

Stochastic Analysis of Insurance Products

by
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Dedicated to my family.

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CHAPTER I

Introduction

In this dissertation, we study two insurance products: (i) reversible annuities and (ii) mortality derivatives.

1.1 Optimal asset allocation with reversible annuities

For a single premium immediate life annuity (SPIA), in exchange for a lump sum payment, a company guarantees to pay the annuitant a fix amount of money periodically until his or her death. With the assumption that there are only bonds and annuities in the financial market, [Yaari \[1965\]](#), as well as [Davidoff et al. \[2005\]](#) among others, prove that it is optimal for an individual with no bequest motive to fully annuitize. In reality, the volume of voluntary purchases by retirees is much lower than predicted by such models, which is the so-called “annuity puzzle.” According to a recent survey in the United Kingdom by [Gardner and Wadsworth \[2004\]](#), the dominant reason given for not wanting to annuitize is the preference for flexibility.

In Chapter [II](#) and [III](#), we propose an innovative annuity product, namely, a *reversible annuity*. Our goal is to reveal the relation between the reversibility (a type of flexibility) of annuities and retirees’ reluctance to annuitize. The reversible annuity, which is a SPIA with a surrender option, has a surrender value equal to its purchase price less a proportional surrender charge (denoted by p). We assume the existence

of a market of reversible life annuities, a riskless asset (bond or money market), and a risky asset (stock). A retiree is allowed to purchase additional annuity income or to surrender his or her existing annuity income. We investigate the behavior of the retiree under two distinct criteria: (i) minimizing the *probability of lifetime ruin*, the probability that one's wealth will reach zero before his death and (ii) maximizing the *utility of lifetime consumption*.

Mathematically, the reversibility of annuities complicates the optimal decisions of the retirees in both problems. For each problem, the optimal strategies of the retiree include continuous controls, instantaneous controls, and singular controls. We solve the two-dimensional control problems via duality arguments and validate the optimality of these solutions through verification theorems. Taking advantage of a duality argument, we are able to express the value functions and optimal strategies in semi-analytical forms, that is, transforms of explicit functions. In each problem, we find a critical value p^* such that the optimal strategies are divided into two categories depending on whether or not p is larger than p^* .

In Chapter II, we assume that the retiree consumes at an exogenous rate, and we determine the optimal investment strategy, as well as the optimal time to annuitize or to surrender, in order to minimize the probability of lifetime ruin. We find that when $p \geq p^*$, the individual will not buy life annuities unless he or she can buy enough to cover all of consumption. When $p < p^*$, the retiree is more willing to annuitize and purchases annuities to cover part of the exogenously given consumption. This chapter is based on Wang and Young [2009]. Parts of this work has been presented at 44th Actuarial Research Conference, Madison, Wisconsin, July 30, 2009.

In Chapter III, we consider the optimal strategies of a utility-maximizing retiree with constant relative risk aversion. In this setting, the consumption of the retiree

becomes a control variable. Adjusting the consumption level is an important strategy for the retiree to avoid bankruptcy. We find that when $p \geq p^*$, a retiree does not surrender existing annuity income under any circumstance. The retiree stops investing in the risky asset and consumes less than existing annuity income as wealth approaches zero. In other words, an individual behaves as if annuities are not reversible at all when $p \geq p^*$. When $p < p^*$, a retiree surrenders enough annuity income to keep wealth non-negative, as needed, and continues to invest in the risky asset as wealth approaches zero. This chapter is based on Wang and Young [2010b]. Part of this work has been presented at Department of Statistics and Actuarial Science, University of Waterloo, February 11, 2011; Department of Mathematics, University of Pittsburgh, January 6, 2011; SIAM Conference on Financial Mathematics and Engineering, San Francisco, November 19, 2010; Department of Mathematics, University of Michigan, November 29, 2009.

1.2 Pricing and hedging mortality derivatives

Mortality risk, which is due to the uncertain development of future hazard rates, has attracted much attention in recent years. Many capital market instruments have been proposed to deal with mortality risk for annuity providers and pension funds; see Dowd et al. [2006], Blake and Burrows [2001], and Blake et al. [2006]. However, few researchers have focused on the effectiveness of hedging mortality risk with the proposed mortality-linked derivatives; one notable exception is the work of Lin and Cox [2005].

In Chapter IV, we price pure endowments assuming that the issuing company hedges its contract with a mortality forward in order to minimize the variance of the value of the hedging portfolio and then requires compensation for the unhedge-

able part of the mortality risk in the form of a pre-specified instantaneous Sharpe ratio. We investigate the hedging efficiency by comparing the prices of the hedged pure endowments and the corresponding unhedged ones. By analyzing the partial differential equations whose solutions are values of pure endowments, we identify the properties of the prices of pure endowments and the factors that affect hedging efficiency. This chapter is based on [Wang and Young \[2010a\]](#) . Part of this work has been presented in the 6th International Longevity Risk and Capital Markets Solutions Conference, Sydney, Australia, September 9, 2010.

CHAPTER II

Optimal reversible annuities to minimize the probability of lifetime ruin

2.1 Introduction and motivation

The so-called “annuity puzzle” is that in financial markets for which annuity purchase is not mandatory, the volume of voluntary purchases by retirees is much smaller than predicted by models, such as those proposed by Yaari [1965], Richard [1975] and Davidoff et al. [2005]. Although life annuities provide income security in retirement, very few retirees choose a life annuity over a lump sum. According to a recent survey exploring attitudes towards annuitization among individuals approaching retirement in the United Kingdom by Gardner and Wadsworth [2004], over half of the individuals in the sample chose not to annuitize given the option. Whether the option was 100% annuitization or only partial (50%) annuitization, the attitude was the same. The dominant reason given for not wanting to annuitize in the survey is the preference for flexibility. It is well known that annuity income is *not* reversible. In other words, annuity holders can neither surrender for a refund nor short-sell (borrow against) their earlier purchased annuities, even when such a deal is desirable.

In this paper, we explore a way to add flexibility to life annuities by proposing a financial innovation, specifically a *reversible annuity*, an immediate life annuity with

a surrender option. The option to surrender allows an annuity holder to either borrow against or surrender any portion of her annuities at any time when she is still alive. The purchase value of this reversible annuity is determined by the expected present value of future payments to the annuity holder, which follows the same principle as regular annuities. The surrender value is set as a fixed proportion of its purchase value at the time of surrendering. The surrender value can also be viewed as the purchasing value less a proportional surrender charge, which is a combination of transaction cost, operating charge, and compensation for adverse selection. To explore how this reversible annuity would work for retirees as a reliable flow of income, as well as an asset able to be surrendered under certain personal circumstance, we investigate the optimal investment strategy and optimal annuity purchase and surrender strategies for an individual who seeks to minimize the probability that she outlives her wealth, also called the probability of *lifetime ruin*. In other words, we assume that the retiree consumes at a exogenous level, and we determine the optimal investment strategy, as well as the optimal time to annuitize or to surrender, in order to minimize the probability that wealth will reach zero before her death.

As a risk metric, the probability of lifetime ruin is widely used to investigate optimization problems faced by retirees in a financial market. This metric was first introduced by Milevsky and Robinson [2000] in a static environment and was extended by Young [2004a] to a stochastic environment without immediate life annuities. A recent paper by Milevsky et al. [2006a] determined the optimal dynamic investment policy for an individual who consumes at a specific rate, who invests in a complete financial market, and who can buy irreversible immediate life annuities. Milevsky, Moore, and Young show that the individual will not annuitize any of her wealth until she can fully cover her desired consumption with an immediate life annuity. Addi-

tionally, [Bayraktar and Young \[2009\]](#) investigate the optimal strategy for an retiree in a financial market with deferred (not immediate) life annuities. Within the topic of minimizing probability of lifetime ruin in a complete financial market without life annuities, [Bayraktar et al. \[2008\]](#) consider the case for which the exogenous consumption is random, and in [Bayraktar and Young \[2008\]](#), consumption is ratcheted (that is, it is a non-decreasing function of maximum wealth). [Bayraktar and Young \[2007c\]](#) investigate the optimal strategy when consumption level is deterministic but borrowing is constrained.

In contrast to the literature mentioned above, we allow an individual not only to buy an immediate life annuity, but also to surrender existing immediate life annuities with a proportional surrender charge. This reversibility of life annuities and the incompleteness of the annuity market (due to the proportional surrender charge) creates a more complex optimization environment and makes the problem mathematically challenging. Our model can be viewed as a generalization of the model by [Milevsky et al. \[2006a\]](#) in which annuities are irreversible, and the limiting case for which the surrender value of existing annuity approaches zero is consistent with their study.

Our work is the first to investigate the optimal strategies for a retiree in a market with reversible immediate life annuities. We comprehensively analyze the annuitization and investment strategies for such a retiree. We focus on how the proportional surrender charge, which ranges from 0% to 100% of the purchasing value of annuity, affects an individual's optimal strategies. We predict that, when the surrender charge is low enough, the individual has incentive to annuitize partially. This distinguishes our model from the one with irreversible annuities, in which an individual is only willing to fully annuitize. This difference shows that the flexibility offered by

reversible annuities might be able to resolve the “annuity puzzle.”

The remainder of the paper is organized as follows: In Section 2, we present the financial market in which the individual invests her wealth. In addition to investing in riskless and risky assets, the individual can purchase reversible immediate life annuities. In Section 3, we consider the life annuity as part of her total wealth, thereby allowing her assets to have negative value as long as the imputed surrender value of her annuity makes her total wealth positive. We prove a verification theorem for the minimal probability of lifetime ruin in this case, and we obtain the minimal probability, along with optimal investment and annuitization strategies. In Section 4, we consider the case for which individual is forced to keep the value of her riskless and risky assets non-negative (excluding the surrender value of the annuity) by surrendering the annuity when needed. It turns out that the optimal annuitization strategy depends on the size of the proportional surrender charge. We consider the case when the charge is large in Section 4.2.1, and in Section 4.2.2, we discuss the case when the charge is small.

2.2 Minimizing the probability of lifetime ruin

In this section, we describe the financial market in which the individual can invest her wealth, and we formulate the problem of minimizing the probability of lifetime ruin in this market. We allow the individual to purchase and surrender her reversible life annuity at any time.

2.2.1 Financial model

We consider an individual with future lifetime described by the random variable τ_d . Suppose τ_d is an exponential random variable with parameter λ^S , also referred to as the force of mortality or hazard rate; in particular, $\mathbb{E}[\tau_d] = 1/\lambda^S$. The superscript

S indicates that the parameter equals the individual's subjective belief as to the value of her hazard rate.

We assume that the individual consumes wealth at a constant rate of $c \geq 0$; this rate might be given in real or nominal units. One can interpret c as the minimum net consumption level below which the individual cannot (or will not) reduce her consumption further; therefore, the minimum probability of lifetime ruin gives a lower bound for the probability of ruin under any consumption function bounded below by c .

The individual can invest in a riskless asset, which earns interest at the constant rate $r \geq 0$. Also, she can invest in a risky asset whose price satisfies

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad S_0 = S > 0, \quad (2.2.1)$$

in which $\mu > r$, $\sigma > 0$, and B is a standard Brownian motion with respect to a filtration $\mathbb{F} = \{\mathcal{F}_t\}$ of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that B is independent of τ_d , the random time of death of the individual. If c is given as a real rate of consumption (that is, inflation adjusted), then we also express r and μ as real rates.

Moreover, an individual can buy any amount of reversible immediate life annuity or surrender any portion of her existing annuity income and receive some fraction of its value. The purchase price of an immediate life annuity that pays \$1 per year continuously until the insured's death is given by

$$\bar{a} = \int_0^\infty e^{-rs} e^{-\lambda^O s} ds = \frac{1}{r + \lambda^O}, \quad (2.2.2)$$

in which $\lambda^O > 0$ is the constant objective hazard rate that is used to price annuities. In other words, in return for each $\$ \bar{a}$ the individual pays for an immediate life annuity, she receives \$1 per year of continuous annuity income until she dies.

Due to the reversibility of the life annuity, she can surrender any amount of the annuity she owns. The surrender value for \$1 of annuity income is $(1 - p)\bar{a}$ with $0 < p \leq 1$. The factor p is the proportional surrender charge. In other words, the individual can get $\$(1 - p)\bar{a}$ dollars from the issuer by giving up \$1 of annuity income. Notice that the surrender value is less than the purchase price, and the difference is the surrender charge (in dollars).

Let W_t denote the amount of wealth the individual has invested in the risky and riskless assets at time t , with π_t in the risky asset and $W_t - \pi_t$ in the riskless. Let A_t^+ denote the cumulative amount of annuity income bought on or before time t , and let A_t^- denote the cumulative amount of annuity income surrendered on or before time t . Then, $A_t = A_t^+ - A_t^-$ represents the cumulative amount of immediate life annuity income at time t . The investment and annuitization strategy $\{\pi_t, A_t\}_{t \geq 0}$ is said to be *admissible* if the processes $\{\pi_t\}_{t \geq 0}$ and $\{A_t^\pm\}_{t \geq 0}$ are adapted to the filtration \mathbb{F} , if $\int_0^t \pi_s^2 ds < \infty$, almost surely, for all $t \geq 0$, and if $A_t \geq 0$, almost surely, for all $t \geq 0$. The wealth dynamics of the individual for a given admissible strategy are given by

$$dW_t = [rW_{t-} + (\mu - r)\pi_{t-} - c + A_{t-}]dt + \sigma\pi_{t-}dB_t - \bar{a}dA_t^+ + \bar{a}(1 - p)dA_t^-, \quad W_0 = w \geq 0. \quad (2.2.3)$$

By “lifetime ruin,” we mean that the individual’s wealth reaches the line $w = -(1 - p)\bar{a}A$ before she dies. We denote the time of ruin by $\tau_0 \triangleq \{t \geq 0 : W_t^{\pi, A} + (1 - p)\bar{a}A_t^{\pi, A} \leq 0\}$. In Section 3, we allow wealth (namely, the value of the riskless and risky assets) to be negative with the individual effectively borrowing against her annuity income. Then, in Section 4, we require that wealth remain non-negative. Note that τ_0 is independent of τ_d . The minimum probability of lifetime ruin ψ for the individual at time 0 is defined by

$$\psi(w, A) \triangleq \inf_{\{\pi_t, A_t\}} \mathbb{P} \left[\tau_0 < \tau_d \mid W_0 = w, A_0 = A, \tau_d > 0, \tau_0 > 0 \right]. \quad (2.2.4)$$

Remark 2.2.1. Notice that because we assume that the hazard rates λ^S and λ^O , as well as the financial parameters r , μ , and σ , are constant, ψ only depends on the state variables w and A and not upon time.

Remark 2.2.2. We can derive an equivalent form for the minimum probability of ruin due to the independence of the τ_d from τ_0 :

$$\begin{aligned} \psi(w, A) &= \inf_{\{\pi_t, A_t\}} \mathbb{P} \left[\tau_0 < \tau_d \mid W_0 = w, A_0 = A, \tau_d > 0, \tau_0 > 0 \right] \\ &= \inf_{\{\pi_t, A_t\}} \mathbb{E} \left[\int_0^\infty \lambda^S e^{-\lambda^S t} \mathbf{1}_{\{0 \leq \tau_0 \leq t\}} dt \mid W_0 = w, A_0 = A, \tau_d > 0, \tau_0 > 0 \right] \\ &= \inf_{\{\pi_t, A_t\}} \mathbb{E} \left[\int_{\tau_0}^\infty \lambda^S e^{-\lambda^S t} dt \mid W_0 = w, A_0 = A, \tau_d > 0, \tau_0 > 0 \right] \\ &= \inf_{\{\pi_t, A_t\}} \mathbb{E} \left[e^{-\lambda^S \tau_0} \mid W_0 = w, A_0 = A, \tau_d > 0, \tau_0 > 0 \right]. \end{aligned} \quad (2.2.5)$$

We will use this expression in our proof of verification theorem in next section.

Remark 2.2.3. *Milevsky et al. [2006a]* show that if one only allows irreversible life annuities, then the individual will not buy a life annuity until her wealth is large enough to cover all her consumption. Specifically, if $w \geq (c - A)\bar{a}$, then it is optimal for the individual to spend $(c - A)\bar{a}$ to buy an immediate annuity that will pay at the continuous rate $c - A$ for the rest of her life. This income, together with the prior income of A , will cover her desired consumption rate of c . In this case, the individual will not ruin, under the convention that if her net consumption rate becomes c , then she is not considered ruined even if her wealth is 0. (The latter occurs if her wealth is identically $(c - A)\bar{a}$ immediately before buying the annuity.)

2.3 No borrowing restriction

In this section, we consider the case in which the individual's wealth w is allowed to be negative, as long as $w + (1 - p)\bar{a}A$ is positive. Effectively, the individual is allowed to borrow against her life annuity income.

2.3.1 Motivation for the Hamilton-Jacobi-Bellman variational inequality

Let us first consider the strategies one can choose to minimize the probability of ruin. Before ruin occurs or the individual dies, she can execute one or more of the following strategies: (i) purchase additional annuity income, (ii) surrender existing annuity income, or (iii) do neither.

Now, suppose that at point (w, A) , it is optimal *not* to purchase or surrender any annuity income. In this case, we expect ψ will satisfy the equation

$$\lambda^S \psi = (rw - c + A)\psi_w + \min_{\pi} \left[(\mu - r)\pi\psi_w + \frac{1}{2}\sigma^2\pi^2\psi_{ww} \right]. \quad (2.3.1)$$

Because the above policy is in general suboptimal, (2.3.1) holds as an inequality; that is, for all (w, A) ,

$$\lambda^S \psi \leq (rw - c + A)\psi_w + \min_{\pi} \left[(\mu - r)\pi\psi_w + \frac{1}{2}\sigma^2\pi^2\psi_{ww} \right]. \quad (2.3.2)$$

As we shall prove later, no continuous purchase of lifetime annuity income is optimal; that is, the problem of purchasing or surrendering annuity is one of singular control. Thus, if at the point (w, A) , it is optimal to purchase annuity income instantaneously, then the individual moves instantly from (w, A) to $(w - \bar{a}\Delta A, A + \Delta A)$, for some $\Delta A > 0$. The optimality of this decision implies that

$$\psi(w, A) = \psi(w - \bar{a}\Delta A, A + \Delta A), \quad (2.3.3)$$

which in turn yields

$$\bar{a}\psi_w(w, A) = \psi_A(w, A). \quad (2.3.4)$$

Similarly, if it is optimal to surrender annuity income at the point (w, A) , the following equation holds:

$$\psi(w, A) = \psi(w + (1 - p)\bar{a}\Delta A, A - \Delta A), \quad (2.3.5)$$

which implies

$$(1 - p)\bar{a}\psi_w(w, A) = \psi_A(w, A). \quad (2.3.6)$$

Notice that the surrender value is a portion of the value of annuity determined by the proportional surrender charge p .

In general, such purchasing or surrendering policies are suboptimal; therefore, (2.3.4) and (2.3.6) hold as inequalities and become

$$\bar{a}\psi_w(w, A) \leq \psi_A(w, A), \quad (2.3.7)$$

and

$$(1 - p)\bar{a}\psi_w(w, A) \geq \psi_A(w, A). \quad (2.3.8)$$

Because the individual will either buy additional annuity income, surrender existing annuity income, or neither, we expect that the probability of lifetime ruin is a solution of the following Hamilton-Jacobi-Bellman (HJB) variational inequality

$$\max \left\{ \lambda^S \psi - (rw - c + A)\psi_w - \min_{\pi} \left[(\mu - r)\pi\psi_w + \frac{1}{2}\sigma^2\pi^2\psi_{ww} \right], \right. \\ \left. \bar{a}\psi_w(w, A) - \psi_A(w, A), \psi_A(w, A) - (1 - p)\bar{a}\psi_w(w, A) \right\} = 0. \quad (2.3.9)$$

Define $w_s(A) \triangleq \frac{pc}{\frac{1}{\bar{a}} - (1 - p)r} - \bar{a}A$, in which A is the existing annuity income. At the point $(w_s(A), A)$, suppose an individual borrows $\tilde{w}(A) \triangleq \frac{(c - A)\bar{a} - w_s(A)}{1 - r\bar{a}}$ at the interest rate r . She, then, has wealth $w_s(A) + \tilde{w}(A)$, which she spends to buy $\frac{1}{\bar{a}}(w_s(A) + \tilde{w}(A))$ additional life annuity income. Therefore, the total annuity income

she has is $A + \frac{1}{\bar{a}}(w_s(A) + \tilde{w}(A)) = r\tilde{w}(A) + c$, which is just enough to cover the interest for the debt and the consumption and thereby ensure that lifetime ruin is impossible. Note that $w_s(A)$ is the minimum required wealth to execute this strategy, so we call it the *safe level* for the case in which we allow wealth w to be negative. If asset and annuity income initially satisfy $w \geq w_s(A)$, then the individual will immediately execute this strategy to guarantee that her probability of lifetime ruin is zero. It follows that

$$\psi(w, A) = 0, \quad (2.3.10)$$

for $w \geq w_s(A)$.

Recall that ruin occurs when $w + (1 - p)\bar{a}A \leq 0$, from which it follows that

$$\psi(w, A) = 1, \quad (2.3.11)$$

for $w \leq \underline{w}(A) \triangleq -(1 - p)\bar{a}A$.

The two boundaries $w_s(A)$ and $\underline{w}(A)$ meet at $A = \frac{c}{1 - (1 - p)r\bar{a}} > 0$ as in Figure

2.1. Thus, it remains to solve the minimum probability of ruin in the region

$$\mathcal{D} \triangleq \left\{ (w, A) : \underline{w}(A) \leq w \leq w_s(A), 0 \leq A < \frac{c}{1 - (1 - p)r\bar{a}} \right\}.$$

2.3.2 Verification theorem

The discussion in Section 3.1 motivates the following verification theorem:

Theorem 2.3.1. *For any $\pi \in \mathbb{R}$, define the functional operator \mathcal{L}^π through its action on a test function f by*

$$\mathcal{L}^\pi f = [rw + (\mu - r)\pi - c + A]f_w + \frac{1}{2}\sigma^2\pi^2 f_{ww} - \lambda^S f, \quad (2.3.12)$$

Let $v = v(w, A)$ be a non-increasing, non-negative, convex function of w that is twice-differentiable with respect to w , except possibly at $w = w_s(A)$ where we assume that

it has right- and left-derivatives, and that is differentiable with respect to A . Suppose v satisfies the following conditions on \mathcal{D} :

1. $\mathcal{L}^\pi v(w, A) \geq 0$ for any $\pi \in \mathbb{R}$.
2. $\bar{a}v_w(w, A) - v_A(w, A) \leq 0$.
3. $(1 - p)\bar{a}v_w(w, A) - v_A(w, A) \geq 0$.
4. $v(\underline{w}(A), A) = 1$, where $\underline{w}(A)$ is the lower boundary of wealth for the problem.

Then,

$$v(w, A) \leq \psi(w, A), \quad (2.3.13)$$

on \mathcal{D} .

Proof. Suppose $\{\pi_t\}$ is an admissible investment strategy, and define $\tau_n \triangleq \{t \geq 0 : \int_0^t \pi_t^2 dt \geq n\}$ and $\tau \triangleq \tau_0 \wedge \tau_n$, which is a stopping time with respect to the filtration \mathbb{F} . Then, by using Itô's formula for semi-martingales, we can write

$$\begin{aligned} e^{-\lambda S_\tau} v(W_\tau, A_\tau) &= v(w, A) + \int_0^\tau e^{-\lambda S_t} v_w(W_t, A_t) \sigma \pi_t dB_t + \int_0^\tau e^{-\lambda S_t} \mathcal{L}^{\pi_t} v(W_t, A_t) dt \\ &\quad + \int_0^\tau e^{-\lambda S_t} [v_A(W_t, A_t) - \bar{a}v_w(W_t, A_t)] d(A_t^+)^{(c)} \\ &\quad + \int_0^\tau e^{-\lambda S_t} [(1 - p)\bar{a}v_w(W_t, A_t) - v_A(W_t, A_t)] d(A_t^-)^{(c)} \\ &\quad + \sum_{0 \leq t \leq \tau} e^{-\lambda S_t} [v(W_t, A_t) - v(W_{t-}, A_{t-})]. \end{aligned} \quad (2.3.14)$$

Here, $(A^\pm)^{(c)}$ is the continuous part of A^\pm ; that is,

$$(A_t^\pm)^{(c)} \triangleq A_t^\pm - \sum_{0 \leq s \leq t} (A_s^\pm - A_{s-}^\pm). \quad (2.3.15)$$

Since v is non-increasing and convex in w , $v_w^2(w, A) \leq v_w^2(\underline{w}(A), A)$ for $w \geq \underline{w}(A)$.

Therefore,

$$\mathbb{E} \left[\int_0^\tau e^{-2\lambda S_t} v_w^2(W_t, A_t) \sigma^2 \pi_t^2 dt \mid W_0 = w, A_0 = A \right] < \infty, \quad (2.3.16)$$

which implies that

$$\mathbb{E} \left[\int_0^\tau e^{-\lambda^S t} v_w(W_t, A_t) \sigma \pi_t dB_t \middle| W_0 = w, A_0 = A \right] = 0. \quad (2.3.17)$$

By taking expectations of equation (2.3.14), as well as using (2.3.17) and Conditions 1, 2, and 3 in the statement of the theorem, we obtain

$$\mathbb{E} \left[e^{-\lambda^S \tau} v(W_\tau, A_\tau) \middle| W_t = w, A_t = A \right] \geq v(w, A). \quad (2.3.18)$$

In deriving (2.3.18), we also use the fact that

$$\sum_{0 \leq t \leq \tau} e^{-\lambda^S t} [v(W_t, A_t) - v(W_{t-}, A_{t-})] \geq 0, \quad (2.3.19)$$

because Assumptions 2 and 3 imply that v is non-decreasing in the direction of purchase and surrender.

Since $\tau_n \nearrow \infty$ and v is bounded, applying the dominated convergence theorem to (2.3.18) yields

$$\mathbb{E} \left[e^{-\lambda^S \tau_0} v(W_{\tau_0}, A_{\tau_0}) \middle| W_0 = w, A_0 = A, \tau_d > 0, \tau_0 > 0 \right] \geq v(w, A). \quad (2.3.20)$$

By using Assumption 4, one can rewrite (2.3.20) as

$$v(w, A) \leq \mathbb{E} \left[e^{-\lambda^S \tau_0} \middle| W_0 = w, A_0 = A, \tau_d > 0, \tau_0 > 0 \right]. \quad (2.3.21)$$

From this expression and from (2.2.5), we infer that

$$\begin{aligned} v(w, A) &\leq \inf_{\{\pi_t, A_t\}} \mathbb{E} \left[e^{-\lambda^S \tau_0} \middle| W_0 = w, A_0 = A, \tau_d > 0, \tau_0 > 0 \right] \\ &= \psi(w, A). \end{aligned} \quad (2.3.22)$$

□

We will use the following corollary of Theorem 2.3.1 to determine ψ , the minimum probability of ruin, along with an optimal investment and annuitization strategy.

Corollary 2.3.1. *Suppose v satisfies the conditions in Theorem 2.3.1 and additionally is the probability of ruin associated with an admissible strategy, then $v = \psi$ on D and the associated strategy is optimal.*

2.3.3 Linearizing the equation for ψ via duality arguments

We hypothesize that in the region $\mathcal{D} \setminus \{w = w_s(A) \text{ or } w = \underline{w}(A)\}$ as defined in Section 2.3.1, the optimal strategy for minimizing the probability of ruin is neither to purchase nor to surrender any life annuity income. In other words, the individual does not buy any additional annuity income until her wealth reaches the safe level $w_s(A)$, which is consistent with the results of Milevsky et al. [2006a]. Additionally, the individual never surrenders her annuity income. Intuitively, this makes sense because we count the annuity income's wealth equivalence in the ruin level $\underline{w}(A)$ and thereby allow the individual to borrow against future annuity income without actually forcing her to surrender the annuity.

Under this hypothesis, the first inequality in the HJB variational inequality (2.3.9) holds with equality in the region $\mathcal{D} \setminus \{w = w_s(A) \text{ or } w = \underline{w}(A)\}$, and the minimum probability of ruin ψ is the solution to the following boundary-value problem (BVP)

$$\lambda^S \psi = (rw - c + A)\psi_w + \min_{\pi} \left[(\mu - r)\pi\psi_w + \frac{1}{2}\sigma^2\pi^2\psi_{ww} \right], \quad (2.3.23)$$

with the boundary conditions

$$\psi(\underline{w}(A), A) = 1, \quad (2.3.24)$$

and

$$\psi(w_s(A), A) = 0. \quad (2.3.25)$$

After solving this BVP, we will show that its solution satisfies the conditions of the Verification Theorem 2.3.1 to verify our hypothesis.

To solve the BVP, we transform the nonlinear boundary value problem above into a linear free-boundary problem (FBP) via the Legendre transform. Assume $\psi(w, A)$ is convex with respect to w , which we verify later; therefore, we can define the concave dual $\hat{\psi}$ of ψ by

$$\hat{\psi}(y, A) = \min_{w \geq \underline{w}(A)} [\psi(w, A) + wy]. \quad (2.3.26)$$

The critical value $w^*(A)$ solves the equation $\psi_w(w, A) + y = 0$; thus, $w^* = I(-y, A)$, in which I is the inverse of ψ_w with respect to w . It follows that

$$\hat{\psi}_y(y, A) = I(-y, A), \quad (2.3.27)$$

$$\hat{\psi}_{yy}(y, A) = -\frac{1}{\psi_{ww}(w, A)} \Big|_{w=\psi_w^{-1}(-y, A)}, \quad (2.3.28)$$

$$\hat{\psi}_A(y, A) = \psi_A(w, A) \Big|_{w=\psi_w^{-1}(-y, A)}, \quad (2.3.29)$$

and

$$\hat{\psi}_{Ay}(y, A) = \psi_{Aw}(w, A) \hat{\psi}_{yy}(y, A) \Big|_{w=\psi_w^{-1}(-y, A)}. \quad (2.3.30)$$

Rewrite the differential equation (2.3.23) in terms of $\hat{\psi}$ to get

$$-\lambda^S \hat{\psi} - (r - \lambda^S) y \hat{\psi}_y + m y^2 \hat{\psi}_{yy} + y(c - A) = 0, \quad (2.3.31)$$

in which $m = \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2$. The general solution of (2.3.31) is

$$\hat{\psi}(y, A) = D_1(A) y^{B_1} + D_2(A) y^{B_2} + \frac{c - A}{r} y, \quad (2.3.32)$$

in which

$$B_{1,2} = \frac{1}{2m} \left((r - \lambda^S + m) \pm \sqrt{(r - \lambda^S + m)^2 + 4m\lambda^S} \right), \quad (2.3.33)$$

with $B_1 > 1$ and $B_2 < 0$. It remains for us to determine the coefficients $D_1(A)$ and $D_2(A)$ via the two boundary conditions.

To that end, consider the boundary conditions (2.3.24) and (2.3.25). Define

$$\underline{y}(A) = -\psi_w(\underline{w}(A), A), \quad (2.3.34)$$

and

$$y_s(A) = -\psi_w(w_s(A), A). \quad (2.3.35)$$

We will show later that $y_s(A) \leq \underline{y}(A)$, which is obvious if ψ is decreasing and convex with respect to w . Then, for the free boundaries $\underline{y}(A)$ and $y_s(A)$, we obtain from (2.3.24) and (2.3.34)

$$\begin{cases} \hat{\psi}(\underline{y}(A), A) = \psi(\underline{w}(A), A) + \underline{w}(A)\underline{y}(A) = 1 - (1-p)\bar{a}A\underline{y}(A), \\ \hat{\psi}_y(\underline{y}(A), A) = \underline{w}(A) = -(1-p)\bar{a}A; \end{cases} \quad (2.3.36)$$

and from (2.3.25) and (2.3.35)

$$\begin{cases} \hat{\psi}(y_s(A), A) = \psi(w_s(A), A) + w_s(A)y_s(A) = \left(\frac{pc}{\frac{1}{\bar{a}} - (1-p)r} - \bar{a}A \right) y_s(A), \\ \hat{\psi}_y(y_s(A), A) = w_s(A) = \frac{pc}{\frac{1}{\bar{a}} - (1-p)r} - \bar{a}A. \end{cases} \quad (2.3.37)$$

Next, we find $D_1(A)$ and $D_2(A)$ along with $\underline{y}(A)$ and $y_s(A)$. To do so, we use the four equations in (2.3.36) and (2.3.37) to find these four unknowns in terms of A . Substitute (2.3.32) into (2.3.36) and (2.3.37) to get

$$D_1(A)\underline{y}(A)^{B_1} + D_2(A)\underline{y}(A)^{B_2} + \frac{c-A}{r}\underline{y}(A) = 1 - \frac{1-p}{r+\lambda^O}A\underline{y}(A), \quad (2.3.38)$$

$$D_1(A)B_1\underline{y}(A)^{B_1-1} + D_2(A)B_2\underline{y}(A)^{B_2-1} + \frac{c-A}{r} = -\frac{1-p}{r+\lambda^O}A, \quad (2.3.39)$$

$$D_1(A)y_s(A)^{B_1} + D_2(A)y_s(A)^{B_2} + \frac{c-A}{r}y_s(A) = \left(\frac{pc}{pr+\lambda^O} - \frac{A}{r+\lambda^O} \right) y_s(A), \quad (2.3.40)$$

and

$$D_1(A)B_1y_s(A)^{B_1-1} + D_2(A)B_2y_s(A)^{B_2-1} + \frac{c-A}{r} = \frac{pc}{pr+\lambda^O} - \frac{A}{r+\lambda^O}. \quad (2.3.41)$$

From (2.3.40) and (2.3.41), solve for $D_1(A)$ and $D_2(A)$ to obtain

$$D_1(A) = \frac{1-B_2}{B_1-B_2} \frac{1}{y_s(A)^{B_1-1}} \left(\frac{pc}{pr+\lambda^O} - \frac{A}{r+\lambda^O} - \frac{c-A}{r} \right) < 0, \quad (2.3.42)$$

$$D_2(A) = \frac{B_1-1}{B_1-B_2} \frac{1}{y_s(A)^{B_2-1}} \left(\frac{pc}{pr+\lambda^O} - \frac{A}{r+\lambda^O} - \frac{c-A}{r} \right) < 0. \quad (2.3.43)$$

Substituting $D_1(A)$ and $D_2(A)$ into (2.3.39) gives

$$\begin{aligned} -\frac{1-p}{r+\lambda^O}A &= \frac{B_1(1-B_2)}{B_1-B_2} x(A)^{B_1-1} \left(\frac{pc}{pr+\lambda^O} - \frac{A}{r+\lambda^O} - \frac{c-A}{r} \right) \\ &\quad + \frac{B_2(B_1-1)}{B_1-B_2} x(A)^{B_2-1} \left(\frac{pc}{pr+\lambda^O} - \frac{A}{r+\lambda^O} - \frac{c-A}{r} \right) + \frac{c-A}{r} \end{aligned} \quad (2.3.44)$$

with $x(A) \triangleq \underline{y}(A)/y_s(A)$ as a function of A .

With A fixed, (i) if $x(A) = 1$, the right-hand side of (2.3.44) equals $\frac{pc}{pr+\lambda^O} - \frac{A}{r+\lambda^O} = w_s(A) > -\frac{1-p}{r+\lambda^O}A = \underline{w}(A)$; (ii) if $x(A) \rightarrow +\infty$, then the right-hand side of (2.3.44) approaches $-\infty$; and (iii) one can show that the right-hand side is strictly decreasing with respect to $x(A)$. Therefore, there exists a unique $x(A) > 1$ that satisfies equation (2.3.44).

Substitute for $D_1(A)$ and $D_2(A)$ into (2.3.38) to get

$$\begin{aligned} \frac{1}{\underline{y}(A)} &= \frac{c-A}{r} + \frac{1-p}{r+\lambda^O}A + \left(\frac{pc}{pr+\lambda^O} - \frac{A}{r+\lambda^O} - \frac{c-A}{r} \right) \\ &\quad \cdot \left(\frac{1-B_2}{B_1-B_2} x(A)^{B_1-1} + \frac{B_1-1}{B_1-B_2} x(A)^{B_2-1} \right). \end{aligned} \quad (2.3.45)$$

Then, by the definition of $x(A)$, the solution for $y_s(A)$ is simply

$$y_s(A) = \frac{y(A)}{x(A)}. \quad (2.3.46)$$

Thus, we have solved the FBP given in (2.3.31), (2.3.36), and (2.3.37), and we state this formally in the following proposition.

Proposition 2.3.1. *The solution of the FBP (2.3.31) with conditions (2.3.36) and (2.3.37) is given by (2.3.32), with $D_1(A)$, $D_2(A)$, $\underline{y}(A)$, $y_s(A)$, and $x(A)$ defined in (2.3.42), (2.3.43), (2.3.45), (2.3.46), and (2.3.44), respectively.*

Next, we determine some properties of $\hat{\psi}(y, A)$; in particular, we show that it is concave. Also, notice that we can rewrite the inequalities (2.3.7) and (2.3.8) in terms of $\hat{\psi}$ as

$$\hat{\psi}_A(y, A) \geq -\frac{1}{r + \lambda^O} y, \quad (2.3.47)$$

$$\hat{\psi}_A(y, A) \leq -\frac{1-p}{r + \lambda^O} y, \quad (2.3.48)$$

for $y_s(A) \leq y \leq \underline{y}(A)$, and we show below that these inequalities hold for our solution $\hat{\psi}$.

For notational simplicity, we drop the argument A in $\underline{w}(A)$, $w_s(A)$, $\underline{y}(A)$, and $y_s(A)$ in much of the remainder of this subsection. By taking the derivative of (2.3.44) with respect to A , we get

$$\begin{aligned} & \left(w_s - \frac{c-A}{r} \right) \frac{(B_1-1)(1-B_2)}{B_1-B_2} \{ B_1 x(A)^{B_1-1} - B_2 x(A)^{B_2-1} \} \frac{dx(A)/dA}{x(A)} \\ &= \left(\frac{1}{r} - \frac{1-p}{r + \lambda^O} \right) - \left(\frac{1}{r} - \frac{1}{r + \lambda^O} \right) \frac{w - \frac{c-A}{r}}{w_s - \frac{c-A}{r}}. \end{aligned} \quad (2.3.49)$$

It is easy to check that the right-hand side of the equation above is 0, which implies that

$$\frac{dx(A)}{dA} = 0. \quad (2.3.50)$$

In other words, $x(A) = x$ is a constant, independent of A , and the equation (2.3.44) holds for all A with the same value $x > 1$.

By taking the derivative of (2.3.45) and (2.3.46) with respect to A , we get

$$\frac{dy_s(A)}{dA} = -y_s(A) \frac{\lambda^O}{r(r + \lambda^O)} \frac{1}{w_s(A) - \frac{c - A}{r}}. \quad (2.3.51)$$

Also, after substituting for $D_1(A)$ and $D_2(A)$ in (2.3.32), we differentiate $\hat{\psi}(y, A)$ with respect to A to get

$$\begin{aligned} \hat{\psi}_A(y, A) = & -y \left\{ \left(\frac{1}{r + \lambda^O} - \frac{1}{r} \right) \left[\frac{1 - B_2}{B_1 - B_2} \left(\frac{y}{y_s} \right)^{B_1 - 1} + \frac{B_1 - 1}{B_1 - B_2} \left(\frac{y}{y_s} \right)^{B_2 - 1} \right] + \frac{1}{r} \right\} \\ & - \frac{dy_s(A)}{dA} \frac{(B_1 - 1)(1 - B_2)}{B_1 - B_2} \left(w_s(A) - \frac{c - A}{r} \right) \left[\left(\frac{y}{y_s} \right)^{B_1} - \left(\frac{y}{y_s} \right)^{B_2} \right]. \end{aligned} \quad (2.3.52)$$

Proposition 2.3.2. $\hat{\psi}(y, A)$ given by Proposition 2.3.1 is concave with respect to y and satisfies inequalities (2.3.47) and (2.3.48).

Proof. First, it is straightforward to show the positivity of $\underline{y}(A)$ from (2.3.45). This confirms that $\underline{y}(A) = y_s(A)x > y_s(A) > 0$ because $x > 1$. It follows that $\hat{\psi}(y, A)$ is concave with respect to y since both $D_1(A) < 0$ and $D_2(A) < 0$, and both $B_1(B_1 - 1) > 0$ and $B_2(B_2 - 1) > 0$.

To prove the inequalities, we first substitute (2.3.51) into (2.3.52) to get

$$\hat{\psi}_A(y, A) = y \frac{\lambda^O}{r(r + \lambda^O)} \left[\frac{B_1(1 - B_2)}{B_1 - B_2} \left(\frac{y}{y_s} \right)^{B_1 - 1} + \frac{(B_1 - 1)B_2}{B_1 - B_2} \left(\frac{y}{y_s} \right)^{B_2 - 1} \right] - \frac{y}{r}, \quad (2.3.53)$$

Substitute the expression for $\hat{\psi}_A(w, A)$ from (2.3.53) into inequalities (2.3.47) and (2.3.48) to obtain the equivalent inequalities

$$1 \geq \frac{r + \lambda^O}{r} - \frac{\lambda^O}{r} \left[\frac{B_1(1 - B_2)}{B_1 - B_2} \left(\frac{y}{y_s} \right)^{B_1 - 1} + \frac{(B_1 - 1)B_2}{B_1 - B_2} \left(\frac{y}{y_s} \right)^{B_2 - 1} \right] \geq 1 - p. \quad (2.3.54)$$

Notice that the first inequality holds with equality if $y = y_s(A)$ and the second inequality holds with equality if $y = \underline{y}(A)$. Define the auxiliary function

$$f(z) = \frac{B_1(1 - B_2)}{B_1 - B_2} z^{B_1-1} + \frac{(B_1 - 1)B_2}{B_1 - B_2} z^{B_2-1}, \quad (2.3.55)$$

which is increasing for $1 = \frac{y_s(A)}{y_s(A)} \leq z \leq \frac{\underline{y}(A)}{y_s(A)} = x$. Indeed, in this interval,

$$f'(z) = \frac{(B_1 - 1)(1 - B_2)}{B_1 - B_2} [B_1 z^{B_1-2} - B_2 z^{B_2-2}] > 0. \quad (2.3.56)$$

It follows that, for $y_s(A) \leq y \leq \underline{y}(A)$, the inequality (2.3.54), and equivalently (2.3.47) and (2.3.48) hold. \square

In the next section, we rely on the work in this section to show that the convex dual of $\hat{\psi}(y, A)$ equals the minimum probability of ruin $\psi(w, A)$.

2.3.4 Relation between the FBP and the minimum probability of ruin

In this section, we show that the Legendre transform of the solution to the FBP given in (2.3.31), (2.3.36), and (2.3.37) is in fact the minimum probability of ruin ψ . Since $\hat{\psi}$ is concave from Propostion 2.3.2, we can define its convex dual via the Legendre transform for $w \geq \underline{w}(A)$ as

$$\Psi(w, A) = \max_{y \geq 0} [\hat{\psi}(y, A) - wy]. \quad (2.3.57)$$

Given A , the critical value y^* solves the equation $\hat{\psi}_y(y, A) - w = 0$. Thus $y^*(A) = I(w, A)$, in which I is the inverse of $\hat{\psi}_y$. In this case, we also have expressions similar to those in (2.3.27)-(2.3.30).

Given $\hat{\psi}$, we proceed to find the boundary-value problem that Ψ solves. In the partial differential equation for $\hat{\psi}$ in (2.3.31), let $y = I(w, A) = -\Psi_w(w, A)$ to obtain

$$\lambda^S \Psi(w, A) = (rw - c) \Psi_w(w, A) - m \frac{\Psi_w^2(w, A)}{\Psi_{ww}(w, A)}. \quad (2.3.58)$$

Notice that we can rewrite (2.3.58) as

$$\min_{\pi} \mathcal{L}^{\pi} \Psi = 0, \quad (2.3.59)$$

with the minimizing strategy π^* given in feedback form by

$$\pi^*(w, A) = -\frac{\mu - r}{\sigma^2} \frac{\Psi_w(w, A)}{\Psi_{ww}(w, A)}. \quad (2.3.60)$$

Therefore, Ψ satisfies Condition 1 in Verification Theorem 2.3.1.

Next, consider the boundary conditions for (2.3.31). First, the boundary conditions at $y_s(A)$, namely $\hat{\psi}(y_s(A), A) = w_s(A)y_s(A)$ and $\hat{\psi}_y(y_s(A), A) = w_s(A)$, imply that the corresponding dual value of w is $w_s(A)$ and that

$$\Psi(w_s(A), A) = 0. \quad (2.3.61)$$

Similarly, the boundary conditions at $\underline{y}(A)$, namely $\hat{\psi}(\underline{y}(A), A) = 1 + \underline{w}(A)\underline{y}(A)$ and $\hat{\psi}_y(\underline{y}(A), A) = \underline{w}(A)$, imply that the corresponding dual value of w is $\underline{w}(A)$ and that

$$\Psi(\underline{w}(A), A) = 1. \quad (2.3.62)$$

Finally, Propostion 2.3.2 implies that

$$\bar{a}\Psi_w(w, A) - \Psi_A(w, A) \leq 0, \quad (2.3.63)$$

and

$$(1 - p)\bar{a}\Psi_w(w, A) - \Psi_A(w, A) \geq 0. \quad (2.3.64)$$

Therefore, $\Psi(w, A)$ satisfies Conditions 2 and 3 in Theorem 2.3.1.

From $\Psi_w(w, A) = -y^*(A)$ and the fact that $y \geq y_s(A) > 0$, $\Psi(w, A)$ is decreasing with respect to w , and consequently $0 \leq \Psi(w, A) \leq 1$ for $(w, A) \in \mathcal{D}$ due to (2.3.61) and (2.3.62). Thus, Ψ is the minimum probability of ruin by Corollary 2.3.1, and we state this formally in the next theorem.

Theorem 2.3.2. *The minimum probability of ruin $\psi(w, A)$ for $(w, A) \in \mathcal{D}$, in which \mathcal{D} is defined by $\mathcal{D} = \left\{ (w, A) : \underline{w}(A) \leq w \leq w_s(A), 0 \leq A < \frac{c}{1 - (1-p)\frac{r}{r+\lambda\sigma}} \right\}$ equals $\Psi(w, A)$ in (2.3.57). The associated optimal annuitization and investment strategies are given by*

1. *never to surrender existing annuity income;*
2. *to purchase additional annuity income only when wealth reaches $w_s(A)$, the safe level;*
3. *for $w \in \mathcal{D} \setminus \{w = w_s(A) \text{ or } w = \underline{w}(A)\}$, to invest the following amount of wealth in the risky asset:*

$$\pi^*(w, A) = -\frac{\mu - r}{\sigma^2} \frac{\psi_w(w, A)}{\psi_{ww}(w, A)}.$$

2.3.5 Numerical examples

In this section, we present numerical examples to demonstrate the results of Section 2.3.4. We calculate the probabilities of lifetime ruin $\psi(w, A)$, as well as the associated investment strategies $\pi^*(w, A)$ for different values of the existing annuity income A and the surrender charge p . We use the following values of the parameters for our calculation:

- $\lambda^S = \lambda^O = 0.04$; the hazard rate is such that the expected future lifetime is 25 years.
- $r = 0.02$; the riskless rate of return is 2% over inflation.
- $\mu = 0.06$; the drift of the risky asset is 6% over inflation.
- $\sigma = 0.20$; the volatility of the risky asset is 20%.
- $c = 1$; the individual consumes one unit of wealth per year.

We focus on how the surrender penalty affects the probability of ruin and the optimal investment strategy.

Figures 2.2-2.5 show the ruin probability $\psi(w, A)$ and the associated optimal investment $\pi^*(w, A)$ in the risky asset with the parameters described above, as well as with values for A and p as indicated in the figures. Each curve gives values from $w = \underline{w}(A)$ to $w = w_s(A)$, in which $\underline{w}(A)$ and $w_s(A)$ vary with respect to A and p . This is the reason why each curve lies in a distinct domain. From the figures, we can see that the proved properties are verified in these examples: the probability of ruin is decreasing and convex with respect to w . We also observe that investment in the risky asset increases as wealth increases for each case.

2.4 Borrowing restriction

In this section, we consider the case in which the individual is forced to keep her wealth non-negative by surrendering the life annuity when needed. With this restriction, the situation is different from the one we studied in the previous section because in this section, the individual cannot borrow against future life annuity income. It is reasonable to apply this restriction because if the individual were to die, then the annuity income ceases. Therefore, if the individual were to borrow against future annuity income and die, there might be insufficient assets available to pay the debt.

Therefore, ruin occurs when both an individual's annuity income A and wealth w are 0 since she has no more annuity income to surrender to raise her wealth. It follows that τ_0 in this case reduces to the hitting time of $(w, A) = (0, 0)$ because on the line $w = -(1-p)\bar{a}A$, $(0, 0)$ is the only point at which wealth w is non-negative. Notice that the probability of lifetime ruin is not 1 when wealth reaches 0 if an individual

still has existing annuity income, which differs from the case of irreversible annuities.

2.4.1 HJB variational inequality and verification theorem

As the preceding case without a borrowing restriction, we have the same HJB variational inequality because the individual still has only three options to minimize the probability of ruin: purchasing additional annuity income, surrendering existing annuity income, and doing neither. Suboptimality of each strategy, in general, is represented by an inequality, while the optimality of one's executed strategy at all time requires that at least one of the three inequalities holds as an equality.

We need only consider when $A < c$; otherwise the individual already has enough annuity income to cover her consumption and lifetime ruin is impossible. In this case, the safe level is given by $w_s(A) \triangleq (c - A)\bar{a}$. When the individual's wealth reaches the safe level, she is able to purchase $(c - A)$ of additional annuity income and, thereby, ensure that lifetime ruin is impossible. Therefore, we have the condition

$$\psi(w_s(A), A) = 0. \tag{2.4.1}$$

Notice that, for a given existing annuity income A , more wealth is needed to reach the safe level if borrowing against the annuity is not allowed; that is, $w_s(A)$ in this section is greater than $w_s(A)$ in the previous section.

When the individual's wealth reaches 0, she is forced to surrender her life annuity to keep her wealth non-negative. In this case, an annuitization strategy $\{A_t\}$ is admissible if the associated wealth process $W_t \geq 0$ almost surely, for all $t \geq 0$. Inspired by the optimal annuitization strategy obtained in Theorem 2.3.2 for the case in which borrowing is not restricted, we hypothesize that the individual will only surrender enough annuity income to keep wealth non-negative. This means that on the boundary $w = 0$, she executes instantaneous control, so we expect the

following Neumann condition:

$$(1 - p)\bar{a}\psi_w(0, A) = \psi_A(0, A). \quad (2.4.2)$$

Moreover, if both her wealth and annuity income are 0, ruin occurs; that is,

$$\psi(0, 0) = 1. \quad (2.4.3)$$

Therefore, we need to solve for $\psi(w, A)$ in the region $\mathcal{D} \triangleq \{(w, A) : 0 \leq w \leq w_s(A), 0 \leq A < c\}$. Notice that the safe level $w_s(A) = (c - A)\bar{a}$ is different from the previous case. With \mathcal{D} thus redefined, we obtain the same verification theorem and corollary as Theorem 2.3.1 and Corollary 2.3.1, respectively. Please refer to the previous section for details.

2.4.2 Solving for ψ via duality arguments

Through the course of our study, we determined that the optimal annuitization strategy for the individual to minimize her probability of lifetime ruin depends on the value of p . We will show that when the penalty for surrendering is greater than p^* , a critical value to be determined later, the individual will not purchase any annuity until her wealth reaches the safe level $w_s(A)$, at which point she buys annuity income to cover the shortfall $c - A$. On the other hand, if the penalty is low enough, namely $p < p^*$, the individual has incentive to annuitize partially; that is, the individual purchases additional annuity to cover part of the shortfall $c - A$ when her wealth is strictly below the safe level. In this case, the individual will keep some wealth to invest in the risky financial market and spend the surplus to purchase annuity income. We solve for the minimum probability of lifetime ruin ψ for the first case $p \geq p^*$ in Section 2.4.2 and for the second case $p < p^*$ in Section 2.4.2. We also obtain the corresponding optimal annuitization and investment strategies.

$p \geq p^*$

When $p \geq p^*$, we hypothesize that in the domain $\mathcal{D} \setminus \{w = w_s(A) \text{ or } w = 0\}$, the optimal strategy for minimizing the probability of ruin is neither to purchase nor to surrender any annuity income. Under this hypothesis, the first inequality in the HJB variational inequality (2.3.9) holds with equality, and the minimum probability of ruin ψ is the solution to the following BVP

$$\lambda^S \psi = (rw - c + A)\psi_w + \min_{\pi} \left[(\mu - r)\pi\psi_w + \frac{1}{2}\sigma^2\pi^2\psi_{ww} \right], \quad (2.4.4)$$

with boundary conditions

$$\psi(w_s(A), A) = 0, \quad (2.4.5)$$

$$(1 - p)\bar{a}\psi_w(0, A) = \psi_A(0, A), \quad (2.4.6)$$

and

$$\psi(0, 0) = 1. \quad (2.4.7)$$

After solving the BVP, we will show that its solution satisfies the conditions of the Verification Theorem 2.3.1 to verify our hypothesis.

As in Section 2.3.3, we can define a related linear free-boundary problem via the Legendre transform. Specifically, for $(w, A) \in \mathcal{D}$, define

$$\hat{\psi}(y, A) = \min_{w \geq 0} [\psi(w, A) + wy]. \quad (2.4.8)$$

We can rewrite (2.4.4) as

$$-\lambda^S \hat{\psi} - (r - \lambda^S)y\hat{\psi}_y + my^2\hat{\psi}_{yy} + y(c - A) = 0. \quad (2.4.9)$$

Its general solution is

$$\hat{\psi}(y, A) = D_1(A)y^{B_1} + D_2(A)y^{B_2} + \frac{c - A}{r}y, \quad (2.4.10)$$

with $B_1 > 1$ and $B_2 < 0$ defined in (2.3.33).

