POSITIVITY IN REAL GRASSMANNIANS: COMBINATORIAL FORMULAS

by

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This dissertation is dedicated to the memory of my mother, Kathleen Marie Harter.

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CHAPTER 1

Introduction

The purpose of this dissertation is to provide explicit formulas for a combinatorial approach to studying *totally nonnegative Grassmannians.* A totally nonnegative Grassmannian consists of the points in a real Grassmannian where all Plücker coordinates can be taken to be simultaneously nonnegative.

From an elementary perspective, the study of totally nonnegative Grassmannians is a natural step up from the study of *totally nonnegative matrices*, i.e., those matrices whose minors are all nonnegative; in fact, totally nonnegative matrices can be viewed as a special case of the Grassmannian theory. Totally nonnegative matrices have a rich history, and they have significant applications in many areas of mathematics and related fields. The developing theory of total positivity in Grassmannians aspires to make a similar impact, via its emerging connections to algebra, combinatorics, geometry, and topology. In both cases, total positivity can be studied in a very concrete way, namely via path enumeration in planar networks.

Both matrices and Grassmannians fit into a broader notion of positivity, framed in terms of cluster algebras. The cluster variables play the role that minors play in the matrix setting; the totally positive part of a space carrying a cluster algebra structure consists of the points at which all cluster variables take positive values.

This introduction summarizes the two key results of the dissertation and explains how the remaining chapters are organized.

1.1 Overview of the main results

We begin with a rough sketch of the main results. The first major theorem is an explicit combinatorial formula describing Postnikov's main construction in [Pos07]: the *boundary measurement map* assigning a point in the totally nonnegative Grassmannian to each planar directed network with positive edge weights:

$$\left\{\begin{array}{l}\text{planar directed networks}\\ \text{with } k \text{ boundary sources}\\ \text{and } n-k \text{ boundary sinks}\end{array}\right\} \to \left\{\begin{array}{l}\text{points in the}\\ \text{totally nonnegative}\\ \text{Grassmannian } (\mathrm{Gr}_{kn})_{\geq 0}\end{array}\right\}.$$

Informally, the boundary measurement map constructs a matrix that encodes all ways to get from each boundary source to each boundary sink. If the edges of the network have positive weights, then this boundary measurement matrix can be viewed as a point in the Grassmannian (of k-dimensional subspaces of $\mathbb{R}^n$). Through recursive methods, Postnikov showed that boundary measurement matrices indeed lie in the totally nonnegative part of the Grassmannian. We provide a direct proof of this fact by giving an explicit combinatorial formula for the maximal minors of the boundary measurement matrix, writing each of them as a ratio of two polynomials in the edge weights, with positive integer coefficients.

To state our result, we will need to quickly recall the basic features of Postnikov's construction; for examples and complete details, see Chapter 3.

The construction begins with a planar directed graph G properly embedded in a disk. Every vertex of G lying on the boundary of the disk is assumed to be a source or a sink. Each edge of G is assigned a *weight*, which we treat as a formal variable. Postnikov defines the *boundary measurement matrix* A with columns labeled by the boundary vertices and rows labeled by the set I of boundary sources, as follows. Each matrix entry of A is, up to a sign that accounts for how the sources and sinks interlace along the boundary, a weight generating function for directed walks from a given boundary source to a given boundary vertex, where each walk is counted with a sign reflecting the parity of its topological winding index.

The maximal minors $\Delta_J(A)$ of the boundary measurement matrix A (here J is a subset of boundary vertices with $|J| = |I|$) are formal power series in the edge weights of the network (possibly with infinitely many nonzero terms). Remarkably, each minor $\Delta_J(A)$ can be rewritten as a *subtraction-free* rational expression in the edge weights [Pos07]. This allows us to specialize to positive edge weights without worrying about convergence. The minors $\Delta_J(A)$ can then be interpreted as Plücker coordinates of a point in a Grassmannian. Since they are all nonnegative, we obtain a point in the totally nonnegative Grassmannian.

Our formula is easiest to state in the case when G is *perfectly oriented*, i.e., every interior vertex of G has exactly one incoming edge or exactly one outgoing edge (or both). Every graph can be easily transformed into a perfectly oriented one without changing the boundary measurement matrix, so this simple set-up essentially covers all cases. Complete details and proofs for both the perfectly oriented case and the general case can be found in Chapter 4.

To state our formula, we need the following notions. A *conservative flow* in a perfectly oriented graph G is a (possibly empty) collection of pairwise vertex-disjoint oriented cycles. (Each cycle is self-avoiding, i.e., it is not allowed to pass through a vertex more than once. For perfectly oriented graphs G, this is equivalent to not repeating an edge.) For $|J| = |I|$, a *flow from I to J* is a collection of self-avoiding walks and cycles, all pairwise vertex-disjoint, such that each walk connects a source in I to a boundary vertex in J. (The sets I and J may overlap, in which case a boundary source may be connected to itself by a walk with no edges.) The *weight* of a flow (conservative or not) is the product of the weights of all its edges. A flow with no edges has weight 1.

Theorem 1.1. *The maximal minor $\Delta_J(A)$ is given by $\Delta_J = \dfrac{f}{g}$, where f and g are nonnegative polynomials in the edge weights, defined as follows:*

- *f is the weight generating function for all flows from I to J;*
- *g is the weight generating function for all conservative flows in G.*

If the underlying graph G is acyclic, then $g = 1$, and Theorem 1.1 reduces to a well known result [KM59, Lin73, GV85] expressing the determinant of a matrix associated with a planar acyclic network in terms of non-intersecting path families; see, e.g., [FZ00a] and references therein.

The boundary measurement map is (infinitely) many-to-one. However, the restriction of the boundary measurement map to a special class of networks, called Γ-networks, is one-to-one, and its image is the entire totally nonnegative Grassmannian. In this thesis, we also give combinatorial formulas for the inverse map to this restriction. Our second major theorem is the resulting explicit combinatorial bijection between Γ-networks and points in the totally nonnegative Grassmannian:

$$\left\{\begin{array}{l}\Gamma\text{-networks}\\ \text{with } k \text{ boundary sources}\\ \text{and } n-k \text{ boundary sinks}\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\text{points in the}\\ \text{totally nonnegative}\\ \text{Grassmannian } (\mathrm{Gr}_{kn})_{\geq 0}\end{array}\right\}.$$

Finding the Γ-network associated to a point in the totally nonnegative Grassmannian is a two-step process. First, we give a simple algorithm for determining the underlying graph of the network. Postnikov [Pos07] has described a cell decomposition of a totally nonnegative Grassmannian into *positroid cells*, each characterized by requiring that certain Plücker coordinates vanish and the rest do not. The first part of our process is a matter of checking whether particular Plücker coordinates vanish or not at a given point, or equivalently, determining which positroid cell contains the point. The networks corresponding to points in the same positroid cell of a totally nonnegative Grassmannian will have the same underlying graph.

Postnikov also introduced a parametrization of each positroid cell using a collection of parameters which we call Le-coordinates. In the context of Γ-networks, this translates into finding appropriate weights for the graph found in the first step. In the second step, we give combinatorial expressions for the weights in the network: each one is a ratio of products of Plücker coordinates. In fact, we provide two such expressions. In the first, the Möbius function for the set of faces partially ordered by "northwest-ness" determines which Plücker coordinates appear in the numerator or denominator of any given weight. The second expression involves tracing out paths which form the north and south boundaries of faces in the network, keeping track of where these paths change direction. The second formulation is especially nice in that it uses a minimal subset of the Plücker coordinates to express the entire set of weights. This minimal subset forms a totally positive base (in the sense of Fomin and Zelevinsky [FZ99]) for the set of Plücker coordinates which do not vanish on the specified cell; i.e., every non-vanishing Plücker coordinate can be formed by taking products, sums, and ratios of those Plücker coordinates appearing in the totally positive base.

1.2 Organization

The remainder of the thesis is organized as follows. Chapter 2 is meant to provide motivation for the study of total positivity in Grassmannians. It begins with a discussion of total positivity for matrices, with a bit of history in Section 2.1 and a discussion of the combinatorial approach using acyclic planar networks in Section 2.2. In Section 2.3, we look at total positivity from the perspective of cluster algebras, and in Section 2.4, we provide some context for the study of Grassmannians in particular.

Chapter 3 is a rigorous description of Postnikov's combinatorial approach to total positivity in Grassmannians. Section 3.1 is a brief introduction to Grassmannians. Section 3.2 contains all the details on Postnikov's construction of the boundary measurement matrix of a network. A running example illustrates each step of the process. Section 3.3 introduces Γ-networks, which are in bijection with points in the totally nonnegative Grassmannian, and their corresponding Le-tableaux. Section 3.4 describes Postnikov's positroid stratification of the totally nonnegative Grassmannian and provides explicit formulas for the restriction of the boundary measurement map to Γ-networks, using classical results from the matrix case.

