$$w_{n-j} \neq i, i' \qquad \text{if } 0 \leq j < n \tag{4.1a}$$

$$w_{j-n+1} \neq i-1, i' \qquad \text{if } n \leq j < 2n \tag{4.1b}$$

Note that w_{n-j} or w_{j-n+1} is the next letter read after the jth step. We are now able to state the shifted analogue of the Littlewood-Richardson rule:

Theorem IV.10. *(Shifted LR Rule) For any shifted diagram $D'(\lambda/\mu)$*

$$Q_{\lambda/\mu} = \sum_{\nu} f^{\lambda}_{\mu\nu} Q_{\nu} \tag{4.2}$$

summed over all strict partitions ν, where the coefficient $f^{\lambda}_{\mu\nu}$ is the number of shifted tableaux T with diagram $D'(\lambda/\mu)$ and content ν such that:

(i) $w = w(T)$ satisfies the lattice property;

(ii) the leftmost i^* in w is unmarked, for $1 \leq i \leq l(\nu)$.

In Chapter 2, we discussed that the leading term of $Q_{\lambda/\mu}$ has content equal to the conjugate of the diagonal set partition. We showed that this leading term content is greatest in dominance order and that, up to the marking of the free entries, there is a single shifted tableau with this content. Thus, the shifted LR coefficient of the Q-function for this leading term partition in $Q_{\lambda/\mu}$ is non-zero, specifically the ratio of the leading coefficients which is a power of two where the power is the difference between $l(\lambda) - b(\lambda, \mu)$ and the length of this leading term partition. Note also that unless $\mu, \nu \subseteq \lambda$, $|\nu| = |\lambda/\mu|$, and the leading term partition dominates ν, we have $f^{\lambda}_{\mu\nu} = 0$.

4.2 The Flip of $D'(\lambda/\mu)$

In this section we discuss a transformation of shifted shapes that is well known to lead to equal skew Q-functions.

Definition IV.11. For a strict partition $\lambda \subseteq \delta(k)$, the *$k$-complement* λ^* is the strict partition that is the conjugate of the (not necessarily strict) partition which is equal to $\delta(k) - \lambda$ as sets. For example, in Figure 4.1, 6421 and 53 are 6-complements.

Definition IV.12. For a shifted diagram $D'(\lambda/\mu)$, the *flip* $\phi(\lambda/\mu)$ is the (possibly skew) shifted shape whose diagram is formed by reflection across a line perpendicular to the main diagonal, so that the top row and rightmost column are interchanged. Thus, the leftmost position in the first row of $D'(\lambda/\mu)$ becomes the bottom position

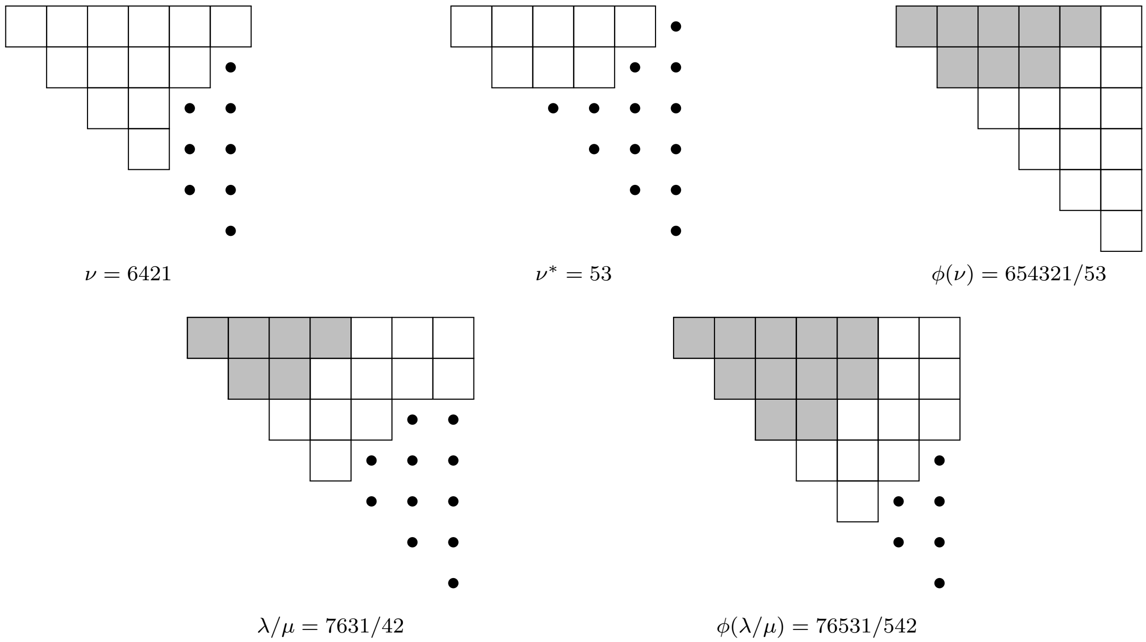


Figure 4.1: Complements and Flips of Shifted Diagrams

in the last column of $D'(\phi(\lambda/\mu))$. For μ^* and λ^* the λ_1-complements of μ and λ respectively, $\phi(\lambda/\mu) = \mu^*/\lambda^*$. Figure 4.1 provides two pairs of diagrams related by flipping: 6421 with $\delta(6)/53$ and 7631/42 with 76531/542.

Proposition IV.13. *For any shifted diagram $D'(\lambda/\mu)$, $Q_{\phi(\lambda/\mu)} = Q_{\lambda/\mu}$.*

Proof. Let T be any unmarked shifted tableau, as in Definition II.9, with diagram $D'(\lambda/\mu)$ and let n be the minimum entry of T and let N be the maximum entry of T. To create the unmarked shifted tableau $\phi(T)$ of shape $\phi(\lambda/\mu)$, flip the diagram $D'(\lambda/\mu)$ and replace each entry a with $(n + N - a)$. Since the rows of T are weakly increasing, the columns of $\phi(T)$ are weakly increasing, and since the columns of T are weakly increasing, the rows of $\phi(T)$ are weakly increasing. Since each diagonal is reflected on itself to form $\phi(T)$ and since the diagonals of T are strictly increasing, the diagonals of $\phi(T)$ are strictly increasing. Therefore, $\phi(T)$ is an unmarked shifted tableau. Now if the content of T is $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, then the content of $\phi(T)$

is the reverse permutation of α, $(\alpha_n, \ldots, \alpha_2, \alpha_1)$. The image of the left-adjacent (respectively, bottom-adjacent) position to (i, j) in T is bottom-adjacent (respectively, left-adjacent) to the image of (i, j) in $\phi(T)$, so that (i, j) is free in T if and only if its image is free in $\phi(T)$. Thus, $fr(\phi(T)) = fr(T)$. Therefore, since Q-functions are symmetric, the terms of $Q_{\phi(\lambda/\mu)}$ are simply a reordering of the terms of $Q_{\lambda/\mu}$, and so the two Q-functions are equal. □

This provides examples relevant for Question IV.1, as well as in a special case for Question IV.2. For any staircase partition $\delta(k)$, the k-complement is $\emptyset$, leading to the following corollary of Proposition IV.13.