Define

$$y_0(A) = -\psi_w(0, A), \quad (2.4.11)$$

and

$$y_s(A) = -\psi_w(w_s(A), A). \quad (2.4.12)$$

We get from (2.4.6) and (2.4.11) that

$$\begin{cases} \hat{\psi}_A(y_0(A), A) = -(1-p)\bar{a}y_0(A), \\ \hat{\psi}_y(y_0(A), A) = 0; \end{cases} \quad (2.4.13)$$

from (2.4.5) and (2.4.12) that

$$\begin{cases} \hat{\psi}(y_s(A), A) = (c-A)\bar{a}y_s(A), \\ \hat{\psi}_y(y_s(A), A) = (c-A)\bar{a}; \end{cases} \quad (2.4.14)$$

and from (2.4.7) and (2.4.11) that

$$\hat{\psi}(y_0(0), 0) = 1. \quad (2.4.15)$$

Next, we determine $D_1(A)$ and $D_2(A)$ along with $y_0(A)$ and $y_s(A)$. Rewrite (2.4.13), (2.4.14), and (2.4.15) using (2.4.10) to get

$$D_1(A)B_1y_0(A)^{B_1-1} + D_2(A)B_2y_0(A)^{B_2-1} + \frac{c-A}{r} = 0, \quad (2.4.16)$$

$$D_1'(A)y_0(A)^{B_1-1} + D_2'(A)y_0(A)^{B_2-1} = \frac{1}{r} - \frac{1-p}{r+\lambda^O}, \quad (2.4.17)$$

$$D_1(A)B_1y_s(A)^{B_1-1} + D_2(A)B_2y_s(A)^{B_2-1} = \frac{c-A}{r+\lambda^O} - \frac{c-A}{r}, \quad (2.4.18)$$

$$D_1(A)y_s(A)^{B_1-1} + D_2(A)y_s(A)^{B_2-1} = \frac{c-A}{r+\lambda^O} - \frac{c-A}{r}, \quad (2.4.19)$$

$$D_1(0)y_0(0)^{B_1} + D_2(0)y_0(0)^{B_2} + \frac{c}{r}y_0(0) = 1. \quad (2.4.20)$$

From (2.4.18) and (2.4.19), we get

$$D_1(A) = -\frac{1-B_2}{B_1-B_2} \frac{\lambda^O}{r(r+\lambda^O)} (c-A) \frac{1}{y_s(A)^{B_1-1}} < 0, \quad (2.4.21)$$

$$D_2(A) = -\frac{B_1-1}{B_1-B_2} \frac{\lambda^O}{r(r+\lambda^O)} (c-A) \frac{1}{y_s(A)^{B_2-1}} < 0. \quad (2.4.22)$$

Then, substitute $D_1(A)$ and $D_2(A)$ into (2.4.16) to get

$$\frac{\lambda^O}{r+\lambda^O} \left[\frac{B_1(1-B_2)}{B_1-B_2} \left(\frac{y_0(A)}{y_s(A)} \right)^{B_1-1} + \frac{B_2(B_1-1)}{B_1-B_2} \left(\frac{y_0(A)}{y_s(A)} \right)^{B_2-1} \right] = 1. \quad (2.4.23)$$

It is clear that $\frac{y_0(A)}{y_s(A)}$ is independent of A , and one can show that it is greater than 1 through an argument similar to the one following (2.3.44). So, we define the constant

$$x \triangleq \frac{y_0(A)}{y_s(A)}. \quad (2.4.24)$$

Now, differentiate (2.4.21) and (2.4.22) with respect to A and substitute into (2.4.17)

to get

$$\begin{aligned} & \frac{dy_s(A)}{dA} (c-A) \frac{\lambda^O}{r(r+\lambda^O)} \frac{(B_1-1)(1-B_2)}{B_1-B_2} (x^{B_1} - x^{B_2}) \\ &= -\frac{1-p}{r+\lambda^O} x y_s(A) - y_s(A) \left\{ \frac{\lambda^O}{r(r+\lambda^O)} \left[\frac{1-B_2}{B_1-B_2} x^{B_1} + \frac{B_1-1}{B_1-B_2} x^{B_2} \right] - \frac{x}{r} \right\}. \end{aligned} \quad (2.4.25)$$

Solve (2.4.23) for x^{B_2-1} to simplify (2.4.25) and obtain

$$\frac{1}{y_s(A)} \frac{dy_s(A)}{dA} = \frac{K}{c-A}, \quad (2.4.26)$$

in which

$$K = \frac{-\frac{B_2}{1-B_2} \frac{1-p}{r+\lambda^O} + \frac{\lambda^O}{r(r+\lambda^O)} x^{B_1-1} - \frac{1}{r}}{\frac{\lambda^O}{r(r+\lambda^O)} x^{B_1-1} - \frac{1}{r}} \quad (2.4.27)$$

Define the critical value p^* as follows:

$$p^* \triangleq \frac{1}{B_2} - \frac{1-B_2}{B_2} \frac{\lambda^O}{r} (x^{B_1-1} - 1). \quad (2.4.28)$$

It is straightforward to show that $K \geq 0$ iff $p \geq p^*$. As we mentioned, we only consider the case $p \geq p^*$ here and leave the discussion for $p < p^*$ in Section 2.4.2.

The expressions in (2.4.24) and (2.4.26) imply that

$$y_0(A) = \left(\frac{c}{c-A} \right)^K y_0(0), \quad (2.4.29)$$

and

$$y_s(A) = \frac{y_0(A)}{x}. \quad (2.4.30)$$

We determine the value of $y_0(0)$ by substituting (2.4.21) and (2.4.22) into (2.4.20):

$$\frac{1}{y_0(0)} = \frac{c}{r} \left[1 - \frac{\lambda^O}{r + \lambda^O} \frac{1 - B_2}{B_1 - B_2} x^{B_1-1} - \frac{\lambda^O}{r + \lambda^O} \frac{B_1 - 1}{B_1 - B_2} x^{B_2-1} \right]. \quad (2.4.31)$$

By solving for x^{B_2-1} from (2.4.23) and substituting it into (2.4.31), we get

$$\frac{1}{y_0(0)} = \frac{c}{r} \left(-\frac{1 - B_2}{B_2} \right) \left(1 - \frac{\lambda^O}{r + \lambda^O} x^{B_1-1} \right) > 0. \quad (2.4.32)$$

The inequality in (2.4.32) holds because $x^{B_1-1} < (r + \lambda^O) / \lambda^O$, which is straightforward to show from equation (2.4.23) and the fact that the left-hand of that equation is increasing with respect to x . From this inequality, we conclude that both $y_s(A)$ and $y_0(A)$ are positive for $(w, A) \in \mathcal{D}$.

Proposition 2.4.1. *The solution $\hat{\psi}(y, A)$ for the FBP (2.4.9) with conditions (2.4.13), (2.4.14), and (2.4.15) is given by (2.4.10), with $D_1(A)$, $D_2(A)$, $y_0(0)$, $y_0(A)$, $y_s(A)$, x , and K defined in (2.4.21), (2.4.22), (2.4.31), (2.4.29), (2.4.30), (2.4.23), and (2.4.27), respectively.*

Notice that we can rewrite the inequalities (2.3.7) and (2.3.8) in terms of $\hat{\psi}$ as

$$\hat{\psi}_A(y, A) \geq -\frac{1}{r + \lambda^O} y, \quad (2.4.33)$$

$$\hat{\psi}_A(y, A) \leq -\frac{1-p}{r + \lambda^O} y. \quad (2.4.34)$$

Proposition 2.4.2. $\hat{\psi}(y, A)$ given by Proposition 2.4.1 is concave and satisfies inequalities (2.4.33) and (2.4.34).

Proof. The proof of the concavity of $\hat{\psi}$ with respect to y follows from the observations that both $D_1(A) < 0$ and $D_2(A) < 0$ and that both $B_1(B_1 - 1) > 0$ and $B_2(B_2 - 1) > 0$.

To prove the inequalities, differentiate (2.4.21) and (2.4.22) with respect to A , substitute those expressions into $\hat{\psi}_A(y, A)$, use (2.4.26) to simplify, and obtain

$$\begin{aligned} \hat{\psi}_A(y, A) = & yK \frac{\lambda^O}{r(r + \lambda^O)} \frac{(B_1 - 1)(1 - B_2)}{B_1 - B_2} \left[\left(\frac{y}{y_s(A)} \right)^{B_1 - 1} - \left(\frac{y}{y_s(A)} \right)^{B_2 - 1} \right] \\ & + y \left\{ \frac{\lambda^O}{r(r + \lambda^O)} \left[\frac{1 - B_2}{B_1 - B_2} \left(\frac{y}{y_s(A)} \right)^{B_1 - 1} + \frac{B_1 - 1}{B_1 - B_2} \left(\frac{y}{y_s(A)} \right)^{B_2 - 1} \right] - \frac{1}{r} \right\}. \end{aligned} \quad (2.4.35)$$

Then, rewrite inequalities (2.4.33) and (2.4.34) in the equivalent form as

$$\begin{aligned} 1 \geq & -K \frac{\lambda^O}{r} \frac{(B_1 - 1)(1 - B_2)}{B_1 - B_2} \left[\left(\frac{y}{y_s(A)} \right)^{B_1 - 1} - \left(\frac{y}{y_s(A)} \right)^{B_2 - 1} \right] \\ & - \frac{\lambda^O}{r} \left[\frac{1 - B_2}{B_1 - B_2} \left(\frac{y}{y_s(A)} \right)^{B_1 - 1} + \frac{B_1 - 1}{B_1 - B_2} \left(\frac{y}{y_s(A)} \right)^{B_2 - 1} \right] + \frac{r + \lambda^O}{r} \geq 1 - p. \end{aligned} \quad (2.4.36)$$

To prove (2.4.36), define the function g by

$$g(z) = -K \frac{(B_1 - 1)(1 - B_2)}{B_1 - B_2} [z^{B_1 - 1} - z^{B_2 - 1}] - \left[\frac{1 - B_2}{B_1 - B_2} z^{B_1 - 1} + \frac{B_1 - 1}{B_1 - B_2} z^{B_2 - 1} \right].$$

For $z \geq 1$, g is decreasing because

$$\begin{aligned} g'(z) = & -K \frac{(B_1 - 1)(1 - B_2)}{B_1 - B_2} [(B_1 - 1)z^{B_1 - 2} + (1 - B_2)z^{B_2 - 2}] \\ & - \frac{(B_1 - 1)(1 - B_2)}{B_1 - B_2} [z^{B_1 - 2} - z^{B_2 - 2}] \leq 0. \end{aligned} \quad (2.4.37)$$

Also, the first inequality in (2.4.36) holds with equality when $y = y_s(A)$, and the second inequality holds with equality when $y = y_0(A)$. Therefore, (2.4.36) holds for $y_s(A) \leq y \leq y_0(A)$. \square

Since $\hat{\psi}$ is concave, we can define its convex dual via the Legendre transform:

$$\Psi(w, A) = \max_{y \geq y_s(A)} \left[\hat{\psi}(y, A) - wy \right]. \quad (2.4.38)$$

As in Section 2.3.4, we can prove that Ψ is the minimum probability of ruin ψ , and we have the following theorem.

Theorem 2.4.1. *When $p \geq p^*$ and the borrowing restriction is enforced, the minimum probability of ruin $\psi(w, A)$ for $(w, A) \in \mathcal{D}$, in which \mathcal{D} is defined by $\mathcal{D} = \{(w, A) : 0 \leq w \leq w_s(A), 0 \leq A < c\}$, is given by $\Psi(w, A)$ in (2.4.38). The associated optimal annuitization and investment strategies are given by*

1. *to surrender existing annuity income instantaneously to keep wealth non-negative as needed;*
2. *to purchase additional annuity income only when wealth reaches $w_s(A)$, the safe level;*
3. *for $w \in \mathcal{D} \setminus \{w = w_s(A)\}$, to invest the following amount of wealth in the risky asset:*

$$\pi^*(w, A) = -\frac{\mu - r}{\sigma^2} \frac{\psi_w(w, A)}{\psi_{ww}(w, A)}. \quad (2.4.39)$$

It is clear from Theorem 2.4.1 that the optimal annuitization strategy is independent of the surrender charge p as long as $p \geq p^*$. However, it is not clear how the optimal investment strategy and the minimum probability of ruin vary with p . We investigate this in the next proposition.

Proposition 2.4.3. *$\pi^*(w, A)$ given in (2.4.39) is independent of the surrender charge p , and the probability of ruin $\psi(w, A)$ increases with respect to p .*

Proof. Fix w and A . Given w , the corresponding y is defined by (2.4.38) as

$$w = \hat{\psi}_y(y, A), \quad (2.4.40)$$

which implies that $\psi_w(w, A) = -y$ and $\psi_{ww}(w, A) = -1/\hat{\psi}_{yy}(y, A)$. Thus, we can write the optimal investment amount as

$$\pi^*(w, A) = -\frac{\mu - r}{\sigma^2} y \hat{\psi}_{yy}(y, A). \quad (2.4.41)$$

By substituting (2.4.10), (2.4.21), (2.4.22), (2.4.29), and (2.4.30) into (2.4.40) and (2.4.41), we get the following two expressions, respectively:

$$w = \tilde{D}_1(A) \left[\left(\frac{c-A}{c} \right)^K y \right]^{B_1-1} + \tilde{D}_2(A) \left[\left(\frac{c-A}{c} \right)^K y \right]^{B_2-1} + \frac{c-A}{r}, \quad (2.4.42)$$

and

$$\begin{aligned} \pi^*(w, A) = & -\frac{\mu - r}{\sigma^2} (B_1 - 1) \tilde{D}_1(A) \left[\left(\frac{c-A}{c} \right)^K y \right]^{B_1-1} \\ & -\frac{\mu - r}{\sigma^2} (B_2 - 1) \tilde{D}_2(A) \left[\left(\frac{c-A}{c} \right)^K y \right]^{B_2-1}, \end{aligned} \quad (2.4.43)$$

in which

$$\tilde{D}_1(A) = -\frac{B_1(1-B_2)}{B_1-B_2} \frac{\lambda^O}{r(r+\lambda^O)} (c-A) \left(\frac{x}{y_0(0)} \right)^{B_1-1}, \quad (2.4.44)$$

$$\tilde{D}_2(A) = -\frac{B_2(B_1-1)}{B_1-B_2} \frac{\lambda^O}{r(r+\lambda^O)} (c-A) \left(\frac{x}{y_0(0)} \right)^{B_2-1}. \quad (2.4.45)$$

The numbers x and $y_0(0)$ are independent of p by (2.4.23) and (2.4.31), respectively.

Thus, $\tilde{D}_1(A)$ and $\tilde{D}_2(A)$ are also independent of p . From (2.4.42), we deduce that $z = \left(\frac{c-A}{c} \right)^K y$, which determines $\pi^*(w, A)$ via (2.4.43), does not depend on p .

Indeed, differentiate (2.4.42) with respect to p to obtain

$$\begin{aligned} 0 &= \left[\tilde{D}_1(A)(B_1-1)z^{B_1-2} + \tilde{D}_2(A)(B_2-1)z^{B_2-2} \right] \frac{\partial z}{\partial p} \\ &= \hat{\psi}_{yy}(y, A) \frac{x}{y_0(0)} \frac{\partial z}{\partial p}. \end{aligned} \quad (2.4.46)$$

Because $\hat{\psi}$ is strictly concave with respect to y for $y_s(A) \leq y \leq y_0(A)$, it follows that

$\frac{\partial z}{\partial p} = 0$, from which we deduce that $z = \left(\frac{c-A}{c} \right)^K y$ is independent of p . Therefore,

the optimal investment strategy $\pi^*(w, A)$ does not depend on p .

Next, we show that $\partial\psi(w, A)/\partial p > 0$. To this end, recall from (2.4.38) that

$$\begin{aligned}\psi(w, A) &= \hat{\psi}(y, A) - wy \\ &= \frac{x}{y_0(0)} \left[\tilde{D}_1(A)z^{B_1} + \tilde{D}_2(A)z^{B_2} \right] + \left(\frac{c-A}{r} - w \right) y,\end{aligned}\tag{2.4.47}$$

in which y is given by (2.4.40). Differentiate this expression with respect to p to obtain

$$\begin{aligned}\frac{\partial\psi(w, A)}{\partial p} &= \left(\frac{c-A}{r} - w \right) \frac{\partial y}{\partial p} \propto \frac{\partial y}{\partial p} \\ &= -\ln\left(\frac{c-A}{c}\right) \frac{\partial K}{\partial p} y > 0,\end{aligned}\tag{2.4.48}$$

in which we use the fact that $z = \left(\frac{c-A}{c}\right)^K y$ is independent of p in order to compute $\partial y/\partial p$, and we use the definition of K in (2.4.27) to deduce that $\partial K/\partial p$ is positive. Thus, the probability of ruin $\psi(w, A)$ increases as p increases. \square

Remark 2.4.1. *Proposition 2.4.3 indicates that, when borrowing is restricted and $p \geq p^*$, an individual follows exactly the same investment and annuitization strategies regardless of the value of $p \geq p^*$. The individual makes her decision based on her wealth and existing annuity income only. It is not surprising that for given values of w and A , the probability of ruin is smaller for a smaller p because with a smaller surrender charge p , one receives more wealth when surrendering a given amount of annuity income.*

In this section, we determined the optimal annuitization and investment strategies and the corresponding minimum probability of ruin under the condition $p \geq p^*$. The latter is equivalent to the condition $K \geq 0$, which plays a critical role in the proof of Proposition 2.4.2. If K were negative, then inequality (2.4.33) would not hold for y just above $y_s(A)$. Consequently, $\Psi(w, A)$ would not satisfy Condition 2 in the Verification Theorem 2.3.1. From this, we infer that buying additional annuity

income before reaching the safe level $w = w_s(A)$ might be optimal when $p < p^*$. With this in mind, we proceed to the next section.

$p < p^*$

In this section, we consider the case for which $p < p^*$. Define $\mathcal{D}_1 \triangleq \{(w, A) : 0 \leq w \leq w_b(A), 0 \leq A < c\}$ with $w_b(A) \in [0, w_s(A)]$ to be specified later. Also, define $\mathcal{D}_2 \triangleq \{(w, A) : w_b(A) < w \leq w_s(A), 0 \leq A < c\}$, and note that $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. As in the case for which $p \geq p^*$ in Section 2.4.2, we only need to determine the minimum probability $\psi(w, A)$ for $(w, A) \in \mathcal{D}$.

We hypothesize that the following annuitization strategy is optimal: If $(w, A) \in \mathcal{D}_1 \setminus \{w = 0 \text{ or } w = w_b(A)\}$, the individual neither purchases or surrenders any life annuity income. If $(w, A) \in \mathcal{D}_2$, the individual purchases just enough annuity income to reach the region \mathcal{D}_1 . That is to say, if she starts with $(w, A) \in \mathcal{D}_2$, the optimal strategy is to purchase ΔA of annuity income such that $w - \Delta A / (r + \lambda^O) = w_b(A + \Delta A)$. Thereafter, whenever wealth reaches the barrier $w_b(A)$, she keeps her portfolio of wealth and annuity income (w, A) in the region \mathcal{D}_1 by instantaneously purchasing enough annuity income. On the other hand, when wealth reaches 0, the individual instantaneously surrenders enough annuity income to keep her wealth non-negative as we hypothesized in Section 2.4.2.

Ruin occurs only when $(w, A) = (0, 0)$, at which point one has no existing annuity income to surrender to keep wealth non-negative. Under the hypothesis for the optimal annuitization strategy, we anticipate that the associated minimum probability of ruin ψ satisfies the following boundary-value problem. After we solve this BVP, we will verify our hypothesis via Verification Theorem 2.3.1.

1. For $(w, A) \in \mathcal{D}_1$, $\psi(w, A)$ solves the following BVP:

$$\lambda^S \psi = (rw - c + A)\psi_w + \min_{\pi} \left[(\mu - r)\pi\psi_w + \frac{1}{2}\sigma^2\pi^2\psi_{ww} \right], \quad (2.4.49)$$

with boundary conditions

$$\bar{a}\psi_w(w_b(A), A) = \psi_A(w_b(A), A), \quad (2.4.50)$$

$$(1 - p)\bar{a}\psi_w(0, A) = \psi_A(0, A), \quad (2.4.51)$$

and

$$\psi(0, 0) = 1. \quad (2.4.52)$$

2. For $(w, A) \in \mathcal{D}_2$, we have

$$\psi(w, A) = \psi \left(w - \frac{\Delta A}{r + \lambda^O}, A + \Delta A \right), \quad (2.4.53)$$

in which $w - \Delta A/(r + \lambda^O) = w_b(A + \Delta A)$. Notice that $(w - \Delta A/(r + \lambda^O), A + \Delta A) \in \mathcal{D}_1$, and thus $\psi(w - \Delta A/(r + \lambda^O), A + \Delta A)$ is determined by the BVP (2.4.49)-(2.4.52).

3. To solve for ψ in the entire region \mathcal{D} , as well as to determine the purchase boundary $w_b(A)$, we also rely on a smooth fit condition across the boundary $w_b(A)$, namely,

$$\bar{a}\psi_{ww}(w_b(A), A) = \psi_{wA}(w_b(A), A). \quad (2.4.54)$$

We first consider $\psi(w, A)$ in the region \mathcal{D}_1 by solving the related BVP (2.4.49)-(2.4.52). Hypothesize that ψ is convex with respect to w , and define its concave dual via the Legendre transform by

$$\hat{\psi}(y, A) = \min_{w \geq 0} [\psi(w, A) + wy]. \quad (2.4.55)$$

As before, rewrite (2.4.49) as

$$-\lambda^S \hat{\psi} - (r - \lambda^S)y \hat{\psi}_y + my^2 \hat{\psi}_{yy} + y(c - A) = 0. \quad (2.4.56)$$

Its general solution is

$$\hat{\psi}(y, A) = D_1(A)y^{B_1} + D_2(A)y^{B_2} + \frac{c - A}{r}y, \quad (2.4.57)$$

in which $B_1 > 1$ and $B_2 < 0$ are defined in (2.3.33). Define

$$y_0(A) = -\psi_w(0, A), \quad (2.4.58)$$

and

$$y_b(A) = -\psi_w(w_b(A), A). \quad (2.4.59)$$

We get the following free-boundary conditions from (2.4.50), (2.4.51), (2.4.52), (2.4.58),

and (2.4.59):

$$\begin{cases} \hat{\psi}_A(y_0(A), A) = -(1 - p)\bar{a}y_0(A), \\ \hat{\psi}_y(y_0(A), A) = 0; \end{cases} \quad (2.4.60)$$

$$\begin{cases} \hat{\psi}_A(y_b(A), A) = -\bar{a}y_b(A), \\ \hat{\psi}_y(y_b(A), A) = w_b(A); \end{cases} \quad (2.4.61)$$

and

$$\hat{\psi}(y_0(0), 0) = 1. \quad (2.4.62)$$

The smooth fit condition on the boundary $w = w_b(A)$ implies

$$\hat{\psi}_{Ay}(y_b(A), A) = -\bar{a}. \quad (2.4.63)$$

Use (2.4.57) to rewrite (2.4.60), (2.4.61), (2.4.62), and (2.4.63) as follows:

$$D_1(A)B_1y_0(A)^{B_1-1} + D_2(A)B_2y_0(A)^{B_2-1} + \frac{c-A}{r} = 0, \quad (2.4.64)$$

$$D_1'(A)y_0(A)^{B_1-1} + D_2'(A)y_0(A)^{B_2-1} = \frac{1}{r} - \frac{1-p}{r+\lambda^O}, \quad (2.4.65)$$

$$D_1(A)B_1y_b(A)^{B_1-1} + D_2(A)B_2y_b(A)^{B_2-1} + \frac{c-A}{r} = w_b(A), \quad (2.4.66)$$

$$D_1'(A)y_b(A)^{B_1-1} + D_2'(A)y_b(A)^{B_2-1} = \frac{1}{r} - \frac{1}{r+\lambda^O}, \quad (2.4.67)$$

$$D_1(0)y_0(0)^{B_1} + D_2(0)y_0(0)^{B_2} + \frac{c}{r}y_0(0) = 1, \quad (2.4.68)$$

$$D_1'(A)B_1y_b(A)^{B_1-1} + D_2'(A)B_2y_b(A)^{B_2-1} = \frac{1}{r} - \frac{1}{r+\lambda^O}. \quad (2.4.69)$$

Solve (2.4.64) and (2.4.66) for $D_1(A)$ and $D_2(A)$:

$$D_1(A) = \frac{1}{B_1}y_b(A)^{1-B_1} \frac{1}{x^{B_1-B_2}-1} \left[-w_b(A) + \frac{c-A}{r} (1-x^{1-B_2}) \right], \quad (2.4.70)$$

$$D_2(A) = \frac{1}{B_2}y_b(A)^{1-B_2} \frac{1}{x^{B_2-B_1}-1} \left[-w_b(A) + \frac{c-A}{r} (1-x^{1-B_1}) \right], \quad (2.4.71)$$

in which

$$x \triangleq \frac{y_0(A)}{y_b(A)}. \quad (2.4.72)$$

Recall that $w_b(A)$ is to be determined. We solve for $D_1'(A)$ and $D_2'(A)$ from (2.4.67) and (2.4.69) to get:

$$D_1'(A) = \frac{\lambda^O}{r(r+\lambda^O)} \frac{1-B_2}{B_1-B_2} y_b(A)^{1-B_1}, \quad (2.4.73)$$

$$D_2'(A) = \frac{\lambda^O}{r(r+\lambda^O)} \frac{B_1-1}{B_1-B_2} y_b(A)^{1-B_2}. \quad (2.4.74)$$

By substituting (2.4.73) and (2.4.74) into (2.4.65), we get

$$\frac{1-B_2}{B_1-B_2} \frac{\lambda^O}{r(r+\lambda^O)} x^{B_1-1} + \frac{B_1-1}{B_1-B_2} \frac{\lambda^O}{r(r+\lambda^O)} x^{B_2-1} = \frac{1}{r} - \frac{1-p}{r+\lambda^O}, \quad (2.4.75)$$

which has a unique solution for $x > 1$; the argument is similar to the corresponding one in Section 2.3.3 for the solution of (2.3.44). It is clear from (2.4.75) that x is independent of A .

Differentiate $D_1(A)$ and $D_2(A)$ in (2.4.70) and (2.4.71) with respect to A to get a second expression for $D'_1(A)$ and $D'_2(A)$; set equal the two expressions for each of $D'_1(A)$ and $D'_2(A)$ to get

$$\frac{dy_b(A)/dA}{y_b(A)} = \frac{\frac{\lambda^O}{r(r+\lambda^O)} \frac{B_1(1-B_2)}{B_1-B_2} (x^{B_1-B_2} - 1) + w'_b(A) + \frac{1}{r} (1-x^{1-B_2})}{(1-B_1) \left[-w_b(A) + \frac{c-A}{r} (1-x^{1-B_2}) \right]} \quad (2.4.76)$$

$$\frac{dy_b(A)/dA}{y_b(A)} = \frac{\frac{\lambda^O}{r(r+\lambda^O)} \frac{B_2(B_1-1)}{B_1-B_2} (x^{B_2-B_1} - 1) + w'_b(A) + \frac{1}{r} (1-x^{1-B_1})}{(1-B_2) \left[-w_b(A) + \frac{c-A}{r} (1-x^{1-B_1}) \right]} \quad (2.4.77)$$

Set equal the right-hand sides of the two equations above to get a non-linear ODE for $w_b(A)$:

$$\alpha_1(c-A)w'_b(A) + \alpha_2w_b(A) + \alpha_3w'_b(A)w_b(A) + \alpha_4(c-A) = 0, \quad (2.4.78)$$

in which

$$\left\{ \begin{array}{l} \alpha_1 = -\frac{1}{r} [(B_1-1)(1-x^{1-B_2}) + (1-B_2)(1-x^{1-B_1})], \\ \alpha_2 = (B_1-1) \left[\frac{\lambda^O}{r(r+\lambda^O)} \frac{(B_1-1)B_2}{B_1-B_2} (x^{B_2-B_1} - 1) + \frac{1}{r} (1-x^{1-B_1}) \right] \\ \quad + (1-B_2) \left[\frac{\lambda^O}{r(r+\lambda^O)} \frac{B_1(1-B_2)}{B_1-B_2} (x^{B_1-B_2} - 1) + \frac{1}{r} (1-x^{1-B_2}) \right], \\ \alpha_3 = B_1 - B_2 > 0, \\ \alpha_4 = -\frac{1}{r} \left\{ (B_1-1) \left[\frac{\lambda^O}{r(r+\lambda^O)} \frac{(B_1-1)B_2}{B_1-B_2} (x^{B_2-B_1} - 1) + \frac{1}{r} (1-x^{1-B_1}) \right] (1-x^{1-B_2}) \right. \\ \quad \left. + (1-B_2) \left[\frac{\lambda^O}{r(r+\lambda^O)} \frac{B_1(1-B_2)}{B_1-B_2} (x^{B_1-B_2} - 1) + \frac{1}{r} (1-x^{1-B_2}) \right] (1-x^{1-B_1}) \right\}. \end{array} \right. \quad (2.4.79)$$

Also, we have the boundary condition $w_b(c-) = 0$ because $0 \leq w_b(A) \leq w_s(A)$ for all $0 \leq A < c$ and $w_s(c-) = 0$. A solution of the ODE, together with the boundary condition at $A = c$, is given by

$$w_b(A) = b \cdot (c - A), \quad (2.4.80)$$

in which

$$b = \frac{(\alpha_2 - \alpha_1) + \sqrt{(\alpha_2 - \alpha_1)^2 + 4\alpha_3\alpha_4}}{2\alpha_3}. \quad (2.4.81)$$

Note that this solution for the purchase boundary $w_b(A)$ is linear with respect to A .

From the expression on the right-hand side of (2.4.76) and from (2.4.80), define

$$K \triangleq \frac{\frac{\lambda^O}{r(r + \lambda^O)} \frac{B_1(1 - B_2)}{B_1 - B_2} (x^{B_1 - B_2} - 1) - b + \frac{1}{r} (1 - x^{1 - B_2})}{(1 - B_1) \left[-b + \frac{1}{r} (1 - x^{1 - B_2}) \right]}. \quad (2.4.82)$$

Solve (2.4.76) and (2.4.72) to obtain

$$y_0(A) = \left(\frac{c}{c - A} \right)^K y_0(0), \quad (2.4.83)$$

and

$$y_b(A) = \frac{y_0(A)}{x}. \quad (2.4.84)$$

To finish solving the FBP, we substitute (2.4.70), (2.4.71), and (2.4.80) into (2.4.68)

to get

$$\frac{1}{y_0(0)} = \frac{c}{B_1} \frac{x^{B_1 - 1}}{x^{B_1 - B_2} - 1} \left[-b + \frac{1}{r} (1 - x^{1 - B_2}) \right] + \frac{c}{B_2} \frac{x^{B_2 - 1}}{x^{B_2 - B_1} - 1} \left[-b + \frac{1}{r} (1 - x^{1 - B_1}) \right] + \frac{c}{r}. \quad (2.4.85)$$

Proposition 2.4.4. *The solution of the FBP (2.4.56) with conditions (2.4.60), (2.4.61), and (2.4.62) is given by (2.4.57), with $D_1(A)$, $D_2(A)$, $y_0(0)$, $y_0(A)$, $y_b(A)$, x , and K defined in (2.4.70), (2.4.71), (2.4.85), (2.4.83), (2.4.84), (2.4.75), and (2.4.82), respectively.*

Notice that we can rewrite the inequalities (2.3.7) and (2.3.8) in terms of $\hat{\psi}$ as

$$\hat{\psi}_A(y, A) \geq -\frac{1}{r + \lambda^O} y, \quad (2.4.86)$$

$$\hat{\psi}_A(y, A) \leq -\frac{1 - p}{r + \lambda^O} y. \quad (2.4.87)$$

Next, we prove that $\hat{\psi}$ is concave with respect to y and satisfies inequalities (2.4.86) and (2.4.87).

Proposition 2.4.5. $\hat{\psi}(y, A)$ given by Proposition 2.4.4 is concave with respect to y and satisfies inequalities (2.4.86) and (2.4.87).

Proof. The proof that $\hat{\psi}$ is concave with respect to y is not obvious (unlike the previous two cases), so we relegate that (long) proof to the Appendix.

Substitute $D'_1(A)y^{B_1} + D'_2(A)y^{B_2} - \frac{y}{r}$ for $\hat{\psi}_A(y, A)$ to rewrite the inequalities (2.4.86) and (2.4.87) in the equivalent form as

$$-1 \leq \frac{\lambda^O}{r} \left[\frac{1 - B_2}{B_1 - B_2} \left(\frac{y}{y_b(A)} \right)^{B_1-1} + \frac{B_1 - 1}{B_1 - B_2} \left(\frac{y}{y_b(A)} \right)^{B_2-1} \right] - \frac{r + \lambda^O}{r} \leq -(1 - p). \quad (2.4.88)$$

To prove the inequality above, define

$$h(z) = \frac{1 - B_2}{B_1 - B_2} z^{B_1-1} + \frac{B_1 - 1}{B_1 - B_2} z^{B_2-1}, \quad (2.4.89)$$

and note that

$$h'(z) = \frac{(1 - B_2)(B_1 - 1)}{B_1 - B_2} [z^{B_1-2} - z^{B_2-2}] \geq 0, \quad z \geq 1. \quad (2.4.90)$$

Also, the first inequality in (2.4.88) holds with equality when $y = y_b(A)$, and the second inequality holds with equality when $y = y_0(A)$. Thus, because $h(z)$ is non-decreasing for $z \geq 1$, inequality (2.4.88) holds for $y_b(A) \leq y \leq y_0(A)$. \square

As before, we define the convex dual of $\hat{\psi}$ via the Legendre transform for $(w, A) \in \mathcal{D}_1$ as

$$\Psi(w, A) = \max_{y \geq y_b(A)} \left[\hat{\psi}(y, A) - wy \right]. \quad (2.4.91)$$

For $(w, A) \in \mathcal{D}_2$, we define

$$\Psi(w, A) = \Psi(w - \bar{a}\Delta A, A + \Delta A), \quad (2.4.92)$$

in which ΔA solves $w - \bar{a}\Delta A = b(c - (A + \Delta A))$; that is, $\Delta A = \frac{w - b(c - A)}{\bar{a} - b}$. Notice that since $(w - \bar{a}\Delta A, A + \Delta A) \in \mathcal{D}_1$, $\Psi(w - \bar{a}\Delta A, A + \Delta A)$ is given through (2.4.91).

Now we proceed to the following lemma, which demonstrates that Ψ is the minimum probability of ruin by Verification Theorem 2.3.1.

Lemma 2.4.1. $\Psi(w, A)$ defined in (2.4.91) and (2.4.92) satisfies Conditions 1-4 of the Verification Theorem 2.3.1.

Proof. First, consider $(w, A) \in \mathcal{D}_1$. In terms of $\Psi(w, A)$, we rewrite (2.4.56) as follows:

$$\lambda^S \Psi(w, A) = (rw - c) \Psi_w(w, A) - m \frac{\Psi_w^2(w, A)}{\Psi_{ww}(w, A)}, \quad (2.4.93)$$

as well as (2.4.86) and (2.4.87)

$$\bar{a} \Psi_w(w, A) \leq \Psi_A(w, A), \quad (2.4.94)$$

$$(1 - p) \bar{a} \Psi_w(w, A) \geq \Psi_A(w, A). \quad (2.4.95)$$

Expressions (2.4.93)-(2.4.95) show that $\Psi(w, A)$ satisfies Conditions 1-3 of the Verification Theorem 2.3.1 on \mathcal{D}_1 . It is clear by construction that Ψ satisfies Condition 4, namely, $\Psi(0, 0) = 1$.

Now, consider $(w, A) \in \mathcal{D}_2$. By definition,

$$\Psi(w, A) = \Psi(w', A'), \quad (2.4.96)$$

with $w' = w - \frac{w - b(c - A)}{1 - b/\bar{a}}$ and $A' = A + \frac{w - b(c - A)}{\bar{a} - b}$. From (2.4.96), we get the following relations

$$\Psi_w(w, A) = -\frac{b}{\bar{a} - b} \Psi_w(w', A') + \frac{1}{\bar{a} - b} \Psi_A(w', A'), \quad (2.4.97)$$

$$\Psi_{ww}(w, A) = \left(\frac{b}{\bar{a} - b}\right)^2 \Psi_{ww}(w', A') - \frac{2b}{(\bar{a} - b)^2} \Psi_{wA}(w', A') + \left(\frac{1}{\bar{a} - b}\right)^2 \Psi_{AA}(w', A'), \quad (2.4.98)$$

and

$$\Psi_A(w, A) = -\frac{b\bar{a}}{\bar{a} - b} \Psi_w(w', A') + \frac{\bar{a}}{\bar{a} - b} \Psi_A(w', A'). \quad (2.4.99)$$

Since (w', A') is on the boundary $w = w_b(A)$, we have

$$\bar{a}\Psi_w(w', A') = \Psi_A(w', A'). \quad (2.4.100)$$

This along with (2.4.97) leads to

$$\Psi_w(w, A) = \Psi_w(w', A'). \quad (2.4.101)$$

Differentiate (2.4.101) with respect to w to get

$$\Psi_{ww}(w, A) = -\frac{b}{\bar{a}-b}\Psi_{ww}(w', A') + \frac{1}{\bar{a}-b}\Psi_{wA}(w', A'). \quad (2.4.102)$$

From (2.4.102) and from the smooth fit condition on the purchase boundary, namely

$$\bar{a}\Psi_{ww}(w', A') = \Psi_{wA}(w', A'), \quad (2.4.103)$$

we obtain

$$\Psi_{ww}(w, A) = \Psi_{ww}(w', A'). \quad (2.4.104)$$

We know that for $(w', A') \in \mathcal{D}_1$ and for $\pi \in \mathbb{R}$,

$$\mathcal{L}^\pi \Psi(w', A') = [rw' + (\mu - r)\pi - c + A']\Psi_w(w', A') + \frac{1}{2}\sigma^2\pi^2\Psi_{ww}(w', A') - \lambda^S\Psi(w', A') \geq 0. \quad (2.4.105)$$

It follows that for $(w, A) \in \mathcal{D}_2$,

$$\begin{aligned} \mathcal{L}^\pi \Psi(w, A) &= \left[r \left(w' + \frac{w - b(c - A)}{1 - b/\bar{a}} \right) + (\mu - r)\pi - c + \left(A' - \frac{w - b(c - A)}{\bar{a} - b} \right) \right] \Psi_w(w', A') \\ &\quad + \frac{1}{2}\sigma^2\pi^2\Psi_{ww}(w', A') - \lambda^S\Psi(w', A') \\ &= \mathcal{L}^\pi \Psi(w', A') + \left[r \frac{w - b(c - A)}{1 - b/\bar{a}} - \frac{w - b(c - A)}{\bar{a} - b} \right] \Psi_w(w', A') \geq 0, \end{aligned} \quad (2.4.106)$$

because $\mathcal{L}^\pi \Psi(w', A') \geq 0$, $\Psi_w(w', A') \leq 0$, and $r \frac{w - b(c - A)}{1 - b/\bar{a}} - \frac{w - b(c - A)}{\bar{a} - b} \leq 0$.

Thus, $\Psi(w, A)$ satisfies Condition 1 of the Verification Theorem for $(w, A) \in \mathcal{D}_2$.

Next, consider Conditions 2 and 3 for $(w, A) \in \mathcal{D}_2$. From (2.4.97) and (2.4.99), Condition 2 holds for (w, A) if and only if

$$\bar{a} \left[-\frac{b}{\bar{a}-b} \Psi_w(w', A') + \frac{1}{\bar{a}-b} \Psi_A(w', A') \right] \leq -\frac{b\bar{a}}{\bar{a}-b} \Psi_w(w', A') + \frac{\bar{a}}{\bar{a}-b} \Psi_A(w', A'), \quad (2.4.107)$$

which is true with equality. Thus, we conclude that Condition 2 holds with equality for $(w, A) \in \mathcal{D}_2$. Finally, because $0 < p < p^* \leq 1$, $\Psi_w \leq 0$, and $\Psi_A \leq 0$, it follows that Condition 3 also holds on \mathcal{D}_2 . \square

Therefore, $\Psi(w, A)$ is the minimum probability of ruin by the Verification Theorem 2.3.1, and we present the following theorem that summarizes the work of this section.

Theorem 2.4.2. *When $p < p^*$ and the borrowing restriction is enforced, the minimum probability of ruin for $(w, A) \in \mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, with $\mathcal{D}_1 = \{(w, A) : 0 \leq w \leq w_b(A), 0 \leq A < c\}$ and $\mathcal{D}_2 = \{(w, A) : w_b(A) < w < w_s(A), 0 \leq A < c\}$, is given by $\Psi(w, A)$ defined above. The associated optimal strategy is:*

1. to purchase additional annuity income so that wealth and annuity income lie on the boundary $w = b \cdot (c - A)$ of the region \mathcal{D}_1 when $(w, A) \in \mathcal{D}_2$;
2. to purchase additional annuity income instantaneously to keep (w, A) in the region \mathcal{D}_1 when $w = w_b(A)$;
3. to surrender existing annuity income instantaneously to keep w non-negative when needed;
4. to invest in the risky asset with amount

$$\pi^*(w, A) = -\frac{\mu - r}{\sigma^2} \frac{\Psi_w(w, A)}{\Psi_{ww}(w, A)},$$

when $(w, A) \in \mathcal{D}_1$.

2.4.3 Numerical examples

In this section, we present numerical examples to demonstrate the results of Section 2.4.2 and 2.4.2. The basic scenario is the same as in Section 2.3.5, and we focus on the role of the surrender penalty p .

$$p \geq p^*$$

Figures 2.6-2.9 show the ruin probabilities and associated optimal investment strategies when $p \geq p^*$. We fix all the parameters except for annuity income A and surrender penalty p . Note that the smallest p value of the selected is 0.258, which is the value of p^* for the scenario we chose. The boundary $w = 0$ and w_s does not depend on p . Therefore, for each figure, all four curves have the same domain. By showing the ruin probabilities and investment strategies for different A and p , we see some common patterns as well as the effect of p . Within each figure, the probabilities of ruin are decreasing and convex. On the surrender boundary $w = 0$, the curves of the ruin probabilities begin with different values, not necessarily 0. This occurs because if the individual has some annuity income, she surrenders some of it to avoid ruin when reaching that boundary. We also observe that bigger p results in higher probability of ruin. This is consistent with the financial intuition that an individual receives less wealth from surrendering annuity income when the penalty p is bigger, as we also show in Proposition 2.4.3.

If A is not 0, reversibility makes difference in the ruin probability on the boundary $w = 0$, and consequently on the whole ruin probability curve. Reversibility of the annuity offers an extra chance to avoid bankruptcy. This is demonstrated by the difference of ruin probabilities between $p = 0.258$ and $p = 1$ for given values of (w, A) . Note that at $w = 0$, the difference increases dramatically as A increases.

When $A = 0$, both ruin probabilities are 1. On the other hand, when $A = 0.75$, the individual with the reversible annuity ($p = 0.258$) has only about a 25% chance to ruin when her wealth is 0 if she follows the optimal strategy. By contrast, if the annuity is irreversible ($p = 1$), she ruins immediately when wealth is 0 because the annuity is effectively worthless. This gap in the ruin probabilities shrinks as wealth w increases, and the ruin probabilities associated with different p 's converge to 0 at $w = w_s(A)$, the safe level.

The interesting phenomenon that the investment in the risky asset does not depend on p is demonstrated in all figures, as we also show in Proposition 2.4.3. This indicates that the individual invests in the risky asset as if the annuity is irreversible when $p \geq p^*$. That is, we see a type of separation result: optimal investment in the risky asset is independent of the optimal annuitization strategy when $p \geq p^*$.

$p < p^*$

Figures 2.10-2.13 show the ruin probabilities and associated optimal investment strategies when $p < p^*$. Recall from Section 2.4.2 that it is optimal to purchase immediate life annuities before wealth reaches the safe level. Note that the largest value of p we can choose is 0.258. It is natural to believe that one's behavior changes smoothly as penalty p changes. This belief is confirmed in these figures. By observing the curves associated with $p = 0.258$ in Figures 2.6-2.9 and in Figures 2.10-2.13, we conclude that the optimal investment strategies and ruin probabilities from the two different sets of equations are the same. (We can also demonstrate this fact algebraically, but in the interest of space, we omit that computation.)

We see that the ruin probabilities in Figures 2.10-2.13 are all decreasing and convex. The wealth domain for a given function in these figures is $[0, b \cdot (c - A)]$, and note that b decreases as p decreases because for a smaller surrender charge, the

individual has more incentive to annuitize at a lower wealth level. It remains true that, with all else equal, a smaller surrender charge p results in a smaller probability of ruin. Also, note that investment in risky asset increases as wealth increases, as in the case for which $p \geq p^*$. However, different from what we see for $p \geq p^*$ case, the investment strategy is no longer independent of p . More cash is invested in the risky asset if one can get a larger portion of her annuity value back by surrendering.

Figure 2.14 demonstrates the relation between b and the proportional surrender penalty p . The sign * in the figure indicates the b value of $\frac{1}{r + \lambda^O}$. We see that b increases monotonically and continuously from 0 to $\frac{1}{r + \lambda^O}$ as p increases from 0 to p^* , as we expect.

2.5 Conclusion

The annuity puzzle has been widely noted both in practice and in theoretical work; see Milevsky and Young [2007] and Milevsky et al. [2006a] and the references therein. In this paper, we considered a financial innovation that might encourage more retirees to purchase immediate life annuities, namely the option to surrender one's annuity for cash. We explained the relation between the irreversibility of annuitization and the retirees' reluctance to purchase. We investigated how reversibility would affect the decision of a retiree seeking to minimize her lifetime probability of ruin. We analyzed the optimal investment and annuitization strategies for such a retiree when borrowing against the surrender value of the annuity is prohibited. We found that the individuals annuity purchasing strategy depends on the size of the proportional surrender charge. When the charge is large enough, the individual will not buy a life annuity unless she can cover all her consumption, the so-called safe level. When the charge is small enough, the individual will buy a life annuity at a wealth lower than

this safe level. In both cases, the individual only surrenders annuity income in order to keep her wealth non-negative.

These results confirm the point of view in [Gardner and Wadsworth \[2004\]](#) that the lack of flexibility discourages retirees from purchasing immediate life annuities. In our model, if annuities are irreversible, then retirees will buy annuities only when their wealth reaches the safe level. Moreover, we showed that if annuities are reversible, then a retiree will partially annuitize if the surrender charge is low enough. In numerical examples, we noticed that the threshold value of surrender charge for an individual to consider partial annuitization might be too low for annuity providers. This perhaps explains why reversible immediate life annuities are not offered in the annuity market.

The model in this paper offers a mathematical framework to understand the annuity puzzle. Even though we assumed constant hazard rates and interest rate in our analysis, we believe that the main qualitative insight will be true in general and will be useful to develop better structured annuity products for retirees. Our analysis also implies that a well developed secondary market of annuities would benefit both potential annuity buyers and providers.

2.6 Appendix

In this appendix, we prove that the $\hat{\psi}$ given in [Proposition 2.4.4](#) is concave thereby completing the proof of [Proposition 2.4.5](#).

Take the second derivative of [\(2.4.57\)](#) with respect to y to get

$$\hat{\psi}_{yy}(y, A) = D_1(A)B_1(B_1 - 1)y^{B_1-2} + D_2(A)B_2(B_2 - 1)y^{B_2-2}. \quad (2.6.1)$$

We want to show that $\hat{\psi}_{yy}(y, A) \leq 0$ for $y_b(A) \leq y \leq y_0(A)$. Substitute [\(2.4.70\)](#) and [\(2.4.71\)](#) into [\(2.6.1\)](#), and define $z \triangleq y/y_b(A) \in [1, x]$, with x defined by [\(2.4.72\)](#).

Then, we get

$$\begin{aligned}
\hat{\psi}_{yy}(y, A) \leq 0 &\iff (B_1 - 1) \left[\frac{y}{y_b(A)} \right]^{B_1 - B_2} \frac{1}{x^{B_1 - B_2} - 1} \left[-b + \frac{1}{r} (1 - x^{1 - B_2}) \right] \\
&\quad + (1 - B_2) \frac{x^{B_1 - B_2}}{x^{B_1 - B_2} - 1} \left[-b + \frac{1}{r} (1 - x^{1 - B_1}) \right] \leq 0 \\
&\iff (B_1 - 1) \left[-b + \frac{1}{r} (1 - x^{1 - B_2}) \right] \left(\frac{z}{x} \right)^{B_1 - B_2} \\
&\quad + (1 - B_2) \left[-b + \frac{1}{r} (1 - x^{1 - B_1}) \right] \leq 0.
\end{aligned} \tag{2.6.2}$$

Note that $B_1 - 1 > 0$ and $x^{B_2 - 1} < 1$. It follows that

$$(B_1 - 1) \left[-b + \frac{1}{r} (1 - x^{1 - B_2}) \right] < 0. \tag{2.6.3}$$

Hence the left-hand side of the last inequality in (2.6.2) reaches its maximum value when $z = 1$. So, to prove that $\hat{\psi}$ is concave with respect to y , it is sufficient to show that

$$(B_1 - 1) \left[-b + \frac{1}{r} (1 - x^{1 - B_2}) \right] x^{B_2 - 1} + (1 - B_2) \left[-b + \frac{1}{r} (1 - x^{1 - B_1}) \right] x^{B_1 - 1} \leq 0. \tag{2.6.4}$$

Solve for $x^{B_2 - 1}$ from (2.4.75); then, substitute into (2.6.4), which becomes

$$\begin{aligned}
&(B_1 - 1) \left[\left(\frac{1}{r} - b \right) \frac{B_1 - B_2}{B_1 - 1} \frac{\lambda^O + pr}{\lambda^O} - \left(\frac{1}{r} - b \right) \frac{1 - B_2}{B_1 - 1} x^{B_1 - 1} - \frac{1}{r} \right] \\
&\quad + (1 - B_2) \left[\left(\frac{1}{r} - b \right) x^{B_1 - 1} - \frac{1}{r} \right] \leq 0 \\
&\iff \left(\frac{1}{r} - b \right) (B_1 - B_2) \frac{\lambda^O + pr}{\lambda^O} - \frac{1}{r} (B_1 - B_2) \leq 0 \iff b \geq \frac{p}{\lambda^O + pr}.
\end{aligned} \tag{2.6.5}$$

Therefore, if we show that $b \geq p/(\lambda^O + pr)$, then we are done. To this end, note that

$$b \geq \frac{p}{\lambda^O + pr} \iff \frac{2\alpha_3}{r} \left(1 - \frac{\lambda^O}{\lambda^O + pr} \right) - (\alpha_2 - \alpha_1) \leq \sqrt{(\alpha_2 - \alpha_1)^2 + 4\alpha_3\alpha_4}, \tag{2.6.6}$$

in which the α_i are given in (2.4.79) for $i = 1, \dots, 4$. The second inequality above holds automatically if its left-hand side is less than or equal to 0. Thus, suppose that the left-hand side is positive, and square both sides to get that $b \geq p/(\lambda^O + pr)$ holds if

$$\alpha_4 + \frac{1}{r} \left(1 - \frac{\lambda^O}{\lambda^O + pr}\right) (\alpha_2 - \alpha_1) - \frac{\alpha_3}{r} \left(1 - \frac{\lambda^O}{\lambda^O + pr}\right)^2 \geq 0. \quad (2.6.7)$$

By substituting for the α_i , $i = 1, \dots, 4$, by substituting for $\lambda^O/(\lambda^O + pr)$ via the following expression from (2.4.75)

$$\frac{\lambda^O}{\lambda^O + pr} = \frac{(B_1 - B_2)x^{1-B_1}x^{1-B_2}}{(B_1 - 1)x^{1-B_1} + (1 - B_2)x^{1-B_2}}, \quad (2.6.8)$$

and by simplifying carefully, we learn that (2.6.7) is equivalent to

$$0 \leq \frac{\lambda^O}{\lambda^O + pr} - \frac{\lambda^O}{\lambda^O + r}, \quad (2.6.9)$$

which is true because $0 < p \leq 1$. We have proved that $b \geq p/(\lambda^O + pr)$ and, thereby, that $\hat{\psi}$ is concave with respect to y .