Chapter 4 presents the first major result described above, namely the explicit combinatorial formulation of the boundary measurement map (c.f. Theorem 1.1). Section 4.1 gives precise statements of the theorem, first for perfectly oriented networks, and then for the general case. Section 4.2 contains some auxiliary lemmas needed for the main proofs, which are given in Section 4.3. Section 4.3 mirrors Section 4.1, as the perfectly oriented case is not only easier to state than the general case, but also easier to prove. Section 4.4 is an aside containing a discussion of non-planar networks; it can be skipped without any loss of continuity.

Chapter 5 presents the second major result of this thesis, an explicit construction of the Γ-network associated to a point in the totally nonnegative Grassmannian. Section 5.1 describes the first stage, determining which positroid cell a point belongs to. Section 5.2 gives our first formula for the weights of the network, in terms of the Möbius function of the partially ordered set of faces of the network, and Section 5.3 provides an alternative formula for the weights, using a minimal totally positive base for the non-vanishing Plücker coordinates of the appropriate positroid cell.

CHAPTER 2

Fundamentals of total positivity

This chapter has two key goals. The first is to provide a brief introduction to the study of total positivity for matrices. In the matrix setting, we will make a point of highlighting the combinatorial approach, which has provided the tools for many recent advances in total positivity. Most remarkably, this approach yields a rather simple way to construct every totally nonnegative matrix. Section 2.1 discusses the origins of total positivity, and Section 2.2 provides the combinatorial perspective.

The second goal is to informally introduce a broader notion of total positivity, using cluster algebras. Section 2.3 provides the "big picture" and shows how total positivity for matrices fits into the cluster framework. Section 2.4 specifically addresses Grassmannians, which are the key objects studied in this dissertation.

This chapter does not include any sort of rigorous treatment of total positivity; instead, it should be treated as an informal discussion which aims to provide some background and context. Plenty of references are included for the reader interested in further details.

2.1 The origins and applications of total positivity for matrices

The study of totally positive matrices has its roots in analysis, but it has influenced and been influenced by many other areas of mathematics and other fields as well. To give just a few examples, total positivity has played important roles in stochastic processes and approximation theory [GM96, Kar68], the theory of immanants [Ste91], the representation theory of S_∞ [Edr52, Tho64], unimodality and log-concavity [Sta89], and matrix Poisson varieties [GLL09b, GLL09a].

Definition 2.1. A matrix with real entries is *totally positive* if each of its minors is strictly positive. Similarly, a matrix with real entries is *totally nonnegative* if each of its minors is nonnegative.

Example 2.2. The 2×2 matrix below is totally nonnegative, but not totally positive. The 1×1 minors are simply the entries of the matrix, and each of these is positive. However, the determinant, which is the only 2×2 minor, is zero.

$$\begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix}$$

Prior to the combinatorial approach that we describe in Section 2.2, total positivity was dominated by four mathematicians: Schoenberg, Gantmacher, Krein, and Karlin. Several surveys and books describe this history quite well (see, for example, [And87], [Kar68], [GK02], and [Pin10]). Below, we give a brief and informal summary of some of the key topics from the early history of total positivity.

Beginning with the work of Schoenberg in the 1930's, we find total positivity in the guise of the *variation diminishing property*; a matrix M satisfies this property if the vector $M\mathbf{x}$ has no more sign changes than the vector $\mathbf{x}$. While this characterization seems at first glance to be quite different from our definition above, the two are essentially equivalent. In the late 1930's, the fundamentals of the classical theory were developed in the work of Gantmacher and Krein on oscillations in mechanics. In particular, they showed that every $n \times n$ totally positive matrix has n distinct eigenvalues, each of which is positive [GK02].

Total positivity is a special case of *sign-regularity*, a property requiring all minors of the same size to have the same sign. Totally positive matrices are also the simplest examples of *totally positive kernels*, a class of functions which includes Pólya frequency functions.

In the 1950's, Karlin explored the probabilistic aspects of total positivity; his work was rooted in the observation that transition kernels of one-dimensional diffusion processes are totally positive.

2.2 The combinatorial perspective: planar networks and totally positive matrices

We begin this section by showing how to construct totally nonnegative matrices associated to certain planar networks. It is perhaps not surprising that this combinatorial approach provides an important class of examples in total positivity. What is remarkable is the fact that we obtain *every* totally nonnegative matrix in this way. See Theorem 2.6, Corollary 2.8, and Theorem 2.9 for details.

The following construction makes sense in a much more general setting, but for the sake of simplicity, we present only the material necessary to give the combinatorial characterization of total positivity. The interested reader may consult [Lin73], [GV85], [Bre96], and [FZ00a] for more details.

Suppose that G is a finite acyclic directed graph which has a set S of sources, a set T of sinks, and a planar embedding (in a disk) satisfying the following properties:

- all sources in S and all sinks in T lie on the boundary of the disk,
- all remaining vertices of G lie in the interior of the disk, and
- for any pair of subsets $S' = \{s_1, \ldots, s_k\} \subset S$ and $T' = \{t_1, \ldots, t_k\} \subset T$, there exists a unique bijection $\pi : S' \to T'$ such that no two chords $[s_i, t_{\pi(i)}]$ intersect.

Typical graphs used in studying totally positive matrices will have the sources on one side of the disk and the sinks on the other (with no interlacing), as in Figure 2.1.

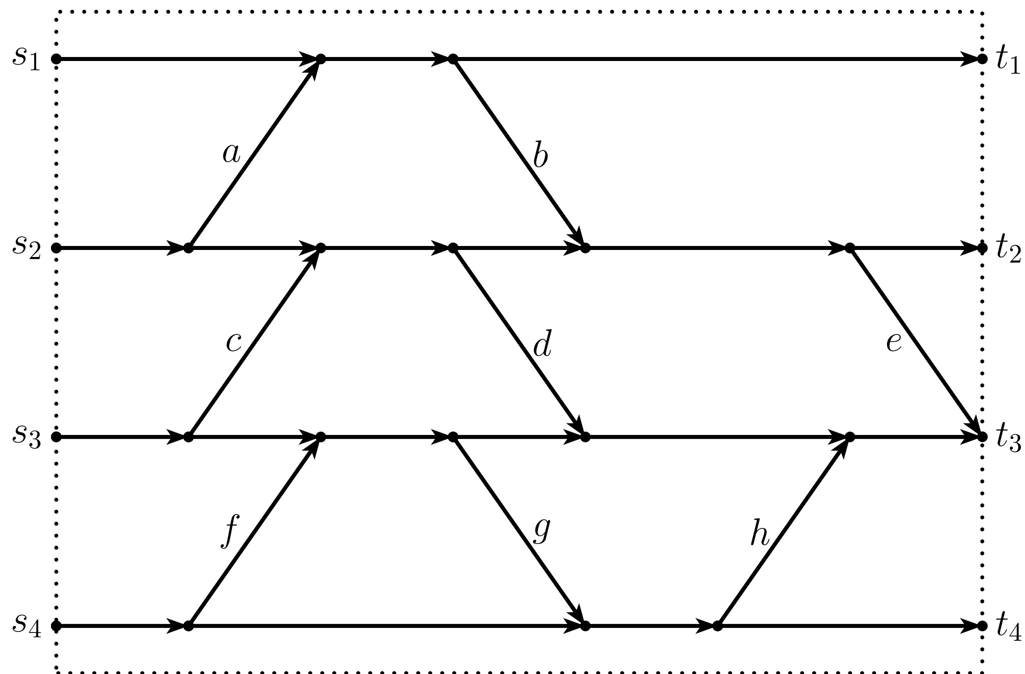


Figure 2.1: An appropriately embedded graph G with source set S and sink set T.

Assume that we assign a weight x_e to each edge e in G. We will refer to this weighted graph as a *network*, denoted $N = (G, x)$. In this setting, the weights may be viewed as formal variables, or we can specialize to positive real edge weights.

A *walk* $P = (e_1, \ldots, e_m)$ in G is formed by traversing the edges $e_1, e_2, \ldots, e_m$ in the specified order. We write $P : s \rightsquigarrow t$ to indicate that P is a walk starting at a vertex s and ending at a vertex t.

Define the *weight* of a walk $P = (e_1, \ldots, e_m)$ to be

$$\mathrm{wt}(P) = x_{e_1} \cdots x_{e_m}.$$

Definition 2.3. For a source $s_i \in S$ and a sink $t_j \in T$ in an acyclic network N, the *weighted path matrix* $A(N)$, with rows indexed by S and columns indexed by T, is defined by its entries:

$$a_{ij} = \sum_{P:s_i \rightsquigarrow t_j} \mathrm{wt}(P),$$

with the sum taken over all directed walks $P : s_i \rightsquigarrow t_j$.

