Sensor Scheduling Under Energy Constraints

by

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To my parents, my teachers and my friends

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ABSTRACT

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Recent advancement of wireless technologies and electronics has enabled the development of low-cost wireless sensor networks (WSN). The development of wireless sensor networks has also been motivated by military applications such as battlefield surveillance and target tracking. They are now used in various application areas, including habitat monitoring, industrial process monitoring and control, environment monitoring, health care applications, home automation, and traffic control.

In this dissertation we investigate sensor scheduling problems under energy constraints through three scenarios: stationary parameter estimation, dynamic parameter tracking and discrete search. We first formulate a stochastic resource allocation problem for the stationary parameter estimation scenario with a sensor-dependent, parameter-dependent observation model. With the Gaussian assumption and linear observation model, the original problem is equivalent to a deterministic resource allocation problem. We propose a greedy algorithm and identify conditions sufficient to guarantee its optimality. Thereafter we formulate the parameter estimation problem with a sensor-dependent parameter-independent observation model as a static allocation problem. We

derive lower bound on the optimal performance and propose a preprocessing algorithm to improve the lower bound. We use the improved lower bound to evaluate the performance of the proposed greedy strategy. Subsequently, we investigate the dynamic parameter tracking problem and discover the structure of an optimal strategy. For the discrete search problem with multiple sensors, we develop an easily implementable greedy strategy and identify conditions sufficient to guarantee its optimality. We discuss the relationship between each problem and the multi-armed bandit problem.

CHAPTER 1

INTRODUCTION

1.1 Motivation

Recent advancement of wireless technologies and electronics has enabled the development of low-cost low-power wireless sensor networks (WSN). A WSN consists of spatially distributed autonomous sensors that can cooperatively monitor various environmental conditions, such as temperature, sound, light, vibration, pressure, motion or pollutants. The development of wireless sensor networks has also been motivated by military applications such as battlefield surveillance and target tracking. WSN are now used in various application areas, including habitat monitoring, industrial process monitoring and control, environment monitoring, health care applications, home automation, and traffic control (see [1]).

Irrespective of the application, a common features underlying all of the above include, (i) a WSN is typically composed of a potentially large number of unattended sensor nodes which may be densely deployed; (ii) each node in a sensor network is typically equipped with sensing hardware, a radio transceiver, a micro-controller unit, memory and an energy source, usually batteries; and (iii) each node needs to communicate with either other sensor nodes or some central controller directly.

Powered by battery, a sensor node has finite energy reserves. At the same time, it may be difficult or infeasible to recharge or replace these batteries due to the large scale of the network or harsh environment. Therefore, energy efficiency is one of the most important factors that need to be considered in the design and operation of sensor networks.

Below are a list of sub-systems of a sensor node that consume energy. (see [2])

- A **computing** sub-system consists of a micro-controller, which is responsible for the control of the sensors and execution of communication protocols. Usually the micro-controller operates under various operation modes for energy management purposes. But switching between these operating modes also consumes power.

- A **communicating** sub-system usually consists of a short range radio which is used to communicate with neighbor nodes or a central controller. Radios can operate under the Transmit, Receive, Idle and Sleep modes. The energy consumption in idle mode is quite significant, often observed to be on the same order as the receiving mode.

- A **sensing** sub-system consists of a group of sensors and actuators that link the node to the outside world. Energy consumption can be reduced by using low power components and trading off less important performance.

- A **power supply** system usually consists of a battery which supplies power to the sensor node.

In a sensor network, energy awareness not only needs to be incorporated in the individual nodes, but also into groups of communicating nodes and the entire network. In order to prolong the lifetime of the sensor network, there is a variety of things one can do, such as (1) deploying redundant sensors so as to maintain connectivity and coverage even as sensor nodes' energy starts to deplete; (2) using hierarchical organization to decrease sensor node's communication distance; (3) spreading the processing and communication loads evenly among or in proportion to their available resources, such as rotating the role of cluster heads or enabling sensor nodes to take turns turning off their transceivers; (4) enabling nodes with more battery, processing or memory resources to participate more in network coordination, data aggregation and data dissemination; and (5) managing the data flow, which is explained next.

Since communication consumes significant amount of energy, data flow in the sensor network is one of the most important factors to be considered. Data acquisition and dissemination in sensor networks may be categorized into time-driven, event-driven and demand-driven. In time-driven networks, sensor nodes collect and report data from the physical environment periodically, such as in a temperature monitoring sensor network, where each sensor periodically takes a temperature measurement and sends it back to the central controller. Here the frequency of the data collection is crucial since it needs to be frequent enough to provide enough information to the controller and in the meantime, it needs to be sparse enough to conserve energy. In intrusion detection or event notification system, event-driven sensing is used in general. For example, in a fire alarm system, if the particulate density is larger than a threshold, a signal will be transmitted back to the central controller. Here the threshold is crucial since we want to keep the detection probability high while the false alarm probability is small. In a demand-driven sensor network, a central controller queries sensors for desired information; only the sensors that satisfy the query condition from the central controller report their sensed data. Here the central controller decides when sensing will be done and which sensors will be activated and report the data back. Some sensor networks combine more than one of the above data acquisition approaches.

Based on the nature of data processing and aggregation, sensor networks can be also categorized into distributed sensor networks and centralized sensor networks. In networks with a distributed processing architecture, individual sensors may make their own sensing, routing or data fusion decisions in order to reduce communication overhead. In networks with a centralized processing architecture, data aggregation and processing occur at the central controller. Some networks have a hybrid processing architecture, which provides a compromise by forming clusters and allowing cluster heads to process data and only cluster heads communicate with the central controller directly.

This dissertation focuses on a demand-driven network with a centralized processing architecture. Specifically, it investigates the following scenario. There is one central controller and a set of sensors, which can communicate directly with the controller. Each sensor can perform some measurement or detection task and can be activated only a limited number of times due to the energy constraint. The time horizon under consideration is finite. At each time instant, the central controller activates a set of sensors and the activated sensors then perform the sensing tasks and report the data back to the central controller. The central controller processes the data and determines the next set of sensors to be activated. This process is repeated until either the time horizon of interest has expired or a certain performance objective is achieved. In this scenario, we focus on a sequential scheduling problem and seek to answer questions including: when to wakeup a sensor to do sensing, when the sensor should report the data back to the central controller, which task each sensor should accomplish, when the sensing process should be terminated, *etc.*

In the following section, we highlight the contributions of this dissertation.

1.2 Main Contributions of this Dissertation

The general centralized sensor scheduling problem with energy constraints does not have a closed form solution, and it is difficult to discover qualitative properties of optimal sensor scheduling strategies. In this dissertation we investigate three specific problems, which are abstract versions of real applications, capture key features of the scheduling problems. In the demand-driven sensor networks, there are three categories of applications: estimation, tracking and detection. Corresponding to each application, we investigate the following problems:

1. Multiple stationary parameters estimation,

2. Single dynamic parameter tracking,

3. Discrete search.

All these problems are centralized stochastic optimization problems and can, in principle, be solved by stochastic dynamic programming (SDP). A SDP approach leads to computationally challenging problems, and does not always provide insight into the nature of the above-mentioned problems. For this reason, in this dissertation, we use the SDP approach to discover qualitative properties of optimal strategies only for the single dynamic parameter tracking problem. For the other two problems we use the following methodology to investigate allocation strategies: For each problem, we propose an easy-to-implement greedy algorithm and analyze the properties of the optimal strategy. Based on the different properties of the optimal strategies, we either derive sufficient conditions to guarantee the optimality of a greedy strategy, or obtain structural results of the nature of the optimal strategy.

Below we introduce the three sensor scheduling problems under energy constraints that we investigate, and discuss the specific contributions of the dissertation to each problems.

1.2.1 Multiple Stationary Parameters Estimation

The problem is described as follows. Multiple sensors are sequentially activated by a central controller to take measurements of one of many stationary parameters. The measurement model is a linear Gaussian observation model, which is sensor- and parameter-dependent. The controller combines successive measurement data to form an estimate for each parameter. A single parameter may be measured multiple times. Each activation incurs a cost (*e.g.,* sensing and communication cost), which is both sensor- and parameter-dependent. Assuming that sensors may be of different qualities (*i.e.,* they may have different signal-to-noise-ratios) and the activation of different sensors may incur different costs, our objective is to determine the sequence in which sensors should be activated and the corresponding sequence of parameters to be measured so as to minimize the sum of the terminal parameter estimation errors and the total sensor activation costs.

The main contributions are summarized as follows.

- A novel formulation of a sensor scheduling problem for multiple stationary parameters estimation under an energy constraint.
- The decomposition of the sensor-parameter scheduling problem into two subproblems: the first one is to determine the order of the sensors to be used; the second one is to determine the order of the parameters to be measured.
- The derivation of conditions sufficient to guarantee the optimality of a greedy strategy.

Furthermore, we consider a special case of the above-mentioned problem, where the measurement model is only sensor-dependent but parameter-independent and each activation incurs a constant cost. The additional contributions are summarized as follows.

- The development of a method to obtain a lower bound on the performance of an optimal scheduling strategy.
- The discovery of a preprocessing procedure that can be used to reduce the solution space in which the optimal strategy lies for any given set of parameters and any set of sensors; such a procedure can thereby potentially improve the lower bound on the performance of an optimal scheduling strategy.
- The use of the lower bound to effectively evaluate the performance of the greedy strategy.

1.2.2 Single Dynamic Parameter Tracking

The problem is described as follows. Multiple sensors are sequentially activated by a central controller to track a dynamic parameter. Each sensor can be activated only once. The dynamic evolution of the parameter and the measurement model are linear Gaussian. The controller decides whether a sensor should be activated to take a measurement at present, estimates the parameter along its evolution trajectory and computes the accuracy of the estimation along a finite time horizon. Each activation of a sensor incurs a cost (*e.g.,* sensing and communication cost), which is a constant. The objective is to determine at which time instants to activate a sensor so as to minimize the sum of the error covariances of the estimates along the finite time horizon and the total activation costs.

The main contributions are summarized as follows.

- The derivation of the conditions under which an optimal sensor activation strategy either hold a threshold property or a "stopping property".

1.2.3 Discrete Search Using Multiple Sensors

The problem is described as follows. Multiple sensors are sequentially activated by a central controller to search an object hidden in an area, which is divided into several cells. The prior probability that the object in certain cell is given. Each sensor can be activated a limited number of times. Each cell can be searched by at most one sensor at each instant of time. If the object is in a certain cell, say cell i, and a sensor searches cell i, it finds the object with probability α_i which is given, and is independent of precious searches of that cell. The false alarm is always 0. The objective is to determine at each time instant, which sensor set should be activated in order to maximize the total time-discounted detection probability.

The main contributions are summarized as follows.

- A novel formulation of the discrete search problem with multiple sensors.

- The investigation of properties of an optimal strategy.
- The development of an easy-to-implement greedy algorithm and the derivation of conditions sufficient to guarantee its optimality.

In the next section, we discuss the difficulties in solving the above-stated problems.

1.3 Difficulties in Solving Sensor Scheduling Problem with Energy Constraints

There is no canonical way to represent the sensor scheduling problem with energy constraints. One way to model the energy constraints is to assume that each sensor node can be activated at most once. This is done without loss of generality because multiple uses of the same sensor can be effectively replaced by multiple identical sensors, each with a single usage. In the mean time since different sensors have different energy profiles while accomplishing different tasks, a cost (which includes the sensing cost and communication cost) is introduced while a sensor does the sensing and communication. Optimality is measured by a combined performance measure that accounts for errors in parameter estimation and for sensor activation costs.

The centralized sensor scheduling problem with energy constraints can be formulated as a stochastic sequential allocation problem, which has been extensively studied in the literature (see [3]). It is in general difficult to explicitly determine optimal strategies or even qualitative properties of optimal strategies for these problems. One exception is the multi-armed bandit problem and its variants (see [4–16]). This is a class of sequential allocation problems where qualitative properties of optimal allocation strategies or even optimal solutions have been explicitly determined.

Unfortunately, the centralized sensor scheduling problem with energy constraints does not belong to the class of multi-armed bandit problems and its variants, and it appears difficult to explicitly/analytically determine the nature of an optimal solution of the general problem under consideration. Wherever appropriate, we discuss the relationship between the problem we investigated and the multi-armed bandit problem.

1.4 Dissertation Organization

The dissertation is organized as follows: In Chapters 2 and 3, we formulate and analyze a sensor scheduling problem under an energy constraint for static parameter estimation. In Chapter 4, we investigate a sensor scheduling problem under an energy constraint for dynamic target tracking. In Chapter 5 we investigate a discrete search problem under an energy constraint. We conclude in Chapter 6. In Appendices A–D we present proofs of results appearing in Chapters 2–5.

CHAPTER 2

MULTIPLE STATIONARY PARAMETERS ESTIMATION: PART I

Advances in integrated sensing and wireless technologies have enabled a wide range of emerging applications, from environmental monitoring to intrusion detection, to robotic exploration. In particular, unattended ground sensors have been increasingly used to enhance situational awareness for surveillance and monitoring purposes.

In this chapter we study the use of sensors for the purpose of parameter estimation. Specifically, we consider the following scheduling problem. Multiple sensors are sequentially activated by a central controller to take measurements of one of many parameters. The controller combines successive measurement data to form an estimate for each parameter. A single parameter may be measured multiple times. Each activation incurs a cost (*e.g.,* sensing and communication cost), which may be both sensor- and parameter-dependent. This process continues until a certain criterion is satisfied, *e.g.,* when the total estimation error is sufficiently small, or when the time period of interest has elapsed, *etc.* Assuming that sensors may be of different quality (*i.e.,* they may have different signal-to-noise-ratios) and the activation of different sensors may incur different costs, our objective is to determine the sequence in which sensors should be activated and the corresponding sequence of parameters to be measured so as to minimize the sum of the terminal parameter estimation errors and the sensor activation cost.

This chapter is organized as follows: In Section 2.1 We present a literature survey and state the contribution of this chapter. In Section 2.2 we formulate the sequential sensor allocation problem. In Section 2.3 we introduce preliminary results used in subsequent analysis. We then present a greedy strategy in Section 2.4 and derive conditions sufficient to guarantee its optimality. In Section 2.5, we present two special cases of the sequential allocation problem and discuss its relation to the multi-armed bandit problem. We present numerical results illustrating the performance of the greedy strategy in Section 2.6.

2.1 Introduction

In this chapter, we restrict our attention to the case of N stationary scalar parameters, modeled by independent Gaussian random variables with known means and variances, measured by M sensors. Each observation is described by a linear Gaussian observation model. We assume that each sensor can only be used once. This is done without loss of generality because multiple uses of the same sensor can be effectively replaced by multiple identical sensors, each with a single use. We formulate the above sensor scheduling problem as a stochastic sequential allocation problem.

Our problem does not belong to the class of multi-armed bandit problems and its variants (see discussion in Section 2.5), and it appears difficult to determine the nature of an optimal solution. To obtain some insight into the nature of this problem, we consider a greedy algorithm and discover conditions sufficient to guarantee its optimality. We then present two special cases of the general problem; in each special case, the greedy algorithm results in an optimal strategy under conditions weaker than the sufficient conditions mentioned above. Furthermore, we discuss the relationship between our problem and the multi-armed bandit problem and its variants. Finally we illustrate the nature of our results through a number of numerical examples.

Sensor scheduling problems associated with stationary parameter estimation have been investigated in [17] and [18]. In [17], the sensor selection problem is formulated as a constrained optimization problem, *i.e.,* to maximize a utility function given a cost budget and the observation model is a general convex polygon of the plane. In [18], an entropy-based sensor selection heuristic for localization is proposed. Our results are different from those of [17] and [18] since our observation model and performance criteria are different. Sensor allocation problems associated with dynamic system estimation were investigated in [19], [20], [21], [22], [23]. The dynamic system in [19], [20], [21] is linear. The model of [23] is nonlinear. The objective in [19], [20], [21], [22], [23] is the tracking of a single dynamic system. The objective in our problem is the estimation of multiple random variables, or in other words, multiple static systems. Thus, our problem is different from those formulated in [19]- [23].

The main contributions of this chapter are: (1) the formulation of a sensor scheduling problem under an energy constraint, (2) the decomposition of the sensor-parameter scheduling problem into two subproblems: the first one is to determine the order of the sensors to be used; the second one is to determine the order of the parameters to be measured, (3)the derivation of conditions sufficient to guarantee the optimality of a greedy policy.

In the following section, we formulate the problem formally.

2.2 Problem Formulation

Consider a set Ω of stationary scalar parameters, indexed by $\{1, 2, \cdots, N\}$, that need to be estimated. Parameter $p \in \Omega$ is modeled as a Gaussian random variable, denoted by X_p, with mean $\mu_p(0)$ and variance $\sigma_p(0)$. The random variables $X_1, X_2, \cdots, X_N$ are mutually independent. There is a set Φ of sensors, indexed by $\{1, 2, \cdots, M\}$, that are used to measure the parameters. The measurement of parameter p taken by sensor s

is described by

$$Z_{p,s} = H_{p,s} X_p + V_{p,s} \ , \tag{2.1}$$

where $H_{p,s}$ is a known gain, and $V_{p,s}$ is a Gaussian random variable with zero mean and a known variance $v_{p,s}$. The random variables $V_{p,s}$, $p = 1, 2, \cdots, N$, $s = 1, 2, \cdots, M$ are mutually independent; they are also independent of $X_1, X_2, \cdots, X_N$. A non-negative observation cost $c_{p,s}$ is incurred by activating and using sensor s to measure parameter p.

As mentioned earlier, without loss of generality we assume that each sensor may be activated only once. The available sensors are activated one at a time by a controller to measure a chosen parameter. The observation is then used to update the estimate of that parameter and the total accumulated observation cost is updated. The controller then decides whether to activate another sensor from the set of remaining available sensors, and if so which parameter to measure, or to terminate the process. This sensor and parameter selection process continues until either all M sensors are used, or until a time period of interest T has elapsed, or until the controller decides to terminate the process. For simplicity and without loss of generality, we assume $M \leq T$, implying that at most M sensors/parameters can be scheduled.

Under any sensor and parameter selection strategy γ, the decision/control action at each time instant t is a random vector $U_t := (p_t, s_t)$, taking values in $\Omega \times \Phi^{\gamma,t} \cup \{\emptyset, \emptyset\}$, where $\Phi^{\gamma,t}$ is the set of sensors available at t under the strategy γ. That is, the action at time t is given by a parameter-sensor pair. $U_t = (\emptyset, \emptyset)$ means that no measurement is taken at t; naturally $c_{\emptyset,\emptyset} = 0$. A measurement strategy γ is defined as $\gamma := (\gamma_1, \gamma_2, \cdots, \gamma_T)$, where γ_t is such that under this control law the action $U_t^\gamma = (p_t^\gamma, s_t^\gamma)$ is a function of the initial error variances, all past observations up to time t, and all past control actions up to time t. Denote by Z_t^γ the measurement taken at time t under strategy γ.

Let Γ be the set of all admissible measurement policies. Our optimization problem is formally stated as follows.

Problem 1 (P1):

$$\min_{\gamma\in\Gamma} J^{\gamma} = \sum_{p=1}^{N} E\left\{\left[X_p - \hat{X}_p^{\gamma}(T)\right]^2\right\} + E\left\{\sum_{t=1}^{T} c_{p_t^{\gamma} s_t^{\gamma}}\right\}$$

$$\text{s.t.} \quad \begin{cases} \hat{X}_p^{\gamma}(T) = E[X_p | Z_t^{\gamma} \cdot 1(\{p_t^{\gamma} = p\}), t = 1, \cdots, T], \\ s^{\gamma}(t) \neq s^{\gamma}(t'), \text{if } t \neq t',\ t, t' = 1, \cdots, T, \end{cases}$$

where J^{γ} is the cost of strategy $\gamma \in \Gamma$, $\hat{X}_p^{\gamma}(T)$ is the terminal estimate of parameter p under strategy γ, and $1(A)$ is the indicator function: $1(A) = 1$ if A is true and 0 otherwise.

Denote by $Z_p^{\gamma,t}$ the observation data set collected for parameter p up to time t under strategy γ. Then the variance of p at time t under strategy γ is given by

$$\sigma_p^{\gamma}(t) := E\left\{[X_p - \hat{X}_p^{\gamma}(T)]^2\right\} = E\left\{\left[X_p - E(X_p|Z_p^{\gamma,t})\right]^2\right\},\ p = 1, \cdots, N.$$

Since X_p is a Gaussian random variable and the observation model is linear, $\sigma_p^{\gamma}(t)$ is data independent (see *e.g.,* [24]). Furthermore, at each time instant, the variance of parameter p evolves as follows.

If at $t+1$, parameter p and sensor s are selected by γ, then

$$\sigma_p^{\gamma}(t+1) = \begin{cases} \sigma_p^{\gamma}(t) - \frac{\left(\sigma_p^{\gamma}(t)\right)^2 H_{p,s}^2}{\sigma_p^{\gamma}(t) H_{p,s}^2 + v_{p,s}}\ , & \text{if } p_{t+1}^{\gamma} = p, s_{t+1}^{\gamma} = s \\ \sigma_p^{\gamma}(t)\ , & \text{if } p_{t+1}^{\gamma} \neq p \end{cases} . \tag{2.2}$$

With the above, problem (P1) can be reformulated as a deterministic optimization problem as follows. Rewrite the scheduling strategy γ as $\gamma := (P^{\gamma}, S^{\gamma})$, where

$$P^{\gamma} = \{p_1^{\gamma}, \cdots, p_T^{\gamma}\} \text{ and } S^{\gamma} = \{s_1^{\gamma}, \cdots, s_T^{\gamma}\}.$$

Note that this is an equivalent representation of the strategy as the one given earlier. We have simply grouped the sequence of sensors (and parameters, respectively) into a single vector. Under strategy γ, parameter $p_t^{\gamma} \in \Omega$ is measured by sensor $s_t^{\gamma} \in \Phi^{\gamma,t} \cup \{\emptyset\}$ at time t. If $s_t^{\gamma} = \emptyset$, then no measurement takes place at time t and $c_{p_t^{\gamma}, s_t^{\gamma}} = 0$.

Since the parameters are assumed to be stationary, not taking a measurement at some time instant will incur zero cost and will leave the parameters and their estimates unchanged. Thus, without loss of optimality, we can restrict our attention to measurement strategies with the following property.

Property 2.1. *For* $\forall t$, $t = 1, \cdots, T-1$, *if* $s_t^\gamma = \emptyset$, *then* $s_{t'}^\gamma = \emptyset$, $\forall t' > t$.

For convenience of notation, we will redefine Γ as the set of all admissible measurement policies that satisfy Property 2.1. Then the optimization Problem (P1) can be equivalently written as

Problem 2 (P2):

$$\min_{\gamma \in \Gamma} \quad J^\gamma = \sum_{p=1}^{N} \sigma_p^\gamma(\tau^\gamma) + \sum_{t=1}^{\tau^\gamma} c_{p_t^\gamma, s_t^\gamma}$$

$$s.t. \quad \begin{cases} p_t^\gamma \in \Omega \ \text{ and } s_t^\gamma \in \Phi^{\gamma,t}, \\ \text{(2.2) holds} \\ s_t^\gamma \neq s_{t'}^\gamma, \ \text{if } t \neq t', \ t, t' \leq \tau^\gamma, \end{cases}$$

where τ^γ denotes the stopping time, *i.e.,* the number of measurements taken under strategy γ.

For the remainder of this paper we will focus on problem (P2). In the next section, we present preliminary results and concepts that are used in the analysis of this problem. Unless otherwise noted, all proofs may be found in Appendix.

2.3 Preliminaries

The following definition characterizes a sensor in terms of its measurement quality.

Definition 2.1. An *index* I is defined for a parameter-sensor pair (p, s): $I_{p,s} = \frac{H_{p,s}^2}{v_{p,s}}$, where as stated earlier $H_{p,s}$ is the gain and $v_{p,s}$ the variance of the Gaussian noise when using sensor s to measure parameter p.

This index $I_{p,s}$ can be viewed as the signal-to-noise-ratio (SNR) of sensor s when measuring parameter p. This quantity reflects the accuracy of the measurement: the higher the index/SNR, the more statistically reliable is the measurement. This quality measurement is reflected in the next lemma.

Lemma 2.1. *Assume that according to some sensor allocation strategy, sensor set A is used to measure parameter p starting with a variance $\sigma_p(t)$ at time t. Denoting by $\sigma_p(t, A)$ parameter p's post-measurement variance, we have*

$$\sigma_p(t, A) = \frac{\sigma_p(t)}{\sigma_p(t)\hat{I}_{p,A} + 1}, \tag{2.3}$$

where $\hat{I}_{p,A} = \sum_{s \in A} I_{p,s}$. Furthermore, $\sigma_{p,A}$ is an increasing function of $\sigma_p(t)$ and a decreasing function of $\hat{I}_{p,A}$. This immediately implies that if $A_1 \subset A_2$, then $\sigma_{p,A_1} > \sigma_{p,A_2}$.

Proof. See Appendix A. □

Note that Lemma 2.1 immediately implies that if $A_1 \subset A_2$, then $\sigma_{p,A_1} > \sigma_{p,A_2}$. From Lemma 2.1, we know that the final variance of each parameter only depends on the sensor set to measure the parameter, does not depend on the order of the sensors measuring or the time of the sensors measuring.

We denote by $R_p(\sigma_p(t), A)$ the *variance reduction* for parameter p through using sensor set A starting at time t, given its variance at time t is $\sigma_p(t)$. That is,

$$R_p(\sigma_p(t), A) := \sigma_p(t) - \sigma_p(t, A) = \frac{\sigma_p(t)^2 \hat{I}_{p,A}}{\sigma_p(t)\hat{I}_{p,A} + 1}. \tag{2.4}$$

Lemma 2.2. *Variance reduction $R_p(\sigma_p(t), A)$ is an increasing function of $\sigma_p(t)$ and $\hat{I}_{p,A}$.*

Proof. See Appendix A. □

We next decompose the objective function of problem (P2) (which is the sum of terminal variances and measurement costs) into variance reductions and measurement costs incurred at each time step.

$$
\begin{aligned}
J^\gamma &= \sum_{t=1}^{\tau^\gamma} \left\{ c_{p_t^\gamma, s_t^\gamma} - \left[\sigma^\gamma_{p_t^\gamma}(t-1) - \sigma^\gamma_{p_t^\gamma}(t) \right] \right\} + \sum_{p=1}^{N} \sigma_p(0) \\
&= \sum_{t=1}^{\tau^\gamma} Q_{p_t^\gamma, s_t^\gamma}(\sigma^\gamma_{p_t^\gamma}(t-1)) + \sum_{p=1}^{N} \sigma_p(0) \ , \qquad (2.5)
\end{aligned}
$$

where $Q_{p,s}(\sigma)$ is given by:

$$
Q_{p,s}(\sigma) = c_{p,s} - R_p(\sigma, s) = c_{p,s} - \frac{\sigma^2 I_{p,s}}{\sigma I_{p,s} + 1} \ . \qquad (2.6)
$$

The quantity $Q_{p,s}(\sigma)$ is referred to as the *step cost* of using sensor s to measure parameter p, when its variance is σ. With the above representation, we see that the total cost can be viewed as the sum over all initial variances and all step costs.

Definition 2.2. A *threshold* TH is defined for a parameter-sensor pair (p, s): $TH_{p,s} = \frac{1}{2} \cdot (c_{p,s} + \sqrt{c_{p,s}^2 + 4 \cdot c_{p,s}/I_{p,s}})$.

With this definition, we have that

$$
\text{when } \sigma = TH_{p,s}, \quad Q_{p,s}(\sigma) = c_{p,s} - \frac{\sigma^2 I_{p,s}}{\sigma I_{p,s} + 1} = 0 \ ; \qquad (2.7)
$$

$$
\text{when } \sigma > TH_{p,s}, \quad Q_{p,s}(\sigma) = c_{p,s} - \frac{\sigma^2 I_{p,s}}{\sigma I_{p,s} + 1} < 0 \ . \qquad (2.8)
$$

In other words, when a parameter's current variance lies above (below) this threshold, we incur negative (positive) step cost, *i.e.,* more (less) variance reduction than observation cost; when the current variance is equal to the threshold, we break even. Thus, $TH_{p,s}$ provides a criterion for assessing whether it pays to measure a parameter p at its current variance level with a particular sensor s.

Furthermore, consider two sensors s_1, s_2 and a parameter p. Assuming $I_{p,s_1} = I_{p,s_2}$, then $TH_{p,s_1} < TH_{p,s_2}$ implies $c_{p,s_1} < c_{p,s_2}$. On the other hand, if $c_{p,s_1} = c_{p,s_2}$, then $TH_{p,s_1} < TH_{p,s_2}$ implies $I_{p,s_1} > I_{p,s_2}$. Therefore, the threshold is a combined measure of sensor quality and its cost with respect to a parameter, and reflects the overall "goodness" of a sensor: the lower the threshold, the better its quality. The following lemma reveals the exact relationship between the step cost, a sensor's index, and a sensor's threshold.

Lemma 2.3. *The step cost $Q_{p,s}(\sigma)$ is a decreasing function of $I_{p,s}$ and σ, and an increasing function of $TH_{p,s}$.*

Proof. See Appendix A. □

2.4 Sufficient Conditions for the Optimality of a Greedy Policy

We now decompose the sensor-selection parameter-estimation decision problem into two subproblems. The first is to determine the order in which sensors should be used regardless of which parameter is measured. The second problem is to determine which parameter should be measured at each time instant given the order in which sensors are used. Such a decomposition is not always optimal. In what follows we present conditions that guarantee the optimality of this decomposition. Specifically, we determine two conditions under which it is optimal to use the sensors in non-increasing order of their indices (regardless of which parameter is measured). We then propose a greedy algorithm for the selection of parameters. We determine a condition sufficient, but not necessary, to guarantee the optimality of the greedy algorithm. Thus, overall we specify a sensor-selection parameter-estimation strategy for problem (P2) and determine a set of conditions, under which this strategy is optimal.

2.4.1 The Optimal Sensor Sequence

We present the following two conditions.

Condition 2.1. *The sensors can be ordered into a sequence $s_1^g, s_2^g, \cdots, s_M^g$ such that*

$$I_{p,s_1^g} \geq I_{p,s_2^g} \geq \cdots \geq I_{p,s_M^g}, \ \forall p = 1, 2, \cdots N \ . \tag{2.9}$$

This condition says that if we order the sensors in non-increasing order of their quality for one particular parameter, the same order holds for all other parameters. For the rest of our discussion we will denote s^g_j as the j-th sensor in this ordered set.

Condition 2.2. *For each parameter p, we have $TH_{p,s^g_1} \leq TH_{p,s^g_2} \leq \cdots \leq TH_{p,s^g_M}$, where s^g_i, $i = 1, \cdots, N$, are defined in Condition 2.1.*

If Conditions 2.1 and 2.2 both hold, then they imply that the ordering of sensors with respect to their measurement quality is the same as their ordering when observation cost is also taken into account. Furthermore, both orderings are parameter invariant.

The next result establishes a property of an optimal sensor selection strategy.

Theorem 2.1. *Under Conditions 2.1 and 2.2, assume that an optimal strategy is $\gamma^* = (P^{\gamma^*}, S^{\gamma^*})$, where $P^{\gamma^*} = \{p^*_1, p^*_2, \cdots, p^*_{\tau^{\gamma^*}}\}$, $S^{\gamma^*} = \{s^*_1, s^*_2, \cdots, s^*_{\tau^{\gamma^*}}\}$, and τ^{γ^*} is the number of measurements taken by γ^*. Then $\forall p \in P^{\gamma^*}$, $\forall s \in S^{\gamma^*}$, and $\forall s' \in \Phi - S^{\gamma^*}$, we have $I_{p,s} \geq I_{p,s'}$.*

Proof. See Appendix A. □

The intuition behind this theorem is the following. Although different sensors may incur different costs, so long as the costs are such that they do not change the relative quality of the sensors (represented by their indices), it is optimal to use the best quality sensors.

To proceed further, we note from Lemma 2.1 that the performance of an allocation strategy is completely determined by the set of sensors allocated to each parameter; the order in which the sensors are used for a parameter is irrelevant. Thus, strategies that result in the same association between sensors and parameters may be viewed as *equivalent strategies*. From Theorem 2.1, we conclude that for any optimal strategy, there exists one equivalent strategy, under which sensors are used in non-increasing order of their indices. Therefore, without loss of optimality, we only consider strategies that use sensors in non-increasing order of their indices.

Parameter Selection Algorithm L:

$t := 0$
while $t < T$ **do**
  $k := \arg\min_{p=1,\cdots,N} Q_{p,s_{t+1}}(\sigma_p(t))$
  if $Q_{k,s_{t+1}}(\sigma_k(t)) < 0$ **then**
    $p_{t+1} := k$
    $\sigma_k(t+1) := \frac{\sigma_k(t)}{\sigma_k(t) I_{k,s_{t+1}} + 1}$
    for $p := 1$ to M **do**
      if $p \neq k$ **then**
        $\sigma_p(t+1) := \sigma_p(t)$
        $t := t + 1$
      end if
    end for
  else
    BREAK
  end if
end while
return $\tau := t$ and $P := \{p_1, \cdots, p_\tau\}$

Figure 2.1: A greedy algorithm to determine the parameter sequence.

Consequently, problem (P2) is reduced to determining the stopping time τ^γ and the parameter sequence corresponding to the sensor sequence $S^g = \{s_1^g, s_2^g, \cdots, s_{\tau^\gamma}^g\}$.

2.4.2 A Greedy Algorithm

We consider the parameter selection algorithm L given in Figure 2.1.

Given the ordered sensor sequence $S^g = \{s_1^g, s_2^g, \cdots, s_M^g\}$, this algorithm computes a sequence of parameters, P, by sequentially selecting a parameter that provides the minimum step cost, defined in Equation (2.6), among all parameters. The algorithm terminates when the minimum step cost becomes non-negative, or the time horizon T is reached. The termination time is the stopping time τ^g. The parameter selection strategy resulting from this algorithm, combined with the given sensor sequence, is denoted by $\gamma^g := (P^g, S^g)$, where $P^g = \{p_1^g, \cdots, p_{\tau^{\mathbf{g}}}^g\}$ and $S^g = \{s_1^g, \cdots, s_{\tau^{\mathbf{g}}}^g\}$.

This algorithm is greedy in nature in that it always selects the parameter whose measurement provides the maximum gain for the given sensor sequence. In the next subsection, we investigate conditions under which this algorithm is optimal for problem (P2).

2.4.3 Optimality of Algorithm L

Our objective is to determine conditions sufficient to guarantee the optimality of the greedy algorithm L described in Figure 2.1, given the ordered sensor sequence $\{s_1^g, s_2^g, \cdots, s_M^g\}$.

To proceed with our analysis, we first note that $\sigma_p(t)$, the variance of parameter p at time t, depends on the initial variance $\sigma_p(0)$ and the set of sensors used to measure parameter p up until time t. Recall that $\sigma_p(t, A)$ is parameter p's variance following measurement by the sensor set A starting from time t, $R_p(\sigma_p(t), A)$ is its variance reduction.

Then for any sensor set $E \subseteq \{s_{t+1}^g, \cdots, s_M^g\}$, we define the advantage of using the set $\{s_t^g\} \cup E$ over using the set E to measure parameter p_t at time t as follows.

$$B_t(p_i, E) := R_{p_i}(\sigma_{p_i}(t-1), \{s_t^g\} \cup E) - R_{p_i}(\sigma_{p_i}(t-1), E) - c_{p_i, s_t^g}. \tag{2.10}$$

Using the definition of variance reduction (2.4), $B_t(p_i, E)$ can be rewritten as

$$B_t(p_i, E) = R_{p_i}(\sigma_{p_i}(t-1), \{s_t^g\}) - c_{p_i, s_t^g} + \Delta_{p_i}(E) \ , \tag{2.11}$$

where

$$\Delta_{p_i}(E) := R_{p_i}(\sigma_{p_i}(t-1, \{s_t^g\}), E) - R_{p_i}(\sigma_{p_i}(t-1), E) \tag{2.12}$$

denotes the difference between two variance reductions. The first one is the variance reduction incurred by using sensor subset E when the initial variance is $\sigma_p((t-1),\{s_t^g\})$. The second one is the variance reduction incurred by using sensor subset E when the initial variance is $\sigma_{p_t}(t-1)$. We have the following property on $\Delta_{p_t}(E)$.

Lemma 2.4. *Consider the sensor sets* $A=\{s_{t+1}^g,\cdots,s_M^g\}$, $E_1=\{s_{t+1}^g,s_{t+2}^g,\cdots,s_{k-1}^g,s_k^g\}$, *and* $E_2=\{s_{t+1}^g,s_{t+2}^g,\cdots,s_{j-1}^g,s_j^g\}$, *where* $j<k\le M$. *Consider an arbitrary parameter choice* p_i *at time* $t+1$. *Then* $\Delta_{p_i}(A)\le\Delta_{p_i}(E_1)<\Delta_{p_i}(E_2)\le 0$.

Proof. See Appendix A. □

Based on Lemma 2.4 and Equation (2.11), we can define an upper bound $B_{u,t}(p_i)$ and a lower bound $B_{l,t}(p_i)$ on the aforementioned advantage as follows:

$$\begin{aligned} B_t(p_i,E) &\le B_t(p_i,\emptyset)=R_{p_i}(\sigma_{p_i}(t-1),\{s_t^g\})-c_{p_i,s_t^g} \\ &:= B_{u,t}(p_i)\ , \end{aligned} \tag{2.13}$$

$$\begin{aligned} B_t(p_i,E) &\ge B_t(p_i,A)=R_{p_i}(\sigma_{p_i}(t-1),\{s_t^g\})-c_{p_i,s_t^g}+\Delta_{p_i}(A) \\ &:= B_{l,t}(p_i)\ . \end{aligned} \tag{2.14}$$

$B_{u,t}(p_i)$ represents the total variance reduction in parameter p_i's estimate when p_i is measured by sensor s_t^g at time t and no further measurements of p_i are taken after t. Thus, $B_{u,t}(p_i)$ measures the marginal contribution of sensor s_t^g on the variance reduction of p_i when only s_t^g is used to measure p_i after $t-1$. $B_{l,t}(p_i)$ measures the marginal contribution of sensor s_t^g on the variance reduction of p_i when p_i is measured at all time instants after t.

Intuitively, one expects that the larger the number of sensors used to measure a parameter p_i after a certain time, the smaller is the marginal contribution of any particular sensor in the overall variance reduction of the estimate of p_i. Note that $-B_{u,t}(p_i)$ is the same as the step cost $Q_{p_i,s_t^g}(\sigma_{p_i}(t-1))$ and $B_{l,t}(p_i)$ and $B_{u,t}(p_i)$ are easily computable.

The use of the above upper and lower bounds allows us to obtain the following result.

Lemma 2.5. *Consider two strategies* $\gamma_1 = (P_1, S_1)$ *and* $\gamma_2 = (P_2, S_2)$*, with*

$$S_1 = S_2 = \{s_1^g, s_2^g, \cdots, s_t^g\} \ ,$$

$$P_1 = \{p_1, \cdots, p_{i-1}, p_i, p_{i+1}, \cdots, p_t\} \ ,$$

$$P_2 = \{p_1, \cdots, p_{i-1}, p_i', p_{i+1}, \cdots, p_t\}, \ \textit{where } p_i' \neq p_i \ .$$

If $B_{l,i}(p_i) > B_{u,i}(p_i')$*, then* $J^{\gamma_1} < J^{\gamma_2}$*.*

Proof. See Appendix A. □

The idea behind this result is that regardless of which allocation strategy is used from time t on, under the conditions of Lemma 2.5, using sensor s_t^g to measure parameter p_t at time t will result in better performance than using sensor s_t^g to measure parameter p_t'.