Corollary IV.14. *For any strict partition $\mu \subseteq \delta(k)$ where μ^* is the k-complement of μ, $Q_{\delta(k)/\mu} = Q_{\mu^*}$.*

For example, the Q-function for $\delta(6)/53$, shown in Figure 4.1, is equal to the Q-function of its non-skew flip 6421.

4.3 Transformations leading to equal S-functions

In the previous section, we showed that the flip transformation leads to a shifted shape with identical Q-function. In this section we discuss two additional transformations for the S-function class of skew Q-functions that are well known to lead to an identical S-function.

Proposition IV.15. *For any unshifted diagram $D(\lambda/\mu)$,*

(a) $S_{\lambda/\mu} = S_{\lambda'/\mu'}$

(b) $S_{\lambda/\mu} = S_{Rot(\lambda/\mu)}$

Proof. For (a), let T be an unmarked shifted tableau, as in Definition II.9, with unshifted diagram $D(\lambda/\mu)$. Let T' be the result when we conjugate the diagram and leave the entries unchanged. Since the rows of T are weakly increasing, the columns of T' are weakly increasing; similarly, since the columns of T are weakly increasing, the rows of T' are weakly increasing. Since the diagonals of T' are simply a reordering of the diagonals of T with the last diagonal of T the first diagonal of T', each diagonal remains strictly increasing. Thus T' is indeed an unmarked shifted tableaux of unshifted diagram $D(\lambda'/\mu')$. Since the border strips remain intact, $fr(T) = fr(T')$ and so the term $2^{fr(T')}x^{T'}$ in the unmarked shifted tableau expansion as in (2.2) for Q-function $S_{\lambda'/\mu'}$ is identical to the $2^{fr(T)}x^T$ term in the unmarked shifted tableau expansion for $S_{\lambda/\mu}$. Therefore, the two functions are equal.

Since $Rot(\lambda/\mu) = \phi(\lambda'/\mu')$, (b) is a corollary of (a) and Proposition IV.13. $\square$

4.4 Proof of Theorem IV.3: The Case $\mu = \emptyset$

We continue our focus on S-functions and now begin the proof of the main result of this chapter. In this section we prove Theorem IV.3 for all $S_{\lambda/\mu}$ with $\mu = \emptyset$. For this class of skew Q-functions, whether or not S_λ is equal to a scalar multiple of a single non-skew Q-function is determined by a single characteristic: whether λ is a rectangle.

The proof will rely on showing that particular tableaux satisfy the conditions of the shifted Littlewood-Richardson rule (Theorem IV.10), and so we begin with a pair of lemmas proving that words of particular forms always satisfy the lattice property of Definition IV.9.

Definition IV.16. For partition λ, the *(i,j)-hook* is the set of positions in $D(\lambda)$ directly to the right of and directly below (i, j), including the (i, j)-position itself. The *hook length* $h(i, j)$ is the number of positions in the (i, j)-hook, counting (i, j) exactly once, so that $h(i, j) = \lambda_i + \lambda'_j - i - j + 1$.

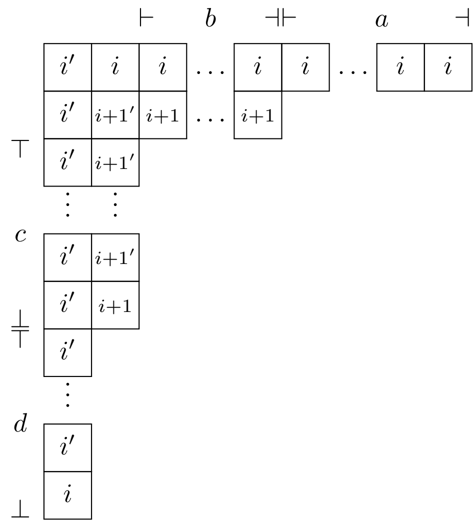


Figure 4.2: The (i, i) and $(i+1, i+1)$ Hooks of Shifted Tableau T_H

Notation IV.17. Given a word w over alphabet $\mathbf{P}'$, we will use $w|_{i,j}$ to denote the (i,j)-subword, which includes only the instances of i^* and j^* and is formed by simply deleting from w all other entries.

Lemma IV.18. *For any unshifted diagram $D(\lambda)$, let T_H be the unique shifted tableau of shape λ where all the entries in the (i, i)-hook are i^* and where the instance of i^* that is leftmost in $w(T_H)$ is unmarked. Then $w(T_H)$ satisfies the lattice property.*

Proof. Let r be the smallest integer such that $(r+1, r+1) \notin D(\lambda)$. To prove that T_H is a shifted tableau whose word satisfies the lattice property, we will show that $w(T_H)|_{i,i+1}$ satisfies the lattice property for each $i < r$ (for $i \geq r$ the lattice property is clearly satisfied). Since $i+1 \leq r$, $(i+1, i+1) \in D(\lambda)$, so that the i and $i+1$ hooks have the form in Figure 4.2 with $a, b, c, d \geq 0$. In that case,

$$w(T_H)|_{i,i+1} = i(i')^d \underline{(i+1)}(i'(i+1)')^c (i+1)^b i' i^{1+b+a}.$$

Note if $c = 0$ then the $(2, 2)$ entry in Figure 4.2 is actually unmarked and if $d = 0$ then the $(c+2, 1)$ entry is unmarked, but the word expression is still valid. During the right-to-left reading $m_i(j) - m_{i+1}(j)$ is smallest as the underlined $i+1$ is read, at which point

$$m_i(j) = 1 + b + a \geq 1 + b = m_{i+1}(j).$$

Even if $a = 0$ and equality holds, the next letter read is i' (if $d > 0$) or i so that the lattice property holds. As we finish the right-to-left reading,

$$m_i(n) = 2 + b + a > 1 + b = m_{i+1}(n)$$

where n is the length of this subword. Thus, since during the left-to-right reading of $w(T_H)|_{i,i+1}$ every instance of $(i+1)'$ is immediately preceded by i', $m_i(j) > m_{i+1}(j)$ for $n \leq j \leq 2n$. Therefore, $w(T_H)|_{i,i+1}$ satisfies the lattice property for all i and thus $w(T_H)$ does. $\square$

The content of T_H is an example of the following partition.