Each entry of the weighted path matrix is simply a polynomial in the edge weights, since there are finitely many paths from any source to any sink.

Remark 2.4. If we set $x_e = 1$ for each edge e in G, then the entry a_{ij} in the weighted path matrix is the number of paths from s_i to t_j.

Example 2.5. If we assume that each unlabeled edge in Figure 2.1 has weight 1, then the weighted path matrix of the network is

$$\begin{pmatrix} 1 & b & be & 0 \\ a & 1+ab & d+e+abe & 0 \\ 0 & c & 1+cd+ce+gh & g \\ 0 & 0 & f+h+fgh & 1+fg \end{pmatrix}.$$

Theorem 2.6, expressing determinants of a weighted path matrix as sums of weights of families of paths in the corresponding network, is a simple but powerful tool. The formula can essentially be found in [KM59] and [Lin73], though its usefulness in combinatorics was first highlighted by Gessel and Viennot [GV85]. Many more applications have been developed since then. For example, the Jacobi-Trudi identities for Schur functions can be proved using this technique [GV85]. Several more examples involving commonly encountered combinatorial objects, including Catalan numbers and rhombus tilings of hexagons, can be found in [Aig01].

To state the theorem, we need to introduce notation for the minors of a matrix. If $I = \{i_1 < \cdots < i_k\} \subset S$ and $J = \{j_1 < \cdots < j_k\} \subset T$, let $\Delta_{I,J}(A(N))$ denote the

I, J-minor of $A(N)$, that is, the determinant of the submatrix of $A(N)$ with rows in I and columns in J.

Theorem 2.6. *[KM59, Lin73, GV85] Given a planar acyclic network N with source set S and sink set T arranged as above, the minors of the weighted path matrix of N are given by the formula*

$$\Delta_{I,J}(A(N)) = \sum_{\mathbf{P}=\{P_1:s_{i_1} \rightsquigarrow t_{\pi(i_1)},\ldots,P_k:s_{i_k} \rightsquigarrow t_{\pi(i_k)}\}} \mathrm{wt}(P_1)\cdots\mathrm{wt}(P_k),$$

where the sum is taken over all collections $\mathbf{P}$ of non-intersecting paths in N connecting the sources in I to the sinks in J (and π is the unique bijection for which non-intersecting collections of paths may exist).

Example 2.7. Using Theorem 2.6, a quick glance at Figure 2.1 implies that the weighted path matrix in Example 2.5 has determinant 1, since there is only one way to choose four non-intersecting paths $P_i : s_i \rightsquigarrow t_i$, namely by taking the four strictly horizontal paths, and each such path has weight 1.

Theorem 2.6 has an elegant and elementary proof involving a sign-reversing involution on the collections of paths in which one or more pairs of paths has an intersection. This "proof from the book" can be found in [AZ10]. Care must be taken to produce an actual involution, of course, but the basic idea is that such path collections can all grouped into pairs which are identical except that two of their paths have had their "tails swapped", as in Figure 2.2.

Corollary 2.8. *If the network $N = (G, x)$ has positive real edge weights, then the weighted path matrix $A(N)$ is a totally nonnegative matrix.*

Anne Whitney's main result in [Whi52] can be restated as follows: every invertible totally nonnegative matrix can be factored into a product consisting of a diagonal

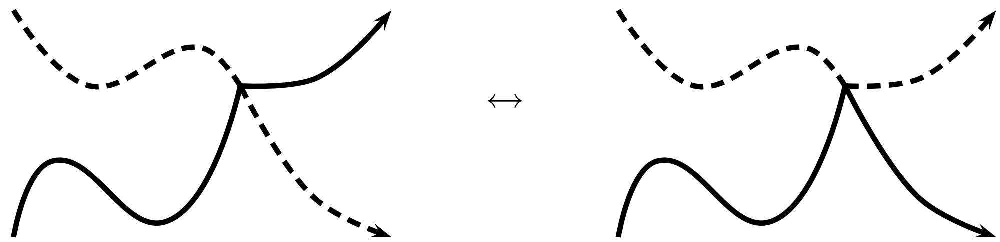


Figure 2.2: Tail swapping.

matrix and $n^2 - n$ *elementary Jacobi matrices*, that is, matrices which differ from the identity matrix in a single (nonnegative) entry on the superdiagonal or subdiagonal. In fact, Whitney's particular factorization parametrizes totally positive matrices. Furthermore, the Whitney-Loewner Theorem [Whi52, Loe55] tells us that every invertible totally nonnegative matrix can be approximated arbitrarily closely by totally positive matrices. The former result hints at the form of a graph for which different choices of weights yield all totally positive matrices, and the latter suggests that we may be able to modify this graph to obtain all invertible totally nonnegative matrices. Theorem 2.9 tells us that the invertibility condition is actually unnecessary.

Theorem 2.9. *[Bre95] Every totally nonnegative matrix is the weighted path matrix of some planar acyclic network.*

Figure 2.3 displays the network corresponding to Whitney's factorization for $n \times n$ totally positive matrices. To obtain the remaining $n \times n$ totally nonnegative matrices, we need to take certain degenerations of this network; it is possible to do so in a way that ensures we encounter each totally nonnegative matrix exactly once in our collection. The poset of graphs for 2×2 matrices is given in Figure 2.4. A natural extension of this construction yields networks corresponding to non-square matrices,

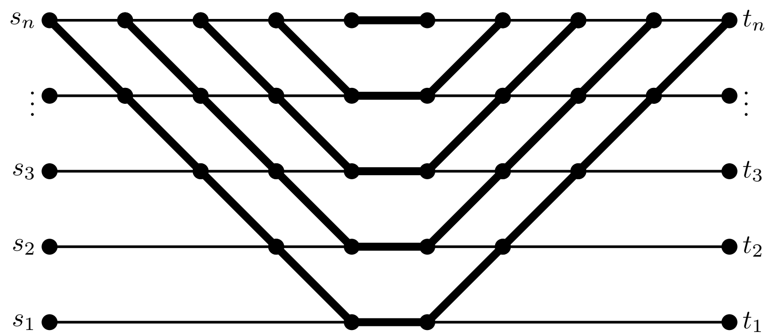


Figure 2.3: The graph corresponding to an arbitrary totally positive $n \times n$ matrix. All edges are directed to the right. The slanted bold edges correspond to elementary Jacobi matrices in Whitney's factorization, and the bold edges in the middle of the network correspond to entries of the diagonal matrix, assuming all other edges have weight 1.

though we will not provide the details here.

2.3 Total positivity and cluster algebras

In the previous sections of this chapter, we explored the totally positive part of the space of matrices of a given size. In this section, we wish to convey the idea that this is just the tip of the iceberg – in fact, there is a much broader notion of positivity in which totally positive matrices simply give us the first interesting example.

Consider $G = SL_n$, the space of $n \times n$ matrices with determinant ± 1. The Loewner-Whitney Theorem [Whi52, Loe55] tells us that the semigroup of totally nonnegative matrices in G is generated by the *Chevalley generators* of the corresponding Lie algebra. This observation lead Lusztig [Lus94] to develop a theory of positivity for semisimple Lie groups, taking the totally nonnegative part $G_{\geq 0}$ of a group G to be the semigroup generated by the corresponding Chevalley generators. Lusztig has shown that, just as in the case of matrices, the totally nonnegative part $G_{\geq 0}$ can be described by a collection of inequalities. Lusztig's characterization, however, involves an infinite set of inequalities of the form $\Delta(x) \geq 0$, with Δ ranging over the elements of the appropriate *dual canonical basis*, which is quite difficult