The result of Lemma 2.5 allows us to obtain the following condition, which, together with Conditions 2.1 and 2.2, are sufficient to guarantee the optimality of the greedy algorithm L described in Figure 2.1.

Condition 2.3. *Consider strategy* γ*, where* $\gamma = (S, P)$*. At some time instant* t*, there exists a parameter* $\hat{p}_t$*, such that for any other parameter* $p_t' \neq \hat{p}_t$*, we have* $B_{l,t}(\hat{p}_t) \geq B_{u,t}(p_t')$*, where* $B_{l,t}(\hat{p}_t)$ *and* $B_{u,t}(p_t')$ *are defined in a manner similar to* (2.14) *and* (2.13)*, respectively.*

Condition 3 at time t says that there is a parameter $\hat{p}$ such that irrespectively of the scheduling strategy followed after t sensors s_t^g's marginal contribution to the variance reduction of $\hat{p}$ is greater than its marginal contribution to the variance reduction of any other parameter.

Furthermore, since

$$B_{u,t}(\hat{p}_t) \geq B_{l,t}(\hat{p}_t) \geq B_{u,t}(p_t') \ , \forall p_t' \neq \hat{p}_t \ ,$$

and $-B_{u,t}(\hat{p}_t)$ is equal to the step cost, $\hat{p}_t$ is the parameter that will result in the smallest step cost when measured by sensor s_t^g.

Theorem 2.2. *Apply Algorithm L to the sequence of sensors in non-increasing order of their indices. If Conditions 2.1 and 2.2 hold and Condition 2.3 is satisfied at each time instant $1 \leq t \leq \tau$, then Algorithm L results in an optimal strategy for problem (P2).*

Proof. See Appendix A. □

2.5 Special Cases and Discussion

In this section, we present two special cases of the general formulation given in Section 2.2. In the first case, there is only one parameter to be estimated. This means the second subproblem in the decomposition of problem (P2) does not exist. In this case, we show that it is optimal to use sensors in non-increasing order of their indices under Conditions 2.1 and 2.2.

In the second case, M sensors are identical, implying that the first subproblem in the decomposition of problem (P2) does not exist. In this case, we show that the problem is a finite horizon multi-armed bandit problem and the greedy algorithm is always optimal.

We end the section with a discussion of the relationship between our problem and the multi-armed bandit problem and its variants.

2.5.1 A Single Parameter and M Different Sensors

Consider problem (P2) with only one static parameter to be estimated. Then the cost of using sensor s is c_s, and the observation model of sensor s reduces to

$$Z_s = H_s X + V_s \ . \tag{2.15}$$

In this case we only need to determine which sensors should be used to measure the parameter. Thus, the second subproblem of the decomposition in Section 2.4 does not exist. Furthermore, Condition 2.1 is satisfied automatically. If Condition 2.2 is also satisfied, then Theorem 2.1 implies that it is optimal to use the sensors according to non-increasing order of their indices. Note that if the observation cost for every sensor is equal, i.e. $c_s = c$, $\forall s = 1, \cdots, M$, then Condition 2.2 is equivalent to Condition 2.1. Thus in this case, it is optimal to use the sensors according to non-increasing order of their indices.

2.5.2 N Parameters and M Identical Sensors

Consider problem (P2) in the case where the M sensors are identical. Then the cost of measuring parameter p by any sensor is c_p, and the observation model for parameter p is sensor-independent:

$$Z_p = HX_p + V \ . \tag{2.16}$$

Since the sensors are identical, Conditions 2.1 and 2.2 are satisfied automatically. Therefore, in this case we are only concerned with the second subproblem of the decomposition described in Section 2.4. We can view the M identical sensors as one processor which can be used at most M times, and the N different parameters as N independent machines. The state of every machine/parameter is its variance. At every time instant t, we must select one machine/parameter p_t to process/estimate. The variance of machine/parameter p_t is updated and all the other machines'/parameters' states/variances are frozen. The reward at each time instant t is the variance reduction of parameter p_t minus the observation cost c_{p_t}. Viewed this way, problem (P2) is a finite horizon multi-armed bandit problem with discount factor of 1.

For finite horizon multi-armed bandit problems, the Gittins index rule (see [3]) is not generally optimal. However, in the problem under consideration, the reward sequence for each machine/parameter is deterministic and non-increasing with time. Thus, for each machine/parameter, the Gittins index is always achieved at $\tau = 1$. Therefore, in this case the Gittins index rule coincides with the one-step look-ahead strategy resulting from Algorithm L described in Section 2.4. Consequently, since Conditions 2.1 and 2.2

are automatically satisfied, the Gittins index rule is optimal for this special case.

2.5.3 Discussion

We now compare problem (P2) with the multi-armed bandit problem and its variants.

In general, our problem does not belong to the class of multi-armed bandit problems, for the reasons we explain below. The main features of the multi-armed bandit problem are: (1) there are N machines and one processor; (2) each time the processor is allocated to only one machine; (3) the state of the machine to which the processor is allocated evolves according to a known probabilistic rule; all other machines are frozen; (4) machines evolve independently of one another (*i.e.,* the N random processes describing the evolution of the N machines are mutually independent); (5) at any time instant the machine operated by the processor results in a reward that depends on the machine's state; all other machines do not contribute any reward; and (6) the objective is to determine a processor allocation strategy so as to maximize an infinite horizon expected discounted reward.

There are several similarities between the multi-armed bandit problem and ours. Specifically: (1) each machine in the multi-armed bandit problem can be associated with a parameter in our problem; (2) the processor in the multi-armed bandit problem corresponds to all sensors (taken together and considered as one sensor that can be used M times) in our problem; (3) the reward obtained by allocating the processor to a particular machine(parameter) corresponds to the variance reduction of the parameter minus the cost incurred by using a particular sensor to measure the parameter; (4) machines not operated by the processor at a particular time instant remain frozen; the variance of parameters not measured by a sensor at a particular time instant remains unchanged; and (5) the N parameters are mutually independent random variables.

The fundamental differences between our problem and the multi-armed bandit prob-

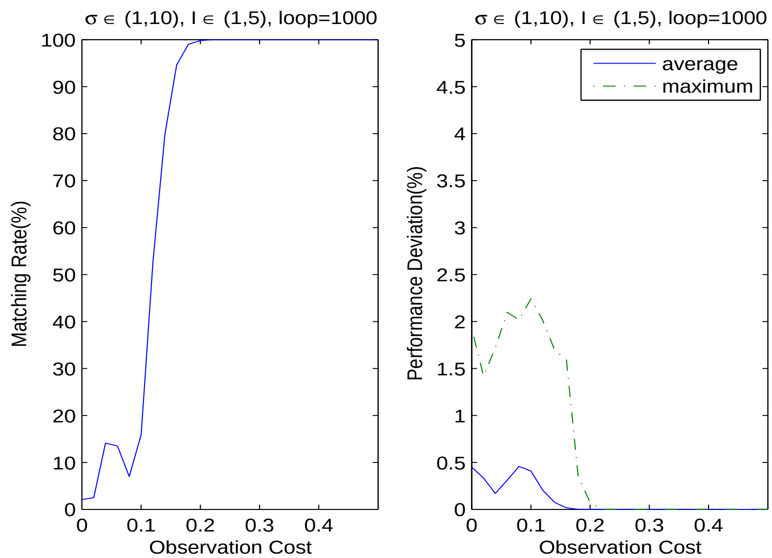


Figure 2.2: Performance of the Greedy Algorithm.

lem are: (1) we consider a finite horizon problem, and (2) the sensors we consider may not be of the same quality, thus, our objective is not only to determine which parameter to measure at each time instant but also which sensor to use. Because of these differences, problem (P2) is not a standard multi-armed bandit problem. Thus, Gittins index policies (see [3], [25]) are not, in general, optimal sensor allocation and measurement strategies.

Furthermore, our problem is not a superprocess problem (see [6]). Even if we can view all sensors as a processor with different modes, a sensor used to measure a parameter is not available after the measurement. Thus, the processor's control action set changes (is reduced) with time. If all sensors could be operated an unlimited number of times, then our problem would reduce to a superprocess problem.

2.6 Numerical Examples

We illustrate the performance of Algorithm L with a number of numerical examples when Condition 1 and Condition 2 are satisfied.

The setup of the numerical experiment is as follows. We consider 7 sensors and 3 parameters, and an observation cost c that is parameter- and sensor-independent. We vary the observation cost c from 0 to 0.5 with increments of size 0.01; thus we consider 51 different values of the observation cost. For each of the 51 possible values of c we run an experiment 1000 times. In each run we randomly select the index I_s of sensor s, $s = 1, 2, \cdots, 7$, according to a uniform distribution over the region $(1, 5)$. Also in each run we randomly select the variance $\sigma_p(0)$ of parameter p, $p = 1, 2, 3$, according to a uniform distribution over the region $(1, 10)$. Finally, in each run we determine the performance J^{γ^g} of the greedy algorithm, and, by exhaustive search, the optimal performance J^{γ^*}.

We consider the following performance criteria:

1. Matching rate $:= \frac{\#\text{ of times } \gamma^g=\gamma^*}{1000}$;

2. Average deviation $:= \frac{1}{1000}\sum_{i=1}^{1000} \frac{J^{\gamma^g}(i)-J^{\gamma^*}(i)}{J^{\gamma^*}(i)}$, where $J^{\gamma^g}(i)$ (respectively, $J^{\gamma^*}(i)$) denotes the performance of the greedy strategy (respectively, the optimal strategy) in the ith run;

3. Maximum deviation $:= \max\limits_{i=1,2,\cdots,1000} \frac{J^{\gamma^g}(i)-J^{\gamma^*}(i)}{J^{\gamma^*}(i)}$.

As a result of our experimental setting, Condition 1 is always satisfied (because each sensor's index is parameter-independent). Furthermore, Condition 2 is also satisfied (because both the index and the observation cost are parameter-independent). Conditions 1 and 2 imply that the sensors can be ordered in terms of their quality measured by their indices.

Under the setting described above, the results of our experiment are shown in Figure 2.2. From Figure 2.2 we observe that when the observation cost is sufficiently large strategy γ^g is always optimal. This observation can be intuitively explained as follows. When the observation cost is large, we expect that each parameter will be measured at most once. This happens because the variance reduction $\sigma_p(t-1)-\sigma_p(t)$ of parameter p,

$p = 1, 2, 3$ after the tth measurement is taken is a decreasing function of t. Thus, when c is large, one expects that after the first measurement the future variance reduction of any parameter will fall below the observation cost. Then using the sensor with the largest index to measure the parameter with the largest variance results in an optimal strategy. This fact together with the observation that each parameter can be measured at most once leads to a heuristic explanation of the optimality of the greedy strategy. From the same results we also observe that even when strategy γ^g is not optimal, the average deviation and the maximum deviation are always below 2.5%.

We then repeat the same numerical experiment described above but now use different distributions to select the indices and the initial variances. Specifically, we maintain the same 51 different values of the observation cost c. For each value of c we run an experiment 1000 times. For each run we consider two cases. In the first case we randomly select I_s, $s = 1, 2, \cdots, 7$, from the uniform distribution over $(1, 5)$, and $\sigma_p(0)$, $p = 1, 2, 3$, from the uniform distribution over $(0, 1)$. In the second case we randomly select both I_s, $s = 1, 2, \cdots, 7$, and $\sigma_p(0)$, $p = 1, 2, 3$, from the uniform distribution over $(0, 1)$. The results for these two cases are shown in Figure 2.3 and 2.4, respectively. We observe qualitatively that these results are similar to these of Figure 2.2.

These results suggest that the greedy algorithm appears to produce satisfactory performance especially when the observation cost is large compared to the initial variance $\sigma_p(0)$.

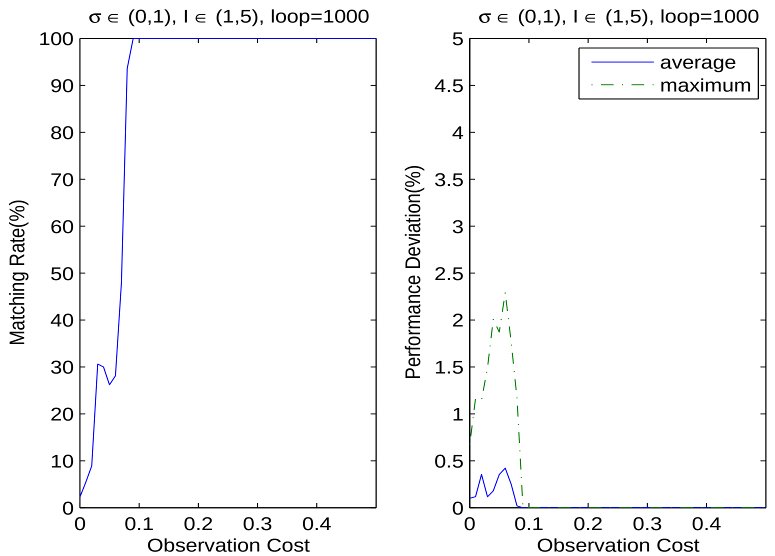


Figure 2.3: Performance of the Greedy Algorithm.

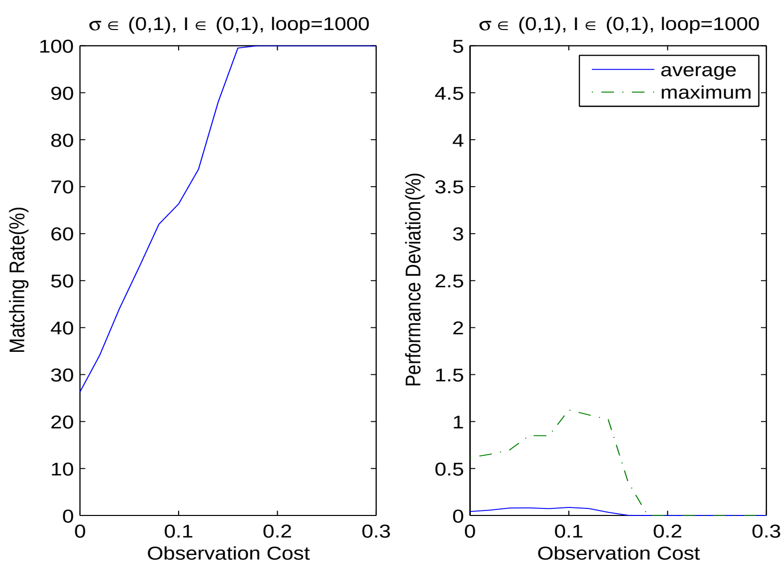


Figure 2.4: Performance of the Greedy Algorithm.

CHAPTER 3

MULTIPLE STATIONARY PARAMETERS ESTIMATION: PART II

In this chapter we restrict attention to the case of P stationary scalar parameters, modeled by independent Gaussian random variables with known mean and variance, measured by S sensors. Each observation is described by a linear Gaussian observation model that is sensor-dependent but parameter-independent, which is a special case of the observation model in Chapter 2.

The chapter is organized as follows: Section 3.1 lists all the notation used in this chapter. In Section 3.2 we demonstrate the motivation of this problem and state the contribution of this chapter. In Section 3.3 we formulate the sensor allocation problem. In Section 3.4 we derive a lower bound on the performance of an optimal sensor allocation strategy. In Section 3.5 we evaluate the performance of the greedy strategy by comparing its performance to the lower bound of Section 3.4. We consider the two-parameter problem in Section 3.6 and show that it is equivalent to a classical Knapsack problem.

3.1 Notation

- $\mathcal{S} = \{1, \cdots, S\}$: initial sensor set.

- $\mathcal{P} = \{1, \cdots, P\}$: initial parameter set.
- X_p: a Gaussian random variable.
- $Z_{p,s} = H_s X_p + V_s$: the measurement of parameter p taken by sensor s, where H_s is a known gain, V_s is a Gaussian random variable with zero mean and known variance v_s.
- $\sigma_p(t)$: the variance of parameter p at time t, $t = 0, 1, 2, \cdots$.
- $\sigma_p(A)$: the post-measurement variance of parameter p measured by sensor set A.
- I_s: the index of sensor s, which is the signal-to-noise ratio of sensor s, $I_s = \frac{H_s^2}{V_s}$.
- TH_s: the threshold of sensor s.
- I_A: the index of the sensor set A, $I_A = \sum_{s \in A} I_s$.
- $J^\lambda(\mathcal{S}, \mathcal{P})$: the system performance under allocation policy λ (λ does not necessarily use all sensors in $\mathcal{S}$).
- $\hat{J}^\lambda(\mathcal{S}, \mathcal{P})$: the system performance under partition policy λ (λ use all sensors in $\mathcal{S}$).
- $\lambda^* = \{A_1^*, \cdots, A_P^*\}$: an optimal policy.
- $A(\lambda^*) = \cup_{i=1}^P A_i^*$: the set of sensors used by λ^*.
- $\tau^* = |A(\lambda^*)|$: the number of sensors used by λ^*.
- $\lambda^g = \{A_1^g, \cdots, A_P^g\}$: a greedy policy.
- $A(\lambda^g) = \cup_{i=1}^P A_i^g$: the set of sensors used by λ^g.
- $\tilde{\sigma}(\mathcal{S}, \mathcal{P})$: the harmonic mean function of the sensor set $\mathcal{S}$ and parameter set $\mathcal{P}$.
- t_p: the number of the parameters which can be identified as not being measured.

- t_s: the number of the sensors which can be identified as being used alone.
- $\Omega_s = \{s_1, \cdots, s_N\}$: an arbitrary sensor set where $I_{s_1} \geq \cdots \geq I_{s_N}$.
- $\Omega_p = \{p_1, \cdots, p_M\}$: an arbitrary parameter set where $\sigma_{p_1}(0) \geq \cdots \geq \sigma_{p_M}(0)$.
- Ψ_s: the sensor set in Algorithm PL.
- Ψ_p: the parameter set in Algorithm PL.

3.2 Introduction

As we discuss in Chapter 2, this sensor scheduling problem can be in general formulated as a stochastic sequential allocation problem. In Chapter 2 we study a more general version of this problem that has an observation model that is both sensor- and parameter-dependent, and a sensor activation cost that is also both sensor- and parameter-dependent. We considered a simple greedy scheduling strategy, and derived conditions under which it is optimal.

In this chapter our model is more restrictive than that studied in Chapter 2, as all parameter dependencies are removed. This restriction allows us to analyze the performance of an optimal strategy and derive a lower bound on its performance. Furthermore, this lower bound can be used to evaluate the performance of the same greedy scheduling strategy studied in Chapter 2. Thus, while the present model is more restrictive than that of Chapter 2, we are able to obtain results stronger than those obtained in Chapter 2. Attempting to do the same for the more general case quickly becomes intractable.

In Chapter 2 the stochastic sequential allocation problem described above was reduced to a deterministic sequential allocation problem due to the Gaussian assumption and the linearity of the observations. In this chapter the same problem (with parameter-independent observation model and constant activation cost) is further reduced to a static sensor allocation and partition problem; the special case of 2-parameter estimation is shown to be equivalent to a 0-1 knapsack problem.

The main contributions of this paper are: (1) the development of a method to obtain a lower bound on the performance of an optimal strategy, (2) the discovery of a preprocessing procedure that can be used to reduce the solution space in which the optimal strategy lies for any given set of parameters and any set of sensors; such a procedure can thereby potentially improve the lower bound, and (3) the use of the lower bound to effectively evaluate the performance of the greedy strategy.

The rest of the paper is organized as follows. In Section

3.3 Problem Formulation

Consider a set $\mathcal{P}$ of stationary scalar parameters indexed by $\{1, 2, \cdots, P\}$, that need to be estimated. Parameter p is modeled as a Gaussian random variable, denoted by X_p, with mean $\mu_p(0)$ and variance $\sigma_p(0)$. The random variables $X_1, X_2, \cdots, X_P$ are mutually independent. There is a set $\mathcal{S}$ of sensors, indexed by $\{1, 2, \cdots, S\}$, which are used to measure the parameters. The measurement of any parameter p taken by sensor s is described by

$$Z_{p,s} = H_s X_p + V_s \ , \tag{3.1}$$

where H_s is a known gain, and V_s is a Gaussian random variable with zero mean and known variance $v_{p,s}$. The random variables V_s, $s = 1, 2, \cdots, S$, are mutually independent; they are also independent of $X_1, X_2, \cdots, X_P$. A non-negative constant measurement cost c is incurred by activating and using any sensor.

As mentioned earlier, we assume that each sensor may be activated only once. Sensors are activated sequentially to take a measurement of a chosen parameter. This process continues till either all S sensors are used, or until a time period of interest T has elapsed. For simplicity, we assume $S \leq T$, implying that at most S sensors/parameters can be scheduled. The objective is to determine how to allocate sensors to parameters so as to minimize the sum of the final error variances of all parameters and the total measurement cost.

3.3.1 Reduction to a static allocation problem

As detailed in Chapter 2, the above problem is in general a stochastic sequential resource allocation problem. However, due to the Gaussian assumption as well as the linearity of the observation model, it is equivalent to a deterministic sequential resource allocation problem; the evolution of the error variance of parameter p at time $t+1$, $\sigma_p(t+1)$, is observation-independent and governed by the following deterministic relationship [24]:

$$\sigma_p(t+1) = \begin{cases} \sigma_p(t) - \frac{\sigma_p^2(t)H_s^2}{\sigma_p(t)H_s^2+v_s}, & \text{if parameter } p \text{ is measured by sensor } s \text{ at } t+1; \\ \sigma_p(t), & \text{if parameter } p \text{ is not measured at } t+1. \end{cases} \tag{3.2}$$

Furthermore, the order in which a set of sensors measure a parameter does not affect the final error variance or the observation cost. This is formally expressed in the following lemma.

Lemma 3.1. *Suppose we use a set of sensors $A \subset S$ to measure parameter p with an initial variance $\sigma_p(0)$. Then parameter p's post-measurement variance, denoted by $\sigma_p(A)$, regardless of the order in which the sensors in A are used, is given by:*

$$\sigma_p(A) = \frac{\sigma_p(0)}{\sigma_p(0)I_A + 1}, \tag{3.3}$$

where $I_A = \sum_{s \in A} \frac{H_s^2}{v_s}$. Furthermore, $\sigma_p(A)$ is an increasing function of $\sigma_p(0)$ and a decreasing function of I_A. This immediately implies that if $A_1 \subset A_2$, then $\sigma_p(A_1) > \sigma_p(A_2)$.

Proof. See Appendix B. □

The quantity $I_s = \frac{H_s^2}{v_s}$ will be referred to as the *index* of sensor s, and the quantity $I_A = \sum_{s \in A} \frac{H_s^2}{v_s} = \sum_{s \in A} I_s$ as the *index* of the set A. An index I_s can be viewed as the signal-to-noise ratio (SNR) of sensor s when measuring a parameter: the higher the index/SNR, the more statistically reliable the measurement.

The above lemma immediately suggests that the final error variance of a given parameter is completely determined by the set of sensors assigned to measure it. Since each sensor can only be used once, this allows us to further reduce the deterministic sequential allocation problem to a *static* allocation problem, whereby a strategy λ is a specification of P subsets, each assigned to a parameter; that is $\lambda = (A_1, \cdots, A_P)$, where $A_i \cap A_j = \emptyset, \forall i \neq j$, $\cup_{i=1}^{P} A_i \subseteq \mathcal{S}$.

Let Λ be the set of all admissible measurement polices. The optimization problem is formally stated as follows.

Problem (Q1) – Allocation:

$$
\begin{aligned}
\min_{\lambda \in \Lambda} J^{\lambda} &= \sum_{p=1}^{P} \left(\sigma_p(A_p) + c|A_p| \right) \\
&= \sum_{p=1}^{P} \left(\frac{\sigma_p(0)}{\sigma_p(0) I_{A_p} + 1} + c|A_p| \right)
\end{aligned}
\tag{3.4}
$$

For reasons that will become clear in the sequel, we next introduce a problem (Q2), the same minimization as in (Q1) except over a more restrictive (smaller) admissible strategy space. Specifically, consider the set of admissible policies $\Lambda' := \{\lambda = (A_1, \cdots, A_P) | A_i \cap A_j = \emptyset, \forall i \neq j, \cup_{i=1}^{P} A_i = \mathcal{S}\}$. That is, any allocation strategy in Λ' has to use all the sensors, effectively *partitioning* the set $\mathcal{S}$ into P subsets. For this reason, problem (Q2) will be referred to as the *partition* problem as opposed to the *allocation* problem (Q1).

Problem (Q2) – Partition:

$$
\min_{\lambda \in \Lambda'} \hat{J}^{\lambda} = \sum_{p=1}^{P} \left(\sigma_p(A_p) + c|A_p| \right) \ . \tag{3.5}
$$

It is obvious that $\Lambda' \subseteq \Lambda$, and that (Q2) is an instance of (Q1). Denoting by $J^*(\mathcal{S}, \mathcal{P})$

and $\hat{J}^*(\mathcal{S}, \mathcal{P})$ the optimal performance of problems (Q1) and (Q2), respectively, we then have $J^*(\mathcal{S}, \mathcal{P}) \leq \hat{J}^*(\mathcal{S}, \mathcal{P})$.

We will also subsequently denote by $J^\lambda(\mathcal{S}, \mathcal{P})$ the performance attained in (Q1) by an arbitrary allocation strategy $\lambda(\mathcal{S}, \mathcal{P}) \in \Lambda$.

3.3.2 Preliminaries

We next introduce a lemma that effectively reduces the set of feasible policies we need to consider. Let $\lambda^* := \{A_1^*, A_2^*, \cdots, A_P^*\}$ denote an optimal strategy for Problem (Q1), and $A(\lambda^*) := \cup_{i=1}^P A_i^*$ denote the set of sensors used by the strategy λ^*.

Lemma 3.2. *If $s \in A(\lambda^*)$ and $I_{s'} > I_s$, then $s' \in A(\lambda^*)$. [If $s \notin A(\lambda^*)$ and $I_{s'} < I_s$, then $s' \notin A(\lambda^*)$.]*

Proof. See Appendix B. □

This lemma says that sensors with high SNR should be used before those with low SNR, which is a highly intuitive result. Consequently, we may limit our attention to policies that follow this order. Accordingly, we will relabel the sensors in the set so that they are in decreasing order of their indices: $\mathcal{S} = \{1, 2, \cdots, S\}$, such that $I_1 \geq I_2 \geq \cdots \geq I_S$.

For convenience and for reasons soon to be clear, we will also relabel the set of parameters so that they appear in decreasing order of their initial variances: $\mathcal{P} = \{1, 2, \cdots, P\}$, such that $\sigma_1(0) \geq \sigma_2(0) \geq \cdots \geq \sigma_P(0)$.

For the remainder of this chapter, I_i and $\sigma_j(0)$ will refer to sensor i and parameter j in the above relabeled, ordered sets, respectively.

By Lemma 3.2, an optimal strategy uses the first τ sensors in the set $\mathcal{S}$. This number τ may not be unique, i.e., two policies may be both optimal with one using more sensors than the other (this implies that the additional variance reduction equals the additional cost incurred). However, without loss of optimality, for the remainder of this paper we will only consider the optimal strategy that uses the smallest number of sensors, among all optimal policies. This effectively makes the optimal strategy under consideration unique.

3.3.3 Complexity of problems (Q1) and (Q2)

We end this section with a brief discussion on the computational complexity of the two problems defined above. Even with the assumptions that the observation model is parameter-independent and that the observation cost is constant for each measurement, problem (Q1) is still a complicated problem. Consider the special case with only 2 parameters to estimate, and further assume that we know exactly the set of sensors to use between the 2 parameters (this would be (Q2)). This special case is then equivalent to a $0-1$ knapsack problem (see proof in Section 3.6). Furthermore, if the initial variances of these two parameters are equal, the problem is equivalent to **the optimization version of a partition problem**, as well as a special case of the subset sum problem. As the knapsack problem and partition problem are both NP-complete, it follows that problem (Q1) is NP-hard.

For knapsack problems, several kinds of relaxations have been investigated and corresponding upper bounds have been derived, see e.g., [26], [27], [28]. But all of them highly depend on the linearity of the objective function,thus they don't hold for Problem (Q2) where the objective is a nonlinear function of the allocation.

In principle we can solve problem (Q1) using dynamic programming. Such an approach would produce an optimal strategy in numerical form. A dynamic programming solution becomes computationally prohibitive if the sensor set and parameter set are large. A reasonable thing to do in such cases is to consider an easy-to-implement suboptimal strategy. In Chapter 2 we proposed a greedy strategy and derived conditions under which it is optimal. In this chapter, we focus on obtaining a lower bound on the performance of an optimal strategy and use it to evaluate the performance of the greedy strategy.

3.4 A Lower Bound on Problem (Q1)

In this section we derive a lower bound on the optimal performance attained in problem (Q1). We first present our bounding method in Section 3.4.1. This is followed by a set of key properties of an optimal strategy for (Q1) in Section 3.4.2. These properties are then used to develop an algorithm PL in Section 3.4.3, which improves the lower bound for (Q1) in Section 3.4.4.

3.4.1 A Bounding Method Through Harmonic Mean

Definition 3.1. The *harmonic mean function* of the sensor set $\mathcal{S}$ and parameter set $\mathcal{P}$ is given by

$$\tilde{\sigma}(\mathcal{S}, \mathcal{P}) := \frac{P}{\sum_{i=1}^{S} I_i + \sum_{j=1}^{P} \frac{1}{\sigma_j(0)}} \ . \tag{3.6}$$

The following property is an immediate consequence of the above definition.

Property 3.1. *$\tilde{\sigma}(\mathcal{S}, \mathcal{P})$ is strictly decreasing w.r.t. $\mathcal{S}$,* i.e., *if $\mathcal{S}_1 \subset \mathcal{S}_2$, then $\tilde{\sigma}(\mathcal{S}_1, \mathcal{P}) > \tilde{\sigma}(\mathcal{S}_2, \mathcal{P})$.*

Proof. See Appendix B. □

Using the arithmetic-harmonic mean inequality (see e.g., [29]), which states that for a set of positive real numbers, their harmonic mean is no more than their arithmetic mean), we obtain the following result.

Lemma 3.3. *Consider problem (Q2). Given the pair $(\mathcal{S}, \mathcal{P})$, the performance attained by an optimal partition strategy $\hat{\lambda}^* = \{A_1^*, A_2^*, \cdots, A_P^*\}$ is lower bounded by*

$$\begin{aligned} \hat{J}^*(\mathcal{S}, \mathcal{P}) &= \sum_{i=1}^{P} \frac{1}{I_{A_i^*} + \frac{1}{\sigma_i(0)}} + S \cdot c \\ &\geq P \cdot \tilde{\sigma}(\mathcal{S}, \mathcal{P}) + S \cdot c \ , \end{aligned} \tag{3.7}$$

where the equality holds if and only if $I_{A_i^*} + \frac{1}{\sigma_i(0)} = I_{A_j^*} + \frac{1}{\sigma_j(0)}$, $\forall i, j \in \mathcal{P}$. *Equivalently, since* $\sigma_i(A_i^*) = \frac{1}{I_{A_i^*} + \frac{1}{\sigma_i(0)}}$, *the equality in* (3.7) *holds if and only if* $\sigma_i(A_i^*) = \tilde{\sigma}(\mathcal{S}, \mathcal{P})$ *for all* $i \in \mathcal{P}$.

Proof. See Appendix B. □

Theorem 3.1. *Consider problem (Q1) with the pair* $(\mathcal{S}, \mathcal{P})$*, and an optimal allocation strategy* $\lambda^* = \{A_1^*, A_2^*, \cdots, A_P^*\}$. *Denote by* τ^* *the size of the set* $A(\lambda^*) := \cup_{i=1}^{P} A_i^*$. *Then, the optimal performance attained by* λ^* *is lower bounded by*

$$J^*(\mathcal{S}, \mathcal{P}) = J^*(A(\lambda^*), \mathcal{P}) \tag{3.8}$$

$$= \hat{J}^*(A(\lambda^*), \mathcal{P}) \tag{3.9}$$

$$\geq \frac{P^2}{\sum_{j=1}^{\tau^*} I_j + \sum_{i=1}^{P} \frac{1}{\sigma_i(0)}} + \tau^* \cdot c \tag{3.10}$$

$$\geq \frac{P^2}{\sum_{j=1}^{t^*} I_j + \sum_{i=1}^{P} \frac{1}{\sigma_i(0)}} + t^* \cdot c \ , \tag{3.11}$$

where t^* *is a minimizer of the lower bound function*

$$L(t) := \frac{P^2}{\sum_{i=1}^{t} I_i + \sum_{i=1}^{P} \frac{1}{\sigma_i(0)}} + t \cdot c \ , \quad t = 0, 1, \cdots, S. \tag{3.12}$$

Proof. See Appendix B. □

Theorem 3.1 is an immediate consequence of Lemma 3.3, and provides a lower bound $L(t^*)$ on the optimal performance attained in (Q1). Note that τ^* and t^* may or may not be the same. When $\tau^* = t^*$ and each parameter has the same final variance, the bound given by (3.11) is attained. It is not difficult to construct examples where this lower bound is indeed reached.

In the next two sections we derive a sequence of properties of an optimal strategy and use them to improve the above lower bound. The main idea behind this improvement is the discovery of conditions that identify parameters that will never be measured under an optimal strategy and sensors that will be singleton sets in an optimal partition.

3.4.2 Properties of an Optimal Policy

The first lemma below says that under an optimal strategy for problem (Q1), the allocation of sensors between any two parameters is also pairwise optimal.

Lemma 3.4. *Consider the optimal strategy* $\lambda^* = \{A_1^*, A_2^*, \cdots, A_P^*\}$ *for problem (Q1), and the set of sensors* $A(\lambda^*) = \cup_{i=1}^{P} A_i^*$ *it uses. For any two parameters* $i, j \in \mathcal{P}$, $i \neq j$, *denote by* $A_{i,j}^* := A_i^* \cup A_j^*$. *Then for any possible partition* $A_i \cup A_j = A_{i,j}^*$, *we have*

$$(I_{A_i} + \frac{1}{\sigma_i(0)})^{-1} + (I_{A_j} + \frac{1}{\sigma_j(0)})^{-1} \geq (I_{A_i^*} + \frac{1}{\sigma_i(0)})^{-1} + (I_{A_j^*} + \frac{1}{\sigma_j(0)})^{-1} \ , \tag{3.13}$$

and equivalently,

$$\left|(I_{A_i} + \frac{1}{\sigma_i(0)}) - (I_{A_j} + \frac{1}{\sigma_j(0)})\right| \geq \left|(I_{A_i^*} + \frac{1}{\sigma_i(0)}) - (I_{A_j^*} + \frac{1}{\sigma_j(0)})\right| \ . \tag{3.14}$$

Proof. See Appendix B. □

Lemma 3.5. *If* $\sigma_1(0) \geq \cdots \geq \sigma_P(0)$, *then* $I_{A_1^*} \geq \cdots \geq I_{A_P^*}$.

Proof. See Appendix B. □

This lemma confirms the intuition that along an optimal allocation strategy the overall sensing quality of a parameter with high initial variance is better than that of a parameter with low initial variance.

We next introduce the notion of *threshold*, a break-even point in the variance where its further reduction (by taking a measurement) is exactly the same as the measurement cost.

Definition 3.2. The *threshold* of a sensor s, denoted by TH_s, is given by $TH_s = \sigma$, such that $\sigma - \frac{1}{I_s + \frac{1}{\sigma}} = c$.

Following the above definition, we know that if $I_1 \geq \cdots \geq I_S$, then $TH_1 \leq \cdots TH_S$. Furthermore, if a parameter's variance falls below a sensor's threshold, then taking a measurement incurs a net cost. In other words, it's only beneficial to take a measurement when the parameter's variance is above the sensor's threshold. This leads to the following lemmas.

Lemma 3.6. *If there exists an integer i, such that $\sigma_1(0) \leq TH_i$, then under an optimal strategy of problem (Q1), sensors j, $i \leq j \leq S$, will not be used to measure any parameter. In particular, if $\sigma_1(0) \leq TH_1$, then no parameter will be measured by any sensor.*

Proof. See Appendix B. □

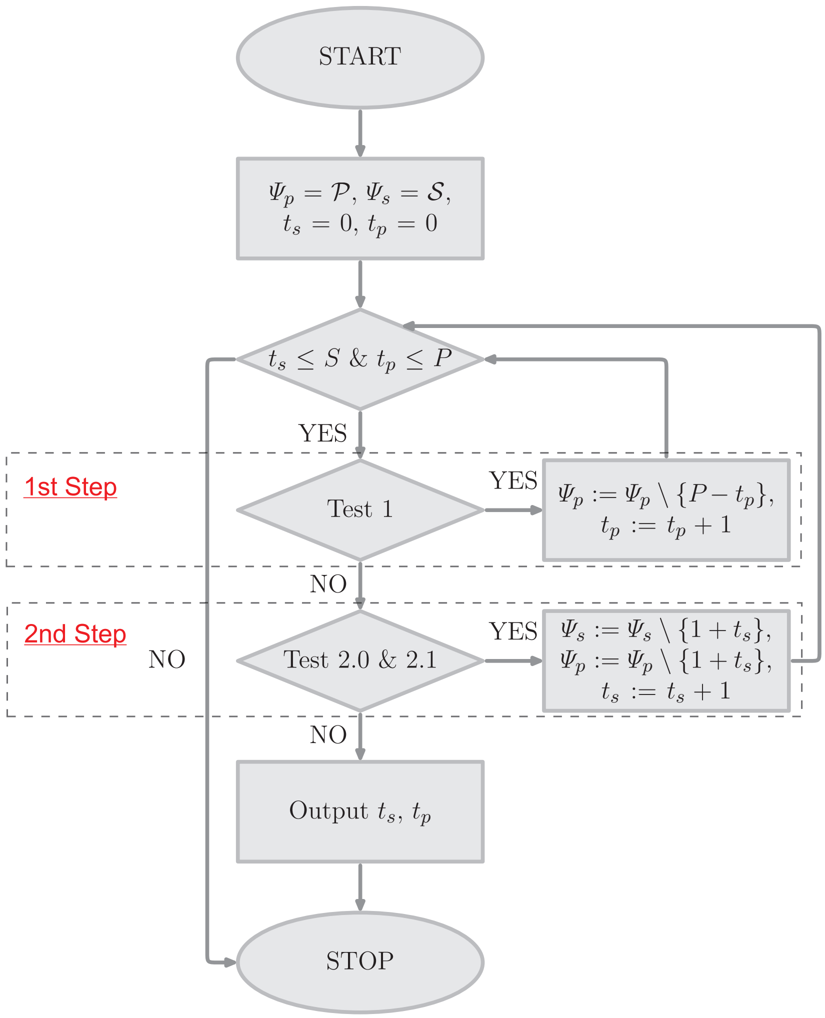
3.4.3 An Algorithm PL to Improve the Lower Bound

Based on the properties of an optimal strategy, we introduce an algorithm PL in Figure 3.1.

The main idea behind Algorithm PL is to sequentially test whether a parameter may be eliminated (not measured), and whether a sensor may be a singleton in an optimal partition. The details are following. In Algorithm PL, numbers t_s and t_p are initialized to 0. The sensor set to be considered, denoted as Ψ_s, and the parameter set to be considered, denoted as Ψ_p, are initially set to be $\mathcal{S}$ and $\mathcal{P}$, respectively. The algorithm proceeds in cycles. Each cycle has two main steps as shown in Figure 3.1. In the first step, Test 1 is performed. If it is passed, t_p and Ψ_p are updated. Test 1 is repeatedly performed while t_p and Ψ_p are repeatedly updated until Test 1 is not passed anymore. The first step ends here. In the second step Test 2.0 and 2.1 are checked. If they are passed, t_s, Ψ_s and Ψ_p are updated and PL goes to the next cycle. Otherwise, PL stops and the output of PL is t_p, which counts the total number of times that Test 1 is passed and t_s, which counts the total number of times that Test 2.0 and 2.1 are passed. Also t_s is the number of cycles that PL has completed.