Notation IV.19. For any unshifted diagram $D(\lambda)$, we will use $H(\lambda)$ to denote the partition whose ith part is the (i,i)-hook length:

$$H(\lambda) = (h(1,1), h(2,2), \ldots, h(r,r))$$

where r is the smallest i such that $(i+1, i+1) \notin D(\lambda)$.

Lemma IV.20. *If a word $w = w_1 w_2 \cdots w_{n-1}$ in i^* and $(i+1)^*$ satisfies the lattice property, then the word formed by prepending an i, $w' = i w_1 w_2 \cdots w_{n-1}$, also satisfies the lattice property.*

Proof. During the right-to-left reading up to the prepended i, w' is identical to w and so satisfies the lattice property to this point. As each letter is read during the left-to-right reading of w, m_i must be at least as great as m_{i+1} since w satisfies the lattice property. The prepended i in w' increases m_i by one and leaves m_{i+1} unchanged, so that m_i is strictly greater than m_{i+1} during the left-to-right reading of w', and therefore w' satisfies the lattice property. $\square$

1^*	1	1	1	1	1	1	1
$\circ$	2^*	2	2	2	2	2	2
$\circ$	$\circ$	3^*	3	3	3	3	3
$\circ$	$\circ$	$\circ$	4^*	4	4	4	4
$\circ$	$\circ$	$\circ$	$\circ$	5^*	5	5	5

Partial

$1'$	1	1	1	1	1	1	1
$1'$	$2'$	2	2	2	2	2	2
$1'$	$2'$	$3'$	3	3	3	3	3
$1'$	$2'$	$3'$	$4'$	4	4	4	4
1	2	3	4	5	5	5	5

Complete

Figure 4.3: The Shifted Tableau of Unshifted Shape $\lambda/\mu = 8^5/\emptyset$

We now proceed to the proof of the theorem in the case of $\mu = \emptyset$. We have split this case into two subcases. First we will show that for any rectangular $\lambda = m^k$ $S_{(m^k)} = Q_{(m+k-1,m+k-2,\ldots,|m-k|+1)}$.

Proof of Theorem IV.3*, The Case* $\mu = \emptyset$ *and* $\lambda = m^k$: By Proposition IV.15, $S_{(m^k)} = S_{(k^m)}$ and so it is sufficient to show the result holds for $m \geq k$. We will show that the conditions in the shifted LR rule allow only a single shifted tableau of unshifted shape m^k. Let T be such a tableau and $w = w(T) = w_1 w_2 \cdots w_{mk}$. Since $m_i(0) = 0$ for all i, the lattice property (4.1a) requires that $w_{mk} = 1^*$, and since the row must be weakly increasing left-to-right with at most one $1'$, the first row of T is 1^*1^{m-1}. Since (4.1a) forces the rightmost entry of the second row to be $< 3'$ and since there can be at most one 1 in each column, the second row must be $\circ 2^* 2^{m-2}$, where the $\circ$ indicates a position whose entry is yet to be determined. Following the same logic for the lower rows, we know that the ith row of T, from the (i,i)-position of $D(m^k)$ to the right must have entries $i^* i i \cdots i = i^* i^{m-i}$. See the partial tableau of shape 8^5 in Figure 4.3 for an example of the entries deduced thus far.

Now if in T the number of unmarked i entries is equal to the number of unmarked $(i-1)$ entries, then in terms of the statistics $m_{i-1}(mk) = m_i(mk)$ after completing the right-to-left reading. Since condition (ii) of the shifted LR rule requires the leftmost instance of $(i-1)^*$ to be unmarked, during the left-to-right reading of w we would read an $(i-1)$ before m_{i-1} is increased by reading the first $(i-1)'$, and so w would fail the lattice property condition in (4.1b). Thus for all i, $1 < i \leq k$, it must be that the number of unmarked $(i-1)$ entries is strictly greater than the number of unmarked i entries. Since the leftmost instance of 1^* in w must be unmarked and there is at most one 1 per column, there must be a 1 in the first column so that the number of 1 entries is m. Then for all $i \geq 1$ the number of unmarked i entries is no more than $m+1-i$.

Since $i' < i$ and the leftmost instance of i^* in w must be unmarked, we know that there are no marked entries in the lowest row. Thus, as we see in the partial tableau in Figure 4.3 where the 5^* entry must be an unmarked 5, the tableau already includes the maximum number of k entries, $m+1-k$. Then $T(k, k-1)$, the entry in the bottom row immediately left of the entries k^{m+1-k}, must be unmarked and $(k-1)$ or smaller; since it is also below a $(k-1)^*$, $T(k, k-1) = k-1$ and thus $T(k-1, k-1) = (k-1)'$. But now the number of $k-1$ entries is $m-k+2$, the maximum allowed, so that $T(k, k-2)$ must be unmarked and

$$(k-2)^* \leq T(k-2, k-2) \leq T(k, k-2) \leq T(k, k-1) = (k-2),$$

forcing $T(k, k-2) = k-2$. Therefore, continuing this logic, the kth row of T must be $123 \cdots (k-1)k^{m+1-k}$. The definition of shifted tableaux (I.21) then uniquely determines the remaining entries, including the marking of the entries $T(i, i)$, so that all entries in the (i, i)-hook are i^* with the entry in the lowest row unmarked, as seen in the complete tableau for 8^5 in Figure 4.3.

In this T all $m+k-2i+1$ entries in the (i, i)-hook are i^*, so that $w(T)$ does satisfy the lattice property by Lemma IV.18 and the content of T is

$$(m+k-1, m+k-3, \ldots, m-k+1).$$

Thus for $\lambda = (m^k)$ there is exactly one shifted tableau T satisfying the conditions of the shifted LR rule. Therefore, in the Q-expansion as in (4.2) for $S_{(m^k)}$, $f^{\lambda}_{\mu\nu} = 1$ for $v = (m+k-1, m+k-3, \ldots, m-k+1)$ while $f^{\lambda}_{\mu\nu} = 0$ for all other ν and so the theorem holds in this case. $\square$

We now proceed to the remaining cases where $\mu = \emptyset$. We will prove that for all other $\lambda \neq m^k$ there is no ν and c such that $S_\lambda = cQ_\nu$ by showing there are at least two tableaux with different content partitions satisfying the conditions of the shifted LR rule (Theorem IV.10).