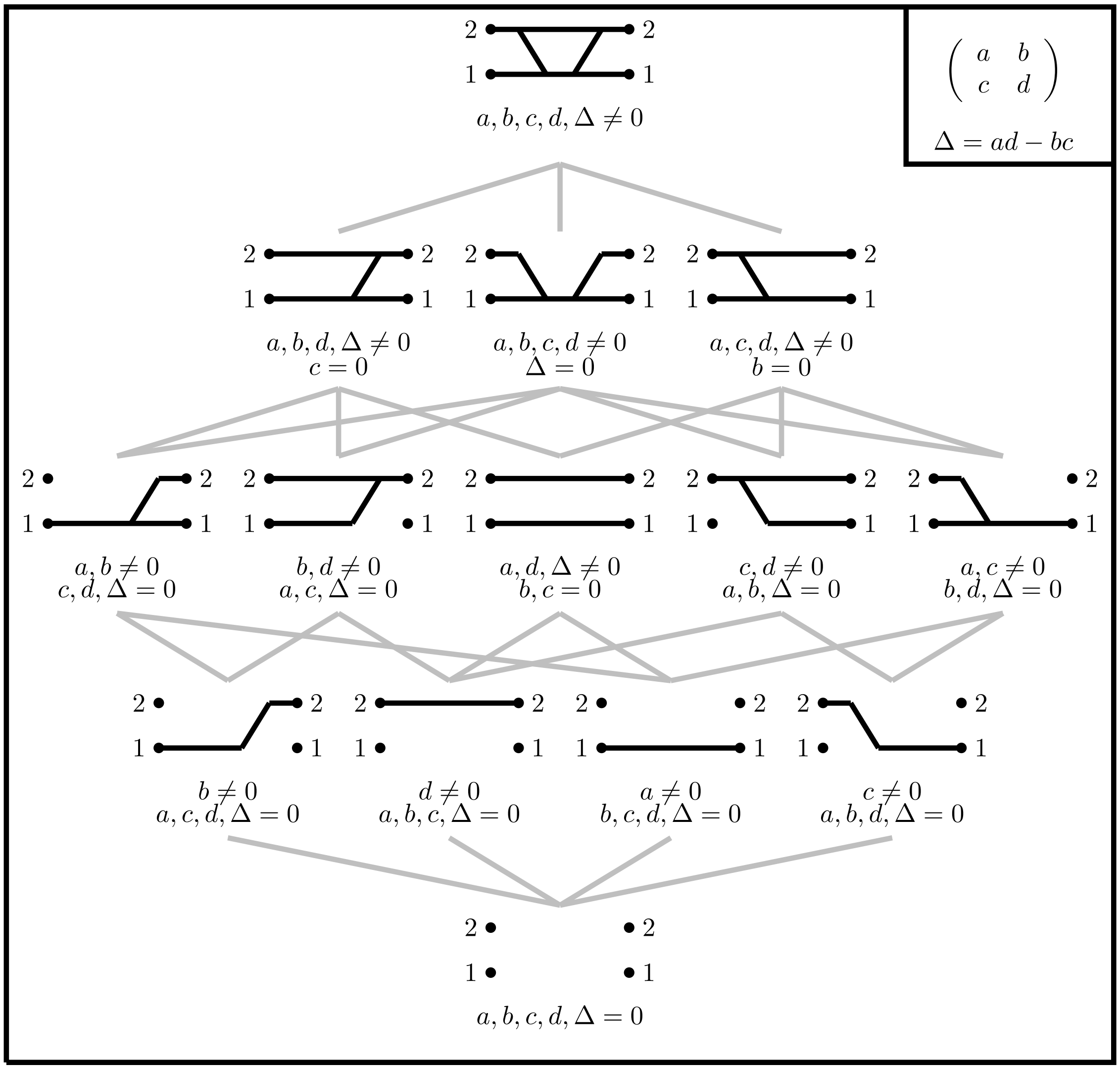


Figure 2.4: Graphs corresponding to 2×2 totally nonnegative matrices.

to understand. Later work by Fomin and Zelevinsky [FZ99, FZ00b] shows that a finite set of such inequalities will suffice; each such inequality specifies that a given *generalized minor* is nonnegative.

Here we begin to see a larger pattern – for certain algebraic varieties, we have been able to specify a set of regular functions which play the role of minors, and the totally positive part of such a variety consists of those points for which all of these "minors" are positive. We can ask which varieties have such a notion of positivity, and for those that do, which families of regular functions should be used to define the positive part. The theory of cluster algebras can be viewed as an attempt to provide a general answer to this question. (See [Fom10] for further details.)

For the purposes of positivity, we will consider cluster algebras over $\mathbb{R}$. A *cluster algebra* is an $\mathbb{R}$-algebra with a distinguished set of generators, called *cluster variables*, which are grouped into non-disjoint sets called *clusters*, all of the same cardinality[1]. The cluster algebra carries some additional information, namely a collection of *exchange relations*, each of which describes how to transform one cluster into another by exchanging a single cluster variable. An exchange relation swapping one cluster variable x for another cluster variable x' must have the form

$$xx' = M_1 + M_2, \tag{2.1}$$

where M_1 and M_2 are monomials in the cluster variables of the current cluster.

Although the precise definition is quite technical (see [FZ02]), the following example illustrates many of the key properties of cluster algebras.

Example 2.10. Let us examine a cluster structure in the ring $\mathcal{A}$ of regular functions on $SL_3(\mathbb{R})$. We have

$$SL_3(\mathbb{R}) = \left\{ X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} : \begin{array}{c} x_{11}, x_{12}, \ldots, x_{33} \in \mathbb{R} \\ \text{and } \det(X) = 1 \end{array} \right\},$$

so $\mathcal{A}$ is the ring of polynomials in the nine matrix entries x_{ij}, modulo the relation $\det(X) - 1 = 0$.

Viewed as a cluster algebra, $\mathcal{A}$ is generated by twenty cluster variables, including the eighteen nontrivial minors of the matrix $X \in SL_3(\mathbb{R})$. These cluster variables

[1]Experts will note that what we call clusters are usually called "extended clusters". In particular, we do not make a distinction between cluster variables and frozen variables.

are grouped into fifty overlapping clusters, each containing eight cluster variables. For example, one of the clusters is

$$\{x_{11}, x_{12}, x_{13}, x_{21}, x_{31}, \Delta_{12,12}, \Delta_{12,23}, \Delta_{23,12}\}, \tag{2.2}$$

corresponding to the *initial minors* of the matrix, i.e. those in which we take the first k rows and any block of k consecutive columns, or vice versa.

The minors of a matrix satisfy certain identities, the (generalized) Plücker relations. For example, a direct calculation verifies that

$$x_{21}\Delta_{13,12} = x_{11}\Delta_{23,12} + x_{31}\Delta_{12,12}. \tag{2.3}$$

This equation is one of our exchange relations, and its right-hand side involves only cluster variables which appear in the cluster in (2.2). Thus, we may replace the cluster variable x_{21} with the cluster variable $\Delta_{13,12} = \dfrac{x_{11}\Delta_{23,12} + x_{31}\Delta_{12,12}}{x_{21}}$ to obtain another cluster:

$$\{x_{11}, x_{12}, x_{13}, \Delta_{13,12}, x_{31}, \Delta_{12,12}, \Delta_{12,23}, \Delta_{23,12}\}. \tag{2.4}$$

By performing sequences of exchanges corresponding to Plücker relations, we can obtain all fifty clusters of this cluster algebra.

This example illustrates why cluster algebras are a powerful tool for understanding total positivity. Indeed, if all of the cluster variables in our initial cluster (2.2) are positive, then the Plücker relation (2.3) forces $\Delta_{13,12}$ to be positive as well, so that all cluster variables in the resulting cluster (2.4) are positive. This is a general phenomenon due to the fact that all exchange relations take the form in (2.1) and hence preserve positivity.

This gives us an efficient way to test for total positivity. If all of the cluster variables of a given cluster are positive, then we can conclude that *all* cluster variables are positive. For matrices in $SL_3(\mathbb{R})$, this means that it is sufficient to test the eight minors in a particular cluster, instead of all eighteen minors, to verify that a matrix is totally positive.

There are many benefits to the cluster algebra perspective, but two stand out above the rest. First of all, many well-studied algebras arising throughout mathematics have cluster algebra structures, particularly those associated to highly symmetric geometric situations. These include algebraic groups, such as SL_n; homogeneous spaces on which they act, e.g., Grassmannians, flag varieties, and spaces of quiver representations; certain nice moduli spaces, such as decorated Teichmüller spaces and spaces of laminations on surfaces; and so on.

Second, the nature of cluster mutations ensures that if the cluster variables of some initial cluster are all positive, then *all* cluster variables are positive. If cluster variables are indeed the right functions to play the role of minors, this positivity property implies that we only need a finite number of inequalities to define the totally positive part of a variety.

This emerging notion of positivity via cluster algebras is still in flux, but there are many indications that this is the most natural way to think about positivity in a broader sense. The classical notion of positivity for matrices and Lusztig's positivity for semisimple groups both fit into this framework, as does total positivity in Grassmannians, to which this thesis is dedicated.

2.4 Total positivity and Grassmannians

Total positivity for matrices is by now well understood. A natural next step is to look at positivity in Grassmannians, which will be the focus of this dissertation. Chapter 3 carefully describes Postnikov's combinatorial approach to the totally nonnegative Grassmannian; this section is meant to provide some context before diving into details.