Based on Algorithm PL, we have the following result.

Lemma 3.7. *If $\sigma_1(0) \geq TH_S$, then under an optimal strategy λ^* the first t_s sensors with the largest indices,* i.e., $\{1, \cdots, t_s\}$, *will each be used alone in measuring a parameter and the last t_p parameters with the smallest initial variances,* i.e., $\{P - t_p + 1, \cdots, P\}$,



Test 1 : $\sigma_{N-t_p}(0) \leq \tilde{\sigma}(\Psi_s, \Psi_p)$,

Test 2.0: $\sigma_{1+t_s}(0) > TH_{1+t_s}$,

Test 2.1: $I_{1+t_s} \geq \dfrac{1}{\tilde{\sigma}(\Psi_s, \Psi_p)} - \dfrac{1}{\sigma_{1+t_s}(0)}$.

Figure 3.1: The flowchart of Algorithm PL

will not be measured. That is,

$$
\begin{aligned}
J^*(\mathcal{S},\mathcal{P}) =& J^*(\mathcal{S}\setminus\{1,\cdots,t_s\},\mathcal{P}\setminus\{k_1,\cdots,k_{t_s},P-t_p+1,\cdots,P\})) \\
&+\sum_{i=1}^{t_s}\frac{1}{I_i+\frac{1}{\sigma_{k_i}(0)}}+\sum_{i=P-t_p+1}^{P}\sigma_i(0)+t_s\cdot c\ ,
\end{aligned}
\tag{3.15}
$$

where k_i denotes the parameter measured by sensor i under strategy λ^*. Furthermore, the numbers t_s and t_p are determined by an algorithm PL, shown in Figure 3.1. The input of Algorithm PL is the sets $\mathcal{S}$ and $\mathcal{P}$; the output of this algorithm are the numbers t_s and t_p.

Proof. See Appendix B. □

3.4.4 A Lower Bound on Problem (Q1)

Using Lemma 3.4-3.7 along with Theorem 3.1 we obtained the main result of this section.

Theorem 3.2. *Consider problem (Q1). The optimal performance attained in (Q1) is lower bounded by*

$$
J^*(\mathcal{S},\mathcal{P}) \geq \max\{L_1(t_1), L_2(t_2)\}\ . \tag{3.16}
$$

where

$$
L_1(t) := \frac{(P-t_p)^2}{\sum_{i=1}^{t} I_i+\sum_{i=1}^{P-t_p}\frac{1}{\sigma_i(0)}}+\sum_{i=P-t_p+1}^{P}\sigma_i(0)+t\cdot c\ , \tag{3.17}
$$

$$
L_2(t) := \frac{(P-t_p-t_s)^2}{\sum_{i=t_s+1}^{t} I_i+\sum_{i=t_s+1}^{P-t_p}\frac{1}{\sigma_i(0)}}+\sum_{i=1}^{t_s}\frac{1}{I_i+\frac{1}{\sigma_{P-t_p-(t_s-i)}(0)}}+\sum_{i=P-t_p+1}^{P}\sigma_i(0)+t\cdot c \tag{3.18}
$$

are convex functions, and t_1 and $t_2 \in \{1,2,\cdots,S\}$ are the minimizers of $L_1(t)$ and $L_2(t)$, respectively.

Proof. See Appendix B. □

Since $L_1(t_1) \geq L(t_1) \geq L(t^*)$, the bound of Theorem 3.2 (referred to as LB in the sequel) is uniformly better (for all pairs $\mathcal{S}$ and $\mathcal{P}$) than the bound in Theorem 3.1 (referred to as $LB1$ in the sequel). As a special case, when $t_s = 0$ and $t_p = 0$, $LB = LB1$.

3.5 Evaluation of the Greedy Policy

We next use the lower bound derived in the previous section to evaluate the performance of a simple greedy strategy. This greedy strategy was first introduced in Chapter 2 for the more general case of parameter-dependent measurement model, and sufficient conditions for its optimality were derived. In this section we first describe how this greedy strategy works for the present problem, and then analyze its key properties; these properties are shown to be shared by the optimal strategy. We then compare its performance against the lower bound.

The greedy strategy works as follows. It takes as input the sets $\mathcal{S}$ and $\mathcal{P}$, and works in discrete steps. In each step it allocates/assigns a sensor to a parameter and removes that sensor from the available set of sensors. At time step t, we test the condition $\sigma_{k^*}(t) > TH_t$, where $k^* = \arg\max_{k=1,\cdots,P} \sigma_k(t)$. If this is true, then we assign sensor t to measure parameter k^*, the maximizer of the LHS of this inequality, and update all variances to $\sigma_k(t+1)$ according to Equation (2.2). If this condition does not hold, the algorithm terminates. The output of the greedy algorithm is an allocation of sensors to each parameter, given by $\lambda^g = \{A_1^g, \cdots, A_P^g\}$

Having described how the greedy algorithm works, below we evaluate its performance.

3.5.1 Properties of the Greedy Policy

The greedy strategy has a number of properties similar to those of an optimal strategy. These are summarized in the next two lemmas.

Lemma 3.8. *If there exists an integer i, such that $\sigma_1(0) < TH_i$, then under the greedy strategy described above, all sensors j, $i \leq j \leq S$, will not be used to measure any parameter. In particular, if $\sigma_1(0) < TH_1$, then no parameter will be measured by any sensor.*

Proof. See Appendix B. □

Corollary 3.1. *If there exists m such that $TH_m < \sigma_1(0) \leq TH_{m+1}$ and $m < 3$, then the greedy strategy is optimal.*

Proof. See Appendix B. □

Lemma 3.9. *If $\sigma_1(0) \geq TH_S$, then under the greedy strategy λ^g, sensor i, $1 \leq i \leq t_s$, is used alone in measuring parameter i, and the last t_p parameters with the smallest initial variances will not be measured. That is,*

$$\begin{aligned} J^g =& J^g(\mathcal{S} \setminus \{1, \cdots, t_s\}, \mathcal{P} \setminus \{1, \cdots, t_s, P - t_p + 1, \cdots, P\}) \\ &+ \sum_{i=1}^{t_s} \frac{1}{I_i + \frac{1}{\sigma_i(0)}} + \sum_{i=P-t_p+1}^{P} \sigma_i(0) + t_s \cdot c \ . \end{aligned} \tag{3.19}$$

Here the numbers t_s and t_p are determined by the same algorithm PL used in the previous section and given in Figure 3.1.

Proof. See Appendix B. □

Lemmas 3.8 and 3.6, and Lemmas 3.9 and 3.7, respectively, present certain properties shared by the greedy strategy and the optimal strategy. This suggests that we may expect the greedy strategy to perform quite well. We next present a more detailed performance evaluation through a number of numerical examples.

3.5.2 Numerical Examples

In this section, we first investigate $LB1$ (the lower bound resulting from Theorem 3.1), LB (the lower bound resulting from Theorem 3.2) and the performance of the greedy strategy, $J(\lambda^g)$, through a single example. Then we compare $J(\lambda^g)$ with LB through a sequence of numerical experiments.

- **$LB1$, LB and $J(\lambda^g)$ in a single run**

  We consider a set of 30 sensors with I_s, $s = 1, \cdots, 30$, randomly chosen from a uniform distribution over $(1, 5)$ and 15 parameters with $\sigma_p(0)$, $p = 1, \cdots, 10$, randomly chosen from a uniform distribution over $(0, 1)$. The measurement cost c varies from 0 to 1 with an increment of size 0.02. We define c^* as the threshold of c, such that when $c \geq c^*$, no measurement is taken under the greedy strategy. Accordingly, we set the corresponding ratios $Greedy/LB$ and $Greedy/LB1$ to 1. In Figure 3.2(a), we show the performance of the greedy strategy and the lower bounds ($LB1$ and LB). The ratios $Greedy/LB$ and $Greedy/LB1$ are plotted in Figure 3.2(b). We show the number of sensors used under the greedy strategy and the number of sensors used to compute LB and $LB1$ in Figure 3.2(c). t_p and t_s (determined by Algorithm PL) are shown in Figure 3.2(d). In the example $c^* = 0.64$.

  From Figure 3.2, we make the following observations.

  (O1) LB is tighter than $LB1$ when $t_s \neq 0$ or $t_p \neq 0$ (see Fig. 3.2(b)).

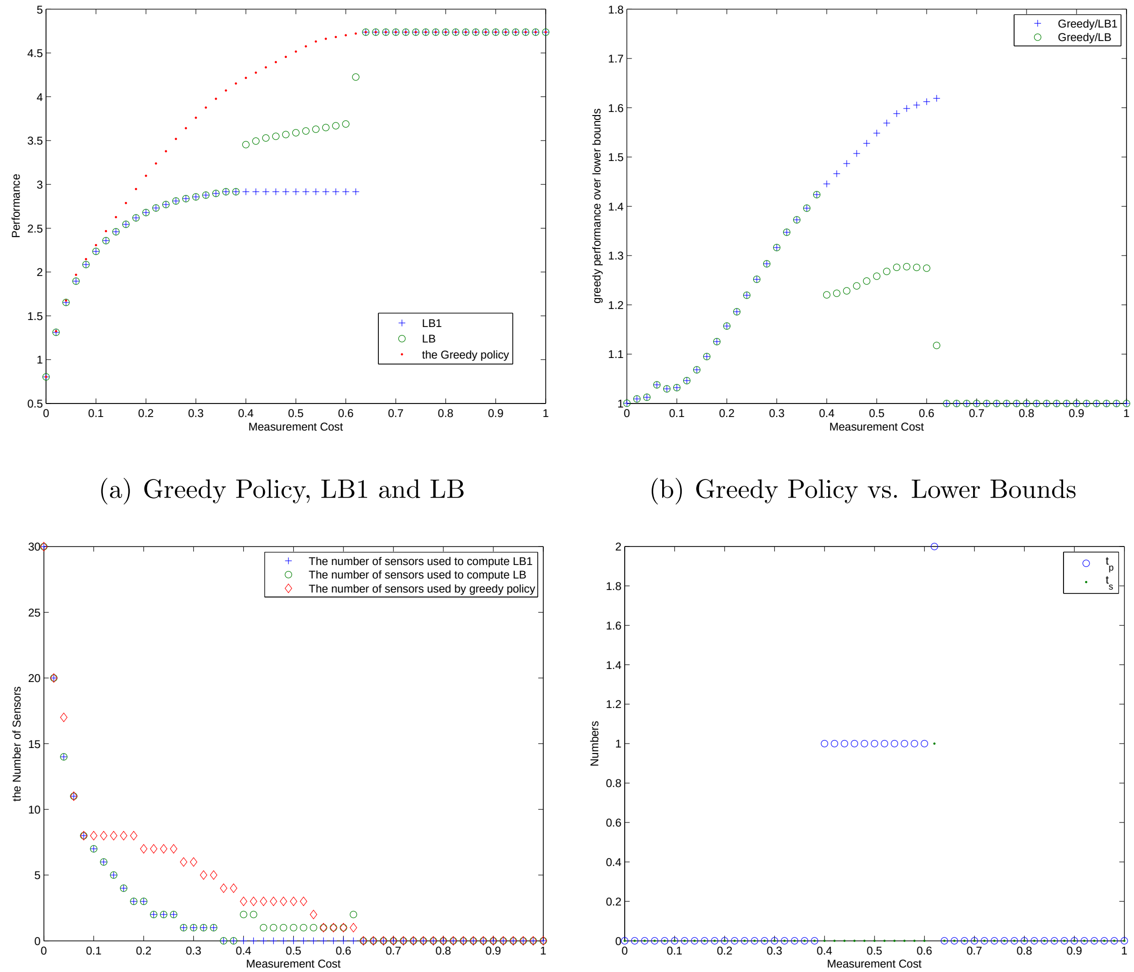
  (O2) Whenever t_s or t_p increases, LB increases significantly (see Fig. 3.2(a)).

  (O3) When $c = 0$, the ratios $Greedy/LB$ and $Greedy/LB1$ are very close to 1 (see Fig. 3.2(b)).

  (O4) The number of sensors used by the greedy strategy or the number of sensors used to compute $LB1$ is a non-increasing step function of c (see Fig. 3.2(c)).

  (O5) When neither t_s nor t_p changes, the number of sensors used to compute LB is a non-increasing step function of c (see Fig. 3.2(c)).

  Observations (O1) and (O2) are consistent with the analysis in Section 3.4.4. Observation (O3) is due to the fact that when $c = 0$, all the sensors are used under



(a) Greedy Policy, LB1 and LB

(b) Greedy Policy vs. Lower Bounds

(c) the Number of Sensors

(d) t_p and t_s

Figure 3.2: A specific experiment when $\sigma \in (0, 1), I \in (1, 5), P = 10, S = 30$

the greedy strategy and all the sensors are used to compute LB and $LB1$. Meanwhile, in this particular example, under the greedy strategy the final variances are well-balanced among different parameters. Observations (O4) and (O5) are true for the following reasons. First, the number of sensors are integers. So they are step functions of c. Secondly, under the greedy strategy, fewer sensors are used as c increases. Therefore the number of sensors used under the greedy strategy is a non-increasing function of c. Thirdly, the lower bound function $L(t)$ (defined in Eq. (3.12), such that $L(t) := \frac{P^2}{\sum_{i=1}^{t} I_i + \sum_{i=1}^{P} \frac{1}{\sigma_i(0)}} + t \cdot c$) is composed of two parts; the first part being a decreasing function of t and the second part is an increasing function of t. The minimizer of $L(t)$, denoted as t^*, balances the two parts. When c increases, the second part gets more weight. Then t^* will not increase in order to re-balance the two parts when c increases. Therefore the number of sensor used to computer $LB1$, which is t^* here, is a non-increasing function of c. LB is similar to $LB1$ while t_p and t_s do not change.

Based on observation (O4) and Figure 3.2(a), we argue that $J(\lambda^g)$ is approximately piecewise linear and non-decreasing in c. Based on observation (O5) and Figure 3.2(a), we argue that when t_p and t_s do not change, LB is approximately piecewise linear and non-decreasing in c.

Using the above observations and arguments we can intuitively explain the behavior of $Greedy/LB$ as follows. When t_p and t_s do not change and the number of sensors used by the greedy strategy is the same as the number of sensors used by LB (such as when $c \in [0.06, 0.08]$ and $c \in [0.56, 0.6]$), $J^(\lambda^g)$ and LB increase at the same rate with c. Since $J(\lambda^g)$ is larger than LB in general, the ratio of $Greedy/LB$ is decreasing in such cases. When t_p or t_s increases (such as when $c = 0.4$ or $c = 0.62$) LB increases significantly and $Greedy/LB$ decreases.

We next compare the performance of greedy strategy $J(\lambda^g)$ and LB through a sequence of numerical experiments to get the average ratio of $Greedy/LB$.

- **Comparison between $J(\lambda^g)$ and LB**

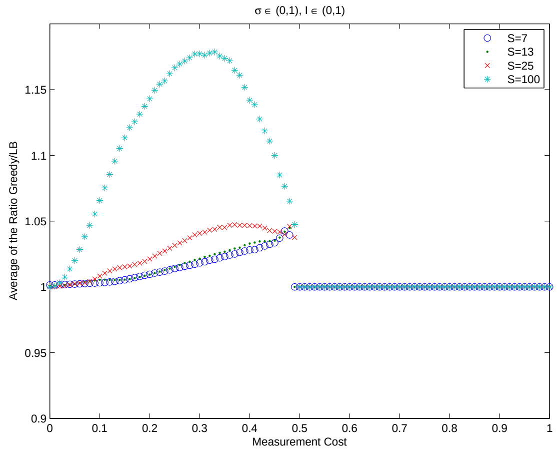
We consider a set of 7 sensors and 10 parameters, with a measurement cost c that is both parameter-independent and sensor-independent. For a given cost c we run an experiment 1000 times; each time the index I_s of sensor s, $s = 1, 2, \cdots, 7$, is randomly chosen from a uniform distribution over $(0, 1)$, while the variance $\sigma_p(0)$ of parameter p, $p = 1, 2, \cdots, 10$, is randomly chosen from a uniform distribution over $(0, 1)$. For the ith run the performance $J_i(\lambda^g)$ of the greedy algorithm, as well as the lower bound LB_i, are calculated.

Our goal is to compare the performance of the greedy algorithm against LB. Thus, we form the ratio $\frac{J_i(\lambda^g)}{LB_i}$. We then calculate the average of this ratio over the 1000 random runs. To assess more accurately the greedy strategy's performance, we exclude from this calculation the instances where the greedy strategy is known to be optimal. Specifically, from Corollary 3.1, we know that when only $m = 0, 1, 2$ sensors are used, the greedy strategy is optimal. We therefore ignore such cases whenever they occur during these random experiments. An average ratio is thus generated over the 1000 random runs less these exclusions, for a given value of c. We repeat the above experiment by varying c, from 0 to 1 with an increment of size 0.01. This results in 101 values of average performance ratio as a function of c. In order for this ratio to be well defined, in the event that all 1000 runs are excluded (i.e., this can occur when the measurement cost is larger than 0.5, making it optimal to never take a measurement), we set this average performance ratio to 1. This threshold value is dependent on the distribution of the parameter initial variances and the sensor indices.

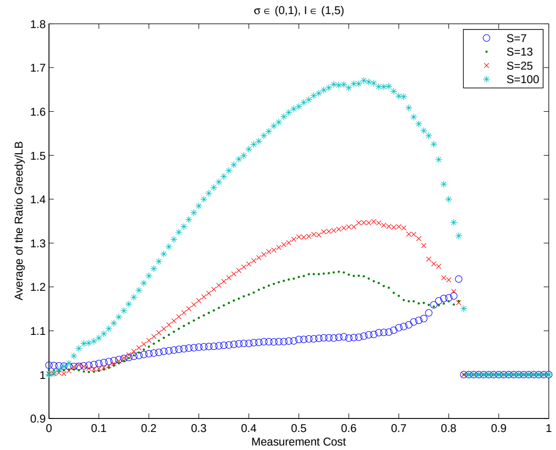
We then repeat the entire procedure described above by varying the number of sensors to be activated; we take $S = 13, 25, 100$, and obtain three corresponding curves with average performance ratio as a function of c, which are shown in Figure 3.3(a). Figures 3.3(b), 3.3(c) and 3.3(d) are obtained similarly, with different distributions for the the sensor indices and parameter initial variances, as indicated in the caption of each figure. For instance in Figure 3.3(c) and 3.3(d), the measurement cost is varied from 0 to 5 with an increment of size 0.05.

Corresponding to the above setup, the number of instances out of 1000 in which $t_p \neq 0$ or $t_s \neq 0$ is shown in Figures 3.4 and 3.5, respectively. Figure 3.6 further shows the average values of t_p and t_s, given $t_p \neq 0$ and $t_s \neq 0$, when $\sigma \in (0, 1)$ and $I \in (1, 5)$.

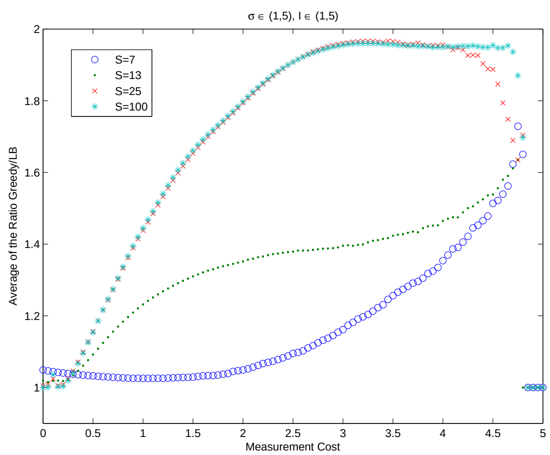
Based on Figure 3.3, we make the following observations. Firstly, the greedy strat-



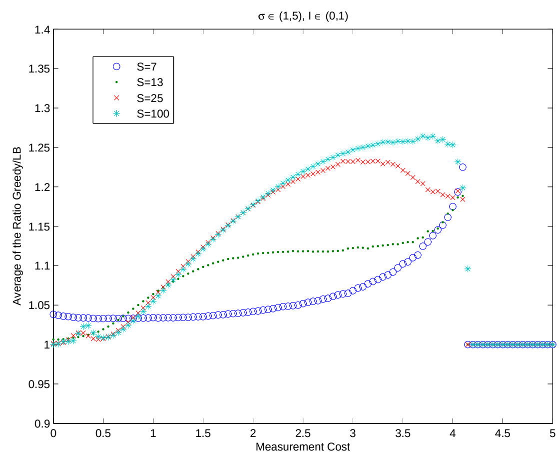
(a) $\sigma \in (0,1), I \in (0,1)$



(b) $\sigma \in (0,1), I \in (1,5)$



(c) $\sigma \in (1,5), I \in (1,5)$

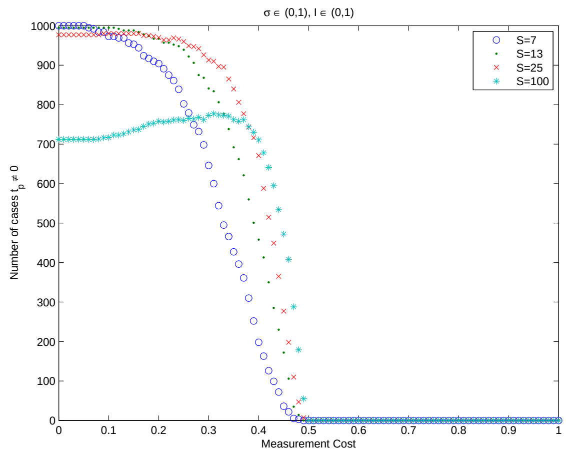


(d) $\sigma \in (1,5), I \in (0,1)$

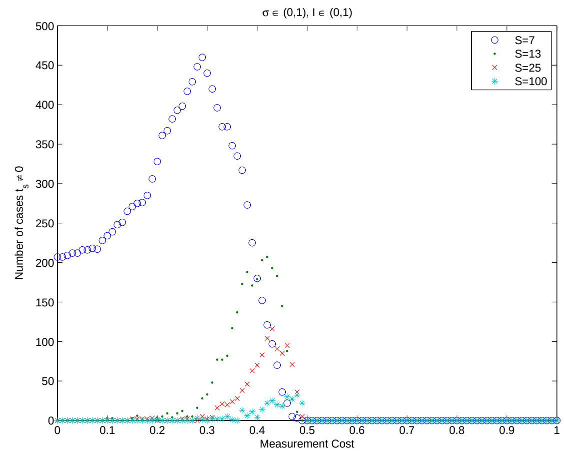
Figure 3.3: Average of Greedy/LB.

egy has a good performance in general. The average performance ratio between the greedy strategy and LB is always below 2 no matter which distributions are used to draw the sensor indices and initial parameter variances. As a function of c, this ratio behaves similarly to the ratio $Greedy/LB$ in the single run experiment, but more smoothly.

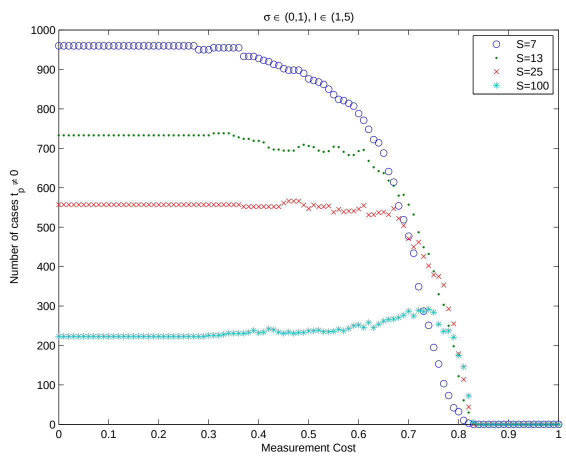
Secondly, when $c = 0$, the average ratio is very close to 1 irrespectively of the number of sensors or the distribution of the sensor indices. Furthermore, when $c = 0$, the average ratio decreases when the number of available sensors S increases.



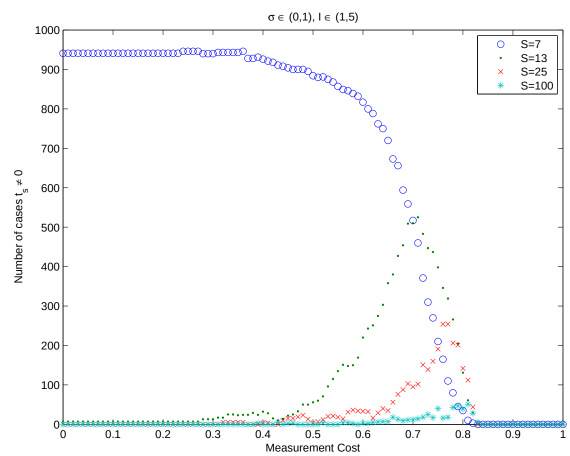
(a) $t_p \neq 0$ when $\sigma \in (0,1), I \in (0,1)$



(b) $t_s \neq 0$ when $\sigma \in (0,1), I \in (0,1)$



(c) $t_p \neq 0$ when $\sigma \in (0,1), I \in (1,5)$



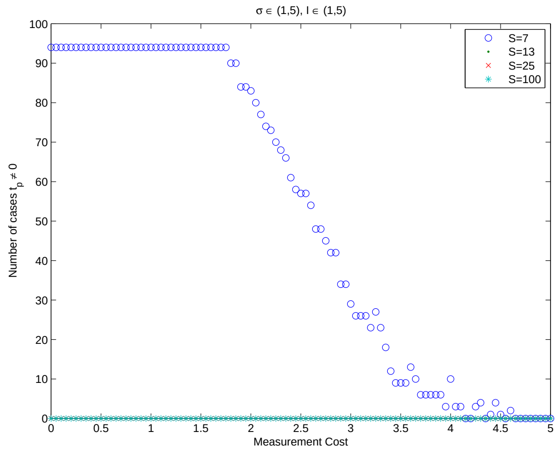
(d) $t_s \neq 0$ when $\sigma \in (0,1), I \in (1,5)$

Figure 3.4: the number of instances that $t_p \neq 0$ and $t_s \neq 0$.

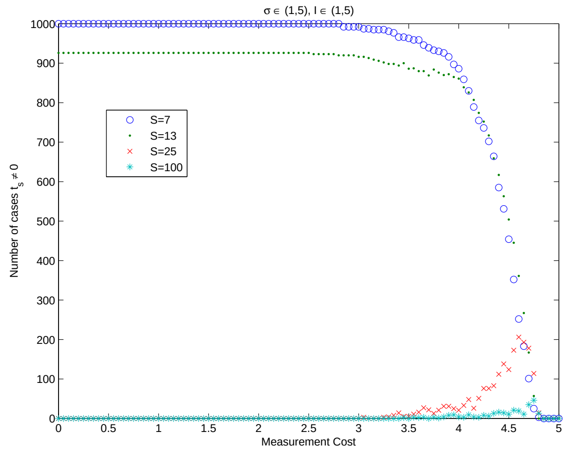
This is consistent with the intuition that in general more sensors can balance the final variances among different parameters better.

From Figures 3.4 and 3.5, we can see the number of runs (out of 1000) in which LB is tighter than $LB1$. Furthermore, Figure 3.6 shows the average values of t_p and t_s when $\sigma \in (0,1)$ and $I \in (1,5)$. The figures related with t_p and t_s show the improvement in the lower bound resulting from algorithm PL.

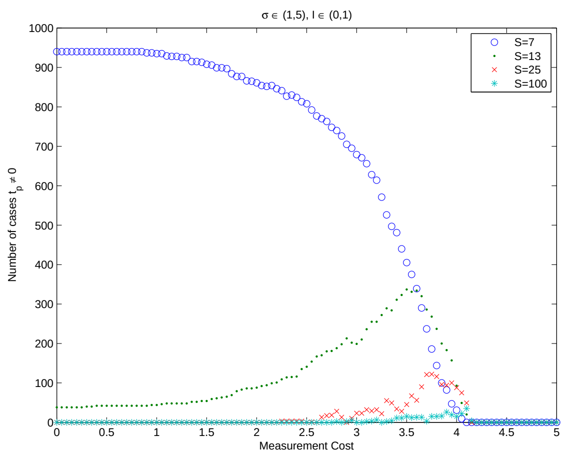
From the comparison between $LB1$ and LB and the analysis of the average ratio of the greedy strategy performance over LB, we conclude that the greedy strategy



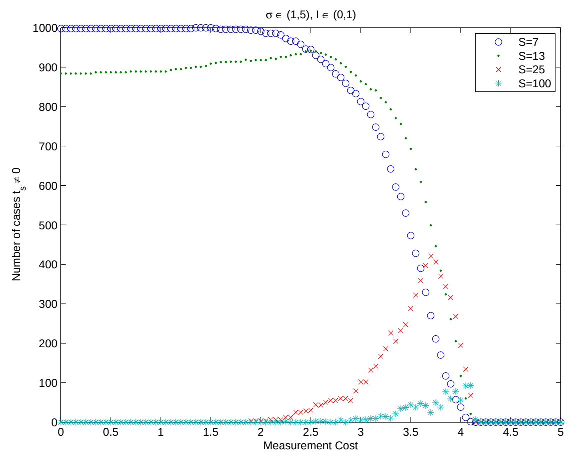
(a) $t_p \neq 0$ when $\sigma \in (1,5), I \in (1,5)$



(b) $t_s \neq 0$ when $\sigma \in (1,5), I \in (1,5)$



(c) $t_p \neq 0$ when $\sigma \in (1,5), I \in (0,1)$



(d) $t_s \neq 0$ when $\sigma \in (1,5), I \in (0,1)$

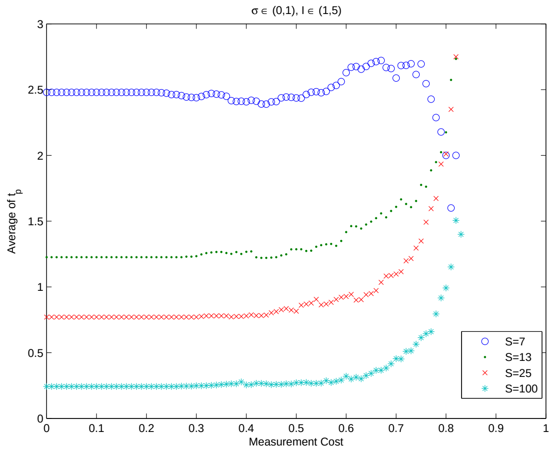
Figure 3.5: the number of instances that $t_p \neq 0$ and $t_s \neq 0$ (continued).

performs well; LB, which is obtained through Algorithm PL and Theorem 3.2, is a tighter lower bound than $LB1$.

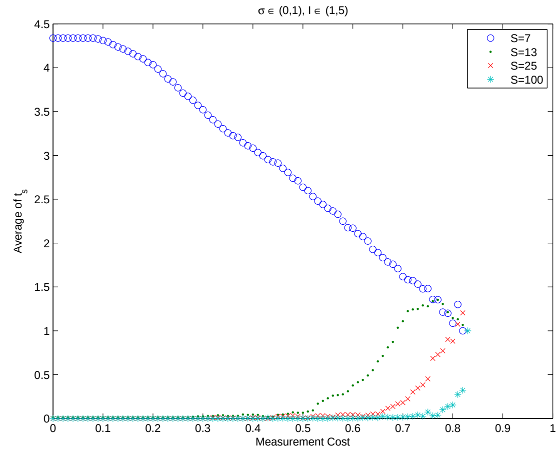
3.6 A Special Case When $P = 2$

In this section, we show that when there are only two identical parameters to be estimated, Problem (Q2) is equivalent to a 0-1 knapsack problem.

Consider the classical 0-1 knapsack problem stated as follows. There is one knapsack and L items available. Each item has a value $v_i > 0$ and weight $w_i > 0$. The objective



(a) the average value of t_p when $t_p \neq 0$



(b) the average value of t_s when $t_s \neq 0$

Figure 3.6: the average value of t_p and t_s when $\sigma \in (0,1)$, $I \in (1,5)$.

is to find a selection of items ($\delta_i = 1$ if i is selected, 0 if not) that fit, i.e., $\sum_{i=1}^{N} \delta_i w_i \leq b$ for some $b > 0$, while the total value, $\sum_{i=1}^{N} \delta_i v_i$, is maximized.

Consider now problem (Q2) with $P = 2$ identical parameters (i.e., same initial error variance), and we wish to determine A_1, A_2 such that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = \mathcal{S}$, so as to minimize

$$\hat{J} = \frac{1}{I_{A_1} + \frac{1}{\sigma_1(0)}} + \frac{1}{I_{A_2} + \frac{1}{\sigma_2(0)}} + c \cdot S \ . \tag{3.20}$$

Since the two parameters are identical, without loss of generality we will assume that $I_{A_1} \leq I_{A_2}$. By Lemma 3.4 and the assumption that the two parameters are identical, we know that the optimal partition of $\mathcal{S}$ minimizes $I_{A_2} - I_{A_1}$, which is equivalent to minimizing $\frac{1}{2} I_{\mathcal{S}} - I_{A_1}$. Therefore Problem $(Q2)$ can be reformulated in a knapsack style as follows. By viewing each sensor as an item and its index as its weight, we can rewrite Problem $(Q2)$ as

$$\max_{A_1 \subset \mathcal{S}} K^{A_1} = \sum_{i=1}^{M} \delta_i \cdot I_i \tag{3.21}$$

$$s.t. \quad \sum_{i=1}^{M} \delta_i \cdot I_i \leq \frac{1}{2} I_{\mathcal{S}} \ . \tag{3.22}$$

We have thus shown that a special case of problem (Q2) is a knapsack problem. This implies that if problem (Q2) can be solved in polynomial time, then a knapsack problem can be solved in polynomial time. Therefore, (Q2) is at least as hard as a knapsack problem which is NP-complete. We thus conclude that (Q2) is an NP-hard problem. We now return to problem (Q1). Note that in the special case where all measurement costs are zero, every sensor will be used in problem (Q1), effectively reducing it to (Q2). Problem (Q2) is thus a special case of (Q1), and hence problem (Q1) is also NP-hard.

CHAPTER 4

SINGLE DYNAMIC PARAMETER TRACKING

In this chapter, we investigate the problem of estimating a dynamic parameter with multiple sensors. Specifically, multiple identical sensors are sequentially activated by a central controller to track a dynamic parameter. Each sensor can be activated only once. The model describing the dynamic evolution of the parameter and the measurement model are linear Gaussian. Each activation of a sensor incurs a cost (*e.g.,* sensing and communication cost) which is a constant. At each time instant, a central controller should determine whether a sensor should be activated to take a measurement. The objective is to determine a sensor activation strategy so as to minimize, over a fixed finite time horizon, the sum of the error covariances associated with estimation of the parameter and the total activation cost.

We proceed as follows. In Section 4.1 we formulate the sensor activation problem. We define the error covariance evolution and related functions in Section 4.2. In Section 4.3.1, we present several properties of the functions defined in Section 4.2. In Section 4.3.2, we prove the threshold property and "stopping property" of an optimal sensor activation strategy. We conclude in section 4.3.3.

4.1 Problem Formulation

4.1.1 The Measurement Model and Problem Formulation

The evolution of the parameter we want to track is describe by a linear Gaussian system

$$X_{t+1} = AX_t + W_t \ . \tag{4.1}$$

We assume that X_0 is Gaussian with mean μ_0 and variance σ_0. The random variables $W_0, W_1, \cdots, W_T$ are Gaussian with zero mean and variance Q; they are independent of each other and also independent of X_0. There is a sensor set $\Omega = \{1, 2, \cdots, m\}$ which can be used to take measurements of the parameter. Every sensor is identical with sensing cost C.

At every time instant t, $t = 1, 2, \cdots, T$, the sensing model is described by

$$Z_t = HX_t + V_t, \tag{4.2}$$

where the sensor's measurement noise, $\{V_t\}_{t=1}^{T}$ are i.i.d. random variables with Gaussian distribution $\mathcal{N}(0, R)$. We assume that the random variables $V_0, V_1, \cdots, V_T$ are independent of $X_0, W_0, W_1, \cdots, W_T$. We define the index for every sensor as $K = \frac{R}{H^2}$. We assume that each sensor can be used only once.

We assume that a central controller, gathers the measurement information, decides whether a sensor should be used to take measurement at present, estimates the parameter and computes the accuracy of the estimation along a time horizon T. Each sensor's usage incurs a cost C. The objective is to determine a sensor activation strategy $g \in G$ to minimize the performance criterion J_T^g,

$$J_T^g := \sum_{t=1}^{T} \left\{ E^g[(X_t - \hat{X}_t)^2] + C \cdot 1_{\{u_t=1\}} \right\} \tag{4.3}$$

$$s.t. \sum_{t=1}^{t=T} u_t \leq m$$

where

$$g := (g_1, g_2, \cdots, g_T),$$

$$u_t = g_t(Z_1 \cdot u_1, Z_2 \cdot u_2, \cdots, Z_{t-1} \cdot u_{t-1}) \ , \tag{4.4}$$

$$u_t = \begin{cases} 1, & \text{if a measurement is taken at time } t, \\ 0, & \text{otherwise,} \end{cases}$$

$$\hat{X}_t = E(X_t | Z_1 \cdot u_1, Z_2 \cdot u_2, \cdots, Z_t \cdot u_t) \ , \tag{4.5}$$

and 1_A denotes the indicator function of event A.

4.2 Preliminaries

In this section, we first present some facts about the error covariance evolution and define certain functions related with it.

The estimate $\hat{X}_t$ of the parameter at time t, $t = 1, \cdots, T$ and its error covariance are

$$\hat{X}_t^g = E^g[X_t | Z_1 \cdot u_1, Z_2 \cdot u_2, \cdots, Z_t \cdot u_t] \ , \tag{4.6}$$

$$\sigma_t^g = E[(X_t - \hat{X}_t^g)^2 | Z_1 \cdot u_1, Z_2 \cdot u_2, \cdots, Z_t \cdot u_t] = E[(X_t - \hat{X}_t^g)^2] \ . \tag{4.7}$$

The last equality in (4.7) follows from the fact that we have a linear Gaussian system and linear Gaussian observations.

From estimation theory [30], we know that:

1. If no measurement is taken at time t, i.e. $u_t = 0$,

$$\sigma_t = \sigma_{t|t-1} = A^2 \sigma_{t-1} + Q := L_2(\sigma_{t-1}) \tag{4.8}$$

2. If a measurement is taken at time t, i.e. $u_t = 1$,

$$\sigma_t = \frac{\sigma_{t|t-1} \cdot K}{\sigma_{t|t-1} + K} := L_1(\sigma_{t|t-1}) \ , \tag{4.9}$$

where

$$\sigma_{t|t-1} = A^2\sigma_{t-1} + Q = L_2(\sigma_{t-1}) \ . \tag{4.10}$$

That is

$$\sigma_t = \begin{cases} \sigma_{t|t-1} := L_2(\sigma_{t-1}) \ , & \text{if } u_t = 0 \ , \\ \sigma_{t|t} := L_1 L_2(\sigma_{t-1}) \ , & \text{if } u_t = 1 \ . \end{cases} \tag{4.11}$$

4.3 Analysis of the Dynamic Parameter Tracking Problem

The problem formulated in Section 4.1.1 is a stochastic control problem with imperfect information. Because the error covariance is independent of the data, as the statistics involved are Gaussian, this stochastic control problem is equivalent to a deterministic control problem where the objective is to determine a sensor activation strategy so as to control the error covariances σ_t, $t = 1, 2, \cdots, T$ and minimize the objective cost described by (4.3).