Proof of Theorem IV.3*, The Case* $\mu = \emptyset$ *and* $\lambda \neq m^k$: Since $S_\lambda = S_{\lambda'}$ by Lemma IV.15, it is sufficient to show the theorem holds for either. Therefore, for the smallest i such that $h(i,i) - 2 > h(i+1, i+1)$, which is the smallest i such that $\lambda_i > \lambda_{i+1}$ or $\lambda'_i > \lambda'_{i+1}$, we will assume that $\lambda_i > \lambda_{i+1}$.

First, consider the unique shifted tableau T_H where all the entries of the (i,i)-hook of $D(\lambda)$ are i^* and with the leftmost instance of i^* in $w(T_H)$ unmarked for each $i \geq 1$ to satisfy condition (ii) in the shifted LR rule. The content of T_H is the partition $H(\lambda)$. By Lemma IV.18, we know that T_H satisfies the lattice property. Therefore, by the shifted LR rule the coefficient of $Q_{H(\lambda)}$ in S_λ is nonzero.

Next we will form a tableau T_a whose content is a partition distinct from $H(\lambda)$. Although the proof is valid for any value of i, to simplify the argument we will assume that the smallest i such that $h(i,i) - 2 > h(i+1, i+1)$ is $i = 2$. In the case that λ is a hook, so that $i = 1$, the proof is even simpler since, when we consider whether the tableau satisfies the lattice property below, we will not need to compare 0 and 1. Then, given the assumptions in the first paragraph, $\lambda_1 = \lambda_2 > \lambda_3$, else we would have chosen to show the result for $S_{\lambda'}$. Also, $\lambda'_1 = \lambda'_2 \geq \lambda'_3$, so that every row except the first includes a position in the $(2,2)$-hook. Since by assumption λ is not a rectangle,

$1'$	1	1
$1'$	2	2
1	3	

$\lambda = 332$

$1'$	1	1	1
$1'$	$2'$	2	2
$1'$	2	3	
1	3		

$\lambda = 4432$

$1'$	1	1	1	1
$1'$	$2'$	2	2	2
$1'$	$2'$	$3'$	3	
$1'$	2	$3'$	$4'$	
1	3	3	4	

$\lambda = 55444$

Figure 4.4: Examples of Tableaux of Form T_a

$\lambda_3 > 0$ so that both of the bottom two rows of T_H include a 2^*. To form T_a we begin with T_H and make the following changes:

(i) replace the 2 in the lowest row, which is the leftmost 2^* in $w(T_H)$, with a 3; and

(ii) replace the $2'$ in the row second from the bottom with a 2, which is now the leftmost 2^* in $w(T_a)$, in order to satisfy condition (ii) of the shifted LR rule.

Figure 4.4 shows T_a for several small examples for which 2 is the smallest i such that $h(i,i) - 2 > h(i+1,i+1)$.

We have added an unmarked 3 and changed the new leftmost 2^* entry in $w(T_a)$ to be unmarked, so that we have already ensured that condition (ii) of the shifted LR rule is satisfied. Since in T_a the only changes from T_H involve 2^* and 3^* entries, $w(T_a)$ will satisfy the lattice property as long as the subwords $w(T_a)|_{1,2}$, $w(T_a)|_{2,3}$, and $w(T_a)|_{3,4}$ each do. If $(3,3) \in D(\lambda)$, then the $(1,1)$-, $(2,2)$-, and $(3,3)$-hooks of T_a form the general diagram in Figure 4.5 where $a > 0$ and $b, c, d \geq 0$. If $(3,3) \notin D(\lambda)$, then the $(3,3)$-hook is empty so $b = c = 0$ in Figure 4.5, but since λ is not a rectangle it must be that $(2,3) \in D(\lambda)$, and so the number of unmarked 2 entries in the second row is $1 + a$ with $a \geq 0$. A small example of T_a for a diagram with empty $(3,3)$-hook is $\lambda = 332$ in Figure 4.4.

First, we consider the subword containing only 1^* and 2^* entries. In the diagram formed by the 1^* and 2^* entries, all the entries in the $(1,1)$-hook are 1^* and all the

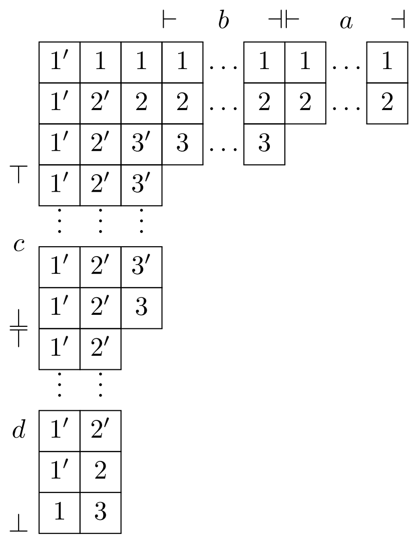


Figure 4.5: The $(1,1)$-, $(2,2)$-, and $(3,3)$-Hooks of a General Tableau of Form T_a

entries in the $(2,2)$-hook are 2^*, and so $w(T_a)|_{1,2}$ satisfies the lattice property by Lemma IV.18. Second, we consider the subword containing only 3^* and 4^* entries. Since in T_H the 3^* entries form the $(3,3)$-hook and the 4^* entries form the $(4,4)$-hook, by Lemma IV.18 we know that the $w(T_H)|_{3,4}$ satisfies the lattice property. In forming T_a, the only change we make to $w(T_H)|_{3,4}$ to form $w(T_a)|_{3,4}$ is prepending a 3, so $w(T_a)|_{3,4}$ satisfies the lattice property by Lemma IV.20.

Finally, we consider the subword containing only 2^* and 3^* entries. Now if the $(3,3)$-hook is empty, $w(T_a)|_{2,3}$ consists of a 3 followed by 2^* entries at least two of which are unmarked, for example $\lambda = 332$ in Figure 4.4 with subword 322, and so the lattice property is clearly satisfied. Thus we may assume that $(3,3) \in D(\lambda)$. Then

$$w(T_H)|_{2,3} = 2(2')^d 3(2'3')^c 3^b 2' 2^{1+b+a}$$

with $a > 0$ and $b, c, d \geq 0$. The algorithm changes the leftmost 2 to a 3 and the leftmost $2'$ to a 2, so that

$$w(T_a)|_{2,3} = \begin{cases} \underline{32}(2')^{d-1}3(2'3')^{c}3^{b}2'2^{1+b+a} & \text{if } d > 0 \\ \underline{3}3\underline{2}3'(2'3')^{c-1}3^{b}2'2^{1+b+a} & \text{if } d = 0, c > 0 \\ \underline{3}33^{b}\underline{2}2^{1+b+a} & \text{if } d = c = 0 \end{cases}$$

where the entries that have changed from $w(T_H)|_{2,3}$ are underlined. In each case since $a > 0$, $w(T_a)|_{2,3}$ satisfies the lattice property.