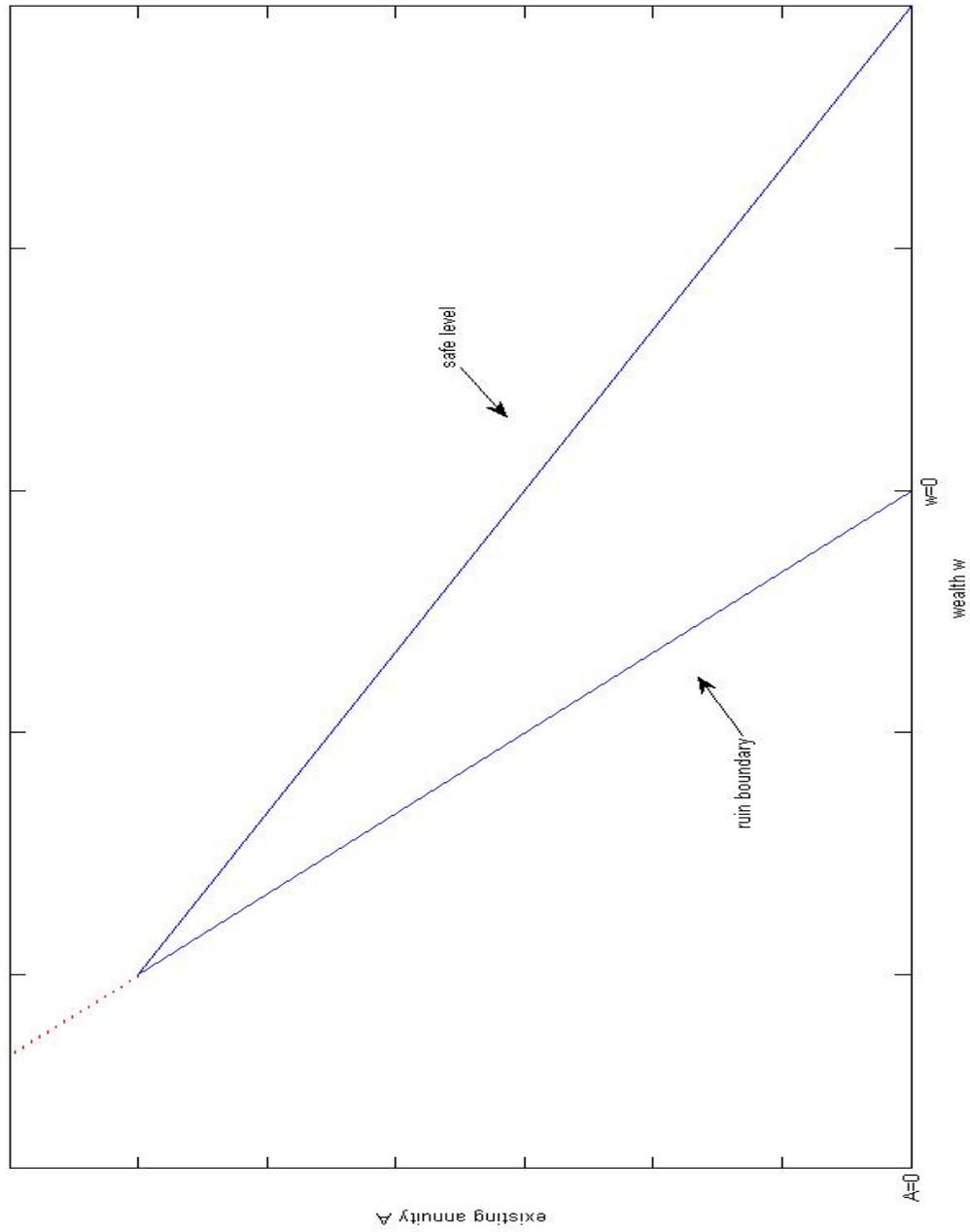


Figure 2.1: The region for solving minimum probability of ruin when borrowing against annuity is allowed

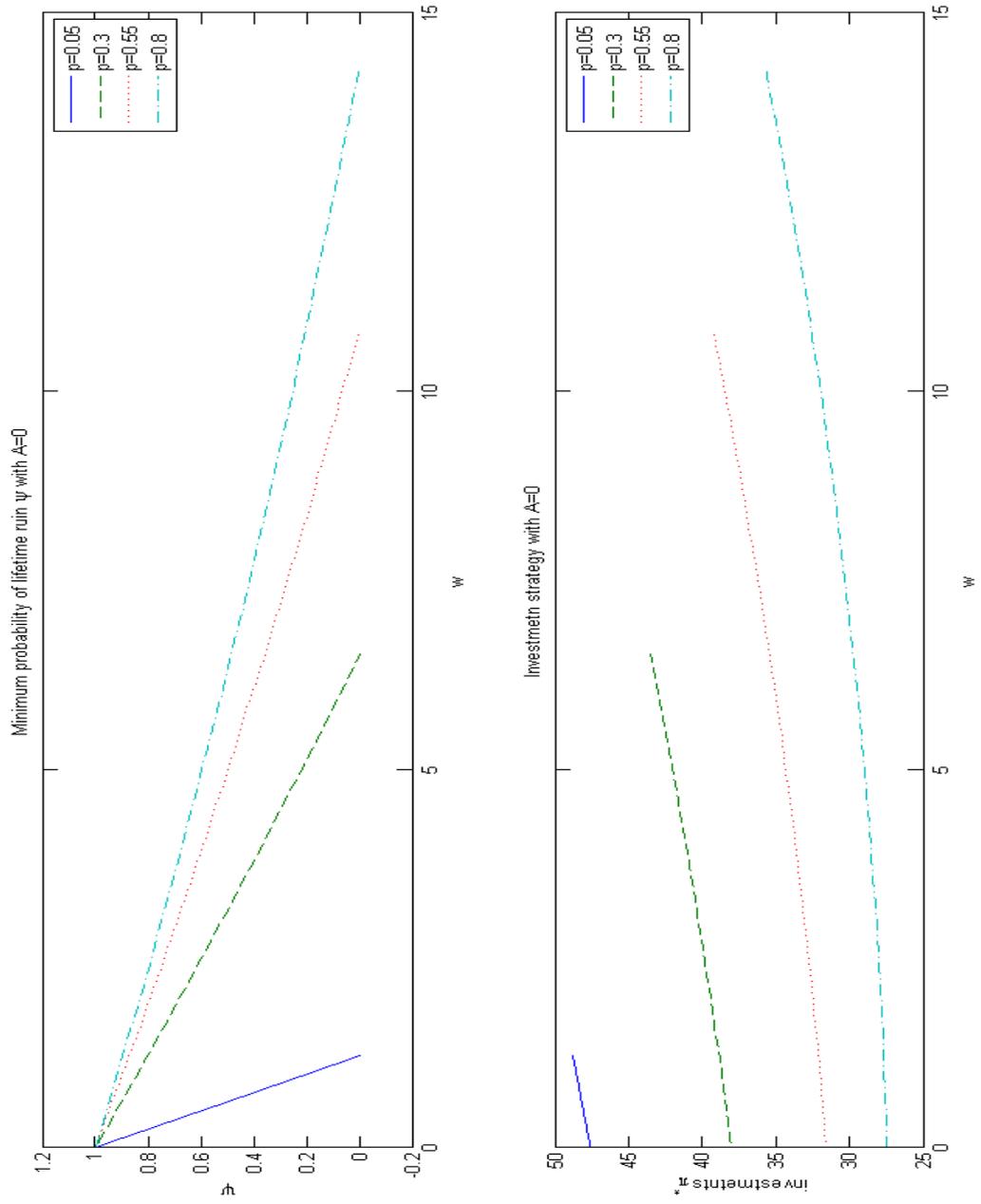


Figure 2.2: Ruin probabilities and optimal investment strategies for different p when A is 0

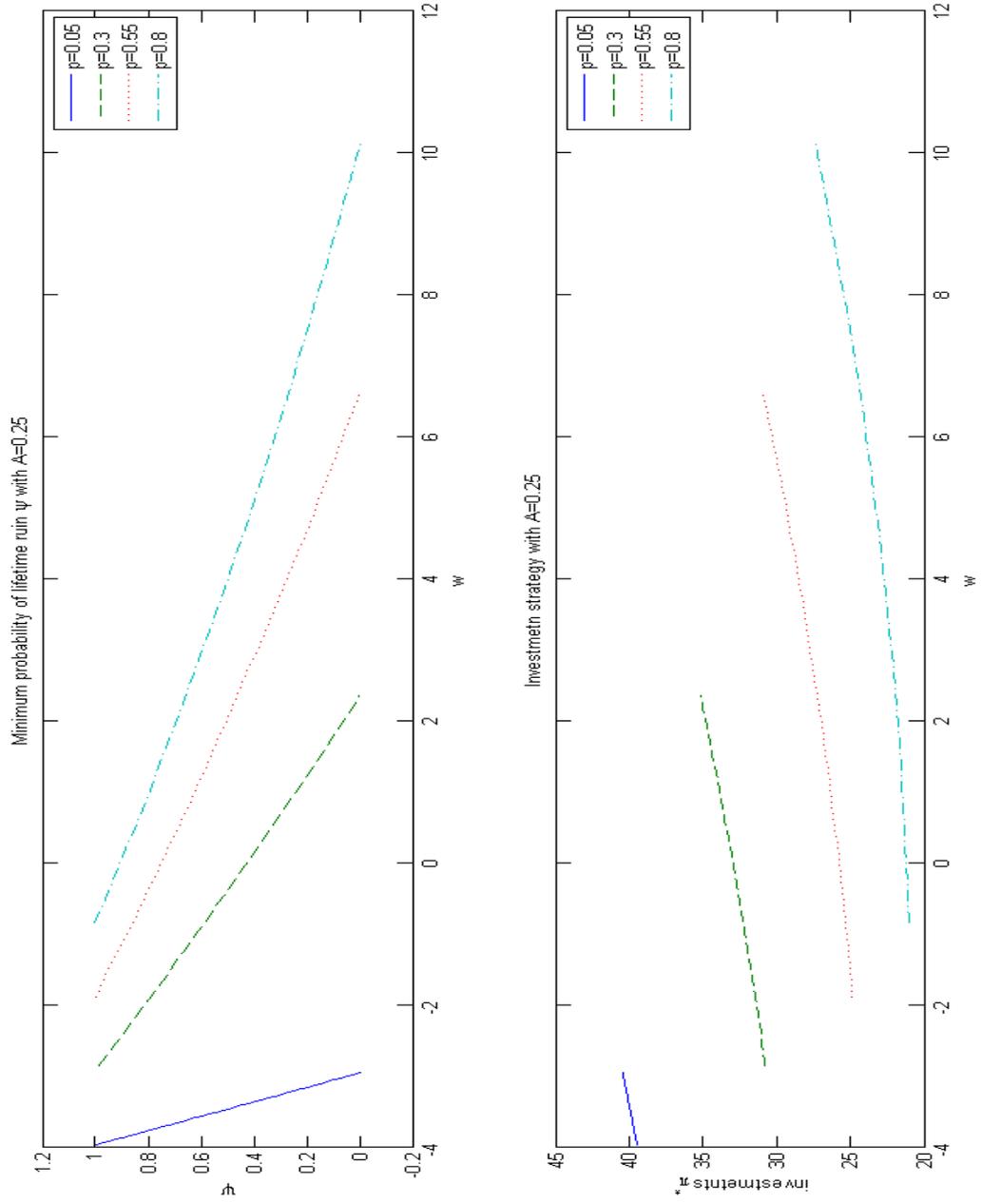


Figure 2.3: Ruin probabilities and optimal investment strategies for different p when A is 0.25

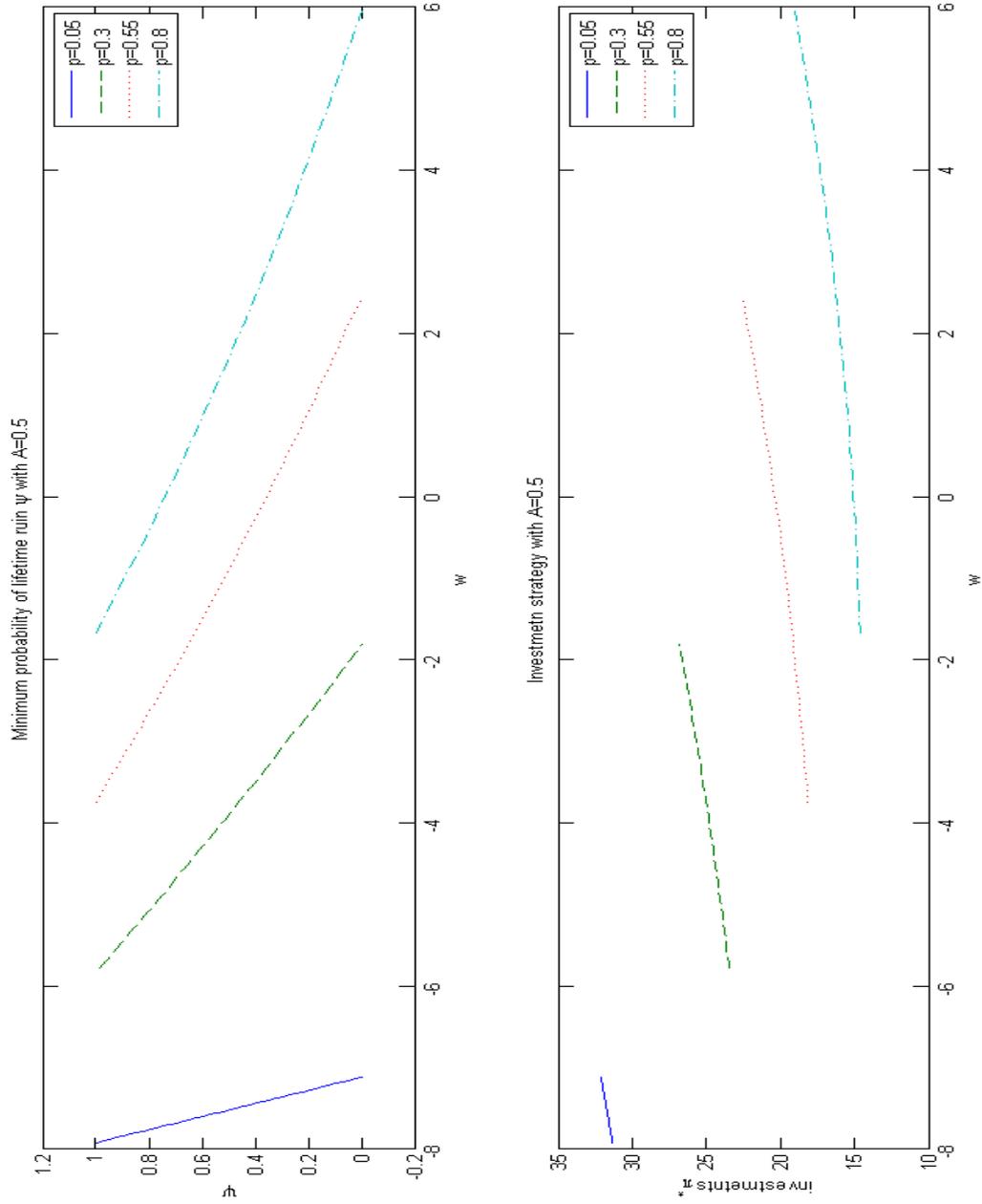


Figure 2.4: Ruin probabilities and optimal investment strategies for different p when A is 0.5

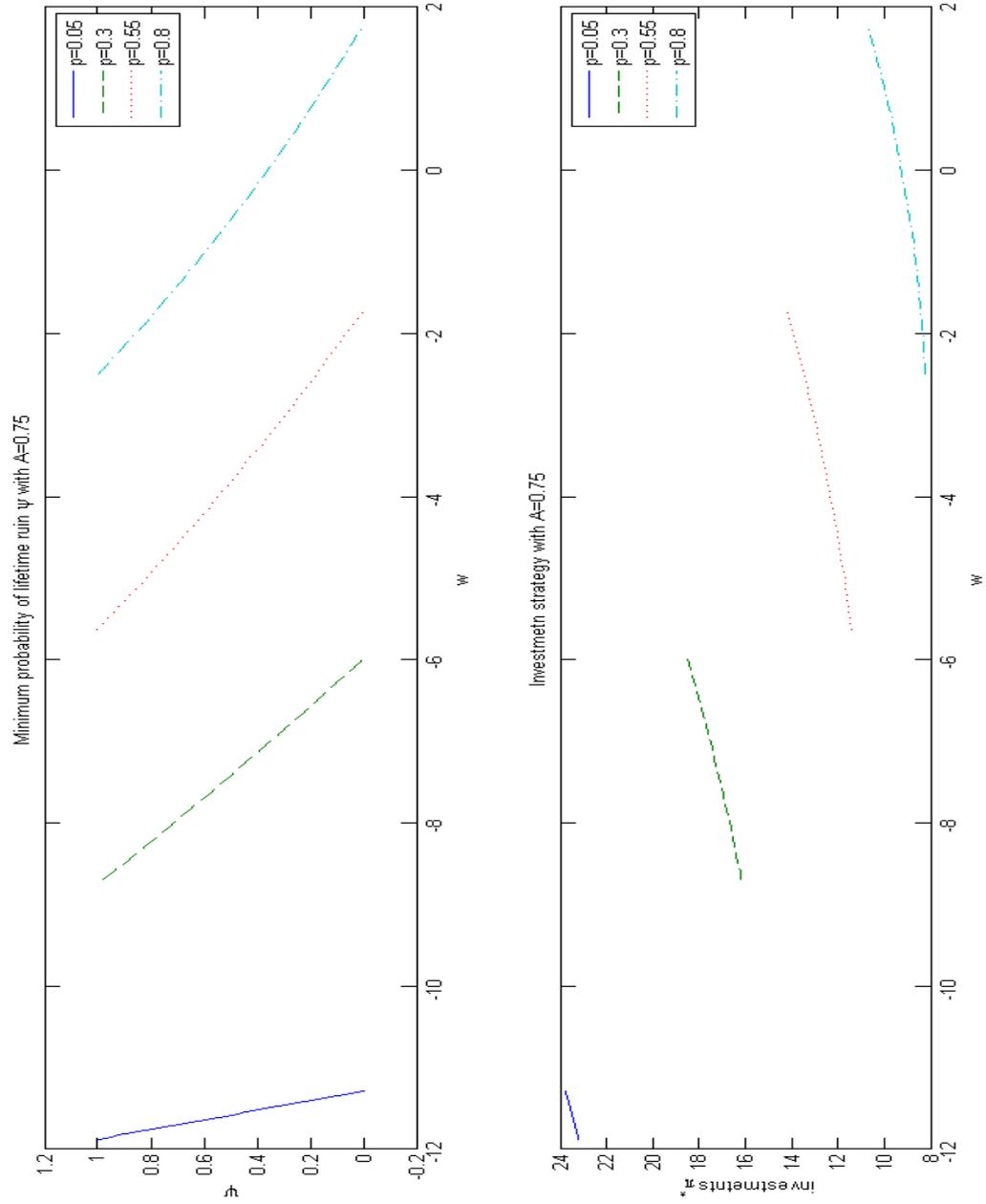


Figure 2.5: Ruin probabilities and optimal investment strategies for different p when A is 0.75

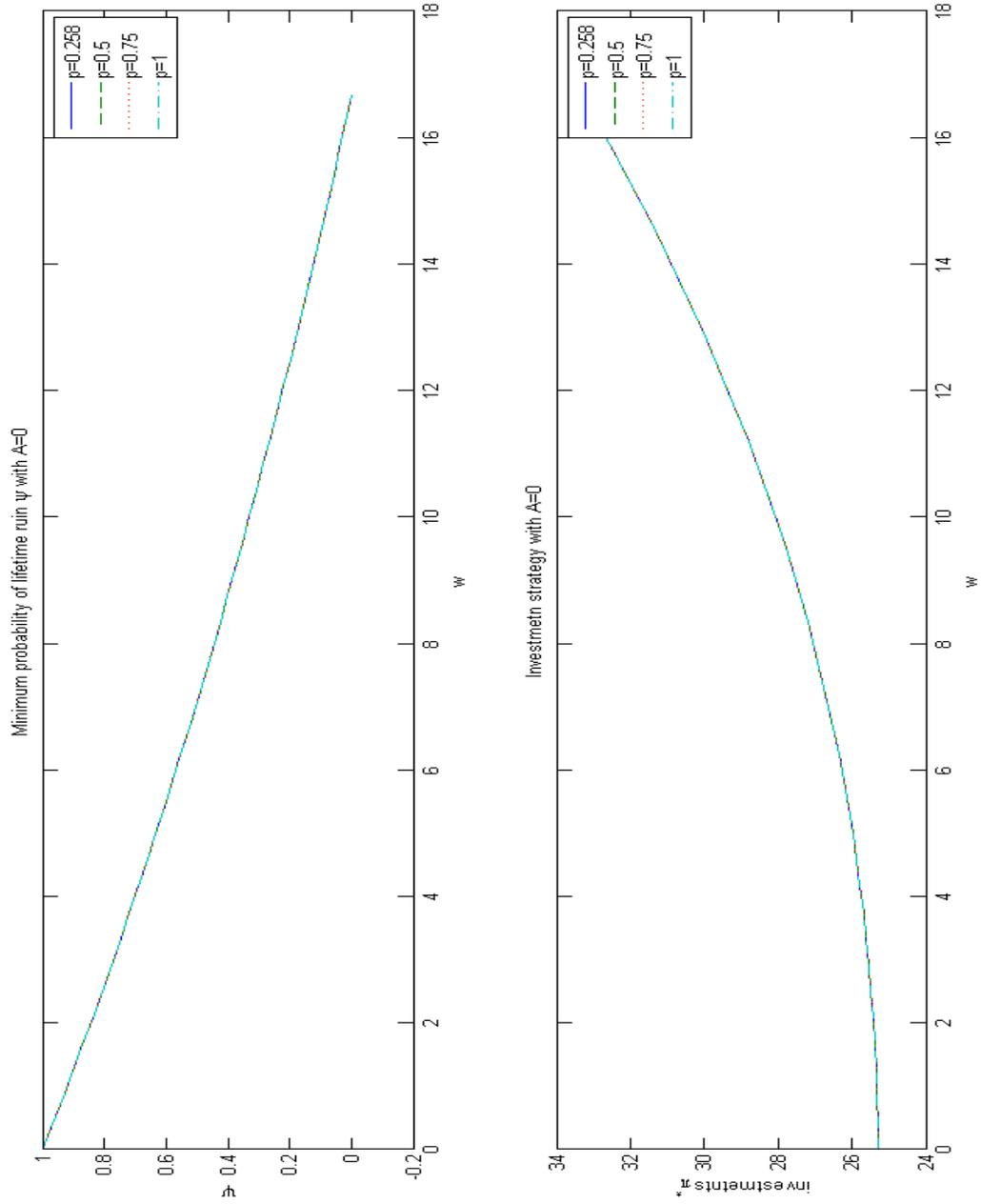


Figure 2.6: Ruin probabilities and optimal investment strategies for different p when A is 0

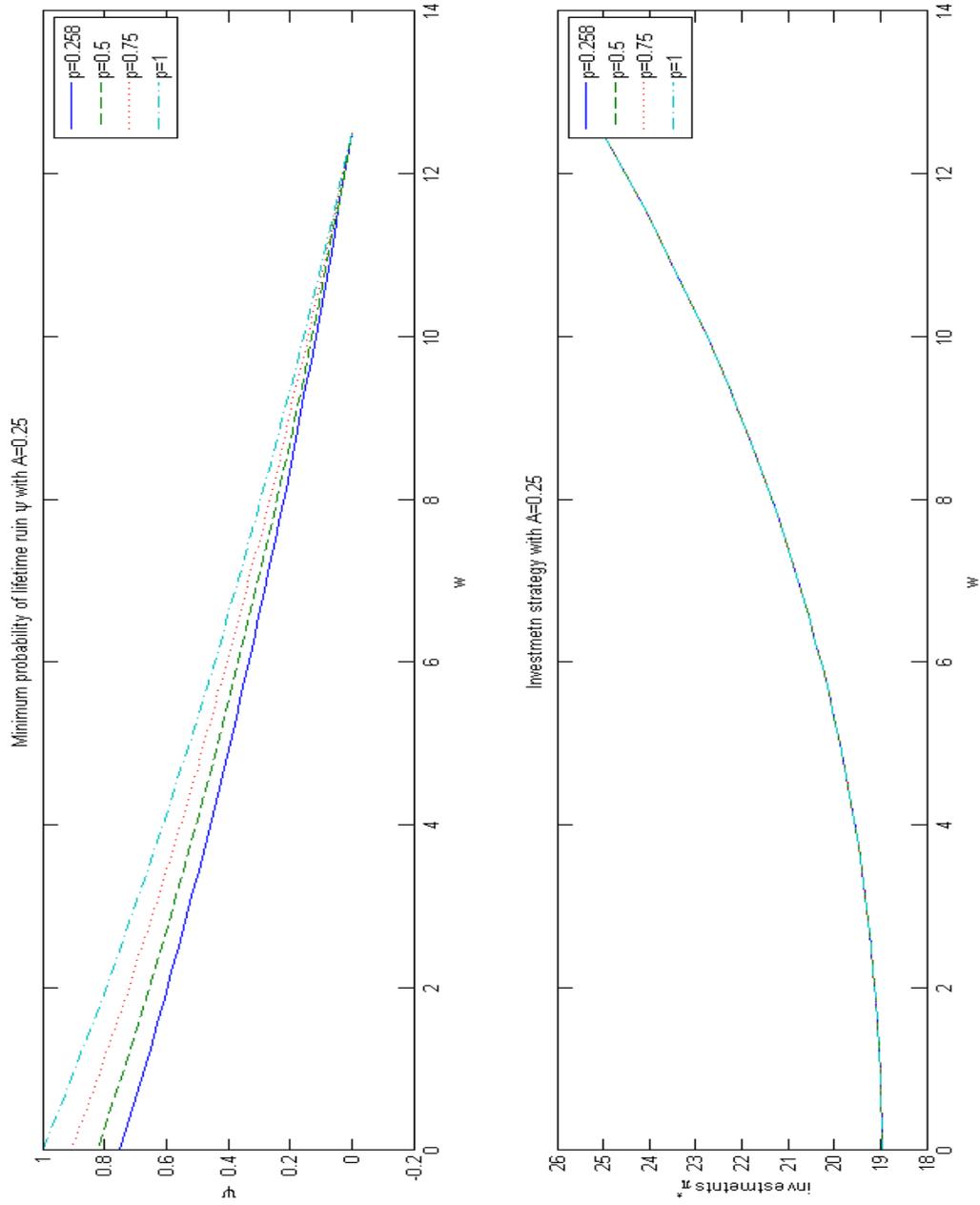


Figure 2.7: Ruin probabilities and optimal investment strategies for different p when A is 0.25

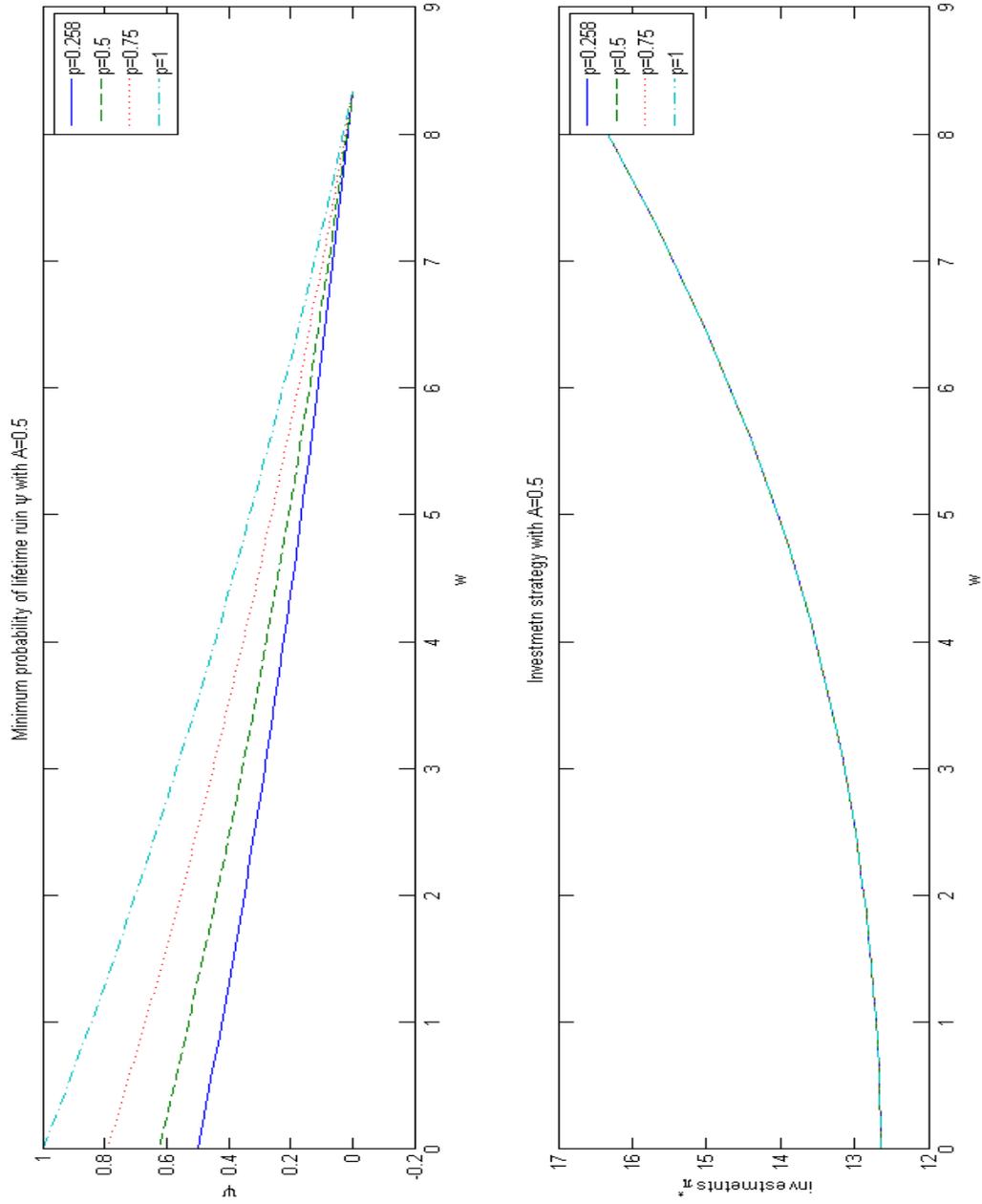


Figure 2.8: Ruin probabilities and optimal investment strategies for different p when A is 0.5

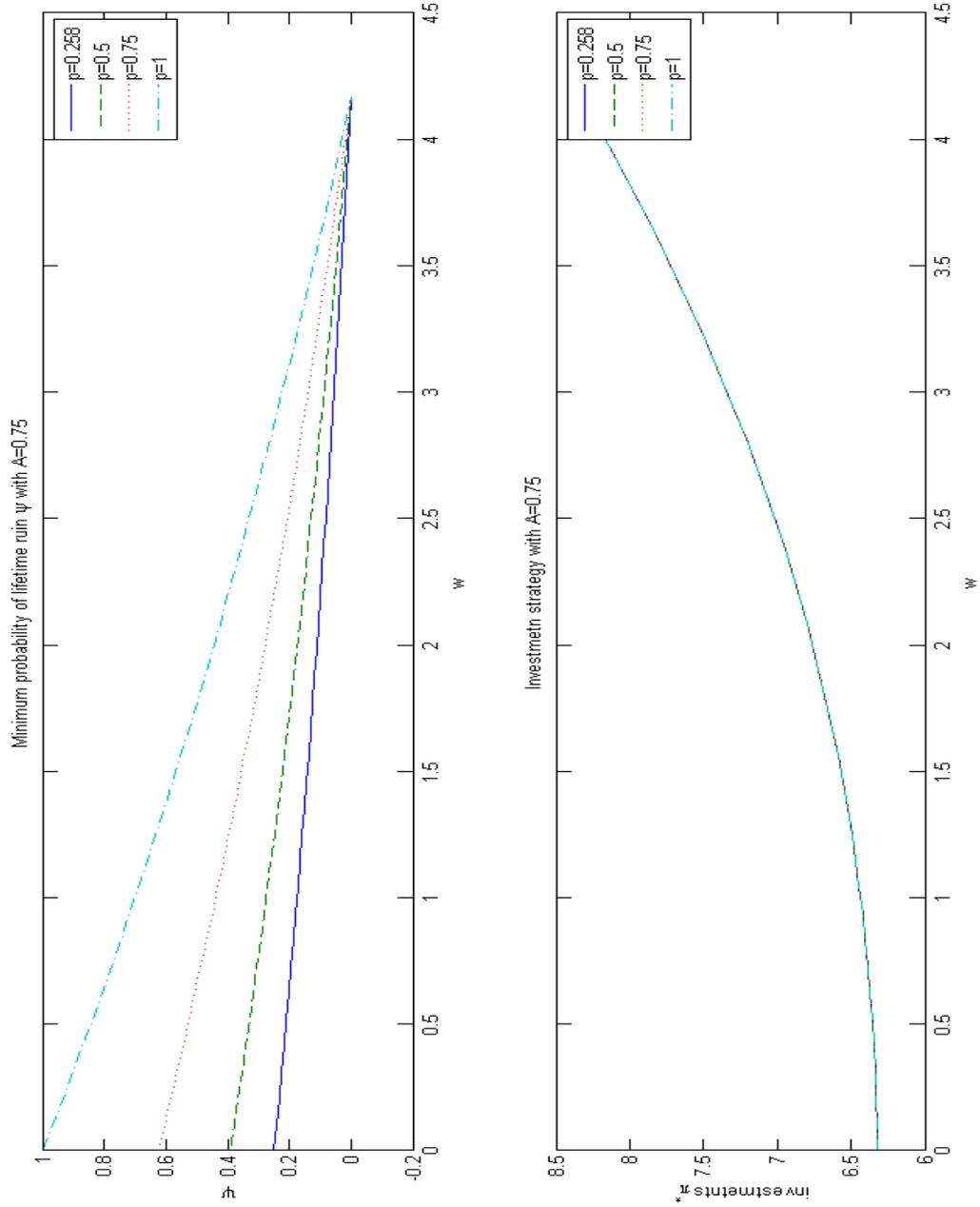


Figure 2.9: Ruin probabilities and optimal investment strategies for different p when A is 0.75

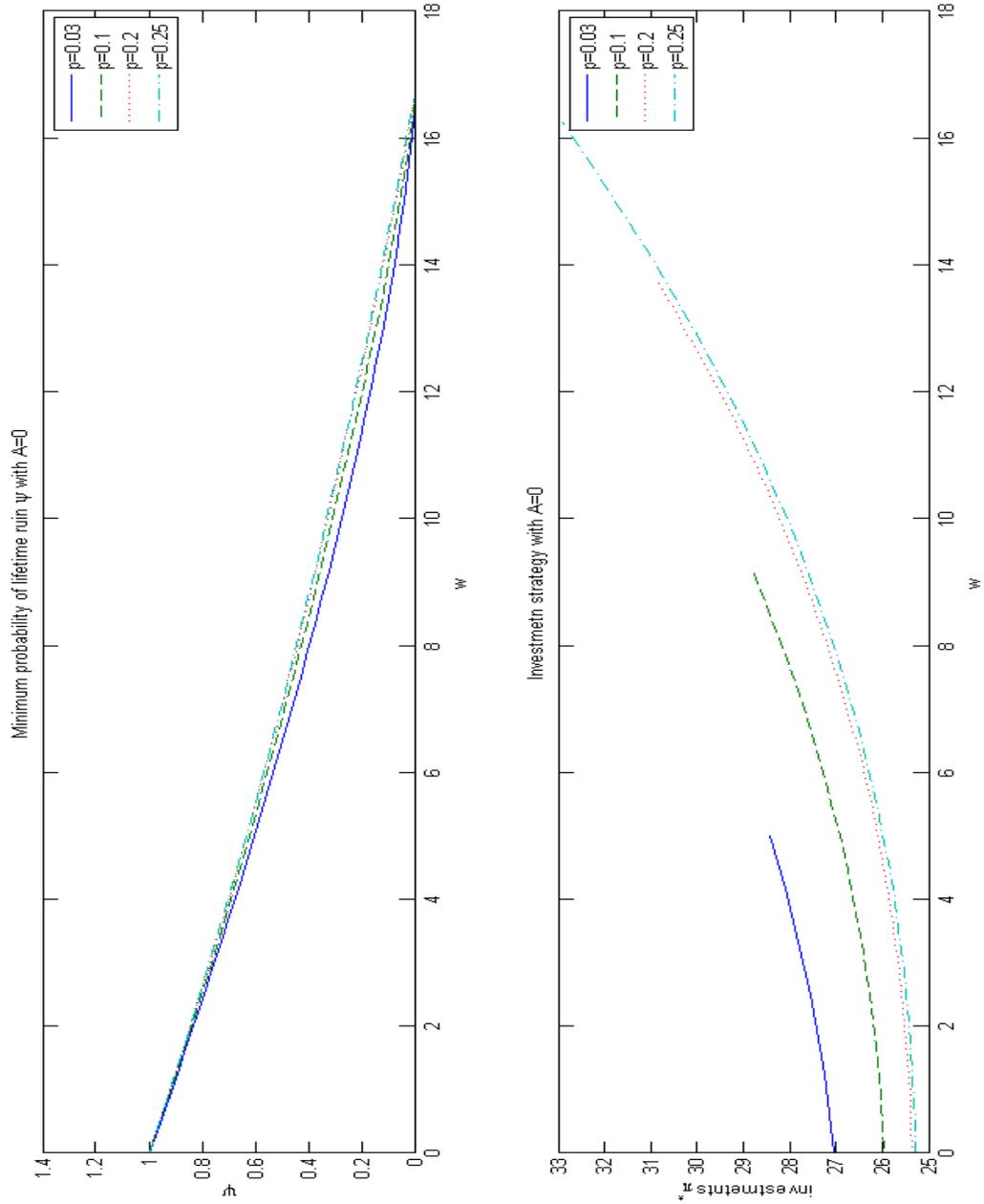


Figure 2.10: Ruin probabilities and optimal investment strategies for different p when A is 0

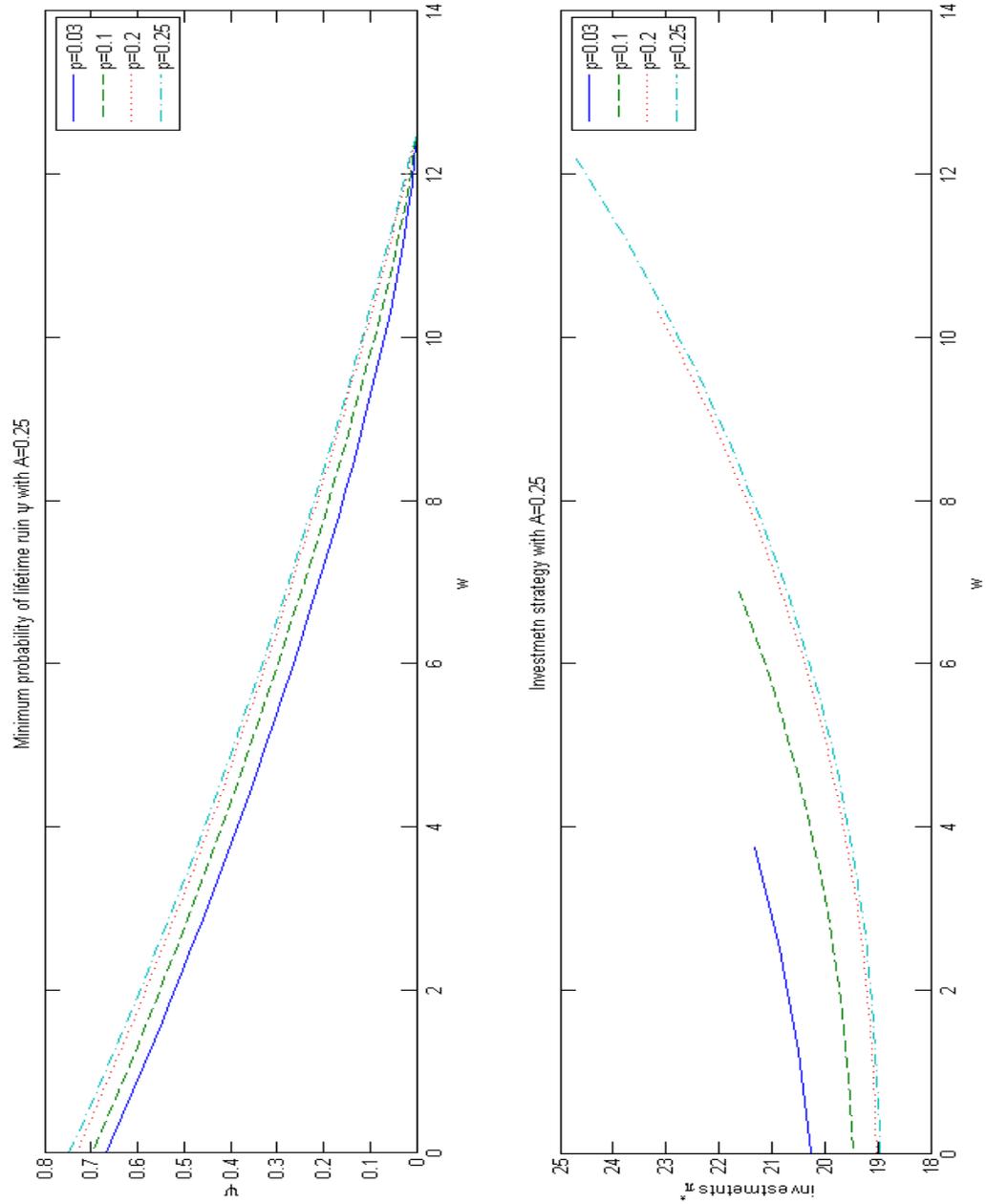


Figure 2.11: Ruin probabilities and optimal investment strategies for different p when A is 0.25

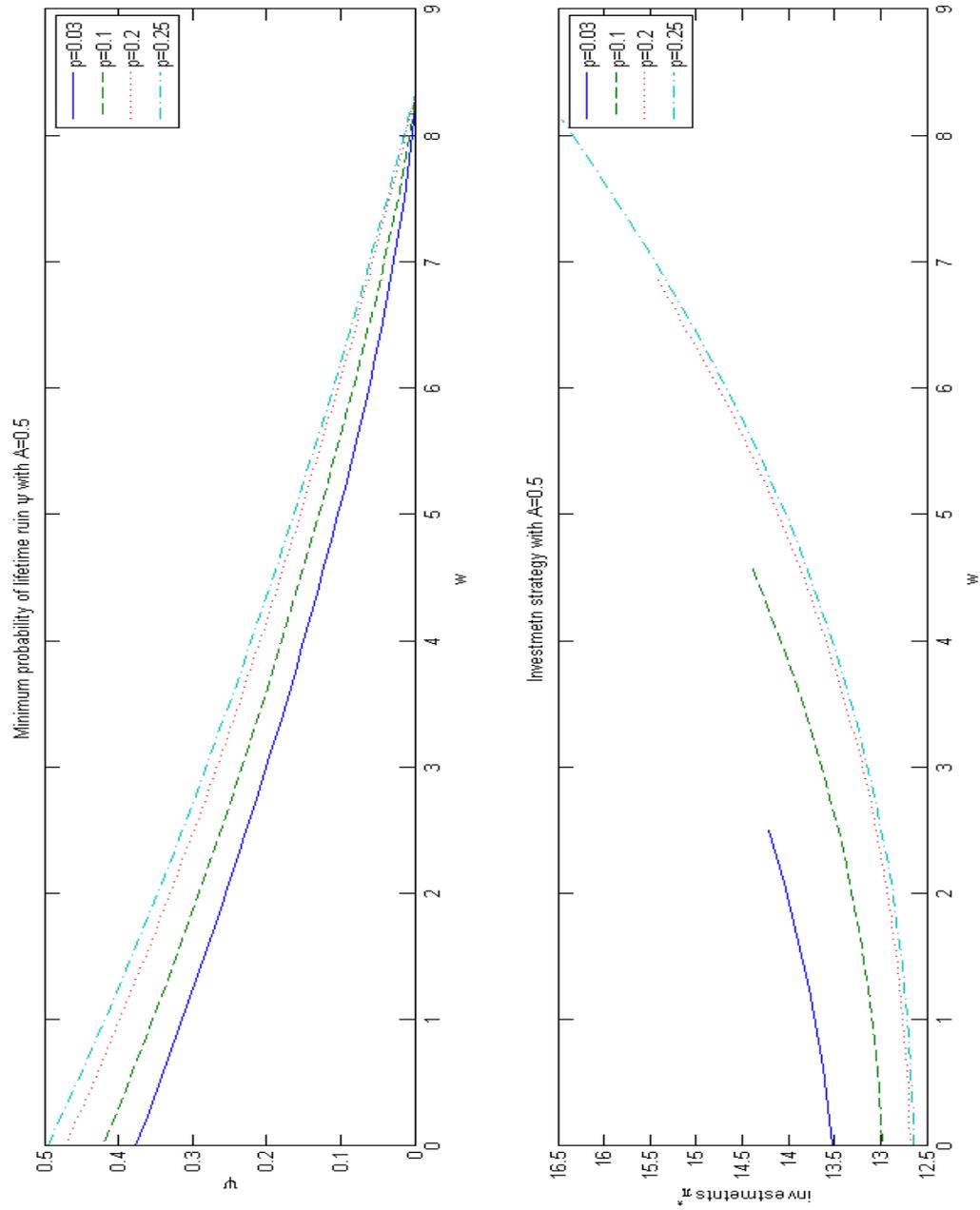


Figure 2.12: Ruin probabilities and optimal investment strategies for different p when A is 0.5

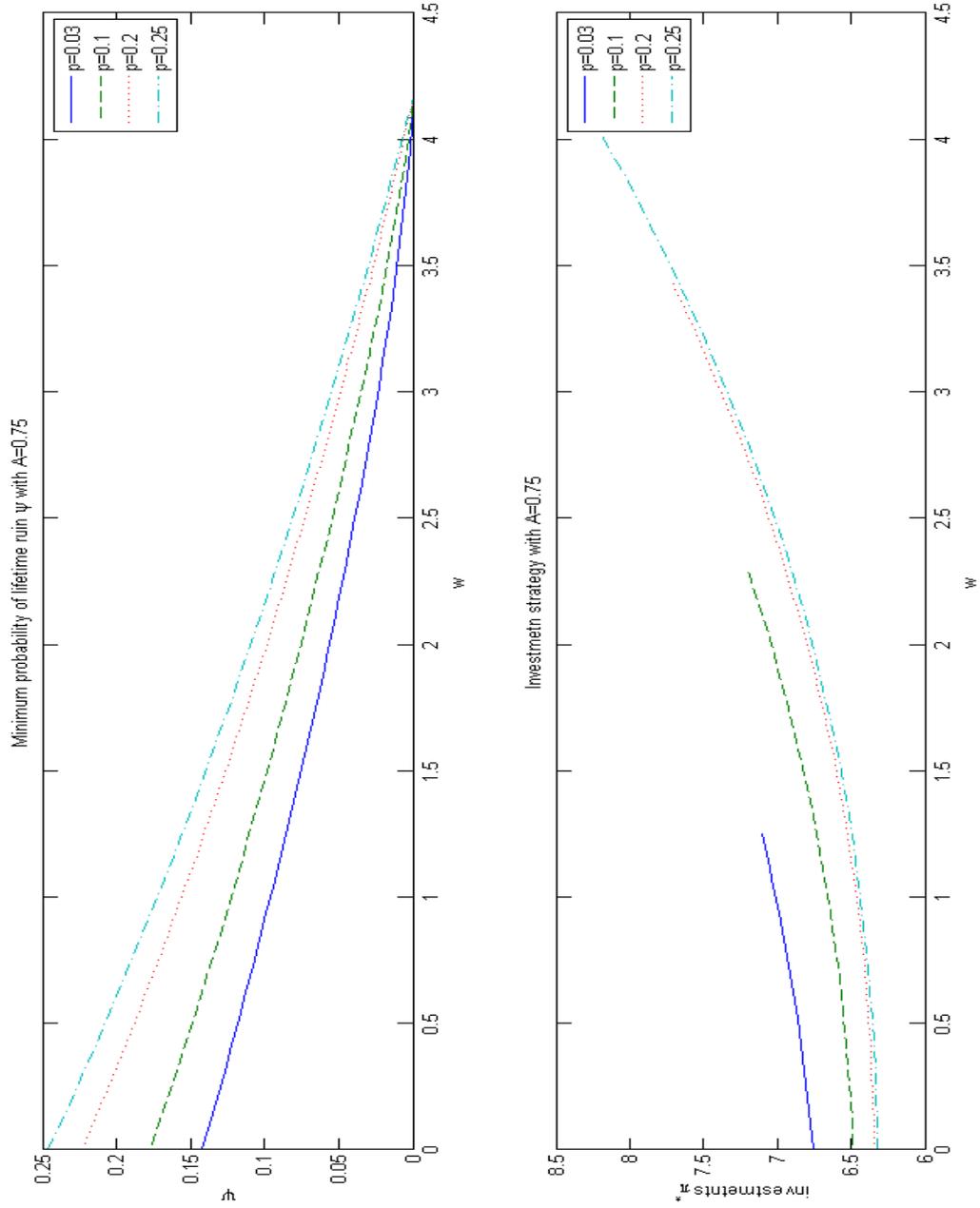


Figure 2.13: Ruin probabilities and optimal investment strategies for different p when A is 0.75

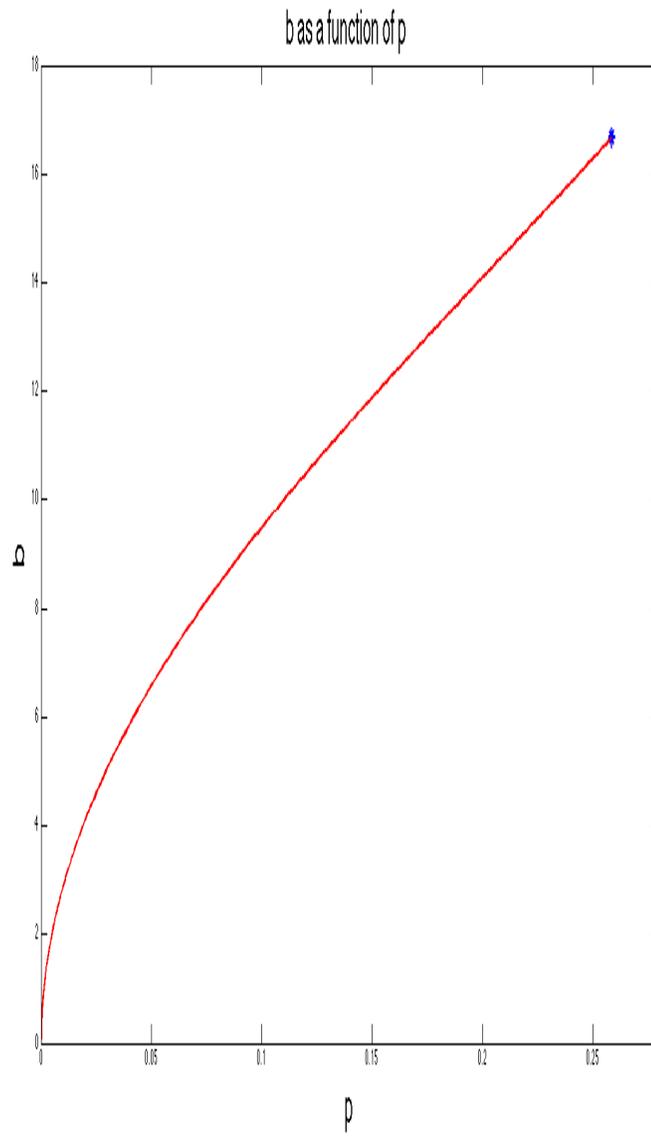


Figure 2.14: b as a function of p

CHAPTER III

Maximizing the utility of consumption with reversible annuities

3.1 Introduction and motivation

As a financial product designed for hedging lifetime uncertainty, a life annuity is a contract between an annuitant and an insurance company. For a single premium immediate annuity (SPIA), in exchange for a lump sum payment, the company guarantees to pay the annuitant a fix amount of money periodically until her death. Optimal investment problems in a market with life annuities have been extensively studied since the seminal paper of Yaari [1965]; see, for example, the references in Milevsky and Young [2007]. With the assumption that there are only bonds and annuities in the financial market, Yaari [1965], as well as Davidoff et al. [2005] among others, prove that it is optimal for an individual with no bequest motive to fully annuitize. In reality, the volume of voluntary purchases by retirees is much lower than predicted by such models, which is the so-called “annuity puzzle.” According to a recent survey in the United Kingdom by Gardner and Wadsworth [2004], over half of the individuals in the sample chose not to annuitize given the option. The dominant reason given for not wanting to annuitize is the preference for flexibility. It is well known that annuity income is *not* reversible. Annuity holders can neither surrender for a refund nor short-sell (borrow against) their purchased annuities, even

when they are in urgent need of money. This paper is motivated by the potential relation between the irreversibility of life annuities, specifically SPIAs, and retirees' reluctance to annuitize.

In this paper, we investigate how reversibility (a type of flexibility) of an annuity affects the annuitization, consumption, and investment strategies of an retiree. To this end, we assume the existence of a market of *reversible life annuities*, a riskless asset (bond or money market), and a risky asset (stock). The reversible annuity, which is a SPIA with a surrender option, has both a purchase price and a surrender value. The purchase price of this reversible annuity is equal to the present value of expected future payments to the annuity holder. The surrender value is the purchase price less a proportional surrender charge (denoted by p). A retiree is allowed to purchase additional annuity income or to surrender her existing annuity income. She can invest in the other assets in the market as well. To model the behavior of a utility-maximizing retiree in such a financial market, we formulate a continuous-time optimal consumption and asset allocation problem. We assume that the utility function of the retiree exhibits constant relative risk aversion (CRRA), and we determine the optimal strategy that maximizes the expected utility of lifetime consumption. We are especially interested in the relation between the optimal annuitization strategy and the size of the proportional surrender charge, the factor that determines the financial flexibility of the annuities.

Our model is an extension of the classical asset allocation framework of [Merton \[1971\]](#). Merton considers the problem of optimal consumption and investment in a complete market with a riskless asset and a risky asset. [Cox and Huang \[1989\]](#) first extended the model to the case of an incomplete market. [He and Pages \[1993\]](#) considered the case with the presence of labor income. [Koo \[1998\]](#) considered the case

in which labor income is subject to uninsurable risk and a liquidity constraint. [Davis and Norman \[1990\]](#) extended the model to an imperfect market in which buying and selling of the risky asset is subject to proportional transaction costs. [Øksendal and Sulem \[2002\]](#) considered the case with the presence of both fixed and proportional transaction costs. See also [Elie and Touzi \[2008\]](#), [Karatzas et al. \[1997\]](#), [Tahar et al. \[2005\]](#), and [Egami and Iwaki \[2008\]](#) for other extensions. The problem treated in our paper is a direct generalization of the one in [Milevsky and Young \[2007\]](#), in which the life annuity is irreversible.

The reversibility of annuities in our model complicates the optimal decisions of the retiree. It leads to a two-dimensional optimal control problem in an incomplete market. The optimal strategy depends on two state variables, wealth and existing annuity income. Taking advantage of the homogeneity of CRRA utility, we simplify our problem to a one-dimensional equivalent problem. Via a duality argument, we solve for the maximized utility and the optimal strategies in feedback form. We prove the optimality of these solutions through a verification theorem. The advantage of the duality method we use is that it is not necessary to formulate a control problem that is “dual” to the original one. [Milevsky et al. \[2006a\]](#) and [Milevsky and Young \[2007\]](#) also apply this duality argument.

In this paper, we find that when the proportional surrender charge is smaller than a critical value, an individual keeps wealth to one side of a separating ray in wealth-annuity space by purchasing more annuity income. The slope of this ray increases as the the proportional surrender charge decreases; that is, an individual is more willing to annuitize as p decreases. When her wealth reaches zero, the individual continues to invest in the risky asset by borrowing from the riskless account and surrenders just enough annuity income to keep her wealth non-negative when needed

In contrast, when the proportional surrender charge is larger than this critical value, an individual does not invest in the risky asset when her wealth is zero. Additionally, the retiree does not surrender her annuity income; instead, she reduces her consumption to a level lower than her annuity income in order to accumulate wealth. More surprisingly, we find that in the case when the surrender charge is larger than the critical value, the optimal annuitization, investment, consumption strategies do not depend on the size of the surrender charge. An individual behaves as if the annuity is not reversible and does not surrender existing annuity income under any circumstance. We use a variety of numerical examples to demonstrate our results.

The remainder of this paper is organized as follows: In Section 3.2, we present the financial market in which the individual invests her wealth. In addition to investing in riskless and risky assets, the individual can purchase or surrender reversible life annuities. In Section 3.3.1, we consider two special cases: $p = 0$ and $p = 1$. We solve the case $p = 0$ in the primal space by connecting it to the classical Merton problem. By analyzing the retirees' optimal strategies in these two special cases, we gain insight in solving the more general cases. We consider the case when the proportional surrender charge is smaller than some critical value in Section 3.3.2, and in Section 3.3.3, we discuss the case when the proportional surrender charge is larger than some critical value. We present properties of the optimal strategies in Section 3.4 both analytically and numerically. Section 3.5 concludes our paper.