For the problem under consideration, we define the information state as $\pi_t = (\sigma_{t-1}, n_{t-1})$, where σ_{t-1} is the error covariance associated with the estimation after the decision and corresponding action at time $t-1$ are taken and n_{t-1} is the number of available sensors at time $t-1$ are taken. Then the corresponding dynamic program for the problem formulated in Section 4.1.1 is

$$V_T(\sigma, n) = min\{C + L_1 L_2(\sigma), L_2(\sigma)\} \ , \tag{4.12}$$

$$V_t(\sigma, n) = min\{C + L_1 L_2(\sigma) + V_{t+1}(L_1 L_2(\sigma), n-1), L_2(\sigma) + V_{t+1}(L_2(\sigma), n)\} \ , \tag{4.13}$$

and $\min_{g \in G} J_T^g = V_1(\sigma_0, m)$.

Before we present qualitative properties of an optimal strategy, we discuss properties of $L_1(\cdot)$, $L_2(\cdot)$, $L_1L_2(\cdot)$ and $L_2L_1(\cdot)$. These properties will be extensively used to prove the main result of this chapter.

4.3.1 Properties of $L_1(\cdot)$, $L_2(\cdot)$, $L_1L_2(\cdot)$ and $L_2L_1(\cdot)$

Property 4.1. *$L_1(\cdot), L_1L_2(\cdot), L_2L_1(\cdot)$, defined on $\mathbb{R}^+$, are increasing concave functions and $L_2(\cdot)$, defined on $\mathbb{R}^+$, is an increasing function.*

Proof. See Appendix C. $\square$

Property 4.2. *$x - L_1(x)$ is a positive and increasing function of x, $x > 0$, i.e. $x > L_1(x)$ when $x > 0$; for any positive x_1, x_2, $x_1 - L_1(x_1) < x_2 - L_1(x_2)$ when $x_1 < x_2$.*

Proof. See Appendix C. $\square$

The positivity of $x - L_1(x)$ means that taking a measurement can always reduce the error covariance. The fact that $x - L_1(x)$ is increasing in x means that the error covariance reduction due to a measurement increases as the error covariance increases.

Property 4.3. *For any $x > 0$, when $|A|^2 \geq 1$, $L_2(x) > x$; when $|A|^2 < 1$ and $x \geq \sigma^*$, $L_2(x) \leq x$, otherwise $L_2(x) > x$; σ^* is defined as $\sigma^* = L_2(\sigma^*)$.*

Proof. See Appendix C. $\square$

The result that $|A|^2 < 1$ and $x \geq \sigma^*$ implies $L_2(x) \leq x$ means that even when we do not take a measurement, the error covariance will still decrease. This result can be used to establish the following properties of $L_1L_2(\cdot)$ and $L_2L_1(\cdot)$.

Property 4.4. *For any positive x_1, x_2, such that $L_2(x_1) < x_1$, $L_2(x_2) < x_2$, and $x_1 < x_2$, we have $L_1L_2(x_2) - L_1L_2(x_1) > L_2L_1(x_2) - L_2L_1(x_1)$.*

Proof. See Appendix C. $\square$

Property 4.5. *For any positive x, if $K \leq \sigma^*$, $L_1L_2(x) < L_2L_1(x)$. If $K > \sigma^*$ and $x \leq \hat{\sigma}$, where $\hat{\sigma}$ is uniquely defined by $L_1L_2(\hat{\sigma}) = L_2L_1(\hat{\sigma})$, then $L_1L_2(x) \leq L_2L_1(x)$. Furthermore, if $K > \sigma^*$ and $x \geq \hat{\sigma}$, then $L_1L_2(x) \geq L_2L_1(x)$.*

Proof. See Appendix C. □

4.3.2 The Main Results

We now present the two main results of this chapter and prove them using properties 4.1-4.5. The first result is the threshold property of an optimal strategy; the second result is the "stopping property" of an optimal strategy. Each result holds under different conditions. First, we need the following definition and the lemma that follows it.

Definition 4.1. Define $S_0(m,T) = \{\sigma \in \mathbb{R}^+ :$ for all $t = 1, \cdots, T$ and for all sensor activation strategies $g \in G$, if $\sigma_0 = \sigma$, then $\sigma_t^g \geq \sigma^*$, i.e. $L_2(\sigma_t^g) \leq \sigma_t^g$ and $L_1L_2(\sigma_t^g) \leq L_2L_1(\sigma_t^g)$, where σ_t^g is the error covariance at time t under the strategy g and σ^* is defined in Property 4.3$\}$.

This definition means if the initial error covariance belongs to $S_0(m,T)$, then no matter what kind of sensor activation strategy we use, we can guarantee at any time t that σ_t^g satisfies the inequalities, $L_2(\sigma_t^g) \leq \sigma_t^g$ and $L_1L_2(\sigma_t^g) \leq L_2L_1(\sigma_t^g)$.

Lemma 4.1. *If $K \leq \sigma^*$ and $|A|^2 < 1$, for any m and T, the set $S_0(m,T)$ is nonempty.*

Proof. See Appendix C. □

The following result establishes the threshold property of an optimal sensor activation strategy.

Theorem 4.1. *If $K \leq \sigma^*$, $|A|^2 < 1$ and $\sigma_0 \in S_0(m,T)$, an optimal strategy $g^* = \{u_1^{g^*}, u_2^{g^*}, \cdots, u_T^{g^*}\}$ is described by thresholds $l_t(n_{t-1})$, $t = 1, 2, \cdots, T$, $n = 1, 2, \cdots, m$,* *as follows. At any time instant t,*

$$u_t^{g^*} = \begin{cases} 1 & \text{if } \sigma_{t-1} > l_t(n_{t-1}), \\ 0 & \text{otherwise.} \end{cases} \tag{4.14}$$

To proceed further, we need the following definition and the lemma that follows it.

Definition 4.2. $S_1(m,T) = \{\sigma \in \mathbb{R}^+ :$ for all $t = 1, \cdots, T$ and for all measurement selection strategies $g \in G$, if $\sigma_0 = \sigma$, then $L_1 L_2(\sigma_t^g) \geq L_2 L_1(\sigma_t^g)$, where σ_t^g is the error covariance at time t under the strategy g $\}$.

This definition means if the initial error covariance belongs to $S_1(m,T)$, then no matter what kind of sensor activation strategy we will use, we can guarantee at any time t that σ_t^g satisfies the inequality, $L_1 L_2(\sigma_t^g) \geq L_2 L_1(\sigma_t^g)$.

Lemma 4.2. *If $K > \sigma^*$, for any m and T, the set $S_1(m,T)$ is non-empty.*

Proof. See Appendix C. □

The following result establishes the "stopping property" of an optimal sensor activation strategy.

Theorem 4.2. *If $K > \sigma^*$ and $\sigma_0 \in S_1(m,T)$, then an optimal sensor activation strategy $g^* = \{g_1^*, g_2^*, \cdots, g_T^*\}$ has the following "stopping property": if $u_t^{g^*} = 0$, then $u_{t'}^{g^*} = 0$ for all $t' > t$.*

Proof. See Appendix C. □

4.3.3 Discussion

We formulated a sensor activation problem associated with the tracking of a dynamic parameter and identified conditions on the parameter, initial covariance and the sensors' sensing quality (i.e. the signal-noise-ratio) under which an optimal sensor activation strategy is characterized by a set of thresholds or possesses the "stopping property" (defined in Theorem 4.2). The explicit characterization of the sets $S_0(m,T)$ and $S_1(m,T)$, as well as the determination of the thresholds $\{l_t(n_{t-1})\}_{t=1}^{T}$ of Theorem 4.1 and the stopping time τ of Theorem 4.2 are open challenging problems.

CHAPTER 5

DISCRETE SEARCH USING MULTIPLE SENSORS

In this chapter we investigate a discrete search problem using multiple sensors. Specifically, there are S sensors to monitor an area, which is divided into L cells, $L > S$ and one object is hidden in on of the L cells with probability p_i, $i = 1, \cdots, L$. Each cell can be searched by at most one sensor at each instant of time and at each time instant, no more than S sensors can be activated. The probability to find the object in cell i, given the object is in cell i, is α_i, which is independent of previous search. The false alarm is always 0 and the sensor switches among different cells without delay. Due to energy constraints, each sensor can be used only T times. We want to determine a search strategy to maximize the total time-discounted detection probability given $S \cdot T$ usage of sensors available.

This chapter is organized as follows: In Section 5.1, we introduce the search problem and present the contribution of this chapter. Section 5.2 presents the notation and formulation of the problem. The properties of an optimal strategy are presented in Section 5.3. Algorithm G is proposed in Section 5.4 and an example is presented to illustrate the process of Algorithm G. Section 5.5 presents the sufficient conditions under which Algorithm G results in an optimal search strategy. In Section 5.6 the relationship between this search problem and the multi-armed bandit problem is discussed.

5.1 Introduction

The problem we consider is the following. Suppose we have S sensors that monitor an area, divided into L cells, $L > S$. An object is hidden in any one of these L cells with probability p_i, $i = 1, 2, \cdots, L$. We would like to find out where this object is located with those sensors. Each of the S sensors can be used to scan any cell within a time slot and each cell can be scanned by at most one sensor in a given time slot. At any time slot, no more than S sensors can be activated and the result of each sensor's scanning is either positive or negative. The probability that a sensor finds the object in cell i, given the object is in cell i, is α_i, $i = 1, \cdots, L$. We assume that this probability is independent of how many times this cell has been searched before and the false alarm probability for any sensor on any cell is zero. We further assume that due to certain energy constraint each sensor can only be used T times. We would like to derive a sequential search policy (i.e., how many sensors to use at each time slot and which cells will be scanned) so as to maximize the total time-discounted detection probability of the object given the number of sensors we have and the number of times we can use these sensors. This objective can be viewed as one that tries to maximize the likelihood of finding the object, at the same time finding the object as soon as possible.

Discrete search problems with the assumption that only one sensor available for conducting search has been investigated in [31–51]. To the best of our knowledge, [52] is the first one to investigate a discrete search problem with multiple sensors and determine optimal search strategies for this problem. It also discussed an implementation of an optimal search strategy and specified conditions under which an optimal search strategy can be obtained by forward induction. The discrete search problem investigated by [52] is different from the problem we investigated in the following aspects: (1) It assumes that at each time instant, all the sensors have to search some cell; (2) The time-discount factor equals to one. The search problem investigated in [52] is simpler than our problem since the first aspect makes the following scenario impossible, such that at some time instant, one sensor does not search any cell, and the second aspect makes when to search a cell or in which order to search the cells not important any more.

The main contributions of this chapter are: (1) the formulation of a time-discounted discrete search problem with multiple sensors; (2) the proposal of an easily implementable algorithm to find the search strategy; (3) the development of sufficient conditions to guarantee the optimality of the search strategy results from the proposed algorithm.

5.2 Problem Formulation and Notation

First we list the notation which will be used in this chapter.

- $\mathcal{L} := \{1, \cdots, L\}$, the set of cells that constitute the entire search area, where L is the total number of cells.
- $\mathcal{S} := \{1, \cdots, S\}$, the set of sensors that we have for use, where S is the total number of sensors.
- T : the total number of times a sensor can be used.
- p_i : the probability that the object is in cell i, $i = 1, \cdots, L$.
- α_i : the probability that a sensor finds an object in cell i given that the object is in cell i, $i = 1, \cdots, L$.
- β : the discount factor, such that $0 < \beta < 1$.
- n_t : the number of sensors used at time t, $0 \leq n_t \leq S$.
- $g^i(t)$: the number of times that cell i has been searched up to and including time t ($g^i(0) := 0, i = 1, \cdots, L$).
- $r_{i,m} = p_i\alpha_i(1 - \alpha_i)^{m-1}$: the probability that a sensor finds the object in cell i at the m^{th} search.

- $\pi = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,S\cdot T} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,S\cdot T} \\ \vdots & \vdots & \vdots & \vdots \\ a_{L,1} & a_{L,2} & \cdots & a_{L,S\cdot T} \end{pmatrix}$: a search strategy, where $a_{i,t} \in \{0,1\}$ and $a_{i,t} = 1$ indicates cell i is searched at time t.

Our objective is to find sequences of sensor allocations at time t, $t = 1, 2, \cdots$ until all sensors have been used T times, in order to maximize the total time-discounted detection probabilities (TDDPs).

Let Π be the set of all admissible search strategies. Our optimization problem is formally stated as follows.

Problem (P4)

$$\max_{\pi\in\Pi} J(\pi) = \sum_{t=1}^{S\cdot T} \beta^{t-1} \sum_{i=1}^{L} r_{i,g^i(t)} \cdot a_{i,t} \tag{5.1}$$

$$s.t. \begin{cases} \sum_{i=1}^{L} a_{i,t} = n_t \leq S, \quad t = 1, \cdots, S\cdot T\ , \\ \sum_{t=1}^{S\cdot T} n_t = S \cdot T, \\ \sum_{j=1}^{t} a_{i,j} = g^i(t), \qquad i = 1, \cdots, L, \quad j = 1, \cdots, S\cdot T. \end{cases} \tag{5.2}$$

5.3 Properties of an Optimal Strategy

In this section, we show two properties of an optimal strategy. Assume π^* is an optimal strategy. Under π^*, at each time instant t, denote by $\mathcal{L}_t^*$ the cell set searched by π^* and let $|\mathcal{L}_t^*| = n_t^*$. Let τ^* denote the time at which the process ends under policy π^*. Then we have the following property.

Property 5.1. $S \geq n_1^* \geq n_2^* \geq \cdots \geq n_{\tau^*}^*$.

Property 5.1 implies that under an optimal search strategy, if there are sensors not activated at time instant t, $t = 1, \cdots, S\cdot T$, then it is impossible to activate them again at time s, $s > t$.

Property 5.2. *If* $n_1^* = \cdots = n_{t-1}^* = S$ *and* $n_t^* < S$, *then* $\mathcal{L}_t^* \supseteq \mathcal{L}_{t+1}^* \supseteq \cdots \supseteq \mathcal{L}_{\tau^*}^*$.

According to Property 5.2, any optimal search strategy π^* has the following feature. If π^* does not activate all S sensors at some time instant t, the locations which are not searched at t will never be searched again.

5.4 Greedy Algorithm G

In this section, we propose a search algorithm, called Algorithm G, and present an example to illustrate how Algorithm G works.

Table 5.1: The TDDP Table of Cell i

$r_{i,1}$	$\beta r_{i,1}$	$\cdots$	$\beta^{ST-2} r_{i,1}$	$\beta^{ST-1} r_{i,1}$
$\beta r_{i,2}$	$\beta^2 r_{i,2}$	$\cdots$	$\beta^{ST-1} r_{i,2}$	
$\vdots$	$\vdots$	$\iddots$		
$\beta^{ST-2} r_{i,ST-1}$	$\beta^{ST-1} r_{i,ST-1}$			
$\beta^{ST-1} r_{i,ST}$				

First we define the TDDP table for each cell i, $1 \le i \le L$, in Table 5.1. Each table lists the time-discounted probability of finding the object in all possible situations. The (m, n) entry of this table is the probability to find the object at the m^{th} search at time $m + n - 1$. The first column shows the rewards if cell i is searched continuously until time ST. Similarly the n^{th} column shows the rewards if the search in cell i begins at time n, and the cell is searched continuously until time $n + ST$. Since the largest possible time instant is ST, we do not have a well-defined TDDP entry at (m, n), *s.t.*, $m + n - 1 > ST$. Therefore each table is a $ST \times ST$ upper triangular matrix, which shows all the possible TDDPs. The m^{th} row shows the set of all possible rewards when cell i is searched for the m^{th} time.

Consider Table 5.1: Imagine drawing a diagonal line from the lower left entry to the upper right entry, and then draw all lines parallel to this diagonal line. Define the uppermost left line to be the 1^{st} parallel line and enumerate sequentially the remaining parallel lines (the diagonal line is the $(ST)^{th}$ line). Do the same enumeration for the tables associated with the remaining cells. Then determining a search strategy is equivalent to choosing ST TDDPs out of the L TDDP tables with the following constraints: (i) in each table, at most one TDDP can be chosen from each row (which means that if some cell is searched for the m^{th} time, it cannot be searched for the m^{th} time again in the future); (ii) in each table, at most one TDDP is chosen from one parallel line (which means that at each time instant, each cell can be searched by at most one sensor); (iii) the total number of TDDPs with the same time discount from different tables (*i.e.,* the total number of TDDPs from the j^{th} parallel line of different tables, $1 \leq j \leq S \cdot T$), cannot be more than S (which means that for each time instant, no more than S cells can be searched).

Under the above constraints, Algorithm G proceeds as follows. Initially, the TDDPs at the top left corner from each table are compared and the cell with the largest TDDP at that corner is searched at time 1. The first row of the table of the above-mentioned cell is discarded and the new top left corner of this table will be the left most entry in the second row. Then the TDDPs at the top left corners from each table are compared again. The above process is repeated until the number of TDDPs, chosen from the 1^{st} parallel lines of different tables, reaches S. At this point the first columns of all the tables, in which no TDDP have been chosen from the 1^{st} parallel line, are discarded; furthermore, all cells that are searched at time 1 have been determined. Afterwards, the above process is repeated until the number of TDDPs, chosen from the 2^{nd} parallel lines of different tables, reaches S. Then, the first columns of all the tables, in which no TDDP have been chosen from the 2^{st} parallel line, are discarded. The above process is repeated and terminated when the total number of TDDPs, chosen from different tables, reaches ST.

The following example illustrates how Algorithm G works.

0.2500	0.2250	0.2025	0.1823	0.1640	0.1476	✓1
0.1125	0.1013	0.0911	0.0820	0.0738	0	✓3
0.0506	0.0456	0.0410	0.0369	0	0	✓4
0.0228	0.0205	0.0185	0	0	0	
0.0103	0.0092	0	0	0	0	
0.0046	0	0	0	0	0	

(a) Cell 1

0.1200	0.1080	0.0972	0.0875	0.0787	0.0709	✓2
0.0432	0.0389	0.0350	0.0315	0.0283	0	✓6
0.0156	0.0140	0.0126	0.0113	0	0	
0.0056	0.0050	0.0045	0	0	0	
0.0020	0.0018	0	0	0	0	
0.0007	0	0	0	0	0	

(b) Cell 2

0.0500	0.0450	0.0405	0.0365	0.0328	0.0295	✓5
0.0113	0.0101	0.0091	0.0082	0.0074	0	
0.0025	0.0023	0.0021	0.0018	0	0	
0.0006	0.0005	0.0005	0	0	0	
0.0001	0.0001	0	0	0	0	
ε	0	0	0	0	0	

(c) Cell 3

	t=1	*t*=2	*t*=3	*t*=4	*t*=5	*t*=6
Cell 1	✓1	✓3	✓4			
Cell 2	✓2		✓6			
Cell 3		✓5				

(d) Search Strategy

Figure 5.1: TDDP Tables for Example 1

Example 5.1. Consider $L = 3$, $S = 2$, $T = 3$, $\beta = 0.9$. Suppose for each cell,

$$p_1 = 0.5, \alpha_1 = 0.5 \ , \tag{5.3}$$

$$p_2 = 0.3, \alpha_2 = 0.4 \ , \tag{5.4}$$

$$p_3 = 0.2, \alpha_3 = 0.25 \ . \tag{5.5}$$

The TDDP tables are given in Figures 5.1(a)-5.1(c), where ϵ means that the TDDP is extremely small and $\epsilon < 0.00005$. Figure 5.1(d) records the order in which the cells are assigned to different time slots by Algorithm G; in this figure, $\checkmark_l$, $1 \leq l \leq 6$ represents that the assignment is the l^{th} one determined under Algorithm G.

At first, the TDDPs $\{0.25, 0.12, 0.05\}$ at the top left corner of each table are compared. Since cell 1 has the largest TDDP (0.25) at the top left corner, cell 1 is assigned at time 1 and the first row of the table in Figure 5.1(a) (corresponding to cell 1) is discarded. Then the TDDP at the top left corner of cell 1 is 0.1125 and the TDDPs at the top left corner of other cells are the same. At this stage $\{0.1125, 0.12, 0.05\}$ are compared. Since cell 2 has the largest TDDP (0.12), cell 2 is assigned at time 1 and the first row of the table in Figure 5.1(b) (corresponding to cell 2) is discarded. Since the number of TDDPs chosen from the 1^{st} parallel line reaches 2, cells 1 and 2 are searched at time 1 and the first column of the table in Figure 5.1(c) (corresponding to cell 3) is discarded. Then the new TDDPs at the top left corner of each table are $\{0.1125, 0.0432, 0.045\}$. Since cell 1 has the largest TDDP (0.1125), cell 1 is assigned at time 2 and the second row of the table in Figure 5.1(a) (corresponding to cell 1) is discarded. Subsequently, the TDDPs $\{0.05060.0432, 0.045\}$ are compared. Since cell 1 has the largest TDDP (0.0506) at the top left corner again, cell 1 is assigned at time 3 and the third row of the table in Figure 5.1(a) (corresponding to cell 1) is discarded. At the next step, the TDDPs at the top left corner are $\{0.0228, 0.0432, 0.045\}$. Since cell 3 has the largest TDDP (0.045)at the top left corner, cell 3 is assigned at time 2 and the first row of the table in Figure 5.1(c) (corresponding to cell 3) is discarded. Since the number of TDDPs chosen from the 2^{nd} parallel line reaches 2, cells 1 and 3 are searched at time 2 and the first column of the table in Figure 5.1(b) (corresponding to cell 2) is discarded. The new TDDPs are $\{0.0228, 0.0389, 0.0101\}$. Cell 2 has the largest TDDP (0.0389) and the second row of the table in Figure 5.1(b) (corresponding to cell 2)is discarded. At this stage, since the total number of searches reaches 6, the process terminates.

The above example demonstrates that Algorithm G is easily implementable. It also shows that the total number of rows discarded from the L tables throughout the whole process is ST. In Example 5.1, 6 rows are discarded. We would like to point out that Algorithm G completely determines the set of cells that are searched at each time instant only when it terminates. In the above example, Algorithm G first determines that cells 1 and 2 are searched at time 1; then it determines that cell 1 is searched at time 2 and 3; afterwards it determines that cell 3 is searched at time 2 and cell 2 is searched at time 3.

In the following section, we show two different conditions which are sufficient to guarantee the optimality of the greedy policy.

5.5 the Optimality of the Greedy Algorithm G

In this section we determine two different conditions sufficient to guarantee the optimality of the greedy strategy π^g, resulting from the greedy algorithm G. We denote the number of sensors used at time t under the greedy policy π^g by n_t^g.

Theorem 5.1. *If $n_1^g < S$, then the greedy strategy π^g resulting from Algorithm G is optimal.*

When $n_1^g < S$, all $S \cdot T$ TDDPs are only chosen from the first column of each table. The original problem is thus equivalent to a un-discounted finite horizon classical multi-armed bandit problem. The TDDPs in the first column can be seen as the rewards incurred for the successive plays on different arms. For finite horizon multi-armed bandit problems, the Gittins index rule (see [3]) is not generally optimal. However, in this search problem, the reward sequence for each cell is deterministic and strictly decreasing with time. Thus, for each cell, the Gittins index is always achieved at stopping time 1. Therefore, in this case the Gittins index rule is optimal, which searches the cell with the largest TDDP at each step.

Consider the greedy strategy π^g. Denote by $\mathcal{L}_t^g$ the set of cells searched at time t under π^g. Let $h_t := \sum_{i=1}^S h_t^i$, where h_t^i denotes the number of times sensor i still available for use at time t, that is, $h_t^i = T - g^i(t-1)$, where $g^i(t-1)$ is the number of times cell i has been searched up to and including time $t-1$. Let $v_{i,t}$ denote the TDDP of cell i at time t ($v_{i,t} = \beta^{t-1} p_i \alpha_i (1-\alpha_i)^{g^i(t)}$).

The following lemma provides an intermediate result that leads to another condition sufficient to guarantee the optimality of the greedy strategy π^g.

Lemma 5.1. *Consider the greedy strategy π^g. If $n_t^g = S$ at some time instant t, and $\forall i \in \mathcal{L}_t^g$ and $\forall j \notin \mathcal{L}_t^g$, we have*

$$v_{i,t} \cdot \frac{1 - \beta + \alpha_i \cdot \beta^{h_t} \cdot (1-\alpha_i)^{h_t - 1}}{1 - \beta \cdot (1-\alpha_i)} > v_{j,t}, \tag{5.6}$$

then (given that π^g was used up to time $t-1$) it is optimal to search all the cells in $\mathcal{L}_t^g$ at time t.

Note that when $h_t = 1$, (5.6) is equivalent to $v_{i,t} > v_{j,t}$, which is consistent with the intuition that if there is only one sensor use left, it is better to search the cell with larger TDDP. When h_t is larger, (5.6) is stricter than $v_{i,t} > v_{j,t}$ since the full effect of future rewards has to be taken into account.

From Theorem 5.1 and Lemma 5.1, we can conclude

Theorem 5.2. *Assume $n_1^g = n_2^g = \cdots = n_s^g = S > n_{s+1}^g \geq \cdots > n_{\tau^g}^g$. If at any time instant $t \leq s$, for $\forall i \in \mathcal{L}_t^g$ and $\forall j \notin \mathcal{L}_t^g$,*

$$v_{i,t} \cdot \frac{1-\beta+\alpha_i \cdot \beta^{h_t} \cdot (1-\alpha_i)^{h_t-1}}{1-\beta \cdot (1-\alpha_i)} > v_{j,t} \ , \tag{5.7}$$

then π^g is optimal.

5.6 Discussion

We now compare Problem (P4) with the multi-armed bandit problem.

This search problem is similar to a time-discounted deterministic multi-armed bandit problem with multiple plays (see [15]) in the following way:(1) there are S processors and L projects; (2) at each time t, no more than one processor can work on the same project; (3) the deterministic reward process associated with each project/cell i is $r_{i,g^i(t)}$; (4) the objective is to determine the search strategy that maximize the total β-discounted rewards. The key difference between Problem (P4) and the problem investigated in [15] is the following: in the time-discounted deterministic multi-armed bandit problem with multiple plays (in [15]), the time horizon is infinite and at each time instant, each processor must work on exactly one project; in problem (P4), each processor/sensor can be used at most T times and at some time instant, it is possible that some processor/sensor does not work on any project/cell. Consequently, Problem (P4) is distinctly different from that of [15]. Furthermore, an optimal strategy for

Problem (P4) can only be determined in general by backward induction. Nevertheless, under the condition of Theorem 5.2 (that are distinctly different from those of [15]) the greedy strategy described in Section 5.4 is optimal for Problem (P4).

When $n_t^g = S$, at each time instant, every sensor has to search one cell. Since the reward sequence for each cell is deterministic and strictly decreasing with time, for each cell, the Gittins index is always achieved at $\tau = 1$, which is $r_{i,g^i(t)}$. Similar to the condition in [15], we develop a sufficient condition (5.6) to guarantee that the Gittins index rule is optimal. Here the full effect of future rewards are taken into account in determining an optimal search strategy and (5.6) guarantees the Gittins indices of different cells are sufficiently separated.

CHAPTER 6

CONCLUSION

6.1 Summary and Philosophy of Approaches

In this dissertation we investigated sensor scheduling problems under energy constraints. We concentrated on three classes of problems: stationary parameter estimation, dynamic parameter estimation and discrete search. In Chapter 2 we first formulated a stochastic resource allocation problem for stationary parameter estimation with a sensor-dependent parameter-dependent observation model. With the Gaussian assumption and linear observation model, the problem is equivalent to a deterministic resource allocation problem. We proposed a greedy algorithm for the solution of the problem and identified conditions sufficient to guarantee the optimality of the greedy strategy. In Chapter 3, we formulated the same parameter estimation problem with a sensor-dependent parameter-independent observation model as a static allocation problem. We derived a lower bound on the optimal performance and developed a preprocessing algorithm to obtain an improvement of the lower bound. We used the improved lower bound to evaluate the performance of the greedy strategy proposed in Chapter 2. In Chapter 4, we investigated a dynamic parameter estimation problem under an energy constraint, and discovered the structure of an optimal sensor measurement strategy. In Chapter 5 we proposed an easily implementable greedy strategy for the search problem with multiple sensors under an energy constraint, and identified conditions sufficient to guarantee the optimality of this greedy strategy. We discussed the relationship between each problem and the multi-armed bandit problem.

In general stochastic sequential decision problems can be solved numerically through dynamic programing. In this dissertation, in order to obtain insight into the nature of the problems and to investigate and discover the structure of optimal strategies, we employed both stochastic dynamic programing and a methodology that uses approximate algorithm development, along with the analysis of optimal strategies and the identification of conditions sufficient to guarantee the optimality of the proposed algorithms. In doing so, we were able to analytically explain why in general the greedy strategy performs well.

6.2 Future Directions

In this section, we illustrate some future directions for research.

- **Investigate situations where the sufficient conditions for optimality discovered for the parameter estimation and discrete search problem are satisfied.** In this dissertation, for the parameter estimation and discrete search problem, we derived sufficient conditions to guarantee the optimality of the proposed greedy algorithms. We must further investigate when these conditions are satisfied or how often they are satisfied.

- **Investigate tracking problem where the parameter changes in space and the cost depends on the parameter's position.** In this dissertation, for the tracking problem, we assumed that the measurement accuracy only changes in time, but not in space. In many tracking problems, it is more likely that the measurement accuracy and the measurement cost are dependent of the position of the parameter as the sensing and transmission distance changes. The nature of the optimal solution of such problems is currently unknown.

- **Investigate search problems where multiple sensors can measure the same cell at the same time or there are multiple hidden targets to be searched.** In this dissertation, for the discrete search problem, we assumed that there is only one hidden target to be searched, and at any time instant at most one sensor can be used to search one cell. The nature/solution of search problems where there are several hidden targets to be found/detected and multiple sensors can be used together to search one cell at any time instant is currently unknown.

APPENDICES

APPENDIX A

PROOFS FOR CHAPTER 2

Proof of Lemma 2.1.

We prove this lemma by induction.

First we prove that when sensor set A consists of two sensors s_1 and s_2, the lemma is true. Denote by $\sigma_p(t, \{s_1\})$, the variance after the parameter p is measured by sensor s_1 given the initial variance as $\sigma_p(t)$ and by $\sigma_p(t, \{s_1, s_2\})$, the variance after the parameter p is measured by sensor s_1 and sensor s_2 given the initial variance as $\sigma_p(t)$. Then from equation (2.2) we have

$$\sigma_p(t, \{s_1\}) = \frac{\sigma_p(t)}{\sigma_p(t) \cdot I_{p,s_1} + 1} \tag{A.1}$$

$$\sigma_p(t, \{s_1, s_2\}) = \frac{\sigma_p(t, \{s_1\})}{\sigma_p(t, s_1) \cdot I_{p,s_2} + 1} \tag{A.2}$$

$$= \frac{\frac{\sigma_p(t)}{\sigma_p(t) \cdot I_{p,s_1} + 1}}{\frac{\sigma_p(t)}{\sigma_p(t) \cdot I_{p,s_1} + 1} \cdot I_{p,s_2} + 1} \tag{A.3}$$

$$= \frac{\sigma_p(t)}{\sigma_p(t)(I_{p,s_1} + I_{p,s_2}) + 1} \tag{A.4}$$

Equations (A.1) and (A.4) establish the basis of induction.

Assume for any sensor set A_n, s.t. $|A_n| = n$, $|A|$ denotes the cardinality of A_n and $A_n = \{s_1, s_2, \cdots, s_n\}$, it is true that

$$\sigma_p(t, A_n) = \frac{\sigma_p(t)}{\sigma_p(t)\hat{I}_{p,A_n} + 1}, \tag{A.5}$$

where $\hat{I}_{p,A} = \sum_{k=1}^{n} I_{p,s_k}$.

Then for sensor set $A_{n+1} = \{s_1, s_2, \cdots, s_{n+1}\}$, the post-measurement variance is

$$\sigma_p(t, A_{n+1}) = \frac{\sigma_p(t, A_n,)}{\sigma_p(t, A_n) \cdot I_{p,s_{n+1}} + 1} \tag{A.6}$$

$$= \frac{\frac{\sigma_p(0)}{\sigma_p(t)\hat{I}_{p,A_n}+1}}{\frac{\sigma_p(t)}{\sigma_p(t)\hat{I}_{p,A_n}+1} \cdot I_{p,s_{n+1}} + 1} \tag{A.7}$$

$$= \frac{\sigma_p(t)}{\sigma_p(t)(\hat{I}_{p,A_n} + I_{p,s_{n+1}}) + 1} \tag{A.8}$$

$$= \frac{\sigma_p(t)}{\sigma_p(t)\hat{I}_{p,A_{n+1}} + 1}. \tag{A.9}$$

Equation (A.7) follows from the induction hypothesis (A.5). Equations (A.6)-(A.9) establish the induction step. From Equation (A.9), it is easily verified that $\sigma_p(t, A)$ is an increasing function of $\sigma_p(t)$ and a decreasing function of of $\hat{I}_{p,A}$. □

Proof of Lemma 2.2.

From Equation (2.4), it is easily verified that $R_p(\sigma_p(t), A)$ is an increasing function of $\sigma_p(t)$ and $\hat{I}_{i,A}$. □

Proof of Lemma 2.3.

We note that

$$Q_{p,s}(\sigma) = c_{p,s} - R_p(\sigma, \{s\}) \tag{A.10}$$

$$= c_{p,s} - \frac{\sigma^2 I_{p,s}}{\sigma I_{p,s} + 1} \tag{A.11}$$

From Lemma 2.2, we know $R_p(\sigma, s)$ is an increasing function of σ and $I_{p,s}$. Thus $Q_{p,s}(\sigma)$ is a decreasing function of σ and $I_{p,s}$.

From the definition of $TH_{p,s}$, $Q_{p,s}(\sigma)$ can be rewritten as

$$Q_{p,s}(\sigma) = \left(c_{p,s} - \frac{\sigma^2 I_{p,s}}{\sigma I_{p,s} + 1}\right) - \left(c_{p,s} - \frac{TH_{p,s}^2 I_{p,s}}{TH_{p,s} I_{p,s} + 1}\right) \tag{A.12}$$

$$= \frac{TH_{p,s}^2 I_{p,s}}{TH_{p,s} I_{p,s} + 1} - \frac{\sigma^2 I_{p,s}}{\sigma I_{p,s} + 1} \tag{A.13}$$

Since $\frac{TH_{p,s}^2 I_{p,s}}{TH_{p,s} I_{p,s}+1}$ is an increasing function of $TH_{p,s}$, it follows that

$$Q_{p,s}(\sigma) = \frac{TH_{p,s}^2 I_{p,s}}{TH_{p,s} I_{p,s}+1} - \frac{\sigma^2 I_{p,s}}{\sigma I_{p,s}+1} \tag{A.14}$$

is an increasing function of $TH_{p,s}$. □

Proof of Theorem 2.1.

We prove this theorem by contradiction.

Assume

$$\exists s \in S^{\gamma}, \quad \exists s' \in \Omega_s \setminus S^{\gamma} \tag{A.15}$$

such that

$$I_{p_k^*,s} < I_{p_k^*,s'} \quad \text{for some parameter } p_k^* \ . \tag{A.16}$$

Since Conditions 2.1 and 2.2 are satisfied, we have

$$TH_{p_k^*,s} < TH_{p_k^*,s'}. \tag{A.17}$$

Define $\hat{\gamma} := (P^{\hat{\gamma}}, S^{\hat{\gamma}})$, where

$$P^{\hat{\gamma}} = P^{\gamma^*} \ , \tag{A.18}$$

$$S^{\hat{\gamma}} = \{s_1^*, \cdots, s_{k-1}^*, s', s_{k+1}^*, \cdots, s_{\tau\gamma^*}^*\} \ . \tag{A.19}$$

Then, there exists a strategy $\hat{\gamma}' := (P^{\hat{\gamma}'}, S^{\hat{\gamma}'})$, which is equivalent to $\hat{\gamma}$, with

$$P^{\hat{\gamma}'} = \{p_1^*, \cdots, p_{k-1}^*, p_{k+1}^*, \cdots, p_{\tau\gamma^*}^*, p_k^*\} \ , \tag{A.20}$$

$$S^{\hat{\gamma}'} = \{s_1^*, \cdots, s_{k-1}^*, s_{k+1}^*, \cdots, s_{\tau\gamma^*}^*, s'\} \ . \tag{A.21}$$

There is also a strategy $\gamma' := (P^{\gamma'}, S^{\gamma'})$, which is equivalent to strategy γ^*, with

$$P^{\gamma'} = \{p_1^*, \cdots, p_{k-1}^*, p_{k+1}^*, \cdots, p_{\tau\gamma^*}^*, p_k^*\} \ , \tag{A.22}$$

$$S^{\gamma'} = \{s_1^*, \cdots, s_{k-1}^*, s_{k+1}^*, \cdots, s_{\tau\gamma^*}^*, s_k^*\} \ . \tag{A.23}$$

Define strategy $\tilde{\gamma} := (P^{\tilde{\gamma}}, S^{\tilde{\gamma}})$, where

$$P^{\tilde{\gamma}} = \{p_1^*, \cdots, p_{k-1}^*, p_{k+1}^*, \cdots, p_{\tau^{\gamma^*}}^*\} \ , \tag{A.24}$$

$$S^{\tilde{\gamma}} = \{s_1^*, \cdots, s_{k-1}^*, s_{k+1}^*, \cdots, s_{\tau^{\gamma^*}}^*\} \ . \tag{A.25}$$

Assume the variance of parameter p_k^* after strategy $\tilde{\gamma}$ has been executed is $\sigma_{p_k^*}^{\tilde{\gamma}}$. Then

$$J(\gamma^*) = J(\gamma') = J(\tilde{\gamma}) + Q_{p_k^*,s}(\sigma_{p_k^*}^{\tilde{\gamma}}) \ , \tag{A.26}$$

$$J(\hat{\gamma}) = J(\hat{\gamma}') = J(\tilde{\gamma}) + Q_{p_k^*,s'}(\sigma_{p_k^*}^{\tilde{\gamma}}) \ . \tag{A.27}$$

where $Q_{p_k^*,s}(\sigma_{p_k^*}^{\tilde{\gamma}})$ and $Q_{p_k^*,s'}(\sigma_{p_k^*}^{\tilde{\gamma}})$ are defined in Equation (2.6). Since $I_{p_k^*,s} < I_{p_k^*,s'}$, it follows from Lemma 2.3 that

$$Q_{p_k^*,s}(\sigma_{p_k^*}^{\tilde{\gamma}}) > Q_{p_k^*,s'}(\sigma_{p_k^*}^{\tilde{\gamma}}) \ . \tag{A.28}$$

Hence

$$J(\gamma^*) > J(\hat{\gamma}), \tag{A.29}$$

which contradicts the optimality of γ^*. Thus we must have

$$I_{p,s} \geq I_{p,s'}, \ \forall p \in P^{\gamma^*}, \forall s \in S^{\gamma^*}, \forall s' \in \Phi - S^{\gamma^*} \ . \tag{A.30}$$

$\square$

Proof of Lemma 2.4.