Therefore, T_a is a tableau satisfying the shifted LR rule with content

$$\nu = (h(1,1), h(2,2) - 1, h(3,3) + 1, h(4,4), \ldots, h(r,r)).$$

Since by assumption 2 is the smallest i such that $h(i,i) - 2 > h(i+1, i+1)$, $h(2,2) - 1 > h(3,3) + 1$ and so ν is indeed a strict partition distinct from $H(\lambda)$. Thus there is a second term Q_ν in the Q-expansion of S_λ with a non-zero coefficient:

$$S_\lambda = a_1 Q_{H(\lambda)} + a_2 Q_\nu + \cdots$$

with $a_1, a_2 \neq 0$ and, therefore, the theorem holds for all $\lambda \neq m^k$ and $\mu = \emptyset$. □

4.5 Proof of Theorem IV.3: The Case $\mu \neq \emptyset$

To finish the proof of Theorem IV.3, we now turn our focus from non-skew to skew S-functions, and consider the question of whether there are any cases with $\mu \neq \emptyset$ such that $S_{\lambda/\mu} = cQ_\nu$. We claim $S_{21/1}$ is the single example.

This proof relies on different but equivalent expressions for the s- and S-functions than our definitions. As mentioned in Stembridge ([12]):

Proposition IV.21. *There is a (surjective) ring homomorphism $\theta : \Lambda_{\mathbb{Q}} \to \Omega_{\mathbb{Q}}$ such that $S_\lambda = \theta(s_\lambda)$ for any $D(\lambda)$.*

We claim there is a similar expression for skew S-functions.

Lemma IV.22. *For any diagram $D(\lambda/\mu)$, $S_{\lambda/\mu} = \theta(s_{\lambda/\mu})$.*

Proof. By the Jacobi-Trudi Identity (e.g. [10], 7.16), for any diagram $D(\lambda/\mu)$,

$$s_{\lambda/\mu} = det[s_{\lambda_i-\mu_j-i+j}]. \tag{4.3}$$

For any unshifted diagram $D(\lambda/\mu)$, $S_{\lambda/\mu}$ is a skew Q-function with shape combinatorially equivalent to $\lambda+\delta(l(\lambda))/\mu+\delta(l(\lambda))$. Any $S_{\lambda/\mu}$ has the following Jósefiak-Pragacz Pfaffian expression ([5]; [6], 3.8 Example 9):

$$S_{\lambda/\mu} = Pf(M_{\lambda+\delta,\mu+\delta})$$

where $M_{\lambda+\delta,\mu+\delta}$ is the skew symmetric block matrix

$$M_{\lambda+\delta,\mu+\delta} = \begin{pmatrix} M_{\lambda+\delta} & N_{\lambda+\delta,\mu+\delta} \\ -N^T_{\lambda+\delta,\mu+\delta} & 0 \end{pmatrix}$$

in which all blocks are $l(\lambda)\times l(\lambda)$ matrices of Q-functions, and specifically $N_{\lambda+\delta,\mu+\delta} = (Q_{\lambda_i-\mu_j-i+j})$. In this case,

$$Pf(M_{\lambda+\delta,\mu+\delta}) = det(N_{\lambda+\delta,\mu+\delta}) = det[Q_{\lambda_i-\mu_j-i+j}].$$

Applying θ to the expansion of $s_{\lambda/\mu}$ in (4.3), we see

$$\begin{aligned}\theta(s_{\lambda/\mu}) &= \theta(det[s_{\lambda_i-\mu_j-i+j}]) \\ &= det[S_{\lambda_i-\mu_j-i+j}] \\ &= det[Q_{\lambda_i-\mu_j-i+j}] \\ &= S_{\lambda/\mu}\end{aligned}$$

□

Applying θ to the expansion

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu} \tag{4.4}$$

from the LR rule (Theorem IV.6), we have

$$S_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} S_{\nu}$$

summed over all partitions ν with the $c^{\lambda}_{\mu\nu}$ the LR coefficients and hence non-negative integers. From the proof of Theorem IV.3 for the case $\mu = \emptyset$, we know that if there is a $\nu \neq m^k$ with $c^{\lambda}_{\mu\nu} > 0$ then there are at least two Q-functions with non-zero coefficients in the expansion of S_{ν} and hence in the expansion of $S_{\lambda/\mu}$. Thus it is sufficient to show that in the s-expansion of $s_{\lambda/\mu}$ there is a non-zero $c^{\lambda}_{\mu\nu}$ such that ν is not a rectangle.

As previously mentioned, when ν is the leading term partition of $s_{\lambda/\mu}$, the LR coefficient is 1. This leading term partition is a rectangle if and only if all the columns have the same length. Recall that for the ring involution ω of Definition I.29, $\omega(s_{\lambda/\mu}) = s_{\lambda'/\mu'}$. By applying ω and then θ to the expansion of $s_{\lambda/\mu}$ in (4.4), we have

$$S_{\lambda'/\mu'} = \sum_{\nu} c^{\lambda}_{\mu\nu} S_{\nu'}$$

which, as with $S_{\lambda/\mu}$, will have at least two terms in its Q-expansion if the column set of $D(\lambda'/\mu')$ is not a rectangle. Since $S_{\lambda/\mu} = S_{\lambda'/\mu'}$ by Proposition IV.15.(a), $S_{\lambda/\mu}$ will have at least two terms in its Q-expansion unless all the columns have length k and all rows have length m for some constants k and m. Thus, if $S_{\lambda/\mu} = cQ_{\nu}$, each connected component of $D(\lambda/\mu)$ must be a k by m rectangle; for example, see Figure 4.6 where each connected component is rectangular with two rows of length three.

First, if $D(\lambda/\mu)$ has a single connected component, then $\mu = \emptyset$, which is the case we have already demonstrated. Second, for $\lambda/\mu = 21/1$, since the only strict partition of size two is $\nu = 2$, $S_{21/1}$ is a constant multiple of Q_2. We can determine the

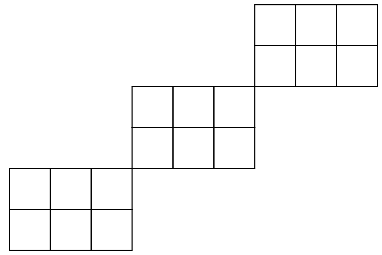


Figure 4.6: A Skew λ/μ with Constant Row and Column Lengths

scale factor of the leading terms, and thus of the functions, by counting the number of marked shifted tableaux of each shape with all 1^* entries, so that $S_{21/1} = 2Q_2$.