3.2 Problem formulation

In this section, we first introduce the assets in the financial market: a riskless asset (bond or money market account), a risky asset (stock) and reversible life annuities.

Then, we define the maximized utility function, which is the objective function for our optimal control problem. After that, we preliminarily discuss a retiree's optimal strategy. Finally, we construct a verification theorem, which we will use to validate our solution in the next section.

3.2.1 The financial market and reversible life annuities

We consider an individual with future lifetime described by the random variable τ_d . We assume that τ_d is an exponential random variable with parameter λ^S , also referred to as the force of mortality or hazard rate; in particular, $\mathbb{E}[\tau_d] = 1/\lambda^S$. The superscript S indicates that the parameter equals the individual's subjective belief as to the value of her hazard rate.

We assume a *frictionless* financial market, which has no transaction costs, no taxes and no restrictions on borrowing or short selling. In this financial market, the individual can invest in or borrow from a riskless asset at interest rate $r > 0$. Also, she can buy or short sell a risky asset whose price follows geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad S_0 = S > 0, \quad (3.2.1)$$

in which $\mu > r$, $\sigma > 0$, and B is a standard Brownian motion with respect to a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that B is independent of τ_d , the random time of death of the individual.

Moreover, we assume an *unrestricted* life annuity market in which an individual can purchase any amount of reversible life annuity income or surrender any portion of existing annuity income at anytime. The price of a life annuity (specifically, a SPIA) that pays \$1 per year continuously until the individual dies is given by

$$\bar{a} = \int_0^\infty e^{-rs} e^{-\lambda^O s} ds = \frac{1}{r + \lambda^O}, \quad (3.2.2)$$

in which $\lambda^O > 0$ is the (constant) objective hazard rate that is used to price annuities. In other words, in return for each \bar{a} the individual pays for a life annuity, she receives \$1 per year of continuous annuity income until she dies. Due to the reversibility of the life annuity, she can surrender any amount of the annuity income she has already bought to get some money back from the issuer of the annuity. The surrender value of \$1 of annuity income is $(1 - p)\bar{a}$ with $0 \leq p \leq 1$, in which p is the *proportional surrender charge*. In other words, the individual can get $$(1 - p)\bar{a}$ dollars back from the issuer by giving up \$1 of annuity income. Notice that the surrender value is less than the purchase price, and the difference is the surrender charge (in dollars).

3.2.2 Utility of lifetime consumption

Following Yaari [1965], we consider a retiree without a bequest motive; therefore, utility only comes from her consumption. She chooses to consume at a rate of c_t at time t . Let π_t denote the amount invested in the risky asset at time t . Let A_t^+ denote the cumulative amount of annuity income bought on or before time t , and A_t^- the cumulative amount of annuity income surrendered on or before time t . Then, $A_t = A_t^+ - A_t^-$ equals the cumulative amount of immediate life annuity income at time t . The wealth dynamics of the individual for a given *admissible* strategy are given by

$$dW_t = [rW_{t-} + (\mu - r)\pi_{t-} - c_t + A_{t-}] dt + \sigma\pi_{t-} dB_t - \bar{a} dA_t^+ + \bar{a}(1 - p) dA_t^-, \quad W_0 = w \geq 0, \quad (3.2.3)$$

whereby the investment, consumption and annuitization strategies $\{\pi_t, c_t \geq 0, A_t \geq 0\}_{t \geq 0}$ are said to be *admissible* if

1. The processes $\{\pi_t\}_{t \geq 0}$, $\{c_t\}_{t \geq 0}$, and $\{A_t^\pm\}_{t \geq 0}$ are adapted to the filtration \mathbb{F} .
2. $\int_0^t \pi_s^2 ds < \infty$, $\int_0^t c_s ds < \infty$, and $A_t \geq 0$ *a.s.* for all $t \geq 0$.

3. The associated wealth process $W_t \geq 0$ a.s. for all $t \geq 0$.

We denote by $\mathbb{A}^p(w, A)$ the collection of all admissible strategies when the initial wealth and annuity is (w, A) , and when the corresponding surrender charge is p .

Remark 3.2.1. *We highlight our assumption that wealth does not include the imputed value $(1 - p)\bar{a}A$ of the individual's annuity. In order for the individual to include any of that amount in her wealth, she must physically surrender the corresponding annuity income, so that her income is reduced in future. In particular, we prevent the retiree from borrowing against her annuity income. This assumption is reasonable because annuity income will cease when she dies, so we do not allow her to die with negative wealth.*

We assume that the individual is risk averse and that her preferences exhibit constant relative risk aversion (CRRA); that is, the utility function for the individual is given by

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0 \text{ and } \gamma \neq 1, \quad (3.2.4)$$

in which γ is the (constant) relative risk aversion. In this paper, we assume that an individual seeks to maximize her expected utility of discounted consumption over admissible strategies $\{\pi_t, c_t, A_t\}$. In addition, we assume that an individual discounts the utility at the riskless rate r . Therefore, the maximized utility for such an individual is defined as

$$U(w, A; p) = \sup_{\{\pi_t, c_t, A_t\} \in \mathbb{A}^p(w, A)} \mathbb{E} \left[\int_0^{\tau_d} e^{-rt} u(c_t) dt \mid w_0 = w, A_0 = A, \tau_d > 0 \right]. \quad (3.2.5)$$

Remark 3.2.2. *For the rest of the this paper, we will simply write $U(w, A; p)$ as $U(w, A)$ or even U when it is appropriate.*

Remark 3.2.3. *Because we assume that the hazard rates λ^S and λ^O , as well as the financial parameters r , μ , σ and p , are constant, U only depends on the state variables w and A and not upon time.*

Remark 3.2.4. *We derive an equivalent form for the maximized utility as follows:*

$$\begin{aligned}
U(w, A; p) &= \sup_{\{\pi_t, c_t, A_t\} \in \mathbb{A}^p(w, A)} \mathbb{E} \left[\int_0^{\tau_d} e^{-rt} u(c_t) dt \mid w_0 = w, A_0 = A, \tau_d > 0 \right] \\
&= \sup_{\{\pi_t, c_t, A_t\} \in \mathbb{A}^p(w, A)} \mathbb{E} \left[\int_0^\infty \mathbf{1}_{\{\tau_d > t\}} e^{-rt} u(c_t) dt \mid w_0 = w, A_0 = A, \tau_d > 0 \right] \\
&= \sup_{\{\pi_t, c_t, A_t\} \in \mathbb{A}^p(w, A)} \mathbb{E} \left[\int_0^\infty e^{-\lambda^S t} e^{-rt} u(c_t) dt \mid w_0 = w, A_0 = A, \tau_d > 0 \right] \\
&= \sup_{\{\pi_t, c_t, A_t\} \in \mathbb{A}^p(w, A)} \mathbb{E} \left[\int_0^\infty e^{-(r+\lambda^S)t} u(c_t) dt \mid w_0 = w, A_0 = A, \tau_d > 0 \right]
\end{aligned} \tag{3.2.6}$$

We will use this expression in the proof of our verification theorem.

Proposition 3.2.1. *For an arbitrary $p \in [0, 1]$, $\mathbb{A}^0(w, A) \supseteq \mathbb{A}^p(w, A) \supseteq \mathbb{A}^1(w, A)$; thus, $U(w, A; 0) \geq U(w, A; p) \geq U(w, A; 1)$.*

Proof. If $\{\pi_t, c_t \geq 0, A_t \geq 0\}_{t \geq 0}$ is admissible when the surrender charge is p_1 , it is also admissible when surrender charge is $p_2 < p_1$. In other words, $\mathbb{A}^{p_2}(w, A) \supseteq \mathbb{A}^{p_1}(w, A)$ for $p_2 < p_1$. It follows from the definition of U that

$$\begin{aligned}
U(w, A; p_2) &= \sup_{\{\pi_t, c_t, A_t\} \in \mathbb{A}^{p_2}(w, A)} \mathbb{E} \left[\int_0^{\tau_d} e^{-rt} u(c_t) dt \mid w_0 = w, A_0 = A, \tau_d > 0 \right] \\
&\geq \sup_{\{\pi_t, c_t, A_t\} \in \mathbb{A}^{p_1}(w, A)} \mathbb{E} \left[\int_0^{\tau_d} e^{-rt} u(c_t) dt \mid w_0 = w, A_0 = A, \tau_d > 0 \right] \\
&= U(w, A; p_1),
\end{aligned} \tag{3.2.7}$$

□

3.2.3 A preliminary discussion

We now formally discuss the utility maximization problem described in previous section. Consider the strategies the retiree can choose to maximize her utility of consumption. Before her death, an individual can execute one or more of the following strategies: (i) purchase additional annuity income, (ii) surrender existing annuity income, or (iii) do neither.

Suppose that at the point (w, A) , it is optimal *not* to purchase or surrender any annuity income. In this case, it follows from Itô's lemma that the maximum utility U satisfies the following equation:

$$(r + \lambda^S) U = (rw + A)U_w + \max_{\pi} \left[\frac{1}{2} \sigma^2 \pi^2 U_{ww} + (\mu - r) \pi U_w \right] + \max_{c \geq 0} [-cU_w + u(c)]. \quad (3.2.8)$$

Because the above policy is in general suboptimal, (3.2.8) holds as an inequality; that is, for all (w, A) ,

$$(r + \lambda^S) U \geq (rw + A)U_w + \max_{\pi} \left[\frac{1}{2} \sigma^2 \pi^2 U_{ww} + (\mu - r) \pi U_w \right] + \max_{c \geq 0} [-cU_w + u(c)]. \quad (3.2.9)$$

One can show that no absolutely continuous purchasing (or surrendering) policy with bounded rate is optimal; that is, the policy of purchasing or surrendering annuity income is one of singular control. This can be verified through the same argument as in [Davis and Norman \[1990\]](#). If it is optimal to purchase annuity income instantaneously at the point (w, A) , then the individual moves instantly from (w, A) to $(w - \bar{a} \Delta A, A + \Delta A)$, for some $\Delta A > 0$. The optimality of this decision implies that

$$U(w, A) = U(w - \bar{a} \Delta A, A + \Delta A), \quad (3.2.10)$$

which in turn yields

$$\bar{a}U_w(w, A) = U_A(w, A). \quad (3.2.11)$$

Similarly, if it is optimal to surrender annuity income at the point (w, A) , the following equation holds:

$$U(w, A) = U(w + (1 - p)\bar{a} \Delta A, A - \Delta A), \quad (3.2.12)$$

which implies

$$(1 - p)\bar{a}U_w(w, A) = U_A(w, A). \quad (3.2.13)$$

Notice that the surrender value is a portion of the value of annuity determined by the proportional surrender charge p .

In general, such purchasing or surrendering policies are suboptimal; therefore, (3.2.11) and (3.2.13) hold as inequalities and become

$$\bar{a}U_w(w, A) \geq U_A(w, A), \quad (3.2.14)$$

and

$$(1 - p)\bar{a}U_w(w, A) \leq U_A(w, A). \quad (3.2.15)$$

Because the individual will either buy additional annuity income, surrender existing annuity income, or do neither, we expect that the maximum utility solves the following Hamilton-Jacobi-Bellman variational inequality (HJBVI):

$$\max \left\{ \begin{aligned} & -(\lambda^S + r)U + (rw + A)U_w + \max_{\pi} \left[(\mu - r)\pi U_w + \frac{1}{2}\sigma^2\pi^2 U_{ww} \right] \\ & + \max_{c \geq 0} \left(\frac{c^{1-\gamma}}{1-\gamma} - cU_w \right), \quad U_A - \bar{a}U_w, (1-p)\bar{a}U_w - U_A \end{aligned} \right\} = 0. \quad (3.2.16)$$

3.2.4 A verification theorem

Inspired by the discussion in Section 3.2.3, we present the following verification theorem:

Theorem 3.2.1. For any $(\pi, c) \in \mathbb{R} \times \mathbb{R}_+$, define the functional operator $\mathcal{L}^{\pi, c}$ through its action on a test function f as

$$\mathcal{L}^{\pi, c} f = -(r + \lambda^S) f + (rw + A) f_w + \left[(\mu - r) \pi f_w + \frac{1}{2} \sigma^2 \pi^2 f_{ww} \right] + \left[\frac{c^{1-\gamma}}{1-\gamma} - c f_w \right]. \quad (3.2.17)$$

Let the function $v = v(w, A)$ be non-decreasing, twice-differentiable, and concave with respect to $w > 0$ and non-decreasing and differentiable with respect to A . Suppose v satisfies the following conditions on $\mathcal{D} \triangleq \{(w, A) : w \geq 0, A \geq 0\}$:

1. $\mathcal{L}^{\pi, c} v(w, A) \leq 0$ for $(\pi, c) \in \mathbb{R} \times \mathbb{R}_+$.
2. $\bar{a} v_w(w, A) - v_A(w, A) \geq 0$.
3. $(1 - p) \bar{a} v_w(w, A) - v_A(w, A) \leq 0$.

Then,

$$v(w, A) \geq U(w, A),$$

on \mathcal{D} .

Remark 3.2.5. Note that for the utility function $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, when $0 < \gamma < 1$, the utility $u(c) > 0$ for all $c > 0$ and $u(0) = 0$. In contrast, when $\gamma > 1$, the utility $u(c) < 0$ for all $c > 0$ and $\lim_{c \rightarrow 0} u(c) = -\infty$. Thus, $U(w, A) \geq 0$ (< 0) when $0 < \gamma < 1$ ($\gamma > 1$). In particular, we have no reason to expect that $U(w, A)$ is bounded from below when $\gamma > 1$.

Proof of Theorem 3.2.1. We prove the theorem in two steps. First, we prove the theorem with two additional assumptions:

1. $v(w, A)$ is bounded from below; that is, $v(w, A) \geq \underline{V} > -\infty$ for all $(w, A) \in \mathcal{D}$.
2. $v_w(0, A) < +\infty$ for all $A \geq 0$.

Then, we remove these assumptions and show that the conclusion still holds.

Let $\tau_n^a \triangleq \inf\{s \geq 0 : \int_0^s \pi_s^2 ds \geq n\}$ and $\tau_n^b \triangleq \inf\{s \geq 0 : A \geq n\}$. Define $\tau_n = n \wedge \tau_n^a \wedge \tau_n^b$, which is a stopping time with respect to the filtration \mathbb{F} ; then, using Itô's formula for semi-martingales (see Protter [2004]), we can write, for any admissible strategy $\{\pi_t, c_t, A_t\}$,

$$\begin{aligned}
e^{-(\lambda^S+r)\tau_n} v(W_{\tau_n}, A_{\tau_n}) &= v(w, A) + \int_0^{\tau_n} e^{-(\lambda^S+r)t} v_w(W_t, A_t) \sigma \pi_t dB_t \\
&\quad + \int_0^{\tau_n} e^{-(\lambda^S+r)t} \left[\mathcal{L}^{\pi_t, c_t} v(W_t, A_t) - \frac{c_t^{1-\gamma}}{1-\gamma} \right] dt \\
&\quad + \int_0^{\tau_n} e^{-(\lambda^S+r)t} [v_A(W_t, A_t) - \bar{a} v_w(W_t, A_t)] d(A_t^+)^{(c)} \\
&\quad + \int_0^{\tau_n} e^{-(\lambda^S+r)t} [(1-p)\bar{a} v_w(W_t, A_t) - v_A(W_t, A_t)] d(A_t^-)^{(c)} \\
&\quad + \sum_{0 \leq t \leq \tau_n} e^{-(\lambda^S+r)t} [v(W_t, A_t) - v(W_{t-}, A_{t-})].
\end{aligned} \tag{3.2.18}$$

Here, $(A^\pm)^{(c)}$ is the continuous part of A^\pm , respectively; that is,

$$(A_t^\pm)^{(c)} \triangleq A_t^\pm - \sum_{0 \leq s \leq t} (A_s^\pm - A_{s-}^\pm). \tag{3.2.19}$$

Since v is non-decreasing and concave in w , $v_w^2(w, A) \leq v_w^2(0, A)$ for $w \geq 0$. Therefore,

$$\mathbb{E}^{w,A} \left[\int_0^{\tau_n} e^{-2(\lambda^S+r)t} v_w^2(W_t, A_t) \sigma^2 \pi_t^2 dt \mid W_0 = w, A_0 = A \right] < \infty, \tag{3.2.20}$$

which implies that

$$\mathbb{E}^{w,A} \left[\int_0^{\tau_n} e^{-(\lambda^S+r)t} v_w(W_t, A_t) \sigma \pi_t dB_t \mid W_0 = w, A_0 = A \right] = 0. \tag{3.2.21}$$

Here, $\mathbb{E}^{w,A}$ denotes conditional expectation given $W_0 = w$ and $A_0 = A$.

By taking expectations of equation (3.2.18), as well as using (3.2.21), assumptions (i)-(iii) in the statement of the Verification Theorem, and the additional assumptions

(i) and (ii), we obtain

$$\mathbb{E}^{w,A} \left[e^{-(\lambda^S+r)\tau_n} \underline{V} \right] \leq \mathbb{E}^{w,A} \left[e^{-(\lambda^S+r)\tau_n} v(W_{\tau_n}, A_{\tau_n}) \right] \leq v(w, A) - \mathbb{E}^{w,A} \left[\int_0^{\tau_n} e^{-(\lambda^S+r)t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right]. \quad (3.2.22)$$

In deriving (3.2.22), we also use the fact that

$$\sum_{0 \leq t \leq \tau_n} e^{-(\lambda^S+r)t} [v(W_t, A_t) - v(W_{t-}, A_{t-})] \leq 0, \quad (3.2.23)$$

because assumptions (ii) and (iii) in the statement of the Verification Theorem imply that v is non-increasing in the direction of jumps.

Since $\tau_n \nearrow \infty$ as $n \rightarrow \infty$, applying the monotonic convergence theorem to (3.2.22) yields

$$v(w, A) \geq \mathbb{E}^{w,A} \left[\int_0^\infty e^{-(\lambda^S+r)t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right]. \quad (3.2.24)$$

This implies that

$$v(w, A) \geq \sup_{\{\pi_t, c_t, A_t\} \in \mathbb{A}^p(w, A)} \mathbb{E}^{w,A} \left[\int_0^\infty e^{-(\lambda^S+r)t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right] = U(w, A). \quad (3.2.25)$$

Next, we show that the conclusion still holds even when $v(w, A)$ is not bounded from below or when $v_w(0, A)$ is not finite. We follow an argument similar to the one in [Davis and Norman \[1990\]](#). For a sequence of $\epsilon_n \searrow 0$, define $v^{\epsilon_n}(w, A) \triangleq v(w + \epsilon_n, A + \epsilon_n)$. The function v^{ϵ_n} is non-decreasing, twice-differentiable, and concave with respect to w and non-decreasing and differentiable with respect to A . Note that on \mathcal{D} , $v^{\epsilon_n}(w, A)$ is bounded from below by $v(\epsilon_n, \epsilon_n)$ and that $v_w^{\epsilon_n}(0, A) = v_w(\epsilon_n, A + \epsilon_n) < +\infty$. Since $v_w^{\epsilon_n}(w, A) = v_w(w + \epsilon_n, A + \epsilon_n)$ and $v_A^{\epsilon_n}(w, A) = v_A(w + \epsilon_n, A + \epsilon_n)$, we

have

$$\begin{aligned}
0 &\geq \mathcal{L}^{\pi,c}v(w + \epsilon_n, A + \epsilon_n) = - (r + \lambda^S)v^{\epsilon_n}(w, A) + [r(w + \epsilon_n) + (A + \epsilon_n)]v_w^{\epsilon_n}(w, A) \\
&\quad + \left[(\mu - r)\pi v_w^{\epsilon_n}(w, A) + \frac{1}{2}\sigma^2\pi^2 v_{ww}^{\epsilon_n}(w, A) \right] + \left[\frac{c^{1-\gamma}}{1-\gamma} - cv_w^{\epsilon_n}(w, A) \right] \\
&= \mathcal{L}^{\pi,c}v^{\epsilon_n}(w, A) + (r + 1)\epsilon_n v_w^{\epsilon_n}(w, A).
\end{aligned} \tag{3.2.26}$$

Because $v_w^{\epsilon_n}(w, A) \geq 0$ and $\epsilon_n > 0$, we get $\mathcal{L}^{\pi,c}v^{\epsilon_n}(w, A) \leq 0$. Also, we have

$$\bar{a} v_w^{\epsilon_n}(w, A) - v_A^{\epsilon_n}(w, A) = \bar{a} v_w(w + \epsilon_n, A + \epsilon_n) - v_A(w + \epsilon_n, A + \epsilon_n) \geq 0, \tag{3.2.27}$$

and

$$(1-p)\bar{a} v_w^{\epsilon_n}(w, A) - v_A^{\epsilon_n}(w, A) = (1-p)\bar{a} v_w(w + \epsilon_n, A + \epsilon_n) - v_A(w + \epsilon_n, A + \epsilon_n) \leq 0, \tag{3.2.28}$$

which are exactly assumptions (ii) and (iii) in Theorem 3.2.1. Therefore, $v^{\epsilon_n}(w, A) \geq U(w, A)$ for all n . Since $v(w, A)$ is continuous in both w and A , we conclude that $v(w, A) = \lim_{n \rightarrow \infty} v^{\epsilon_n}(w, A) \geq U(w, A)$. \square

We use the following corollary of Theorem 3.2.1 to determine U , the maximized utility, along with an optimal strategy.

Corollary 3.2.1. *Suppose v is the expected utility of lifetime consumption associated with an admissible strategy $\{\pi_t, c_t, A_t\}$. If v satisfies the conditions in Theorem 3.2.1 with equality and the condition that*

$$\lim_{n \rightarrow \infty} \mathbb{E}^{w,A} \left[e^{-(\lambda^S+r)\tau_n} v(W_{\tau_n}, A_{\tau_n}) \right] = 0, \tag{3.2.29}$$

with the stopping time $(\tau_n)_{n \geq 1}$ defined in the proof of Theorem 3.2.1, then $v = U$ on \mathcal{D} and the associated strategy is optimal.

Proof. The proof is similar to the proof for Theorem 3.2.1. Note that when all the inequalities in the Verification Theorem hold as equalities for the specific strategy, the second inequality in (3.2.22) becomes an equality. Take the limit on both sides of that equality and apply the condition (3.2.29) to prove this corollary. \square

3.3 Determining the maximized utility U

To solve the utility maximization problem defined in the previous section, we first consider two special cases: $p = 0$ and $p = 1$, which give hints for the optimal strategy of a retiree in a more general case when p is arbitrary. After analyzing these two special cases, we solve the general case for which $p \in [0, 1]$.

3.3.1 Two special cases: $p = 0$ and $p = 1$

Consider the two special cases: $p = 0$ and $p = 1$. The significance of these two cases is already indicated by Proposition 3.2.1. By investigating the cases for $p = 0$ and $p = 1$, we will find upper and lower bounds for both the maximized utility function and the admissible strategy set for an arbitrary p .

We first consider the case $p = 0$, in which the annuity is completely reversible. We solve this special case by connecting it to a classical Merton problem. When $p = 0$, the life annuity acts as another money market account with a higher risk free rate, namely, $r + \lambda^O$. It is optimal for the individual to annuitize all her wealth immediately and to invest in the risky asset with money borrowed from riskless asset at rate r . As an optimal strategy, all her earnings will be used to purchase more annuity income immediately, and all her losses will be paid back by surrendering existing annuity income. Although there exist two accounts growing with different rates (r versus $r + \lambda^O$), there is no arbitrage opportunity to make an arbitrary amount of money, due to our restriction that wealth be non-negative. Since the imputed value of the

annuity is not included in wealth, the individual is prohibited from purchasing an arbitrary amount of annuity income with money borrowed from the riskless account.

The wealth W_t is always zero when the individual follows the optimal strategy describe above, for $t > 0$, because she fully annuitizes at $t = 0$. Therefore, we only need to consider the dynamics of her annuity income A_t . For convenience of the following discussion, we use \bar{W}_t to represent the imputed value of her annuity income A_t (that is, $\bar{W}_t = \bar{a} A_t$), and we give the dynamics of \bar{W}_t instead. Assume that the individual shorts $\bar{\pi}_t$ in riskless asset, invests this amount of money in the risky asset, and consumes at a rate of \bar{c}_t . The dynamics of \bar{W}_t are given by

$$\begin{aligned} d\bar{W}_t &= [(r + \lambda^O) \bar{W}_t + (\mu - r) \bar{\pi}_t - \bar{c}_t] dt + \sigma \bar{\pi}_t dB_t. \\ &= \{(r + \lambda^O) \bar{W}_t + [(\mu + \lambda^O) - (r + \lambda^O)] \bar{\pi}_t - \bar{c}_t\} dt + \sigma \bar{\pi}_t dB_t. \end{aligned} \quad (3.3.1)$$

with

$$\bar{W}_0 = \bar{w} \triangleq w + \bar{a} A. \quad (3.3.2)$$

Therefore, the utility maximization problem becomes

$$\begin{aligned} U(w, A) &= \sup_{\{\bar{\pi}_t, \bar{c}_t\}} \mathbb{E} \left[\int_0^{\tau_d} e^{-rt} u(\bar{c}_t) dt \mid W_0 = w, A_0 = A, \tau_d > 0 \right] \\ &= \sup_{\{\bar{\pi}_t, \bar{c}_t\}} \mathbb{E} \left[\int_0^{+\infty} e^{-(r+\lambda^S)t} u(\bar{c}_t) dt \mid \bar{W}_0 = w + \bar{a}A, \tau_d > 0 \right]. \end{aligned} \quad (3.3.3)$$

In fact, the stochastic optimization problem defined by (3.3.1)-(3.3.3) is equivalent to Merton's problem of solving for

$$\bar{U}(\bar{w}) = \sup_{\{\bar{\pi}_t, \bar{c}_t\}} \mathbb{E} \left[\int_0^{+\infty} e^{-(r+\lambda^S)t} u(\bar{c}_t) dt \mid \bar{W}_0 = \bar{w} \right], \quad (3.3.4)$$

in a complete market with a riskless asset \bar{P}_t that evolves as

$$d\bar{P}_t = (r + \lambda^O) \bar{P}_t dt, \quad (3.3.5)$$

and a risky asset \bar{S}_t that follows

$$d\bar{S}_t = (\mu + \lambda^O) \bar{S}_t dt + \sigma \bar{S}_t dB_t. \quad (3.3.6)$$

It is well known that, with CRRA utility $u(\bar{c}) = \frac{\bar{c}^{1-\gamma}}{1-\gamma}$, Merton's problem defined by (3.3.4)-(3.3.6) is explicitly solvable. Indeed, define

$$K \triangleq \frac{1}{\gamma} \left[(r + \lambda^S) - (1 - \gamma)(r + \lambda^O) - \frac{1 - \gamma}{\gamma} m \right], \quad (3.3.7)$$

with $m = \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2$. The maximized utility and optimal strategies for this version of Merton's problem are given by

$$\bar{U}(\bar{w}) = K^{-\gamma} \frac{\bar{w}^{1-\gamma}}{1-\gamma}, \quad (3.3.8)$$

$$\bar{c}^*(\bar{w}) = K \bar{w}, \quad (3.3.9)$$

$$\bar{\pi}^*(\bar{w}) = \frac{\mu - r}{\gamma \sigma^2} \bar{w}. \quad (3.3.10)$$

Note that the solutions (3.3.8)-(3.3.10) hold under the following well-posedness condition:

$$K = \frac{1}{\gamma} \left[(r + \lambda^S) - (1 - \gamma)(r + \lambda^O) - m \frac{1 - \gamma}{\gamma} \right] > 0. \quad (3.3.11)$$

If this condition is not satisfied, arbitrarily large discounted utility could be obtained by a strategy of prolonged investment followed by massive consumption. With the result above, we directly obtain the maximized utility for (3.3.3) with $p = 0$ as follows:

$$U(w, A) = K^{-\gamma} \frac{(w + \bar{a}A)^{1-\gamma}}{1-\gamma}. \quad (3.3.12)$$

Remark 3.3.1. *We assume that the condition (3.3.11) holds for rest of this paper. Note that given the same initial wealth w and annuity income A , the maximized utility function for $p > 0$ is less than that for $p = 0$. Therefore, by imposing the condition (3.3.11), we guarantee that maximized utility with any initial w and A is finite for all $p \in [0, 1]$.*

When $p = 1$, we have a utility maximization problem with irreversible annuities. This problem is thoroughly investigated via duality argument by [Milevsky and Young \[2007\]](#). In their paper, the authors found that an individual purchases annuity to keep herself to one side of a separating *ray* in wealth-annuity space and stops investing in risky asset when her wealth is zero. Since the annuity is irreversible, she will never surrender it. The reader can refer to that paper for more details.

These two extreme cases help us to better understand our optimization problem for $p \in (0, 1)$. For an arbitrary value of p , we expect that the maximized utility is bounded by those of the two extreme cases. We also expect that the optimal strategy is similar to one of the two cases above. Inspired by solutions for these two special cases, we solve our problem for an arbitrary proportional surrender charge $p \in [0, 1]$ in the following sections.

3.3.2 The case for which $p < p^*$

Consider the utility maximization problem in the region $\mathcal{D} \triangleq \{(w, A) : w \geq 0, A \geq 0\}$ with an arbitrary value of p . For a strategy to be admissible, the wealth w of the individual is required to be non-negative. In other words, whenever one's wealth reaches zero, she is forced to keep her wealth from further decline. Through our study, we learn that an individual handles this situation in one of two ways, depending on size of the proportional surrender charge p . If $p < p^*$, a critical value to be determined later, the retiree prefers to keep investing in the risky asset by borrowing the same amount from the riskless account. To keep her wealth from declining due to a decline in the price of the risky asset, she surrenders some of her existing annuity income when needed. If $p \geq p^*$, then the surrender charge is large, and she prefers to stop investing. Instead, by saving from the annuity income, she increases her wealth to $w > 0$. We investigate the two cases of $p < p^*$ and $p \geq p^*$ in

this section and in Section 3.3.3, respectively. Figure 3.1 demonstrates the optimal annuitization strategies in both cases on the wealth-annuity $(w - A)$ plane.

When $p < p^*$, we hypothesize that there exists a critical ratio z_0 of wealth-to-annuity income. At a point (w, A) such that $w/A > z_0$, the individual purchases annuity income to raise her annuity to A' and reduces her wealth to w' such that $w'/A' = z_0$. Additionally, we hypothesize that the individual neither purchases nor surrenders annuity income if $0 < w/A \leq z_0$. On the boundary $w = 0$, we hypothesize that she will surrender existing annuity income when needed to keep her wealth w non-negative.

We set up equations and boundary conditions based on our hypotheses above and solve for U and the associated optimal strategies. We also determine the critical proportional surrender charge p^* and critical ratio of wealth-to-annuity income z_0 . Lastly, we verify our hypothesis through the Verification Theorem 3.2.1.

Based on our hypotheses for the optimal strategies, we formulate the problem as follows:

1. In the region $\mathcal{D}_1 \triangleq \{(w, A) : 0 \leq w/A \leq z_0, A \geq 0\}$, U is the solution of the following boundary value problem (BVP)

$$(\lambda^S + r)U = (rw + A)U_w + \max_{\pi} \left[(\mu - r)\pi U_w + \frac{1}{2}\sigma^2\pi^2 U_{ww} \right] + \max_{c \geq 0} \left(\frac{c^{1-\gamma}}{1-\gamma} - cU_w \right), \quad (3.3.13)$$

with boundary conditions

$$U_A(z_0A, A) = \bar{a} U_w(z_0A, A), \quad (3.3.14)$$

$$U_A(0, A) = (1 - p)\bar{a} U_w(0, A), \quad (3.3.15)$$

for $A \geq 0$.

2. In the region $\mathcal{D}_2 \triangleq \{(w, A) : w/A > z_0, A \geq 0\}$, we have

$$U(w, A) = U(w - \bar{a} \Delta A, A + \Delta A), \quad (3.3.16)$$

in which $\Delta A = \frac{w - z_0 A}{z_0 + \bar{a}}$. Notice that $(w - \bar{a} \Delta A, A + \Delta A) \in \partial \mathcal{D}_1$.

3. Since the separating line $w/A = z_0$ for purchasing annuity income is optimally chosen, we apply the smooth fit condition

$$U_{Aw}(z_0 A, A) = \bar{a} U_{ww}(z_0 A, A), \quad A > 0. \quad (3.3.17)$$

This condition is also assumed in [Davis and Norman \[1990\]](#) and [Karatzas et al. \[2000\]](#). An intuitive derivation of the smooth fit condition for optimal regulation problem in a discrete-time setting is given in [Dixit \[1991\]](#). A discussion on the smooth fit condition can be found in [Dumas \[1991\]](#).

Dimension reduction

The value function U defined in [\(3.2.6\)](#) is homogeneous of degree $1 - \gamma$ with respect to wealth w and annuity A due to the homogeneity property of the CRRA utility function. More precisely, $U(\alpha w, \alpha A) = \alpha^{1-\gamma} U(w, A)$ for $\alpha > 0$. We utilize this property to define $V(z) = U(z, 1)$ and to write U in terms of V by

$$U(w, A) = A^{1-\gamma} V(w/A), \quad \text{for } A > 0. \quad (3.3.18)$$

This transform simplifies our problem by reducing it to a one-dimensional problem. Implementations of this transformation in optimal consumption and investment problems can be also found in [Davis and Norman \[1990\]](#) and [Koo \[1998\]](#).

Now, we apply the dimension reduction to equations [\(3.3.13\)](#)-[\(3.3.17\)](#) to get a set of equations for $V(z)$ with $z = w/A$. Then, we will solve for V and recover U from

V . To that end, we obtain the following BVP for V :

$$-(\lambda^S + r)V + (rz + 1)V_z + \max_{\hat{\pi}} \left[(\mu - r)\hat{\pi}V_z + \frac{1}{2}\sigma^2\hat{\pi}^2V_{zz} \right] + \max_{\hat{c} \geq 0} \left(\frac{\hat{c}^{1-\gamma}}{1-\gamma} - \hat{c}V_z \right) = 0, \quad 0 \leq z \leq z_0, \quad (3.3.19)$$

$$(1 - \gamma)V(z_0) = (z_0 + \bar{a})V_z(z_0), \quad (3.3.20)$$

$$(1 - \gamma)V(0) = (1 - p)\bar{a}V_z(0). \quad (3.3.21)$$

In terms of V , the smooth fit condition (3.3.17) is

$$(z_0 + \bar{a})V_{zz}(z_0) + \gamma V_z(z_0) = 0. \quad (3.3.22)$$

Linearization via the Legendre transform

To avoid handling the non-linear PDE in equation (3.3.19) directly, we apply the Legendre transform to $V(z)$, the solution of the above BVP, and obtain a linear PDE with free boundaries for the convex dual $\hat{V}(y)$ of $V(z)$. Recall that the maximized utility U is non-decreasing and concave with respect to w because the CRRA utility function is non-decreasing and concave; thus, $V(z) = U(z, 1)$ is also non-decreasing and concave. However, we do not know *a priori* that the solution V of the BVP is non-decreasing and concave. Despite this shortcoming, we hypothesize that the solution of the BVP is non-decreasing and concave, which allows us to define its convex dual via the Legendre transform. Later, in Proposition 3.3.4, we will show that the convex dual is *indeed* convex, so the solution to the BVP, V , is concave.

Define

$$\hat{V}(y) = \max_{z \geq 0} [V(z) - yz]. \quad (3.3.23)$$

For a given y , the critical z^* that maximizes $V(z) - yz$ solves $V_z(z^*) - y = 0$. Thus,

$$z^* = I(y), \quad (3.3.24)$$

in which I is the inverse of $V_z(z)$. It follows that

$$\hat{V}_y(y) = -V_z^{-1}(y) = -z^* \leq 0, \quad (3.3.25)$$

$$\hat{V}_{yy} = -\frac{1}{V_{zz}(z)} \Big|_{z=V_z^{-1}(y)} \geq 0. \quad (3.3.26)$$

Rewrite (3.3.19) in terms of \hat{V} to obtain a linear PDE for $\hat{V}(y)$:

$$-(r + \lambda^S)\hat{V}(y) + \lambda^S y \hat{V}_y(y) + m y^2 \hat{V}_{yy}(y) + y + \frac{\gamma}{1-\gamma} y^{\frac{\gamma-1}{\gamma}} = 0, \quad (3.3.27)$$

in which $m = \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2$. The general solution of (3.3.27) is given by

$$\hat{V}(y) = D_1 y^{B_1} + D_2 y^{B_2} + \frac{y}{r} + C y^{\frac{\gamma-1}{\gamma}}, \quad (3.3.28)$$

in which

$$B_1 = \frac{1}{2m} \left[(m - \lambda^S) + \sqrt{(m - \lambda^S)^2 + 4m(r + \lambda^S)} \right] > 1, \quad (3.3.29)$$

$$B_2 = \frac{1}{2m} \left[(m - \lambda^S) - \sqrt{(m - \lambda^S)^2 + 4m(r + \lambda^S)} \right] < 0, \quad (3.3.30)$$

$$C = \frac{\gamma}{1-\gamma} \left[r + \frac{\lambda^S}{\gamma} - m \frac{1-\gamma}{\gamma^2} \right]^{-1}. \quad (3.3.31)$$

Remark 3.3.2. Condition (3.3.11) implies that

$$r + \frac{\lambda^S}{\gamma} - m \frac{1-\gamma}{\gamma^2} > 0. \quad (3.3.32)$$

Because $[1 + \gamma(B_1 - 1)] \cdot [1 + \gamma(B_2 - 1)] = -\frac{\gamma^2}{m} \left[r + \frac{\lambda^S}{\gamma} - m \frac{1-\gamma}{\gamma^2} \right]$, the inequality above implies that $1 + \gamma(B_2 - 1) < 0$.

Define

$$y_s = V_z(0), \quad (3.3.33)$$

$$y_b = V_z(z_0). \quad (3.3.34)$$

Thus,

$$\hat{V}_y(y_s) = 0, \quad (3.3.35)$$

$$\hat{V}_y(y_b) = -z_0. \quad (3.3.36)$$

Under the assumption that V is non-decreasing and concave, it follows that $0 \leq y_b \leq y_s$, and we discuss this further when we solve for y_b and y_s .

Rewrite equations (3.3.20), (3.3.21), and (3.3.22) in terms of $\hat{V}(y)$ to get

$$(1 - \gamma)\hat{V}(y_b) + \gamma y_b \hat{V}_y(y_b) = \bar{a} y_b, \quad (3.3.37)$$

$$(1 - \gamma)\hat{V}(y_s) = (1 - p)\bar{a} y_s, \quad (3.3.38)$$

$$\hat{V}_y(y_b) + \gamma y_b \hat{V}_{yy}(y_b) = \bar{a}. \quad (3.3.39)$$

Instead of solving for $V(z)$ directly, we solve for $\hat{V}(y)$ given by the free-boundary problem (FBP) (3.3.27) with boundary conditions (3.3.37) and (3.3.38) and smooth fit condition (3.3.39).

To solve for $\hat{V}(y)$, we substitute (3.3.28) into (3.3.35)-(3.3.39) to get

$$D_1[1 + \gamma(B_1 - 1)]y_b^{B_1} + D_2[1 + \gamma(B_2 - 1)]y_b^{B_2} + \frac{y_b}{r} = \frac{y_b}{r + \lambda^O} \quad (3.3.40)$$

$$(1 - \gamma) D_1 y_s^{B_1} + (1 - \gamma) D_2 y_s^{B_2} + (1 - \gamma) \frac{y_s}{r} + (1 - \gamma) C y_s^{\frac{\gamma-1}{\gamma}} = \frac{(1-p)y_s}{r + \lambda^O} \quad (3.3.41)$$

$$D_1 B_1 [1 + \gamma(B_1 - 1)] y_b^{B_1-1} + D_2 B_2 [1 + \gamma(B_2 - 1)] y_b^{B_2-1} + \frac{1}{r} = \frac{1}{r + \lambda^O} \quad (3.3.42)$$

$$D_1 B_1 y_s^{B_1-1} + D_2 B_2 y_s^{B_2-1} + \frac{1}{r} + \frac{\gamma-1}{\gamma} C y_s^{-\frac{1}{\gamma}} = 0, \quad (3.3.43)$$

$$D_1 B_1 y_b^{B_1-1} + D_2 B_2 y_b^{B_2-1} + \frac{1}{r} + \frac{\gamma-1}{\gamma} C y_b^{-\frac{1}{\gamma}} = -z_0. \quad (3.3.44)$$

We solve for D_1 , D_2 , y_s , y_b , and z_0 from the equations above. From (3.3.40) and (3.3.42), we get

$$D_1 = -\frac{\lambda^O}{r(r + \lambda^O)} \frac{1 - B_2}{B_1 - B_2} \frac{1}{1 + \gamma(B_1 - 1)} y_b^{1-B_1}, \quad (3.3.45)$$

$$D_2 = -\frac{\lambda^O}{r(r + \lambda^O)} \frac{B_1 - 1}{B_1 - B_2} \frac{1}{1 + \gamma(B_2 - 1)} y_b^{1-B_2}. \quad (3.3.46)$$

By substituting (3.3.45) and (3.3.46) into (3.3.41) and (3.3.43) (and eliminating the term with C in it), we obtain

$$\frac{1 - B_2}{B_1 - B_2} x^{B_1-1} + \frac{B_1 - 1}{B_1 - B_2} x^{B_2-1} = 1 + \frac{pr}{\lambda^O}, \quad (3.3.47)$$

with $x \triangleq \frac{y_s}{y_b}$. Note that x has a unique solution in $[1, +\infty)$ because (i) when $x = 1$, LHS = 1 \leq RHS; (ii) when $x \rightarrow \infty$, LHS $\rightarrow \infty >$ RHS; and (iii) $\frac{d\text{LHS}}{dx} > 0$ for $x \in (1, +\infty)$. Given B_1 and B_2 , x is a function of p . In our paper, when we write x , we will mean this unique solution.

We substitute (3.3.45) and (3.3.46) into (3.3.43) to get

$$\frac{\lambda^O}{r(r + \lambda^O)} \frac{B_1(1 - B_2)}{B_1 - B_2} \frac{x^{B_1-1}}{1 + \gamma(B_1 - 1)} - \frac{\lambda^O}{r(r + \lambda^O)} \frac{B_2(B_1 - 1)}{B_1 - B_2} \frac{x^{B_2-1}}{1 + \gamma(B_2 - 1)} + \frac{1}{r} = \frac{1 - \gamma}{\gamma} C y_s^{-\frac{1}{\gamma}}. \quad (3.3.48)$$

This expression gives y_s in terms of x , and y_b is, therefore,

$$y_b = \frac{y_s}{x}. \quad (3.3.49)$$

The purchasing boundary z_0 is obtained by substituting (3.3.45) and (3.3.46) into (3.3.44),

$$\begin{aligned} z_0 &= \frac{\lambda^O}{r(r + \lambda^O)} \frac{1 - \gamma + B_1 B_2 \gamma}{[1 + \gamma(B_1 - 1)][1 + \gamma(B_2 - 1)]} - \frac{1}{r} - \frac{\gamma - 1}{\gamma} C y_b^{-\frac{1}{\gamma}} \\ &= -\frac{\lambda^O}{r(r + \lambda^O)} \frac{m(1 - \gamma) - (r + \lambda^S)\gamma}{\gamma^2 \left[r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right]} - \frac{1}{r} - \frac{\gamma - 1}{\gamma} C y_b^{-\frac{1}{\gamma}}. \end{aligned} \quad (3.3.50)$$

Proposition 3.3.1. *The solution for the FBP (3.3.27) with conditions (3.3.37), (3.3.38), and (3.3.39) is given by (3.3.28), with D_1 , D_2 , y_s , y_b and x defined in (3.3.45), (3.3.46), (3.3.48), (3.3.49), and (3.3.47), respectively.*

Remark 3.3.3. *To make the solution meaningful, we require that $y_s > 0$, which implies that $0 < y_b \leq y_s$ because $x \geq 1$. After we define the critical proportional surrender charge p^* below, we will prove that $y_s > 0$ for $0 \leq p < p^*$.*

The Critical proportional surrender charge p^*

We will demonstrate later that the value of the critical proportional surrender charge is given by

$$p^* = \frac{\lambda^O}{r} \left[\frac{1 - B_2}{B_1 - B_2} \tilde{x}^{B_1-1} + \frac{B_1 - 1}{B_1 - B_2} \tilde{x}^{B_2-1} - 1 \right], \quad (3.3.51)$$

in which $\tilde{x} > 1$ is the unique solution of the following equation:

$$\frac{\lambda^O}{r + \lambda^O} \left[\frac{B_1(1 - B_2)}{B_1 - B_2} \tilde{x}^{B_1-1} + \frac{B_2(B_1 - 1)}{B_1 - B_2} \tilde{x}^{B_2-1} \right] = 1. \quad (3.3.52)$$

Note that p^* is independent of γ . Moreover, $x(p^*) = \tilde{x}$, in which $x = x(p)$ is given by (3.3.47).

In the next proposition, we show that y_s is positive.

Proposition 3.3.2. *The well-posedness condition (3.3.11) implies that y_s as defined in (3.3.48) is positive for $0 \leq p < p^*$.*

Proof. The well-posedness condition implies that $(1 - \gamma)C$ defined in (3.3.31) is positive, so $y_s > 0$ if and only if $h(x) > 0$ for all $1 \leq x < \tilde{x}$, in which

$$h(x) \triangleq 1 - \frac{\lambda^O}{r + \lambda^O} \frac{B_1(1 - B_2)}{B_1 - B_2} \frac{x^{B_1-1}}{1 + \gamma(B_1 - 1)} - \frac{\lambda^O}{r + \lambda^O} \frac{B_2(B_1 - 1)}{B_1 - B_2} \frac{x^{B_2-1}}{1 + \gamma(B_2 - 1)}. \quad (3.3.53)$$

By differentiating h with respect to γ , one can show that h strictly increases with respect to γ . When $\gamma = 0$, we get

$$h(x) \Big|_{\gamma=0} = 1 - \frac{\lambda^O}{r + \lambda^O} \frac{B_1(1 - B_2)}{B_1 - B_2} x^{B_1-1} - \frac{\lambda^O}{r + \lambda^O} \frac{B_2(B_1 - 1)}{B_1 - B_2} x^{B_2-1}. \quad (3.3.54)$$

From the definition of \tilde{x} in (3.3.52) and from the fact that the left-hand side of (3.3.52) strictly increases with respect to \tilde{x} for $\tilde{x} > 1$, the latter expression is positive for $1 \leq x < \tilde{x}$. Thus, $h(x) > 0$ for $1 \leq x < \tilde{x}$ for all $\gamma > 0$. \square

Proposition 3.3.3. $0 < p^* < 1$.

Proof. First, we show that $p^* > 0$. Taking the derivative of p^* in (3.3.51) with respect to \tilde{x} , we have

$$\frac{dp^*}{d\tilde{x}} = \frac{\lambda^O}{r} \frac{(B_1 - 1)(1 - B_2)}{B_1 - B_2} (\tilde{x}^{B_1-2} - \tilde{x}^{B_2-2}) > 0, \quad \forall x > 1. \quad (3.3.55)$$

If $\tilde{x} = 1$, we have $p^* = 0$. Since $\tilde{x} > 1$, we conclude that $p^* > 0$. To show that $p^* < 1$, we write \tilde{x}^{B_1-1} in terms of \tilde{x}^{B_2-1} from (3.3.52) and substitute it into (3.3.51) to get

$$\begin{aligned} p^* &= \frac{1}{B_1} \frac{\lambda^O}{r} \left[\frac{r + \lambda^O}{\lambda^O} + (B_1 - 1)\tilde{x}^{B_2-1} - B_1 \right] \\ &\leq \frac{1}{B_1} \frac{\lambda^O}{r} \left[\frac{r + \lambda^O}{\lambda^O} + (B_1 - 1) - B_1 \right] = \frac{1}{B_1} < 1. \end{aligned} \quad (3.3.56)$$

□

Properties of \hat{V}

Rewrite (3.2.14) and (3.2.15) in terms of \hat{V} as

$$(1 - \gamma)\hat{V} + \gamma y \hat{V}_y \leq \bar{a}y, \quad (3.3.57)$$

$$(1 - \gamma)\hat{V} + \gamma y \hat{V}_y \geq (1 - p)\bar{a}y, . \quad (3.3.58)$$

In the next proposition, we will prove that the FBP's solution, \hat{V} , as determined in Proposition 3.3.1, satisfies these inequalities and satisfies other expected properties.

Remark 3.3.4. *The maximized utility U is non-decreasing and concave with respect to w because the CRRA utility function is concave. Thus, $V(z) = U(z, 1)$ is non-decreasing and concave, and $\hat{V}(y)$ is non-increasing and convex because it is the convex dual of V . However, in Proposition 3.3.1, we determine \hat{V} as the solution to a FBP that is suggested by our ansatz at the beginning of Section 3.3.2. Because we do not know a priori that our ansatz is correct, we must demonstrate that the solution \hat{V} has all the properties that we expect it to have as the Legendre transform of V .*

Proposition 3.3.4. \hat{V} given in Proposition 3.3.1 satisfies inequalities (3.3.57) and (3.3.58) and is decreasing and convex for $0 < y_b \leq y \leq y_s$.

Proof. By substituting (3.3.28) into (3.3.57) and (3.3.58), we rewrite the inequalities as

$$-\frac{pr + \lambda^O}{r(r + \lambda^O)} \leq f(y) \leq -\frac{\lambda^O}{r(r + \lambda^O)}, \quad (3.3.59)$$

in which

$$f(y) = D_1[1 + \gamma(B_1 - 1)]y^{B_1-1} + D_2[1 + \gamma(B_2 - 1)]y^{B_2-1}. \quad (3.3.60)$$

Note that $f(y_s) = -\frac{pr + \lambda^O}{r(r + \lambda^O)}$ and $f(y_b) = -\frac{\lambda^O}{r(r + \lambda^O)}$ by (3.3.41) and (3.3.40), respectively. Also, we check that

$$\begin{aligned} f'(y) &= D_1(B_1 - 1)[1 + \gamma(B_1 - 1)]y^{B_1-2} + D_2(B_2 - 1)[1 + \gamma(B_2 - 1)]y^{B_2-2} \\ &= -\frac{1}{y} \frac{\lambda^O}{r(r + \lambda^O)} \frac{(B_1 - 1)(1 - B_2)}{B_1 - B_2} \left[\left(\frac{y}{y_b} \right)^{B_1-1} - \left(\frac{y}{y_b} \right)^{B_2-1} \right] \leq 0, \end{aligned} \quad (3.3.61)$$

for $0 < y_b \leq y \leq y_s$. Therefore, the inequalities in (3.3.59) hold.

To show that \hat{V} is non-increasing, compute \hat{V}_y from (3.3.28) after substituting for D_1 , D_2 , and C :

$$\begin{aligned} \hat{V}_y(y) &= -\frac{\lambda^O}{r(r + \lambda^O)} \frac{1}{B_1 - B_2} \left[\frac{B_1(1 - B_2)}{1 + \gamma(B_1 - 1)} \left(\frac{y}{y_b} \right)^{B_1-1} + \frac{B_2(B_1 - 1)}{1 + \gamma(B_2 - 1)} \left(\frac{y}{y_b} \right)^{B_2-1} \right] \\ &\quad - \left[r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma} \right]^{-1} y^{-\frac{1}{\gamma}} \\ &< 0, \end{aligned} \quad (3.3.62)$$

in which the inequality follows because all three terms are non-negative for $0 < y_b \leq y \leq y_s$, and the third is strictly negative because of inequality (3.3.32).

As for the convexity of \hat{V} , we see that for $0 < y_b \leq y \leq y_s$,

$$\begin{aligned} & \hat{V}_{yy}(y) \geq 0 \\ \iff & -\frac{\lambda^O}{r(r+\lambda^O)} \frac{(B_1-1)(1-B_2)}{B_1-B_2} \left[\frac{B_1}{1+\gamma(B_1-1)} \left(\frac{y}{y_b}\right)^{B_1-1} - \frac{B_2}{1+\gamma(B_2-1)} \left(\frac{y}{y_b}\right)^{B_2-1} \right] \\ & + \left(1 - \frac{1}{\gamma}\right) \left(-\frac{1}{\gamma}\right) Cy^{-\frac{1}{\gamma}} \geq 0. \end{aligned} \tag{3.3.63}$$

Define

$$\begin{aligned} g(y) = & -\frac{\lambda^O}{r(r+\lambda^O)} \frac{(B_1-1)(1-B_2)}{B_1-B_2} \left[\frac{B_1}{1+\gamma(B_1-1)} \left(\frac{y}{y_b}\right)^{B_1-1} - \frac{B_2}{1+\gamma(B_2-1)} \left(\frac{y}{y_b}\right)^{B_2-1} \right] \\ & + \left(1 - \frac{1}{\gamma}\right) \left(-\frac{1}{\gamma}\right) Cy^{-\frac{1}{\gamma}}. \end{aligned} \tag{3.3.64}$$

We note that

$$\begin{aligned} g'(y) = & -\frac{1}{y_b} \frac{\lambda^O}{r(r+\lambda^O)} \frac{(B_1-1)(1-B_2)}{B_1-B_2} \left[\frac{B_1(B_1-1)}{1+\gamma(B_1-1)} \left(\frac{y}{y_b}\right)^{B_1-2} - \frac{B_2(B_2-1)}{1+\gamma(B_2-1)} \left(\frac{y}{y_b}\right)^{B_2-2} \right] \\ & + \left(1 - \frac{1}{\gamma}\right) \left(-\frac{1}{\gamma}\right)^2 Cy^{-\frac{\gamma+1}{\gamma}} < 0, \end{aligned} \tag{3.3.65}$$

in which the inequality follows because all three terms are non-negative, and the third is strictly negative because of inequality (3.3.32). Therefore, to complete our proof, we just need to show that $g(y_s) \geq 0$. Note that by (3.3.47) and (3.3.48),

$$g(y_s) = \frac{1}{\gamma r} \left[-B_1 \frac{pr + \lambda^O}{r + \lambda^O} + \frac{\lambda^O}{r + \lambda^O} (B_1 - 1) x^{B_2-1} + 1 \right]. \tag{3.3.66}$$

By taking the derivative of (3.3.47) implicitly with respect to p , we obtain

$$\frac{dx}{dp} = \frac{r}{\lambda^O} \frac{B_1 - B_2}{(B_1 - 1)(1 - B_2)} (x^{B_1-2} - x^{B_2-2})^{-1} > 0, \quad \text{for } x > 1. \tag{3.3.67}$$

Therefore,

$$\frac{dg(y_s)}{dp} = \frac{1}{\gamma r} \left[-B_1 \frac{r}{r + \lambda^O} + \frac{\lambda^O}{r + \lambda^O} (B_1 - 1)(B_2 - 1) x^{B_2-2} \frac{dx}{dp} \right] < 0. \tag{3.3.68}$$

When $p = p^*$,

$$\begin{aligned}
g(y_s) &= \frac{1}{\gamma r} \left[-B_1 \frac{p^* r + \lambda^O}{r + \lambda^O} + \frac{\lambda^O}{r + \lambda^O} (B_1 - 1) \tilde{x}^{B_2 - 1} + 1 \right] \\
&= \frac{1}{\gamma r} \left[-\frac{\lambda^O (B_1 - 1) (\tilde{x}^{B_2 - 1} - 1) + r + \lambda^O B_1}{r + \lambda^O} + \frac{\lambda^O}{r + \lambda^O} (B_1 - 1) \tilde{x}^{B_2 - 1} + 1 \right] \\
&= 0,
\end{aligned} \tag{3.3.69}$$

in which the second line follows from expressing p^* in terms of $\tilde{x}^{B_2 - 1}$ by using equations (3.3.51) and (3.3.52). Therefore, $g(y_s) \geq 0$ for any $p \in [0, p^*]$. The convexity of \hat{V} follows. \square

Remark 3.3.5. *It is at this point that we see the importance of the critical value p^* . If $p > p^*$, then \hat{V} as given in Proposition 3.3.1 is not necessarily convex, and our argument hinges on this property. If \hat{V} were not convex, then it would not be the Legendre dual of a concave function, and V must be concave.*

construction of U from \hat{V}

In this section, we construct U from \hat{V} , and verify that it is, indeed, the solution for our utility maximization problem through the Verification Theorem 3.2.1. First, we define the concave dual of \hat{V} by

$$\tilde{V}(z) = \min_{y \geq 0} [\hat{V}(y) + zy]. \tag{3.3.70}$$

For a given z , the critical y^* that minimizes $\hat{V}(y) + zy$ is given by $\hat{V}_y(y^*) + z = 0$. Thus, $y^* = \hat{V}_y^{-1}(-z)$. Then,

$$\tilde{V}_z(z) = \hat{V}_y^{-1}(-z) = y^*, \tag{3.3.71}$$

$$\tilde{V}_{zz}(z) = -\frac{1}{\hat{V}_{yy}(y)} \Big|_{y=\hat{V}_y^{-1}(-z)}. \tag{3.3.72}$$

We next define the function $\tilde{U}(w, A)$, with z_0 given by (3.3.50):

1. In the region $\mathcal{D}_1 = \{(w, A) : 0 \leq w/A \leq z_0, A \geq 0\}$,

$$\tilde{U}(w, A) \triangleq A^{1-\gamma} \tilde{V}(w/A). \quad (3.3.73)$$

2. In the region $\mathcal{D}_2 = \{(w, A) : w/A > z_0, A \geq 0\}$, define

$$\tilde{U}(w, A) \triangleq \tilde{U}(w', A'), \quad (3.3.74)$$

in which $w' = w - \bar{a}\Delta A$, $A' = A + \Delta A$, and $\Delta A = \frac{w - z_0 A}{z_0 + \bar{a}}$. It is easy to check that $w' = z_0 A'$.