For any $E = \{s_{i+1}^g, \cdots, s_{l-1}^g, s_l^g\}$, such that $l \leq M$, according to Equation (2.12) , we have

$$\begin{aligned}\Delta_{p_i}(E) = &[\sigma_{p_i}(i-1, \{s_i^g\}) - \sigma_{p_i}(i-1, \{s_i^g\} \cup E)] \\ &- [\sigma_{p_i}(i-1) - \sigma_{p_i}(i-1, E)] \ .\end{aligned} \tag{A.31}$$

Furthermore,

$$\begin{aligned}&\sigma_{p_i}(i-1, \{s_i^g\}) - \sigma_{p_i}(i-1, \{s_i^g\} \cup E) \\ &\qquad = \frac{\sigma_{p_i}^2(i-1, \{s_i^g\})\hat{I}_{p_i,E}}{\sigma_{p_i}(i-1, \{s_i^g\})\hat{I}_{p_i,E} + 1} \ ,\end{aligned} \tag{A.32}$$

$$\sigma_{p_i}(i-1) - \sigma_{p_i}(i-1, E) = \frac{\sigma_{p_i}^2(i-1)\hat{I}_{p_i,E}}{\sigma_{p_i}(i-1)\hat{I}_{p_i,E} + 1} \ . \tag{A.33}$$

Since $\sigma_{p_i}(i-1, \{s_i^g\}) < \sigma_{p_i}(i-1)$, Lemma 2.2 implies that

$$\sigma_{p_i}(i, \{s_i^g\}) - \sigma_{p_i}(i, \{s_i^g\} \cup E) < \sigma_{p_i}(i-1) - \sigma_{p_i}(i, E). \tag{A.34}$$

Therefore we have the following inequality,

$$\Delta_{p_i}(E) \leq 0, \ \forall E \subseteq \{s_{i+1}^g, \cdots, s_M^g\} \ . \tag{A.35}$$

According to Equation (A.31), for any E_1, E_2 and $j < k \leq M$, such that

$$E_1 = \{s_{i+1}^g, s_{i+2}^g, \cdots, s_{k-1}^g, s_k^g\} \ , \tag{A.36}$$

$$E_2 = \{s_{i+1}^g, s_{i+2}^g, \cdots, s_{j-1}^g, s_j^g\} \ , \tag{A.37}$$

we have

$$\begin{aligned} &\Delta_{p_i}(E_1) - \Delta_{p_i}(E_2) = \\ &\qquad [\sigma_{p_i}(i-1, E_1) - \sigma_{p_i}(i-1, \{s_i^g\} \cup E_1)] \\ &\qquad - [\sigma_{p_i}(i-1, E_2) - \sigma_{p_i}(i-1, \{s_i^g\} \cup E_2)] \ . \end{aligned} \tag{A.38}$$

Furthermore,

$$\begin{aligned} &\sigma_{p_i}(i-1, E_1) - \sigma_{p_i}(i-1, \{s_i^g\} \cup E_1) \\ &\qquad = \frac{\sigma_{p_i}^2(i-1, E_1) I_{p_i, s_i^g}}{\sigma_{p_i}(i-1, E_1) I_{p_i, s_i^g} + 1} \ , \end{aligned} \tag{A.39}$$

$$\begin{aligned} &\sigma_{p_i}(i-1, E_2) - \sigma_{p_i}(i-1, \{s_i^g\} \cup E_2) \\ &\qquad = \frac{\sigma_{p_i}^2(i-1, E_2) I_{p_i, s_i^g}}{\sigma_{p_i}(i-1, E_2) I_{p_i, s_i^g} + 1} \ . \end{aligned} \tag{A.40}$$

Since $j < k$, $E_2 \subset E_1$, therefore

$$\sigma_{p_i}(i-1, E_1) < \sigma_{p_i}(i-1, E_2) \ . \tag{A.41}$$

Then Lemma 2.2 implies that

$$\begin{aligned}&\sigma_{p_i}(i-1,E_1)-\sigma_{p_i}(i-1,\{s_i^g\}\cup E_1)\\&<\sigma_{p_i}(i-1,E_2)-\sigma_{p_i}(i-1,\{s_i^g\}\cup E_2)\ .\end{aligned}\tag{A.42}$$

Consequently, from (A.35), (A.38) and (A.42), we obtain

$$\Delta_{p_i}(\{s_{i+1}^g,\cdots,s_M^g\})\leq\Delta_{p_i}(E_1)<\Delta_{p_i}(E_2)\leq 0\ .\tag{A.43}$$

□

Proof of Lemma 2.5.

For strategy γ_1, defined in the statement of the lemma, there exists an equivalent strategy $\gamma_1' := (P^{\gamma_1'}, S^{\gamma_1'})$, where

$$P^{\gamma_1'}=\{p_1,\cdots,p_{i-1},p_{i+1},\cdots,p_t,p_i\}\ ,\tag{A.44}$$

$$S^{\gamma_1'}=\{s_1^g,\cdots,s_{i-1}^g,s_{i+1}^g,\cdots,s_t^g,s_i^g\}\ .\tag{A.45}$$

For strategy γ_2, defined in the statement of the lemma, there exists an equivalent strategy $\gamma_2' := (P^{\gamma_2'}, S^{\gamma_2'})$, where

$$P^{\gamma_2'}=\{p_1,\cdots,p_{i-1},p_{i+1},\cdots,p_t,p_i'\}\ ,\tag{A.46}$$

$$S^{\gamma_2'}=S^{\gamma_1'}=\{s_1^g,\cdots,s_{i-1}^g,s_{i+1}^g,\cdots,s_t^g,s_i^g\}\ .\tag{A.47}$$

Define strategy $\gamma := (P^\gamma, S^\gamma)$, where

$$P^{\gamma}=\{p_1,\cdots,p_{i-1},p_{i+1},\cdots,p_t\}\ ,\tag{A.48}$$

$$S^{\gamma}=\{s_1^g,\cdots,s_{i-1}^g,s_{i+1}^g,\cdots,s_t^g\}\ .\tag{A.49}$$

Assume the variances of parameter p_i and p_i', after the strategy γ has been executed, are $\sigma_{p_i}^\gamma$ and $\sigma_{p_i'}^\gamma$, respectively. Then

$$J(\gamma_1)=J(\gamma_1')=J(\gamma)+Q_{p_i,s_i^g}(\sigma_{p_i}^\gamma)\ ,\tag{A.50}$$

$$J(\gamma_2)=J(\gamma_2')=J(\gamma)+Q_{p_i',s_i^g}(\sigma_{p_i'}^\gamma)\ ,\tag{A.51}$$

where $Q_{p_i,s_i^g}(\sigma_{p_i}^{\gamma})$ and $Q_{p_i',s_i^g}(\sigma_{p_i'}^{\gamma})$ are defined in equation (2.6).

From Lemma 2.4 and Equation (2.11) and (2.14), we have

$$\begin{aligned} B_{l,i}(p_i) &= R_{p_i}(i-1,\{s_i^g, s_{i+1}^g, s_{i+2}^g, \cdots, s_M^g\}) \\ &\quad - R_{p_i}(i-1,\{s_{i+1}^g, s_{i+2}^g, \cdots, s_M^g\}) - c_{p_i,s_i^g} \\ &= \sigma_{p_i}(i-1,\{s_{i+1}^g, s_{i+2}^g, \cdots, s_M^g\}) \\ &\quad - \sigma_{p_i}(i-1,\{s_i^g, s_{i+1}^g, s_{i+2}^g, \cdots, s_M^g\}) - c_{p_i,s_i^g} \\ &\leq \sigma_{p_i}^{\gamma} - \sigma_{p_i}^{\gamma}(t-1,\{s_i^g\}) - c_{p_i,s_i^g} \\ &= -Q_{p_i,s_i^g}(\sigma_{p_i}^{\gamma}) \ . \end{aligned} \tag{A.52}$$

Equality in (A.52) holds if and only if every sensor in the set $\{s_{i+1}^g, s_{i+2}^g, \cdots, s_t^g\}$ is used to measure parameter p_i after time instant i.

Similarly, from Lemma 2.4 and Equation (2.13), we have

$$\begin{aligned} B_{u,i}(p_i') &= R_{p_i'}(i-1,\{s_i^g\}) - c_{p_i',s_i^g} \\ &= \sigma_{p_i'}(i-1) - \sigma_{p_i'}(i-1,\{s_i^g\}) - c_{p_i',s_i^g} \\ &\geq \sigma_{p_i'}^{\gamma} - \sigma_{p_i'}^{\gamma}(t-1,\{s_i^g\}) - c_{p_i',s_i^g} \\ &= -Q_{p_i',s_i^g}(\sigma_{p_i'}^{\gamma}) \ . \end{aligned} \tag{A.53}$$

Equality in (A.53) holds if and only if no sensor in the set $\{s_{i+1}^g, s_{i+2}^g, \cdots, s_t^g\}$ is used to measure parameter p_i' after time instant i.

From (A.52), (A.53) and the assumption $B_{l,i}(p_i) \geq B_{u,i}(p_i')$, we have

$$-Q_{p_i',s_i^g}(\sigma_{p_i'}^{\gamma}) \leq B_{u,i}(p_i') \leq B_{l,i}(p_i) \leq -Q_{p_i,s_i^g}(\sigma_{p_i}^{\gamma}) \ . \tag{A.54}$$

Therefore,

$$J(\gamma_1) = J(\gamma) + Q_{p_i,s_i^g}(\sigma_{p_i}^{\gamma}) \tag{A.55}$$

$$\leq J(\gamma) + Q_{p_i',s_i^g}(\sigma_{p_i'}^{\gamma}) \tag{A.56}$$

$$= J(\gamma_2) \ . \tag{A.57}$$

□

Proof of Theorem 2.2.

We will prove that Conditions 2.1, 2.2 and 2.3 are sufficient to establish the optimality of the greedy algorithm by contradiction.

Consider the strategy $\gamma^g = (P^g, S^g)$, with

$$P^g = \{p_1^g, \cdots, p_\tau^g\} \ , \tag{A.58}$$

$$S^g = \{s_1^g, \cdots, s_\tau^g\} \ , \tag{A.59}$$

where P^g is generated by Algorithm L. Assume Conditions 2.1, 2.2 hold and Condition 2.3 holds for $t = 1, \cdots, \tau$. Suppose that strategy $\gamma^g = (P^g, S^g)$ is not optimal; instead, there exists a strategy $\gamma = (P, S)$ with

$$P = \{p_1', \cdots, p_t'\} \ , \tag{A.60}$$

$$S = \{s_1^g, \cdots, s_t^g\} \ , \tag{A.61}$$

which is optimal and $P^g \neq P$. Thus,

$$J(\gamma^g) > J(\gamma). \tag{A.62}$$

We examine two different cases.

Case 1: $t \leq \tau$

If $P = \{p_1', \cdots, p_t'\} = \{p_1^g, \cdots, p_t^g\}$, then $t < \tau$ since $P \neq P^g$. From Algorithm L and $t < \tau$, we know there exists at least one parameter p_{t+1}^g, such that

$$\sigma_{p_{t+1}^g}(t) > TH_{p_{t+1}^g, s_{t+1}^g} \ . \tag{A.63}$$

Define a strategy $\gamma' := (P', S')$, with

$$P' = \{p_1^g, \cdots, p_t^g, p_{t+1}^g\} \ , \tag{A.64}$$

$$S' = \{s_1^g, \cdots, s_t^g, s_{t+1}^g\} \ , \tag{A.65}$$

The cost of strategy γ' is

$$J(\gamma') = J(\gamma) + \{c_{p_{t+1}^g, s_{t+1}^g} - \frac{\sigma^2_{p_{t+1}^g}(t) I_{p_{t+1}^g, s_{t+1}^g}}{\sigma_{p_{t+1}^g}(t) I_{p_{t+1}^g, s_{t+1}^g} + 1}\} \ . \tag{A.66}$$

Because of (A.63), (2.3), (2.5) and (2.8), (A.66) gives

$$J(\gamma') < J(\gamma) \ . \tag{A.67}$$

which contradicts the optimality of γ.

If $P = \{p'_1, \cdots, p'_t\} \neq \{p^g_1, \cdots, p^g_t\}$, denote p'_i to be the first parameter in P, which is different from p^g_i, i.e. $p'_j = p_j$, for $j = 1, \cdots, i-1$, $p'_i \neq p_i$. Then,

$$P = \{p^g_1, \cdots, p^g_{i-1}, p'_i, p'_{i+1}, \cdots, p'_t\} \ . \tag{A.68}$$

Define a strategy $\gamma' := (P', S')$, with

$$\begin{aligned} P' &= \{p'_1, \cdots, p'_{i-1}, p^g_i, p'_{i+1}, \cdots, p'_t\} \quad &(A.69)\\ &= \{p^g_1, \cdots, p^g_{i-1}, p^g_i, p'_{i+1}, \cdots, p'_t\} \ , \\ S' &= S \ . \quad &(A.70) \end{aligned}$$

Since Condition 3 for parameter p^g_i holds at time instant i, by Lemma 2.5, we have

$$J(\gamma) \geq J(\gamma') \ , \tag{A.71}$$

which contradicts the optimality of γ.

Case 2: $t > \tau$

If $P = \{p^g_1, \cdots, p^g_\tau, p'_{\tau+1}, \cdots, p'_t\}$, from algorithm L we know for any parameter $p'_{\tau+1}$,

$$\sigma_{p'_{\tau+1}}(\tau) \leq TH_{p'_{\tau+1}, s^g_{\tau+1}} \ . \tag{A.72}$$

Furthermore, there exists a strategy $\hat{\gamma} := (\hat{P}, \hat{S})$ that is equivalent to γ, with

$$\hat{P} = \{p^g_1, \cdots, p^g_\tau, p'_{\tau+2}, \cdots, p'_t, p'_{\tau+1}\} \ , \tag{A.73}$$

$$\hat{S} = \{s^g_1, \cdots, s^g_\tau, s^g_{\tau+2}, \cdots, s^g_t, s^g_{\tau+1}\} \ . \tag{A.74}$$

From Algorithm L, we know for any parameter $p'_{\tau+1}$,

$$\sigma_{p'_{\tau+1}}(t-1) \leq \sigma_{p'_{\tau+1}}(\tau) \leq TH_{p'_{\tau+1}, s^g_{\tau+1}} \ . \tag{A.75}$$

Define a strategy $\gamma' := (P', S')$, with

$$P' = \{p_1^g, \cdots, p_\tau^g, p'_{\tau+2}, \cdots, p'_t\} \ , \tag{A.76}$$

$$S' = \{s_1^g, \cdots, s_\tau^g, s_{\tau+2}^g, \cdots, s_t^g\} \ . \tag{A.77}$$

Then

$$\begin{aligned} J(\gamma') &= J(\hat{\gamma}) - [c_{p^g_{\tau+1}, s^g_{\tau+1}} - \frac{\sigma^2_{p^g_{\tau+1}}(t-1) I_{p^g_{\tau+1}, s^g_{\tau+1}}}{\sigma_{p^g_{\tau+1}}(t-1) I_{p^g_{\tau+1}, s^g_{\tau+1}} + 1}] \\ &< J(\hat{\gamma}) \\ &= J(\gamma), \end{aligned} \tag{A.78}$$

which contradicts the optimality of γ.

If $P \neq \{p_1^g, \cdots, p_\tau^g, p'_{\tau+1}, \cdots, p'_t\}$, denote p'_i to be the first parameter in P, which is different from p_i^g, i.e. $p'_j = p_j^g$, for $j = 1, \cdots, i-1$ and $p'_i \neq p_i^g$, where $i \leq \tau$, and

$$P = \{p_1^g, \cdots, p_{i-1}^g, p'_i, \cdots, p'_\tau, \cdots, p'_t\} \ . \tag{A.79}$$

Define a strategy $\gamma' := (P', S')$, with

$$P' = \{p'_1, \cdots, p'_{i-1}, p_i^g, p'_{i+1}, \cdots, p'_t\} \tag{A.80}$$

$$= \{p_1^g, \cdots, p_{i-1}^g, p_i^g, p'_{i+1}, \cdots, p'_t\} \ , \tag{A.81}$$

$$S' = S \ . \tag{A.82}$$

Since by assumption Condition 3 holds for parameter p_i^g at time instant i, by Lemma 2.5, we have

$$J(\gamma) \geq J(\gamma') \ , \tag{A.83}$$

which contradicts the optimality of γ.

By combing the above two cases, we conclude that if Condition 2.1, 2.2 are satisfied and Condition 2.3 holds at every time instant t the greedy algorithm L generates an optimal strategy for Problem $P2$.

$\square$

APPENDIX B

PROOFS FOR CHAPTER 3

Proof of Lemma 3.1.

We prove (3.3) by induction.

First we prove that when the sensor set $A = \{1, 2\}$, the lemma is true. Denote by $\sigma_p(\{1\})$, the variance after parameter p is measured by sensor 1, and by $\sigma_p(\{1, 2\})$, the variance after parameter p is measured by sensors 1 and 2. Then from Equation (3.2), we have

$$\sigma_p(\{1\}) = \frac{\sigma_p(0)}{\sigma_p(0) \cdot \frac{H_1^2}{v_1} + 1} \tag{B.1}$$

$$\sigma_p(\{1,2\}) = \frac{\sigma_p(\{1\})}{\sigma_p(\{1\}) \cdot \frac{H_2^2}{v_2} + 1} \tag{B.2}$$

$$= \frac{\frac{\sigma_p(0)}{\sigma_p(0) \cdot \frac{H_1^2}{v_1} + 1}}{\frac{\sigma_p(0)}{\sigma_p(0) \cdot \frac{H_1^2}{v_1} + 1} \cdot \frac{H_2^2}{v_2} + 1} \tag{B.3}$$

$$= \frac{\sigma_p(0)}{\sigma_p(0)(\frac{H_1^2}{v_1} + \frac{H_2^2}{v_2}) + 1} \; . \tag{B.4}$$

Equations (B.1) and (B.4) establish the induction basis.

Assume for any sensor set $A_n = \{1, 2, \cdots, n\}$, we have

$$\sigma_p(A_n) = \frac{\sigma_p(0)}{\sigma_p(0) I_{A_n} + 1}, \tag{B.5}$$

where $I_A = \sum_{k=1}^{n} I_k$.

Then for sensor set $A_{n+1} = \{1, 2, \cdots, n+1\}$, the post-measurement variance is

$$\sigma_p(A_{n+1}) = \frac{\sigma_p(A_n)}{\sigma_p(A_n) \cdot \frac{H_{n+1}^2}{v_{n+1}} + 1} \tag{B.6}$$

$$= \frac{\frac{\sigma_p(0)}{\sigma_p(0)I_{A_n}+1}}{\frac{\sigma_p(0)}{\sigma_p(0)I_{A_n}+1} \cdot \frac{H_{n+1}^2}{v_{n+1}} + 1} \tag{B.7}$$

$$= \frac{\sigma_p(0)}{\sigma_p(0)(I_{A_n} + \frac{H_{n+1}^2}{v_{n+1}}) + 1} \tag{B.8}$$

$$= \frac{\sigma_p(0)}{\sigma_p(0)I_{A_{n+1}} + 1}. \tag{B.9}$$

Equation (B.7) follows from the induction hypothesis (B.5). Equations (B.6)-(B.9) establish the induction step.

From Equation (B.9), it is easily verified that $\sigma_p(A)$ is an increasing function of $\sigma_p(0)$ and a decreasing function of of I_A. $\square$

Proof of Lemma 3.2.

We prove this lemma by contradiction.

Suppose $\lambda^* = \{A_1^*, A_2^*, \cdots, A_P^*\}$ is an optimal sensor allocation policy, and $A(\lambda^*) = \cup_{i=1}^N A_i^*$. Assume there exists sensor i and parameter k, *s.t.*, $i \in A_k^* \subseteq A(\lambda^*)$ and sensor $j \in \mathcal{S} \setminus A(\lambda^*)$, such that

$$I_i < I_j \ . \tag{B.10}$$

Define a strategy $\lambda := \{A_1^*, A_2^*, \cdots, A_{k-1}^*, A_k, A_{k+1}^*, \cdots, A_N^*\}$, where

$$A_k = A_k^* \setminus \{i\} \cup \{j\} \ . \tag{B.11}$$

By (B.10) and (B.11), we have

$$I_{A_k^*} < I_{A_k} \tag{B.12}$$

The last inequality (B.12) and Lemma 3.1 imply that

$$\sigma_k(A_k^*) > \sigma_k(A_k) \ . \tag{B.13}$$

Since all the sensor subsets used to measure parameters other than parameter k are the same under strategies λ^* and λ, we have

$$J^* > J^\lambda \ , \tag{B.14}$$

which contradicts the optimality of λ^*. $\square$

Proof of Property 3.1.

From Definition 3.1, we observe that if $\mathcal{S}_1 \subset \mathcal{S}_2$, then $\sum_{i=1}^{S_1} I_i < \sum_{i=1}^{S_2} I_i$ whereas $\sum_{j=1}^{P} \frac{1}{\sigma_j(0)}$ remains unchanged. Hence $\tilde{\sigma}(\mathcal{S}_1, \mathcal{P}) > \tilde{\sigma}(\mathcal{S}_2, \mathcal{P})$. $\square$

Proof of Lemma 3.3.

The performance achieved by $\hat{\lambda}^*$ is

$$\hat{J}^*(\mathcal{S}, \mathcal{P}) = \sum_{i=1}^{P} \frac{1}{I_{A_i^*} + \frac{1}{\sigma_i(0)}} + S \cdot c \ . \tag{B.15}$$

By the arithmetic-harmonic-mean inequality, which is a special case of Theorem 16 in [29] Chapter II,

$$\sum_{i=1}^{P} \frac{1}{I_{A_i^*} + \frac{1}{\sigma_i(0)}} + S \cdot c \geq P \cdot \left(\frac{\sum_{i=1}^{P} \left(I_{A_i^*} + \frac{1}{\sigma_i(0)} \right)}{P} \right)^{-1} + S \cdot c \tag{B.16}$$

$$= \frac{P^2}{\sum_{i=1}^{S} I_i + \sum_{j=1}^{P} \frac{1}{\sigma_j(0)}} + S \cdot c \tag{B.17}$$

$$= P \cdot \tilde{\sigma}(\mathcal{S}, \mathcal{P}) + S \cdot c \ , \tag{B.18}$$

where equality in (B.16) holds if and only if

$$I_{A_i^*} + \frac{1}{\sigma_i(0)} = I_{A_j^*} + \frac{1}{\sigma_j(0)} \text{ for all } i, j \in \mathcal{P} \ , \tag{B.19}$$

which is equivalent to

$$\sigma_i(A_i^*) = \frac{1}{I_{A_i^*} + \frac{1}{\sigma_i(0)}} = \tilde{\sigma}(\mathcal{S}, \mathcal{P}) \ , \text{ for all } i \in \mathcal{P} \ . \tag{B.20}$$

Equality in (B.18) follows from the definition of harmonic mean function in equation (3.6). $\square$

Proof of Theorem 3.1.

Since λ^* is optimal for (P1), (3.8) and (3.9) immediately follow from the definitions of Problem (P1) and Problem (P2) and $A(\lambda^*) := \cup_{i=1}^{P} A_i^* = \{s_1, s_2, \cdots, s_{\tau^*}\}$.

By Lemma 3.3, replacing $\mathcal{S}$ with $A(\lambda^*)$, we have

$$\hat{J}^*(A(\lambda^*), \mathcal{P}) = \sum_{i=1}^{P} \frac{1}{I_{A_i^*} + \frac{1}{\sigma_i(0)}} + \tau^* \cdot c \tag{B.21}$$

$$\geq P \cdot \tilde{\sigma}(A(\lambda^*), \mathcal{P}) + \tau^* \cdot c \tag{B.22}$$

$$= \frac{P^2}{\sum_{i=1}^{\tau^*} I_{s_i} + \sum_{i=1}^{N} \frac{1}{\sigma_i(0)}} + \tau^* \cdot c \ , \tag{B.23}$$

where the equality in (B.23) follows from Equation (3.6). Define $L(t) := \frac{P^2}{\sum_{i=1}^{t} I_i + \sum_{i=1}^{P} \frac{1}{\sigma_i(0)}} + t \cdot c$. Below we prove that $L(t)$ is a strictly convex function of t, $t = 1, 2, \cdots, S$, i.e. $L(t) - L(t+1) > L(t+1) - L(t+2)$ for all $t = 1, 2, \cdots, S-2$.

$$\begin{aligned} L(t) - L(t+1) &= \frac{P^2}{\sum_{i=1}^{t} I_i + \alpha} - \frac{P^2}{\sum_{i=1}^{t+1} I_i + \alpha} - c \\ &= \frac{P^2 \cdot I_{t+1}}{(\sum_{i=1}^{t} I_i + \alpha) \cdot (\sum_{i=1}^{t+1} I_i + \alpha)} - c \ , \end{aligned} \tag{B.24}$$

$$\begin{aligned} L(t+1) - L(t+2) &= \frac{P^2}{\sum_{i=1}^{t+1} I_i + \alpha} - \frac{P^2}{\sum_{i=1}^{t+2} I_i + \alpha} - c \\ &= \frac{P^2 \cdot I_{t+2}}{(\sum_{i=1}^{t+1} I_i + \alpha) \cdot (\sum_{i=1}^{t+2} I_i + \alpha)} - c \ . \end{aligned} \tag{B.25}$$

Since

$$I_t \geq I_{t+1} \ , \tag{B.26}$$

and

$$(\sum_{i=1}^{t+1} I_i + \alpha) \cdot (\sum_{i=1}^{t+2} I_i + \alpha) > (\sum_{i=1}^{t} I_i + \alpha) \cdot (\sum_{i=1}^{t+1} I_i + \alpha) \ , \tag{B.27}$$

it follows that

$$L(t) - L(t+1) > L(t+1) - L(t+2) \ , \tag{B.28}$$

which means $L(t) - L(t+1)$ is a strictly decreasing function of t and $L(t)$ is strictly convex in t.

Hence there exists t^*, which is a minimizer of $L(t)$. $\square$

Proof of Lemma 3.4.

We prove (3.13) by contradiction.

Suppose $\lambda^* = \{A_1^*, A_2^*, \cdots, A_P^*\}$ is an optimal partition but (3.13) is not true. Let $i < j$ and consider the partition strategy

$$\lambda = \{A_1^*, \cdots, A_{i-1}^*, A_i, A_{i+1}^*, \cdots, A_{j-1}^*, A_j, A_{j+1}^*, \cdots, A_P^*\} \ . \tag{B.29}$$

Then

$$\begin{aligned} J^* - J^\lambda = &\left[(I_{A_i^*} + \frac{1}{\sigma_i(0)})^{-1} + (I_{A_j^*} + \frac{1}{\sigma_j(0)})^{-1} \right] \\ &- \left[(I_{A_i} + \frac{1}{\sigma_i(0)})^{-1} + (I_{A_j} + \frac{1}{\sigma_j(0)})^{-1} \right] \geq 0 \ , \end{aligned} \tag{B.30}$$

which contradicts the optimality of λ^*.

To prove (3.14) we note that

$$\arg \min_{A_i \subseteq A_{i,j}^*} \left(\frac{1}{I_{A_i} + \frac{1}{\sigma_i(0)}} + \frac{1}{I_{A_j} + \frac{1}{\sigma_j(0)}} \right) \tag{B.31}$$

$$= \arg \max_{A_i \subseteq A_{i,j}^*} \left(I_{A_i} + \frac{1}{\sigma_i(0)} \right) \times \left(I_{A_j} + \frac{1}{\sigma_j(0)} \right) \tag{B.32}$$

$$= \arg \max_{A_i \subseteq A_{i,j}^*} \left(I_{A_i} + \frac{1}{\sigma_i(0)} \right) \times \left(I_{A_{i,j}^*} - I_{A_i} + \frac{1}{\sigma_j(0)} \right) \tag{B.33}$$

$$= \arg \max_{A_i \subseteq A_{i,j}^*} \left[-(I_{A_i})^2 + (I_{A_{i,j}^*} + \frac{1}{\sigma_j(0)} - \frac{1}{\sigma_i(0)}) I_{A_i} + (I_{A_{i,j}^*} + \frac{1}{\sigma_j(0)}) \frac{1}{\sigma_i(0)} \right] \tag{B.34}$$

$$\begin{aligned} = \arg \max_{A_i \subseteq A_{i,j}^*} \Bigg\{ &- \left[I_{A_i} - \frac{1}{2}(I_{A_{i,j}^*} + \frac{1}{\sigma_j(0)} - \frac{1}{\sigma_i(0)}) \right]^2 \\ &+ \left[\frac{1}{4}(I_{A_{i,j}^*} + \frac{1}{\sigma_j(0)} - \frac{1}{\sigma_i(0)})^2 + (I_{A_{i,j}^*} + \frac{1}{\sigma_j(0)}) \frac{1}{\sigma_i(0)} \right] \Bigg\} \end{aligned} \tag{B.35}$$

$$= \arg \min_{A_i \subseteq A_{i,j}^*} \left| I_{A_i} - \frac{1}{2}(I_{A_{i,j}^*} + \frac{1}{\sigma_j(0)} - \frac{1}{\sigma_i(0)}) \right| \tag{B.36}$$

$$= \arg \min_{A_i \subseteq A_{i,j}^*} \left| (I_{A_i} + \frac{1}{\sigma_i(0)}) - (I_{A_j} + \frac{1}{\sigma_j(0)}) \right| \ . \tag{B.37}$$

By the first part of this lemma, A_i^* minimizes (B.31). Therefore (B.37) gives

$$A_i^* = \arg\min_{A_i \subseteq A_{i,j}^*} \left|(I_{A_i} + \frac{1}{\sigma_i(0)}) - (I_{A_j} + \frac{1}{\sigma_j(0)})\right| . \tag{B.38}$$

Consequently,

$$\left|(I_{A_i} + \frac{1}{\sigma_i(0)}) - (I_{A_j} + \frac{1}{\sigma_j(0)})\right| \geq \left|(I_{A_i^*} + \frac{1}{\sigma_i(0)}) - (I_{A_j^*} + \frac{1}{\sigma_j(0)})\right| . \tag{B.39}$$

$\square$

Proof of Lemma 3.5.

We prove the lemma by contradiction.

Suppose $\lambda^* = \{A_1^*, A_2^*, \cdots, A_P^*\}$ is an optimal policy. Assume that under λ^*, there exists parameters $i, j \in \mathcal{P}$, such that

$$\sigma_i(0) \geq \sigma_j(0), \tag{B.40}$$

and

$$I_{A_i^*} < I_{A_j^*}. \tag{B.41}$$

Define a strategy $\lambda := \{A_1, \cdots, A_P\}$, such that

$$A_i = A_j^* \ , A_j = A_i^* \ , \tag{B.42}$$

$$A_k = A_k^* \ , \forall k \neq i, j \ . \tag{B.43}$$

By (B.40) and (B.41), we have

$$I_{A_i^*} - I_{A_j^*} < 0 \ , \text{ and } \frac{1}{\sigma_i(0)} - \frac{1}{\sigma_j(0)} < 0 \ , \tag{B.44}$$

$$\left|(\frac{1}{\sigma_i(0)} - \frac{1}{\sigma_j(0)}) + (I_{A_i^*} - I_{A_j^*})\right| > \left|(\frac{1}{\sigma_i(0)} - \frac{1}{\sigma_j(0)}) - (I_{A_i^*} - I_{A_j^*})\right| \ , \tag{B.45}$$

or, equivalently,

$$\left|(I_{A_i^*} + \frac{1}{\sigma_i(0)}) - (I_{A_j^*} + \frac{1}{\sigma_j(0)})\right| > \left|(I_{A_j^*} + \frac{1}{\sigma_i(0)}) - (I_{A_i^*} + \frac{1}{\sigma_j(0)})\right| . \tag{B.46}$$

Equation (B.42) along with (B.46) imply that

$$\left|(I_{A_i^*} + \frac{1}{\sigma_i(0)}) - (I_{A_j^*} + \frac{1}{\sigma_j(0)})\right| > \left|(I_{A_i} + \frac{1}{\sigma_i(0)}) - (I_{A_j} + \frac{1}{\sigma_j(0)})\right| \ , \tag{B.47}$$

which, because of Lemma 3.4, contradicts the optimality of λ^*. $\square$

Proof of Lemma 3.6.

The assumption $\sigma_1(0) < TH_i$ along with the definition of the threshold of a sensor implies that it is not optimal to measure parameter 1 with sensor i. Since $\sigma_1(0) \geq \sigma_j(0)$, $\forall j > 1$, no parameter should be measured by sensor i. Furthermore, since $TH_i < TH_j$, when $j > i$, which means $I_i > I_j$, Lemma 3.2 implies that no parameter should be measured by any sensor j, $j \geq i$. $\square$

Proof of Lemma 3.7.

Preliminaries:

For convenience, we repeat Algorithm PL in Figure B.1 and describe its mth cycle. We assume that in the ith cycle, $i = 1, \cdots, m-1$, Test 1 is passed l_i times, where $l_i \geq 0$.

the mth cycle:

Step 1 of the mth cycle begins with

$$t_s = m - 1 \ , t_p = \sum_{i=1}^{m-1} l_i \ , \tag{B.48}$$

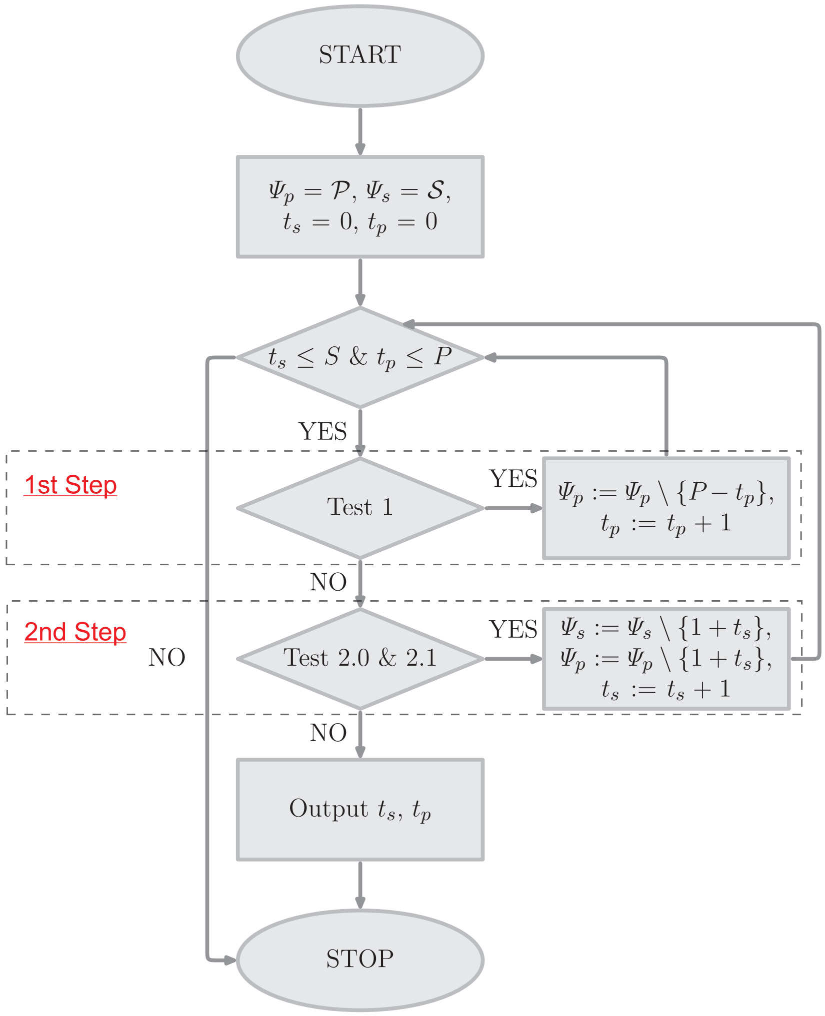
$$\Psi_s = \mathcal{S} \setminus \{1, \cdots, m-1\} \ , \tag{B.49}$$

$$\Psi_p = \mathcal{P} \setminus \{1, \cdots, m-1, P - \sum_{i=1}^{m-1} l_i + 1, \cdots, P\} \ . \tag{B.50}$$

It is implicitly assumed here that $P - \sum_{i=1}^{m-1} l_i + 1 > m - 1$.

PL performs Test 1, which is

$$\sigma_{P-\sum_{i=1}^{m-1} l_i}(0) < \tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m-1\}, \mathcal{P} \setminus \{1, \cdots, m-1, P - \sum_{i=1}^{m-1} l_i + 1, \cdots, P\}) \ . \tag{B.51}$$



Test 1 : $\sigma_{N-t_p}(0) \leq \tilde{\sigma}(\Psi_s, \Psi_p)$,

Test 2.0: $\sigma_{1+t_s}(0) > TH_{1+t_s}$,

Test 2.1: $I_{1+t_s} \geq \dfrac{1}{\tilde{\sigma}(\Psi_s, \Psi_p)} - \dfrac{1}{\sigma_{1+t_s}(0)}$.

Figure B.1: The flowchart of Algorithm PL

If Test 1 is passed, t_p and Ψ_p are updated as follows.

$$t_p := \sum_{i=1}^{m-1} l_i + 1 \ , \tag{B.52}$$

$$\Psi_p := \mathcal{P} \setminus \{1, \cdots, m-1, P - \sum_{i=1}^{m-1} l_i, \cdots, P\} \ . \tag{B.53}$$

Then Test 1 is repeatedly performed until it is not passed anymore. With the assumption that Test 1 is passed l_m times, the mth cycle, Step 1 ends with

$$t_p := \sum_{i=1}^{m} l_i \ , \tag{B.54}$$

$$\Psi_p := \mathcal{P} \setminus \{1, \cdots, m-1, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\} \ . \tag{B.55}$$

In Step 2 of the mth cycle, Test 2.0 and 2.1 are performed. Test 2.0 is

$$\sigma_m(0) > TH_m \ , \tag{B.56}$$

and Test 2.1 is

$$I_m > \frac{1}{\tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m-1\}, \mathcal{P} \setminus \{1, \cdots, m-1, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\})} - \frac{1}{\sigma_m(0)} \ . \tag{B.57}$$

If Test 2.0 and 2.1 are passed, t_s, Ψ_s and Ψ_p are updated as follows.

$$t_s = m \ , \tag{B.58}$$

$$\Psi_s = \mathcal{S} \setminus \{1, \cdots, m\} \ , \tag{B.59}$$

$$\Psi_p = \mathcal{P} \setminus \{1, \cdots, m, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\} \ . \tag{B.60}$$

The mth cycle, Step 2 ends here.

Proof: We prove (3.15) based on Algorithm PL and Lemmas B.1-B.3, stated below.

We denote an arbitrary parameter set by $\Omega_p = \{p_1, p_2, \cdots, p_N\}$, and an arbitrary sensor set by $\Omega_s = \{s_1, s_2, \cdots, s_M\}$, such that $\sigma_{p_1}(0) \geq \sigma_{p_2}(0) \geq \cdots \geq \sigma_{p_N}(0)$ and $I_{s_1} \geq I_{s_2} \geq \cdots \geq I_{s_M}$. For Problem (P1) with the pair (Ω_p, Ω_s), assume that an optimal policy is $\lambda^* := \{A^*_{p_1}, \cdots, A^*_{p_N}\}$.