Various aspects of totally nonnegative Grassmannians have already been studied. They naturally appear in the work by Rietsch and coauthors ([Rie99],[Rie06], [MR04], and [RW08]) on cell decompositions and parametrizations for flag varieties, studied from the perspective of Lusztig's work on total positivity. Geometric and topological properties of the totally nonnegative Grassmannian have garnered a great deal of interest as well. Although matroid strata in a large enough Grassmannian can have arbitrarily bad singularities [Vak06], the positroid strata of the totally nonnegative part are all expected to be well-behaved; for evidence towards this claim, see work by Fomin-Shapiro [FS00], Williams [Wil05], Postnikov [Pos07], Postnikov-Speyer-Williams [PSW09], and Hersh [Her10]. Speyer and Williams have also investigated the tropicalization of the totally nonnegative part of the Grassmannian [SW05].

Scott has shown that every Grassmannian can be given the structure of a cluster algebra [Sco06]. The set of cluster variables includes the Plücker coordinates, together with some much more complicated functions. Every Grassmannian has clusters consisting entirely of Plücker coordinates.

Postnikov [Pos07] has provided the foundation for a combinatorial approach to totally nonnegative Grassmannians which is similar in spirit to that found in the study of totally positive matrices. This required developing an analogue of the weighted path matrix for planar networks which are not necessarily acyclic. These *boundary measurement matrices* represent points in the totally nonnegative part of the Grassmannian, and furthermore, every such point comes from a planar network [Pos07]. A critical step in Postnikov's construction is a proof that every Plücker coordinate of a boundary measurement matrix is a ratio of two polynomials in the edge weights of the network, each with positive coefficients. Though his proof is recursive, the fact that the Plücker coordinates can be written in this form suggests that explicit combinatorial formulas may be found.

There are two key results in this dissertation. In Chapter 4, we provide combinatorial formulas of the kind alluded to above, that is, explicit subtraction-free formulas for the Plücker coordinates of the boundary measurement matrices of networks which are not necessarily acyclic. This gives a constructive proof that such networks yield points in the totally nonnegative Grassmannian.

In Chapter 5, we go in reverse, providing a constructive proof that every point in the totally nonnegative Grassmannian is realized by some network. We explicitly describe how to construct the underlying graph of a network and give combinatorial formulas for its weights. The result gives a bijection between points in the totally nonnegative Grassmannian and certain planar networks, called Γ-*networks*. These Γ-networks are in obvious bijection with certain fillings of Young diagrams, called *Le-tableaux*. As a result, we are able to give a parametrization of the totally nonnegative Grassmannian using very simple combinatorial objects.

CHAPTER 3

Positivity in real Grassmannians

In this chapter, we will present Postnikov's construction assigning a point in the Grassmannian to each planar network satisfying certain properties. This will allow us to carefully state two key results from [Pos07], which are summarized informally below:

- the construction always produces points in the *totally nonnegative Grassmannian*, i.e. the part of the Grassmannian in which all Plücker coordinates have the same sign (see Theorem 3.10 and Corollary 3.11 for details), and
- every point in the totally nonnegative Grassmannian can be realized using this construction (see Theorem 3.12 and its refinement, Theorem 3.14).

Note that these results are the analogues of Corollary 2.8 and Theorem 2.9 for the Grassmannian setting. Chapters 4 and 5 will provide explicit formulas, completing the picture for Grassmannians.

3.1 Grassmannians

Let Gr_{kn} denote the appropriate real *Grassmannian*: the space whose points are k-dimensional linear subspaces of $\mathbb{R}^n$. Grassmannians have a great deal of structure; they can be viewed as smooth manifolds or projective varieties, for example, but we will not need such heavy machinery. We are primarily interested in elementary ways to manipulate points in Gr_{kn}. We will describe points in Gr_{kn} in two ways: first using matrix representatives, and then using Plücker coordinates.

The most elementary way to write down an element of Gr_{kn} is as follows. Fix an ordered basis $\{w_1, w_2, \ldots, w_n\}$ of $\mathbb{R}^n$. Suppose that V is the k-dimensional subspace of $\mathbb{R}^n$ such that $V = \mathrm{span}\langle v_1, v_2, \ldots, v_k\rangle$, where for each i, we have

$$v_i = a_{i1}w_1 + a_{i2}w_2 + \cdots + a_{in}w_n.$$

Then the matrix $A = (a_{ij})$ encodes all the information about V, and we call it a *matrix representative of* V.

Example 3.1. Suppose $\{e_1, e_2, e_3, e_4, e_5\}$ is the standard basis for $\mathbb{R}^5$, and let $V = \mathrm{span}\{v_1, v_2\}$, where

$$\begin{aligned} v_1 &= 2e_1 - 4e_2 + 8e_4 + 17e_5, \text{ and} \\ v_2 &= 3e_1 - 6e_2 + 7e_3 + e_4. \end{aligned}$$

Then one matrix representative for W is

$$M = \begin{pmatrix} 2 & -4 & 0 & 8 & 17 \\ 3 & -6 & 7 & 1 & 0 \end{pmatrix},$$

with rows indexed by the v_i and columns indexed by the e_i.

We can also write $V = \mathrm{span}\{v_1, 2v_2 - 3v_1\}$, which yields another matrix representative for V:

$$M' = \begin{pmatrix} 2 & -4 & 0 & 8 & 17 \\ 0 & 0 & 14 & -22 & -51 \end{pmatrix}.$$

The matrix point of view is useful because we can easily manipulate matrix representatives. On the other hand, matrix representatives are quite far from unique. If

$M \in GL_n(\mathbb{R})$, then left multiplication of A by M is equivalent to a change of basis for V, so that the every matrix in the collection $\{MA : M \in GL_n(\mathbb{R})\}$ is a matrix representative for the same subspace.

A point $x \in \mathrm{Gr}_{kn}$ can also be described by a collection of (projective) *Plücker coordinates* $(P_J(x))$, indexed by k-element subsets $J \subset [n]$. These Plücker coordinates satisfy certain quadratic relations, called *Plücker relations*. The simplest Plücker relations are *three-term Plücker relations*. They take the form

$$P_{S\cup\{a,c\}}P_{S\cup\{b,d\}} = P_{S\cup\{a,b\}}P_{S\cup\{c,d\}} + P_{S\cup\{a,d\}}P_{S\cup\{b,c\}},$$

where $a < b < c < d$, and S is a set which does not contain any of the elements a, b, c, d.

The Plücker coordinates can be easily obtained from a matrix representative; they are simply the $k \times k$ minors of our $k \times n$ matrix representative, and the subsets J simply tell us which columns to choose. In the other direction, it is also possible to construct explicit matrix representatives given a valid collection of Plücker coordinates, but this is not quite as simple.

Example 3.2. Let us find the Plücker coordinates for the matrix representatives M and M' given for V in Example 3.1.

J	$1,2$	$1,3$	$1,4$	$1,5$	$2,3$	$2,4$	$2,5$	$3,4$	$3,5$	$4,5$
$P_J(M)$	0	14	-22	-51	-28	44	102	-56	-119	-17
$P_J(M')$	0	28	-44	-102	-56	88	204	-112	-238	-34

Notice that for every J, we have $P_J(M') = 2P_J(M)$. This is because the change of basis matrix producing M' from M, shown below, has determinant 2.

$$\begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 2v_2 - 3v_1 \end{pmatrix}$$

This is an explicit example of the fact that different matrix representatives for V will yield the same collection of Plücker coordinates, up to multiplication by a common scalar.

Using Plücker coordinates, we can give a simple definition for the main object we will study in this chapter.

Definition 3.3. The *totally nonnegative Grassmannian* $(\mathrm{Gr}_{kn})_{\geq 0}$ is the subset of points $x \in \mathrm{Gr}_{kn}$ whose nonzero Plücker coordinates all have the same sign. That is,

$$(\mathrm{Gr}_{kn})_{\geq 0} = \left\{ x \in \mathrm{Gr}_{kn} : \frac{P_J(x)}{P_{J'}(x)} \geq 0 \text{ whenever } P_{J'}(x) \neq 0 \right\}.$$

In other words, a point $x \in \mathrm{Gr}_{kn}$ is *totally nonnegative* if and only if it has a matrix representative for which all $k \times k$ minors are nonnegative.