Finally, consider the remaining cases, for which $D(\lambda/\mu)$ has $a \geq 2$ connected components each a rectangle of k rows of length m. The LR coefficient $c^{\lambda}_{\mu\nu}$ counts the number of unshifted tableaux of shape λ/μ and content ν. There is a single tableau T_L with the leading term content $(am)^k$, in which all entries in the ith row of each of the a connected components are i. For this tableau $w(T_L) = (1^m 2^m \cdots k^m)^a$, which clearly satisfies the lattice property of Definition IV.5. If we change the rightmost k in the bottom row of $D(\lambda/\mu)$ to $k+1$ to form tableau T_a, then

$$w(T_a) = (1^m 2^m \cdots k^m)^{a-1} 1^m 2^m \cdots (k-1)^m (k+1) k^{m-1},$$

which also satisfies the lattice property when the number of connected components is $a \geq 2$. Thus, s_ν with $\nu = (am, am, \ldots, am, am-1, 1)$ has non-zero coefficient in $s_{\lambda/\mu}$, and so S_ν has non-zero coefficient in $S_{\lambda/\mu}$. Except when $m = k = 1$ and $a = 2$ which is the case $\lambda/\mu = 21/1$ discussed above, ν is not a rectangle. Therefore, there are at least two Q-functions with non-zero coefficients in the expansion S_ν, and hence in the expansion of $S_{\lambda/\mu}$ as claimed. $\square$

4.6 Alternate Proof of Theorem IV.3: The Case $m^k/\emptyset$

In this section, we show an alternate proof for the case

$$S_{(m^k)} = Q_{(m+k-1,m+k-3,\ldots,|m-k|+1)} \tag{4.5}$$

that does not require finding specific tableaux. Yet a third proof is provided by Worley ([14]).

Definition IV.23. For any unshifted diagram $D(\lambda/\mu)$, a *standard tableau* of shape λ/μ is an assignment to the positions in $D(\lambda/\mu)$ of symbols $1, 2, \ldots, n$ with $n = |\lambda/\mu|$ such that each appears exactly once with both columns and rows strictly increasing.

Definition IV.24. For any shifted diagram $D'(\lambda/\mu)$, a *shifted standard tableau* of shape λ/μ is an assignment to the positions in $D'(\lambda/\mu)$ of (unmarked) symbols $1, 2, \ldots, n$ with $n = |\lambda/\mu|$ such that each appears exactly once with both columns and rows strictly increasing.

For non-skew shapes, there are closed formulas for the number of standard tableaux and the number of shifted standard tableaux:

Lemma IV.25.

(a) *For any partition λ of n, if f^λ represents the number of standard tableaux of shape λ, then:*

$$f^\lambda = \frac{n!}{\prod_i(\lambda_i + l - i)!}\prod_{i<j}(\lambda_i - \lambda_j + j - i) \tag{4.6}$$

for $1 \leq i, j \leq l(\lambda) = l$ (e.g. [10], 7.21).

(b) *For any strict partition ν of n, if g^ν represents the number of shifted standard tableaux of shape ν, then:*

$$g^\nu = \frac{n!}{\prod_i \nu_i!}\prod_{i<j}\frac{\nu_i - \nu_j}{\nu_i + \nu_j} \tag{4.7}$$

for $1 \leq i, j \leq l(\lambda) = l$ *(e.g. [6], 3.8 Example 12).*

Proposition IV.26. *For any* m *and* k*,*

$$f^{(m^k)} = g^{H(m^k)}$$

where $H(m^k) = (m+k-1, m+k-3, \ldots, |m-k|+1)$.

Proof. Substituting $\lambda_i = m$ and $\lambda'_i = k$ for all i into (4.6):

$$\begin{aligned} f^{(m^k)} &= \frac{n!}{\prod_i (m+k-i)!} \prod_{i<j} (m-m+j-i) \\ &= \frac{n!}{\prod_i (m+k-i)!} \prod_{i<j} (j-i). \end{aligned} \tag{4.8}$$

For $\nu = H(m^k)$, since ν_i is the (i,i)-hook length of $D(m^k)$:

$$\nu_i = \lambda_i + \lambda'_i - i - i + 1 = m + k - 2i + 1,$$

$$\nu_i - \nu_j = (m+k-2i+1) - (m+k-2j+1) = 2(j-i), \text{ and}$$

$$\nu_i + \nu_j = (m+k-2i+1) + (m+k-2j+2) = 2(m+k-i-j+1).$$

Substituting these into (4.7):

$$\begin{aligned} g^{H(m^k)} &= \frac{n!}{\prod_i (m+k-2i+1)!} \prod_{i<j} \frac{2(j-i)}{2(m+k-i-j+1)} \\ &= \frac{n!}{\prod_i (m+k-2i+1)!} \prod_{i<j} \frac{j-i}{m+k-i-j+1}. \end{aligned} \tag{4.9}$$

To show that this is equivalent to $f^{(m^k)}$, we regroup the factors of the denominator:

$$\begin{aligned}
\prod_i (m+k-2i+1)! \prod_{i<j} (m+k-i-j+1) \\
&= \prod_j \Big((m+k-2j+1)! \prod_{i<j} (m+k-i-j+1) \Big) \\
&= \prod_j (m+k-2j+1)!(m+k-(j-1)-j+1) \\
&\qquad (m+k-(j-2)-j+1)\cdots(m+k-1-j+1) \\
&= \prod_j (m+k-2j+1)!(m+k-2j+2)(m+k-2j+3)\cdots(m+k-j) \\
&= \prod_j (m+k-j)!
\end{aligned}$$

so that (4.9) becomes

$$g^{H(m^k)} = \frac{n!}{\prod_i (m+k-i)!} \prod_{i<j} (j-i),$$

which is the expression we found for $f^{(m^k)}$ in (4.8). □

Remark IV.27. Haiman has given a bijective proof of this identity for $k \leq m$ ([4], Proposition 8.11).

We may now proceed with the alternate proof of Theorem IV.3 for the case $\lambda/\mu = m^k/\emptyset$, for any m and k. From Definition I.31 of S_λ and Definition II.9 of unmarked shifted tableaux, we know that $S_\lambda = \sum 2^{fr(T)} x^T$ summed over all unmarked shifted tableaux T of unshifted diagram $D(\lambda)$. Let $n = |\lambda|$. The unmarked shifted tableaux whose entries are $1, 2, \ldots, n$ without repetition are the shifted standard tableaux of unshifted shape λ, but are also the unshifted standard tableaux of shape λ. Thus, the coefficient of $x_1 x_2 \cdots x_n$ in S_λ is $2^n f^\lambda$.