In the remainder of this section, we show that the \tilde{U} defined above equals the maximized utility U for $0 \leq p \leq p^*$. First, consider $(w, A) \in \mathcal{D}_1$. Note that $\tilde{U}_w(w, A) = A^{-\gamma} \tilde{V}_z(w/A) = A^{-\gamma} y$, with $y = V_y^{-1}(-w/A)$. Because $0 \leq w/A \leq z_0$, we have that $0 < y_b \leq y \leq y_s$. It follows that $\tilde{U}_w(w, A) > 0$. Similarly, $\tilde{U}_{ww}(w, A) = A^{-(\gamma+1)} \tilde{V}_{zz}(w/A) = -A^{-(\gamma+1)} / \hat{V}_{yy}(y) \leq 0$. Therefore, $\tilde{U}(w, A)$ is increasing and concave with respect to w in the region \mathcal{D}_1 .

By rewriting (3.3.27) in terms of $\tilde{U}(w, A)$ in the region \mathcal{D}_1 , we get

$$-(\lambda^S + r)\tilde{U} + (rw + A)\tilde{U}_w - m \frac{\tilde{U}_w^2}{\tilde{U}_{ww}} + \frac{\gamma}{1-\gamma} \tilde{U}_w^{\frac{\gamma-1}{\gamma}} = 0, \quad (3.3.75)$$

or equivalently

$$\max_{\{\pi, c\}} \mathcal{L}^{\pi, c} \tilde{U} = 0, \quad (3.3.76)$$

with the optimal strategies π^* and c^* given, respectively, in feedback form by

$$\pi_t^* = -\frac{\mu - r}{\sigma^2} \frac{\tilde{U}_w(W_t^*, A_t^*)}{\tilde{U}_{ww}(W_t^*, A_t^*)}, \quad (3.3.77)$$

$$c_t^* = \tilde{U}_w^{-\frac{1}{\gamma}}(W_t^*, A_t^*), \quad (3.3.78)$$

in which W^* and A^* are the optimally controlled wealth and annuity income processes, respectively. Thus, in \mathcal{D}_1 , \tilde{U} solves the same HJB equation as the one given in equation (3.3.13).

For $(w, A) \in \mathcal{D}_1$, rewrite (3.3.57) and (3.3.58) in terms of \tilde{U} to get

$$\tilde{U}_A(w, A) \leq \bar{a} \tilde{U}_w(w, A), \quad (3.3.79)$$

$$\tilde{U}_A(w, A) \geq (1-p)\bar{a} \tilde{U}_w(w, A). \quad (3.3.80)$$

Inequality (3.3.80) implies that \tilde{U} is increasing with respect to A in \mathcal{D}_1 . From (3.3.37), (3.3.38), and (3.3.39), we have the following conditions for \tilde{U} on the boundary of \mathcal{D}_1 :

$$\tilde{U}_A(z_0A, A) = \bar{a} \tilde{U}_w(z_0A, A), \quad (3.3.81)$$

$$\tilde{U}_A(0, A) = (1-p)\bar{a} \tilde{U}_w(0, A), \quad (3.3.82)$$

$$\tilde{U}_{Aw}(z_0A, A) = \bar{a} \tilde{U}_{ww}(z_0A, A). \quad (3.3.83)$$

Based on our construction of $\tilde{U}(w, A)$ for $(w, A) \in \mathcal{D}_2$, define $(w', A') \in \mathcal{D}_1$ by $w' = z_0 \cdot \frac{w + \bar{a}A}{z_0 + \bar{a}}$ and $A' = \frac{w + \bar{a}A}{z_0 + \bar{a}}$. Since $w'/A' = z_0$, we get the following equations from (3.3.81) and (3.3.83):

$$\tilde{U}_A(w', A') = \bar{a} \tilde{U}_w(w', A'), \quad (3.3.84)$$

$$\tilde{U}_{Aw}(w', A') = \bar{a} \tilde{U}_{ww}(w', A'). \quad (3.3.85)$$

Then, from (3.3.74), (3.3.84), and (3.3.85), we obtain the following conditions for $(w, A) \in \mathcal{D}_2$:

$$\tilde{U}_w(w, A) = \tilde{U}_w(w', A') \geq 0, \quad (3.3.86)$$

$$\tilde{U}_A(w, A) = \tilde{U}_A(w', A') \geq 0, \quad (3.3.87)$$

$$\tilde{U}_{ww}(w, A) = \tilde{U}_{ww}(w', A') \leq 0. \quad (3.3.88)$$

Thus, \tilde{U} is non-decreasing and concave with respect to w and is non-decreasing with respect to A in \mathcal{D}_2 . Moreover, (3.3.86) and (3.3.87) lead to the inequalities (3.3.79) and (3.3.80) for $(w, A) \in \mathcal{D}_2$.

Next, consider $\max_{\{\pi, c\}} \mathcal{L}^{\pi, c} \tilde{U}(w, A)$ for $(w, A) \in \mathcal{D}_2$. Due to the concavity of $U(w, A)$, we have that

$$\begin{aligned}
\max_{\{\pi, c\}} \mathcal{L}^{\pi, c} \tilde{U}(w, A) &= -(\lambda^S + r) \tilde{U}(w, A) + (rw + A) \tilde{U}_w(w, A) \\
&\quad - m \frac{\tilde{U}_w^2(w, A)}{\tilde{U}_{ww}(w, A)} + \frac{\gamma}{1 - \gamma} \tilde{U}_w(w, A)^{\frac{\gamma-1}{\gamma}} \\
&= -(\lambda^S + r) \tilde{U}(w', A') + (rw' + A') \tilde{U}_w(w', A') \\
&\quad - m \frac{\tilde{U}_w^2(w', A')}{\tilde{U}_{ww}(w', A')} + \frac{\gamma}{1 - \gamma} \tilde{U}_w(w', A')^{\frac{\gamma-1}{\gamma}} \\
&\quad + [r(w - w') + (A - A')] \tilde{U}_w(w', A') \\
&= \max_{\{\pi, c\}} \mathcal{L}^{\pi, c} \tilde{U}(w', A') + [r(w - w') + (A - A')] \tilde{U}_w(w', A') \\
&= -\frac{\lambda^O}{r + \lambda^O} \Delta A U_w(w', A') \leq 0.
\end{aligned} \tag{3.3.89}$$

Here, we use the fact that $\max_{\{\pi, c\}} \mathcal{L}^{\pi, c} \tilde{U}(w', A') = 0$ for $(w', A') \in \mathcal{D}_1$.

To prove that the utility function \tilde{U} defined in equations (3.3.73) and (3.3.74) is the maximized utility U , we require the following lemma:

Lemma 3.3.1. *For every sequence of bounded stopping times $(\tau_n)_{n \geq 1}$ with $\tau_n \rightarrow \infty$ a.s.,*

$$\lim_{n \rightarrow \infty} \mathbb{E}^{w, A} \left[e^{-(\lambda^S + r)\tau_n} \tilde{U}(W_{\tau_n}^*, A_{\tau_n}^*) \right] = 0, \tag{3.3.90}$$

in which W^* and A^* are the optimally controlled wealth and annuity processes, respectively.

Proof. Consider $\tilde{U}(w, A)$ defined in (3.3.73). First, observe that for any $(w, A) \in \mathcal{D}_1$, we have:

- When $\gamma < 1$, then $\tilde{U}(w, A) \neq 0 \iff (w, A) \neq (0, 0)$.
- When $\gamma > 1$, then $\tilde{U}(w, A) > -\infty \iff (w, A) \neq (0, 0)$.

Without loss of generality, we assume that $(W_0, A_0) \in \mathcal{D}_1 \setminus \{(0, 0)\}$. Define $\tau \triangleq \inf\{t > 0 : (W_t^*, A_t^*) = (0, 0)\}$. For the optimally controlled wealth and annuity income processes, we have that $\tau > 0$ *a.s.*

Let $T > 0$ be fixed but arbitrary. We follow [Davis and Norman \[1990\]](#) and apply Itô's formula to $\ln \left[e^{-(\lambda^S+r)(T \wedge \tau)} \tilde{U}(W_{T \wedge \tau}^*, A_{T \wedge \tau}^*) \right]$ under the optimal consumption, investment, and annuitization strategies to get

$$\begin{aligned}
& \ln \left[e^{-(\lambda^S+r)(T \wedge \tau)} \tilde{U}(W_{T \wedge \tau}^*, A_{T \wedge \tau}^*) \right] - \ln \left[\tilde{U}(W_0, A_0) \right] \\
&= \int_0^{T \wedge \tau} \frac{\tilde{U}_w}{\tilde{U}} \sigma \pi_t^* dB_t + \int_0^{T \wedge \tau} \frac{1}{\tilde{U}} \left[\mathcal{L}^{\pi_t^*, c_t^*} \tilde{U} - \frac{(c_t^*)^{1-\gamma}}{1-\gamma} \right] dt \\
&\quad + \int_0^{T \wedge \tau} \frac{1}{\tilde{U}} \left(\tilde{U}_A - \bar{a} \tilde{U}_w \right) d(A_t^+)^{(c)} + \int_0^{T \wedge \tau} \frac{1}{\tilde{U}} \left[(1-p)\bar{a} \tilde{U}_w - \tilde{U}_A \right] d(A_t^-)^{(c)} \\
&\quad + \sum_{0 \leq t \leq T \wedge \tau} \left\{ \ln \left[\tilde{U}(W_t^*, A_t^*) \right] - \ln \left[\tilde{U}(W_{t-}^*, A_{t-}^*) \right] \right\} - \frac{1}{2} \int_0^{T \wedge \tau} \frac{\tilde{U}_w^2}{\tilde{U}^2} \sigma^2 (\pi_t^*)^2 dt \\
&= \int_0^{T \wedge \tau} \left[-\frac{1}{\tilde{U}} \frac{(c_t^*)^{1-\gamma}}{1-\gamma} - \frac{\tilde{U}_w^2}{2\tilde{U}^2} \sigma^2 (\pi_t^*)^2 \right] dt + \int_0^{T \wedge \tau} \frac{\tilde{U}_w}{\tilde{U}} \sigma \pi_t^* dB_t,
\end{aligned} \tag{3.3.91}$$

in which π_t^* and c_t^* are defined in [\(3.3.77\)](#) and [\(3.3.78\)](#) and the annuitization strategy is to buy or surrender annuities only in order to keep $(W_t^*, A_t^*) \in \mathcal{D}_1$. In this equation, \tilde{U} , \tilde{U}_w , etc. are evaluated at (W_t^*, A_t^*) unless noted otherwise. We simplify [\(3.3.91\)](#) to

$$\begin{aligned}
e^{-(\lambda^S+r)(T \wedge \tau)} \tilde{U}(W_{T \wedge \tau}^*, A_{T \wedge \tau}^*) &= \tilde{U}(W_0, A_0) \cdot \exp \left(\int_0^{T \wedge \tau} -\frac{1}{\tilde{U}} \frac{(c_t^*)^{1-\gamma}}{1-\gamma} dt \right) \\
&\quad \cdot \exp \left(\int_0^{T \wedge \tau} -\frac{\tilde{U}_w^2}{2\tilde{U}^2} \sigma^2 (\pi_t^*)^2 dt + \frac{\tilde{U}_w}{\tilde{U}} \sigma \pi_t^* dB_t \right).
\end{aligned} \tag{3.3.92}$$

We wish to show that \tilde{U} is bounded away from 0 on $\mathcal{D}_1 \setminus \{(0, 0)\}$. Recall that $\tilde{U}(w, A) = A^{1-\gamma} \tilde{V}(w/A)$; thus, it is equivalent to show that \tilde{V} is bounded away from 0 for $0 \leq z \leq z_0$. Equations [\(3.3.20\)](#) and [\(3.3.21\)](#) hold with V replaced by

\tilde{V} ; therefore, from these equations, it follows that it is sufficient to show that \tilde{V}_z is bounded away from 0 for $0 \leq z \leq z_0$. The latter condition holds due to equations (3.3.33), (3.3.34), and (3.3.71).

From equation (3.3.72), it follows that $1/\tilde{V}_{zz}$ is uniformly bounded for $0 \leq z \leq z_0$. Thus, because $\frac{\tilde{U}_w}{\tilde{U}} \sigma \pi_t^* = -\frac{\mu - r}{\sigma} \frac{\tilde{V}_z^2}{\tilde{V} \tilde{V}_{zz}}$, the integral $\int_0^{T \wedge \tau} \frac{\tilde{U}_w}{\tilde{U}} \sigma \pi_t^* dB_t$ is finite for any $0 < T < \infty$. Similarly, we have that $\frac{1}{\tilde{U}} \frac{(c_t^*)^{1-\gamma}}{1-\gamma} = \frac{1}{1-\gamma} \frac{\tilde{V}_z^{-(1-\gamma)/\gamma}}{\tilde{V}} > 0$ is bounded away from zero for $0 \leq z \leq z_0$ by, say, $K > 0$.

Suppose $\tau < \infty$ on some set Δ with positive measure. Choose a sample path in the set Δ , and let $T \rightarrow \tau$. Due to the continuity of (3.3.92), we have that $\lim_{T \rightarrow \tau} e^{-(\lambda^S + r)(T \wedge \tau)} \tilde{U}(W_{T \wedge \tau}^*, A_{T \wedge \tau}^*)$ does not equal 0 nor $-\infty$. On the other hand, by the definition of τ ,

$$\lim_{T \rightarrow \tau} e^{-(\lambda^S + r)(T \wedge \tau)} \tilde{U}(W_{T \wedge \tau}^*, A_{T \wedge \tau}^*) = 0 \text{ or } -\infty. \quad (3.3.93)$$

By contradiction, we conclude that $\tau = \infty$ almost surely. In other words, the optimal strategy does not lead an individual to bankruptcy within finite time.

Since $\frac{1}{\tilde{U}} \frac{(c_t^*)^{1-\gamma}}{1-\gamma} \geq K > 0$ as we showed above, $\int_0^{\tau_n} -\frac{1}{\tilde{U}} \frac{(c_t^*)^{1-\gamma}}{1-\gamma} dt \leq -K\tau_n$. Thus, for every sequence of bounded stopping times $(\tau_n)_{n \geq 1}$ with $\tau_n \rightarrow \infty$ *a.s.*,

$$e^{-(\lambda^S + r)\tau_n} \tilde{U}(W_{\tau_n}^*, A_{\tau_n}^*) \leq \tilde{U}(W_0, A_0) e^{-K\tau_n} \exp \left(\int_0^{\tau_n} -\frac{\tilde{U}_w^2}{2\tilde{U}^2} \sigma^2 (\pi_t^*)^2 dt + \frac{\tilde{U}_w}{\tilde{U}} \sigma \pi_t^* dB_t \right), \quad (3.3.94)$$

$\forall n \geq 1$. As $n \rightarrow \infty$, (3.3.94) leads to the transversality condition

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \mathbb{E}^{W_0, A_0} \left[e^{-(\lambda^S + r)\tau_n} \tilde{U}(W_{\tau_n}^*, A_{\tau_n}^*) \right] \\
& \leq \lim_{n \rightarrow \infty} \tilde{U}(W_0, A_0) \operatorname{ess\,sup} (e^{-K\tau_n}) \mathbb{E}^{W_0, A_0} \left[\exp \left(\int_0^{\tau_n} -\frac{\tilde{U}_w^2}{2\tilde{U}^2} \sigma^2 (\pi_t^*)^2 dt + \frac{\tilde{U}_w}{\tilde{U}} \sigma \pi_t^* dB_t \right) \right] \\
& = \tilde{U}(W_0, A_0) \lim_{n \rightarrow \infty} \operatorname{ess\,sup} (e^{-K\tau_n}) \\
& = 0.
\end{aligned} \tag{3.3.95}$$

To get from the second to the third line in (3.3.95), we apply the optional stopping theorem to the exponential martingale $\left\{ \exp \left(\int_0^t -\frac{\tilde{U}_w^2}{2\tilde{U}^2} \sigma^2 (\pi_s^*)^2 ds + \frac{\tilde{U}_w}{\tilde{U}} \sigma \pi_s^* dB_s \right) \right\}$ with bounded stopping times $(\tau_n)_{n \geq 1}$. \square

Remark 3.3.6. *In the proof of Lemma 3.3.1 above, we obtain a non-bankruptcy condition, that is, $(w_t^*, A_t^*) \neq (0, 0)$ almost surely for all $t \geq 0$. This condition is implicit in the proof of Corollary 3.2.1. Indeed, we proved Corollary 3.2.1 using the second equality in (3.2.22), which comes from (3.2.18), and we derived (3.2.18) assuming that $v(w, A)$ and $v_w(w, A)$ are bounded from below. But, this assumption does not necessarily hold in the case for which $\gamma > 1$. To get (3.2.18) in the case for which $\gamma > 1$, we need to guarantee that each term in (3.2.18) is finite, that is, $(w_t, A_t) \neq (0, 0)$ almost surely for $t \in (0, \tau_n)$. To this end, one can define $\tau_n^c \triangleq \inf\{s \geq 0 : \sqrt{w_s^2 + A_s^2} \leq \frac{1}{n}\}$ and set $\tau_n = n \wedge \tau_n^a \wedge \tau_n^b \wedge \tau_n^c$ with the stopping times τ_n^a and τ_n^b defined in the proof of Theorem 3.2.1. With this modified definition of τ_n , all our deductions in the proof of Theorem 3.2.1 are valid in the case for which $\gamma > 1$. Moreover, the non-bankruptcy condition guarantees that $\tau_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$.*

So far we have shown that \tilde{U} is non-decreasing and concave with respect to w , non-decreasing with respect to A , and satisfies the conditions (i)-(iii) of the Verification Theorem 3.2.1. More precisely, (3.3.76) and (3.3.89) prove condition (i), and (3.3.79),

(3.3.80), (3.3.81), and (3.3.82) prove conditions (ii) and (iii). Therefore, with Lemma 3.3.1 and Corollary 3.2.1, we get the following theorem.

Theorem 3.3.1. *When $p < p^*$, the maximized utility U defined in equation (3.2.5) in the region $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, with $\mathcal{D}_1 = \{(w, A) : 0 \leq w/A \leq z_0, A \geq 0\}$ and $\mathcal{D}_2 = \{(w, A) : w/A > z_0, A > 0\}$, is given by \tilde{U} defined in equations (3.3.73) and (3.3.74). The associated optimal strategy is as follows:*

1. *When $(w, A) \in \mathcal{D}_2$, purchase additional annuity income of $\Delta A = \frac{w - z_0 A}{z_0 + \bar{a}}$;*
2. *When $w/A = z_0$, purchase additional annuity income instantaneously to keep (w, A) in the region \mathcal{D}_1 ;*
3. *Surrender existing annuity income instantaneously to keep w non-negative when needed;*
4. *Invest in the risky asset with the dollar amount*

$$\pi^*(w, A) = -\frac{\mu - r}{\sigma^2} \frac{\tilde{U}_w(w, A)}{\tilde{U}_{ww}(w, A)},$$

and consume continuously at the rate

$$c^*(w, A) = \tilde{U}_w^{-\frac{1}{\gamma}}(w, A),$$

when $(w, A) \in \mathcal{D}_1$.

Remark 3.3.7. *When $p = 0$, the solution given in this section is same as the one given in Section 3.3.1. To show this, we write y_s in terms of K defined in (3.3.7) to obtain*

$$y_s = (\bar{a} K)^{-\gamma}. \quad (3.3.96)$$

When $p = 0$, the solution of (3.3.47) is $x = 1$; that is, $y_s = y_b$. From (3.3.43) and (3.3.44), we conclude that $z_0 = 0$. Also, note that $\tilde{V}(0) = \hat{V}(y_s)$. Based on the

construction of $U(w, A)$ given in (3.3.73) and (3.3.74), as well as equation (3.3.38), we obtain the maximized utility with initial wealth w and annuity income A :

$$U(w, A) = U\left(0, \frac{w}{\bar{a}} + A\right) = \left(\frac{w}{\bar{a}} + A\right)^{1-\gamma} \tilde{V}(0) = \left(\frac{w}{\bar{a}} + A\right)^{1-\gamma} \frac{\bar{a}y_s}{1-\gamma} = K^{-\gamma} \frac{(w + \bar{a}A)^{1-\gamma}}{1-\gamma}. \quad (3.3.97)$$

This expression is identical to the one in equation (3.3.8). Similarly, we obtain

$$c^*(w, A) = K(w + \bar{a}A), \quad (3.3.98)$$

$$\pi^*(w, A) = \frac{\mu - r}{\gamma\sigma^2}(w + \bar{a}A), \quad (3.3.99)$$

which equal to the expressions in (3.3.9) and (3.3.10), respectively.

3.3.3 The case for which $p \geq p^*$

The difference between this case and the case for which $p < p^*$ is in the different optimal strategy that an individual uses when wealth reaches zero. As we showed in the previous section, when $p < p^*$, a retiree keeps her wealth non-negative by surrendering existing annuity income. When the size of the proportional surrender charge p is larger, namely, when the cost of surrendering is higher, the retiree has more incentive not to surrender annuity income. Instead, we will show in this section that, as her wealth reaches zero, the optimal strategy is for the retiree to consume less than her existing annuity income and to invest nothing in the risky asset. As before, we also hypothesize the existence of a ratio of wealth-to-annuity income z_0 such that, at a point (w, A) with $w/A > z_0$, the individual purchases annuity income to reach (w', A') such that $w'/A' = z_0$. Additionally, the retiree does not purchase any annuity income when $0 < w/A < z_0$, and she does not surrender existing annuity income under any circumstance.

One can see that our hypothesized optimal strategy is different from the one in the previous section only at the boundary $w = 0$. We solve the problem the same

way as we did in the previous section, and the details of the solution in this case are very similar to the case for which $p < p^*$. For this reason, we omit some of the details in the following derivations and proofs.

Based on our hypotheses, $U = U(w, A)$ satisfies the following conditions:

1. In the region $\mathcal{D}_1 \triangleq \{(w, A) : 0 \leq w/A \leq z_0, A \geq 0\}$, U is the solution of the following BVP:

$$(\lambda^S + r)U = (rw + A)U_w + \max_{\pi} \left[(\mu - r)\pi U_w + \frac{1}{2}\sigma^2\pi^2 U_{ww} \right] + \max_{c \geq 0} \left(\frac{c^{1-\gamma}}{1-\gamma} - cU_w \right), \quad (3.3.100)$$

with boundary conditions

$$U_A(z_0A, A) = \bar{a} U_w(z_0A, A), \quad (3.3.101)$$

$$\pi(0, A) = -\frac{\mu - r}{\sigma^2} \frac{U_w(0, A)}{U_{ww}(0, A)} = 0, \quad (3.3.102)$$

for $A \geq 0$. The expression in (3.3.102) imposes the condition that the optimal investment strategy is not to invest in risky asset when $w = 0$.

2. In the region $\mathcal{D}_2 \triangleq \{(w, A) : w/A > z_0, A \geq 0\}$, we have

$$U(w, A) = U(w', A'), \quad (3.3.103)$$

in which $w' = w - \bar{a}\Delta A$, $A' = A + \Delta A$, and $\Delta A = \frac{w - z_0A}{z_0 + \bar{a}}$.

3. The smooth fit condition holds on the line $w/A = z_0$:

$$U_{Aw}(z_0A, A) = \bar{a} U_{ww}(z_0A, A), \quad A > 0. \quad (3.3.104)$$

As in the previous section, we rewrite the BVP (3.3.100) and the conditions (3.3.101), (3.3.102), and (3.3.104) in terms of $V(z) = U(z, 1)$. Recall that we can

recover U from V via $U(w, A) = A^{1-\gamma}V(w/A)$.

$$-(\lambda^S + r)V + (rz + 1)V_z + \max_{\hat{\pi}} \left[(\mu - r)\hat{\pi}V_z + \frac{1}{2}\sigma^2\hat{\pi}^2V_{zz} \right] + \max_{\hat{c} \geq 0} \left(\frac{\hat{c}^{1-\gamma}}{1-\gamma} - \hat{c}V_z \right) = 0, \quad (3.3.105)$$

with $0 \leq z \leq z_0$, and

$$(1 - \gamma)V(z_0) - (z_0 + \bar{a})V_z(z_0) = 0, \quad (3.3.106)$$

$$\hat{\pi}(0, A) = -\frac{\mu - r}{\sigma^2}A \frac{V_z(0)}{V_{zz}(0)} = 0, \quad (3.3.107)$$

$$(z_0 + \bar{a})V_{zz}(z_0) + \gamma V_z(z_0) = 0. \quad (3.3.108)$$

Next, we proceed to solve the V 's BVP via a duality argument. Because V is concave, then we define its convex dual \hat{V} by

$$\hat{V}(y) = \max_{z \geq 0} [V(z) - yz]. \quad (3.3.109)$$

and because \hat{V} solves the same linear differential equation as in the previous section, namely, (3.3.27), its general form is

$$\hat{V}(y) = D_1y^{B_1} + D_2y^{B_2} + \frac{y}{r} + Cy^{\frac{\gamma-1}{\gamma}}, \quad (3.3.110)$$

with B_1 , B_2 , and C given in equations (3.3.29), (3.3.30), and (3.3.31), respectively.

D_1 and D_2 are to be determined.

As before, define

$$y_s = V_z(0), \quad (3.3.111)$$

$$y_b = V_z(z_0). \quad (3.3.112)$$

Notice that the conditions in the case $p \geq p^*$ are same as in the case $p < p^*$, except for the boundary condition at $z = 0$. Rewrite (3.3.107) in terms of \hat{V} as

$$y_s \hat{V}_{yy}(y_s) = 0. \quad (3.3.113)$$

Substitute (3.3.110) into (3.3.113), and restate the other conditions from the previous section, namely, (3.3.40) and (3.3.42)-(3.3.44), to obtain the following:

$$D_1[1 + \gamma(B_1 - 1)]y_b^{B_1} + D_2[1 + \gamma(B_2 - 1)]y_b^{B_2} + \frac{y_b}{r} = \frac{y_b}{r + \lambda^O} \quad (3.3.114)$$

$$D_1B_1(B_1 - 1)y_s^{B_1-1} + D_2B_2(B_2 - 1)y_s^{B_2-1} - \frac{\gamma - 1}{\gamma^2}Cy_s^{-\frac{1}{\gamma}} = 0, \quad (3.3.115)$$

$$D_1B_1[1 + \gamma(B_1 - 1)]y_b^{B_1} + D_2B_2[1 + \gamma(B_2 - 1)]y_b^{B_2} + \frac{y_b}{r} = \frac{y_b}{r + \lambda^O} \quad (3.3.116)$$

$$D_1B_1y_s^{B_1-1} + D_2B_2y_s^{B_2-1} + \frac{1}{r} + \frac{\gamma - 1}{\gamma}Cy_s^{-\frac{1}{\gamma}} = 0, \quad (3.3.117)$$

$$D_1B_1y_b^{B_1-1} + D_2B_2y_b^{B_2-1} + \frac{1}{r} + \frac{\gamma - 1}{\gamma}Cy_b^{-\frac{1}{\gamma}} = -z_0. \quad (3.3.118)$$

The proportional surrender charge p does not appear in these equations. Its absence indicates that the solution is the same for any $p \in [p^*, 1]$.

Solve for D_1 and D_2 from (3.3.114) and (3.3.116) to obtain

$$D_1 = -\frac{\lambda^O}{r(r + \lambda^O)} \frac{1 - B_2}{B_1 - B_2} \frac{1}{1 + \gamma(B_1 - 1)} y_b^{1-B_1}, \quad (3.3.119)$$

$$D_2 = -\frac{\lambda^O}{r(r + \lambda^O)} \frac{B_1 - 1}{B_1 - B_2} \frac{1}{1 + \gamma(B_2 - 1)} y_b^{1-B_2}. \quad (3.3.120)$$

Then, substitute (3.3.119) and (3.3.120) into (3.3.115) and (3.3.117) to get the following equation for $\tilde{x} \triangleq \frac{y_s}{y_b}$:

$$\frac{\lambda^O}{r + \lambda^O} \left[\frac{B_1(1 - B_2)}{B_1 - B_2} \tilde{x}^{B_1-1} + \frac{B_2(B_1 - 1)}{B_1 - B_2} \tilde{x}^{B_2-1} \right] = 1. \quad (3.3.121)$$

We conclude that \tilde{x} has a unique solution in $(1, \infty)$ through the same argument as the one following equation (3.3.47). Note that equation (3.3.121) is identical to (3.3.52); thus, we use the same notation for its solution, namely, \tilde{x} .

Substitute (3.3.119) and (3.3.120) into (3.3.117) and (3.3.118) to get expressions for y_s , y_b , and z_0 , respectively,

$$\frac{1 - \gamma}{\gamma} Cy_s^{-\frac{1}{\gamma}} = -\frac{\lambda^O}{r(r + \lambda^O)} \left[\frac{B_1(1 - B_2)}{B_1 - B_2} \frac{\tilde{x}^{B_1-1}}{1 + \gamma(B_1 - 1)} + \frac{B_2(B_1 - 1)}{B_1 - B_2} \frac{\tilde{x}^{B_2-1}}{1 + \gamma(B_2 - 1)} \right] + \frac{1}{r}, \quad (3.3.122)$$

$$y_b = \frac{y_s}{\tilde{x}}, \quad (3.3.123)$$

$$z_0 = -\frac{\lambda^O}{r(r + \lambda^O)} \frac{m(1 - \gamma) - (r + \lambda^S)\gamma}{\gamma^2 \left[r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right]} - \frac{1}{r} - \frac{\gamma - 1}{\gamma} C y_b^{-\frac{1}{\gamma}}, \quad (3.3.124)$$

One can prove that $y_s > 0$ with an argument parallel to the one in Proposition 3.3.2, so we state the following proposition without proof.

Proposition 3.3.5. *The well-posedness condition (3.3.11) implies that y_s as defined in (3.3.122) is positive for $p \geq p^*$.*

As in the case of Proposition 3.3.1, the next proposition follows from the derivations above.

Proposition 3.3.6. *The solution \hat{V} for the FBP (3.3.27) with conditions (3.3.37), (3.3.113), and (3.3.39) is given by (3.3.110), with D_1 , D_2 , y_s , y_b , and \tilde{x} defined in (3.3.119), (3.3.120), (3.3.122), (3.3.123), and (3.3.121), respectively.*

As before, we wish to show that \hat{V} , indeed, has the properties that we expect, so we present the following proposition.

Proposition 3.3.7. *\hat{V} given in Proposition 3.3.6 satisfies inequalities (3.3.57) and (3.3.58) and is decreasing and convex for $0 < y_b \leq y \leq y_s$.*

Proof. By substituting (3.3.110) into (3.3.57) and (3.3.58), we rewrite the inequalities as

$$-\frac{pr + \lambda^O}{r(r + \lambda^O)} \leq f(y) \leq -\frac{\lambda^O}{r(r + \lambda^O)}, \quad (3.3.125)$$

in which

$$f(y) = D_1[1 + \gamma(B_1 - 1)]y^{B_1 - 1} + D_2[1 + \gamma(B_2 - 1)]y^{B_2 - 1}. \quad (3.3.126)$$

One can show that $f(y_s) = -\frac{p^*r + \lambda^O}{r(r + \lambda^O)}$ and $f(y_b) = -\frac{\lambda^O}{r(r + \lambda^O)}$. Also, f is non-increasing for $0 < y_b \leq y \leq y_s$, as in the proof of Proposition 3.3.4. Therefore, the inequalities in (3.3.125) hold for all $p \in [p^*, 1]$.

That \hat{V} is decreasing follows as in the proof of Proposition 3.3.4. As for the convexity of \hat{V} , we see that for $0 < y_b \leq y \leq y_s$,

$$\hat{V}_{yy}(y) \geq 0 \iff g(y) \geq 0, \quad (3.3.127)$$

in which

$$g(y) = -\frac{\lambda^O}{r(r+\lambda^O)} \frac{(B_1-1)(1-B_2)}{B_1-B_2} \left[\frac{B_1}{1+\gamma(B_1-1)} \left(\frac{y}{y_b}\right)^{B_1-1} - \frac{B_2}{1+\gamma(B_2-1)} \left(\frac{y}{y_b}\right)^{B_2-1} \right] + \left(1 - \frac{1}{\gamma}\right) \left(-\frac{1}{\gamma}\right) C y^{-\frac{1}{\gamma}}. \quad (3.3.128)$$

As in the proof of Proposition 3.3.4, $g'(y) < 0$; therefore, it is enough to show that $g(y_s) \geq 0$. By (3.3.122) and (3.3.121), $g(y_s) = 0$. \square

We will use the following proposition at the end of this section to show that when wealth equals zero, the retiree consumes at a rate less than the rate of her annuity income. Thereby, she is able (instantaneously) to return to positive wealth.

Proposition 3.3.8. $y_s^{-1/\gamma} < 1$.

Proof.

$$\begin{aligned} & y_s^{-\frac{1}{\gamma}} < 1 \\ \iff & \left\{ -\frac{\lambda^O}{r(r+\lambda^O)} \left[\frac{B_1(1-B_2)}{B_1-B_2} \frac{1}{1+\gamma(B_1-1)} \tilde{x}^{B_1-1} + \frac{B_2(B_1-1)}{B_1-B_2} \frac{1}{1+\gamma(B_2-1)} \tilde{x}^{B_2-1} \right] + \frac{1}{r} \right\} \\ & \times \frac{\gamma}{(1-\gamma)C} < 1 \\ \iff & \left\{ -\frac{\lambda^O}{r(r+\lambda^O)} \left[\frac{B_1(1-B_2)}{B_1-B_2} \frac{1}{1+\gamma(B_1-1)} \tilde{x}^{B_1-1} + \frac{B_2(B_1-1)}{B_1-B_2} \frac{1}{1+\gamma(B_2-1)} \tilde{x}^{B_2-1} \right] + \frac{1}{r} \right\} \\ & \times \left[r + \frac{\lambda^S}{\gamma} - m \frac{1-\gamma}{\gamma^2} \right] < 1 \\ \iff & -\frac{\lambda^O}{r+\lambda^O} \left[\frac{B_1(1-B_2)}{B_1-B_2} \frac{1}{1+\gamma(B_1-1)} \tilde{x}^{B_1-1} + \frac{B_2(B_1-1)}{B_1-B_2} \frac{1}{1+\gamma(B_2-1)} \tilde{x}^{B_2-1} \right] \\ & \cdot \left[r + \frac{\lambda^S}{\gamma} - m \frac{1-\gamma}{\gamma^2} \right] + \left[\frac{\lambda^S}{\gamma} - m \frac{1-\gamma}{\gamma^2} \right] < 0. \end{aligned} \quad (3.3.129)$$

Recall that $[1 + \gamma(B_1 - 1)] \cdot [1 + \gamma(B_2 - 1)] = -\frac{\gamma^2}{m} \left[r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right] < 0$; multiply the above inequality by this factor to get

$$\begin{aligned}
& y_s^{-\frac{1}{\gamma}} < 1 \\
\iff & -\frac{\lambda^O}{r + \lambda^O} \left\{ \frac{B_1(1 - B_2)}{B_1 - B_2} [1 + \gamma(B_2 - 1)] \tilde{x}^{B_1-1} + \frac{B_2(B_1 - 1)}{B_1 - B_2} [1 + \gamma(B_1 - 1)] \tilde{x}^{B_2-1} \right\} \\
& \cdot \left[r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right] + \left[\frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right] \cdot [1 + \gamma(B_1 - 1)] \cdot [1 + \gamma(B_2 - 1)] > 0 \\
\iff & -\frac{\lambda^O}{r + \lambda^O} (1 - \gamma) \left[\frac{B_1(1 - B_2)}{B_1 - B_2} \tilde{x}^{B_1-1} + \frac{B_2(B_1 - 1)}{B_1 - B_2} \tilde{x}^{B_2-1} \right] \cdot \left[r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right] \\
& - \frac{\lambda^O}{r + \lambda^O} \gamma B_1 B_2 \left[\frac{1 - B_2}{B_1 - B_2} \tilde{x}^{B_1-1} + \frac{B_1 - 1}{B_1 - B_2} \tilde{x}^{B_2-1} \right] \cdot \left[r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right] \\
& + \left[\frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right] \cdot [1 + \gamma(B_1 - 1)] \cdot [1 + \gamma(B_2 - 1)] > 0.
\end{aligned} \tag{3.3.130}$$

Use (3.3.121) to simplify the above expression to obtain

$$\begin{aligned}
& y_s^{-\frac{1}{\gamma}} < 1 \\
\iff & -(1 - \gamma) \left[r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right] \\
& - \frac{\lambda^O}{r + \lambda^O} \gamma B_1 B_2 \left[\frac{1 - B_2}{B_1 - B_2} \tilde{x}^{B_1-1} + \frac{B_1 - 1}{B_1 - B_2} \tilde{x}^{B_2-1} \right] \cdot \left[r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right] \\
& + \left[\frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right] \cdot [1 + \gamma(B_1 - 1)] \cdot [1 + \gamma(B_2 - 1)] > 0 \\
\iff & -\frac{\lambda^O}{r + \lambda^O} \gamma B_1 B_2 \left[\frac{1 - B_2}{B_1 - B_2} \tilde{x}^{B_1-1} + \frac{B_1 - 1}{B_1 - B_2} \tilde{x}^{B_2-1} \right] \cdot \left[r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right] \\
& - \gamma \frac{\lambda^S}{m} \left[r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right] > 0 \\
\iff & -\frac{\lambda^O}{r + \lambda^O} B_1 B_2 \left[\frac{1 - B_2}{B_1 - B_2} \tilde{x}^{B_1-1} + \frac{B_1 - 1}{B_1 - B_2} \tilde{x}^{B_2-1} \right] - \frac{\lambda^S}{m} > 0.
\end{aligned} \tag{3.3.131}$$

We demonstrate this last inequality by using (3.3.121) and the equality $B_1 + B_2 - 1 =$

$$-\frac{\lambda^S}{m}:$$

$$\begin{aligned} & -\frac{\lambda^O}{r+\lambda^O}B_1B_2\left[\frac{1-B_2}{B_1-B_2}\tilde{x}^{B_1-1}+\frac{B_1-1}{B_1-B_2}\tilde{x}^{B_2-1}\right]-\frac{\lambda^S}{m} \\ & = -B_2-\frac{\lambda^O}{r+\lambda^O}B_2(B_1-1)\tilde{x}^{B_2-1}-\frac{\lambda^S}{m} \\ & = (B_1-1)\left[1-\frac{\lambda^O}{r+\lambda^O}B_2\tilde{x}^{B_2-1}\right]>0. \end{aligned} \tag{3.3.132}$$

□

We now construct the maximized utility U from \hat{V} as we did for the case $p < p^*$.

First, define the concave dual \tilde{V} of \hat{V} by

$$\tilde{V}(z) = \min_{y \geq 0} [\hat{V}(y) + zy]. \tag{3.3.133}$$

Then, define \tilde{U} , with z_0 given by (3.3.124), as follows:

1. In the region $\mathcal{D}_1 \triangleq \{(w, A) : 0 \leq w/A \leq z_0, A \geq 0\}$,

$$\tilde{U}(w, A) \triangleq A^{1-\gamma}\tilde{V}(w/A). \tag{3.3.134}$$

2. In the region $\mathcal{D}_2 \triangleq \{(w, A) : w/A > z_0, A \geq 0\}$, define

$$\tilde{U}(w, A) \triangleq \tilde{U}(w', A'),$$

$$\text{in which } w' = w - \bar{a}\Delta A, A' = A + \Delta A, \text{ and } \Delta A = \frac{w - z_0A}{z_0 + \bar{a}}.$$

That \tilde{U} is the maximized utility U follows as in Section 3.3.2. Thus, we obtain the following theorem.

Theorem 3.3.2. *When $p \geq p^*$, the maximized utility U defined in equation (3.2.5) in the region $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, with $\mathcal{D}_1 = \{(w, A) : 0 \leq w/A \leq z_0, A \geq 0\}$ and $\mathcal{D}_2 = \{(w, A) : w/A > z_0, A > 0\}$, is given by \tilde{U} defined in equations (3.3.134) and (2). The associated optimal strategy is:*

1. When $(w, A) \in \mathcal{D}_2$, purchase additional annuity income of $\Delta A = \frac{w - z_0 A}{z_0 + \bar{a}}$;
2. When $w/A = z_0$, purchase additional annuity income instantaneously to keep (w, A) in the region \mathcal{D}_1 ;
3. Never surrender existing annuity income;
4. Invest in the risky asset with the dollar amount

$$\pi^*(w, A) = -\frac{\mu - r}{\sigma^2} \frac{\tilde{U}_w(w, A)}{\tilde{U}_{ww}(w, A)},$$

and consume continuously at the rate

$$c^*(w, A) = \tilde{U}_w^{-\frac{1}{\gamma}}(w, A).$$

when $(w, A) \in \mathcal{D}_1$. Specifically, $\pi^*(0, A) = 0$.

Remark 3.3.8. When $p \geq p^*$, the optimal annuitization, investment, consumption strategies as well as the maximized utility of an retiree do not depend on the size of surrender charge. An individual behaves as if the annuity is not reversible at all and does not surrender existing annuities under any circumstance.

Corollary 3.3.1. When wealth is zero, the optimal rate of consumption is less than the rate of annuity income; that is, $c^*(0, A) < A$.

Proof. From Theorem 3.3.2 and Proposition 3.3.8, it follows that

$$c^*(0, A) = U_w^{-\frac{1}{\gamma}}(0, A) = (A^{-\gamma} V_z(0))^{-\frac{1}{\gamma}} = A y_s^{-\frac{1}{\gamma}} < A. \quad (3.3.135)$$

□

3.4 Properties of the optimal strategies

In this section, we analyze the optimal strategies of a retiree under a variety of conditions. We demonstrate the relation between the proportional surrender charge

and the retiree's optimal strategies via numerical examples. We also analyze the retiree's optimal strategies as she becomes very risk averse.

3.4.1 Properties of z_0

The (derived) parameter z_0 is a measure of the willingness of the retiree to annuitize. Indeed, if w/A is greater than z_0 , then she immediately annuitizes enough of her wealth to bring that ratio equal to z_0 . The smaller the value of z_0 , the lower her wealth has to be in order for her to annuitize. Thus, we associate smaller values of z_0 with greater willingness to annuitize.

We expect that a retiree will be less willing to annuitize if the proportional surrender charge p is larger. It turns out that z_0 is increasing with respect to p for $p < p^*$, and we prove fact this in the next proposition. Recall that z_0 is independent of p for $p \geq p^*$.

Proposition 3.4.1. z_0 , given in (3.3.50), increases with respect to $p < p^*$.

Proof. From equations (3.3.50), (3.3.31), and (3.3.49), it follows that

$$\frac{\partial z_0}{\partial p} \propto -\frac{\partial y_b}{\partial p} \propto y_s \frac{\partial x}{\partial p} - x \frac{\partial y_s}{\partial p}. \quad (3.4.1)$$

From (3.3.47), we get

$$\frac{(B_1 - 1)(1 - B_2)}{B_1 - B_2} (x^{B_1-1} - x^{B_2-1}) \frac{\partial x}{\partial p} = \frac{r}{\lambda^O}; \quad (3.4.2)$$

which implies that $\frac{\partial x}{\partial p} > 0$ for $x > 1$. Next, from (3.3.48), we obtain

$$\begin{aligned} -\frac{1}{\gamma} y_s^{-\frac{1}{\gamma}-1} \frac{\partial y_s}{\partial p} = & -\frac{\lambda^O}{r(r + \lambda^O)} \left[r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right] \frac{(B_1 - 1)(1 - B_2)}{(B_1 - B_2)} \\ & \cdot \left[\frac{B_1 x^{B_1-2}}{1 + \gamma(B_1 - 1)} - \frac{B_2 x^{B_2-2}}{1 + \gamma(B_2 - 1)} \right] \frac{\partial x}{\partial p}. \end{aligned} \quad (3.4.3)$$

Thus,

$$\begin{aligned}
\frac{\partial z_0}{\partial p} &\propto y_s - \gamma x y_s^{\frac{1}{\gamma}+1} \frac{\lambda^O}{r(r + \lambda^O)} \left[r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right] \frac{(B_1 - 1)(1 - B_2)}{(B_1 - B_2)} \\
&\cdot \left[\frac{B_1 x^{B_1-2}}{1 + \gamma(B_1 - 1)} - \frac{B_2 x^{B_2-2}}{1 + \gamma(B_2 - 1)} \right] \\
&\propto \left[\frac{1}{r} - \frac{\lambda^O}{r(r + \lambda^O)} \frac{B_1(1 - B_2)}{B_1 - B_2} \frac{x^{B_1-1}}{1 + \gamma(B_1 - 1)} - \frac{\lambda^O}{r(r + \lambda^O)} \frac{B_2(B_1 - 1)}{B_1 - B_2} \frac{x^{B_2-1}}{1 + \gamma(B_2 - 1)} \right] \\
&\quad - \gamma \frac{\lambda^O}{r(r + \lambda^O)} \frac{(B_1 - 1)(1 - B_2)}{B_1 - B_2} \left[\frac{B_1 x^{B_1-1}}{1 + \gamma(B_1 - 1)} - \frac{B_2 x^{B_2-1}}{1 + \gamma(B_2 - 1)} \right] \\
&\propto 1 - \frac{\lambda^O}{r + \lambda^O} \left[\frac{B_1(1 - B_2)}{B_1 - B_2} x^{B_1-1} + \frac{B_2(B_1 - 1)}{B_1 - B_2} x^{B_2-1} \right].
\end{aligned} \tag{3.4.4}$$

This last expression is positive for $1 \leq x < \tilde{x}$ because the left-hand side of equation (3.3.52) is increasing with respect to \tilde{x} for $\tilde{x} > 1$. \square

As the retiree becomes more risk averse, we expect her to be more willing to annuitize her wealth in order to guarantee a particular income to fund her consumption. It turns out that z_0 is decreasing with respect to γ , and we prove fact this in the next proposition.

Proposition 3.4.2. z_0 , given in (3.3.50), decreases with respect to γ .

Proof. First, consider the case for which $p < p^*$. From equations (3.3.50), (3.3.49), and (3.3.48), we can express z_0 as

$$\begin{aligned}
z_0 &= \frac{\lambda^O}{r(r + \lambda^O)} \left[1 + r \left(\frac{1}{\gamma} - 1 \right) \frac{1}{r + \frac{\lambda^S}{\gamma} - m \frac{1-\gamma}{\gamma^2}} \right] - \frac{1}{r} \\
&\quad + x^{\frac{1}{\gamma}} \left[-\frac{\lambda^O}{r(r + \lambda^O)} \frac{B_1(1 - B_2)}{B_1 - B_2} \frac{x^{B_1-1}}{1 + \gamma(B_1 - 1)} - \frac{\lambda^O}{r(r + \lambda^O)} \frac{B_2(B_1 - 1)}{B_1 - B_2} \frac{x^{B_2-1}}{1 + \gamma(B_2 - 1)} + \frac{1}{r} \right].
\end{aligned} \tag{3.4.5}$$

It follows that

$$\begin{aligned}
\frac{\partial z_0}{\partial \gamma} &= \frac{\lambda^O}{r + \lambda^O} \left[\left(\frac{1}{\gamma} - 1 \right) \frac{-1}{\left[r + \frac{\lambda^S}{\gamma} - m \frac{1-\gamma}{\gamma^2} \right]^2} \left(-\frac{\lambda^S}{\gamma^2} - m \left(-\frac{2}{\gamma^3} + \frac{1}{\gamma^2} \right) \right) - \frac{1}{\gamma^2} \frac{1}{r + \frac{\lambda^S}{\gamma} - m \frac{1-\gamma}{\gamma^2}} \right] \\
&\quad - \frac{1}{\gamma^2} x^{\frac{1}{\gamma}} \ln x \left[-\frac{\lambda^O}{r(r + \lambda^O)} \frac{B_1(1 - B_2)}{B_1 - B_2} \frac{x^{B_1-1}}{1 + \gamma(B_1 - 1)} + \frac{1}{r} \right] \\
&\quad + \frac{1}{\gamma^2} x^{\frac{1}{\gamma}} \ln x \frac{\lambda^O}{r(r + \lambda^O)} \frac{B_2(B_1 - 1)}{B_1 - B_2} \frac{x^{B_2-1}}{1 + \gamma(B_2 - 1)} \\
&\quad + x^{\frac{1}{\gamma}} \frac{\lambda^O}{r(r + \lambda^O)} \frac{(B_1 - 1)(1 - B_2)}{B_1 - B_2} \left[\frac{B_1 x^{B_1-1}}{(1 + \gamma(B_1 - 1))^2} - \frac{B_2 x^{B_2-1}}{(1 + \gamma(B_2 - 1))^2} \right] \\
&\propto - \left[(r + \lambda^S) + m \left(\frac{1 - \gamma}{\gamma} \right)^2 \right] - y_s^{-\frac{1}{\gamma}} x^{\frac{1}{\gamma}} \ln x \left(r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right) \frac{r + \lambda^O}{\lambda^O} \\
&\quad + \frac{x^{\frac{1}{\gamma}}}{\gamma^2} \frac{m}{B_1 - B_2} [B_1 x^{B_1-1} (1 + \gamma(B_2 - 1))^2 - B_2 x^{B_2-1} (1 + \gamma(B_1 - 1))^2] =: g(x)
\end{aligned} \tag{3.4.6}$$

It is straightforward to show that $g(1) = 0$. Thus, if we prove that $g'(x) < 0$ for $1 < x < \tilde{x}$, then it follows that $\frac{\partial z_0}{\partial \gamma} < 0$ for $p < p^*$. Additionally, because z_0 is independent of p for $p \geq p^*$, from continuity of $\frac{\partial z_0}{\partial \gamma}$, we conclude that $\frac{\partial z_0}{\partial \gamma} < 0$ for all p .

To that end,

$$\begin{aligned}
g'(x) &= \frac{1}{\gamma} y_s^{-\frac{1}{\gamma}-1} \frac{\partial y_s}{\partial x} x^{\frac{1}{\gamma}} \ln x \frac{r + \lambda^O}{\lambda^O} \left(r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right) \\
&\quad - y_s^{-\frac{1}{\gamma}} \left[\frac{1}{\gamma} x^{\frac{1}{\gamma}-1} \ln x + x^{\frac{1}{\gamma}-1} \right] \frac{r + \lambda^O}{\lambda^O} \left(r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right) \\
&\quad + \frac{m}{\gamma^3 (B_1 - B_2)} x^{\frac{1}{\gamma}-1} (1 + \gamma(B_1 - 1))^2 (1 + \gamma(B_2 - 1))^2 \left[\frac{B_1 x^{B_1-1}}{1 + \gamma(B_1 - 1)} - \frac{B_2 x^{B_2-1}}{1 + \gamma(B_2 - 1)} \right] \\
&\propto \left(\frac{\ln x}{\gamma} + 1 \right) \left[\frac{B_1(1 - B_2)}{B_1 - B_2} x^{B_1-1} + \frac{B_2(B_1 - 1)}{B_1 - B_2} x^{B_2-1} - \frac{r + \lambda^O}{\lambda^O} \right].
\end{aligned} \tag{3.4.7}$$

From the definition of \tilde{x} in (3.3.52) and from the fact that the left-hand side of (3.3.52) strictly increases with respect to \tilde{x} for $\tilde{x} > 1$, the latter expression is negative for $1 \leq x < \tilde{x}$. \square

3.4.2 Numerical examples

We provide numerous numerical examples to illustrate the analytical results of Section 3.3. We focus our attention on the effects of the proportional surrender charge p and risk aversion γ on the optimal annuitization, consumption, and investment strategies of an individual. We use the following parameter values for our computations:

- $\lambda^S = \lambda^O = 0.04$; the hazard rate is such that the expected future lifetime is 25 years.
- $r = 0.04$; the riskless rate of return is 4% over inflation.
- $\mu = 0.08$; the drift of the risky asset is 8% over inflation.
- $\sigma = 0.20$; the volatility of the risky asset is 20%.