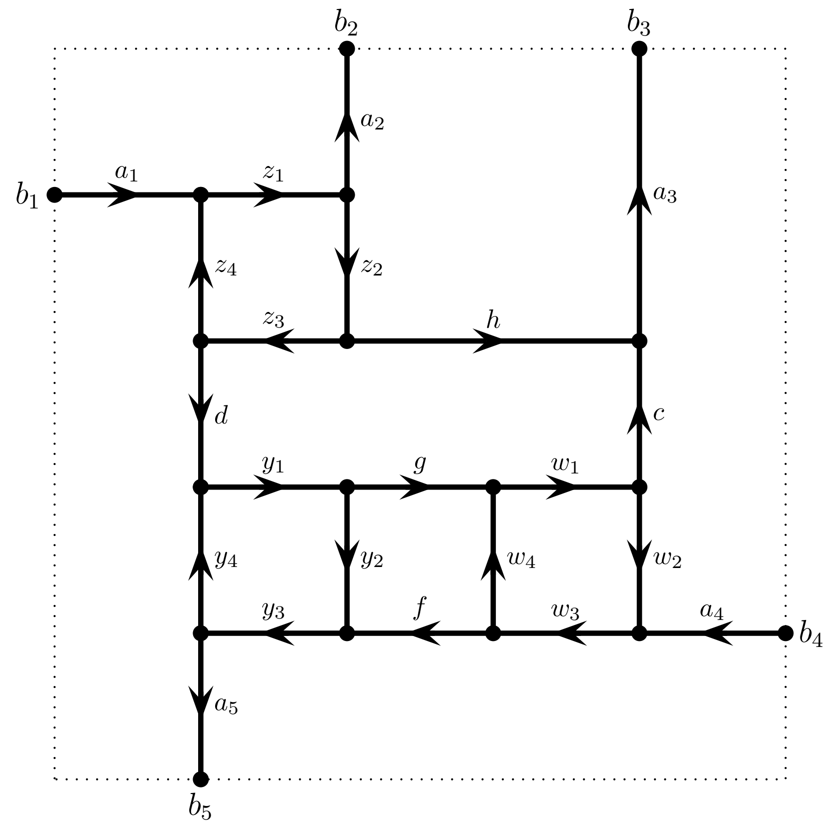
3.2 The boundary measurement matrix of a planar circular network

Definition 3.4. A *planar circular directed graph* is a finite directed graph G properly embedded in a closed oriented disk (so that its edges intersect only at the appropriate vertices), together with a distinguished labeled subset $\{b_1 \ldots, b_n\}$ of *boundary vertices* such that

- $b_1, \ldots, b_n$ appear in clockwise order around the boundary of the disk,
- all other vertices of G lie in the interior of the disk, and
- each boundary vertex b_i is incident to at most one edge.

A non-boundary vertex in G is called an *interior vertex*. Loops and multiple edges are permitted. Each boundary vertex is designated a source or a sink, even if it is an isolated vertex. We denote by $I \subseteq [n] = \{1, \ldots, n\}$ the indexing set for the boundary sources of G, so that these sources form the set $\{b_i : i \in I\}$.

A *planar circular network* $N = (G, x)$ is a planar circular directed graph G together with a collection $x = (x_e)$ of (commuting) formal variables x_e labeled by the edges e in G. We call x_e the *weight* of e.



The cycles of N have weights
$W = w_1w_2w_3w_4$,
$Y = y_1y_2y_3y_4$,
$Z = z_1z_2z_3z_4$, and
$T = y_1gw_1w_2w_3fy_3y_4$.

Figure 3.1: A planar circular network N with boundary vertices b_1, b_2, b_3, b_4, b_5. Edges are labeled by their weights.

A *walk* $P = (e_1, \ldots, e_m)$ in G is formed by traversing the edges $e_1, e_2, \ldots, e_m$ in the specified order. (The head of e_i is the tail of e_{i+1}.) We write $P : u \rightsquigarrow v$ to indicate that P is a walk starting at a vertex u and ending at a vertex v.

Define the *weight* of a walk $P = (e_1, \ldots, e_m)$ to be

$$\mathrm{wt}(P) = x_{e_1} \cdots x_{e_m}.$$

A walk $P : u \rightsquigarrow u$ with no edges is called a *trivial walk* and has weight 1.

Definition 3.5 ([Pos07])**.** Let $P : u \rightsquigarrow v$ be a non-trivial walk in a planar circular directed graph G connecting boundary vertices u and v. Performing an isotopy if necessary, we may assume that the tangent vector to P at u has the same direction as the tangent vector to P at v. The *winding index* $\mathrm{wind}(P)$ is the signed number of full $360°$ turns the tangent vector makes as we travel along P, counting counterclockwise turns as positive and clockwise turns as negative. For a trivial walk P, we set $\mathrm{wind}(P) = 0$.

Definition 3.6 ([Pos07])**.** For boundary vertices b_i and b_j in a planar circular network N, the *(formal) boundary measurement* M_{ij} is the formal power series

$$M_{ij} = \sum_{P : b_i \rightsquigarrow b_j} (-1)^{\mathrm{wind}(P)} \, \mathrm{wt}(P), \tag{3.1}$$

the sum over all directed walks $P : b_i \rightsquigarrow b_j$.

Example 3.7. In the circular network N shown in Figure 3.1, any walk P from b_1 to b_2 consists of the edges with weights a_1, z_4, a_2, together with some number of repetitions of the cycle of weight $Z = z_1 z_2 z_3 z_4$. Consequently,

$$M_{12} = a_1 z_4 a_2 - a_1 Z z_4 a_2 + a_1 Z^2 z_4 a_2 - a_1 Z^3 z_4 a_2 + \ldots = \frac{a_1 z_4 a_2}{1 + Z}.$$

Definition 3.8 ([Pos07])**.** Let N be a planar circular network, and suppose that $I = \{i_1 < \cdots < i_k\}$, so that the boundary sources, listed in clockwise order, are $b_{i_1}, b_{i_2}, \ldots, b_{i_k}$. The *boundary measurement matrix* $A(N) = (a_{tj})$ is the $k \times n$ matrix defined by

$$a_{tj} = (-1)^{s(i_t, j)} M_{i_t j}\,,$$

where $s(i_t, j)$ denotes the number of elements of I strictly between i_t and j.

Let $\Delta_J(A(N))$ denote the $k \times k$ minor of $A(N)$ whose columns are indexed by the subset $J \in \binom{[n]}{k}$. That is, $\Delta_J(A(N)) = \det(a_{tj})_{t\in[1,k],j\in J}$. When no confusion will arise, we may simply write Δ_J.

With this notation, we can rephrase our definition as follows. The matrix $A(N)$ is the unique matrix which has an identity submatrix in the I columns and which has entries $a_{tj} = \pm M_{i_t j}$ in the remaining columns, with signs chosen so that for each $t \in [k]$ and $j \in [n] \setminus I$, we have $\Delta_{I\setminus\{i_t\}\cup\{j\}} = M_{i_t j}$.

Example 3.9. Suppose that N is the planar circular network in Figure 3.1. Then the boundary source set is indexed by $I = \{1, 4\}$, and we have

$$A(N) = \begin{pmatrix} 1 & M_{12} & M_{13} & 0 & -M_{15} \\ 0 & M_{42} & M_{43} & 1 & M_{45} \end{pmatrix}.$$

The 10 minors $\Delta_J(A(N))$ are listed below.

$\Delta_{\{1,2\}}(A(N)) = M_{42}$ $\quad$ $\Delta_{\{2,4\}}(A(N)) = M_{12}$

$\Delta_{\{1,3\}}(A(N)) = M_{43}$ $\quad$ $\Delta_{\{2,5\}}(A(N)) = M_{12}M_{45} + M_{15}M_{42}$

$\Delta_{\{1,4\}}(A(N)) = 1$ $\quad$ $\Delta_{\{3,4\}}(A(N)) = M_{13}$

$\Delta_{\{1,5\}}(A(N)) = M_{45}$ $\quad$ $\Delta_{\{3,5\}}(A(N)) = M_{13}M_{45} + M_{15}M_{43}$

$\Delta_{\{2,3\}}(A(N)) = M_{12}M_{43} - M_{13}M_{42}$ $\quad$ $\Delta_{\{4,5\}}(A(N)) = M_{15}$

Theorem 3.10 ([Pos07]). *If $N = (G, x)$ is a planar circular network, then each maximal minor $\Delta_J(A(N))$ of the boundary measurement matrix can be written as a subtraction-free rational expression in the edge weights x_e.*

We have already seen Theorem 3.10 in action. Example 3.7 shows that the infinite formal power series for the boundary measurement M_{12}, which is precisely the minor

$\Delta_{\{2,4\}}$, can be rewritten as a subtraction-free rational expression in the edge weights of the network.

Postnikov's proof of Theorem 3.10 is inductive. In Section 4.1, we give an explicit combinatorial formula for the boundary measurements in a planar circular network, providing a constructive proof.

Since each minor $\Delta_J(A(N))$ can be written as a subtraction-free rational expression, it is now possible to consider networks with positive real weights. To make this precise, let $E(G)$ denote the edge set of G, and suppose that $\alpha : \{x_e\}_{e\in E(G)} \to \mathbb{R}^+$ assigns a positive real weight α_e to each edge e. That is, $x_e \mapsto \alpha_e$. Then $\Delta_J(A(N))|_\alpha$ denotes the evaluation of the subtraction-free rational expression $\Delta_J(A(N))$ under the specialization $x_e = \alpha_e$ for all $e \in E(G)$. Let $M_{ij}|_\alpha$ denote the specialization of the corresponding minor. Each entry of $A(N)$ is some M_{ij}, so we can also take the specialization $A(N)|_\alpha$.