Lemma B.1. *(i) If*

$$\sigma_{p_N}(0) < \tilde{\sigma}(\Omega_s, \Omega_p) \ , \tag{B.61}$$

(where $\tilde{\sigma}(\Omega_s, \Omega_p)$ is defined by (B.23)*), then parameter p_N is not measured by any sensor under λ^*, i.e. $A^*_{p_N} = \emptyset$, and*

$$J^*(\Omega_s, \Omega_p) = J^*(\Omega_s, \Omega_p \setminus \{p_N\}) + \sigma_{p_N}(0). \tag{B.62}$$

(ii) Furthermore,

$$\tilde{\sigma}(\Omega_s, \Omega_p \setminus \{p_N\}) > \tilde{\sigma}(\Omega_s, \Omega_p) \ . \tag{B.63}$$

Lemma B.2. *(i) If*

$$\sigma_{p_1}(0) > TH_{s_1} \ , \tag{B.64}$$

$$\text{and } I_{s_1} \geq \frac{1}{\tilde{\sigma}(\Omega_s, \Omega_p)} - \frac{1}{\sigma_{p_1}(0)} \ , \tag{B.65}$$

then $\exists p_k \in \Omega_p$, $k < N$, such that parameter p_k is measured only by sensor s_1 under λ^*, i.e., *$A^*_{p_k} = \{s_1\}$, and*

$$J^*(\Omega_s, \Omega_p) = J^*(\Omega_s \setminus \{s_1\}, \Omega_p \setminus \{p_k\}) + \frac{1}{I_{s_1} + \frac{1}{\sigma_{p_k}(0)}} + c \ . \tag{B.66}$$

(ii) Furthermore,

$$\tilde{\sigma}(\Omega_s \setminus \{s_1\}, \Omega_p \setminus \{p_k\}) \geq \tilde{\sigma}(\Omega_s \setminus \{s_1\}, \Omega_p \setminus \{p_1\}) > \tilde{\sigma}(\Omega_s, \Omega_p) \ . \tag{B.67}$$

Lemma B.3. *If sensors $s_1, \cdots, s_j$ are each used alone, and $A^*_{p_{k_i}} = \{s_i\}$, then*

$$(i)\ 1 \leq k_1 < k_2 < \cdots < k_j \leq N, \text{ i.e., } 1 \leq i \leq k_i \leq N - j + i, i = 1, \cdots, j \ ; \tag{B.68}$$

$$(ii)\ \tilde{\sigma}(\Omega_s \setminus \{s_1, \cdots, s_j\}, \Omega_p \setminus \{p_{k_1}, \cdots, p_{k_j}\}) \geq \tilde{\sigma}(\Omega_s \setminus \{s_1, \cdots, s_j\}, \Omega_p \setminus \{p_1, \cdots, p_j\}) \tag{B.69}$$

We prove these lemmas at the end of the proof of Lemma 3.7. The proof of (B.66) proceeds by induction.

Basis of Induction: We want to prove that (3.15) with $t_s = 1$ and $t_p = l_1$ holds true, which means in the first cycle, if Test 1 is passed l_1 times, parameters $P - l_1 + 1, \cdots, P$ are not measured and if Tests 2.0 and 2.1 are passed, sensor 1 should be used alone.

The first step of the first cycle starts with $t_s = 0$, $t_p = 0$ and $\Psi_p = \mathcal{P}$, $\Psi_s = \mathcal{S}$. PL performs Test 1, that is, it checks whether or not

$$\sigma_P < \tilde{\sigma}(\mathcal{S}, \mathcal{P}) \ . \tag{B.70}$$

(B.70) is exactly the same as condition (B.61) in Lemma B.1 with $\Omega_p = \mathcal{P}$ and $\Omega_s = \mathcal{S}$. If Test 1 is passed, by Lemma B.1, parameter P is identified as the one not to be measured. Ψ_p is updated as $\mathcal{P} \setminus \{P\}$ and $t_p = 1$. Test 1 with $\Psi_p = \mathcal{P} \setminus \{P - t_p + 1, \cdots, P\}$ is repeatedly performed as t_p increases. For each t_p, Test 1 is exactly condition (B.61) in Lemma B.1 with $\Omega_p = \mathcal{P} \setminus \{P - t_p + 1, \cdots, P\}$. The parameters not to be measured are repeatedly identified by Lemma B.1 until Test 1 is not passed anymore. Assuming that in the first cycle Test 1 is passed l_1 times, parameters $P - l_1 + 1, \cdots, P$ are not measured. Ψ_p is updated as $\mathcal{P} \setminus \{P - l_1 + 1, \cdots, P\}$ and $t_p = l_1$, $t_s = 0$. The first step of the first cycle ends here.

In the second step of the first cycle, PL performs Tests 2.0 and 2.1, that is, it checks if

$$\sigma_1(0) > TH_1 \ , \tag{B.71}$$

$$I_1 > \frac{1}{\tilde{\sigma}(\mathcal{S}, \mathcal{P})} - \frac{1}{\sigma_1(0)} \ ; \tag{B.72}$$

(B.71) and (B.72) are exactly the same as conditions (B.64) and (B.65) in Lemma B.2 with $\Omega_p = \Psi_p = \mathcal{P} \setminus \{P - l_1 + 1, \cdots, P\}$ and $\Omega_s = \Psi_s = \mathcal{S}$. If Tests 2.0 and 2.1 are passed, by Lemma B.2 sensor 1 must be used alone. The parameter measured by sensor 1 under λ^* is, in general not known. Let k_1 be this parameter ($k_1 \neq P - l_1$ by Lemma B.2). Then (3.15) holds true with $t_s = 1$ and $t_p = l_1$.

The proof of the induction basis is now complete.

Induction step: Assume that in the ith cycle, $1 \leq i \leq m$, Test 1 is passed l_i times, Test 2.0 and Test 2.1 are passed and Equation (3.15), with $t_s - i$, $t_p - \sum_{j=1}^{i} l_j$, holds true. We want to prove that if in the $(m+1)$th cycle Test 1 is paused Equation (3.15), with $t_s - (m-1)$, $t_p - \sum_{j=1}^{m+1} l_j$ will hold true.

The proof follows the two steps of the $(m+1)$th cycle. In the first step, first we show that parameter $P - \sum_{i=1}^{m} l_i$ is not measured by any sensor among $1, \cdots, m$. Furthermore we show that if Test 1 is passed for parameter $P - \sum_{i=1}^{m} l_i$, it will not be measured by any sensor. Secondly we show that parameter $P - \sum_{i=1}^{m} l_i - 1$ is not measured by any sensor among $1, \cdots, m$. Furthermore we show that if Test 1 is passed for parameter $P - \sum_{i=1}^{m} l_i - 1$, this parameter will not be measured by any sensor. This process continues until Test 1 is not passed anymore. In the second step, we show that if Tests 2.0 and 2.1 are passed, sensor $m+1$ should be used alone.

The first step of the $(m+1)$th cycle begins with

$$t_s = m, \quad t_p = \sum_{i=1}^{m} l_i \ , \tag{B.73}$$

$$\Psi_s = \mathcal{S} \setminus \{1, \cdots, m\} \ , \tag{B.74}$$

$$\Psi_p = \mathcal{P} \setminus \{1, \cdots, m, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\} \ . \tag{B.75}$$

First we want to show that parameter $P - \sum_{i=1}^{m} l_i$ is not measured by any sensor among $1, \cdots, m$.

Since sensor m passes Test 2.0 and 2.1 (which means that (B.56) and (B.57) hold), by assumption sensor m should be used alone and will not measure parameter $P - \sum_{i=1}^{m} l_i$ according to λ^*. By Lemma B.3 and (B.69), any sensor $1, \cdots, m-1$ will not measure parameter $P - \sum_{i=1}^{m} l_i$. Therefore parameter $P - \sum_{i=1}^{m} l_i$ is not measured by any sensor among $1, \cdots, m$.

Now Test 1, namely

$$\sigma_{P-\sum_{i=1}^{m} l_i}(0) < \tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{1, \cdots, m, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\}) \ . \tag{B.76}$$

is performed for parameter $P - \sum_{i=1}^{m} l_i$. If it is passed, (B.76) holds true and we want to show that parameter $P - \sum_{i=1}^{m} l_i$ is not measured.

By Lemma B.3 and (B.69), we have

$$\begin{aligned} &\tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{1, \cdots, m, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\}) \\ &\quad \leq \tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{k_1, \cdots, k_m, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\}) \ . \end{aligned} \tag{B.77}$$

Combining (B.76) and (B.77), we have

$$\sigma_{P-\sum_{i=1}^{m} l_i}(0) < \tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{k_1, \cdots, k_m, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\}) \ , \tag{B.78}$$

which is exactly condition (B.61) in Lemma B.1, with $\Omega_s = \mathcal{S} \setminus \{1, \cdots, m\}$ and $\Omega_p = \mathcal{P} \setminus \{k_1, \cdots, k_m, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\}$. By Lemma B.1, parameter $P - \sum_{i=1}^{m} l_i$ should not be measured according to λ^*, (3.15) holds true with $t_s = m$ and $t_p = \sum_{i=1}^{m} l_i + 1$, and

$$\begin{aligned} &\tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{1, \cdots, m, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\}) \\ &\quad < \tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{1, \cdots, m, P - \sum_{i=1}^{m} l_i, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\}) \ . \end{aligned} \tag{B.79}$$

Secondly we want to show that parameter $P - \sum_{i=1}^{m} l_i - 1$ is not measured by any sensor among $1, \cdots, m$.

Since when running PL, Test 2.1 is passed in the second step of the mth cycle, we have

$$I_m \geq \frac{1}{\tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m-1\}, \mathcal{P} \setminus \{1, \cdots, m-1, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\})} - \frac{1}{\sigma_m(0)} \ , \tag{B.80}$$

and by Lemma B.2

$$\tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m-1\}, \mathcal{P} \setminus \{1, \cdots, m-1, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\})$$
$$\leq \tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{1, \cdots, m, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\}) \ . \tag{B.81}$$

Then, by (B.80), (B.81) and (B.79), we obtain

$$I_m > \frac{1}{\tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{1, \cdots, m, P - \sum_{i=1}^{m} l_i, \cdots, P\})} - \frac{1}{\sigma_m(0)} \ , \tag{B.82}$$

which is equivalent to

$$I_m > \frac{\sum_{i=m+1}^{S} I_i + \sum_{j=m+1}^{P-\sum_{i=1}^{m} l_i - 1} \frac{1}{\sigma_j(0)}}{P - \sum_{i=1}^{m} l_i - 1 - m} - \frac{1}{\sigma_m(0)} \ . \tag{B.83}$$

From (B.83), we have

$$(P - \sum_{i=1}^{m} l_i - 1 - m) \cdot (I_m + \frac{1}{\sigma_m(0)}) > \sum_{i=m+1}^{S} I_i + \sum_{j=m+1}^{P-\sum_{i=1}^{m} l_i - 1} \frac{1}{\sigma_j(0)} \ , \tag{B.84}$$

$$(P - \sum_{i=1}^{m} l_i - m) \cdot (I_m + \frac{1}{\sigma_m(0)}) > \sum_{i=m+1}^{S} I_i + \sum_{j=m+1}^{P-\sum_{i=1}^{m} l_i - 1} \frac{1}{\sigma_j(0)} + (I_m + \frac{1}{\sigma_m(0)}) \ , \tag{B.85}$$

$$= \sum_{i=m}^{S} I_i + \sum_{j=m}^{P-\sum_{i=1}^{m} l_i - 1} \frac{1}{\sigma_j(0)} \ . \tag{B.86}$$

From (B.86), we have

$$I_m > \frac{\sum_{i=m}^{S} I_i + \sum_{j=m}^{P-\sum_{i=1}^{m} l_i - 1} \sigma_j(0)}{P - \sum_{i=1}^{m} l_i - 1 - m} - \frac{1}{\sigma_m(0)} \ , \tag{B.87}$$

$$= \frac{1}{\tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m-1\}, \mathcal{P} \setminus \{1, \cdots, m-1, P - \sum_{i=1}^{m} l_i, \cdots, P\})} - \frac{1}{\sigma_m(0)} \ . \tag{B.88}$$

By Lemma B.3, we have

$$\tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m-1\}, \mathcal{P} \setminus \{1, \cdots, m-1, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\})$$
$$\leq \tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m-1\}, \mathcal{P} \setminus \{k_1, \cdots, k_{m-1}, P - \sum_{i=1}^{m} l_i + 1, \cdots, P\}) \ . \tag{B.89}$$

Combining (B.87) and (B.89), we have

$$I_m > \frac{1}{\tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m-1\}, \mathcal{P} \setminus \{k_1, \cdots, k_{m-1}, P - \sum_{i=1}^{m} l_i, \cdots, P\})} - \frac{1}{\sigma_m(0)} \ , \tag{B.90}$$

which is exactly condition (B.65) in Lemma B.2 with $\Omega_s = \mathcal{S} \setminus \{1, \cdots, m-1\}$ and $\Omega_p = \mathcal{P} \setminus \{k_1, \cdots, k_{m-1}, P - \sum_{i=1}^{m} l_i, \cdots, P\}$. By Lemma B.2, sensor m will not measure parameter $P - \sum_{i=1}^{m} -1$ according to λ^*. By Lemma B.3 and (B.68), any sensor among $1, \cdots, m-1$ will not measure parameter $P - \sum_{i=1}^{m} -1$. Therefore parameter $P - \sum_{i=1}^{m} -1$ is not measured by any sensor among $1, \cdots, m$.

Now Test 1,

$$\sigma_{P - \sum_{i=1}^{m} l_i - 1}(0) < \tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{1, \cdots, m, P - \sum_{i=1}^{m} l_i, \cdots, P\}) \ , \tag{B.91}$$

is performed for parameter $P - \sum_{i=1}^{m} l_i - 1$. If it is passed, (B.91) holds true and we want to show that parameter $P - \sum_{i=1}^{m} l_i - 1$ is not measured.

By Lemma B.3 and (B.69), we have

$$\begin{aligned} &\tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{1, \cdots, m, P - \sum_{i=1}^{m} l_i, \cdots, P\}) \\ &\quad \leq \tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{k_1, \cdots, k_m, P - \sum_{i=1}^{m} l_i, \cdots, P\}) \ . \end{aligned} \tag{B.92}$$

Combining (B.91) and (B.92), we have

$$\sigma_{P - \sum_{i=1}^{m} l_i - 1}(0) < \tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{k_1, \cdots, k_m, P - \sum_{i=1}^{m} l_i, \cdots, P\}) \ , \tag{B.93}$$

which is exactly condition (B.61) in Lemma B.1, with $\Omega_s = \mathcal{S} \setminus \{1, \cdots, m\}$ and $\Omega_p = \mathcal{P} \setminus \{k_1, \cdots, k_m, P - \sum_{i=1}^{m} l_i, \cdots, P\}$. By Lemma B.1, parameter $P - \sum_{i=1}^{m} l_i - 1$ should not be measured according to λ^*, and (3.15) holds true with $t_s = m$ and $t_p = \sum_{i=1}^{m} l_i + 2$.

Assume the above process (proving that the parameter with the smallest variance is not measured by any sensor which should be used alone, and showing that if Test 1 is passed, the parameter with the smallest variance is not measured), repeats for l_{m+1} times. Then (3.15) holds true with $t_s = m$ and $t_p = \sum_{i=1}^{m+1} l_i$.

Step 1 of the $(m+1)$th cycle terminates when Test 1 fails.

The second step of $(m+1)$th cycle begins with

$$t_s = m \ , \ t_p = \sum_{i=1}^{m+1} l_i \ , \tag{B.94}$$

$$\Psi_s = \mathcal{S} \setminus \{1, \cdots, m\} \ , \tag{B.95}$$

$$\Psi_p = \mathcal{P} \setminus \{1, \cdots, m, P - \sum_{i=1}^{m+1} l_i + 1, \cdots, P\} \ . \tag{B.96}$$

If Tests 2.0 and 2.1 are passed, then,

$$\sigma_{m+1}(0) > TH_{m+1} \ , \tag{B.97}$$

$$\text{and } I_{m+1} > \frac{1}{\tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{1, \cdots, m, P - \sum_{i=1}^{m+1} l_i + 1, \cdots, P\})} - \frac{1}{\sigma_{m+1}(0)} \ . \tag{B.98}$$

By Lemma B.3 and (B.67), we have

$$\begin{aligned} &\tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{1, \cdots, m, P - \sum_{i=1}^{m+1} l_i + 1, \cdots, P\}) \\ &\quad \leq \tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{k_1, \cdots, k_m, P - \sum_{i=1}^{m+1} l_i + 1, \cdots, P\}) \ . \end{aligned} \tag{B.99}$$

Combining (B.98) and (B.99) we obtain

$$I_{m+1} > \frac{1}{\tilde{\sigma}(\mathcal{S} \setminus \{1, \cdots, m\}, \mathcal{P} \setminus \{k_1, \cdots, k_m, P - \sum_{i=1}^{m+1} l_i + 1, \cdots, P\})} - \frac{1}{\sigma_{m+1}(0)} \ , \tag{B.100}$$

which is exactly condition (B.65) in Lemma B.2, with $\Omega_s = \mathcal{S} \setminus \{1, \cdots, m\}$ and $\Omega_p = \mathcal{P} \setminus \{k_1, \cdots, k_{m+1}, P - \sum_{i=1}^{m+1} l_i + 1, \cdots, P\}$. By Lemma B.2, sensor $m+1$ should be used alone and can not measure $P - \sum_{i=1}^{m+1} l_i$. The second step of $m+1$th cycle ends. Equation (3.15) holds true with $t_s = m+1$ and $t_p = \sum_{i=1}^{m+1} l_i$.

The proof of the induction step is now complete.

□

Discussion: The intuition behind (B.68) in Lemma B.3 is that sensors that are used alone are allocated according to the following logic: under λ^*, sensors of higher quality are allocated to parameters with high variance.

The following features of the algorithm are noteworthy.

1. The updating process of t_s, (which identifies the sensor that should be used alone but does not identify the parameter the sensor should measure), will not interrupt the updating process of t_p, (which potentially excludes from the allocation process the parameter with the smallest initial variance).

2. In the first step of the $(m+1)$th cycle, $m \geq 1$, if parameter $P - \sum_{i=1}^{m} l_i$ passes Test 1, $P - \sum_{i=1}^{m} l_i - 1$ will not be measured by any sensor among $1, \cdots, m$.

Proof of Lemma B.1.

We prove (i) of Lemma B.1 by contradiction.

Assume under an optimal policy λ^*, $\sigma_{p_N}(0) < \tilde{\sigma}(\Omega_s, \Omega_p)$ and $\exists s_k \in \Omega_s$, *s.t.,*, $s_k \in A^*_{p_N}$. Since $\tilde{\sigma}(\Omega_s, \Omega_p)$ is a strictly decreasing function with respect to Ω_s and $A(\lambda^*) \subseteq \Omega_s$, we have

$$\tilde{\sigma}(A(\lambda^*), \Omega_p) \geq \tilde{\sigma}(\Omega_s, \Omega_p) \ . \tag{B.101}$$

From Lemma 3.3, we know that $\exists p_i \in \Omega_p$, such that

$$\frac{1}{\sigma_{p_i}(A^*_{p_i})} I_{A^*_{p_i}} + \frac{1}{\sigma_{p_i}(0)} \leq \frac{1}{\tilde{\sigma}(A(\lambda^*), \Omega_p)} \ . \tag{B.102}$$

Define the allocation strategy λ by

$$A^{\lambda}_{p_i} := A^*_{p_i} \cup \{s_k\} \ , \tag{B.103}$$

$$A^{\lambda}_{p_N} := A^*_{p_N} \setminus \{s_k\} \ , \tag{B.104}$$

$$A^{\lambda}_{p_j} := A^*_{p_j}, \forall j \neq i \text{ or } N \ . \tag{B.105}$$

From the condition $\frac{1}{\tilde{\sigma}(\Omega_s, \Omega_p)} < \frac{1}{\sigma_{p_N}(0)}$ and $I_{A^*_{p_N}} \geq I_{s_k}$, since $s_k \in A^*_{p_N}$, we have

$$I_{A^*_{p_N}} - I_{s_k} + \frac{1}{\sigma_{p_N}(0)} \geq \frac{1}{\sigma_{p_N}(0)} > \frac{1}{\tilde{\sigma}(\Omega_s, \Omega_p)} \ . \tag{B.106}$$

Combining Equations (B.106) (B.101) and (B.102), we obtain

$$I_{A^*_{p_N}} - I_{s_k} + \frac{1}{\sigma_{p_N}(0)} > \frac{1}{\tilde{\sigma}(\Omega_s, \Omega_p)} \geq \frac{1}{\tilde{\sigma}(A(\lambda^*), \Omega_p)} \geq I_{A^*_{p_i}} + \frac{1}{\sigma_{p_i}(0)} \ , \tag{B.107}$$

which means

$$(I_{A^*_{p_N}} + \frac{1}{\sigma_{p_N}(0)}) - (I_{A^*_{p_i}} + \frac{1}{\sigma_{p_i}(0)}) > I_{s_k} > 0 \ , \tag{B.108}$$

$$\text{i.e.,} \ -\left[(I_{A^*_{p_N}} + \frac{1}{\sigma_{p_N}(0)}) - (I_{A^*_{p_i}} + \frac{1}{\sigma_{p_i}(0)}) - 2 \times I_{s_k}\right] < I_{s_k} \ . \tag{B.109}$$

Combining (B.108) and (B.109) we obtain

$$(I_{A^*_{p_N}} + \frac{1}{\sigma_{p_N}(0)}) - (I_{A^*_{p_i}} + \frac{1}{\sigma_{p_i}(0)}) > I_{s_k} \tag{B.110}$$

$$> (I_{A^*_{p_i}} + I_{s_k} + \frac{1}{\sigma_{p_i}(0)}) - (I_{A^*_{p_N}} - I_{s_k} + \frac{1}{\sigma_{p_N}(0)}) \ , \tag{B.111}$$

$$= (I_{A^{\lambda}_{p_i}} + \frac{1}{\sigma_{p_i}(0)}) - (I_{A^{\lambda}_{p_N}} + \frac{1}{\sigma_{p_N}(0)}) \ , \tag{B.112}$$

$$> (I_{A^*_{p_i}} + \frac{1}{\sigma_{p_i}(0)}) - (I_{A^*_{p_N}} + \frac{1}{\sigma_{p_N}(0)}) \ . \tag{B.113}$$

From (B.110), (B.112) and (B.113) we have

$$\left|(I_{A^*_{p_N}} + \frac{1}{\sigma_{p_N}(0)}) - (I_{A^*_{p_i}} + \frac{1}{\sigma_{p_i}(0)})\right| > \left|(I_{A^{\lambda}_{p_i}} + \frac{1}{\sigma_{p_i}(0)}) - (I_{A^{\lambda}_{p_N}} + \frac{1}{\sigma_{p_N}(0)})\right| \ , \tag{B.114}$$

which contradicts the optimality of λ^* according to Lemma 3.4. Therefore, parameter p_N should not be measured and (B.62) holds.

(ii) By Definition 3.1 and the assumption $\frac{1}{\tilde{\sigma}(\Omega_s,\Omega_p)} < \frac{1}{\sigma_{p_N}(0)}$,

$$\frac{1}{\tilde{\sigma}(\Omega_s, \Omega_p)} = \frac{\sum_{i=1}^{M} I_{s_i} + \sum_{j=1}^{N} \frac{1}{\sigma_{p_j}(0)}}{N} > \frac{\sum_{i=1}^{M} I_{s_i} + \sum_{j=1}^{N-1} \frac{1}{\sigma_{p_j}(0)} + \frac{1}{\tilde{\sigma}(\Omega_s,\Omega_p)}}{N} \ , \tag{B.115}$$

Hence,

$$\frac{1}{\tilde{\sigma}(\Omega_s, \Omega_p)} > \frac{\sum_{i=1}^{M} I_{s_i} + \sum_{j=1}^{N-1} \frac{1}{\sigma_{p_j}(0)}}{N-1} = \frac{1}{\tilde{\sigma}(\Omega_s, \Omega_p \setminus \{p_N\})} \ . \tag{B.116}$$

Therefore, $\tilde{\sigma}(\Omega_s, \Omega_p \setminus \{p_N\}) > \tilde{\sigma}(\Omega_s, \Omega_p)$. □

Proof of Lemma B.2.

(i) By Definition 3.2, $\sigma_{p_1}(0) > TH_{s_1}$ means measuring parameter p_1 by sensor s_1 results negative cost. By Lemma 3.2, if sensor s_1 is not used under λ^*, all the other sensors are not used. Therefore sensor s_1 is certainly used to measure some parameter under λ^*. Assume s_1 measures parameter p_k. Then $s_1 \in A^*_{p_k}$.

Second we show that $A^*_{p_k} \subseteq \{s_1\}$ by contradiction. Assume $\{s_1\} \subset A^*_{p_k}$; then $\exists s_i$, *s.t.*, $A^*_{p_k} \supseteq \{s_1, s_i\}$.

From Lemma 3.3, $\exists p_j \in \Omega_p$, such that

$$I_{A^*_{p_j}} + \frac{1}{\sigma_{p_j}(0)} \leq \frac{1}{\tilde{\sigma}(A(\lambda^*), \Omega_p)} \tag{B.117}$$

Define the allocation strategy λ by

$$A^\lambda_{p_k} := A^*_{p_k} \setminus \{s_i\} \supseteq \{s_1\} \ , \tag{B.118}$$

$$A^\lambda_{p_j} := A^*_{p_j} \cup \{s_i\} \ , \tag{B.119}$$

$$A^\lambda_{p_l} := A^*_{p_l}, \forall l \neq i \text{ or } j \ . \tag{B.120}$$

Then we get the following series of inequalities.

$$\begin{aligned}
I_{A^*_{p_k}} - I_{s_i} + \frac{1}{\sigma_{p_k}(0)} &\geq I_{A^*_{p_k}} - I_{s_i} + \frac{1}{\sigma_{p_1}(0)} && \text{(B.121)}\\
&> I_{s_1} + \frac{1}{\sigma_{p_1}(0)} && \text{(B.122)}\\
&\geq \frac{1}{\tilde{\sigma}(\Omega_s, \Omega_p)} && \text{(B.123)}\\
&\geq \frac{1}{\tilde{\sigma}(A(\lambda^*), \Omega_p)} && \text{(B.124)}\\
&\geq I_{A^*_{p_j}} + \frac{1}{\sigma_{p_j}(0)} \ . && \text{(B.125)}
\end{aligned}$$

The first inequality follows from the fact that $\sigma_{p_1}(0) \geq \sigma_{p_k}(0)$; the second inequality follows from $\{s_1\} \subseteq A^*_{p_k} \setminus \{s_i\}$. The third inequality follows from (B.65); the fourth inequality follows from the fact that $\tilde{\sigma}(\Omega_s, \Omega_p)$ is strictly increasing with respect to Ω_s and $A(\lambda^*) \subseteq \Omega_s$; the last inequality is a result of (B.117).

From (B.125) we obtain

$$(I_{A^*_{p_k}} + \frac{1}{\sigma_{p_k}(0)}) - (I_{A^*_{p_j}} + \frac{1}{\sigma_{p_j}(0)}) > I_{s_i} > 0 \ , \tag{B.126}$$

$$-\left[(I_{A^*_{p_k}} + \frac{1}{\sigma_{p_k}(0)}) - (I_{A^*_{p_j}} + \frac{1}{\sigma_{p_j}(0)}) - 2 \times I_{s_i}\right] < I_{s_i} \ . \tag{B.127}$$

Combining (B.126) and (B.127) we obtain

$$(I_{A^*_{p_k}} + \frac{1}{\sigma_{p_k}(0)}) - (I_{A^*_{p_j}} + \frac{1}{\sigma_{p_j}(0)}) > I_{s_i} \tag{B.128}$$

$$> (I_{A^*_{p_j}} + I_{s_i} + \frac{1}{\sigma_{p_j}(0)}) - (I_{A^*_{p_k}} - I_{s_i} + \frac{1}{\sigma_{p_k}(0)}) \ , \tag{B.129}$$

$$= (I_{A^\lambda_{p_j}} + \frac{1}{\sigma_{p_j}(0)}) - (I_{A^\lambda_{p_k}} + \frac{1}{\sigma_{p_k}(0)}) \ , \tag{B.130}$$

$$> (I_{A^*_{p_j}} + \frac{1}{\sigma_{p_j}(0)}) - (I_{A^*_{p_k}} + \frac{1}{\sigma_{p_k}(0)}) \ . \tag{B.131}$$

The last inequality follows from (B.118) and (B.119).

From (B.128), (B.130) and (B.131), we have

$$\left|(I_{A^*_{p_k}} + \frac{1}{\sigma_{p_k}(0)}) - (I_{A^*_{p_j}} + \frac{1}{\sigma_{p_j}(0)})\right| > \left|(I_{A^\lambda_{p_j}} + \frac{1}{\sigma_{p_j}(0)}) - (I_{A^\lambda_{p_k}} + \frac{1}{\sigma_{p_k}(0)})\right| \ , \tag{B.132}$$

which contradicts the optimality of λ^* according to Lemma 3.4. Hence $A^*_{p_k} \subseteq \{s_1\}$.

Combining $s_1 \in A^*_{p_k}$ and $A^*_{p_k} \subseteq \{s_1\}$, we obtain $A^*_{p_k} = \{s_1\}$.

Therefore, sensor s_1 should be used alone, *i.e.*, $\exists A^*_{p_k} = \{s_1\}$, and

$$J^*(\Omega_s, \Omega_p) = J^*(\Omega_s \setminus \{s_1\}, \Omega_p \setminus \{p_k\}) + \frac{1}{I_{s_1} + \frac{1}{\sigma_{p_k}(0)}} + c \ ,$$

We prove that $p_k \neq p_N$ by contradiction.

Assume $I_{s_1} > \frac{1}{\tilde{\sigma}(\Omega_s, \Omega_p)} - \frac{1}{\sigma_{p_1}(0)}$ and $k = N$, *i.e.*, $A^*_{p_N} = \{s_1\}$.

By Lemma 3.5 and the assumption $I_{s_1} \geq \frac{1}{\tilde{\sigma}(\Omega_s, \Omega_p)} - \frac{1}{\sigma_{p_1}(0)}$, we have

$$I_{A^*_{p_1}} \geq \cdots \geq I_{A^*_{p_N}} = I_{s_1} > \frac{1}{\tilde{\sigma}(\Omega_s, \Omega_p)} - \frac{1}{\sigma_{p_1}(0)} \ . \tag{B.133}$$

Then for each parameter p_i, the final variance satisfies

$$\sigma_{p_i}(A^*_{p_i}) = (I_{A^*_{p_i}} + \frac{1}{\sigma_{p_i}(0)})^{-1} \tag{B.134}$$
$$\leq (I_{A^*_{p_i}} + \frac{1}{\sigma_{p_1}(0)})^{-1} \tag{B.135}$$
$$\leq (I_{A^*_{p_N}} + \frac{1}{\sigma_{p_1}(0)})^{-1} = (I_{s_1} + \frac{1}{\sigma_{p_1}(0)})^{-1} \tag{B.136}$$
$$\leq \tilde{\sigma}(\Omega_s, \Omega_p) \ , \tag{B.137}$$

which contradicts Lemma 3.3.

(ii) By Definition 3.1 and assumption $I_{s_1} > \frac{1}{\tilde{\sigma}(\Omega_s,\Omega_p)} - \frac{1}{\sigma_{p_1}(0)}$,

$$\frac{1}{\tilde{\sigma}(\Omega_s, \Omega_p)} = \frac{\sum_{i=1}^{M} I_{s_i} + \sum_{j=1}^{N} \frac{1}{\sigma_{p_j}(0)}}{N} \tag{B.138}$$
$$= \frac{\sum_{i=2}^{M} I_{s_i} + \sum_{j=2}^{N} \frac{1}{\sigma_{p_j}(0)} + I_{s_1} + \frac{1}{\sigma_{p_1}(0)}}{N} \tag{B.139}$$
$$> \frac{\sum_{i=2}^{M} I_{s_i} + \sum_{j=2}^{N} \frac{1}{\sigma_{p_j}(0)} + \frac{1}{\tilde{\sigma}(\Omega_s,\Omega_p)}}{N} \tag{B.140}$$

Recalling that $\forall k, \sigma_{p_1}(0) \geq \sigma_{p_k(0)}$, we have

$$\frac{1}{\tilde{\sigma}(\Omega_s, \Omega_p)} > \frac{\sum_{i=2}^{M} I_{s_i} + \sum_{j=2}^{N} \frac{1}{\sigma_{p_j}(0)}}{N-1} \geq \frac{\sum_{i=2}^{M} I_{s_i} + \sum_{j=1,j\neq k}^{N} \frac{1}{\sigma_{p_j}(0)}}{N-1} \ . \tag{B.141}$$

By Definition 3.1 and (B.141), we have

$$\frac{1}{\tilde{\sigma}(\Omega_s, \Omega_p)} > \frac{1}{\tilde{\sigma}(\Omega_s \setminus \{s_1\}, \Omega_p \setminus \{p_1\})} \geq \frac{1}{\tilde{\sigma}(\Omega_s \setminus \{s_1\}, \Omega_p \setminus \{p_k\})} \ . \tag{B.142}$$

Therefore, $\tilde{\sigma}(\Omega_s \setminus \{s_1\}, \Omega_p \setminus \{p_k\}) \geq \tilde{\sigma}(\Omega_s \setminus \{s_1\}, \Omega_p \setminus \{p_1\}) > \tilde{\sigma}(\Omega_s, \Omega_p)$. □

Proof of Lemma B.3.

(i) If sensors $s_1, \cdots, s_j$ are each used alone to measure parameter $p_{k_1}, \cdots, p_{k_j}$ respectively, then we have $I_{A^*_{p_{k_i}}} = I_{s_i}$, $i = 1, \cdots, j$. Since $I_{s_1} \geq \cdots \geq I_{s_j}$, it follows that

$$I_{A^*_{p_{k_1}}} \geq \cdots \geq I_{A^*_{p_{k_j}}} \ . \tag{B.143}$$

By Lemma 3.5 and (B.143), it follows that

$$1 \leq k_1 < \cdots < k_j \leq N \; . \tag{B.144}$$

(ii) We have

$$\frac{\sum_{i=j+1}^{M} I_{s_i} + \sum_{i=j+1}^{N} \frac{1}{\sigma_{p_i}(0)}}{N-j} \geq \frac{\sum_{i=j+1}^{M} I_{s_i} + \sum_{i=1, i\neq k_1,\cdots,k_j}^{N} \frac{1}{\sigma_{p_i}(0)}}{N-j} \; , \tag{B.145}$$

where the inequality comes from the fact that parameter $p_1, \cdots, p_j$ have the largest j variances. By Definition 3.1 and (B.145), it follows that

$$\frac{1}{\tilde{\sigma}(\Omega_s \setminus \{s_1, \cdots, s_j\}, \Omega_p \setminus \{p_1, \cdots, p_j\})} \geq \frac{1}{\tilde{\sigma}(\Omega_s \setminus \{s_1, \cdots, s_j\}, \Omega_p \setminus \{p_{k_1}, \cdots, p_{k_j}\})} \; . \tag{B.146}$$

Therefore

$$\tilde{\sigma}(\Omega_s \setminus \{s_1, \cdots, s_j\}, \Omega_p \setminus \{p_{k_1}, \cdots, p_{k_j}\}) \leq \tilde{\sigma}(\Omega_s \setminus \{s_1, \cdots, s_j\}, \Omega_p \setminus \{p_1, \cdots, p_j\}) \; . \tag{B.147}$$

□

Proof of Theorem 3.2.

Suppose $\lambda^* = \{A_1^*, A_2^*, \cdots, A_P^*\}$ is an optimal allocation policy for Problem (P1) and $A(\lambda^*) = \cup_{i=1}^{P} A_i^* = \{1, 2, \cdots, \tau^*\}$. From Lemma 3.7, we know that the sensors with the largest t_s indices, *i.e.,* $\{1, \cdots, t_s\}$, should be used alone and the parameters with the smallest t_p initial variances, *i.e.,* $\{P - t_p + 1, \cdots, P\}$, should not be measured.

When $t_p = 0$ and $t_s = 0$, it can be easily verified that the result of this theorem is the same as that of Theorem 3.1. Now we derive a lower bound for Problem (P1) when $t_p > 0$ or $t_s > 0$. There are two methods to derive a lower bound: (i) using only t_p; and (ii) using both t_p and t_s.

- **Using only t_p:** By Lemma 3.7, we know that

$$J^*(\mathcal{S},\mathcal{P}) = J^*(A(\lambda^*),\mathcal{P}) \tag{B.148}$$

$$= J^*(A(\lambda^*),\mathcal{P}\setminus\{P-t_p+1,\cdots,P\}) + \sum_{i=P-t_p+1}^{P}\sigma_i(0) + \tau^*\cdot c \ . \tag{B.149}$$

By Theorem 3.1 and Equation (B.148), we have

$$J^*(\mathcal{S},\mathcal{P}) \geq \frac{(P-t_p)^2}{\sum_{i=1}^{\tau^*} I_i + \sum_{i=1}^{P-t_p}\frac{1}{\sigma_i(0)}} + \sum_{i=P-t_p+1}^{P}\sigma_i(0) + \tau^*\cdot c = L_1(\tau^*) \ . \tag{B.150}$$

Since $t_1 \in \{1,2,\cdots,S\}$ is a minimizer of the convex function $L_1(t)$, which is defined by the right hand side of (B.150),

$$J^*(\mathcal{S},\mathcal{P}) \geq L_1(\tau^*) \geq L_1(t_1) \ . \tag{B.151}$$

- **Using both t_p and t_s:**

Denote $p_{k_1}, p_{k_2}, \cdots, p_{k_{t_s}}$ as the parameters measured by sensors $s_1, s_2, \cdots, s_{t_s}$, respectively. From Lemma B.3, we know that

$$1 \leq k_1 < \cdots < k_{t_s} < P - t_p + 1 \ , \tag{B.152}$$

that is

$$i < k_i < P - t_p + 1 - (t_s - i) \ , \ i = 1,\cdots,t_s \ . \tag{B.153}$$

From (B.153) we get

$$\frac{1}{\sigma_i(0)} \leq \frac{1}{\sigma_{k_i}(0)} \leq \frac{1}{\sigma_{P-t_p+1-(t_s-i)}(0)} \ , i = 1,\cdots,t_s \ . \tag{B.154}$$

The inequalities in (B.154) imply that

$$\sum_{i=1}^{t_s}\frac{1}{I_i + \frac{1}{\sigma_{k_i}(0)}} \geq \sum_{i=1}^{t_s}\frac{1}{I_i + \frac{1}{\sigma_{P-t_p+1-(t_s-i)}(0)}} \ , \tag{B.155}$$

$$\text{and} \quad \sum_{i=1}^{t_s}\frac{1}{\sigma_i(0)} \leq \sum_{i=1}^{t_s}\frac{1}{\sigma_{k_i}(0)} \ . \tag{B.156}$$

By the definition of the harmonic mean function and (B.156), we obtain

$$\begin{aligned}
&\tilde{\sigma}(A(\lambda^*) \setminus \{1, \cdots, t_s\}, \mathcal{P} \setminus \{1, \cdots, t_s, P - t_p + 1, \cdots, P\}) \\
&\quad \leq \tilde{\sigma}(A(\lambda^*) \setminus \{1, \cdots, t_s\}, \mathcal{P} \setminus \{k_1, \cdots, k_{t_s}, P - t_p + 1, \cdots, P\}) \ . \qquad \text{(B.157)}
\end{aligned}$$

From Lemma 3.7, we also have

$$J^*(\mathcal{S}, \mathcal{P}) = J^*(A(\lambda^*), \mathcal{P}) \qquad \text{(B.158)}$$

$$\begin{aligned}
&= J^*(A(\lambda^*) \setminus \{1, \cdots, t_s\}, \mathcal{P} \setminus \{k_1, \cdots, k_{t_s}, P - t_p + 1, \cdots, P\}) \\
&\quad + \sum_{i=1}^{t_s} \frac{1}{I_i + \frac{1}{\sigma_{k_i}(0)}} + \sum_{i=P-t_p+1}^{P} \sigma_i(0) + \tau^* \cdot c \ . \qquad \text{(B.159)}
\end{aligned}$$

Using Theorem 3.1 along with (B.159), we obtain

$$\begin{aligned}
J^*(\mathcal{S}, \mathcal{P}) &\geq (P - t_p - t_s) \cdot \tilde{\sigma}(A(\lambda^*) \setminus \{1, \cdots, t_s\}, \mathcal{P} \setminus \{k_1, \cdots, k_{t_s}, P - t_p + 1, \cdots, P\}) \\
&\quad + \sum_{i=1}^{t_s} \frac{1}{I_i + \frac{1}{\sigma_{k_i}(0)}} + \sum_{i=P-t_p+1}^{P} \sigma_i(0) + \tau^* \cdot c \qquad \text{(B.160)} \\
&\geq (P - t_p - t_s) \cdot \tilde{\sigma}(A(\lambda^*) \setminus \{1, \cdots, t_s\}, \mathcal{P} \setminus \{1, \cdots, t_s, P - t_p + 1, \cdots, P\}) \\
&\quad + \sum_{i=1}^{t_s} \frac{1}{I_i + \frac{1}{\sigma_{P-t_p+1-(t_s-i)}(0)}} + \sum_{i=P-t_p+1}^{P} \sigma_i(0) + \tau^* \cdot c \qquad \text{(B.161)} \\
&= \frac{(P - t_p - t_s)^2}{\sum_{i=t_s+1}^{\tau^*} I_i + \sum_{i=t_s+1}^{P-t_p} \frac{1}{\sigma_i(0)}} + \sum_{i=1}^{t_s} \frac{1}{I_i + \frac{1}{\sigma_{P-t_p+1-(t_s-i)}(0)}} \\
&\quad + \sum_{i=P-t_p+1}^{P} \sigma_i(0) + \tau^* \cdot c \qquad \text{(B.162)} \\
&= L_2(\tau^*) \ , \qquad \text{(B.163)}
\end{aligned}$$

where (B.161) follows from (B.155) and (B.157).