With the parameter values given above, the critical proportional surrender charge is $p^* = 0.308$. Recall that this is the critical value of p above which the optimal annuitization, investment, consumption strategies, as well as the maximized utility, of a retiree do not depend on the size of surrender charge. This feature along with others are demonstrated in the tables and figures in this section.

In Table 3.1, we give the value of $z_0 = w/A$ for various values of surrender charge p and risk aversion γ . The value of z_0 is the critical ratio of wealth to annuity income above which an individual will purchase more annuities; it indicates the willingness of an individual to annuitize. For example, assuming $\gamma = 2.5$ and $p = 0.3$, a retiree with \$100,000 of assets and with \$25,000 of existing annuity income would immediately trade \$47,116 of assets for \$3,773 of additional annuity. By so doing, the critical ratio of wealth-to-annuity income becomes $z_0 = 1.8362$. By comparison, a retiree

Table 3.1: How do the proportional surrender charge p and risk aversion γ affect annuitization?

p ($p^* = 0.308$)	z_0 for various levels of p and γ					
	$\gamma = 0.8$	$\gamma = 1.5$	$\gamma = 2.0$	$\gamma = 2.5$	$\gamma = 3.0$	$\gamma = 5.0$
0.01	1.5794	0.8173	0.6078	0.4838	0.4018	0.2394
0.02	2.2400	1.1443	0.8478	0.6733	0.5584	0.3318
0.04	3.1723	1.5904	1.1722	0.9280	0.7680	0.4545
0.08	4.4593	2.1759	1.5919	1.2548	1.0355	0.6093
0.10	4.9542	2.3914	1.7444	1.3726	1.1314	0.6642
0.20	6.5875	3.0593	2.2088	1.7276	1.4184	0.8263
0.30	7.1591	3.2723	2.3530	1.8362	1.5052	0.8743
0.40	7.1622	3.2734	2.3537	1.8367	1.5057	0.8746
0.60	7.1622	3.2734	2.3537	1.8367	1.5057	0.8746
1.00	7.1622	3.2734	2.3537	1.8367	1.5057	0.8746

with $\gamma = 0.8$ and all other parameters the same, would not purchase additional annuities since her critical ratio of wealth-to-annuity income of 4.0 is already below the level of 7.1591.

A higher value of z_0 indicates less interest in annuitization. In table 3.1, we observe that z_0 increases with respect to p for a fixed value of γ and decreases with respect to γ for a fixed value of p , as expected from Propositions 3.4.1 and 3.4.2, respectively. In other words, a lower proportional surrender charge encourages retirees to annuitize, and those who are more risk averse are more willing to purchase annuities. This result is consistent with our position that lack of flexibility discourages retirees from purchasing (irreversible) immediate life annuities. In the case for which $p \geq p^*$, z_0 does *not* change with p . An individual treats reversible annuities with a surrender charge greater than p^* as an irreversible one ($p = 1$) regardless of the size of $p \in [p^*, 1]$.

Tables 3.2 and 3.3 demonstrate, respectively, the impact of the surrender charge on the individual's optimal investment and consumption strategies when wealth is zero. In these cases, we assume that the existing annuity income is two units, that is, $A = 2$. This assumption of existing annuity income is reasonable due to the fact that most American retirees have annuity income at the time of retirement, say, from

Table 3.2: How do the proportional surrender charge p and risk aversion γ affect investment?

p ($p^* = 0.308$)	Investment in the risky asset for various levels of p and γ when $w = 0$ ($A = 2$)					
	$\gamma = 0.8$	$\gamma = 1.5$	$\gamma = 2.0$	$\gamma = 2.5$	$\gamma = 3.0$	$\gamma = 5.0$
0.01	25.2800	13.4827	10.1120	8.0896	6.7413	4.0448
0.02	22.9429	12.2362	9.1772	7.3417	6.1181	3.6709
0.04	19.7310	10.5232	7.8924	6.3139	5.2616	3.1570
0.08	15.2923	8.1559	6.1169	4.8935	4.0780	2.4468
0.10	13.5110	7.2059	5.4044	4.3235	3.6029	2.1618
0.20	6.3519	3.3877	2.5408	2.0326	1.6938	1.0163
0.30	0.4460	0.2378	0.1784	0.1427	0.1189	0.0714
0.40	0	0	0	0	0	0
0.60	0	0	0	0	0	0
1.00	0	0	0	0	0	0

Social Security or a private pension.

From the tables, we observe that the optimal strategy is divided into two categories: In the cases for which $p < p^*$, an individual continues to invest in the risky asset; in the cases for which $p \geq p^*$, it is optimal not to invest in the risky asset and to consume less than the annuity income. When $p < p^*$, the optimal strategy is to continue investing in the risky asset by borrowing from the riskless account and to surrender just enough annuity income to keep wealth non-negative when needed. In contrast, when $p \geq p^*$, it is optimal not to surrender the annuity income under any circumstance, as we showed in Section 3.3.3. Therefore, when $w = 0$, the corresponding investment and consumption strategies guarantee that wealth will not decrease farther. These numerical results are consistent with our analytical results in Theorems 3.3.1 and 3.3.2.

Note that the amount invested in the risky asset at $w = 0$ decreases with respect to p (and γ) other parameters fixed. In fact, we can show that investment decreases with respect to p at any level of wealth, but for the sake of brevity, we omitted that proposition and corresponding proof from this paper. Similarly, we can show that investment decreases with respect to γ when $w = 0$.

Table 3.3: How do the proportional surrender charge p and risk aversion γ affect consumption?

p	Rate of consumption for various levels of p and γ when $w = 0$ ($A = 2$)					
	$\gamma = 0.8$	$\gamma = 1.5$	$\gamma = 2.0$	$\gamma = 2.5$	$\gamma = 3.0$	$\gamma = 5.0$
$(p^* = 0.308)$						
0.01	1.8486	2.0766	2.0911	2.0891	2.0832	2.0607
0.02	1.8353	2.0549	2.0718	2.0721	2.0682	2.0507
0.04	1.8013	2.0168	2.0389	2.0438	2.0435	2.0345
0.08	1.7235	1.9477	1.9812	1.9947	2.0010	2.0071
0.10	1.6824	1.9147	1.9540	1.9719	1.9814	1.9946
0.20	1.4682	1.7559	1.8254	1.8644	1.8893	1.9363
0.30	1.2478	1.6016	1.7018	1.7617	1.8016	1.8811
0.40	1.2300	1.5893	1.6920	1.7536	1.7947	1.8768
0.60	1.2300	1.5893	1.6920	1.7536	1.7947	1.8768
1.00	1.2300	1.5893	1.6920	1.7536	1.7947	1.8768

Also, the rate of consumption at $w = 0$ decreases with p given γ . However, the relationship between the rate of consumption and risk aversion is not monotonic. Both Tables 3.2 and 3.3 confirm our analytical conclusion that when $p \geq p^*$, an individual behaves as if the annuity is not reversible at all; compare with Theorem 3.3.2.

Figures 3.2-3.4 considers the problem from a different point of view. In these figures, we plot the optimal investment, optimal consumption, and maximized utility, respectively, for wealth w ranging from 0 to 1. We choose the parameters as described above and assume that $\gamma = 2.5$ and $A = 1$. In each figure, we plot two curves representing the cases $p = 0.05$ and $p = 0.6$. The graphs show that a larger p value leads to lower investment, consumption, and utility for all w . Namely, both the investment and consumption strategies are more conservative when the surrender charge is higher. Lastly, Figure 3.5 displays the maximized utility as a function of $p \in [0, 1]$ for fixed $(w, A) = (100, 0)$ and $\gamma = 2.5$. This representative graph shows that the maximized utility is a monotonically decreasing function of p and is of the same value for all $p \geq p^*$.

3.4.3 Optimal strategies as $\gamma \rightarrow \infty$

In this section, we examine the optimal consumption, investment, and annuitization strategies as the individual becomes highly risk averse.

Proposition 3.4.3. *As $\gamma \rightarrow \infty$, $c^*(0, A) \rightarrow A$ for all $p \in [0, 1]$.*

Proof. Recall that $c^*(w, A) = U_w^{-\frac{1}{\gamma}}(w, A) = AV_z^{-\frac{1}{\gamma}}(w/A)$; thus, $c^*(0, A) = AV_z^{-\frac{1}{\gamma}}(0) = Ay_s^{-\frac{1}{\gamma}}$, and it sufficient to show that $\lim_{\gamma \rightarrow \infty} y_s^{-\frac{1}{\gamma}} = 1$. The latter follows from equations (3.3.31), (3.3.48), and (3.3.122). \square

Proposition 3.4.4. *As $\gamma \rightarrow \infty$, $\pi^*(0, A) \rightarrow 0$ for all $A > 0$ and $p \in [0, 1]$.*

Proof. When $p \geq p^*$, from Theorem 3.3.2, we know that $\pi^*(0, A) = 0$ regardless of the value of γ . When $p < p^*$, from Theorem 3.3.1 and the work in Section 3.3.2, we have

$$\pi^*(0, A) = \frac{\mu - r}{\sigma^2} A \cdot y_s \hat{V}_{yy}(y_s). \quad (3.4.8)$$

Using (3.3.28), (3.3.45), and (3.3.46), we obtain

$$\begin{aligned} & \lim_{\gamma \rightarrow \infty} y_s \hat{V}_{yy}(y_s) \\ &= \lim_{\gamma \rightarrow \infty} \left\{ -\frac{\lambda^O}{r(r + \lambda^O)} \frac{(B_1 - 1)(1 - B_2)}{B_1 - B_2} \left[\frac{B_1}{1 + \gamma(B_1 - 1)} x^{B_1 - 1} - \frac{B_2}{1 + \gamma(B_2 - 1)} x^{B_2 - 1} \right] \right\} \\ & \quad + \lim_{\gamma \rightarrow \infty} \left[\frac{1 - \gamma}{\gamma} \left(-\frac{1}{\gamma} \right) C y_s^{-\frac{1}{\gamma}} \right] \\ &= 0. \end{aligned} \quad (3.4.9)$$

This completes the proof. \square

Proposition 3.4.5. *As $\gamma \rightarrow \infty$, $z_0 \rightarrow 0$ for all $p \in [0, 1]$.*

Proof. We prove this property for the case $p < p^*$. The proof for the case $p \geq p^*$ is similar. First, consider the definition of z_0 in (3.3.50). Recall that $y_b = y_s/x$, in

which x is independent of γ . Thus, $\lim_{\gamma \rightarrow \infty} x^{1/\gamma} = 1$, and as shown in Proposition 3.4.3, $\lim_{\gamma \rightarrow \infty} y_s^{-1/\gamma} = 1$. By taking the limit in (3.3.50), we obtain

$$\begin{aligned} \lim_{\gamma \rightarrow \infty} z_0 &= - \lim_{\gamma \rightarrow \infty} \frac{\lambda^O}{r(r + \lambda^O)} \frac{m(1 - \gamma) + (r + \lambda^S)\gamma}{\gamma^2 \left[r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2} \right]} - \lim_{\gamma \rightarrow \infty} \frac{\gamma - 1}{\gamma} C y_b^{-\frac{1}{\gamma}} - \frac{1}{r} \\ &= 0 + \frac{1}{r} - \frac{1}{r} = 0. \end{aligned} \tag{3.4.10}$$

□

From Propositions 3.4.3, 3.4.4, and 3.4.5, we deduce that the optimal strategy for a highly risk averse retiree is to annuitize fully and to consume her annuity income thereafter. Her wealth is always zero because $z_0 = 0$ in the limit.

3.5 Conclusion

In this paper, we considered a utility maximization problem with reversible life annuities. In an incomplete financial market with a riskless asset, a risky asset, and reversible annuities, we investigated a retiree's optimal annuitization, consumption, and investment strategies. In our model, the reversibility of an annuity is quantified by a proportional surrender charge, which ranges from 0% to 100%. A retiree's willingness to annuitize is indicated by the critical ratio of wealth-to-annuity income above which she would immediately purchase more annuities. We proved that a smaller surrender charge leads to a smaller critical ratio of wealth-to-annuity income. This result indicates that the reversibility of annuities encourages retirees to annuitize.

We found that the individual's optimal strategy depends on the size of the proportional surrender charge. When the surrender charge is larger than the so-called critical value, a retiree does not surrender her existing annuity income under any circumstance. She stops investing in the risky asset and consumes less than her annuity

income as her wealth approaches zero. When the surrender charge is smaller than the critical value, a retiree surrenders enough annuity income to keep her wealth non-negative whenever needed. She continues to invest in the risky asset as her wealth approaches zero.

One might think that a smaller surrender charge is *always* better than a larger one. But surprisingly, we found that in the case for which the surrender charge is larger than the critical value, the optimal strategies and maximized utility of a retiree do *not* depend on the size of surrender charge. A reversible annuity with a proportional surrender charge above the critical value is equivalent to an irreversible one for an optimally behaving retiree. We also found that, for a retiree without a bequest motive, full annuitization is optimal when the surrender charge is zero or as the risk aversion of the retiree approaches infinity.

In our paper, we assumed a constant hazard rate and a constant interest rate in our analysis as simplifications of reality. However, we believe that the main qualitative insights will hold in general.

The relation between reversibility of annuities and retirees' willingness to annuitize revealed in this paper offers an explanation for the "annuity puzzle" and suggests a way to better structure life annuities. Also, a well developed second market for life annuities could partially function as a surrender option and, hence, should encourage voluntary annuitization. We note that there are other possible reasons that cause the the "annuity puzzle," such as the default risk of the insurance companies investigated by [Jang et al. \[2009\]](#) and bequest motive investigated by [Lockwood \[2009\]](#). Based on our model, we expect that the existence of the bequest motive will further discourage a retiree from annuitization, but the mathematical tractability of our model will be lost if a bequest motive is added. This occurs because with a bequest motive, we

can no longer reduce the dimension of the problem. The critical ratio of wealth-to-annuity income would likely be a non-linear curve in the wealth-annuity plane.

To investigate reversible annuities, an alternative to the utility metric is the probability of lifetime ruin. As a risk metric, it is some times used to investigate optimization problems faced by retirees in a financial market. An investigation of reversible annuities within that framework can be found in [Wang and Young \[2009\]](#). Wang and Young assume that a retiree consumes at an exogenously given level, and they determine the optimal investment strategy, as well as the optimal time to annuitize or to surrender, in order to minimize the probability that wealth reaches zero before her death. By contrast, in this paper, consumption is determined by the retiree herself. Results in [Wang and Young \[2009\]](#) are consistent with what we found in this work.

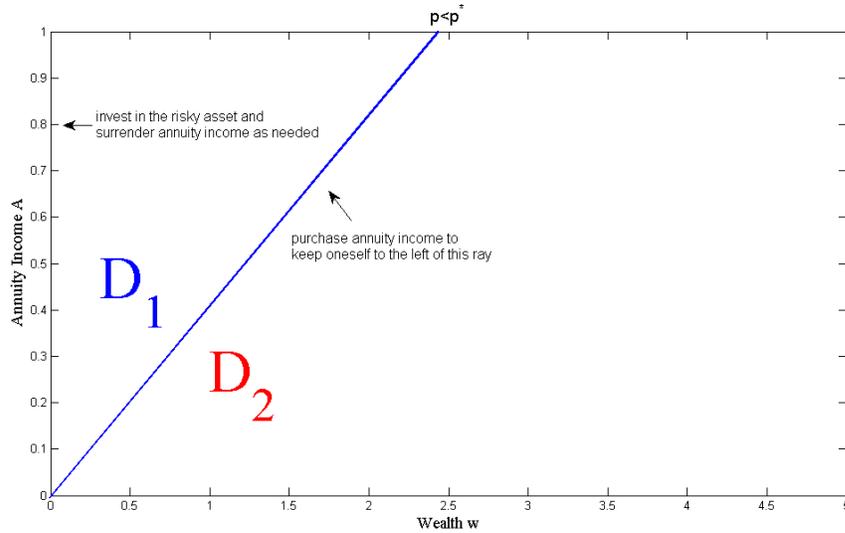
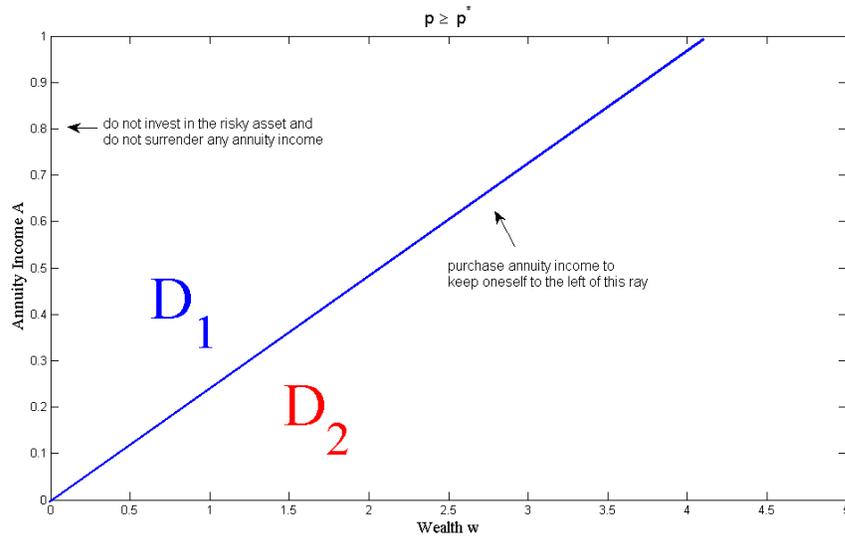
(a) Optimal annuitization with a small proportional surrender charge ($p < p^*$)(b) Optimal annuitization with a large proportional surrender charge ($p \geq p^*$)

Figure 3.1: Optimal annuitization strategies for small and large proportional surrender charges: In each case, there exists a critical ratio of wealth-to-annuity income z_0 indicated by the blue ray in the graph. It is optimal for a retiree to keep herself to the left of the blue ray. Different investment strategies are applied in the two different cases when $w = 0$, as indicated in each graph.

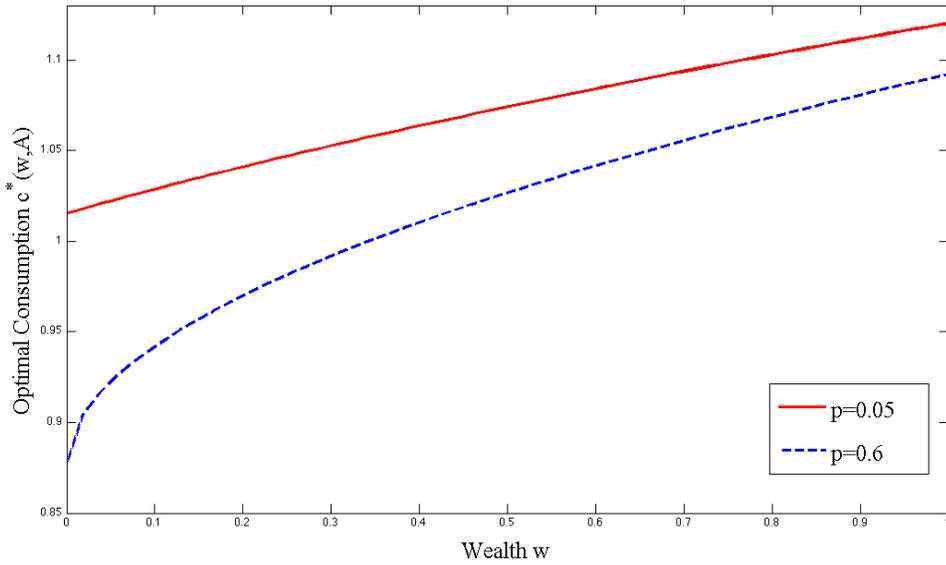


Figure 3.2: Optimal consumption strategies for wealth equal to zero: We set the existing annuity income A to be 1 in this example. The blue dotted line represents the case $p \geq p^*$; recall that in this case, the optimal rate of consumption at $w = 0$ is less than the rate of annuity income. By contrast, the optimal rate of consumption is allowed to be above A when $p < p^*$. In this example, we set $r = 0.04$, $\mu = 0.08$, $\lambda^S = \lambda^O = 0.04$, $\sigma = 0.2$, $\gamma = 2.5$, and $A = 1$.

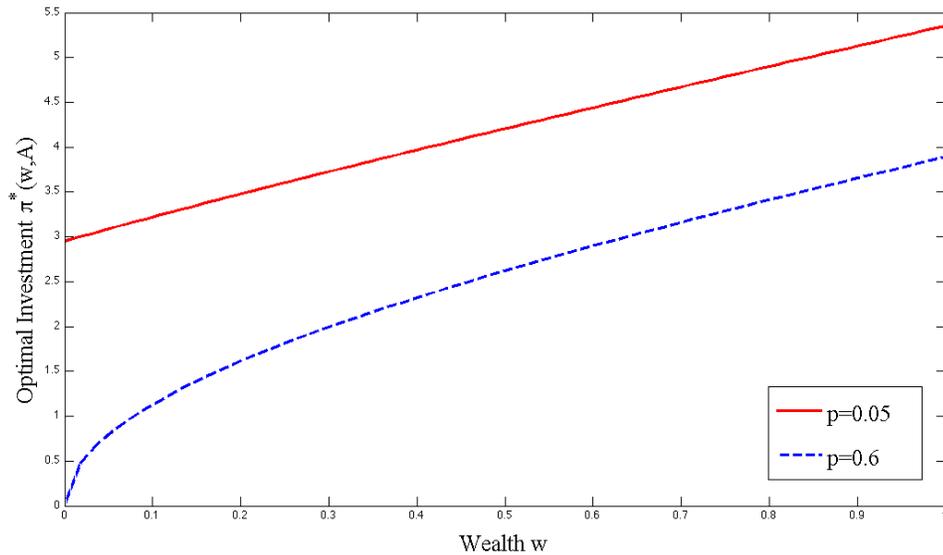


Figure 3.3: Optimal investment strategies for wealth equal to zero: When $p \geq p^*$, a retiree invests nothing in the risky asset when $w = 0$. This case is represented by the blue dotted line. By contrast, when $p < p^*$, it is optimal to invest in risky asset when $w = 0$. In this example, we set $r = 0.04$, $\mu = 0.08$, $\lambda^S = \lambda^O = 0.04$, $\sigma = 0.2$, $\gamma = 2.5$, and $A = 1$.

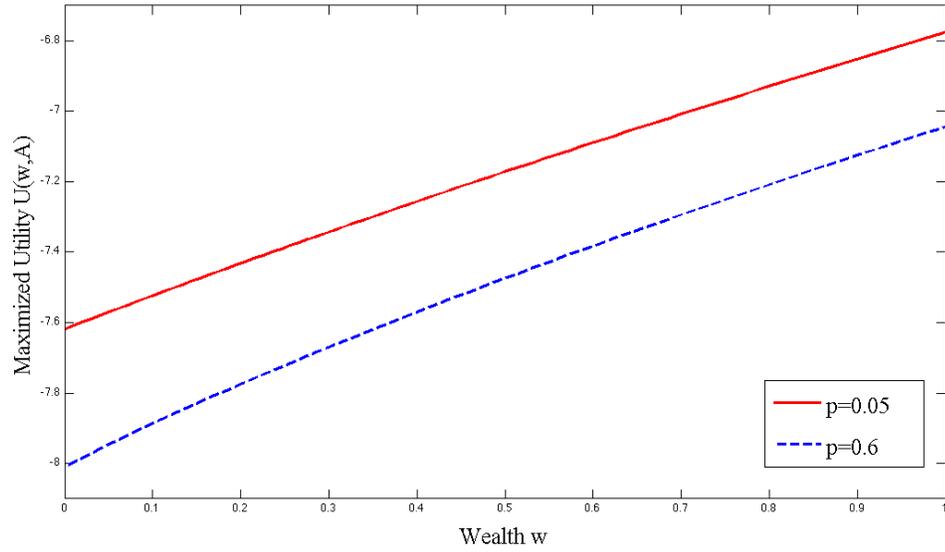


Figure 3.4: Maximized utilities when the proportional surrender charges differ: In this example, we set $r = 0.04$, $\mu = 0.08$, $\lambda^S = \lambda^O = 0.04$, $\sigma = 0.2$, $\gamma = 2.5$, and $A = 1$.

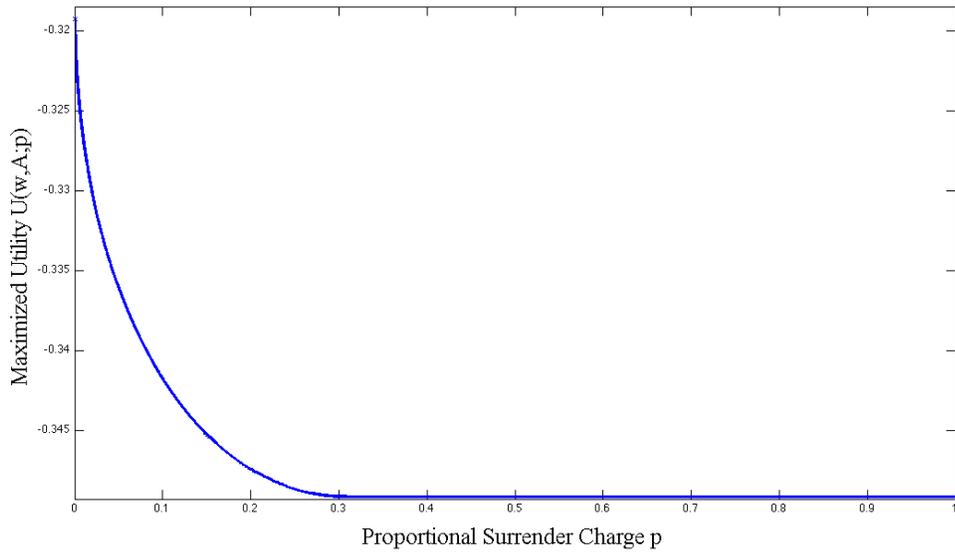


Figure 3.5: How the proportional surrender charge p affects the maximized utility: The shape of the curve above is representative for situations with different given initial wealth and annuity income. In this example, we set $r = 0.04$, $\mu = 0.08$, $\lambda^S = \lambda^O = 0.04$, $\sigma = 0.2$, $\gamma = 2.5$, $w = 100$, and $A = 0$.

CHAPTER IV

Hedging pure endowments with mortality derivatives

4.1 Introduction and motivation

A basic assumption in many actuarial texts is that mortality risk can be eliminated based on the law of large number. It is believed that the standard deviation per insurance policy vanishes as the number of policies sold becomes large enough. However, this assumption is valid only when the mortality intensity is deterministic, and a number of recent researchers argue that mortality intensity, or *hazard rate*, is stochastic; see, for example, [Dowd et al. \[2006\]](#) and the references therein. The uncertainty of hazard rates is significant enough that stochastic mortality risk has to be considered in the valuation of life insurance and annuity contracts and in pension fund management. A concrete example of stochastic mortality risk is longevity risk, namely, the risk that future lifetimes will be greater than expected. Longevity risk has attracted much attention in recent years, and many capital market instruments have been proposed to deal with this risk for annuity providers and pension funds; see [Dowd et al. \[2006\]](#), [Blake and Burrows \[2001\]](#), and [Blake et al. \[2006\]](#) for more details.

However, few researchers have focused on the effectiveness of hedging mortality risk with the proposed mortality-linked derivatives; one notable exception is

the work of [Lin and Cox \[2005\]](#). In this paper, we investigate the application of mortality-linked derivatives for hedging mortality risk and offer suggestions for further mortality-linked innovation based on our analysis. To this end, we select a stochastic model to describe the mortality dynamics. Several stochastic mortality models have been proposed in the recent literature. [Milevsky and Promislow \[2001\]](#), [Biffis \[2005\]](#), [Schrager \[2006\]](#), and [Dahl \[2004\]](#) use continuous-time diffusion processes to model the hazard rate, as we do in this paper. Alternatively, [Miltersen and Persson \[2005\]](#) and [Cairns et al. \[2006a\]](#) model the forward mortality. [Milidonis et al. \[2010\]](#) incorporate mortality state changes into the mortality dynamics with a discrete-time Markov regime-switching model. Also, see [Cairns et al. \[2006b\]](#) for a detailed overview of various modeling frameworks. In this paper, we use the model proposed by [Bayraktar et al. \[2009\]](#) to describe the dynamics of both hazard rates: λ_t^P , the one inherent in the insurance contract to be hedged, and λ_t^I , the one referenced by the mortality-linked derivative.

Another issue is the choice of pricing paradigm. Different methods for pricing mortality risk have been proposed in recent literatures, and [Bauer et al. \[2010\]](#) extensively discusses them. Among these methods, [Bayraktar et al. \[2009\]](#) developed a dynamic pricing theory, which can be considered as a continuous version of the actuarial standard deviation premium principle. In our paper, we extend their pricing mechanism to a market that includes mortality-linked derivatives. We price a pure endowment assuming that the issuing company hedges its contract with a mortality forward in order to minimize the variance of the value of the hedging portfolio and then requires compensation for the unhedgeable part of the mortality risk in the form of a pre-specified instantaneous Sharpe ratio.

The main purpose of this paper is to investigate the hedging of life insurance

and annuity contracts with mortality-linked derivatives. To this end, we develop a partial differential equation (PDE) whose solution is the value of the hedged insurance contract. We compare the values of the hedged contract under different market prices of mortality risk. We also analyze how the correlation between λ_t^P and λ_t^I affects the values of the hedged contract. The main contribution of our paper is to show that hedging can reduce the price of the insurance contract only under certain conditions on the correlation of the hazard rates and on the market price of mortality risk. As part of the procedure, we also show that the desired features of the pricing mechanism by Milevsky et al. [2005] and Bayraktar et al. [2009] still hold in our extension.

The remainder of this paper is organized as follows: In Section 4.2, we present our financial market, describe the pricing mechanism of the pure endowment in a market with mortality-linked derivatives, and derive a non-linear PDE whose solution is the value of the hedged pure endowment. In Section 4.3, we analyze the value $P^{(n)}$ of n pure endowments on conditionally independent and identically distributed lives, with the emphasis on how the correlation of the hazard rates and the market price of mortality risk affect the price of the hedged pure endowments. We then present the PDE that gives the limiting value of $\frac{1}{n}P^{(n)}$ as n goes to infinity in Section 4.4. We show that this limiting value solves a *linear* PDE and represent this value as an expectation with respect to an equivalent martingale measure. In Section 4.5, we demonstrate our results with numerical examples, discuss whether and when the hedging with mortality-risk derivatives reduces the price of pure endowments, and provide suggestions on the application of mortality-linked derivatives for insurance companies. We describe a numerical scheme to compute the value of a pure endowment in Section 4.7. Section 4.6 concludes the paper.

4.2 Incomplete market of financial and mortality derivatives

In this section, we describe the pure endowment contract and the financial market in which the issuer of the contract invests to hedge the risk. In the financial market, there are three products: a money market fund, a bond, and a mortality derivative. We obtain the optimal strategy to hedge the risk of the contract with bonds and mortality derivatives in order to minimize the variance of the value of the investment portfolio. We, then, price the pure endowment using the instantaneous Sharpe ratio.

4.2.1 Mortality model and financial market

First, we set up the model for the dynamics of hazard rates—either the hazard rate for the pure endowment or the one for the mortality derivative. We assume that a hazard rate λ_t follows a diffusion process with some positive lower bound $\underline{\lambda}$. Thus, we require that as λ_t goes to $\underline{\lambda}$, the drift of λ_t is positive and the volatility of λ_t approaches 0. Biologically, the lower bound $\underline{\lambda}$ represents the remaining hazard rate after all accidental or preventable causes of death have been removed. Mathematically, the need for such a lower bound appears later in this paper.

Specifically, we use the following diffusion model for a hazard rate:

$$d\lambda_t = a(\lambda_t, t) (\lambda_t - \underline{\lambda}) dt + b(t) (\lambda_t - \underline{\lambda}) dW_t, \quad (4.2.1)$$

in which W is a standard Brownian motion on a filtered probability space $(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t \geq 0}, \mathbb{P})$.

We require that the volatility $b(t)$ is a continuous function of t and is bounded from below by a positive constant κ in $[0, T]$. We also assume that $a(\lambda_t, t)$ is Hölder continuous with respect to λ and t , and that $a(\lambda_t, t) > 0$ when $0 < \lambda_t - \underline{\lambda} < \epsilon$ for some $\epsilon > 0$.

In this paper, we consider two different but correlated hazard rates. One is the hazard rate of the insured population; namely, the hazard rate of the people who

purchase the pure endowments. For simplicity, when we consider a portfolio of n pure endowment contracts in this paper, we assume that all individuals are of the same age and are subject to the same hazard rate. We denote as λ_t^P the hazard rate of insured population, and the dynamics of λ_t^P is given by

$$d\lambda_t^P = a^P(\lambda_t^P, t) (\lambda_t^P - \underline{\lambda}^P) dt + b^P(t) (\lambda_t^P - \underline{\lambda}^P) dW_t^P. \quad (4.2.2)$$

We also consider a second hazard rate on which the mortality derivatives are based, namely, the hazard rate of an indexed population. We denote this hazard rate as λ_t^I , whose dynamics is given by

$$d\lambda_t^I = a^I(\lambda_t^I, t) (\lambda_t^I - \underline{\lambda}^I) dt + b^I(t) (\lambda_t^I - \underline{\lambda}^I) dW_t^I. \quad (4.2.3)$$

The uncertainties of the two hazard rates are correlated such that $dW_t^I dW_t^P = \rho dt$ with $\rho \in [-1, 1]$.

Suppose, at time $t = 0$, an insurer issues a *pure endowment* to an individual that pays \$1 at time T if the individual is alive at that time. To price this contract, we will create a portfolio composed of the obligation to pay this pure endowment and investment in the financial market.

In the financial market, the dynamics of the short rate r_t is given by

$$dr_t = \mu(r_t, t) dt + \sigma(r_t, t) dW_t^r \quad (4.2.4)$$

in which μ and $\sigma \geq 0$ are deterministic functions of the short rate and time, and W^r is a standard Brownian motion adapted to $(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t \geq 0}, \mathbb{P})$. We assume that W^r is independent of W^P and W^I , and that μ and σ are such that $r_t > 0$ almost surely for all $t \geq 0$ and such that (4.2.4) has a unique solution.

Both the T -bond and the mortality derivative are priced based on the principle of no-arbitrage. Thus, for the short rate r , there exists a market price of risk q^r that

is adapted to the filtration generated by W^r ; and for the hazard rate λ_t^I , there exists a market price of risk q^{λ^I} that is adapted to the filtration generated by W^I . We, therefore, write either q_t^r or $q^r(r_t, t)$ for the market price of the short rate risk at time t ; similarly, we write either $q_t^{\lambda^I}$ or $q^{\lambda^I}(\lambda_t^I, t)$ for the market price of the hazard rate risk at time t .

Define an equivalent martingale measure \mathbb{Q} whose Radon-Nikodym derivative with respect to \mathbb{P} is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left\{ - \int_0^T \left[q^r(r_s, s) dW_s^r + q^{\lambda^I}(\lambda_s^I, s) dW_s^I \right] - \frac{1}{2} \int_0^T \left[(q^r(r_s, s))^2 + (q^{\lambda^I}(\lambda_s^I, s))^2 \right] ds \right\}. \quad (4.2.5)$$

In the \mathbb{Q} -space, the dynamics of the hazard rates and the short rate are given by

$$\begin{cases} d\lambda_t^P = a^{P,Q}(\lambda_t^P, \lambda_t^I, t) (\lambda_t^P - \underline{\lambda}^P) dt + b^P(t) (\lambda_t^P - \underline{\lambda}^P) dW_t^{P,Q}, \\ d\lambda_t^I = a^{I,Q}(\lambda_t^I, t) (\lambda_t^I - \underline{\lambda}^I) dt + b^I(t) (\lambda_t^I - \underline{\lambda}^I) dW_t^{I,Q}, \\ dr_t = \mu^Q(r_t, t) dt + \sigma(r_t, t) dW_t^{r,Q}, \end{cases} \quad (4.2.6)$$

in which

$$\begin{cases} W_t^{P,Q} = W_t^P + \rho \int_0^t q^{\lambda^I}(\lambda_s^I, s) ds, \\ W_t^{I,Q} = W_t^I + \int_0^t q^{\lambda^I}(\lambda_s^I, s) ds, \\ W_t^{r,Q} = W_t^r + \int_0^t q^r(r_s, s) ds, \end{cases} \quad (4.2.7)$$

and

$$\begin{cases} a^{P,Q}(\lambda_t^I, \lambda_t^P, t) = a^P(\lambda_t^P, t) - \rho q^{\lambda^I}(\lambda_t^I, t) b^P(t), \\ a^{I,Q}(\lambda_t^I, t) = a^I(\lambda_t^I, t) - q^{\lambda^I}(\lambda_t^I, t) b^I(t), \\ \mu^Q(r_t, t) = \mu(r_t, t) - q^r(r_t, t) \sigma(r_t, t). \end{cases} \quad (4.2.8)$$

The time- t price of the T -bond is given by

$$F(r, t; T) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} \middle| r_t = r \right], \quad (4.2.9)$$

and the bond price F solves the following partial differential equation (PDE), Björk [2004]:

$$\begin{cases} F_t + \mu^Q(r, t) F_r + \frac{1}{2}\sigma^2(r, t) F_{rr} - rF = 0, \\ F(r, T; T) = 1. \end{cases} \quad (4.2.10)$$

From this PDE, we obtain the dynamics of F for $t \leq s \leq T$:

$$\begin{cases} dF(r_s, s) = [r_s F(r_s, s) + q^r(r_s, s) \sigma(r_s, s) F_r(r_s, s)] ds + \sigma(r_s, s) F_r(r_s, s) dW_s^r \\ F(r_t, t) = F(r, t). \end{cases} \quad (4.2.11)$$

Without loss of generality, we specify the mortality derivative as a q -forward. Define the cumulative hazard rate process by $\Lambda_t^I = \int_0^t \lambda_s^I ds$ for $0 \leq t \leq T$. Then, the time- t value of a q -forward with delivery time T is given by

$$S(r, \lambda^I, \Lambda^I, t; T) = \mathbb{E}^Q \left[e^{-\int_t^T r_s ds} \left(e^{-\int_0^T \lambda_s^I ds} - K \right) \middle| r_t = r, \lambda_t^I = \lambda^I, \Lambda_t^I = \Lambda^I \right]. \quad (4.2.12)$$

in which $K = \mathbb{E}^Q \left[e^{-\int_0^T \lambda_t^I dt} \middle| \mathcal{F}_0 \right]$ is the delivery price. As for the T -bond, we have the following PDE for the mortality derivative:

$$\begin{cases} S_t + \mu^Q S_r + \frac{1}{2}\sigma^2 S_{rr} + a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) S_{\lambda^I} + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 S_{\lambda^I \lambda^I} + \lambda^I S_{\Lambda^I} - rS = 0, \\ S(r, \lambda^I, \Lambda^I, T; T) = e^{-\Lambda^I} - K. \end{cases} \quad (4.2.13)$$

From the PDE above, we obtain the dynamics of S for $t \leq s \leq T$:

$$\begin{cases} dS_s = \left[r_s S + q_s^r \sigma S_r + q_s^{\lambda^I} b^I \cdot (\lambda_s^I - \underline{\lambda}^I) S_{\lambda^I} \right] ds + \sigma S_r dW_s^r + b^I \cdot (\lambda_s^I - \underline{\lambda}^I) S_{\lambda^I} dW_s^I, \\ S(r_t, \lambda_t^I, \Lambda_t^I, t) = S(r, \lambda^I, \Lambda^I, t). \end{cases} \quad (4.2.14)$$

Since we fixed the maturity T for both the bond and the q -forward in this paper, we drop the notation T when appropriate.

4.2.2 Pricing the pure endowment via the instantaneous sharpe ratio

Recipe for valuation

Even with the mortality derivatives, the market for insurance is incomplete when $\rho \neq \pm 1$ due to the fact that the mortality of the insured population and the mortality of the indexed population are not perfectly correlated. This mismatch is called *basis risk*; see [Coughlan et al. \[2007\]](#) for more details. Therefore, there is no unique method of pricing for insurance contracts, and to value contracts in this market, one has to choose a pricing mechanism. For example, [Bayraktar and Ludkovski \[2009\]](#) used indifference pricing and [Dahl and Moller \[2006\]](#) consider the set of equivalent martingale measures when pricing the unhedgeable mortality risk. We use the instantaneous Sharpe ratio proposed by [Milevsky et al. \[2005\]](#) and [Bayraktar et al. \[2009\]](#) to price the risk due to the desirable properties of the resulting price. We will show these properties in Section 4.3. Moreover, as the number of contracts approaches infinity, the limiting price per contract can be represented as an expectation with respect to an equivalent martingale measure. In a market without mortality derivatives, this pricing methodology has been proved useful for pricing pure endowments ([Milevsky et al. \[2005\]](#)), life insurance ([Young \[2008\]](#)), life annuities ([Bayraktar et al. \[2009\]](#)), and financial derivatives ([Bayraktar and Young \[2007b\]](#)). In this paper, we extend this pricing mechanism to incorporate mortality derivatives in the financial market. Our method for pricing in an incomplete market with mortality derivatives is as follows:

1. First, we set up a portfolio composed of two parts: (1) the obligation to underwrite the pure endowment, and (2) a self-financing sub-portfolio of T -bonds, q -forwards maturing at T , and money market funds to partially hedge the pure endowment contract.

2. Second, we find the optimal investments in bonds and mortality derivatives to minimize the local variance of the portfolio. This method is called local risk minimization by Schweizer [2001]. In case of a complete market, the minimized local volatility is zero. However, the incompleteness of the insurance market leads to residual risk, as measured by the local variance.
3. Third, we assume that the insurance provider requires compensations for the unhedgeable risk. The price of the contingent claim is set to make the instantaneous Sharpe ratio of the total portfolio equal to a pre-specified value. This is equivalent to setting the price of the contingent claim such that the drift of the portfolio equals the short rate times the portfolio value plus the pre-specified Sharpe ratio times the local standard deviation of the portfolio. Thus, our pricing method is a type of local standard deviation premium principle, Young [2004b].

Hedging and pricing a single pure endowment

Denote by $P(r, \lambda^I, \lambda^P, t; T)$ the time- t value of a pure endowment that pays \$1 at maturity T if the individual is alive at that time. Here, we explicitly recognize that the price of the pure endowment depends on the short rate r , the hazard rate λ^P of the insured individual, and the hazard rate λ^I of the indexed population. Since the maturity T is fixed, we simplify the notation to $P(r, \lambda^I, \lambda^P, t)$. (By writing P to represent the value of the pure endowment, we assume that the individual is alive. If the individual dies before T , the value of the pure endowment jumps to \$0.)

Suppose the insurer creates a portfolio Π as described in Step (i) in Section 4.2.2. This portfolio is consist of two parts: (1) the obligation to underwrite the pure endowment with value $-P$, and (2) a self-financing sub-portfolio V_t of T -bonds, q -

forwards, and money market funds to hedge the risk of the pure endowment. Thus, $\Pi_t = -P(r_t, \lambda_t^I, \lambda_t^P, t) + V_t$. Let π_t^r equal the number T -bonds and $\pi_t^{\lambda^I}$ the number of q -forwards in the self-financing sub-portfolio at time t with the rest, namely, $V_t - \pi_t^r F(r_t, t) - \pi_t^{\lambda^I} S(r_t, \lambda_t^I, t)$, in money market funds.

By Itô's lemma, the dynamics of the value of the pure endowment $P(r, \lambda^I, \lambda^P, t)$ in the *physical* probability space is given by

$$\begin{aligned}
dP(r, \lambda^I, \lambda^P, t) &= P_t dt + P_r dr_t + P_{\lambda^I} d\lambda_t^I + P_{\lambda^P} d\lambda_t^P + \frac{1}{2} P_{rr} d[r, r]_t \\
&\quad + \frac{1}{2} P_{\lambda^I \lambda^P} d[\lambda^I, \lambda^P]_t + P_{\lambda^I \lambda^P} d[\lambda^I, \lambda^P]_t + \frac{1}{2} P_{\lambda^P \lambda^P} d[\lambda^P, \lambda^P]_t - P dN_t \\
&= [P_t + \mu P_r + a^I \cdot (\lambda_t^I - \underline{\lambda}^I) P_{\lambda^I} + a^P \cdot (\lambda_t^P - \underline{\lambda}^P) P_{\lambda^P}] dt \\
&\quad + \left[\frac{1}{2} \sigma^2 P_{rr} + \frac{1}{2} (b^I)^2 (\lambda_t^I - \underline{\lambda}^I)^2 P_{\lambda^I \lambda^I} + \frac{1}{2} (b^P)^2 (\lambda_t^P - \underline{\lambda}^P)^2 P_{\lambda^P \lambda^P} \right] dt \\
&\quad + \rho b^I b^P (\lambda_t^I - \underline{\lambda}^I) (\lambda_t^P - \underline{\lambda}^P) P_{\lambda^I \lambda^P} dt - P dN_t \\
&\quad + \sigma P_r dW_t^r + b^I \cdot (\lambda_t^I - \underline{\lambda}^I) P_{\lambda^I} dW_t^I + b^P \cdot (\lambda_t^P - \underline{\lambda}^P) P_{\lambda^P} dW_t^P,
\end{aligned} \tag{4.2.15}$$

in which $[\cdot, \cdot]_t$ represents the quadratic variation at time t , and N_t is a time-inhomogeneous Poisson process with intensity λ_t^P that indicates when the individual dies. Recall that the value of P jumps to \$0 when the individual dies; thus, we have the term $-P dN_t$ to account for this drop.

Since the sub-portfolio V_t is self-financing, its dynamics are given by

$$\begin{aligned}
dV_t &= \pi_t^r dF(r_t, t) + \pi_t^{\lambda^I} dS_t(r_t, \lambda_t^I, t) + r_t \left[V_t - \pi_t^r F(r_t, t) - \pi_t^{\lambda^I} S_t(r_t, \lambda_t^I, t) \right] dt \\
&= \left[\pi_t^r q_t^r \sigma F_r + \pi_t^{\lambda^I} q_t^r \sigma S_r + \pi_t^{\lambda^I} q_t^{\lambda^I} b^I \cdot (\lambda_t^I - \underline{\lambda}^I) S_{\lambda^I} + r_t V_t \right] dt \\
&\quad + \left(\pi_t^r \sigma F_r + \pi_t^{\lambda^I} \sigma S_r \right) dW_t^r + \pi_t^{\lambda^I} b^I \cdot (\lambda_t^I - \underline{\lambda}^I) S_{\lambda^I} dW_t^I,
\end{aligned} \tag{4.2.16}$$

in which the second equality follows from equations (4.2.11) and (4.2.14), and we suppress the dependence of the functions on the underlying variables.

It follows from equations (4.2.15) and (4.2.16) that the value of the portfolio Π_{t+h} at time $t+h$ for $h > 0$, given $\Pi_t = \Pi$, is

$$\begin{aligned}
\Pi_{t+h} &= \Pi - \int_t^{t+h} dP(r_s, \lambda_s^P, \lambda_s^I, s) + \int_t^{t+h} dV_s \\
&= \Pi - \int_t^{t+h} \mathcal{D}P(r_s, \lambda_s^P, \lambda_s^I, s) ds + \int_t^{t+h} r_s \Pi_s ds \\
&\quad + \int_t^{t+h} \left[\pi_s^r q_s^r \sigma F_r + \pi_s^{\lambda^I} q_s^r \sigma S_r + \pi_s^{\lambda^I} q_s^{\lambda^I} b^I \cdot (\lambda_s^I - \underline{\lambda}^I) S_{\lambda^I} \right] ds \\
&\quad + \int_t^{t+h} \left(\pi_s^r \sigma F_r + \pi_s^{\lambda^I} \sigma S_r - \sigma P_r \right) dW_s^r - \int_t^{t+h} b^P \cdot (\lambda_s^P - \underline{\lambda}^P) P_{\lambda^P} dW_s^P \\
&\quad + \int_t^{t+h} b^I \cdot (\lambda_s^I - \underline{\lambda}^I) \left(\pi_s^{\lambda^I} S_{\lambda^I} - P_{\lambda^I} \right) dW_s^I + \int_t^{t+h} P (dN_s - \lambda_s^P ds),
\end{aligned} \tag{4.2.17}$$

in which \mathcal{D} is the operator defined on the set of appropriately differentiable functions on $\mathbb{R}_+ \times (\underline{\lambda}^I, \infty) \times (\underline{\lambda}^P, \infty) \times [0, T]$ by

$$\begin{aligned}
\mathcal{D}v &= - (r + \lambda^P) v + v_t + \mu v_r + a^I \cdot (\lambda^I - \underline{\lambda}^I) v_{\lambda^I} + a^P \cdot (\lambda^P - \underline{\lambda}^P) v_{\lambda^P} + \frac{1}{2} \sigma^2 v_{rr} \\
&\quad + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 v_{\lambda^I \lambda^I} + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) v_{\lambda^I \lambda^P} \\
&\quad + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 v_{\lambda^P \lambda^P}.
\end{aligned} \tag{4.2.18}$$

Note that the compensated counting process $N_t - \int_0^t \lambda_s^P ds$ is a (local) martingale .

When we consider the single life case, the value of P_t becomes zero immediately after the individual's death, so the value of the portfolio increase by P . If we consider the price $P^{(n)}$ of n conditionally independent and identically distributed lives as in Section 4.2.2, the intensity of the counting process N_t is $n\lambda_t^P$ at time t . As one of the n individual dies, the value of the portfolio increases by $P^{(n)} - P^{(n-1)}$. We will consider $P^{(n)}$ later and continue with the single-life case for now.

The second step as stated in Section 4.2.2 is to choose π_t^r and $\pi_t^{\lambda^I}$ to minimize the local variance of the portfolio. To this end, we calculate the conditional expectation

and variance of Π_{t+h} at time t given $\Pi_t = \Pi$. First, we define a stochastic process Y_h for $h > 0$ by

$$\begin{aligned} Y_h = & \Pi - \int_t^{t+h} \mathcal{D}P(r_s, \lambda_s^I, \lambda_s^P, s) ds + \int_t^{t+h} r_s \Pi_s ds \\ & + \int_t^{t+h} \left[\pi_s^r q_s^r \sigma F_r + \pi_s^{\lambda^I} q_s^r \sigma S_r + \pi_s^{\lambda^I} q_s^{\lambda^I} b^I \cdot (\lambda_s^I - \underline{\lambda}^I) S_{\lambda^I} \right] ds. \end{aligned} \quad (4.2.19)$$

Thus, $\mathbb{E}(\Pi_{t+h} | \mathcal{F}_t) = \mathbb{E}^{r, \lambda^I, \lambda^P, t}(Y_h)$, in which $\mathbb{E}^{r, \lambda^I, \lambda^P, t}$ denotes the conditional expectation given $r_t = r$, $\lambda_t^I = \lambda^I$, and $\lambda_t^P = \lambda^P$. From (4.2.17) and (4.2.19) we have

$$\begin{aligned} \Pi_{t+h} = & Y_h + \int_t^{t+h} \left(\pi_s^r \sigma F_r + \pi_s^{\lambda^I} \sigma S_r - \sigma P_r \right) dW_s^r - \int_t^{t+h} b^P \cdot (\lambda_s^P - \underline{\lambda}^P) P_{\lambda^P} dW_s^P \\ & + \int_t^{t+h} b^I \cdot (\lambda_s^I - \underline{\lambda}^I) \left(\pi_s^{\lambda^I} S_{\lambda^I} - P_{\lambda^I} \right) dW_s^I + \int_t^{t+h} P (dN_s - \lambda_s^P ds). \end{aligned} \quad (4.2.20)$$

It follows that

$$\begin{aligned} \text{Var} [\Pi_{t+h} | \mathcal{F}_t] = & \mathbb{E} \left[\left(\Pi_{t+h} - \mathbb{E}^{r, \lambda^I, \lambda^P, t}(Y_h) \right)^2 \middle| \mathcal{F}_t \right] \\ = & \mathbb{E}(Y_h - \mathbb{E}Y_h)^2 + \mathbb{E} \int_t^{t+h} \left(\pi_s^r \sigma F_r + \pi_s^{\lambda^I} \sigma S_r - \sigma P_r \right)^2 ds \\ & + \mathbb{E} \int_t^{t+h} (b^I)^2 (\lambda_s^I - \underline{\lambda}^I)^2 \left(\pi_s^{\lambda^I} S_{\lambda^I} - P_{\lambda^I} \right)^2 ds \\ & - 2 \mathbb{E} \int_t^{t+h} \rho b^I b^P (\lambda_s^I - \underline{\lambda}^I) (\lambda_s^P - \underline{\lambda}^P) P_{\lambda^P} \left(\pi_s^{\lambda^I} S_{\lambda^I} - P_{\lambda^I} \right) ds \\ & + \mathbb{E} \int_t^{t+h} (b^P)^2 (\lambda_s^P - \underline{\lambda}^P)^2 (P_{\lambda^P})^2 ds + \mathbb{E} \int_t^{t+h} \lambda^P P^2 ds + o(h), \end{aligned} \quad (4.2.21)$$

in which all the expectations are conditional on the information available at time t .