Corollary 3.11. *If $\alpha : \{x_e\}_{e\in E(G)} \to \mathbb{R}^+$ is a positive specialization map, then each $\Delta_J(A(N))|_\alpha$ is a nonnegative real number. In particular, each $M_{ij}|_\alpha$ is nonnegative. Thus, $\left(\Delta_J(A(N))|_\alpha : J \in \binom{[n]}{k}\right)$ is a point in the totally nonnegative Grassmannian $(\mathrm{Gr}_{kn})_{\geq 0}$, given by its Plücker coordinates.*

Just as every totally nonnegative matrix is the weighted path matrix of some planar acyclic network, we see in Theorem 3.12 that every point in the totally nonnegative Grassmannian has a matrix representative which is the boundary measurement matrix of some planar circular network.

Theorem 3.12 ([Pos07])**.** *For every $V \in (\mathrm{Gr}_{kn})_{\geq 0}$, there exists a planar circular network $N = (G, x)$ and a positive specialization map $\alpha : E(G) \to \mathbb{R}^+$ such that the Plücker coordinates of V are given by $\left(\Delta_J(A(N))|_\alpha : J \in \binom{[n]}{k}\right)$.*

In fact, Postnikov shows much more, providing a (CW complex) decomposition of the totally nonnegative Grassmannian into *positroid cells*, each defined by its vanishing pattern of Plücker coordinates. Furthermore, there is a bijection between certain planar networks, called Γ-networks, and points of the totally nonnegative Grassmannian. Sections 3.3 and 3.4 will provide further details on this decomposition, and the bijection will be made explicit in Chapter 5.

3.3 Le-diagrams and Γ-networks

In this section, we will describe a special subset of planar circular directed networks called Γ-networks. These networks are relatively simple, in that they are acyclic and can be described by purely combinatorial information. Every point in the totally nonnegative Grassmannian can be realized as the boundary measurement matrix of a Γ-network, and the structure of this bijection will be discussed in Section 3.4.

Definition 3.13. A *Le-diagram* is a partition λ together with a filling of the boxes of the Young diagram of λ with entries 0 and + satisfying the Le-property: there is no 0 which has both a + above it (in the same column) and a + to its left (in the same row).

Replacing the boxes labeled + in a Le-diagram with positive real numbers, called *Le-coordinates*, we obtain a *Le-tableau.* Let $\mathbf{T}_L$ denote the set of Le-tableaux whose vanishing pattern is given by the Le-diagram L. Note that $\mathbf{T}_L$ is an affine space whose dimension is equal to the number of "+" entries in L, which we denote by $|L|$.

For a box B in λ, we let L_B and T_B denote the labels of the box B in the Le-diagram L and the Le-tableau T, respectively.

For each Le-diagram L of shape λ which fits inside a $k \times (n-k)$ rectangle, we

+	0	0	0	0	0	0
0	0	0	+	0	0	0
0	+	0	+	+	0	+
0	0	+	+	+	0	
+	+	+	+			

T_{17}	0	0	0	0	0	0
0	0	0	T_{24}	0	0	0
0	T_{36}	0	T_{34}	T_{33}	0	T_{31}
0	0	T_{45}	T_{44}	T_{43}	0	
T_{57}	T_{56}	T_{55}	T_{54}			

Figure 3.2: A Le-diagram L of shape $\lambda = (7, 7, 7, 6, 4)$ and a Le-tableau $T \in \mathbf{T}_L$.

will construct a Γ-graph G_L corresponding to L. For each Le-tableau $T \in \mathbf{T}_L$, we will then assign weights to the faces of G_L to obtain a Γ-network N_T.

We begin by establishing the boundary of the planar network. First, we draw a disk whose boundary consists of the north and west edges of the $k \times (n-k)$ box and the path determining the southeast boundary of λ, all shifted slightly northwest. Place a vertex, called a *boundary source*, at the right end of each row (including empty rows) of λ, and a vertex, called a *boundary sink*, at the bottom of each column of λ (including empty columns). Label these in sequence with the integers $\{1, 2, \ldots, n\}$, following the path from the northeast corner to the southwest corner which determines λ. Let $I = \{i_1 < i_2 < \cdots < i_k\} \subset [n]$ be the set of boundary sources, so that the complement of I, $[n] \setminus I = \{j_1 < j_2 < \cdots < j_{n-k}\}$, is the set of boundary sinks.

Whenever $L_B = +$, we draw the B *hook*, i.e., the hook whose corner is the northwest corner of the box $B = (r, c)$ (in the r^{th} row from the top and the c^{th} column from the right) and which has a horizontal path directed from the boundary source i_r to the corner and a vertical path directed from the corner to the boundary sink j_c. This process yields a Γ-graph G_L.

To obtain the Γ-network N_T from G_L, we would expect to assign weights to each of the edges of G_L. Instead, we will assign weights to the *faces* of G_L; the rationale for this will be explained immediately afterwards.

Note that there is exactly one face for each box B in λ satisfying $L_B = +$ (and this face has a portion of the B hook as its northwest boundary), and in addition, there is one face whose northwest boundary is the boundary of the disk. For each box B with $L_B = +$, we assign to the corresponding face the positive real weight T_B. To the face whose northwest boundary is the boundary of the disk, we assign the weight $\prod \frac{1}{T_B}$, taking the product over boxes satisfying $L_B = +$, so that the product of all face weights in N is exactly 1.

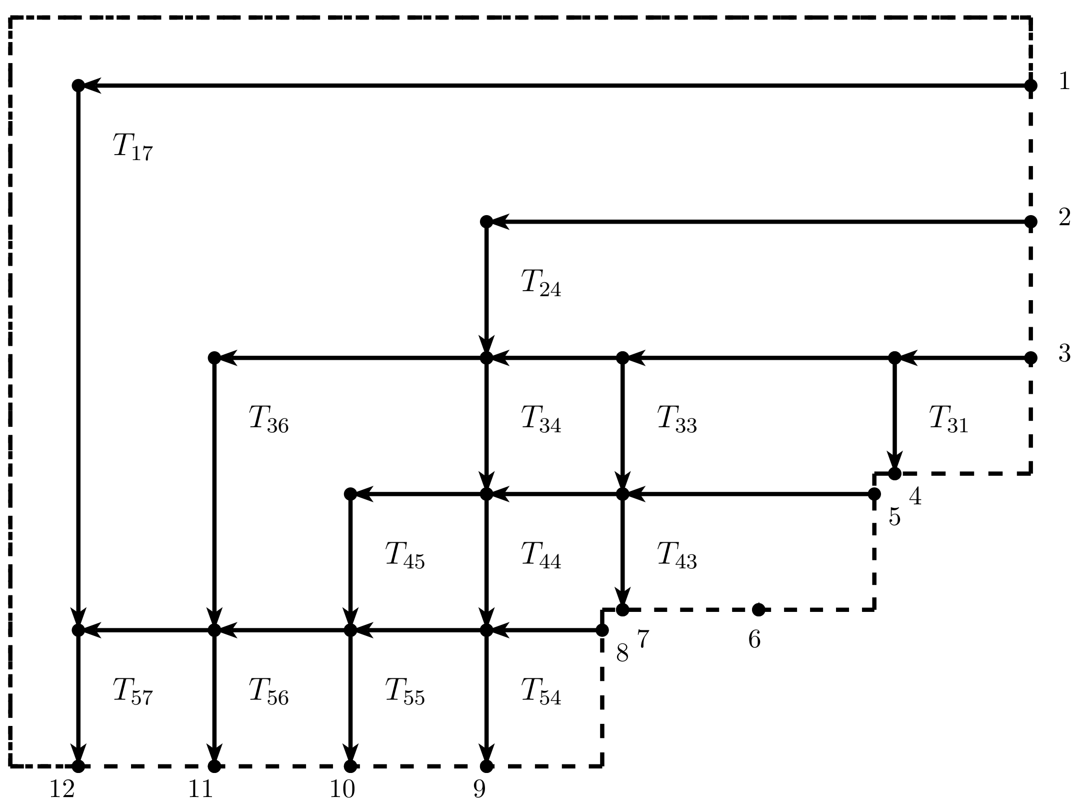


Figure 3.3: Face weights for the Γ-network N_T of the Le-tableau T given in Figure 3.2.