Since $t_2 \in \{1, 2, \cdots, S\}$ is a minimizer of the convex function $L_2(t)$, which is defined by (B.163), we have

$$J^*(\mathcal{S}, \mathcal{P}) \geq L_2(\tau^*) \geq L_2(t_2) \ . \qquad \text{(B.164)}$$

Combining the two bounds, given by (B.151) and (B.164), we obtain

$$J^* \geq \max\{L_1(t_1), L_2(t_2)\} \ . \tag{B.165}$$

$\square$

Proof of Lemma 3.8.

According to the greedy algorithm, at any step t, the variance $\sigma_k(t)$ of any parameter p_k satisfies

$$\sigma_k(t) \leq \sigma_k(0) \tag{B.166}$$

Combining (B.166) with the assumptions

$$\sigma_1(0) < TH_i, \text{ for some sensor } i, \tag{B.167}$$

and

$$TH_i < TH_j, \quad \forall j > i \ , \tag{B.168}$$

we obtain

$$\sigma_k(t) \leq \sigma_k(0) \leq \sigma_1(0) < TH_i < TH_j, \forall k = 1, \cdots, P \text{ and } i < j < S. \tag{B.169}$$

Therefore no parameter should be measured by sensor j, $j \geq i$, under the greedy allocation policy. Consequently, if $\sigma_1(0) < TH_1$, the greedy algorithm will stop at $t = 0$. $\square$

Proof of Corollary 3.1.

The assumption $TH_m < \sigma_1(0) \leq TH_{m+1}$ and Lemma 3.6 imply that under an optimal policy at most m sensors could be activated. Let $\lambda^g = \{A_1^g, \cdots, A_P^g\}$ denote the greedy policy and $\lambda^* = \{A_1^*, \cdots, A_P^*\}$ denote an optimal policy. We prove that $\lambda^g = \lambda^*$ when $m = 0, 1, 2$.

- **Case 1:** $m = 0$, *i.e.,* $\sigma_1 \leq TH_1$.

  In this case, the result follows from Lemmas 3.6 and 3.8.

- **Case 2:** $m = 1$, *i.e.*, $TH_1 \leq \sigma_1 < TH_2$.

  In this case $\sigma_1 \leq TH_2$ and Lemma 3.6 and 3.8 imply that under an optimal policy and the greedy policy, no parameter will be measured by any sensor j, $j \geq 2$. Furthermore, $TH_1 < \sigma_1(0)$ and Lemma 3.5 imply that, under an optimal policy, parameter 1 will be measured by sensor 1. Under the greedy policy, parameter 1 is always measured by sensor 1. Therefore, $\lambda^g = \lambda^*$.

- **Case 3:** $m = 2$, *i.e.*, $TH_2 < \sigma_1 \leq TH_3$.

  In this case, $\sigma_1 \leq TH_3$ along with Lemma 3.6 and 3.8 imply that no parameter will be measured by any of the sensors j, $j > 2$ under both an optimal policy and the greedy policy. There are three feasible greedy policies,

$$\lambda_1 = \{\{1\}, \emptyset, \cdots, \emptyset\} \ , \tag{B.170}$$

$$\lambda_2 = \{\{1, 2\}, \emptyset, \cdots, \emptyset\} \ , \tag{B.171}$$

$$\lambda_3 = \{\{1\}, \{2\}, \emptyset, \cdots, \emptyset\} \ . \tag{B.172}$$

  By Lemma 3.2 and Lemma 3.5, the above policies are also the candidate optimal policies, We prove that $\lambda^g = \lambda^*$.

  1. $\lambda^g = \lambda_1$

     In this case, according to the greedy algorithm, we know that

$$\max\{\sigma_1(\{1\}), \sigma_2(0)\} < TH_2 \ . \tag{B.173}$$

     (B.173) along with the definition of a sensor's threshold implies that under an optimal policy, sensor 2 should not be activated. Therefore $\lambda^g = \lambda^*$.

  2. $\lambda^g = \lambda_2$

     In this case, according to the greedy algorithm, we know that

$$\sigma_1(\{1\}) > TH_2 \ , \tag{B.174}$$

$$\sigma_1(\{1\}) > \sigma_2(0) \ . \tag{B.175}$$

Inequality (B.174) along with the definition of a threshold imply that $J^{\lambda_1} > J^{\lambda_2}$, which means $\lambda^* \neq \lambda_1$. Inequality (B.175) implies that $J^{\lambda_3} > J^{\lambda_2}$, which means $\lambda^* \neq \lambda_3$. Therefore $\lambda^* = \lambda_2 = \lambda^g$.

3. $\lambda^g = \lambda_3$

   In this case, according to the greedy algorithm, we know that

$$\sigma_2(0) > TH_2 \ , \tag{B.176}$$

$$\sigma_2(0) > \sigma_1(\{1\}) \ . \tag{B.177}$$

   The inequality (B.176) along with the definition of a sensor's threshold implies that $J^{\lambda_1} > J^{\lambda_2}$, which means $\lambda^* \neq \lambda_1$. The inequality (B.177) implies that $J^{\lambda_3} > J^{\lambda_2}$, which means $\lambda^* \neq \lambda_3$. Therefore $\lambda^* = \lambda_2 = \lambda^g$.

This completes the proof of Corollary 3.1. $\square$

Proof of Lemma 3.9.

The proof of Lemma 3.9 is the same as that of Lemma 3.7. In Step 2 of any cycle k is used alone and measures parameter k. $\square$

APPENDIX C

PROOFS FOR CHAPTER 4

Proof of Property 4.1.

From Equations (4.9),(4.10), we have

$$L_1(x) = \frac{x \cdot K}{x + K} \ , \tag{C.1}$$

$$L_2(x) = A^2 x + Q \ . \tag{C.2}$$

It is straightforward to show that $L_2(x)$ is an increasing function.

The first order derivative of $L_1(x)$ with respect to x is

$$\frac{dL_1(x)}{dx} = \frac{K^2}{(x + K)^2} > 0 \ , \tag{C.3}$$

which means that $L_1(x)$ is an increasing function.

The second order derivatives of $L_1(x)$ with respect to x is

$$\frac{d^2 L_1(x)}{dx^2} = -\frac{2K^2}{(x + K)^3} < 0 \ , \text{ when } x > 0 \ , \tag{C.4}$$

which means that $L_1(x)$ is a concave function when $x > 0$.

We write $L_1 L_2(x)$ and $L_2 L_1(x)$ as follows.

$$L_1 L_2(x) = \frac{(A^2 x + Q) \cdot K}{A^2 x + Q + K} \ , \tag{C.5}$$

$$L_2 L_1(x) = A^2 \frac{x \cdot K}{x + K} + Q \ . \tag{C.6}$$

The first order derivatives of $L_1L_2(x)$ and $L_2L_1(x)$ with respect to x are:

$$\frac{dL_1L_2(x)}{dx} = \frac{dL_1L_2(x)}{dL_2(x)} \cdot \frac{dL_2(x)}{dx} = \frac{K^2}{(L_2(x)+K)^2} \cdot A^2 > 0 \ , \ \text{when } x > 0 \ , \tag{C.7}$$

$$\frac{dL_2L_1(x)}{dx} = \frac{dL_2L_1(x)}{dL_1(x)} \cdot \frac{dL_1(x)}{dx} = A^2 \cdot \frac{K^2}{(x+K)^2} > 0 \ , \ \text{when } x > 0 \ , \tag{C.8}$$

which means that $L_1L_2(x)$ and $L_2L_1(x)$ are increasing functions when $x > 0$.

The second order derivatives of $L_1L_2(x)$ and $L_2L_1(x)$ with respect to x are

$$\frac{d^2L_1L_2(x)}{dx^2} = -\frac{2A^4K^2}{(L_2(x)+K)^3} < 0 \ , \ \text{when } x > 0 \ , \tag{C.9}$$

$$\frac{d^2L_2L_1(x)}{dx^2} = -\frac{2A^2 \cdot K^2}{(x+K)^3} < 0 \ , \ \text{when } x > 0 \ . \tag{C.10}$$

Thus $L_1L_2(x)$ and $L_2L_1(x)$ are concave functions when $x > 0$. □

Proof of Property 4.2.

From (4.9), we have

$$x - L_1(x) = x - \frac{x \cdot K}{x+K} = \frac{x^2}{x+K} > 0 \ , \ \text{when } x > 0 \ . \tag{C.11}$$

From (C.3) the first order derivative of $x - L_1(x)$ with respect x is

$$\frac{d(x - L_1(x))}{dx} = 1 - \frac{K^2}{(x+K)^2} = \frac{x^2 + 2xK}{(x+K)^2} > 0 \ , \ \text{when } x > 0 \ . \tag{C.12}$$

Thus, $x - L_1(x)$ is an increasing function of x when $x > 0$. □

Proof of Property 4.3.

From (4.10), $L_2(x)$ is an affine function of x while $L_2(x)$ crosses with the y-axis above x-axis. When $|A|^2 > 1$, for any $x > 0$, $L_2(x) > x$. When $|A|^2 = 1$, x and $L_2(x)$ are parallel. Then for all x, $L_2(x) > x$. When $|A|^2 < 1$, define σ^* by $\sigma^* = L_2(\sigma^*)$. Then, if $x \geq \sigma^*$, $L_2(x) \leq x$, otherwise $L_2(x) > x$. □

Proof of Property 4.4.

From (C.7) and (C.8), we have

$$\frac{dL_1L_2(x)}{dx} = \frac{K^2}{(L_2(x)+K)^2} \cdot A^2 \ , \tag{C.13}$$

$$\frac{dL_2L_1(x)}{dx} = A^2 \cdot \frac{K^2}{(x+K)^2} \ . \tag{C.14}$$

Because $L_2(x) < x$, we have

$$\frac{dL_1L_2(x)}{dx} > \frac{dL_2L_1(x)}{dx} \ , \tag{C.15}$$

which implies that $L_1L_2(x_2) - L_1L_2(x_1) > L_2L_1(x_2) - L_2L_1(x_1)$ when $x_1 < x_2$. $\square$

Proof of Property 4.5.

Define $f(x) := L_1L_2(x) - L_2L_1(x)$. Then from (C.5) and (C.6)

$$f(x) = \frac{(A^2x+Q)\cdot K}{A^2x+Q+K} - (A^2\frac{x\cdot K}{x+K} + Q) \ , \tag{C.16}$$

$$= \frac{(A^2K - A^4K - A^2Q)x^2 - (2A^2KQ + Q^2)x - Q^2K}{(A^2x+Q+K)(x+K)} \ . \tag{C.17}$$

When $x > 0$, the denominator of (C.17) is always positive. Therefore we only need to analyze the numerator.

When $A^2K - A^4K - A^2Q \leq 0$, $K \leq A^2K + Q$, which is equivalent to $K \leq \sigma^*$; then, $(A^2K - A^4K - A^2Q)x^2 < 0$. Since $-(2A^2KQ + Q^2)x < 0$ and $-Q^2K < 0$ if $x > 0$, if $K \leq \sigma^*$, $f(x) < 0$ when $x > 0$.

When $A^2K - A^4K - A^2Q > 0$, $K > \sigma^*$. Since $f(0) = -Q^2K < 0$, there exists only one positive solution for $f(x) = 0$. Let $\hat{\sigma}$ be such that $f(\hat{\sigma}) = 0$, *i.e.*, $L_1L_2(\hat{\sigma}) = L_2L_1(\hat{\sigma})$. Then when $x < \hat{\sigma}$, $f(x) < 0$, i.e. $L_1L_2(x) < L_2L_1(x)$; when $x > \hat{\sigma}$, $f(x) > 0$, i.e. $L_1L_2(x) > L_2L_1(x)$. $\square$

Proof of Lemma 4.1.

By Property 4.5, we know that when $K \leq \sigma^*$, for any positive x, $L_1L_2(x) \leq L_2L_1(x)$. By Properties 4.2 and 4.3, we know that when $\sigma_t > \sigma^*$, $\sigma_{t+1} < \sigma_t$ no matter what strategy we use at time $t+1$. Then for σ_0 large enough, we have $\sigma_t^g > \sigma^*$ for all t, $t = 1, \cdots, T$ and for any sensor activation strategy g. Therefore $S_0(m, T)$ is nonempty. $\square$

Proof of Theorem 4.1.

Before we present the proof, we introduce the following definition and lemmas.

Definition C.1. We define

$$D_{t-1}(\sigma, n) := L_2(\sigma) + V_t(L_2(\sigma), n) - [C + L_1L_2(\sigma) + V_t(L_1L_2(\sigma), n-1)] \ . \tag{C.18}$$

$D_{t-1}(\sigma, n)$ is the objective cost's difference between the following two sensor activation strategies: The decisions up to time $t-2$ are the same for these two strategies and there are n sensors available at time instant $t-1$; the first strategy takes a measurement at time $t-1$ and the decisions from time t to T are optimal given the decisions made up to time $t-1$; the second one does not take a measurement at time $t-1$ and the decisions from time t to T are optimal given the decisions made up to time $t-1$.

Lemma C.1. *If $V_t(L_2(\sigma), n) - V_t(L_1L_2(\sigma), n-1)$ is a non-decreasing function of σ for all $n = 1, \cdots, m$ and $t = 1, \cdots, T$, $D_t(\sigma, n)$ is a non-decreasing function of σ for all $n = 1, \cdots, m$ and $t = 1, \cdots, T$.*

Lemma C.2. *If $K \le \sigma^*$, $|A|^2 < 1$ and $\sigma_0 \in S_0(m, T)$, the following three results are true.*

R1: *$V_t(x, n)$ is a non-decreasing function of x, for all $n = 1, 2, \cdots, m$ and $t = 1, 2, \cdots, T$,* i.e., *if $0 < x_1 < x_2$,*

$$V_t(x_1, n) \le V_t(x_2, n) \ . \tag{C.19}$$

R2: *For all $n = 1, 2, \cdots, m$ and $t = 1, 2, \cdots, T$, if $0 < x_1 \le x_2$ and $0 < \epsilon$,*

$$V_t(x_1 + \epsilon, n) - V_t(x_1, n) \ge V_t(x_2 + \epsilon, n) - V_t(x_2, n) \ , \tag{C.20}$$

R3: *$V_t(L_2(x), n) - V_t(L_1L_2(x), n-1)$ is a non-decreasing function of x for all $n = 1, 2, \cdots, m$ and $t = 1, 2, \cdots, T$,* i.e., *if $0 < x$ and $0 < \epsilon$,*

$$V_t(L_2(x + \epsilon), n) - V_t(L_2(x), n) \ge V_t(L_1L_2(x + \epsilon), n-1) - V_t(L_1L_2(x), n-1) \ . \tag{C.21}$$

We prove these lemmas right after the proof of Theorem 4.1. The proof of Theorem 4.1 proceeds as follows.

If $K \le \sigma^*$, $|A|^2 < 1$ and $\sigma_0 \in S_0(m, T)$, Lemma C.2 guarantees that the hypothesis of Lemma C.1 is satisfied. Then Lemma C.1 shows that $D_t(\sigma, n)$ is a non-decreasing function of σ for all $n = 1, 2, \cdots, m$ and $t = 1, 2, \cdots, T$. This property of $D_t(\sigma, n)$ is sufficient to guarantee the threshold property of an optimal sensor activation strategy (see [53]). $\square$

Proof of Lemma C.1.

First, we rewrite D_{t-1} as follows.

$$D_{t-1}(\sigma, n) = L_2(\sigma) + V_t(L_2(\sigma), n) - [C + L_1L_2(\sigma) + V_t(L_1L_2(\sigma), n-1)] \tag{C.22}$$

$$= \{L_2(\sigma) - [C + L_1L_2(\sigma)]\} + \{V_t(L_2(\sigma), n) - V_t(L_1L_2(\sigma), n-1)\} \; . \tag{C.23}$$

By Property 4.2, $x - L_1(x)$ is a non-decreasing function of x. By Property 4.1, $L_2(x)$ is a non-decreasing function of x. Therefore, $L_2(\sigma) - [C + L_1L_2(\sigma)]$ is a non-decreasing function of σ. If $V_t(L_2(\sigma), n) - V_t(L_1L_2(\sigma), n-1)$ is a non-decreasing function of σ, $D_{t-1}(\sigma, n)$ should be a non-decreasing function of σ for all $n = 1, 2, \cdots, m$ and $t = 1, 2, \cdots, T$. $\square$

Proof of Lemma C.2.

We use induction to prove the coupled results R1-R3.

In order to prove R1, we need to compare $V_t(x_1, n)$, $V_t(x_2, n)$. For that matter we consider all combinations of decision choices (taking and not taking a measurement at time instant t) under the two information states (x_1, n) and (x_2, n)).

In order to prove R2, we need to compare $V_t(x_1+\epsilon, n) - V_t(x_1, n)$ and $V_t(x_2+\epsilon, n) - V_t(x_2, n)$. For that matter we consider all 16 combinations of decision choices (taking and not taking a measurement at time instant t) under the four information states (x_1, n), $(x_1 + \epsilon, n)$, $(x_2, n))$ and $(x_2 + \epsilon, n)$.

In order to prove R3, we need to compare $V_t(L_2(x + \epsilon), n) - V_t(L_2(x), n)$ and $V_t(L_1L_2(x+\epsilon), n-1) - V_t(L_1L_2(x), n-1)$. To do this we consider all combinations of decision choices (taking and not taking a measurement at time instant t) under the four information states $(L_2(x), n)$, $(L_2(x+\epsilon), n)$, $(L_1L_2(x), n-1)$ and $(L_1L_2(x+\epsilon), n-1)$.

Basis of Induction: We need to prove

R1: For all $n = 1, 2, \cdots, m$, if $0 < x_1 < x_2$,

$$V_T(x_1, n) \leq V_T(x_2, n) \ . \tag{C.24}$$

R2: For all $n = 1, 2, \cdots, m$, if $0 < x_1 \leq x_2$ and $0 < \epsilon$,

$$V_T(x_1 + \epsilon, n) - V_T(x_1, n) \geq V_T(x_2 + \epsilon, n) - V_T(x_2, n). \tag{C.25}$$

R3: For all $n = 1, 2, \cdots, m$, if $0 < x$ and $0 < \epsilon$,

$$V_T(L_2(x + \epsilon), n) - V_T(L_2(x), n) \geq V_T(L_2 L_1(x + \epsilon), n - 1) - V_T(L_2 L_1(x), n - 1) \ . \tag{C.26}$$

Proof of R1: Note $x_1 \leq x_2$.

Case 1: Assume it is optimal to take a measurement under both of the information states (x_1, n) and (x_2, n) at time instant T. Then

$$V_T(x_1, n) = L_2(x_1) \ , \tag{C.27}$$

$$V_T(x_2, n) = L_2(x_2) \ . \tag{C.28}$$

Since $x_1 < x_2$, by Property 4.1,

$$L_2(x_1) < L_2(x_2) \ . \tag{C.29}$$

Therefore $V_T(x_1, n) \leq V_T(x_2, n)$.

Case 2: Assume it is optimal to take a measurement under (x_2, n) and not to take a measurement under (x_1, n) at time instant T. Then

$$V_T(x_1, n) = L_2(x_1) \ , \tag{C.30}$$

$$V_T(x_2, n) = C + L_1 L_2(x_2) \ , \tag{C.31}$$

and by the definition of $V_T(x, n)$,

$$L_2(x_1) \leq C + L_1 L_2(x_1) \ . \tag{C.32}$$

When $x_1 < x_2$, by Property 4.1 we have

$$L_1 L_2(x_1) < L_1 L_2(x_2) \ . \tag{C.33}$$

Combining (C.32) and (C.33), we get

$$L_2(x_1) \le C + L_1 L_2(x_2) \ , \tag{C.34}$$

that is $V_T(x_1, n) \le V_T(x_2, n)$.

Case 3: Assume it is optimal to take a measurement under (x_1, n) and not to take a measurement under (x_2, n) at time instant T. Then

$$V_T(x_1, n) = C + L_1 L_2(x_1) \ , \tag{C.35}$$

$$V_T(x_2, n) = L_2(x_2) \ . \tag{C.36}$$

We will show that such a combination of decisions is not part of an optimal strategy.

By the definition of $V_T(x, n)$ and (C.35), (C.36),

$$C + L_1 L_2(x_1) \le L_2(x_1) \ , \tag{C.37}$$

$$L_2(x_2) \le C + L_1 L_2(x_2) \ . \tag{C.38}$$

Since $x_1 < x_2$, by Property 4.1

$$L_2(x_1) < L_2(x_2) \ . \tag{C.39}$$

Combining (C.37)–(C.39), we get

$$C + L_1 L_2(x_1) \le L_2(x_1) < L_2(x_2) \le C + L_1 L_2(x_2) \ , \tag{C.40}$$

which implies that

$$L_1 L_2(x_2) - L_1 L_2(x_1) \ge L_2(x_2) - L_2(x_1) \ . \tag{C.41}$$

The last inequality contradicts Property 4.2. Thus, the combination of decisions in Case 3 is not part of an optimal strategy.

Case 4: Assume it is optimal to take a measurement under both information states (x_1, n) and (x_2, n) at time instant T. Then

$$V_T(x_1, n) = C + L_1L_2(x_1) \ , \tag{C.42}$$

$$V_T(x_2, n) = C + L_1L_2(x_2) \ . \tag{C.43}$$

Since $x_1 < x_2$, by Property 4.1, we have

$$L_1L_2(x_1) < L_1L_2(x_2) \ . \tag{C.44}$$

Therefore $V_T(x_1, n) \leq V_T(x_2, n)$.

Combining the four cases above, we conclude that for all $n = 1, \cdots, m$, if $0 < x_1 < x_2$,

$$V_T(x_1, n) \leq V_T(x_2, n) \ . \tag{C.45}$$

Proof of R2: There are 16 possible combinations of decisions made under the four information states (x_1, n), $(x_1 + \epsilon, n)$, (x_2, n) and $(x_2 + \epsilon, n)$ at time T. We present the analysis of one combination of decisions that could be part of an optimal strategy and one combination of decisions that are not part of an optimal strategy. All other combinations can be analyzed in a manner similar to the two cases presented below.

Case 1: Assume it is optimal to take a measurement under $(x_2 + \epsilon, n)$ at time instant T, and to not take a measurement under (x_1, n), $(x_1 + \epsilon, n)$ and (x_2, n) at time instant T. Then we have

$$V_T(x_1, n) = L_2(x_1) \ , \tag{C.46}$$

$$V_T(x_1 + \epsilon, n) = L_2(x_1 + \epsilon) \ , \tag{C.47}$$

$$V_T(x_2, n) = L_2(x_2) \ , \tag{C.48}$$

$$V_T(x_2 + \epsilon, n) = C + L_1L_2(x_2 + \epsilon) \ , \tag{C.49}$$

And

$$V_T(x_1+\epsilon,n) - V_T(x_1,n) = A^2\epsilon \ , \tag{C.50}$$

$$V_T(x_2+\epsilon,n) - V_T(x_2,n) = C + L_1L_2(x_2+\epsilon) - L_2(x_2) \ . \tag{C.51}$$

By the definition $V_T(x_2+\epsilon,n)$ and (C.49), we obtain

$$C + L_1L_2(x_2+\epsilon) \leq L_2(x_2+\epsilon) \ . \tag{C.52}$$

Combining (C.51) and (C.52), we get

$$C + L_1L_2(x_2+\epsilon) - L_2(x_2) < L_2(x_2+\epsilon) - L_2(x_2) = A^2\epsilon \ . \tag{C.53}$$

From (C.50), (C.51) and (C.53), we conclude that

$$V_T(x_1+\epsilon,n) - V_T(x_1,n) \geq V_T(x_2+\epsilon,n) - V_T(x_2,n) \ . \tag{C.54}$$

Case 2: Assume it is optimal to take a measurement under (x_2, n) at time instant T and not to take a measurement under (x_1, n), $(x_1+\epsilon, n)$ and $(x_2+\epsilon, n)$ at time instant T. Then

$$V_T(x_1,n) = L_2(x_1) \ , \tag{C.55}$$

$$V_T(x_1+\epsilon,n) = L_2(x_1+\epsilon) \ , \tag{C.56}$$

$$V_T(x_2,n) = C + L_1L_2(x_2) \ , \tag{C.57}$$

$$V_T(x_2+\epsilon,n) = L_2(x_2+\epsilon) \ . \tag{C.58}$$

We will show that this combination of decisions can not be part of an optimal sensor activation strategy.

Since $V_T(x_2, n)$ is a non-decreasing function of x_2 by R1, we have

$$V_T(x_2,n) \leq V_T(x_2+\epsilon,n) \ , \tag{C.59}$$

which is equivalent to

$$C + L_1L_2(x_2) \leq L_2(x_2+\epsilon) \ . \tag{C.60}$$

By the definition of $V_T(x, n)$ and (C.57), (C.58), it follows that

$$C + L_1L_2(x_2) \leq L_2(x_2) \ , \tag{C.61}$$

$$L_2(x_2 + \epsilon) \leq C + L_1L_2(x_2 + \epsilon) \ . \tag{C.62}$$

According to Property 4.3 and inequalities (C.61), (C.62), we have

$$C + L_1L_2(x_2) \leq L_2(x_2) < L_2(x_2 + \epsilon) \leq C + L_1L_2(x_2 + \epsilon) \ . \tag{C.63}$$

Inequality (C.63) implies that

$$L_1L_2(x_2 + \epsilon) - L_1L_2(x_2) \geq L_2(x_2 + \epsilon) - L_2(x_2) \ , \tag{C.64}$$

which contradicts Property 4.2. Consequently, the combination of decision considered in this case can not be part of an optimal strategy.

Proof of R3: For the situation where $n \geq 2$, similarly to the proof of R2, we will investigate 2 cases among 16 possible cases.

Case 1 : Assume it is optimal to take a measurement under information state $(L_1L_2(x+\epsilon), n-1)$ at time instant T and not to take a measurement under information states $(L_2(x), n)$, $(L_2(x+\epsilon), n)$ and $(L_1L_2(x), n-1)$ at time instant T. Then

$$V_T(L_2(x), n) = L_2L_2(x) \ , \tag{C.65}$$

$$V_T(L_2(x + \epsilon), n) = L_2L_2(x + \epsilon) \ , \tag{C.66}$$

$$V_T(L_1L_2(x), n - 1) = L_2L_1L_2(x) \ , \tag{C.67}$$

$$V_T(L_1L_2(x + \epsilon), n - 1) = C + L_1L_2L_1L_2(x + \epsilon) \ , \tag{C.68}$$

and

$$\begin{aligned} V_T(L_2(x + \epsilon), n) - V_T(L_2(x), n) &= L_2L_2(x + \epsilon) - L_2L_2(x) \ , \\ &= A^2(L_2(x + \epsilon) - L_2(x)) \ , \end{aligned} \tag{C.69}$$

$$V_T(L_1L_2(x + \epsilon), n - 1) - V_T(L_1L_2(x), n - 1) = C + L_1L_2L_1L_2(x + \epsilon) - L_2L_1L_2(x) \ . \tag{C.70}$$

By the definition of $V_T(L_1L_2(x+\epsilon), n-1)$ and (C.68), it follows that

$$C + L_1L_2L_1L_2(x+\epsilon) < L_2L_1L_2(x+\epsilon)\ . \tag{C.71}$$

Combining (C.70) with (C.71), we obtain

$$\begin{aligned} V_T(L_1L_2(x+\epsilon), n-1) - V_T(L_1L_2(x), n-1) &\leq L_2L_1L_2(x+\epsilon) - L_2L_1L_2(x) \\ &= A^2[L_1L_2(x+\epsilon) - L_1L_2(x)]\ . \end{aligned} \tag{C.72}$$

By Property 4.2, we have

$$A^2[L_1L_2(x+\epsilon) - L_1L_2(x)] < A^2(L_2(x+\epsilon) - L_2(x))\ . \tag{C.73}$$

Because of (C.69), (C.72) and (C.73), we obtain

$$V_T(L_2(x+\epsilon), n) - V_T(L_2(x), n) \geq V_T(L_1L_2(x+\epsilon), n-1) - V_T(L_1L_2(x), n-1)\ . \tag{C.74}$$

Case 2 : Assume it is optimal to take a measurement under $(L_1L_2(x), n-1)$, $(L_1L_2(x+\epsilon), n-1)$ at time instant T and not to take a measurement under $(L_2(x), n)$, $(L_2(x+\epsilon), n)$ and $(L_1L_2(x+\epsilon), n-1)$ at time instant T. Then

$$V_T(L_2(x), n) = L_2L_2(x)\ , \tag{C.75}$$

$$V_T(L_2(x+\epsilon), n) = L_2L_2(x+\epsilon)\ , \tag{C.76}$$

$$V_T(L_1L_2(x), n-1) = C + L_1L_2L_1L_2(x)\ , \tag{C.77}$$

$$V_T(L_1L_2(x+\epsilon), n-1) = L_2L_1L_2(x+\epsilon)\ . \tag{C.78}$$

Since $V_T(L_1L_2(x), n-1)$ is a non-decreasing function of x by R1, combining with Property 4.1, we have

$$C + L_1L_2L_1L_2(x) \leq L_2L_1L_2(x+\epsilon)\ . \tag{C.79}$$

By the definition of $V_T(L_1L_2(x), n)$ and (C.78), it follows that

$$C + L_1L_2L_1L_2(x) \leq L_2L_1L_2(x) \ , \tag{C.80}$$

$$L_2L_1L_2(x+\epsilon) \leq C + L_1L_2L_1L_2(x+\epsilon) \ . \tag{C.81}$$

Consequently,

$$C + L_1L_2L_1L_2(x) \leq L_2L_1L_2(x) < L_2L_1L_2(x+\epsilon) \leq C + L_1L_2L_1L_2(x+\epsilon) \ , \tag{C.82}$$

where the second inequality follows from Property 4.1. The above inequality implies that

$$L_1L_2L_1L_2(x+\epsilon) - L_1L_2L_1L_2(x) \geq L_2L_1L_2(x+\epsilon) - L_2L_1L_2(x) \ , \tag{C.83}$$

which conflicts Property 4.2. Therefore, the combination of decisions considered in this case can not be part of an optimal sensor activation strategy.

For the situation where $n = 1$, we can not take a measurement under states $(L_1L_2(x), 0)$ and $(L_1L_2(x+\epsilon), 0)$ at time instant T. Therefore we only need to consider 4 possible decision combinations under states $(L_2(x), 1)$ and $(L_2(x+\epsilon), 1)$. We examine all these cases below.

Case 1 : Assume it is optimal not to take a measurement under $(L_2(x), 1)$ and $(L_2(x+\epsilon), 1)$ at time instant T. Then

$$V_T(L_2(x), 1) = L_2L_2(x) \ , \tag{C.84}$$

$$V_T(L_2(x+\epsilon), 1) = L_2L_2(x+\epsilon) \ , \tag{C.85}$$

$$V_T(L_1L_2(x), 0) = L_2L_1L_2(x) \ , \tag{C.86}$$

$$V_T(L_1L_2(x+\epsilon), 0) = L_2L_1L_2(x+\epsilon) \ , \tag{C.87}$$

and

$$V_T(L_2(x+\epsilon), n) - V_T(L_2(x), n) = L_2L_2(x+\epsilon) - L_2L_2(x) \ , \tag{C.88}$$

$$V_T(L_1L_2(x+\epsilon), n-1) - V_T(L_1L_2(x), n-1) = L_2L_1L_2(x+\epsilon) - L_2L_1L_2(x) \ . \tag{C.89}$$

By (C.88), (C.89) and Property 4.2, it follows that

$$V_T(L_2(x+\epsilon),n) - V_T(L_2(x),n) \geq V_T(L_1L_2(x+\epsilon),n-1) - V_T(L_1L_2(x),n-1) \ . \tag{C.90}$$

Case 2 : Assume it is optimal to take a measurement under $(L_2(x+\epsilon),1)$ and not to take a measurement under $(L_2(x),1)$ at time instant T. Then

$$V_T(L_2(x),1) = L_2L_2(x) \ , \tag{C.91}$$

$$V_T(L_2(x+\epsilon),1) = C + L_1L_2L_2(x+\epsilon) \ , \tag{C.92}$$

$$V_T(L_1L_2(x),0) = L_2L_1L_2(x) \ , \tag{C.93}$$

$$V_T(L_1L_2(x+\epsilon),0) = L_2L_1L_2(x+\epsilon) \ , \tag{C.94}$$

and

$$V_T(L_2(x+\epsilon),n) - V_T(L_2(x),n) = C + L_1L_2L_2(x+\epsilon) - L_2L_2(x) \ , \tag{C.95}$$

$$V_T(L_1L_2(x+\epsilon),n-1) - V_T(L_1L_2(x),n-1) = L_2L_1L_2(x+\epsilon) - L_2L_1L_2(x) \ . \tag{C.96}$$

By the definition of $V_T(L_2(x),n)$ and (C.91), we obtain

$$L_2L_2(x) \leq C + L_1L_2L_2(x) \ . \tag{C.97}$$

Combining (C.95) and (C.97), we get

$$V_T(L_2(x+\epsilon),n) - V_T(L_2(x),n) > C + L_1L_2L_2(x+\epsilon) - [C + L_1L_2L_2(x)] \ . \tag{C.98}$$

From (C.97), (C.98) and Property 4.4, we conclude that

$$V_T(L_2(x+\epsilon),n) - V_T(L_2(x),n) \geq V_T(L_1L_2(x+\epsilon),n-1) - V_T(L_1L_2(x),n-1) \ . \tag{C.99}$$

Case 3 : Assume it is optimal to take a measurement under $(L_2(x), 1)$ and not to take a measurement under $(L_2(x+\epsilon), 1)$ at time instant T. Then

$$V_T(L_2(x), 1) = C + L_1 L_2 L_2(x) \ , \tag{C.100}$$

$$V_T(L_2(x+\epsilon), 1) = L_2 L_2(x+\epsilon) \ , \tag{C.101}$$

$$V_T(L_1 L_2(x), 0) = L_2 L_1 L_2(x) \ , \tag{C.102}$$

$$V_T(L_1 L_2(x+\epsilon), 0) = L_2 L_1 L_2(x+\epsilon) \ . \tag{C.103}$$

We will show that this decision combination is not part of an optimal sensor activation strategy.

Since $V_T(L_2(x), 1)$ is a non-decreasing function of x by R1, combining with Property 4.3, we have

$$C + L_1 L_2 L_2(x) \leq L_2 L_2(x+\epsilon) \ . \tag{C.104}$$

By the definition of $V_T(L_2(x), 1)$ and (C.100), (C.101), we obtain

$$C + L_1 L_2 L_2(x) \leq L_2 L_2(x) \ , \tag{C.105}$$

$$L_2 L_2(x+\epsilon) \leq C + L_1 L_2 L_2(x+\epsilon) \ . \tag{C.106}$$

Combining (C.105), (C.106) and Property 4.1, we obtain

$$C + L_1 L_2 L_2(x) \leq L_2 L_2(x) < L_2 L_2(x+\epsilon) \leq C + L_1 L_2 L_2(x+\epsilon) \ . \tag{C.107}$$

The above inequality implies that

$$L_1 L_2 L_2(x+\epsilon) - L_1 L_2 L_2(x) \geq L_2 L_2(x+\epsilon) - L_2 L_2(x) \ , \tag{C.108}$$

which contradicts Property 4.2. Therefore this decision combination is not part of an optimal sensor activation strategy.

Case 4 : Assume it is optimal to take a measurement under $(L_2(x), 1)$ and $(L_2(x+\epsilon), 1)$

at time instant T. Then

$$V_T(L_2(x),1) = C + L_1L_2L_2(x) \ , \tag{C.109}$$

$$V_T(L_2(x+\epsilon),1) = C + L_1L_2L_2(x+\epsilon) \ , \tag{C.110}$$

$$V_T(L_1L_2(x),0) = L_2L_1L_2(x) \ , \tag{C.111}$$

$$V_T(L_1L_2(x+\epsilon),0) = L_2L_1L_2(x+\epsilon) \ , \tag{C.112}$$

and

$$V_T(L_2(x+\epsilon),n) - V_T(L_2(x),n) = L_1L_2L_2(x+\epsilon) - L_1L_2L_2(x) \ , \tag{C.113}$$

$$V_T(L_1L_2(x+\epsilon),n-1) - V_T(L_1L_2(x),n-1) = L_2L_1L_2(x+\epsilon) - L_2L_1L_2(x) \ . \tag{C.114}$$

By (C.113), (C.114) and Property 4.4, we conclude that

$$V_T(L_2(x+\epsilon),n) - V_T(L_2(x),n) \geq V_T(L_1L_2(x+\epsilon),n-1) - V_T(L_1L_2(x),n-1) \ . \tag{C.115}$$

We have now established that for $n = 1, \cdots, m$,

$$V_T(L_2(x+\epsilon),n) - V_T(L_2(x),n) \geq V_T(L_2L_1(x+\epsilon),n-1) - V_T(L_2L_1(x),n-1) \ .$$

Induction Hypothesis:

Hypothesis 1 (H1): For all $n = 1, \cdots, m$, if $0 < x_1 < x_2$,

$$V_{t+1}(x_1,n) \leq V_{t+1}(x_2,n) \ . \tag{C.116}$$

Hypothesis 2 (H2): For all $n = 1, \cdots, m$, if $0 < x_1 < x_2$,

$$V_{t+1}(x_1+\epsilon,n) - V_{t+1}(x_1,n) \geq V_{t+1}(x_2+\epsilon,n) - V_{t+1}(x_2,n) \ . \tag{C.117}$$

Hypothesis 3 (H3): For all $n = 1, \cdots, m$, if $0 < x$,

$$V_{t+1}(L_2(x+\epsilon),n) - V_{t+1}(L_2(x),n) \geq V_{t+1}(L_1L_2(x+\epsilon),n-1) - V_{t+1}(L_1L_2(x),n-1) \ . \tag{C.118}$$

Induction Step: Assume the induction hypotheses H1-H3 are true. We need to prove R1: For all $n = 1, \cdots, m$, if $0 < x_1 < x_2$,

$$V_t(x_1, n) \le V_t(x_2, n) \ , \tag{C.119}$$

R2: For all $n = 1, \cdots, m$, if $0 < x_1 < x_2$,

$$V_t(x_1 + \epsilon, n) - V_t(x_1, n) \ge V_t(x_2 + \epsilon, n) - V_t(x_2, n) \ . \tag{C.120}$$

R3: For all $n = 1, \cdots, m$, if $0 < x$,

$$V_t(L_2(x + \epsilon), n) - V_t(L_2(x), n) \ge V_t(L_1 L_2(x + \epsilon), n - 1) - V_t(L_1 L_2(x), n - 1) \ . \tag{C.121}$$

Proof of R1: We consider four cases that are the analogues of cases 1-4 in the basis of induction.