Thus, the optimal investments in the q -forward and T -bond to minimize the local variance are given by, respectively,

$$\left(\pi_t^{\lambda^I} \right)^* = \frac{1}{S_{\lambda^I}} \left[P_{\lambda^I} + \rho \frac{b^P \cdot (\lambda_t^P - \underline{\lambda}^P)}{b^I \cdot (\lambda_t^I - \underline{\lambda}^I)} P_{\lambda^P} \right], \quad (4.2.22)$$

$$\left(\pi_t^r \right)^* = \frac{1}{F_r} \left(P_r - \left(\pi_t^{\lambda^I} \right)^* S_r \right). \quad (4.2.23)$$

Equations (4.2.22) and (4.2.23) show that in the self-financing sub-portfolio, the q -forward is used to hedge the mortality risk in the pure endowment, and T -bonds are

used to hedge the interest risk of the portfolio. Under this investment strategy, the drift and local variance of the portfolio become

$$\lim_{h \rightarrow 0} \frac{1}{h} [\mathbb{E}(\Pi_{t+h} | \mathcal{F}_t) - \Pi] = -\mathcal{D}^Q P + r\Pi, \quad (4.2.24)$$

and

$$\lim_{h \rightarrow 0} \frac{1}{h} \text{Var} [\Pi_{t+h} | \mathcal{F}_t] = (1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 P_{\lambda^P}^2 + \lambda^P P^2, \quad (4.2.25)$$

with

$$\begin{aligned} \mathcal{D}^Q P &= - (r + \lambda^P) P + P_t + \mu^Q P_r + a^{I,Q} (\lambda^I - \underline{\lambda}^I) P_{\lambda^I} + a^{P,Q} (\lambda^P - \underline{\lambda}^P) P_{\lambda^P} \\ &\quad + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 P_{\lambda^I \lambda^I} + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) P_{\lambda^I \lambda^P} \\ &\quad + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 P_{\lambda^P \lambda^P} + \frac{1}{2} \sigma^2 P_{rr}. \end{aligned} \quad (4.2.26)$$

Remark 4.2.1. *When $\rho = \pm 1$, the q -forward and the pure endowment bear identical uncertainty risk in the hazard rates. In this case, the mortality risk in the pure endowment can be completely hedged with the q -forward, and the minimum local variance of the portfolio only comes from the random occurrence of death, namely,*

$$\lim_{h \rightarrow 0} \frac{1}{h} \text{Var} \left[\Pi_{t+h} \middle| \mathcal{F}_t \right] = \lambda^P P^2. \quad (4.2.27)$$

Remark 4.2.2. *As we will show in Property 4.3.8 in Section 4.3, $P_{\lambda^I} \equiv 0$ when $\rho = 0$, and the corresponding optimal investment in the q -forward is $\pi_t^{\lambda^I*} \equiv 0$. Intuitively, the q -forward is not used to hedge the mortality risk in the pure endowment when the two underlying hazard rates are not correlated.*

Next, we price the pure endowment via the instantaneous Sharpe ratio as stated in Step (iii) in Section 4.2.2. The minimized local variance of the portfolio in (4.2.25) is positive; therefore, the insurer is not able to hedge all the risk underlying the pure

endowment. The insurer requires an excess return on this unhedgeable risk so that the instantaneous Sharpe ratio of the portfolio equals a pre-specified value α . We could allow α to be a function of say r , λ^I , λ^P , and t to parallel the market price of the risk process $\{q_t^r, q_t^{\lambda^I}\}$. However, for simplicity we choose α to be a constant. (Further discussion of the instantaneous Sharpe ratio is available in [Milevsky et al. \[2006b\]](#).) We assume that $0 \leq \alpha \leq \sqrt{\underline{\lambda}^P}$; as we will see, some of the properties of P rely on this upper bound for α .

To achieve a Sharpe ratio of α and thereby to determine the value P of the pure endowment, we set the drift of the portfolio equal to short rate times the portfolio value plus α times the minimized local standard deviation of the portfolio. Thus, we get the following equation for P from (4.2.24) and (4.2.25):

$$- \mathcal{D}^Q P + r\Pi = r\Pi + \alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 P_{\lambda^P}^2 + \lambda^P P^2}. \quad (4.2.28)$$

If the individual is still alive at time T , then the policy is worth exactly \$1 at that time, that is, $P(r, \lambda^I, \lambda^P, T) = 1$. Thus, $P = P(r, \lambda^I, \lambda^P, t)$ solves the following non-linear PDE on $\mathbb{R}_+ \times (\underline{\lambda}^I, \infty) \times (\underline{\lambda}^P, \infty) \times [0, T]$:

$$\left\{ \begin{array}{l} P_t + \mu^Q P_r + a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) P_{\lambda^I} + a^{P,Q} \cdot (\lambda^P - \underline{\lambda}^P) P_{\lambda^P} \\ \quad + \frac{1}{2} \sigma^2 P_{rr} + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 P_{\lambda^I \lambda^I} + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 P_{\lambda^P \lambda^P} \\ \quad + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) P_{\lambda^I \lambda^P} - (r + \lambda^P) P \\ \quad = -\alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 P_{\lambda^P}^2 + \lambda^P P^2}, \\ P(r, \lambda^I, \lambda^P, T) = 1. \end{array} \right. \quad (4.2.29)$$

We can simplify the solution to (4.2.29) because the uncertainty in the short rate is uncorrelated with the uncertainty in mortality rates. Indeed, note that $P(r, \lambda^I, \lambda^P, t) = F(r, t) \psi(\lambda^I, \lambda^P, t)$, in which F is the price of the T -bond and

solves (4.2.10), and ψ solves the following non-linear PDE:

$$\left\{ \begin{array}{l} \psi_t + a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) \psi_{\lambda^I} + a^{P,Q} \cdot (\lambda^P - \underline{\lambda}^P) \psi_{\lambda^P} + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 \psi_{\lambda^I \lambda^I} \\ \quad + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) \psi_{\lambda^I \lambda^P} + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \psi_{\lambda^P \lambda^P} - \lambda^P \psi \\ = -\alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \psi_{\lambda^P}^2 + \lambda^P \psi^2}, \\ \psi(\lambda^I, \lambda^P, T) = 1. \end{array} \right. \quad (4.2.30)$$

The existence of a solution to (4.2.30) follows from standard techniques; see, for example, Chapter 36 in Walter [1970]. Uniqueness of the solution follows from the comparison result in Section 4.3 of this paper.

Hedging and pricing a portfolio of pure endowments

In this section, we develop the PDE for the price $P^{(n)}$ of n pure endowment contracts. We assume that all the individuals are of the same age and are subject to the same hazard rate given in (4.2.2). We further assume that, given the hazard rate, occurrences of death are independent. As discussed in the paragraph following equation (4.2.18), when an individual dies, the portfolio value Π increases by $P^{(n)} - P^{(n-1)}$. By paralleling the derivation of (4.2.29), one gets the following PDE for $P^{(n)}$:

$$\left\{ \begin{array}{l} P_t^{(n)} + \mu^Q P_r^{(n)} + a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) P_{\lambda^I}^{(n)} + a^{P,Q} \cdot (\lambda^P - \underline{\lambda}^P) P_{\lambda^P}^{(n)} \\ \quad + \frac{1}{2} \sigma^2 P_{rr}^{(n)} + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 P_{\lambda^I \lambda^I}^{(n)} + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 P_{\lambda^P \lambda^P}^{(n)} \\ \quad + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) P_{\lambda^I \lambda^P}^{(n)} - r P^{(n)} - n \lambda^P \cdot (P^{(n)} - P^{(n-1)}) \\ = -\alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \left(P_{\lambda^P}^{(n)} \right)^2 + n \lambda^P (P^{(n)} - P^{(n-1)})^2}, \\ P^{(n)}(r, \lambda^I, \lambda^O, T) = n, \end{array} \right. \quad (4.2.31)$$

with initial value $P^{(0)} \equiv 0$, and $P^{(1)} = P$, as given by (4.2.29).

As in Section 4.2.2, $P^{(n)}(r, \lambda^I, \lambda^P, t) = F(r, t) \psi^{(n)}(\lambda^I, \lambda^P, t)$, in which F solves (4.2.10) and $\psi^{(n)}$ solves the following PDE

$$\left\{ \begin{array}{l} \psi_t^{(n)} + a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) \psi_{\lambda^I}^{(n)} + a^{P,Q} \cdot (\lambda^P - \underline{\lambda}^P) \psi_{\lambda^P}^{(n)} + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 \psi_{\lambda^I \lambda^I}^{(n)} \\ + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) \psi_{\lambda^I \lambda^P}^{(n)} + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \psi_{\lambda^P \lambda^P}^{(n)} \\ - n \lambda^P \cdot (\psi^{(n)} - \psi^{(n-1)}) \\ = -\alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(n)} \right)^2 + n \lambda^P (\psi^{(n)} - \psi^{(n-1)})^2}, \\ \psi^{(n)}(\lambda^I, \lambda^P, T) = n, \end{array} \right. \quad (4.2.32)$$

with initial value $\psi^{(0)} \equiv 0$, and $\psi^{(1)} = \psi$, as given by (4.2.30).

4.3 Properties of $P^{(n)}$

To demonstrate properties of $P^{(n)}$, we need a comparison principle similar to the one in Walter [1970]. To this end, we first state a relevant one-sided Lipschitz condition along with growth conditions. We require that the function $g = g(\lambda^I, \lambda^P, t, v, p_1, p_2)$ satisfies the following one-sided Lipschitz condition: For $v > w$,

$$\begin{aligned} g(\lambda^I, \lambda^P, t, v, p_1, p_2) - g(\lambda^I, \lambda^P, t, w, q_1, q_2) &\leq c(\lambda^I, \lambda^P, t) (v - w) \\ &+ d_1(\lambda^I, \lambda^P, t) |p_1 - q_1| + d_2(\lambda^I, \lambda^P, t) |p_2 - q_2|, \end{aligned} \quad (4.3.1)$$

with growth conditions on c , d_1 and d_2 given by

$$\left\{ \begin{array}{l} 0 \leq c(\lambda^I, \lambda^P, t) \leq K \left[1 + (\ln(\lambda^I - \underline{\lambda}^I))^2 + (\ln(\lambda^P - \underline{\lambda}^P))^2 \right], \\ 0 \leq d_1(\lambda^I, \lambda^P, t) \leq K (\lambda^I - \underline{\lambda}^I) [1 + \ln(\lambda^I - \underline{\lambda}^I) + \ln(\lambda^P - \underline{\lambda}^P)], \\ 0 \leq d_2(\lambda^I, \lambda^P, t) \leq K (\lambda^P - \underline{\lambda}^P) [1 + \ln(\lambda^I - \underline{\lambda}^I) + \ln(\lambda^P - \underline{\lambda}^P)], \end{array} \right. \quad (4.3.2)$$

for some constant $K \geq 0$ and for all $(\lambda^I, \lambda^P, t) \in (\underline{\lambda}^I, \infty) \times (\underline{\lambda}^P, \infty) \times [0, T]$.

To prove Lemma 4.3.2 below, as well as many of the properties of $P^{(n)}$, we rely on the following lemma.

Lemma 4.3.1. $\sqrt{C^2 + A^2} \leq |A - B| + \sqrt{C^2 + B^2}$

Proof. It is clear that the inequality holds if $A \leq B$. For the case $A > B$, see the proof of Lemma 4.5 in Milevsky et al. [2005]. \square

Lemma 4.3.2. Define g_n , for $n \geq 1$, by

$$\begin{aligned} g_n(\lambda^I, \lambda^P, t, v, p_1, p_2) &= a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) p_1 + a^{P,Q} \cdot (\lambda^P - \underline{\lambda}^P) p_2 - n\lambda^P (v - \psi^{(n-1)}) \\ &\quad + \alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 p_2^2 + n\lambda^P (v - \psi^{(n-1)})^2}, \end{aligned} \quad (4.3.3)$$

in which $\psi^{(n-1)}$ solves (4.2.32) with n replaced by $n - 1$. Then, g_n satisfies the one-sided Lipschitz condition (4.3.1) on $(\lambda^I, \lambda^P, t) \in (\underline{\lambda}^I, \infty) \times (\underline{\lambda}^P, \infty) \times [0, T]$.

Furthermore, condition (4.3.2) holds if

$$\begin{cases} |a^{I,Q}| \leq K [1 + \ln(\lambda^I - \underline{\lambda}^I) + \ln(\lambda^P - \underline{\lambda}^P)], \\ |a^{P,Q}| \leq K [1 + \ln(\lambda^I - \underline{\lambda}^I) + \ln(\lambda^P - \underline{\lambda}^P)], \end{cases} \quad (4.3.4)$$

for some constant $K \geq 0$.

Proof. Suppose that $v > w$, then

$$\begin{aligned} &g_n(\lambda^I, \lambda^P, t, v, p_1, p_2) - g_n(\lambda^I, \lambda^P, t, w, q_1, q_2) \\ &= a^{I,Q} (\lambda^I - \underline{\lambda}^I) (p_1 - q_1) + a^{P,Q} (\lambda^P - \underline{\lambda}^P) (p_2 - q_2) - n\lambda^P (v - w) \\ &\quad + \alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 p_2^2 + n\lambda^P (v - \psi^{(n-1)})^2} \\ &\quad - \alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 q_2^2 + n\lambda^P (w - \psi^{(n-1)})^2} \\ &\leq |a^{I,Q}| (\lambda^I - \underline{\lambda}^I) |p_1 - q_1| + \left[|a^{P,Q}| (\lambda^P - \underline{\lambda}^P) + \alpha \sqrt{1 - \rho^2} b^P (\lambda^P - \underline{\lambda}^P) \right] |p_2 - q_2| \\ &\quad - \left(n\lambda^P - \alpha \sqrt{n\lambda^P} \right) (v - w) \\ &\leq |a^{I,Q}| (\lambda^I - \underline{\lambda}^I) |p_1 - q_1| + \left[|a^{P,Q}| + \alpha \sqrt{1 - \rho^2} b^P \right] (\lambda^P - \underline{\lambda}^P) |p_2 - q_2|. \end{aligned} \quad (4.3.5)$$

In the above series of inequalities, we use $\alpha \leq \sqrt{\underline{\lambda}^P} \leq \sqrt{\lambda^P}$ and Lemma 4.3.1. Therefore, (4.3.1) holds with $c = 0$, $d_1 = |a^{I,Q}| (\lambda^I - \underline{\lambda}^I)$ and $d_2 = |a^{P,Q}| (\lambda^P - \underline{\lambda}^P) + \alpha \sqrt{1 - \rho^2} b^P (\lambda^P - \underline{\lambda}^P)$. Notice that d_1 and d_2 satisfy condition (4.3.2) if (4.3.4) holds. \square

Assumption. Henceforth, we assume that the condition (4.3.4) holds for rest of the paper. For later purpose, we also assume that $a_{\lambda^P}^{P,Q} (\lambda^P - \underline{\lambda}^P)$ is Hölder continuous and satisfies the following growth condition

$$\left| a_{\lambda^P}^{P,Q} (\lambda^P - \underline{\lambda}^P) + a^{P,Q} \right| \leq K \left[1 + (\ln (\lambda^P - \underline{\lambda}^P))^2 \right]. \quad (4.3.6)$$

Theorem 4.3.1. Let $G = (\underline{\lambda}^I, \infty) \times (\underline{\lambda}^P, \infty) \times [0, T]$, and denote by \mathcal{G} the collection of functions on G that are twice differentiable in their first two variables and once-differentiable in their third variable. Define an operator \mathcal{L} on \mathcal{G} by

$$\begin{aligned} \mathcal{L}v &= v_t + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 v_{\lambda^I \lambda^I} + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) \psi_{\lambda^I \lambda^P} \\ &\quad + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 v_{\lambda^P \lambda^P} + g_n (\lambda^I, \lambda^P, t, v, v_{\lambda^I}, v_{\lambda^P}), \end{aligned} \quad (4.3.7)$$

in which g_n is given by (4.3.3). Suppose that $v, w \in \mathcal{G}$ are such that there exists a constant $K \geq 0$ with $v \leq e^{K\{(\ln(\lambda^I - \underline{\lambda}^I))^2 + (\ln(\lambda^P - \underline{\lambda}^P))^2\}}$ and $w \geq -e^{K\{(\ln(\lambda^I - \underline{\lambda}^I))^2 + (\ln(\lambda^P - \underline{\lambda}^P))^2\}}$ for large $(\ln(\lambda^I - \underline{\lambda}^I))^2 + (\ln(\lambda^P - \underline{\lambda}^P))^2$. Then, if (a) $\mathcal{L}v \geq \mathcal{L}w$ on G and if (b) $v(\lambda^I, \lambda^P, T) \leq w(\lambda^I, \lambda^P, T)$ for all $\lambda^I > \underline{\lambda}^I$ and $\lambda^P > \underline{\lambda}^P$, then $v \leq w$ on G .

Proof. Define $y_1 = \ln(\lambda^I - \underline{\lambda}^I)$, $y_2 = \ln(\lambda^P - \underline{\lambda}^P)$, and $\tau = T - t$. Write $\tilde{v}(y_1, y_2, \tau) = v(\lambda^I, \lambda^P, t)$, etc. Therefore, $\tilde{v} \leq e^{K(y_1^2 + y_2^2)}$ and $\tilde{w} \geq -e^{K(y_1^2 + y_2^2)}$ for large $y_1^2 + y_2^2$. Under this transformation, (4.3.7) becomes

$$\mathcal{L}\tilde{v} = -\tilde{v}_\tau + \frac{1}{2} (\tilde{b}^I)^2 \tilde{v}_{y_1 y_1} + \rho \tilde{b}^I \tilde{b}^P \tilde{v}_{y_1 y_2} + \frac{1}{2} (\tilde{b}^P)^2 \tilde{v}_{y_2 y_2} + \tilde{h}(y_1, y_2, v, \tilde{v}_{y_1}, \tilde{v}_{y_2}), \quad (4.3.8)$$

in which

$$\tilde{h}(y_1, y_2, \tau, \tilde{v}, \tilde{p}_1, \tilde{p}_2) = -\frac{1}{2} (\tilde{b}^I)^2 \tilde{p}_1 - \frac{1}{2} (\tilde{b}^P)^2 \tilde{p}_2 + \tilde{g}_n(y_1, y_2, \tau, \tilde{v}, \tilde{p}_1, \tilde{p}_2), \quad (4.3.9)$$

and \tilde{v} is a differentiable function defined on $\mathbb{R}^2 \times [0, T]$. The differential operator in (4.3.8) is of the form considered by Walter [1970].

To complete the proof, we consider the Lipschitz and growth conditions in the original variables λ^I , λ^P , and t . From Walter [1970], we know that the conditions on \tilde{h} required for Walter's comparison principle are

$$\begin{aligned} \tilde{h}(y_1, y_2, \tau, \tilde{v}, \tilde{p}_1, \tilde{p}_2) - \tilde{h}(y_1, y_2, \tau, \tilde{w}, \tilde{q}_1, \tilde{q}_2) &\leq \tilde{c}(y_1, y_2, \tau)(\tilde{v} - \tilde{w}) + \tilde{d}_1(y_1, y_2, \tau)|\tilde{p}_1 - \tilde{q}_1| \\ &\quad + \tilde{d}_2(y_1, y_2, \tau)|\tilde{p}_2 - \tilde{q}_2|, \end{aligned} \quad (4.3.10)$$

with

$$\begin{cases} 0 \leq \tilde{c}(y_1, y_2, \tau) \leq K(1 + y_1^2 + y_2^2), \\ 0 \leq \tilde{d}_1(y_1, y_2, \tau) \leq K(1 + |y_1| + |y_2|), \\ 0 \leq \tilde{d}_2(y_1, y_2, \tau) \leq K(1 + |y_1| + |y_2|). \end{cases} \quad (4.3.11)$$

Under the original variables, it follows from (4.3.9) and (4.3.5) that, for $v > w$,

$$\begin{aligned} &\tilde{h}(y_1, y_2, \tau, \tilde{v}, \tilde{p}_1, \tilde{p}_2) - \tilde{h}(y_1, y_2, \tau, \tilde{w}, \tilde{q}_1, \tilde{q}_2) \\ &\leq \left[\frac{1}{2}(\tilde{b}^I)^2 + \left| \widetilde{a^{I,Q}} \right| \right] |\tilde{p}_1 - \tilde{q}_1| + \left[\frac{1}{2}(\tilde{b}^P)^2 + \left| \widetilde{a^{P,Q}} \right| + \alpha\sqrt{1 - \rho^2} \tilde{b}^P \right] |\tilde{p}_2 - \tilde{q}_2|. \end{aligned} \quad (4.3.12)$$

Note that $p_1 = e^{-y_1} \tilde{p}_1$ since $\psi_{\lambda^I} = e^{-y} \tilde{\psi}_{y_1}$; similarly, for \tilde{p}_2 , \tilde{q}_1 , and \tilde{q}_2 . Thus, (4.3.10) is satisfied with $\tilde{c} = c = 0$, $\tilde{d}_1 = (\tilde{b}^I)^2 + \left| \widetilde{a^{I,Q}} \right|$, and $\tilde{d}_2 = (\tilde{b}^P)^2 + \left| \widetilde{a^{P,Q}} \right| + \alpha\sqrt{1 - \rho^2} \tilde{b}^P$, and (4.3.11) is satisfied due to Lemma 4.3.2 and (4.3.4) and the fact that \tilde{b}^I and \tilde{b}^P are continuous on $[0, T]$ and are, thus, bounded. \square

For the remainder of this section, we apply Theorem 4.3.1 to investigate properties of the price $P^{(n)}$ for n pure endowment contracts. For simplicity, we will state and prove properties of $\psi^{(n)}$ and afterwards interpret the results in terms of $P^{(n)}$.

Property 4.3.1. For $n \geq 0$, $0 \leq \psi^{(n)} \leq n e^{-(\lambda^P - \alpha\sqrt{\lambda^P})(T-t)}$ on G .

Proof. For ease of presentation, define h by $h(t) = e^{-(\underline{\lambda}^P - \alpha\sqrt{\underline{\lambda}^P})(T-t)}$ for $t \in [0, T]$. We proceed by induction to prove that $\psi^{(n)} \leq nh$ on G . Note that the inequality holds for $n = 0$ since $\psi^{(0)} \equiv 0$. For $n \geq 1$, assume that $\psi^{(n-1)}(\lambda^I, \lambda^P, t) \leq (n-1)h$, and show that $0 \leq \psi^{(n)}(\lambda^I, \lambda^P, t) \leq nh$.

To apply Theorem 4.3.1, define a differential operator \mathcal{L} on \mathcal{G} by (4.3.7). We have $\mathcal{L}\psi^{(n)} = 0$ due to equation (4.2.32). Apply the operator \mathcal{L} to nh to get

$$\begin{aligned} \mathcal{L}(nh) &= \left(\underline{\lambda}^P - \alpha\sqrt{\underline{\lambda}^P} \right) nh - \left(n\lambda^P - \alpha\sqrt{n\lambda^P} \right) (nh - \psi^{(n-1)}) \\ &\leq \left(\underline{\lambda}^P - \alpha\sqrt{\underline{\lambda}^P} \right) nh - \left(n\lambda^P - \alpha\sqrt{n\lambda^P} \right) (n - (n-1)) h \\ &= \left[n \left(\underline{\lambda}^P - \alpha\sqrt{\underline{\lambda}^P} \right) - \left(n\lambda^P - \alpha\sqrt{n\lambda^P} \right) \right] h \leq 0. \end{aligned} \quad (4.3.13)$$

Because $\mathcal{L}(nh) \leq 0 = \mathcal{L}\psi^{(n)}$ and $nh(T) = \psi^{(n)}(\lambda^I, \lambda^P, T) = n$, Theorem 4.3.1 implies that $\psi^{(n)} \leq ne^{-(\underline{\lambda}^P - \alpha\sqrt{\underline{\lambda}^P})(T-t)}$ on G .

Similarly, we prove that $\psi^{(n)} \geq 0$ by induction. Suppose that $\psi^{(n-1)} \geq 0$ for $n \geq 1$, and show that $\psi^{(n)} \geq 0$. We apply the same operator \mathcal{L} from the first part of this proof to the constant function $\mathbf{0}$ on G . Because $\mathcal{L}\mathbf{0} = \left(n\lambda^P + \alpha\sqrt{n\lambda^P} \right) \psi^{(n-1)} \geq 0 = \mathcal{L}\psi^{(n)}$ and $0 \leq n = \psi^{(n)}(\lambda^I, \lambda^P, T)$, Theorem 4.3.1 implies that $\psi^{(n)} \geq 0$ on G . \square

It follows immediately from Property 4.3.1 that $0 \leq P^{(n)}(r, \lambda^I, \lambda^P, t) \leq nF(r, t)$ for $(r, \lambda^I, \lambda^P, t) \in \mathbb{R}^+ \times G$, in which F is the price of a T -bond with face value of \$1. Thus, the price per risk $\frac{1}{n}P^{(n)}$ lies between 0 and F . This is a no-arbitrage condition since the total payoff of n pure endowments at time T is non-negative and is no more than $\$n$.

Property 4.3.2. For $n \geq 1$, $\psi^{(n)} \geq \psi^{(n-1)}$ on G .

Proof. We prove this property by induction. First, the inequality holds for $n = 1$ since $\psi^{(1)} \geq 0$ by Property 4.3.1 and $\psi^{(0)} \equiv 0$. For $n \geq 2$, assume that $\psi^{(n-1)} \geq$

$\psi^{(n-2)}$, and show that $\psi^{(n)} \geq \psi^{(n-1)}$.

Define a differential operator \mathcal{L} on \mathcal{G} by (4.3.7). We have that $\mathcal{L}\psi^{(n)} = 0$ due to equation (4.2.32). Apply the operator \mathcal{L} to $\psi^{(n-1)}$, and use the fact that $\psi^{(n-1)}$ solves (4.2.32) with n replaced by $n - 1$:

$$\begin{aligned}
\mathcal{L}\psi^{(n-1)} &= (n-1)\lambda^P (\psi^{(n-1)} - \psi^{(n-2)}) + \alpha \sqrt{(1-\rho^2)(b^P)^2(\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(n-1)}\right)^2} \\
&\quad - \alpha \sqrt{(1-\rho^2)(b^P)^2(\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(n-1)}\right)^2 + (n-1)\lambda^P (\psi^{(n-1)} - \psi^{(n-2)})^2} \\
&\geq (n-1)\lambda^P (\psi^{(n-1)} - \psi^{(n-2)}) - \alpha \sqrt{(n-1)\lambda^P} (\psi^{(n-1)} - \psi^{(n-2)}) \\
&= \left[(n-1)\lambda^P - \alpha \sqrt{(n-1)\lambda^P} \right] (\psi^{(n-1)} - \psi^{(n-2)}) \geq 0.
\end{aligned} \tag{4.3.14}$$

Note that the first inequality is due to the fact that $\sqrt{A^2 + B^2} \leq |A| + |B|$. We also use the induction assumption that $\psi^{(n-1)} \geq \psi^{(n-1)}$. Because $\psi^{(n)} = n > n - 1 = \psi^{(n-1)}$ at $t = T$, Theorem 4.3.1 implies that $\psi^{(n)} \geq \psi^{(n-1)}$ on G . \square

We use Property 4.3.2 to prove Property 4.3.5 below; however, Property 4.3.2 is interesting in its own right because it confirms our intuition that $P^{(n)}$ increases with the number of policyholders.

Property 4.3.3. *Suppose $0 \leq \alpha_1 \leq \alpha_2 \leq \sqrt{\underline{\lambda}^P}$, and let $\psi^{(n),\alpha_i}$ be the solution of (4.2.32) with $\alpha = \alpha_i$, for $i = 1, 2$ and for $n \geq 0$. Then, $\psi^{(n),\alpha_1} \leq \psi^{(n),\alpha_2}$ on G .*

Proof. We prove this property by induction. First, the inequality holds for $n = 0$ since $\psi^{(0),\alpha_i} \equiv 0$ for $i = 1, 2$. For $n \geq 1$, assume that $\psi^{(n-1),\alpha_1} \leq \psi^{(n-1),\alpha_2}$, and show that $\psi^{(n),\alpha_1} \leq \psi^{(n),\alpha_2}$.

Define a differential operator \mathcal{L} on \mathcal{G} by (4.3.7) with $\alpha = \alpha_1$. We have that $\mathcal{L}\psi^{(n),\alpha_1} = 0$ since $\psi^{(n),\alpha_1}$ solves (4.2.32) with $\alpha = \alpha_1$. Apply the operator \mathcal{L} to

$\psi^{(n),\alpha_2}$ to get

$$\begin{aligned}
\mathcal{L}\psi^{(n),\alpha_2} &= -n\lambda^P (\psi^{(n),\alpha_2} - \psi^{(n-1),\alpha_1}) + n\lambda^P (\psi^{(n),\alpha_2} - \psi^{(n-1),\alpha_2}) \\
&\quad + \alpha_1 \sqrt{(1-\rho^2)(b^P)^2(\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(n),\alpha_2}\right)^2 + n\lambda^P (\psi^{(n),\alpha_2} - \psi^{(n-1),\alpha_1})^2} \\
&\quad - \alpha_2 \sqrt{(1-\rho^2)(b^P)^2(\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(n),\alpha_2}\right)^2 + n\lambda^P (\psi^{(n),\alpha_2} - \psi^{(n-1),\alpha_2})^2} \\
&= -n\lambda^P (\psi^{(n-1),\alpha_2} - \psi^{(n-1),\alpha_1}) \\
&\quad + \alpha_1 \left\{ \sqrt{(1-\rho^2)(b^P)^2(\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(n),\alpha_2}\right)^2 + n\lambda^P (\psi^{(n),\alpha_2} - \psi^{(n-1),\alpha_1})^2} \right. \\
&\quad \quad \left. - \sqrt{(1-\rho^2)(b^P)^2(\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(n),\alpha_2}\right)^2 + n\lambda^P (\psi^{(n),\alpha_2} - \psi^{(n-1),\alpha_2})^2} \right\} \\
&\quad - (\alpha_2 - \alpha_1) \sqrt{(1-\rho^2)(b^P)^2(\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(n),\alpha_2}\right)^2 + n\lambda^P (\psi^{(n),\alpha_2} - \psi^{(n-1),\alpha_2})^2} \\
&\leq -\left(n\lambda^P - \alpha_1\sqrt{n\lambda^P}\right) (\psi^{(n-1),\alpha_2} - \psi^{(n-1),\alpha_1}) \leq 0 = \mathcal{L}\psi^{(n),\alpha_1}.
\end{aligned} \tag{4.3.15}$$

Here, we use the Lemma 4.3.1 with $A = \sqrt{n\lambda^P} (\psi^{(n),\alpha_2} - \psi^{(n-1),\alpha_1})$, $B = \sqrt{n\lambda^P} (\psi^{(n),\alpha_2} - \psi^{(n-1),\alpha_2})$, and $C = b^P (\lambda^P - \underline{\lambda}^P) \psi_{\lambda^P}^{(n),\alpha_2}$, as well as the induction hypothesis and $\alpha_2 \geq \alpha_1$. Because $\psi^{(n),\alpha_1} = \psi^{(n),\alpha_1} = n$ at $t = T$, Theorem 4.3.1 implies that $\psi^{(n),\alpha_1} \leq \psi^{(n),\alpha_2}$ on G . \square

Property 4.3.3 shows that $P^{(n)}$ increases with the instantaneous Sharpe ratio α . The more that the insurance company wants to be compensated for the unhedgeable portion of the mortality risk, the higher it will set α . We have the following corollary of Property 4.3.3.

Property 4.3.4. *Let $\psi^{(n),\alpha_0}$ be the solution to (4.2.32) with $\alpha = 0$. Then, for $0 \leq \alpha \leq \sqrt{\underline{\lambda}^P}$, $\psi^{(n),\alpha} \geq \psi^{(n),\alpha_0}$ on G , and we can express the lower bound $\psi^{(n),\alpha_0}$ as follows: $\psi^{(n),\alpha_0} = n\psi^{\alpha_0}$, in which ψ^{α_0} is given by*

$$\psi^{\alpha_0}(\lambda^I, \lambda^P, t) = \mathbb{E}^Q \left[e^{-\int_t^T \lambda_s^P ds} \mid \lambda_t^I = \lambda^I, \lambda_t^P = \lambda^P \right], \tag{4.3.16}$$

and the Q -dynamics of $\{\lambda_t^I\}$ and $\{\lambda_t^P\}$ follow, respectively,

$$d\lambda_t^I = a^{I,Q}(\lambda_t^I, t) (\lambda_t^I - \underline{\lambda}^I) dt + b^I(t) (\lambda_t^I - \underline{\lambda}^I) dW_t^{I,Q}, \quad (4.3.17)$$

and

$$d\lambda_t^P = a^{P,Q}(\lambda_t^I, \lambda_t^P, t) (\lambda_t^P - \underline{\lambda}^P) dt + b^P(t) (\lambda_t^P - \underline{\lambda}^P) dW_t^{P,Q}. \quad (4.3.18)$$

Proof. Let $\alpha_1 = 0$ and $\alpha_2 = \alpha \geq 0$ in Property (4.3.3), and the inequality follows. By substituting $\alpha = 0$ in (4.2.30), the Feynman-Kac Theorem leads to the expression of $\psi^{\alpha 0}$ in (4.3.16). Finally, it is straightforward to show that $n\psi^{\alpha 0}$ solves (4.2.32) with $\alpha = 0$; thus, $\psi^{(n),\alpha 0} = n\psi^{\alpha 0}$. \square

Note that $n\psi^{(1),\alpha 0} = n\psi^{\alpha 0}$ is the expected number of survivors under the physical measure, so the lower bound of $\frac{1}{n}P^{(n)}$ (as α approaches zero) is the same as the lower bound of P , namely, $F\psi^{\alpha 0}$.

Property 4.3.5. $\psi_{\lambda^P}^{(n)} \leq 0$ on G for $n \geq 0$.

Proof. We prove this property by induction. First, it is clear that $\psi_{\lambda^P}^{(0)} \equiv 0$. For $n \geq 1$, assume that $\psi_{\lambda^P}^{(n-1)} \leq 0$, and apply a modified version of Theorem 4.3.1 to compare $\psi_{\lambda^P}^{(n)} \leq 0$ and the constant function $\mathbf{0}$. To this end, we first differentiate

$\psi^{(n)}$'s equation, (4.2.32), with respect to λ^P to get an equation for $f^{(n)} = \psi_{\lambda^P}^{(n)}$:

$$\left\{ \begin{aligned} & f_t^{(n)} + \left[a_{\lambda^P}^{P,Q} \cdot (\lambda^P - \underline{\lambda}^P) + a^{P,Q} \right] f^{(n)} + \left[a^{I,Q} + \rho b^I b^P \right] (\lambda^I - \underline{\lambda}^I) f_{\lambda^I}^{(n)} \\ & + \left[a^{P,Q} + (b^P)^2 \right] (\lambda^P - \underline{\lambda}^P) f_{\lambda^P}^{(n)} + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 f_{\lambda^I \lambda^I}^{(n)} \\ & + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) f_{\lambda^I \lambda^P}^{(n)} + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 f_{\lambda^P \lambda^P}^{(n)} \\ & - n (\psi^{(n)} - \psi^{(n-1)}) - n \lambda^P (f^{(n)} - f^{(n-1)}) \\ & = -\alpha \frac{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P) (f^{(n)})^2 + (1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 f^{(n)} f_{\lambda^P}^{(n)}}{\sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 (f^{(n)})^2 + n \lambda^P (\psi^{(n)} - \psi^{(n-1)})^2}} \\ & - \alpha \frac{\frac{1}{2} n (\psi^{(n)} - \psi^{(n-1)})^2 + n \lambda^P (\psi^{(n)} - \psi^{(n-1)}) (f^{(n)} - f^{(n-1)})}{\sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 (f^{(n)})^2 + n \lambda^P (\psi^{(n)} - \psi^{(n-1)})^2}}, \\ & f^{(n)} (\lambda^I, \lambda^P, T) = 0. \end{aligned} \right. \quad (4.3.19)$$

Define a differential operator \mathcal{L} on \mathcal{G} by (4.3.7) with g_n replaced by

$$\begin{aligned} & \tilde{g}_n(\lambda^I, \lambda^P, t, v, p_1, p_2) \\ & = \left[a_{\lambda^P}^{P,Q} \cdot (\lambda^P - \underline{\lambda}^P) + a^{P,Q} \right] v + \left[a^{I,Q} + \rho b^I b^P \right] (\lambda^I - \underline{\lambda}^I) p_1 \\ & + \left[a^{P,Q} + (b^P)^2 \right] (\lambda^P - \underline{\lambda}^P) p_2 - n (\psi^{(n)} - \psi^{(n-1)}) - n \lambda^P (v - f^{(n-1)}) \\ & + \alpha \frac{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P) v^2 + (1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 v p_2}{\sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 v^2 + n \lambda^P (\psi^{(n)} - \psi^{(n-1)})^2}} \\ & + \alpha \frac{\frac{1}{2} n (\psi^{(n)} - \psi^{(n-1)})^2 + n \lambda^P (\psi^{(n)} - \psi^{(n-1)}) (v - f^{(n-1)})}{\sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 v^2 + n \lambda^P (\psi^{(n)} - \psi^{(n-1)})^2}}. \end{aligned} \quad (4.3.20)$$

From Walter [1970], we know that we only need to verify that (4.3.1) holds for

$v > w = 0 = q_1 = q_2$. It is not difficult to show that

$$\begin{aligned} & \tilde{g}_n(\lambda^I, \lambda^P, t, v, p_1, p_2) - \tilde{g}_n(\lambda^I, \lambda^P, t, 0, 0, 0) \leq \left[\left| a_{\lambda^P}^{P,Q} \cdot (\lambda^P - \underline{\lambda}^P) + a^{P,Q} \right| + \alpha \sqrt{1 - \rho^2} b^P \right] v \\ & + \left[\left| a^{I,Q} \right| + \rho b^I b^P \right] (\lambda^I - \underline{\lambda}^I) |p_1| + \left[\left| a^{P,Q} \right| + (b^P)^2 + \alpha \sqrt{1 - \rho^2} b^P \right] (\lambda^P - \underline{\lambda}^P) |p_2|. \end{aligned} \quad (4.3.21)$$

Also, by Assumption 4.3, the corresponding $c = \left| a_{\lambda^P}^{P,Q} (\lambda^P - \underline{\lambda}^P) + a^{P,Q} \right| + \alpha \sqrt{1 - \rho^2} b^P$, $d_1 = |a^{I,Q}| + \rho b^I b^P$, and $d_2 = |a^{P,Q}| + (b^P)^2 + \alpha \sqrt{1 - \rho^2} b^P$ in (4.3.21) satisfy the growth conditions in (4.3.2).

Note that $\mathcal{L}f^{(n)} = 0$ on G . Apply the operator \mathcal{L} to the constant function $\mathbf{0}$ to get $\mathcal{L}\mathbf{0} = \left(n\lambda^P - \alpha\sqrt{n\lambda^P} \right) f^{(n-1)} - \left(n - \alpha/2\sqrt{n/\lambda^P} \right) (\psi^{(n)} - \psi^{(n-1)}) \leq 0$ by the induction assumption, by Property 4.3.2, and by the assumption that $\lambda^P > \underline{\lambda}^P \geq \alpha^2$. Since $f^{(n)}(\lambda^I, \lambda^P, T) = 0$, Theorem 4.3.1 implies that $f^{(n)} = \psi_{\lambda^P}^{(n)} \leq 0$ on G . \square

It is intuitively pleasing that $\psi_{\lambda^P}^{(n)} \leq 0$ because for physical survival probabilities, if the hazard rate increases, then the probability of surviving until time T , and thereby paying the \$1 benefit, decreases. A related result is that $P^{(n)}$ decreases as the risk-adjusted drift of the hazard rate, $a^{P,Q}$, increases because the hazard rate tends to increase with its drift.

Property 4.3.6. *Suppose $a_1^{P,Q} \leq a_2^{P,Q}$ on G , and let $\psi^{(n),a_i}$ denote the solution to (4.2.32) with $a^{P,Q} = a_i^{P,Q}$, for $i = 1, 2$ and for $n \geq 0$. Then, $\psi^{(n),a_1} \geq \psi^{(n),a_2}$ on G .*

Proof. Define a differential operator \mathcal{L} on \mathcal{G} by (4.3.7) with $a^P = a_1^P$; then, it is clear that $\mathcal{L}\psi^{(n),a_1} = 0$. Apply this operator \mathcal{L} to $\psi^{(n),a_2}$ to obtain

$$\mathcal{L}\psi^{(n),a_2} = \left(a_1^{P,Q} - a_2^{P,Q} \right) (\lambda^P - \underline{\lambda}^P) \psi_{\lambda^P}^{(n),a_2} \geq 0. \quad (4.3.22)$$

Since $\psi^{(n),a_1}(\lambda^I, \lambda^P, T) = \psi^{(n),a_2}(\lambda^I, \lambda^P, T) = n$, Theorem 4.3.1 implies that $\psi^{(n),a_1} \geq \psi^{(n),a_2}$ on G . \square

Next, we prove the subadditivity property of $P^{(n)}$. To that end, we use Lemma 4.10 in Milevsky et al. [2005]. We restate the lemma, and one can find its proof in the original paper.

Lemma 4.3.3. *Suppose $A \geq C \geq B$, B_λ , and C_λ are constants; then, for non-negative integers m and n ,*

$$\sqrt{(B_\lambda + C_\lambda)^2 + (m+n)A^2} - \sqrt{n}(A-C) \leq \sqrt{B_\lambda^2 + mB^2} + \sqrt{C_\lambda^2 + nB^2} + \sqrt{m}(A-B). \quad (4.3.23)$$

Property 4.3.7. $\psi^{(m+n)} \leq \psi^{(m)} + \psi^{(n)}$ for $m, n \geq 0$.

Proof. We prove this inequality by induction on $m+n$. When $m+n=0$ or 1 , we know that $\psi^{(0)} = \psi^{(0)} + \psi^{(0)}$ and $\psi^{(1)} = \psi^{(1)} + \psi^{(0)}$ since $\psi^{(0)} = 0$. For $m+n \geq 2$, suppose that $\psi^{(l+k)} \leq \psi^{(l)} + \psi^{(k)}$ for any non-negative integers k and l such that $k+l \leq m+n-1$. We need to show that $\psi^{(m+n)} \leq \psi^{(m)} + \psi^{(n)}$. Define $\xi = \psi^{(m)} + \psi^{(n)}$ and $\eta = \psi^{(m+n)}$ on G . The function ξ solves the PDE given by

$$\left\{ \begin{array}{l} \xi_t + a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) \xi_{\lambda^I} + a^{P,Q} (\lambda^P - \underline{\lambda}^P) \xi_{\lambda^P} + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 \xi_{\lambda^I \lambda^I} \\ \quad + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) \xi_{\lambda^I \lambda^P} + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \xi_{\lambda^P \lambda^P} \\ \quad - n \lambda^P (\psi^{(n)} - \psi^{(n-1)}) - m \lambda^P (\psi^{(m)} - \psi^{(m-1)}) \\ = -\alpha \sqrt{(1-\rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(n)} \right)^2 + n \lambda^P (\psi^{(n)} - \psi^{(n-1)})^2} \\ \quad - \alpha \sqrt{(1-\rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(m)} \right)^2 + m \lambda^P (\psi^{(m)} - \psi^{(m-1)})^2}, \\ \xi(\lambda^I, \lambda^P, T) = m+n. \end{array} \right. \quad (4.3.24)$$

Define a differential operator \mathcal{L} on \mathcal{G} by (4.3.7) with n replaced by $m+n$. It is clear

that $\mathcal{L}\eta = 0$ on G . Apply the operator \mathcal{L} to ξ to get

$$\begin{aligned}
\mathcal{L}\xi &= n\lambda^P (\psi^{(n)} - \psi^{(n-1)}) + m\lambda^P (\psi^{(m)} - \psi^{(m-1)}) \\
&\quad - (m+n)\lambda^P (\xi - \psi^{(m+n-1)}) \\
&\quad - \alpha \sqrt{(1-\rho^2)(b^P)^2(\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(n)}\right)^2 + n\lambda^P (\psi^{(n)} - \psi^{(n-1)})^2} \\
&\quad - \alpha \sqrt{(1-\rho^2)(b^P)^2(\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(m)}\right)^2 + m\lambda^P (\psi^{(m)} - \psi^{(m-1)})^2} \\
&\quad + \alpha \sqrt{(1-\rho^2)(b^P)^2(\lambda^P - \underline{\lambda}^P)^2 \xi_{\lambda^P}^2 + (m+n)\lambda^P (\xi - \psi^{(m+n-1)})^2} \\
&\leq (\psi^{(m+n-1)} - \psi^{(m-1)} - \psi^{(n)}) \left(m\lambda^P - \alpha\sqrt{m\lambda^P}\right) \\
&\quad + (\psi^{(m+n-1)} - \psi^{(m)} - \psi^{(n-1)}) \left(n\lambda^P - \alpha\sqrt{n\lambda^P}\right) \\
&\leq 0.
\end{aligned} \tag{4.3.25}$$

To get the first inequality in (4.3.25), we apply Lemma 4.3.3 after assigning $A = \sqrt{\lambda^P} (\xi - \psi^{(m+n-1)})$, $B = \sqrt{\lambda^P} (\psi^{(m)} - \psi^{(m-1)})$, $C = \sqrt{\lambda^P} (\psi^{(n)} - \psi^{(n-1)})$, $B_\lambda = \sqrt{1-\rho^2} b^P \cdot (\lambda^P - \underline{\lambda}^P) \psi_{\lambda^P}^{(m)}$, and $C_\lambda = \sqrt{1-\rho^2} b^P \cdot (\lambda^P - \underline{\lambda}^P) \psi_{\lambda^P}^{(n)}$. The second inequality in (4.3.25) follows from the induction assumption $\psi^{(m+n-1)} \leq \psi^{(k)} + \psi^{(l)}$ with $k+l = m+n-1$, and from the assumption that $\sqrt{\lambda^P} \geq \alpha$. Since $\xi(\lambda^I, \lambda^P, T) = \eta(\lambda^I, \lambda^P, T) = m+n$, Theorem 4.3.1 implies that $\eta \leq \xi$ on G . \square

Property 4.3.7 states that our pricing mechanism satisfies subadditivity, $P^{(m+n)} \leq P^{(m)} + P^{(n)}$. This is reasonable since if subadditivity did not hold, then buyers of pure endowments could purchase separately and thereby save money.

Property 4.3.8. *Let $\psi^{(n),\rho^0}$ be the solution to (4.2.32) with $\rho = 0$ for $n \geq 0$; then,*

$\psi^{(n),\rho 0} = \psi^{(n),\rho 0}(\lambda^P, t)$ is independent of λ^I and solves the following PDE:

$$\begin{cases} \psi_t^{(n),\rho 0} + a^{P,Q} \cdot (\lambda^P - \underline{\lambda}^P) \psi_{\lambda^P}^{(n),\rho 0} + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \psi_{\lambda^P \lambda^P}^{(n),\rho 0} - n\lambda^P (\psi^{(n),\rho 0} - \psi^{(n-1),\rho 0}) \\ = -\alpha \sqrt{(b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(n),\rho 0}\right)^2 + n\lambda^P (\psi^{(n),\rho 0} - \psi^{(n-1),\rho 0})^2}, \\ \psi^{(n),\rho 0}(\lambda^P, T) = n, \end{cases} \quad (4.3.26)$$

with $\psi^{(0),\rho 0} \equiv 0$ for $n = 0$.

Proof. The solution of (4.3.26) is independent of λ^I and also solves (4.2.32) when $\rho = 0$. Uniqueness of the solutions of (4.3.26) and (4.2.31) implies that the solutions of the two PDEs are equal. \square

When $\rho = 0$, the optimal investment in the mortality derivative is zero, as we discussed in Remark 4.2.2. Also, equation (4.3.26) is identical to equation (4.1) of Milevsky et al. [2005], which determines the price of n pure endowments in a market without mortality derivatives. The coincidence of the two results in the case of $\rho = 0$ shows that the pricing mechanism we apply is consistent.

It is natural to ask if the hedging will reduce the price of pure endowments. To answer this question, we first make an assumption on q^{λ^I} to simplify the equation for $\psi^{(n)}$ as follows.

Property 4.3.9. *When the market price of risk for mortality q^{λ^I} is independent of λ^I , then $\psi^{(n)} = \psi^{(n)}(\lambda^P, t)$ is also independent of λ^I and solves the following PDE:*

$$\begin{cases} \psi_t^{(n)} + a^{P,Q} \cdot (\lambda^P - \underline{\lambda}^P) \psi_{\lambda^P}^{(n)} + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \psi_{\lambda^P \lambda^P}^{(n)} - n\lambda^P (\psi^{(n)} - \psi^{(n-1)}) \\ = -\alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(n)}\right)^2 + n\lambda^P (\psi^{(n)} - \psi^{(n-1)})^2}, \\ \psi^{(n)}(\lambda^P, T) = n. \end{cases} \quad (4.3.27)$$

Proof. The solution of (4.3.27) is independent of λ^I and also solves (4.2.32) when q^{λ^I} is independent of λ^I . Uniqueness of the solutions of (4.2.32) and (4.3.27) implies that the solutions of the two PDEs are equal. \square

Because $P^{(n)} = F\psi^{(n)}$, Property 4.3.9 implies that if q^{λ^I} is independent of λ^I , then $P^{(n)}$ is also independent of λ^I . It follows from this property and Property 4.3.6 that if the q^{λ^I} is independent of λ^I , then $P^{(n)}$ increases with increasing market price of mortality risk q^{λ^I} , as one expects.

Property 4.3.10. *Suppose q^{λ^I} is independent of λ^I and $q_1^{\lambda^I} \leq q_2^{\lambda^I}$. Let $\psi^{(n),q_i^{\lambda^I}}$ be the solution of (4.3.27) with $q^{\lambda^I} = q_i^{\lambda^I}$, for $i = 1, 2$ and for $n \geq 0$. Then, $\psi^{(n),q_1^{\lambda^I}} \leq \psi^{(n),q_2^{\lambda^I}}$ on G .*

Proof. From (4.2.8) we have that $a_1^{P,Q} \geq a_2^{P,Q}$, and we conclude that $\psi^{(n),q_1^{\lambda^I}} \leq \psi^{(n),q_2^{\lambda^I}}$ on G from Property 4.3.6 and Property 4.3.9. \square

Next, we give a condition under which hedging with mortality derivatives reduces the price of pure endowments.

Theorem 4.3.2. *Suppose q^{λ^I} is independent of λ^I . Let $\psi^{(n),-}$ denote the solution of (4.3.27) with $\rho q^{\lambda^I} \leq 0$, and let $\psi^{(n),0}$ denote the solution of (4.3.27) with $\rho = 0$. Then, $\psi^{(n),-} \leq \psi^{(n),0}$ on G .*

Proof. Define a differential operator \mathcal{L} on \mathcal{G} by (4.3.7) with g_n replaced by

$$\begin{aligned} \hat{g}_n(\lambda^P, t, v, p_2) &= a^P \cdot (\lambda^P - \underline{\lambda}^P) p_2 - n\lambda^P (v - \psi^{(n-1),0}) \\ &\quad + \alpha \sqrt{(b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 p_2^2 + n\lambda^P (v - \psi^{(n-1),0})^2}. \end{aligned} \quad (4.3.28)$$

It is straightforward to check that the function \hat{g}_n in (4.3.28) satisfies the one-sided Lipschitz condition (4.3.1) and the growth condition (4.3.2). We have that $\mathcal{L}\psi^{(n),0} = 0$

since $\psi^{(n),0}$ solves (4.3.27) with $\rho = 0$. Apply the operator \mathcal{L} to $\psi^{(n),-}$ to get

$$\begin{aligned}
\mathcal{L}\psi^{(n),-} &= \rho q^{\lambda^I} b^P \cdot (\lambda^P - \underline{\lambda}^P) \psi_{\lambda^P}^{(n),-} \\
&\quad + \alpha \sqrt{(b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(n),-}\right)^2 + n\lambda^P (\psi^{(n),-} - \psi^{(n-1),0})^2} \\
&\quad - \alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \left(\psi_{\lambda^P}^{(n),-}\right)^2 + n\lambda^P (\psi^{(n),-} - \psi^{(n-1),-})^2} \\
&\geq \alpha \sqrt{n\lambda^P} (\psi^{(n-1),0} - \psi^{(n-1),-}) \geq 0 = \mathcal{L}\psi^{(n),0}.
\end{aligned} \tag{4.3.29}$$

The first inequality above follows from $\psi_{\lambda^P}^{(n),-} \leq 0$, $\rho q^{\lambda^I} \leq 0$, and Lemma 4.3.1. The second inequality follows by an induction step; recall that $\psi^{(0),0} = \psi^{(0),-} = 0$. Additionally, $\psi^{(n),-}(\lambda^P, T) = \psi^{(n),0}(\lambda^P, T)$, so Theorem 4.3.1 implies that $\psi^{(n),-} \leq \psi^{(n),0}$ on G . \square

Remark 4.3.1. *One can interpret the price $P^{(n)}$ with $\rho = 0$ as the price for which no hedging with the mortality derivative is allowed because the optimal investment in the mortality derivative when $\rho = 0$ is 0, which follows from Property 4.3.8. Thus, Theorem 4.3.2 asserts that when $\rho q^{\lambda^I} \leq 0$, the price when hedging is allowed is less than the price with no hedging. However, if $\rho q^{\lambda^I} > 0$, then we cannot conclude that hedging necessarily reduces the price of the pure endowment. We discuss this more fully at the end of the next section.*

4.4 Limiting behavior of $\frac{1}{n}P^{(n)}$ as $n \rightarrow \infty$

In this section, we consider the limiting behavior of $\frac{1}{n}P^{(n)}$. First, we show that the price per risk, $\frac{1}{n}P^{(n)}$, decrease as n increases; that is, by increasing the number of pure endowment contracts, we reduce the price per contract. Then, we further explore how far $\frac{1}{n}P^{(n)}$ decreases by determining the limiting value of the decreasing sequence $\{\frac{1}{n}P^{(n)}\}$. Surprisingly, we find in Theorem 4.4.1 that the limiting value

solves a *linear* PDE. The proofs of most results in this section are modifications of the proofs given by Milevsky et al. [2005].

To prove the limiting properties of $\frac{1}{n}P^{(n)}$, we use the Lemma 4.12 in Milevsky et al. [2005]. We restate this lemma without proof.

Lemma 4.4.1. *If $n \geq 2$, and if $A \geq C \geq 0$ and B_λ are constants, then the following inequality holds*

$$\sqrt{B_\lambda^2 + \frac{1}{n}C^2} \leq \sqrt{n-2}(A-C) + \sqrt{B_\lambda^2 + \frac{1}{n-1}[(n-1)C - (n-2)A]^2}. \quad (4.4.1)$$

Proposition 4.4.1. $\frac{1}{n}P^{(n)}$ decreases with respect to n for $n \geq 1$.

Proof. It is sufficient to show that $\frac{1}{n}\psi^{(n)}$ decreases with respect to n . Define $\phi^{(n)} \triangleq \frac{1}{n}\psi^{(n)}$, and we will show that $\phi^{(n-1)} \geq \phi^{(n)}$ for $n \geq 2$ by induction. From (4.2.32), we deduce that $\phi^{(n)}$ solves

$$\left\{ \begin{array}{l} \phi_t^{(n)} + a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) \phi_{\lambda^I}^{(n)} + a^{P,Q} \cdot (\lambda^P - \underline{\lambda}^P) \phi_{\lambda^P}^{(n)} + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 \phi_{\lambda^I \lambda^I}^{(n)} \\ \quad + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) \phi_{\lambda^I \lambda^P}^{(n)} + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \phi_{\lambda^P \lambda^P}^{(n)} \\ \quad - \lambda^P [n\phi^{(n)} - (n-1)\phi^{(n-1)}] \\ = -\alpha \sqrt{(1-\rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \left(\phi_{\lambda^P}^{(n)}\right)^2 + \frac{1}{n} \lambda^P [n\phi^{(n)} - (n-1)\phi^{(n-1)}]^2}, \\ \phi^{(n)}(\lambda^I, \lambda^P, T) = 1, \end{array} \right. \quad (4.4.2)$$

with $\phi^{(1)} = \psi$, in which ψ solves (4.2.30).