Now from Definition I.32 of Q_ν and Definition II.9 of unmarked shifted tableaux, we know that $Q_\nu = \sum 2^{fr(T)} x^T$ summed over all unmarked shifted tableaux of shape

ν. Thus for ν of size n, the coefficient of $x_1x_2\cdots x_n$ in Q_ν is $2^n g^\nu$. Since S_λ is a skew Q-function, $S_\lambda = \sum a_{\lambda\nu}Q_\nu$ with integers $a_{\lambda\nu} \geq 0$ by the shifted LR rule. Thus the coefficient of $x_1x_2\cdots x_n$ in S_λ is:

$$2^n \sum_\nu a_{\lambda\nu} g^\nu$$

summed over all strict partitions ν of size n.

These two expressions for the coefficient of $x_1x_2\cdots x_n$ in S_λ must be equal. The ith part of the leading term partition of S_λ is the (i,i)-hook length of $D(\lambda)$, in this case the partition $H(m^k) = (m+k-1, m+k-3, \ldots, |m-k|+1)$. For both $S_{(m^k)}$ and $Q_{H(m^k)}$, the number of free entries in the leading term tableau is the number of non-zero (i,i)-hook lengths of $D(m^k)$, so $a_{\lambda\nu} = 1$ for $Q_{H(m^k)}$ in $S_{(m^k)}$. By Proposition IV.26, $f^{(m^k)} = g^{H(m^k)}$, and since all the shifted LR coefficients are non-negative, we must have $a_{\lambda\nu} = 0$ in $S_{(m^k)}$ for all other ν. Therefore, (4.5) holds. $\square$

CHAPTER V

Equality between $s_{\delta(n)/\lambda}$ and Q_ν

We now consider equality between a third pairing of symmetric functions:

Question V.1. *When is a (possibly skew) s-function equal to a constant multiple of a (possibly skew) Q-function?*

We focus on the particular case:

Question V.2. *When is a (possibly skew) s-function equal to a constant multiple of a non-skew Q-function?*

From Definition I.32 of P-functions, P- and Q-functions only differ by a power of two, so we could have posed these questions equivalently in terms of P-functions. In this chapter we answer Question V.2 completely.

Theorem V.3. *If $D(\lambda/\mu)$ is a diagram such that*

$$s_{\lambda/\mu} = c\, Q_\nu$$

for some strict partition ν and scalar c, then

$$s_{\lambda/\mu} = s_{\delta(n)/(m^k)} = P_{H(m^k)^*}$$

for some n, where $H(m^k) = (m+k-1, m+k-3, \ldots, |m-k|+1)$ and $H(m^k)^*$ is the n-complement of $H(m^k)$.

$\delta(9)/5^3$ $\qquad$ $H(5^3)^* = 986421$

Figure 5.1: An Example of Theorem V.3, $s_{\delta(9)/(5^3)} = P_{H(5^3)^*}$

5.1 The P-Expansion of $s_{\delta(n)/\lambda}$

We begin with a known special case of Theorem V.3, the case $\lambda = \emptyset$; Macdonald ([6], 3.8 Example 3) describes an alternate proof.

Lemma V.4. *For a staircase partition δ of any size, $s_\delta = P_\delta$.*

Proof. Since $s_\delta \in \Omega_{\mathbb{Q}}$, we know s_δ has a Q-expansion, and hence a P-expansion. The leading term partition of s_δ is δ, which is the strict partition of size $|\delta|$ lowest in dominance order. Hence P_δ is the only P-function with non-zero coefficient in s_δ. The number of tableaux with content δ of diagram $D(\delta)$ and shifted diagram $D'(\delta)$, and thus the leading coefficient of s_δ and P_δ, is 1 in both cases. $\square$

Since the s-functions form a basis of $\Lambda_{\mathbb{Q}}$, we may define coefficients $g_{\nu\lambda}$ for a strict partition ν and a partition λ with $|\lambda| = |\nu|$ by

$$P_\nu = \sum_\lambda g_{\nu\lambda} s_\lambda \tag{5.1}$$

where the sum is over all partitions λ. These coefficients also appear in the Q-expansion of S_λ ([11]):

$$S_\lambda = \sum_\nu g_{\nu\lambda} Q_\nu \tag{5.2}$$

summed over all strict partitions ν. Since S_λ is a skew Q-function, the $g_{\nu\lambda}$ are shifted Littlewood-Richardson coefficients (Theorem IV.10) and thus are non-negative integers. We will show the coefficients $g_{\nu\lambda}$ appear in a third expansion.

Theorem V.5. *For partitions λ and strict partitions ν, define an integer matrix $[g_{\nu\lambda}]$ by (5.1). Then*

$$s_{\delta(n)/\lambda} = \sum_{\nu} g_{\nu\lambda} P_{\nu^*} \tag{5.3}$$

summed over all strict partitions ν with $|\nu| = |\lambda|$ where ν^* is the n-complement of ν.

Proof. If we consider the tableaux contributing terms to $s_{\delta(n)}(x, y)$ as having entries from the ordered alphabet $1_x < 2_x < \ldots < 1_y < 2_y < \ldots$, we may view the tableaux as assignments of $1_x < 2_x < \ldots$ to an inner diagram $D(\lambda)$ and assignments of $1_y < 2_y < \ldots$ to an outer diagram $D(\delta(n)/\lambda)$, so that

$$s_{\delta(n)}(x, y) = \sum_{\lambda} s_{\lambda}(x) s_{\delta(n)/\lambda}(y) \tag{5.4}$$

summed over all partitions λ such that $\lambda \subseteq \delta(n)$. Similarly, considering the tableaux contributing terms to $P_{\delta(n)}(x, y)$ as having entries from the (marked) ordered alphabet $1'_x < 1_x < 2'_x < 2_x < \ldots < 1'_y < 1_y < 2'_y < 2_y < \ldots$, we may view the tableaux as assignments to an inner and an outer shifted diagram, so that:

$$P_{\delta(n)}(x, y) = \sum_{\nu} P_{\nu}(x) P_{\delta(n)/\nu}(y) \tag{5.5}$$

summed over all strict partitions ν such that $\nu \subseteq \delta(n)$. By Proposition IV.13, the Q-functions of a shifted shape and its flip are equal, and since they have an equal number of entries on the main diagonal, so are their P-functions. By Corollary IV.14, the flip of $\delta(n)/\nu$ is the non-skew shifted shape ν^*, the n-complement of ν. Thus (5.5) becomes