Case 1 : Assume it is optimal to take a measurement under both of the information states (x_1, n) and (x_2, n) at time instant t. Then

$$V_t(x_1, n) = L_2(x_1) + V_{t+1}(L_2(x_1), n) \ , \tag{C.122}$$

$$V_t(x_2, n) = L_2(x_2) + V_{t+1}(L_2(x_2), n) \ . \tag{C.123}$$

When $x_1 < x_2$, $L_2(x_1) < L_2(x_2)$ by Property 4.1. Thus

$$V_{t+1}(L_2(x_1), n) < V_{t+1}(L_2(x_2), n) \ , \tag{C.124}$$

by the induction hypothesis H1. Therefore $V_t(x_1, n) \le V_t(x_2, n)$.

Case 2 : Assume it is optimal to take a measurement under (x_2, n) and not to take a measurement under (x_1, n) at time instant t. Then

$$V_t(x_1, n) = L_2(x_1) + V_{t+1}(L_2(x_1), n), \tag{C.125}$$

$$V_t(x_2, n) = C + L_1 L_2(x_2) + V_{t+1}(L_1 L_2(x_2), n - 1) \ . \tag{C.126}$$

By the definition of $V_t(x,n)$ and (C.126), we obtain

$$L_2(x_1) + V_{t+1}(L_2(x_1), n) \leq C + L_1 L_2(x_1) + V_{t+1}(L_1 L_2(x_1), n-1) \; . \tag{C.127}$$

When $x_1 < x_2$, $L_1 L_2(x_1) < L_1 L_2(x_2)$ by Property 4.1. Thus

$$V_{t+1}(L_1 L_2(x_1), n-1) \leq V_{t+1}(L_1 L_2(x_2), n-1) \; , \tag{C.128}$$

by the induction hypothesis H1, and

$$L_2(x_1) + V_{t+1}(L_2(x_1), n) \leq C + L_1 L_2(x_2) + V_{t+1}(L_1 L_2(x_2), n-1) \; . \tag{C.129}$$

Therefore $V_t(x_1, n) \leq V_t(x_2, n)$ by (C.129), (C.125) and (C.126).

Case 3 : Assume it is optimal to take a measurement under (x_1, n) and not to take a measurement under (x_2, n) at time instant t. We show that such a combination of decisions is not part of an optimal sensor activation strategy. Based on the assumption we have

$$V_t(x_1, n) = C + l_1 L_2(x_1) + V_{t+1}(L_1 L_2(x_1), n-1) \; , \tag{C.130}$$

$$V_t(x_2, n) = L_2(x_2) + V_{t+1}(L_2(x_2), n) \; . \tag{C.131}$$

By the definition of $V_t(x,n)$ and (C.130), (C.131) we obtain

$$C + L_1 L_2(x_1) + V_{t+1}(L_1 L_2(x_1), n-1) \leq L_2(x_1) + V_{t+1}(L_2(x_1), n) \; , \tag{C.132}$$

$$L_2(x_2) + V_{t+1}(L_2(x_2), n) \leq C + L_1 L_2(x_2) + V_{t+1}(L_1 L_2(x_2), n-1) \; . \tag{C.133}$$

When $x_1 < x_2$, $L_2(x_1) < L_2(x_2)$ by Property 4.1. Combining (C.132), (C.133) and Property 4.1, we get

$$\begin{aligned} C + L_1 L_2(x_1) + V_{t+1}(L_1 L_2(x_1), n-1) &\leq L_2(x_1) + V_{t+1}(L_2(x_1), n) \\ &< L_2(x_2) + V_{t+1}(L_2(x_2), n) \\ &\leq C + L_1 L_2(x_2) + V_{t+1}(L_1 L_2(x_2), n-1) \; , \end{aligned} \tag{C.134}$$

which implies that

$$L_1L_2(x_2) - L_1L_2(x_1) + V_{t+1}(L_1L_2(x_2), n-1) - V_{t+1}(L_1L_2(x_1), n-1) \geq L_2(x_2) - L_2(x_1) + V_{t+1}(L_2(x_2), n) - V_{t+1}(L_2(x_1), n) \ . \tag{C.135}$$

We know that $L_1L_2(x_2) - L_1L_2(x_1) < L_2(x_2) - L_2(x_1)$ by Property 4.2. Then we must have

$$V_{t+1}(L_1L_2(x_2), n-1) - V_{t+1}(L_1L_2(x_1), n-1) > V_{t+1}(L_2(x_2), n) - V_{t+1}(L_2(x_1), n) \ , \tag{C.136}$$

which contradicts the induction hypothesis H3. Consequently, the combination of the decisions considered in this case can not be part of an optimal strategy.

Case 4 : Assume it is optimal to take a measurement under both information states (x_1, n) and (x_2, n) at time instant t. Then

$$V_t(x_1, n) = C + L_1L_2(x_1) + V_{t+1}(L_1L_2(x_1), n-1) \ , \tag{C.137}$$

$$V_t(x_2, n) = C + L_1L_2(x_2) + V_{t+1}(L_1L_2(x_2), n-1) \ . \tag{C.138}$$

When $x_1 < x_2$, by Property 4.1,

$$L_1L_2(x_1) < L_1L_2(x_2) \ . \tag{C.139}$$

Thus

$$V_{t+1}(L_1L_2(x_1), n-1) < V_{t+1}(L_1L_2(x_2), n-1) \ , \tag{C.140}$$

by the induction hypothesis H1 and Property 4.1. Therefore $V_t(x_1, n) \leq V_t(x_2, n)$, by (C.137)–(C.140).

Combining the four cases above, we conclude that $V_t(x_1, n) \leq V_t(x_2, n)$, $n = 1, 2, \cdots, m$.

Proof of R2: Again there are 16 possible combinations of decisions under the four information states (x_1, n), $(x_1 + \epsilon, n)$, $(x_2, n))$ and $(x_2 + \epsilon, n)$ at time t. We present the analysis of one combination of decisions that could be part of an optimal strategy and one combination of decisions that are not part of an optimal strategy. All other combinations can be analyzed in a manner similar to the two cases presented below.

Case 1 : Assume it is optimal to take a measurement under $(x_2, n))$ at time instant t and not to take a measurement under (x_1, n), $(x_1 + \epsilon, n)$ and $(x_2 + \epsilon, n)$ at time instant t. Then

$$V_t(x_1, n) = L_2(x_1) + V_{t+1}(L_2(x_1), n) \ , \tag{C.141}$$

$$V_t(x_1 + \epsilon, n) = L_2(x_1 + \epsilon) + V_{t+1}(L_2(x_1 + \epsilon), n) \ , \tag{C.142}$$

$$V_t(x_2, n) = C + L_1L_2(x_2) + V_{t+1}(L_1L_2(x_2), n - 1) \ , \tag{C.143}$$

$$V_t(x_2 + \epsilon, n) = L_2(x_2 + \epsilon) + V_{t+1}(L_2(x_2 + \epsilon), n) \ . \tag{C.144}$$

We will show that this combination of decisions can not be part of an optimal sensor activation strategy.

Since $V_t(x_2, n)$ is a non-decreasing function of x_2 by R1, we have

$$C + L_1L_2(x_2) + V_{t+1}(L_1L_2(x_2), n - 1) \leq L_2(x_2 + \epsilon) + V_{t+1}(L_2(x_2 + \epsilon), n) \ . \tag{C.145}$$

By the definition of $V_t(x, n)$ and (C.143), (C.144), we obtain

$$C + L_1L_2(x_2) + V_{t+1}(L_1L_2(x_2), n - 1) \leq L_2(x_2) + V_{t+1}(L_2(x_2), n) \ , \tag{C.146}$$

$$L_2(x_2 + \epsilon) + V_{t+1}(L_2(x_2 + \epsilon), n) \leq C + L_1L_2(x_2 + \epsilon) + V_{t+1}(L_1L_2(x_2 + \epsilon), n - 1) \ . \tag{C.147}$$

Combining (C.146), (C.147) and Property 4.1, we get

$$\begin{aligned} C + L_1L_2(x_2) + &V_{t+1}(L_1L_2(x_2), n - 1) \\ &\leq L_2(x_2) + V_{t+1}(L_2(x_2), n) && \text{(C.148)} \\ &< L_2(x_2 + \epsilon) + V_{t+1}(L_2(x_2 + \epsilon), n) && \text{(C.149)} \\ &\leq C + L_1L_2(x_2 + \epsilon) + V_{t+1}(L_1L_2(x_2 + \epsilon), n - 1) \ , && \text{(C.150)} \end{aligned}$$

which implies that

$$L_1L_2(x_2+\epsilon) - L_1L_2(x_2) + V_{t+1}(L_1L_2(x_2+\epsilon), n-1) - V_{t+1}(L_1L_2(x_2), n-1) \tag{C.151}$$

$$\geq L_2(x_2+\epsilon) - L_2(x_2) + V_{t+1}(L_2(x_2+\epsilon), n) - V_{t+1}(L_2(x_2), n) \ . \tag{C.152}$$

By Property 4.2, we know that

$$L_1L_2(x_2+\epsilon) - L_1L_2(x_2) \leq L_2(x_2+\epsilon) - L_2(x_2) \ . \tag{C.153}$$

From the induction hypothesis H3, we have

$$V_{t+1}(L_1L_2(x_2+\epsilon), n-1) - V_{t+1}(L_1L_2(x_2), n-1)$$
$$\leq V_{t+1}(L_2(x_2+\epsilon), n) - V_{t+1}(L_2(x_2), n) \ . \tag{C.154}$$

Adding (C.153) and (C.154), we obtain an inequality that contradicts (C.152). Consequently, the combination of the decisions considered in this case can not be part of an optimal strategy.

Case 2 : Assume it is optimal to take a measurement under (x_2, n) and $(x_2+\epsilon, n)$ at time instant t and not to take a measurement under (x_1, n) and $(x_1+\epsilon, n)$ at time instant t. Then

$$V_t(x_1, n) = L_2(x_1) + V_{t+1}(L_2(x_1), n) \ , \tag{C.155}$$

$$V_t(x_1+\epsilon, n) = L_2(x_1+\epsilon) + V_{t+1}(L_2(x_1+\epsilon), n) \ , \tag{C.156}$$

$$V_t(x_2, n) = C + L_1L_2(x_2) + V_{t+1}(L_1L_2(x_2), n-1), \tag{C.157}$$

$$V_t(x_2+\epsilon, n) = C + L_1L_2(x_2+\epsilon) + V_{t+1}(L_1L_2(x_2+\epsilon), n-1) \ . \tag{C.158}$$

By the induction hypothesis H2, (C.155) and (C.156) we get

$$V_t(x_1+\epsilon, n) - V_t(x_1, n)$$
$$= L_2(x_1+\epsilon) - L_2(x_1) + V_{t+1}(L_2(x_1+\epsilon), n) - V_{t+1}(L_2(x_1), n)$$
$$> L_2(x_2+\epsilon) - L_2(x_2) + V_{t+1}(L_2(x_2+\epsilon), n) - V_{t+1}(L_2(x_2), n) \ . \tag{C.159}$$

From (C.157) and (C.158), it follows that

$$V_t(x_2+\epsilon,n)-V_t(x_2,n)=L_1L_2(x_2+\epsilon)-L_1L_2(x_2) \\ +V_{t+1}(L_1L_2(x_2+\epsilon),n-1)-V_{t+1}(L_1L_2(x_2),n-1)\ . \tag{C.160}$$

Combining (C.159), (C.160), Property 4.2 and the hypothesis induction H3, we get

$$V_t(x_1+\epsilon,n)-V_t(x_1,n)\geq V_t(x_2+\epsilon,n)-V_t(x_2,n)\ . \tag{C.161}$$

Proof of R3: For the situation where $n\geq 2$, similarly to the proof of R2, we analyze 2 cases. The remaining 14 cases can be analyzed in a manner similar to that of the cases below.

Case 1 : Assume it is optimal not to take a measurement under information states $(L_1L_2(x+\epsilon),n-1)$, $(L_2(x),n)$, $(L_2(x+\epsilon),n)$ and $(L_1L_2(x),n-1)$ at time instant t. Then

$$V_t(L_2(x),n)=L_2L_2(x)+V_{t+1}(L_2L_2(x),n)\ , \tag{C.162}$$

$$V_t(L_2(x+\epsilon),n)=L_2L_2(x+\epsilon)+V_{t+1}(L_2L_2(x+\epsilon),n)\ , \tag{C.163}$$

$$V_t(L_1L_2(x),n-1)=L_2L_1L_2(x)+V_{t+1}(L_2L_1L_2(x),n-1)\ , \tag{C.164}$$

$$V_t(L_1L_2(x+\epsilon),n-1)=L_2L_1L_2(x+\epsilon)+V_{t+1}(L_2L_1L_2(x+\epsilon),n-1)\ , \tag{C.165}$$

and

$$V_t(L_2(x+\epsilon),n)-V_t(L_2(x),n)$$

$$=L_2L_2(x+\epsilon)-L_2L_2(x)+V_{t+1}(L_2L_2(x+\epsilon),n)-V_{t+1}(L_2L_2(x),n)\ , \tag{C.166}$$

$$\geq L_2L_2(x+\epsilon)-L_2L_2(x)+V_{t+1}(L_1L_2L_2(x+\epsilon),n-1)-V_{t+1}(L_1L_2L_2(x),n-1)\ , \tag{C.167}$$

$$\geq L_2L_2(x+\epsilon)-L_2L_2(x)+V_{t+1}(L_2L_1L_2(x+\epsilon),n-1)-V_{t+1}(L_2L_1L_2(x),n-1)\ , \tag{C.168}$$

where (C.167) results in from the induction hypothesis H3 and (C.168) follows from the induction hypothesis H2. From (C.164) and (C.165), it follows that We also know

$$\begin{aligned}&V_t(L_1L_2(x+\epsilon),n-1)-V_t(L_1L_2(x),n-1)\\&=L_2L_1L_2(x+\epsilon)-L_2L_1L_2(x)+V_{t+1}(L_2L_1L_2(x+\epsilon),n-1)\\&\quad -V_{t+1}(L_2L_1L_2(x),n-1)\ .\end{aligned} \tag{C.169}$$

Therefore by Properties 4.1, 4.2 and equations (C.168), (C.169), we obtain

$$V_t(L_2(x+\epsilon),n)-V_t(L_2(x),n)\geq V_t(L_1L_2(x+\epsilon),n-1)-V_t(L_1L_2(x),n-1)\ . \tag{C.170}$$

Case 2 : Assume it is optimal to take a measurement under information state $(L_1L_2(x),n-1)$ at time instant t and not to take a measurement under information states $(L_2(x),n)$, $(L_2(x+\epsilon),n)$ and $(L_1L_2(x+\epsilon),n-1)$ at time instant t. Then

$$V_t(L_2(x),n)=L_2L_2(x)+V_{t+1}(L_2L_2(x),n)\ , \tag{C.171}$$

$$V_t(L_2(x+\epsilon),n)=L_2L_2(x+\epsilon)+V_{t+1}(L_2L_2(x+\epsilon),n)\ , \tag{C.172}$$

$$V_t(L_1L_2(x),n-1)=C+L_1L_2L_1L_2(x)+V_{t+1}(L_1L_2L_1L_2(x),n-2)\ , \tag{C.173}$$

$$V_t(L_1L_2(x+\epsilon),n-1)=L_2L_1L_2(x+\epsilon)+V_{t+1}(L_2L_1L_2(x+\epsilon),n-1)\ . \tag{C.174}$$

Since $V_t(L_1L_2(x),n-1)$ is a non-decreasing function of x by R1 , we have

$$\begin{aligned}&C+L_1L_2L_1L_2(x)+V_{t+1}(L_1L_2L_1L_2(x),n-2)\\&\qquad\leq L_2L_1L_2(x+\epsilon)+V_{t+1}(L_2L_1L_2(x+\epsilon),n-1)\ .\end{aligned} \tag{C.175}$$

By the definition of $V_t(L_1L_2(x), n-1)$ and (C.173), (C.174),

$$C + L_1L_2L_1L_2(x) + V_{t+1}(L_1L_2L_1L_2(x), n-2) \leq L_2L_1L_2(x) + V_{t+1}(L_2L_1L_2(x), n-1)\ , \tag{C.176}$$

$$L_2L_1L_2(x+\epsilon) + V_{t+1}(L_2L_1L_2(x+\epsilon), n-1) \leq C + L_1L_2L_1L_2(x+\epsilon) + V_{t+1}(L_1L_2L_1L_2(x+\epsilon), n-2)\ . \tag{C.177}$$

Combining (C.176), (C.177) and Property 4.1, we get

$$\begin{aligned} C + L_1L_2L_1L_2(x) + V_{t+1}(L_1L_2L_1L_2(x), n-2) & \\ \leq L_2L_1L_2(x) + V_{t+1}(L_2L_1L_2(x), n-1) & \qquad \text{(C.178)} \\ < L_2L_1L_2(x+\epsilon) + V_{t+1}(L_2L_1L_2(x+\epsilon), n-1) & \qquad \text{(C.179)} \\ \leq C + L_1L_2L_1L_2(x+\epsilon) + V_{t+1}(L_1L_2L_1L_2(x+\epsilon), n-2)\ , & \qquad \text{(C.180)} \end{aligned}$$

which implies that

$$\begin{aligned} & L_1L_2L_1L_2(x+\epsilon) - L_1L_2L_1L_2(x) \\ & \qquad + V_{t+1}(L_1L_2L_1L_2(x+\epsilon), n-2) - V_{t+1}(L_1L_2L_1L_2(x), n-2) \\ & \geq L_2L_1L_2(x+\epsilon) - L_2L_1L_2(x) \\ & \qquad + V_{t+1}(L_2L_1L_2(x+\epsilon), n-1) - +V_{t+1}(L_2L_1L_2(x+\epsilon), n-1)\ . \end{aligned} \tag{C.181}$$

Combination of Property 4.2 and the induction hypothesis H3 results in an inequality that contradicts (C.181). Consequently, the combination of decisions considered in this case can not be part of an optimal strategy.

For the situation when $n = 1$, we analyze four cases.

Case 1 : Assume it is optimal not to take a measurement under $(L_2(x), 1)$ and $(L_2(x+$

$\epsilon), 1)$ at time instant t. Then

$$V_t(L_2(x), 1) = L_2L_2(x) + V_{t+1}(L_2L_2(x), 1) \ , \tag{C.182}$$

$$V_t(L_2(x+\epsilon), 1) = L_2L_2(x+\epsilon) + V_{t+1}(L_2L_2(x+\epsilon), 1) \ , \tag{C.183}$$

$$V_t(L_1L_2(x), 0) = L_2L_1L_2(x) + V_{t+1}(L_2L_1L_2(x), 0) \ , \tag{C.184}$$

$$V_t(L_1L_2(x+\epsilon), 0) = L_2L_1L_2(x+\epsilon) + V_{t+1}(L_2L_1L_2(x+\epsilon), 0) \ . \tag{C.185}$$

and

$$\begin{aligned}
&V_t(L_2(x+\epsilon), 1) - V_t(L_2(x), 1) \\
&\quad = L_2L_2(x+\epsilon) - L_2L_2(x) + V_{t+1}(L_2L_2(x+\epsilon), 1) - V_{t+1}(L_2L_2(x), 1) && \text{(C.186)} \\
&\quad \geq L_2L_2(x+\epsilon) - L_2L_2(x) + V_{t+1}(L_1L_2L_2(x+\epsilon), 0) - V_{t+1}(L_1L_2L_2(x), 0) \ , && \text{(C.187)} \\
&\quad \geq L_2L_2(x+\epsilon) - L_2L_2(x) + V_{t+1}(L_2L_1L_2(x+\epsilon), 0) - V_{t+1}(L_2L_1L_2(x), 0) \ , && \text{(C.188)}
\end{aligned}$$

where (C.187) follows from the induction hypothesis H3 and (C.188) follows from the induction hypothesis H2. From (C.184) and (C.185), we obtain

$$\begin{aligned}
&V_t(L_1L_2(x+\epsilon), 0) - V_t(L_1L_2(x), 0) \\
&= L_2L_1L_2(x+\epsilon) - L_2L_1L_2(x) + V_{t+1}(L_2L_1L_2(x+\epsilon), 0) \\
&\quad - V_{t+1}(L_2L_1L_2(x), 0) \ . && \text{(C.189)}
\end{aligned}$$

Combining (C.188), (C.189) and Property 4.2, we conclude

$$V_t(L_2(x+\epsilon), 1) - V_t(L_2(x), 1) \geq V_t(L_1L_2(x+\epsilon), 0) - V_t(L_1L_2(x), 0) \ . \tag{C.190}$$

Case 2 : Assume it is optimal to take a measurement under $(L_2(x+\epsilon), 1)$ and not to take a measurement under $(L_2(x),1)$ at time instant t. Then we have

$$V_t(L_2(x),1) = L_2L_2(x) + V_{t+1}(L_2L_2(x),1) \ , \tag{C.191}$$

$$V_t(L_2(x+\epsilon),1) = C + L_1L_2L_2(x+\epsilon) + V_{t+1}(L_1L_2L_2(x+\epsilon),0) \ , \tag{C.192}$$

$$V_t(L_1L_2(x),0) = L_2L_1L_2(x) + V_{t+1}(L_2L_1L_2(x),0) \ , \tag{C.193}$$

$$V_t(L_1L_2(x+\epsilon),0) = L_2L_1L_2(x+\epsilon) + V_{t+1}(L_2L_1L_2(x+\epsilon),0) \ . \tag{C.194}$$

and

$$\begin{aligned}
&V_t(L_2(x+\epsilon),1) - V_t(L_2(x),1) \\
=&C + L_1L_2L_2(x+\epsilon) - L_2L_2(x) + V_{t+1}(L_1L_2L_2(x+\epsilon),0) - V_{t+1}(L_2L_2(x),1) \qquad \text{(C.195)} \\
\geq&C + L_1L_2L_2(x+\epsilon) - [C + L_1L_2L_2(x)] + V_{t+1}(L_1L_2L_2(x+\epsilon),0) \\
&- V_{t+1}(L_1L_2L_2(x),0) \ , \qquad \text{(C.196)}
\end{aligned}$$

where (C.196) follows from the definition of $V_t(L_2(x),1)$ and (C.191).

Furthermore, from (C.193) and (C.194), we get

$$\begin{aligned}
&V_t(L_1L_2(x+\epsilon),0) - V_t(L_1L_2(x),0) \\
=&L_2L_1L_2(x+\epsilon) - L_2L_1L_2(x) + V_{t+1}(L_2L_1L_2(x+\epsilon),0) \\
&- V_{t+1}(L_2L_1L_2(x),0) \ . \qquad \text{(C.197)}
\end{aligned}$$

Combining (C.196), (C.197) and repeated use of Property 4.1 and 4.4, we obtain

$$V_t(L_2(x+\epsilon),1) - V_t(L_2(x),1) \geq V_t(L_1L_2(x+\epsilon),0) - V_t(L_1L_2(x),0) \ . \tag{C.198}$$

Case 3 : Assume it is optimal to take a measurement under $(L_2(x),1)$ and not to take a measurement under $(L_2(x+\epsilon),1)$ at time instant t. Then

$$V_t(L_2(x),1) = C + L_1L_2L_2(x) + V_{t+1}(L_1L_2L_2(x),0) \ , \tag{C.199}$$

$$V_t(L_2(x+\epsilon),1) = L_2L_2(x+\epsilon) + V_{t+1}(L_2L_2(x+\epsilon),1) \ , \tag{C.200}$$

$$V_t(L_1L_2(x),0) = L_2L_1L_2(x) + V_{t+1}(L_2L_1L_2(x),0) \ , \tag{C.201}$$

$$V_t(L_1L_2(x+\epsilon),0) = L_2L_1L_2(x+\epsilon) + V_{t+1}(L_2L_1L_2(x+\epsilon),0) \ . \tag{C.202}$$

Since $V_t(L_2(x),1)$ is a non-decreasing function of x by R1, we have

$$C + L_1L_2L_2(x) + V_{t+1}(L_1L_2L_2(x),0) \le L_2L_2(x+\epsilon) + V_{t+1}(L_2L_2(x+\epsilon),1) \ . \tag{C.203}$$

By the definition of $V_t(L_2(x),n)$, (C.199) and (C.200), we obtain

$$C + L_1L_2L_2(x) + V_{t+1}(L_1L_2L_2(x),0) \le L_2L_2(x) + V_{t+1}(L_2L_2(x),1) \ , \tag{C.204}$$

$$L_2L_2(x+\epsilon) + V_{t+1}(L_2L_2(x+\epsilon),1) \le C + L_1L_2L_2(x+\epsilon) + V_{t+1}(L_1L_2L_2(x+\epsilon),0) \ . \tag{C.205}$$

Combining (C.204), (C.205), Property 4.1 and R1, we get

$$\begin{aligned} C + L_1L_2L_2(x) &+ V_{t+1}(L_1L_2L_2(x),0) \\ &\le L_2L_2(x) + V_{t+1}(L_2L_2(x),1) \ , && \text{(C.206)} \\ &< L_2L_2(x+\epsilon) + V_{t+1}(L_2L_2(x+\epsilon),1) \ , && \text{(C.207)} \\ &\le C + L_1L_2L_2(x+\epsilon) + V_{t+1}(L_1L_2L_2(x+\epsilon),0) \ , && \text{(C.208)} \end{aligned}$$

which implies that

$$\begin{aligned} L_1L_2L_2(x+\epsilon) - L_1L_2L_2(x) + V_{t+1}(L_1L_2L_2(x+\epsilon),0) - V_{t+1}(L_1L_2L_2(x),0) \\ \ge L_2L_2(x+\epsilon) - L_2L_2(x) + V_{t+1}(L_2L_2(x+\epsilon),1) - V_{t+1}(L_2L_2(x),1) \ . \end{aligned} \tag{C.209}$$

Furthermore, combination of Property 4.2 and the induction hypothesis H3 results in an inequality that contradicts (C.205). Consequently, the combination of decisions considered in this case can not be part of an optimal strategy.

Case 4 : Assume it is optimal to take a measurement under $(L_2(x),1)$ and $(L_2(x+\epsilon),1)$ at time instant t. Then

$$V_t(L_2(x),n) = C + L_1L_2L_2(x) + V_{t+1}(L_1L_2L_2(x),0) \ , \tag{C.210}$$

$$V_t(L_2(x+\epsilon),n) = C + L_1L_2L_2(x+\epsilon) + V_{t+1}(L_1L_2L_2(x+\epsilon),0) \ , \tag{C.211}$$

$$V_t(L_1L_2(x),0) = L_2L_1L_2(x) + V_{t+1}(L_2L_1L_2(x),0) \ , \tag{C.212}$$

$$V_t(L_1L_2(x+\epsilon),0) = L_2L_1L_2(x+\epsilon) + V_{t+1}(L_2L_1L_2(x+\epsilon),0) \ , \tag{C.213}$$

and

$$\begin{aligned}
&V_t(L_2(x+\epsilon),1) - V_t(L_2(x),1) \\
=&C + L_1L_2L_2(x+\epsilon) - [C + L_1L_2L_2(x)] + V_{t+1}(L_1L_2L_2(x+\epsilon),0) \\
&- V_{t+1}(L_1L_2L_2(x),0) \ , \qquad \text{(C.214)} \\
&V_t(L_1L_2(x+\epsilon),0) - V_t(L_1L_2(x),0) \\
=&L_2L_1L_2(x+\epsilon) - L_2L_1L_2(x) + V_{t+1}(L_2L_1L_2(x+\epsilon),0) \\
&- V_{t+1}(L_2L_1L_2(x),0). \qquad \text{(C.215)}
\end{aligned}$$

From (C.214), (C.215) and repeated use of Property 4.2 and 4.4, we conclude that

$$V_t(L_2(x+\epsilon),1) - V_t(L_2(x),1) \geq V_t(L_1L_2(x+\epsilon),0) - V_t(L_1L_2(x),0) \ . \quad \text{(C.216)}$$

From the above four cases, it follows that for all $n = 1, 2, \cdots, m$,

$$V_t(L_2(x+\epsilon),n) - V_t(L_2(x),n) \geq V_t(L_1L_2(x+\epsilon),n-1) - V_t(L_1L_2(x),n-1) \ . \quad \text{(C.217)}$$

The proof of the induction step is complete. Thus, the assertion of Lemma C.2 is true.

□

Proof of Lemma 4.2.

Consider any sensor activation strategy g. Under g, σ_t^g may be either greater than or equal to or smaller than σ_{t+1}^g. If σ_0 is large enough, irrespectively of the choice of g, we will have

$$\sigma_t^g \geq \hat{\sigma} \ , \qquad \text{(C.218)}$$

for all $t = 1, \cdots, T$, where $\hat{\sigma}$ is defined by $L_1L_2(\hat{\sigma}) = L_2L_1(\hat{\sigma})$.

Since $K > \sigma^*$ and $\sigma_t^g \geq \hat{\sigma}$, for all $t = 1, \cdots, T$, it will follow that $L_1L_2(\sigma_t^g) = L_2L_1(\sigma_t^g)$ by Property 4.5. Consequently, the set $S_1(m, T)$ is non-empty. □

Proof of Theorem 4.2.

This theorem essentially states that for any $\sigma_0 \in S_1(m,T)$ an optimal strategy action must have the following form: $\{1, 1, \cdots, 1, 0, 0, \cdots, 0\}$. Here the exact number of 1s and 0s is unspecified.

Assume $\sigma_0 \in S_1(m,T)$ and consider an arbitrary sensor activation strategy $g := (g_1, \cdots, g_T)$, and let $(u_1^g, \cdots, u_T^g)$ be the sequence of actions at time $s = 1, \cdots, T$, resulting from g and (σ_0, m). Suppose

$$u_t^g = 0 \;\text{ and }\; u_{t+1}^g = 1 \; . \tag{C.219}$$

Define another sensor activation strategy $g' := (g_1', \cdots, g_T')$, *s.t.*,

$$u_s^{g'} = u_s^g \;,\; \text{for } s = 1, \cdots, t-1 \;, \tag{C.220}$$

$$u_t^{g'} = 1 \;, \;\text{ and }\; u_{t+1}^{g'} = 0 \;, \tag{C.221}$$

$$u_s^{g'} = u_s^g \;,\; \text{for } s = t+2, \cdots, T \; . \tag{C.222}$$

Then we have

$$\sigma_t^g = L_2(\sigma_{t-1}^g) \;, \tag{C.223}$$

$$\sigma_{t+1}^g = L_1 L_2 L_2(\sigma_{t-1}^g) \;, \tag{C.224}$$

$$\sigma_t^{g'} = L_1 L_2(\sigma_{t-1}^g) \;, \tag{C.225}$$

$$\sigma_{t+1}^{g'} = L_2 L_1 L_2(\sigma_{t-1}^g) \; . \tag{C.226}$$

Since $\sigma_0 \in S_1(m,T)$, it follows that

$$L_1 L_2(\sigma_{t-1}^g) \geq L_2 L_1(\sigma_{t-1}^g) \;, \tag{C.227}$$

$$L_1 L_2(L_2(\sigma_{t-1}^g)) \geq L_2 L_1(L_2(\sigma_{t-1}^g)) \;, \tag{C.228}$$

which implies that

$$\sigma_{t+1}^g \geq \sigma_{t+1}^{g'} \; . \tag{C.229}$$

Furthermore, by the specification of g and g', it follows that

$$\sigma_s^{g'} = \sigma_s^g \ , \ s = 0, 1, \cdots, t-1 \ , \tag{C.230}$$

and

$$\sigma_t^g = L_2(\sigma_{t-1}^g) \geq \sigma_t^{g'} = L_1 L_2(\sigma_{t-1}^{g'}) \ . \tag{C.231}$$

Using (C.222), (C.229) and Property 4.1, we conclude that

$$\sigma_s^g \geq \sigma_s^{g'} \ , \ \text{for all } s = t+2, \cdots, T \ . \tag{C.232}$$

Moreover, by construction, g and g' incur the same activation cost along the evolution path that originates at (σ_0, m). Combining this fact along with (C.229)–(C.232), we conclude that

$$J^g \geq J^{g'}. \tag{C.233}$$

Repeated application of the above argument establishes the "stopping property" of an optimal sensor activation strategy whenever $\sigma_0 \in S_1(m, T)$.

□

APPENDIX D

PROOFS FOR CHAPTER 5

Proof of Property 5.1.

Denote by $v^*_{i,t}$ and $v'_{i,t}$ the TDDP of cell i, $i \in \mathcal{L}$, if cell i is scanned at time t under π^* and π', respectively. Denote by $g^*_{i,t}$ and $g'_{i,t}$ the number of searches happened on cell i until time t, $i \in \mathcal{L}$, under π^* and π', respectively. With this notation, we proceed to prove Property 5.1 by contradiction.

Assume that under the optimal strategy π^*, there exists a time $t < S \cdot T$ such that $n^*_t < n^*_{t+1}$. Then, there exists at least one cell, denoted by l, which is scanned at time $t+1$, but is not scanned at time t, *i.e.,* $a^*_{l,t} = 0$ and $a^*_{l,t+1} = 1$.

Then $g^*_{l,t-1} = g^*_{l,t}$ and

$$v^*_{l,t+1} = \beta^t \cdot r_{l,g^*_{l,t}} = \beta^t \cdot r_{l,g^*_{l,t-1}} \ . \tag{D.1}$$

Define the search strategy π', which is identical to π^* except that it searches location l at time t instead of $t+1$, *i.e.,*

$$a'_{l,t} = 1 \ , \tag{D.2}$$

$$a'_{l,t+1} = 0 \ , \tag{D.3}$$

$$a'_{i,j} = a^*_{i,j} \ , \text{otherwise.} \tag{D.4}$$

Then $g'_{l,t-1} = g^*_{l,t-1}$ and

$$v'_{l,t} = \beta^{t-1} \cdot r_{l,g'_{l,t-1}} = \beta^{t-1} \cdot r_{l,g^*_{l,t-1}} = \frac{v^*_{l,t+1}}{\beta} \ , \tag{D.5}$$

$$\tag{D.6}$$

All the TDDPs of any cell other than l or the ones incurred at any other time instant other than t and $t+1$ are exactly the same under π^* and π'. Therefore

$$J(\pi^*) - J(\pi') = v^*_{l,t+1} - v'_{l,t} = (1 - \frac{1}{\beta})v^*_{l,t+1} < 0 \ , \tag{D.7}$$

which contradicts with the optimality of π^*. $\square$

Proof of Property 5.2.

We prove this property by contradiction.

Assume $\exists t'$, $t \leq t' \leq S \cdot T$, such that $\mathfrak{L}^*_{t'} \not\supseteq \mathfrak{L}^*_{t'+1}$. Then there exists location $l \in \mathfrak{L}^*_{t'+1}$, such that $l \notin \mathfrak{L}^*_{t'}$, *i.e.*, $a^*_{l,t'} = 0$ and $a^*_{l,t'+1} = 1$.

According to Property 1 and the assumption made in the statement of Property 5.2, $S > n^*_t \geq n^*_{t'}$. Define strategy π', which is identical to π^* except that it searches location l at time t' instead of $t'+1$, so that

$$a'_{l,t'} = 1 \ , \tag{D.8}$$

$$a'_{l,t'+1} = 0 \ , \tag{D.9}$$

$$a'_{i,s} = a^*_{i,s} \ , \ \text{otherwise.} \tag{D.10}$$

Then by an argument similar to the one appearing in the proof of Property 5.1, we have

$$J(\pi^*) - J(\pi') = v^*_{i,t+1} - v'_{i,t} = (1 - \frac{1}{\beta})v^*_{i,t+1} < 0 \ , \tag{D.11}$$

which contradicts with the optimality of π^*. $\square$

Proof of Theorem 5.1.

According to the process of Algorithm G, $n^g_1 < S$ means that no column is discarded from the L TDDP tables. Therefore, all $S \cdot T$ TDDPs are only chosen from the first column of each table, and the $S \cdot T$ TDDPs that are chosen are the largest among those in the first column of each table. For any feasible search strategy, at most one TDDP can be chosen from each row and in each row the TDDP in the first column is the largest. Therefore, the TDDPs chosen by Algorithm G are the largest $S \cdot T$ TDDPs among the L tables. Since Algorithm G guarantees the strategy is implementable, it is optimal. $\square$

Proof of Lemma 5.1.

Consider a strategy π does not always give the priority to the cells with the largest TDDPs when every sensor is used at some time instant t, such that cell j is scanned and cell i is not scanned where $v_{i,t} > v_{j,t}$.

Let $j_1, \cdots, j_{k_j}$ be the times that cell j are scanned under π after time t, where $j_1 > t$ and $0 \leq k_j \leq h_t - 1$. $k_j = 0$ means that cell j is not scanned any more under π. Let $i_1, i_2, \cdots, i_{k_i}$ be the times that cell i are scanned under π after time t, where $i_1 > t$ and $0 \leq k_i \leq h_t - 1$. $k_i = 0$ means that cell i is not scanned any more under π. We construct π^1 as follows: π^1 is identical to π except cell i is scanned at time t instead of cell j.

Then in the objective functions $J(\pi)$ and $J(\pi^1)$, only the TDDPs for cell i and j after time $t-1$ are different. Therefore

$$\begin{aligned}
J(\pi^1) - J(\pi) =& (v_{i,t} - v_{j,t}) + \left[\sum_{s=1}^{k_i} v_{i,t} \cdot (1-\alpha_i)^s \cdot \beta^{i_s - t} - \sum_{s=1}^{k_i} v_{i,t} \cdot (1-\alpha_i)^{s-1} \cdot \beta^{i_s - t}\right] \\
&+ \left[\sum_{s=1}^{k_j} v_{j,t} \cdot (1-\alpha_j)^{s-1} \cdot \beta^{j_s - t} - \sum_{s=1}^{k_j} v_{j,t} \cdot (1-\alpha_j)^s \cdot \beta^{j_s - t}\right] \qquad \text{(D.12)} \\
=& \left[v_{i,t} - \alpha_i \cdot v_{i,t} \cdot \sum_{s=1}^{k_i} (1-\alpha_i)^{s-1} \cdot \beta^{i_s - t}\right] - \left[v_{j,t} - \alpha_j \cdot v_{j,t} \cdot \sum_{s=1}^{k_j} (1-\alpha_j)^{s-1} \cdot \beta^{j_s - t}\right].
\end{aligned} \qquad \text{(D.13)}$$

Because of (D.13) and $1 \le k_i \le h_t - 1$, $k_j \ge 0$, we have

$$J(\pi^1) - J(\pi) > \left[v_{i,t} - \alpha_i \cdot v_{i,t} \cdot \sum_{s=1}^{h_t-1} (1-\alpha_i)^{s-1} \cdot \beta^s \right] - v_{j,t} \tag{D.14}$$

$$= v_{i,t} - v_{i,t} \cdot \alpha_i \cdot \frac{\beta - (1-\alpha_i)^{h_t-1}\beta^{h_t}}{1-(1-\alpha_i)\cdot\beta} - v_{j,t} \tag{D.15}$$

$$= v_{i,t} \cdot \frac{1-\beta+\alpha_i \cdot (1-\alpha_i)^{h_t-1} \cdot \beta^{h_t}}{1-\beta\cdot(1-\alpha_i)} - v_{j,t} \ . \tag{D.16}$$

Note that the right side of (D.14) happens when cell j is not scanned any more and all the available sensors will be used to scan cell i after time t. Because of (5.6),

$$J(\pi^1) > J(\pi) \ . \tag{D.17}$$

Therefore, policy π^1 yields higher reward than π because of (D.17). By repeated application of the above modification argument, we conclude that when $n_t^g = S$, it is optimal to search all the cells in $\mathcal{L}_t^g$ at time t given that π^g was used up to time $t-1$. $\square$

Proof of Theorem 5.2.

This theorem can be easily proved by combining Theorem 5.1 and Lemma 5.1. $\square$

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