We first show that $\phi^{(1)} \geq \phi^{(2)}$. To this end, we define a differential operator \mathcal{L} on \mathcal{G} by (4.3.7) with g_n replaced by

$$\begin{aligned} \hat{g}_2(\lambda^I, \lambda^P, t, v, p_1, p_2) &= a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) p_1 + a^{P,Q} \cdot (\lambda^P - \underline{\lambda}^P) p_2 - \lambda^P (2v - \psi) \\ &\quad + \alpha \sqrt{(1-\rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 (p_2)^2 + \frac{1}{2} \lambda^P (2v - \psi)^2}. \end{aligned} \quad (4.4.3)$$

It is clear that \hat{g}_2 satisfies conditions (4.3.1) and (4.3.2); hence, we can apply Theorem 4.3.1. Note that $\mathcal{L}\phi^{(2)} = 0$ since $\phi^{(2)}$ solves (4.4.2) with $n = 2$. By applying the operator \mathcal{L} to $\phi^{(1)} = \psi$, we get

$$\begin{aligned} \mathcal{L}\phi^{(1)} &= \alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \psi_{\lambda^P}^2 + \frac{1}{2} \lambda^P \psi^2} \\ &\quad - \alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \psi_{\lambda^P}^2 + \lambda^P \psi^2} \\ &\leq 0 = \mathcal{L}\phi^{(2)}. \end{aligned} \quad (4.4.4)$$

Since $\phi^{(1)}(\lambda^I, \lambda^P, T) = \phi^{(2)}(\lambda^I, \lambda^P, T) = 1$, Theorem 4.3.1 implies that $\phi^{(1)} \geq \phi^{(2)}$ on G .

Assume that for $n \geq 3$, $\phi^{(n-2)} \geq \phi^{(n-1)}$ on G , and we show that $\phi^{(n-1)} \geq \phi^{(n)}$.

We define a differential operator \mathcal{L} on \mathcal{G} by (4.3.7) with g_n replaced by

$$\begin{aligned} \hat{g}_n(\lambda^I, \lambda^P, t, v, p_1, p_2) &= a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) p_1 + a^{P,Q} \cdot (\lambda^P - \underline{\lambda}^P) p_2 - \lambda^P [nv - (n-1)\phi^{(n-1)}] \\ &\quad + \alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 p_2^2 + \frac{1}{n} \lambda^P [nv - (n-1)\phi^{(n-1)}]^2}. \end{aligned} \quad (4.4.5)$$

It is clear that \hat{g}_n satisfies conditions (4.3.1) and (4.3.2); hence, we can apply Theorem 4.3.1. Note that $\mathcal{L}\phi^{(n)} = 0$ since $\phi^{(n)}$ solves (4.4.2). Apply the operator \mathcal{L} to $\phi^{(n-1)}$ to get

$$\begin{aligned} \mathcal{L}\phi^{(n-1)} &= (n-2)\lambda^P (\phi^{(n-1)} - \phi^{(n-2)}) \\ &\quad + \alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \left(\phi_{\lambda^P}^{(n-1)}\right)^2 + \frac{1}{n} \lambda^P (\phi^{(n-1)})^2} \\ &\quad - \alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \left(\phi_{\lambda^P}^{(n-1)}\right)^2 + \frac{1}{n-1} \lambda^P [(n-1)\phi^{(n-1)} - (n-2)\phi^{(n-2)}]^2} \\ &\leq \left[(n-2)\lambda^P - \alpha \sqrt{(n-2)\lambda^P} \right] (\phi^{(n-1)} - \phi^{(n-2)}) \leq 0 = \mathcal{L}\phi^{(n)}. \end{aligned} \quad (4.4.6)$$

To get the first inequality in (4.4.6), we use Lemma 4.4.1 by assigning $A = \sqrt{\lambda^P} \phi^{(n-2)}$, $C = \sqrt{\lambda^P} \phi^{(n-1)}$, and $B_\lambda = \sqrt{1 - \rho^2} b^P \cdot (\lambda^P - \underline{\lambda}^P) \phi_{\lambda^P}^{(n-1)}$. We also use the induction

assumption that $\phi^{(n-2)} \geq \phi^{(n-1)}$. Additionally, $\phi^{(n-1)}(\lambda^I, \lambda^P, T) = \phi^{(n)}(\lambda^I, \lambda^P, T) = 1$, and Theorem 4.3.1 implies that $\phi^{(n-1)} \geq \phi^{(n)}$ on G . \square

In what follows, we answer the question inspired by Proposition 4.4.1, namely, what is the *limit* of the non-negative, decreasing sequence $\{\frac{1}{n}P^{(n)}\}$? In Theorem 4.4.1 below, we will show the limit equals $F\beta$, in which $\beta = \beta(\lambda^I, \lambda^P, t)$ denote the solution of the following PDE:

$$\left\{ \begin{array}{l} \beta_t + a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) \beta_{\lambda^I} + \left[a^{P,Q} - \alpha \sqrt{1 - \rho^2} b^P \right] (\lambda^P - \underline{\lambda}^P) \beta_{\lambda^P} \\ \quad + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 \beta_{\lambda^I \lambda^I} + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) \beta_{\lambda^I \lambda^P} \\ \quad + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \beta_{\lambda^P \lambda^P} - \lambda^P \beta = 0, \\ \beta(\lambda^I, \lambda^P, T) = 1. \end{array} \right. \quad (4.4.7)$$

By applying the Feynman-Kac Theorem to (4.4.7), we obtain an expression for β as an expectation:

$$\beta(\lambda^I, \lambda^P, t) = \mathbb{E}^{\tilde{Q}} \left[e^{-\int_t^T \lambda_s^P ds} \mid \lambda_t^I = \lambda^I, \lambda_t^P = \lambda^P \right], \quad (4.4.8)$$

in which the \tilde{Q} -dynamics of $\{\lambda_t^I\}$ and $\{\lambda_t^P\}$ follow, respectively,

$$d\lambda_t^I = a^{I,Q}(\lambda_t^I, t) (\lambda_t^I - \underline{\lambda}^I) dt + b^I(t) (\lambda_t^I - \underline{\lambda}^I) dW_t^{I,Q} \quad (4.4.9)$$

and

$$d\lambda_t^P = \left[a^{P,Q}(\lambda_t^I, \lambda_t^P, t) - \alpha \sqrt{1 - \rho^2} b^P(t) \right] (\lambda_t^P - \underline{\lambda}^P) dt + b^P(t) (\lambda_t^P - \underline{\lambda}^P) d\tilde{W}_t^{P,Q}. \quad (4.4.10)$$

Here, $\tilde{W}_t^{P,Q} = W_t^{P,Q} + \alpha \sqrt{1 - \rho^2} t$.

We begin by proving that $\frac{1}{n}P^{(n)}$ is bounded below by $F\beta$, and for that purpose, we need the following lemma.

Lemma 4.4.2. *The function β defined by (4.4.7) is non-increasing with respect to λ^P .*

Proof. Denote $f = \beta_{\lambda^P}$, and we deduce from (4.4.7) that f solves the following PDE:

$$\left\{ \begin{array}{l} f_t + [a^{I,Q} + \rho b^I b^P] (\lambda^I - \underline{\lambda}^I) f_{\lambda^I} + [a^{P,Q} - \alpha \sqrt{1 - \rho^2} b^P] (\lambda^P - \underline{\lambda}^P) f_{\lambda^P} \\ + [a_{\lambda^P}^{P,Q} \cdot (\lambda^P - \underline{\lambda}^P) + a^{P,Q} - \alpha \sqrt{1 - \rho^2} b^P - \lambda^P] f + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 f_{\lambda^I \lambda^I} \\ + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) f_{\lambda^I \lambda^P} + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 f_{\lambda^P \lambda^P} - \beta \\ = 0, \\ f(\lambda^I, \lambda^P, T) = 0. \end{array} \right. \quad (4.4.11)$$

Define a differential operator \mathcal{L} on \mathcal{G} by (4.3.7) with g_n replaced by

$$\begin{aligned} \tilde{g}(\lambda^I, \lambda^P, t, v, p_1, p_2) &= [a^{I,Q} + \rho b^I b^P] (\lambda^I - \underline{\lambda}^I) p_1 + [a^{P,Q} - \alpha \sqrt{1 - \rho^2} b^P] (\lambda^P - \underline{\lambda}^P) p_2 \\ &\quad + [a_{\lambda^P}^{P,Q} \cdot (\lambda^P - \underline{\lambda}^P) + a^{P,Q} - \alpha \sqrt{1 - \rho^2} b^P - \lambda^P] v - \beta \end{aligned} \quad (4.4.12)$$

Because of Assumption 4.3, it is straightforward to check that the function \tilde{g} in (4.4.12) satisfies the one-sided Lipschitz condition (4.3.1) and the growth condition (4.3.2). Because f solves (4.4.11), we have that $\mathcal{L}f = 0$. Because β is clearly non-negative, $\mathcal{L}\mathbf{0} = -\beta \leq 0$, in which $\mathbf{0}$ is the constant function of 0 on G . Additionally, $f(\lambda^I, \lambda^P, T) = 0$, so Theorem 4.3.1 implies that $\beta_{\lambda^P} \leq 0$. \square

Lemma 4.4.3. *For $n \geq 1$, $\frac{1}{n}P^{(n)} \geq F\beta$, in which β is given in (4.4.7)*

Proof. It is sufficient to show that $\frac{1}{n}\psi^{(n)} \geq \beta$ on G . We prove this property by induction. First, for $n = 1$, we show that $\beta \leq \psi^{(1)} = \psi$. Define a differential operator \mathcal{L} on \mathcal{G} by (4.3.7) with $n = 1$. Recall that $\psi^{(0)} = 0$ in (4.3.3). Since ψ solves

(4.2.30), $\mathcal{L}\psi = 0$. Also,

$$\mathcal{L}\beta = \alpha\sqrt{(1-\rho^2)(b^P)^2(\lambda^P - \underline{\lambda}^P)^2\beta_{\lambda^P}^2 + \lambda^P\beta^2} - \alpha\sqrt{1-\rho^2}b^P(\lambda^P - \underline{\lambda}^P)|\beta_{\lambda^P}| \geq 0 = \mathcal{L}\psi. \quad (4.4.13)$$

Additionally, $\beta(\lambda^I, \lambda^P, T) = 1 = \psi(\lambda^I, \lambda^P, T)$, so Theorem 4.3.1 implies that $\beta \leq \psi \geq$ on G .

For $n \geq 1$, assume that $\beta \leq \phi^{(n-1)}$ and show that $\beta \leq \phi^{(n)}$, in which $\phi^{(n)} = \frac{1}{n}\psi^{(n)}$ for $n \geq 1$, as we defined in the proof of Proposition 4.4.1. Define a differential operator \mathcal{L} by (4.3.7) with g_n replaced by \hat{g}_n given by (4.4.5). Since $\phi^{(n)}$ solves (4.4.2), $\mathcal{L}\phi^{(n)} = 0$. By applying this operator on β , we get

$$\begin{aligned} \mathcal{L}\beta &= \alpha\sqrt{(1-\rho^2)(b^P)^2(\lambda^P - \underline{\lambda}^P)^2(\beta_{\lambda^P})^2 + \frac{1}{n}\lambda^P[n\beta - (n-1)\phi^{(n-1)}]^2} \\ &\quad - \alpha\sqrt{1-\rho^2}b^P(\lambda^P - \underline{\lambda}^P)|\beta_{\lambda^P}| + \lambda^P[(n-1)\phi^{(n-1)} - (n-1)\beta] \quad (4.4.14) \\ &\geq 0 = \mathcal{L}\phi^{(n)}. \end{aligned}$$

Also, $\beta(\lambda^I, \lambda^P, T) = \phi^{(n)}(\lambda^I, \lambda^P, T) = 1$; thus, Theorem 4.3.1 implies that $\beta \leq \phi^{(n)} = \frac{1}{n}\psi^{(n)}$ on G . \square

Next, we show that $\lim_{n \rightarrow \infty} \frac{1}{n}P^{(n)} = F\beta$. To this end, we need some auxiliary results. First, we prove that $\psi^{(n)}$ is bounded from above by $\gamma^{(n)} = \gamma^{(n)}(\lambda^I, \lambda^P, t)$ for $n \geq 0$, in which the function $\gamma^{(n)}$ solves the following PDE:

$$\begin{cases} \gamma_t^{(n)} + a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) \gamma_{\lambda^I}^{(n)} + \left[a^{P,Q} - \alpha\sqrt{1-\rho^2}b^P \right] (\lambda^P - \underline{\lambda}^P) \gamma_{\lambda^P}^{(n)} \\ \quad + \frac{1}{2}(b^I)^2(\lambda^I - \underline{\lambda}^I)^2 \gamma_{\lambda^I \lambda^I}^{(n)} + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) \gamma_{\lambda^I \lambda^P}^{(n)} \\ \quad + \frac{1}{2}(b^P)^2(\lambda^P - \underline{\lambda}^P)^2 \gamma_{\lambda^P \lambda^P}^{(n)} - \left(n\lambda^P - \alpha\sqrt{n\lambda^P} \right) (\gamma^{(n)} - \gamma^{(n-1)}) = 0, \\ \gamma^{(n)}(\lambda^I, \lambda^P, T) = n, \end{cases} \quad (4.4.15)$$

in which $\gamma^{(0)} \equiv 0$.

Lemma 4.4.4. *The function $\gamma^{(n)}$ given by (4.4.15) is non-increasing with respect to λ^P , and $\gamma^{(n)} \geq \gamma^{(n-1)}$ for $n \geq 1$ on G .*

Proof. The proof that $\gamma_{\lambda^P}^{(n)} \leq 0$ is similar to the proof that $\psi_{\lambda^P}^{(n)} \leq 0$ in Property 4.3.5. Also, the proof that $\gamma^{(n)} \geq \gamma^{(n-1)}$ is similar to the proof that $\psi^{(n)} \geq \psi^{(n-1)}$ in Property 4.3.2. Therefore, we omit the details of the proof. \square

Lemma 4.4.5. *For $n \geq 0$, $\gamma^{(n)} \geq \psi^{(n)}$ on G .*

Proof. We prove this lemma by induction. For $n = 0$, we have $\gamma^{(0)} = \psi^{(0)} = 0$. Assume that for $n \geq 1$, we have $\gamma^{(n-1)} \geq \psi^{(n-1)}$, and show that $\gamma^{(n)} \geq \psi^{(n)}$. For this purpose, define a differential operator \mathcal{L} on \mathcal{G} by (4.3.7). Then, $\mathcal{L}\psi = 0$, and

$$\begin{aligned} \mathcal{L}\gamma^{(n)} &= \alpha\sqrt{1-\rho^2}b^P(\lambda^P - \underline{\lambda}^P)\gamma_{\lambda^P}^{(n)} + \left(n\lambda^P - \alpha\sqrt{n\lambda^P}\right)(\gamma^{(n)} - \gamma^{(n-1)}) - n\lambda^P(\gamma^{(n)} - \psi^{(n-1)}) \\ &\quad + \alpha\sqrt{(1-\rho^2)(b^P)^2(\lambda^P - \underline{\lambda}^P)^2\left(\gamma_{\lambda^P}^{(n)}\right)^2 + n\lambda^P(\gamma^{(n)} - \psi^{(n-1)})^2} \\ &\leq -\left(n\lambda^P - \alpha\sqrt{n\lambda^P}\right)(\gamma^{(n-1)} - \psi^{(n-1)}) \leq 0 = \mathcal{L}\psi^{(n)}. \end{aligned} \tag{4.4.16}$$

The first inequality above is due to the fact that $\gamma_{\lambda^P}^{(n)} \leq 0$, that $\gamma^{(n)} \geq \gamma^{(n-1)} \geq \psi^{(n-1)}$, and that $\sqrt{A^2 + B^2} \leq |A| + |B|$. Additionally, we have that $\gamma^{(n)}(\lambda^I, \lambda^P, T) = \psi^{(n)}(\lambda^I, \lambda^P, T) = n$; then, Theorem 4.3.1 implies that $\gamma^{(n)} \geq \psi^{(n)}$ on G . \square

Next, we prove the main result of this section.

Theorem 4.4.1. $\lim_{n \rightarrow \infty} \frac{1}{n}P^{(n)}(r, \lambda^I, \lambda^P, t) = F(r, t)\beta(\lambda^I, \lambda^P, t)$ on G .

Proof. By Lemmas 4.4.3 and 4.4.5, it is sufficient to show that $\lim_{n \rightarrow \infty} \left(\frac{1}{n}\gamma^{(n)} - \beta\right) = 0$ since $\frac{1}{n}\gamma^{(n)} - \beta \geq \frac{1}{n}\psi^{(n)} - \beta \geq 0$. For $n \geq 1$, define $\Gamma^{(n)}$ on G by $\Gamma^{(n)} = \frac{1}{n}\gamma^{(n)} - \beta$, so we just need to prove that $\lim_{n \rightarrow \infty} \Gamma^{(n)} = 0$. For $n \geq 1$, the function $\Gamma^{(n)}$ solves

the following PDE:

$$\left\{ \begin{array}{l} \Gamma_t^{(n)} + a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) \Gamma_{\lambda^I}^{(n)} + \left[a^{P,Q} - \alpha \sqrt{1 - \rho^2} b^P \right] (\lambda^P - \underline{\lambda}^P) \Gamma_{\lambda^P}^{(n)} \\ + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 \Gamma_{\lambda^I \lambda^I}^{(n)} + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) \Gamma_{\lambda^I \lambda^P}^{(n)} \\ + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \Gamma_{\lambda^P \lambda^P}^{(n)} - \left(n \lambda^P - \alpha \sqrt{n \lambda^P} \right) \Gamma^{(n)} \\ = -\alpha \sqrt{\frac{\lambda^P}{n}} \beta - (n-1) \left(\lambda^P - \alpha \sqrt{\frac{\lambda^P}{n}} \right) \Gamma^{(n-1)}, \\ \Gamma^{(n)}(\lambda^I, \lambda^P, T) = 0, \end{array} \right. \quad (4.4.17)$$

with $0 \leq \Gamma^{(1)} = \gamma^{(1)} - \beta \leq 1$ on G . By applying the Feynman-Kac Theorem to (4.4.17), we obtain the following expression for $\Gamma^{(n)}$ in terms of $\Gamma^{(n-1)}$:

$$\begin{aligned} \Gamma^{(n)}(\lambda^I, \lambda^P, t) &= \alpha \mathbb{E}^{\tilde{Q}} \left[\int_t^T \sqrt{\frac{\lambda_s^P}{n}} \beta (\lambda_s^I, \lambda_s^P, s) e^{-\int_t^s (n \lambda_u^P - \alpha \sqrt{n \lambda_u^P}) du} ds \Big| \lambda_t^I = \lambda^I, \lambda_t^P = \lambda^P \right] \\ &+ (n-1) \mathbb{E}^{\tilde{Q}} \left[\int_t^T \left(\lambda_s^P - \alpha \sqrt{\frac{\lambda_s^P}{n}} \right) \Gamma^{(n-1)} e^{-\int_t^s (n \lambda_u^P - \alpha \sqrt{n \lambda_u^P}) du} ds \Big| \lambda_t^I = \lambda^I, \lambda_t^P = \lambda^P \right], \end{aligned} \quad (4.4.18)$$

in which the \tilde{Q} -dynamics of $\{\lambda_t^I\}$ and $\{\lambda_t^P\}$ follow, respectively, equations (4.4.9) and (4.4.10).

Suppose $\Gamma^{(n-1)} \leq K_{n-1}$ on G for some $n \geq 2$ and for some constant $K_{n-2} \geq 0$.

Note that $\beta \leq 1$ on G , so we get the following inequality:

$$\begin{aligned} \Gamma^{(n)}(\lambda^I, \lambda^P, t) &\leq \alpha \mathbb{E}^{\tilde{Q}} \left[\int_t^T \sqrt{\frac{\lambda_s^P}{n}} e^{-\int_t^s (n \lambda_u^P - \alpha \sqrt{n \lambda_u^P}) du} ds \Big| \lambda_t^I = \lambda^I, \lambda_t^P = \lambda^P \right] \\ &+ (n-1) K_{n-1} \mathbb{E}^{\tilde{Q}} \left[\int_t^T \left(\lambda_s^P - \alpha \sqrt{\frac{\lambda_s^P}{n}} \right) e^{-\int_t^s (n \lambda_u^P - \alpha \sqrt{n \lambda_u^P}) du} ds \Big| \lambda_t^I = \lambda^I, \lambda_t^P = \lambda^P \right]. \end{aligned} \quad (4.4.19)$$

Equivalently, we can write the inequality (4.4.19) as

$$\Gamma^{(n)}(\lambda^I, \lambda^P, t) \leq \frac{1}{n^{3/2}} A^{(n)}(\lambda^I, \lambda^P, t) + \frac{n-1}{n} K_{n-1} B^{(n)}(\lambda^I, \lambda^P, t), \quad (4.4.20)$$

in which the functions $A^{(n)}$ and $B^{(n)}$ are defined as

$$A^{(n)}(\lambda^I, \lambda^P, t) = \alpha \mathbb{E}^{\tilde{Q}} \left[\int_t^T n \sqrt{\lambda_s^P} e^{-\int_t^s (n\lambda_u^P - \alpha \sqrt{n\lambda_u^P}) du} ds \middle| \lambda_t^I = \lambda^I, \lambda_t^P = \lambda^P \right], \quad (4.4.21)$$

and

$$B^{(n)}(\lambda^I, \lambda^P, t) = \mathbb{E}^{\tilde{Q}} \left[\int_t^T \left(n\lambda_s^P - \alpha \sqrt{n\lambda_s^P} \right) e^{-\int_t^s (n\lambda_u^P - \alpha \sqrt{n\lambda_u^P}) du} ds \middle| \lambda_t^I = \lambda^I, \lambda_t^P = \lambda^P \right]. \quad (4.4.22)$$

After the next two lemmas that give us bounds on $A^{(n)}$ and $B^{(n)}$, respectively, we finish the proof of Theorem 4.4.1. \square

Lemma 4.4.6. For $n \geq 2$, $A^{(n)} \leq J = \frac{\alpha\sqrt{2}}{\sqrt{2\lambda^P} - \alpha}$ on G , in which $A^{(n)}$ is defined in (4.4.21).

Proof. By the Feynman-Kac Theorem, $A^{(n)}$ in (4.4.21) solves the following PDE

$$\begin{cases} A_t^{(n)} + a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) A_{\lambda^I}^{(n)} + \left[a^{P,Q} - \alpha \sqrt{1 - \rho^2} b^P \right] (\lambda^P - \underline{\lambda}^P) A_{\lambda^P}^{(n)} \\ \quad + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 A_{\lambda^I \lambda^I}^{(n)} + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) A_{\lambda^I \lambda^P}^{(n)} \\ \quad + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 A_{\lambda^P \lambda^P}^{(n)} - (n\lambda^P - \alpha \sqrt{n\lambda^P}) A^{(n)} = -\alpha n \sqrt{\lambda^P}, \\ A^{(n)}(\lambda^I, \lambda^P, T) = 0. \end{cases} \quad (4.4.23)$$

For $n \geq 2$, we define a differential operator \mathcal{L} by (4.3.7) with g_n replaced by

$$\begin{aligned} \tilde{g}_n(\lambda^I, \lambda^P, t, v, p_1, p_2) &= a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) p_1 + \left[a^{P,Q} - \alpha \sqrt{1 - \rho^2} b^P \right] (\lambda^P - \underline{\lambda}^P) p_2 \\ &\quad - \left(n\lambda^P - \alpha \sqrt{n\lambda^P} \right) v + \alpha n \sqrt{\lambda^P}. \end{aligned} \quad (4.4.24)$$

Since \tilde{g}_n satisfies conditions (4.3.1) and (4.3.2), we can apply Theorem 4.3.1. It is clear that $\mathcal{L}A^{(n)} = 0$, and by applying the operator \mathcal{L} to \mathbf{J} , the function that is identically equal to J , we get

$$\mathcal{L}\mathbf{J} = - \left(n\lambda^P - \alpha \sqrt{n\lambda^P} \right) J + \alpha n \sqrt{\lambda^P} \leq 0 = \mathcal{L}A^{(n)}. \quad (4.4.25)$$

Since $A^{(n)}(\lambda^I, \lambda^P, T) = 0 \leq J$, Theorem 4.3.1 implies that $A^{(n)} \leq J$ on G . \square

Lemma 4.4.7. For $n \geq 2$, $B^{(n)} \leq 1$ on G , in which $B^{(n)}$ is defined in (4.4.22).

Proof. By the Feynman-Kac Theorem, $B^{(n)}$ in (4.4.22) solves the following PDE

$$\left\{ \begin{array}{l} B_t^{(n)} + a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) B_{\lambda^I}^{(n)} + \left[a^{P,Q} - \alpha \sqrt{1 - \rho^2 b^P} \right] (\lambda^P - \underline{\lambda}^P) B_{\lambda^P}^{(n)} \\ \quad + \frac{1}{2} (b^I)^2 (\lambda^I - \underline{\lambda}^I)^2 B_{\lambda^I \lambda^I}^{(n)} + \rho b^I b^P (\lambda^I - \underline{\lambda}^I) (\lambda^P - \underline{\lambda}^P) B_{\lambda^I \lambda^P}^{(n)} \\ \quad + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 B_{\lambda^P \lambda^P}^{(n)} - \left(n \lambda^P - \alpha \sqrt{n \lambda^P} \right) B^{(n)} \\ \quad = - \left(n \lambda^P - \alpha \sqrt{n \lambda^P} \right), \\ B^{(n)}(\lambda^I, \lambda^P, T) = 0. \end{array} \right. \quad (4.4.26)$$

For $n \geq 2$, we define a differential operator \mathcal{L} on \mathcal{G} by (4.3.7) with g_n replaced by

$$\begin{aligned} \hat{g}_n(\lambda^I, \lambda^P, t, v, p_1, p_2) &= a^{I,Q} \cdot (\lambda^I - \underline{\lambda}^I) p_1 + \left[a^{P,Q} - \alpha \sqrt{1 - \rho^2 b^P} \right] (\lambda^P - \underline{\lambda}^P) p_2 \\ &\quad - \left(n \lambda^P - \alpha \sqrt{n \lambda^P} \right) v + \left(n \lambda^P - \alpha \sqrt{n \lambda^P} \right). \end{aligned} \quad (4.4.27)$$

Since \hat{g}_n satisfies conditions (4.3.1) and (4.3.2), we can apply Theorem 4.3.1. It is clear that $\mathcal{L}B^{(n)} = 0$, and by applying the operator \mathcal{L} to $\mathbf{1}$, we get $\mathcal{L}\mathbf{1} = - \left(n \lambda^P - \alpha \sqrt{n \lambda^P} \right) + \left(n \lambda^P - \alpha \sqrt{n \lambda^P} \right) = 0 = \mathcal{L}B^{(n)}$. Since $B^{(n)}(\lambda^I, \lambda^P, T) = 0 \leq 1$, Theorem 4.3.1 implies that $B^{(n)} \leq 1$ on G . \square

End of Proof of Theorem 4.4.1. By Lemmas 4.4.6 and 4.4.7, we get the following result: for $n \geq 2$, if $\Gamma^{(n-1)} \leq K_{n-1}$, then

$$\Gamma^{(n)} \leq K_n \triangleq \frac{J}{n^{3/2}} + \frac{n-1}{n} K_{n-1}, \quad (4.4.28)$$

with $K_1 = 1$. Define $L_n = nK_n$ and note that $L_n = L_{n-1} + \frac{J}{\sqrt{n}}$ for $n \geq 2$. It follows that

$$L_n = 1 + \sum_{i=2}^n \frac{J}{\sqrt{i}} \leq 1 + J \int_1^n \frac{dx}{\sqrt{x}} \leq 1 + 2J\sqrt{n}, \quad n \geq 2, \quad (4.4.29)$$

which implies that on G ,

$$\Gamma^{(n)} \leq K_n \leq \frac{1}{n} + \frac{2J}{\sqrt{n}}, \quad n \geq 1. \quad (4.4.30)$$

$\lim_{n \rightarrow \infty} \frac{1}{n} + \frac{2J}{\sqrt{n}} = 0$; therefore, $\Gamma^{(n)}$ converges to 0 uniformly on G as n goes to infinity. In other words, $\lim_{n \rightarrow \infty} \frac{1}{n} P^{(n)} = F\beta$ on G . \square

We end this section with some properties of β with the goal of determining the effect of ρ on β .

Property 4.4.1. *If q^{λ^I} is independent of λ^I , then $\beta = \beta(\lambda^P, t)$ is independent of λ^I and solves the following PDE:*

$$\begin{cases} \beta_t + \left[a^{P,Q} - \alpha \sqrt{1 - \rho^2} b^P \right] (\lambda^P - \underline{\lambda}^P) \beta_{\lambda^P} + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \beta_{\lambda^P \lambda^P} - \lambda^P \beta = 0, \\ \beta(\lambda^P, T) = 1. \end{cases} \quad (4.4.31)$$

Proof. The solution of (4.4.31) is independent of λ^I and also solves (4.4.7) when q^{λ^I} is independent of λ^I . Uniqueness of the solutions of (4.4.7) and (4.4.31) implies that solutions of the two PDEs are equal. \square

Theorem 4.4.2. *Suppose q^{λ^I} is independent of λ^I , and define $\hat{a} \triangleq a^P - \left[\rho q^{\lambda^I} + \alpha \sqrt{1 - \rho^2} \right] b^P$.*

Let $\beta^{\hat{a}_i}$ denote the solution of (4.4.31) with $\hat{a} = \hat{a}_i$, for $i = 1, 2$. Then, $\beta^{\hat{a}_1} \geq \beta^{\hat{a}_2}$ on G if $\hat{a}_1 \leq \hat{a}_2$.

Proof. Define a differential operator \mathcal{L} on \mathcal{G} by (4.3.7) with g_n replaced by

$$\hat{g}(\lambda^I, v, p) = \hat{a}_1 \cdot (\lambda^P - \underline{\lambda}^P) p - \lambda^P v. \quad (4.4.32)$$

It is straightforward to check that the function \hat{g} in (4.4.32) satisfies the one-sided Lipschitz condition (4.3.1) and the growth condition (4.3.2). Since $\beta^{\hat{a}_1}$ solves (4.4.31) with $\hat{a} = \hat{a}_1$, we have that $\mathcal{L}\beta^{\hat{a}_1} = 0$. Apply this operator on $\beta^{\hat{a}_2}$ to obtain

$$\mathcal{L}\beta^{\hat{a}_2} = (\hat{a}_1 - \hat{a}_2) (\lambda^P - \underline{\lambda}^P) \beta_{\lambda^P}^{\hat{a}_2} \geq 0 = \mathcal{L}\beta^{\hat{a}_1}. \quad (4.4.33)$$

Since $\beta^{\hat{a}_1}(\lambda^P, T) = \beta^{\hat{a}_2}(\lambda^P, T) = 1$, Theorem 4.3.1 implies that $\beta^{\hat{a}_1} \geq \beta^{\hat{a}_2}$ on G . \square

Remark 4.4.1. *When $\rho = 1$, namely the the insured individuals and the reference population face the same uncertainty in their respective hazard rates, the limiting price per contract is reduced by hedging when q^{λ^I} is less than the pre-specified instantaneous Sharpe ratio α . Indeed, the drift $\hat{a} = a^P - \left[\rho q^{\lambda^I} + \alpha \sqrt{1 - \rho^2} \right] b^P$ from Theorem 4.4.2 equals $a^P - q^{\lambda^I} b^P$ when $\rho = 1$. Also, the effect of not allowing hedging can be achieved by setting $\rho = 0$ throughout our work, as discussed in Remark 4.2.2; in that case, the drift \hat{a} becomes $a^P - \alpha b^P$. Thus, according to Theorem 4.4.2, the limiting price per contract is reduced when hedging is allowed if $q^{\lambda^I} < \alpha$.*

In other words, hedging with mortality derivative benefits the insured, through a reduced price, when the market price of mortality risk is lower than that required by the insurance company. In this limiting case, the risks inherent in the contract can be fully hedged using the interest rate derivative and the mortality derivative. Indeed, the variance of the hedging portfolio goes to 0 as n goes to infinity when $\rho = 1$. Refer to Remark 4.2.1 in which we discuss the mortality risk in the single-life case. So, as $n \rightarrow \infty$, the risk coming from the timing of the deaths disappears; compare with (4.2.27).

The price of the contract is reduced by transferring the mortality risk to a counterparty who requires a lower compensation for the risk than the insurance company does. By contrast, for a single pure endowment contract, the volatility in the contract due to the uncertainty of the individual's time of death is not hedgeable with mortality derivatives even when $\rho = 1$. In the single-life case, even if q^{λ^I} is less than α , hedging does not guarantee a reduction of the contract price.

Corollary 4.4.1. *Suppose q^{λ^I} is independent of λ^I , and let β^{a_i} denote the solution of (4.4.31) with $a^P = a_i$, for $i = 1, 2$. Then, $\beta^{a_1} \geq \beta^{a_2}$ on G if $a_1 \leq a_2$.*

The result above is consistent with our intuition. Indeed, with a higher drift on the hazard rate, the individual is less likely to survive to time T , and, consequently, the (limiting) value of the pure endowment contract is lower.

Corollary 4.4.2. *Suppose q^{λ^I} is independent of λ^I , and let β^{ρ_i} denote the solution of (4.4.31) with $\rho = \rho_i$ for $i = 1, 2$. Then, $\beta^{\rho_1} \leq \beta^{\rho_2}$ on G if $\rho_1 q^{\lambda^I} + \alpha\sqrt{1 - \rho_1^2} \leq \rho_2 q^{\lambda^I} + \alpha\sqrt{1 - \rho_2^2}$ for all $t \in [0, T]$.*

Remark 4.4.2. *A natural question that follows from Corollary 4.4.2 is when is $f(\rho, t) \triangleq \rho q^{\lambda^I}(t) + \alpha\sqrt{1 - \rho^2}$ decreasing with respect to ρ for $t \in [0, T]$? Suppose that $\rho > 0$, which is what one expects between the insured and reference populations. If f is decreasing with respect to $\rho > 0$, then greater positive correlation will lead to a lower per-contract price, an intuitively pleasing result. It is straightforward to show that f decreases with respect to ρ if and only if*

$$\rho > \frac{q^{\lambda^I}}{\sqrt{\alpha^2 + (q^{\lambda^I})^2}}. \quad (4.4.34)$$

This inequality holds automatically if $q^{\lambda^I} < 0$, that is, if the mortality derivative is a so-called natural hedge, which we discuss more fully in Remark 4.4.3 below. When $q^{\lambda^I} > 0$, it holds for ρ in a neighborhood of 1.

We have the following special case of Corollary 4.4.2.

Corollary 4.4.3. *Suppose q^{λ^I} is independent of λ^I . If $\rho q^{\lambda^I} + \alpha\sqrt{1 - \rho^2} < \alpha$, then the limiting price per risk in which hedging is allowed is less than the limiting price with no hedging ($\rho = 0$).*

Remark 4.4.3. *In particular, when q^{λ^I} is negative (and ρ is positive), the unit price of the contract is reduced by hedging, as demonstrated in Corollary 4.4.3. Since the correlation is usually positive, a mortality derivative with a negative market price*

of risk q^{λ^I} , that is, a natural hedge, is preferred. An example of a natural hedge is life insurance, as discussed in Young [2008], although strictly speaking this insurance product is not a mortality derivative traded in the financial market. Both Bayraktar and Young [2007a] and Cox and Lin [2007] proposed hedging pure endowment or life annuity contracts with life insurance.

4.5 Numerical example

In this section, we demonstrate our result with numerical examples. We assume that the risk-free rate of return r is constant and focus on the effect of the correlation ρ and the market price of mortality risk q^{λ^I} . We also assume that the market price of mortality risk is constant, and, thereby, is automatically independent of λ^I . In this case, $P^{(n)}$ and $\lim_{n \rightarrow \infty} \frac{1}{n} P^{(n)}$ do not depend on λ^I , as we prove in Properties 4.3.9 and 4.4.1. Moreover, we assume that the hazard rate λ^P follows the process in (4.2.2) with a^P and b^P constant. We compute the price for a single contract, $P(r, \lambda^P, t) = e^{-r(T-t)} \psi(\lambda^P, t)$, and the limiting price per contract for arbitrarily many insureds, $\lim_{n \rightarrow \infty} \frac{1}{n} P^{(n)}(r, \lambda^P, t) = e^{-r(T-t)} \beta(\lambda^P, t)$, and we use the following parameter values:

- The pure endowment contract matures in $T = 10$ years.
- The constant riskless rate of return is $r = 0.04$.
- The drift of the hazard rate is $a^P = 0.04$.
- The volatility of the hazard rate is $b^P = 0.1$.
- The minimum hazard rate of the insured individuals is $\underline{\lambda}^P = 0.02$.
- The risk parameter is $\alpha = 0.1$.

See Section 4.7 for the algorithm that we use to compute ψ and β .

In Figure 4.1, for a variety of values of the market price of mortality risk q^{λ^I} , we present the price of a single-life contract P and the limiting price per contract $\lim_{n \rightarrow \infty} \frac{1}{n}P^{(n)}$. It follows from Theorem 4.4.2 that, given a positive correlation $\rho > 0$, the limiting unit price of a pure endowment is greater with a greater market price of mortality risk q^{λ^I} , and the second set of graphs in Figure 4.1 demonstrates this result. Notice that the price of the unhedged contract is the price with $\rho = 0$. Since in the pricing mechanism, we hedge the volatility with the mortality derivative as much as possible to reduce the variability of our hedging portfolio, a large value of q^{λ^I} could lead to a higher contract price than that of an unhedged one. Observe this numerically in graphs in Figure 4.1.

In Remark 4.4.1, we concluded that if $q^{\lambda^I} < \alpha$ and if $\rho = 1$, then the limiting price per contract is less than the limiting price per contract of an unhedged portfolio of pure endowments. This result is supported by our numerical work; indeed, the curve for $q^{\lambda^I} = 0.15$ in the second set of graphs lies above the unhedged price of approximately 0.343, the price when $\rho = 0$.

In that same remark, we noted that for a single-life contract, we cannot conclude that the price with hedging will be smaller than the price without hedging, even when $q^{\lambda^I} < \alpha$. This conclusion is also supported by our numerical work; indeed, the curve for $q^{\lambda^I} = 0.09$ in the first set of graphs lies above the unhedged price of approximately 0.435, the price when $\rho = 0$.

Figure 4.1 also demonstrates the relation between the unit price of a contract and the correlation ρ . Take the limiting price per contract $\lim_{n \rightarrow \infty} \frac{1}{n}P^{(n)}$ with $q^{\lambda^I} = 0.05$, for example. When $\rho = 1$, hedging is preferred to not hedging, in terms of reducing the price of the contract. By contrast, when $\rho < 1$, that is, the two mortality

rates λ^P and λ^I are not perfectly correlated, hedging may increase the unit price of the contract such as the case when $\rho = 0.8$. This observation indicates that the population basis risk, which is the risk due to the mismatch of the insured population and the reference population, diminishes the effectiveness of hedging. This mismatch, or equivalently, a correlation $\rho < 1$, may lead to a higher unit price for the hedged contract. See [Coughlan et al. \[2007\]](#) for discussion of population basis risk.

4.6 Conclusion

In this paper, we developed a pricing mechanism for pure endowments, assuming that the issuing company hedges its pure endowment risk with bonds and mortality derivatives, and requires compensation for the unhedged part of mortality risk in the form of a pre-specified instantaneous Sharpe ratio. In our model, we took the hazard rates of the insured population and reference population, as well as the interest rates, to be stochastic. We derived the pricing formulae for the hedged contracts on single life and on multiple conditionally independent lives. Also, we obtained the pricing formula for the limiting price per pure endowment contract as the number of the insureds in the portfolio goes to infinity. In each case, the price solves a PDE, and we analyzed these PDE and thereby determined properties of the prices of the hedged pure endowments. The limiting price per contract solves a linear PDE and represent this value as an expectation with respect to an equivalent martingale measure. We noted that, in the limiting case, the mortality risk inherent in the pure endowment is fully hedged by the mortality derivative when the correlation between the two hazard rates λ^P and λ^I is 1.

To investigate the factors that affect the effectiveness of hedging, we devoted our attention to the market price of the reference mortality risk q^{λ^I} and the correlation

ρ between λ^P and λ^I . Since the correlation ρ is more likely to be positive in reality, we focused on the case for which $\rho \geq 0$ during our discussion (and especially in our numerical work) and assumed that the market price of the mortality risk q^{λ^I} is independent of λ^I . We found that hedging with a mortality derivative requiring a negative market price of mortality risk always reduces the price of the contract. This result is consistent with the conclusions in [Bayraktar and Young \[2007a\]](#) and [Cox and Lin \[2007\]](#) that hedging pure endowments (or life annuities) with life insurance reduces the price of the former.

For the limiting case, we reached a more straightforward conclusion, as we discuss in [Remark 4.4.1](#). Specifically, if $\rho = 1$, the condition that $q^{\lambda^I} < \alpha$ guarantees a reduction in the per-contract price through hedging. However, if $\rho < 1$, it is possible that hedging with the mortality derivatives increases the price of the contract even if this condition is satisfied. This result reflects the significance of ρ on the effectiveness of hedging. We also found that, in our numerical work, hedging with the mortality derivatives is less effective in reducing the variance of the hedging portfolio for pure endowments of a finite number of individuals.

Our results suggest that, to make it efficient for underwriters to hedge mortality risk and thereby benefit the insured, transparent design of mortality indices and mortality derivatives is essential. Reducing the market price of the mortality risk q^{λ^I} is also critical. Therefore, it is important to build up a liquid mortality market and provide more flexible mortality-linked securities in order to reduce q^{λ^I} .

In our paper, we only investigated the prices of pure endowments and assumed that the mortality derivative is a q -forward. However, we believe that the main qualitative insights will hold in general.

4.7 Appendix

In this section, we present an algorithm for numerically computing ψ . Recall that in our numerical example, we assume that q^{λ^I} , the market price of mortality risk λ^I , is a constant, as well as a^P and b^P . Then, equation (4.2.30) becomes

$$\begin{cases} \psi_t + [a^P - \rho q^{\lambda^I} b^P] (\lambda^P - \underline{\lambda}^P) \psi_{\lambda^P} + \frac{1}{2} (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \psi_{\lambda^P \lambda^P} - \lambda^P \psi \\ = -\alpha \sqrt{(1 - \rho^2) (b^P)^2 (\lambda^P - \underline{\lambda}^P)^2 \psi_{\lambda^P}^2 + \lambda^P \psi^2}, \\ \psi(\lambda^P, T) = 1. \end{cases} \quad (4.7.1)$$

Next, we describe our numerical scheme to compute ψ .

Transformation Define $\tau = T - t$, $y = \ln(\lambda^P - \underline{\lambda}^P)$, and $\hat{\psi}(y, \tau) = \psi(\lambda^P, t)$. By

(4.7.1), $\hat{\psi}$ solves

$$\begin{cases} \hat{\psi}_\tau = \hat{a} \hat{\psi}_y + \frac{1}{2} (b^P)^2 \hat{\psi}_{yy} - (e^y + \underline{\lambda}^P) \hat{\psi} + \alpha \sqrt{(1 - \rho^2) (b^P)^2 \hat{\psi}_y^2 + (e^y + \underline{\lambda}^P) \hat{\psi}^2}, \\ \hat{\psi}(y, 0) = 1, \end{cases} \quad (4.7.2)$$

in which $\hat{a} = a^P - \rho q^{\lambda^I} b^P - \frac{1}{2} (b^P)^2$.

Boundary Conditions While equation (4.7.2) for $\hat{\psi}$ is defined in the domain $\mathbb{R} \times [0, T]$, we solve it numerically in the domain $[-M, M] \times [0, T]$ such that e^{-M} is approximately zero. Therefore, we require boundary conditions at $y = \pm M$.

1. If $\lambda_t^P = \underline{\lambda}^P$, then $\lambda_s^P = \underline{\lambda}^P$ for all $s \in [t, T]$. From equation (4.7.1), we have that $\psi(\underline{\lambda}^P, t) = \exp\left\{-\left(\underline{\lambda}^P - \alpha \sqrt{\underline{\lambda}^P}\right)(T - t)\right\}$. Thus, it is reasonable to set the boundary condition at $y = -M$ to be $\hat{\psi}(-M, \tau) = \exp\left\{-\left(\underline{\lambda}^P - \alpha \sqrt{\underline{\lambda}^P}\right)\tau\right\}$.
2. If λ_t^P is very large, we expect the individual to die immediately, so the value

of the pure endowment is approximately 0. Thus, we set the boundary condition at $y = M$ to be $\hat{\psi}(M, \tau) = 0$.

Finite Difference Scheme We discretize the differential equation (4.7.2) and get a corresponding difference equation as follows:

1. Choose the step sizes of y and τ as h and k , respectively, so that $I = 2M/h$ and $J = T/k$ are integers.
2. Define $y_i = -M + ih$, $\tau_j = jk$, and $\hat{\psi}_{i,j} = \hat{\psi}(y_i, \tau_j)$, for $i = 0, 1, \dots, I$ and $j = 0, 1, \dots, J$.
3. We use a backward difference in time, central differences in space, and a forward difference for the square-root term. Therefore, we have the following expressions:

$$\begin{cases} \hat{\psi}_\tau(y_i, \tau_j) = \frac{\hat{\psi}_{i,j+1} - \hat{\psi}_{i,j}}{k} + \mathcal{O}(k), \\ \hat{\psi}_y(y_i, \tau_j) = \frac{\hat{\psi}_{i+1,j+1} - \hat{\psi}_{i-1,j+1}}{2h} + \mathcal{O}(h^2), \\ \hat{\psi}_{yy}(y_i, \tau_j) = \frac{\hat{\psi}_{i+1,j+1} - 2\hat{\psi}_{i,j+1} + \hat{\psi}_{i-1,j+1}}{h^2} + \mathcal{O}(h^2). \end{cases} \quad (4.7.3)$$

Also, for the non-linear term in (4.7.2), we have

$$\begin{aligned} & \sqrt{(1 - \rho^2)(b^P)^2 \hat{\psi}_y^2 + (e^y + \underline{\lambda}^P) \hat{\psi}^2} \\ &= \sqrt{(1 - \rho^2)(b^P)^2 \left(\frac{\hat{\psi}_{i+1,j} - \hat{\psi}_{i-1,j}}{2h} \right)^2 + (e^{y_i} + \underline{\lambda}^P) \hat{\psi}_{i,j}^2} + \mathcal{O}(h). \end{aligned} \quad (4.7.4)$$

Therefore, we approximate (4.7.2) to order $\mathcal{O}(k + h)$ with the following difference equation:

$$\begin{aligned} \frac{\hat{\psi}_{i,j+1} - \hat{\psi}_{i,j}}{k} &= \hat{a} \frac{\hat{\psi}_{i+1,j+1} - \hat{\psi}_{i-1,j+1}}{2h} + \frac{1}{2} (b^P)^2 \frac{\hat{\psi}_{i+1,j+1} - 2\hat{\psi}_{i,j+1} + \hat{\psi}_{i-1,j+1}}{h^2} \\ &\quad - (e^{y_i} + \underline{\lambda}^P) \hat{\psi}_{i,j+1} + \alpha A_{i,j}, \end{aligned} \quad (4.7.5)$$

in which

$$A_{i,j} = \sqrt{(1 - \rho^2) (b^P)^2 \left(\frac{\hat{\psi}_{i+1,j} - \hat{\psi}_{i-1,j}}{2h} \right)^2 + (e^{y_i} + \underline{\lambda}^P) \hat{\psi}_{i,j}^2}. \quad (4.7.6)$$

If we define $a = \hat{a} \frac{k}{2h} - (b^P)^2 \frac{k}{2h^2}$, $b = 1 + (b^P)^2 \frac{k}{h^2} + k \underline{\lambda}^P$, and $c = -\hat{a} \frac{k}{2h} - (b^P)^2 \frac{k}{2h^2}$, then (4.7.5) becomes

$$a\hat{\psi}_{i-1,j+1} + (b + ke^{y_i})\hat{\psi}_{i,j+1} + c\hat{\psi}_{i+1,j+1} = \hat{\psi}_{i,j} + \alpha k A_{i,j}, \quad (4.7.7)$$

for $i = 1, 2, \dots, I - 1$ and $j = 0, 1, \dots, J - 1$, with the following boundary conditions:

$$\begin{cases} (b + ke^{y_1})\hat{\psi}_{1,j+1} + c\hat{\psi}_{2,j+1} = \hat{\psi}_{1,j} + \alpha k A_{1,j} - ae^{-(\underline{\lambda}^P - \alpha \sqrt{\underline{\lambda}^P})(j+1)k} \\ a\hat{\psi}_{I-2,j+1} + (b + ke^{y_{I-1}})\hat{\psi}_{I-1,j+1} = \hat{\psi}_{I-1,j} + \alpha k A_{I-1,j}, \end{cases} \quad (4.7.8)$$

for $j = 0, 1, \dots, J - 1$. It is convenient to write equations (4.7.7)-(4.7.8) in matrix form as

$$\mathbf{M}\hat{\Psi}_{j+1} = \hat{\Psi}_j + \alpha k \mathbf{A}_j - \left[ae^{-(\underline{\lambda}^P - \alpha \sqrt{\underline{\lambda}^P})(j+1)k}, 0, \dots, 0 \right]^t. \quad (4.7.9)$$

for $j = 0, 1, \dots, J - 1$, in which the superscript t represents matrix transpose. In the equation above, $\hat{\Psi}_j = [\hat{\psi}_{1,j}, \hat{\psi}_{2,j}, \dots, \hat{\psi}_{I-1,j}]^t$ and $\mathbf{A}_j = [A_{1,j}, A_{2,j}, \dots, A_{I-1,j}]^t$ with $A_{i,j}$ defined in (4.7.6). The matrix \mathbf{M} is a tri-diagonal matrix with the sub-diagonal identically a , with the main diagonal $b + ke^{y_1}, b + ke^{y_2}, \dots, b + ke^{y_{I-1}}$, and with the super-diagonal identically c .

4. Begin with the initial condition $\hat{\psi}_{i,0} = 1$, for $i = 1, 2, \dots, I - 1$, or equivalently, $\hat{\Psi}_0 = \mathbf{1}$, in which $\mathbf{1}$ is an $(I - 1) \times 1$ column vector of 1s. Then, solve (4.7.9) repeatedly for $j = 0, 1, \dots, J - 1$ until we reach $\hat{\Psi}_J$.

One can modify this algorithm to compute $\psi^{(n)}$ for any $n > 1$ and the limiting result $\beta = \lim_{n \rightarrow \infty} \psi^{(n)}$. Computing the latter is particularly straightforward because β solves a linear PDE.

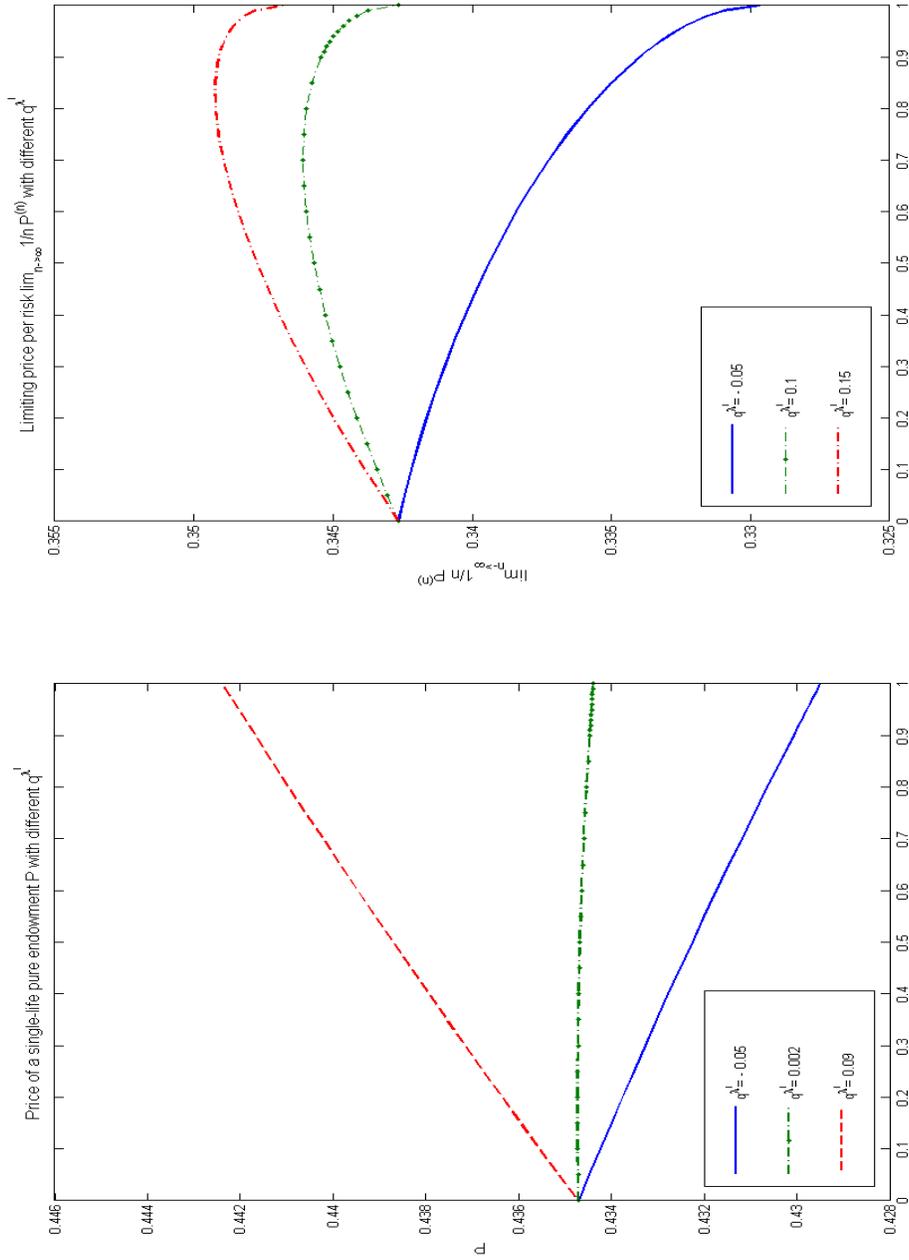


Figure 4.1: Price of hedged pure endowment

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