$$P_{\delta(n)}(x, y) = \sum_{\nu} P_{\nu}(x) P_{\nu^*}(y). \tag{5.6}$$

From the definition of the integers $g_{\nu\lambda}$ in (5.1), the coefficient of $s_{\lambda}(x)$ in each $P_{\nu}(x)$ is $g_{\nu\lambda}$. Thus, from the expansion in (5.6), the coefficient of $s_{\lambda}(x)$ in $P_{\delta(n)}(x, y)$ is

$$\sum_{\nu} g_{\nu\lambda} P_{\nu^*}(y)$$

summed over all strict partitions $\nu \subseteq \delta(n)$ with $|\lambda| = |\nu|$. From (5.4), since non-skew s-functions are linearly independent, in $s_{\delta(n)}(x, y)$ the coefficient of $s_\lambda(x)$ is $s_{\delta(n)/\lambda}(y)$. Since $P_{\delta(n)}(x, y) = s_{\delta(n)}(x, y)$ by Lemma V.4, these coefficients of $s_\lambda(x)$ are equal, and therefore

$$s_{\delta(n)/\lambda}(y) = \sum_{\nu} g_{\nu\lambda} P_{\nu^*}(y).$$

$\square$

Since, as mentioned above, the coefficients $g_{\nu\lambda}$ in the P-expansion of $s_{\delta(n)/\lambda}$ are shifted LR coefficients, we have:

Corollary V.6. *For any n and partition $\lambda \subseteq \delta(n)$, $s_{\delta(n)/\lambda}$ is P-positive; more precisely, the coefficients in the P-expansion of $s_{\delta(n)/\lambda}$ are particular shifted Littlewood-Richardson coefficients.*

Since $P_\nu = 2^{-l(\nu)} Q_\nu$, we can similarly state that $s_{\delta(n)/\lambda}$ is Q-positive and the coefficients in the Q-expansion are, up to a power of two factor, particular shifted LR coefficients. A different combinatorial interpretation of the coefficients and their positivity in the P-expansion of $s_{\delta(n)/\lambda}$ has been proven independently by Ardila and Serrano ([1]).

5.2 Equality of $s_{\delta(n)/\lambda}$ and P_{ν^*}

Proof of Theorem V.3. By Theorem III.5 a (possibly skew) s-function is in $\Omega_{\mathbb{Q}}$ if and only if each of the connected components of its diagram is combinatorially equivalent to δ/α or $Rot(\delta/\alpha)$. Since $s_{Rot(\delta/\alpha)} = s_{\delta/\alpha}$ by Lemma III.3, we see, by rotating each connected component 180 degrees if necessary, that every s-function in $\Omega_{\mathbb{Q}}$ is equal to the s-function for a shape that is combinatorially equivalent to $\delta(n)/\lambda$ for some constant n and partition λ. Thus we need only compare $s_{\delta(n)/\lambda}$ to P_ν. For $\lambda \neq m^k$, Theorem IV.3 states that in the Q-expansion of S_λ there are at least two Q_ν with non-zero coefficient; these coefficients are the $g_{\nu\lambda}$ in (5.2). Therefore, substituting these $g_{\nu\lambda}$ into (5.3), we see there are at least two P_{ν^*} with non-zero coefficients in the P-expansion of $s_{\delta(n)/\lambda}$. By Theorem IV.3, $S_{m^k} = Q_{H(m^k)}$, so that $g_{\nu\lambda} = 1$ for $\nu = H(m^k)$ and $g_{\nu\lambda} = 0$ for all other ν in (5.2). Therefore, substituting the coefficients into (5.3), we have $s_{\delta/m^k} = P_{H(m^k)^*}$. □

Recall from Lemma IV.25 that for a non-skew diagram $D(\lambda)$ there is a closed formula for the number of standard tableaux f^λ; similarly, for a shifted diagram $D'(\nu)$ there is a closed formula for the number of shifted standard tableaux g^ν. In general, for a skew diagram $D(\lambda/\mu)$ there is no known product formula for the number of standard tableaux, which we may denote $f^{\lambda/\mu}$. However, by Theorem V.3 in the particular case of $\lambda/\mu = \delta(n)/m^k$, we may find the number of standard tableaux for this skew shape by counting the number of shifted standard tableaux of the non-skew shifted shape $H(m^k)^*$.

Corollary V.7. *For any n, m and k with $H(m^k)^*$ the n-complement of $H(m^k)$,*

$$f^{\delta(n)/m^k} = g^{H(m^k)^*}.$$

We now focus on the result of Theorem V.3 in the special case when $D(\delta(n)/m^k)$ disconnects. The diagram $D(\delta(n)/m^k)$ is connected for $m + k < n$, but disconnects for $m + k = n$. In the latter case, $s_{\delta(n)/(m^k)}$ factors into $s_{\delta(m)}s_{\delta(k)}$, and thus by Theorem V.3 so does $P_{H(m^k)^*}$, even though $D'(H(m^k)^*)$ is connected and non-skew. From Lemma V.4 $s_{\delta(n)} = P_{\delta(n)}$, so that:

Corollary V.8. *For shifted diagram $D'(H(m^k)^*)$ where $H(m^k)^*$ is the $(m + k)$-complement of $H(m^k) = (m + k - 1, m + k - 3, \ldots, |m - k| + 1)$,*

$$P_{H(m^k)^*} = P_{\delta(m)}P_{\delta(k)}.$$

Worley ([14]) provides an alternate proof of this factorization, since in this case $H(m^k)^* = \delta(m) + \delta(k)$.

We have already seen an example of such a shape in Figure 4.1: 6421 is the 6-complement of $53 = H(4^2)$, and so $P_{6421} = P_{4321}P_{21}$. This is an interesting difference between s-functions and P-functions (equivalently, Q-functions), since the s-function for a connected diagram is an irreducible polynomial (e.g. [2]), and thus suggests the inclusion of an additional question in future examinations of the relationships among the elements of $\Omega_{\mathbb{Q}}$:

Question V.9. *When is a (possibly skew) Q-function for a connected shifted diagram an irreducible polynomial?*

Corollary V.8 provides a class of reducible Q-functions. In fact, we have already seen an example of a reducible Q-function in Theorem IV.3,

$$Q_2 = \frac{1}{2}S_{21/1} = \frac{1}{2}Q_1^2,$$

which is the Q-function version of the case $m = k = 1$ of Corollary V.8. For any strict partitions λ and μ, it is also known that $Q^2_{\lambda/\mu}$ is, up to a power of 2, a skew Q-function, see ([6], 3.8 Example 10).

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