Our previous constructions used weights on the edges of the network. However, there are too many degrees of freedom when using edge weights. One way to deal with this is to fix the weights of certain edge weights. For example, in Γ-networks, we can assume all vertical edges have weight 1, but assign variables to the horizontal edges. (See the discussion on gauge transformations in [Pos07] to see why there is no loss of generality with this restriction.) Another is to use Postnikov's transformation from edge weights to face weights. This can be done for arbitrary planar circular networks (see [Pos07] for details), but we will only present the special case of Γ-networks.

Given a face R in a Γ-network N_T with edge weights $\{x_e\}$, let $NW(R)$ denote the set of edges forming the northwest boundary of R, and let $SE(R)$ denote the set of edges forming the southeast boundary of R. Define the weight of the face R to be the following ratio of edge weights:

$$y_R = \frac{\prod_{e \in NW(R)} x_e}{\prod_{e \in SE(R)} x_e}.$$

Under the assumption that the vertical edges in N_T each have weight 1, this is a birational map from edge weights to face weights. That is, we can easily recover the weights of horizontal edges from the face weights. In fact, the weight of an edge will be the product of the weights of the faces lying directly south of that edge.

With this equivalence in mind, we will typically use face weights when working with Γ-networks. The weight of a walk P between two boundary vertices in a Γ-network is

$$\mathrm{wt}(P) = \prod y_R,$$

with the product taken over all faces lying southeast of P. This notion of the weight of a walk is equivalent to the edge weight definition. It is easiest to see this with an example. See Figures 3.3 and 3.4 for an example of a network in terms of face weights and the corresponding edge weights, respectively.

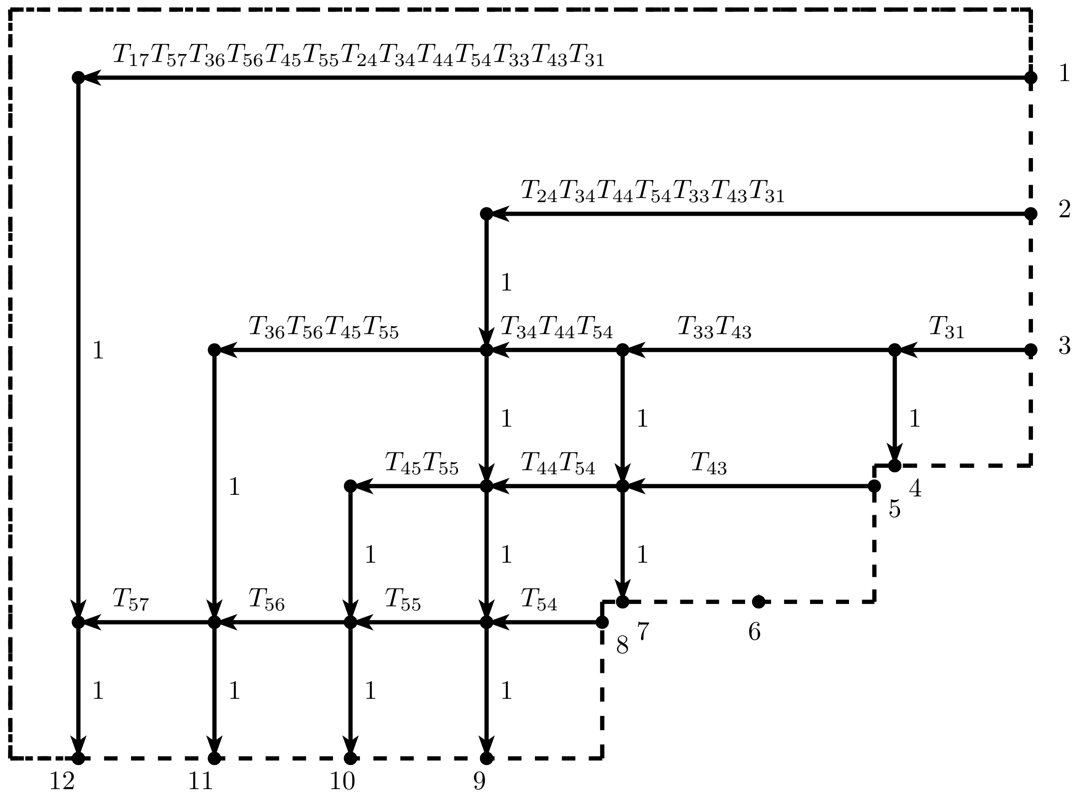


Figure 3.4: A set of edge weights corresponding to the face weights in Figure 3.3.

3.4 The positroid stratification of $(\mathrm{Gr}_{kn})_{\geq 0}$

In [GGMS87], the authors give a decomposition of the Grassmannian Gr_{kn} into *matroid strata*. Each stratum satisfies the property that certain Plücker coordinates are zero for all points in the stratum, and the remaining Plücker coordinates are all nonzero. More precisely, for a matroid $\mathcal{M}$ whose bases are k-element subsets of $[n]$, let $S_{\mathcal{M}}$ denote the stratum consisting of precisely the points $x \in \mathrm{Gr}_{kn}$ such that $P_J(x) \neq 0$ if and only if $J \in \mathcal{M}$. In particular, each possible vanishing pattern of Plücker coordinates is given by a unique (realizable) matroid $\mathcal{M}$. In [Pos07], Postnikov studies a natural analogue of the matroid stratification for the totally nonnegative Grassmannian, a decomposition into disjoint *positroid cells* of the form $(S_{\mathcal{M}})_{\geq 0} = S_{\mathcal{M}} \cap (\mathrm{Gr}_{kn})_{\geq 0}$.

In the positroid cell decomposition of $(\mathrm{Gr}_{kn})_{\geq 0}$ given in [Pos07], the positroid cells are indexed by the Le-diagrams L which fit inside a $k \times (n-k)$ rectangle. Furthermore, the positroid cell corresponding to a fixed Le-diagram L is parametrized by the Le-tableaux $T \in \mathbf{T}_L$, that is, those whose vanishing pattern is given by L.

Recall that for a Le-diagram L, we have the corresponding Γ-graph G_L with source set I. Define the *positroid* $\mathcal{M}_L \subseteq \binom{[n]}{k}$ by the condition that $J \in \mathcal{M}_L$ if and only if there exists a non-intersecting path collection in G_L with sources I and destinations J. It can be shown that $\mathcal{M}_L$ has the structure of a matroid, but this is not necessary for our purposes. It is easily verified that for distinct Le-diagrams L and L^*, we have $\mathcal{M}_L \neq \mathcal{M}_{L^*}$.

Just as realizable matroids characterize the vanishing patterns of points in the full Grassmannian, positroids give the vanishing patterns of points in the totally nonnegative Grassmannian. Of course, each positroid $\mathcal{M}_L$ is a realizable matroid, but there do exist realizable matroids which are not positroids.

In [Pos07], the map Meas_L takes a Le-tableau T with vanishing pattern given by L to the point in the totally nonnegative Grassmannian given by the boundary measurement matrix of its corresponding Γ-network N_T.

Theorem 3.14. *[Pos07] For each Le-diagram L which fits in a $k \times (n-k)$ rectangle, the map $\mathrm{Meas}_L : \mathbf{T}_L \to (\mathrm{Gr}_{kn})_{\geq 0}$ is injective, and the image $\mathrm{Meas}_L(\mathbf{T}_L)$ is precisely the positroid cell $(S_{\mathcal{M}_L})_{\geq 0}$.*

These positroid cells are pairwise disjoint, and the union $\bigcup_L (S_{\mathcal{M}_L})_{\geq 0}$, taken over all Le-diagrams L which fit inside the $k \times (n-k)$ rectangle, is the entire totally nonnegative Grassmannian $(\mathrm{Gr}_{kn})_{\geq 0}$. Each positroid cell $(S_{\mathcal{M}_L})_{\geq 0}$ is a topological *cell; that is,* $(S_{\mathcal{M}_L})_{\geq 0}$ *is isomorphic to* $\mathbb{R}^{|L|}$*, where* $|L|$ *is the number of "+" entries in* L*. Thus, the positroid cells form a cell decomposition of* $(\mathrm{Gr}_{kn})_{\geq 0}$.

In [Pos07], Postnikov's boundary measurement map takes a planar circular network N with a positive specialization map α to the point in the totally nonnegative Grassmannian corresponding to the specialized boundary measurement matrix $A(N)|_\alpha$. In the special case of Γ-networks, we can give formulas for the minors $\Delta_J(A(N))|_\alpha$ using the classical formula of Lindström [Lin73]. (General formulas are given in Chapter 5.) Although Lindström's formula is usually given in terms of weights of edges, we may also view the formula as a polynomial in face weights.

Definition 3.15. For each Le-diagram L which fits in a $k \times (n-k)$ rectangle, the *boundary measurement map* $\mathrm{Meas}_L : \mathbf{T}_L \to (\mathrm{Gr}_{kn})_{\geq 0}$ is defined by

$$P_J(\mathrm{Meas}_L(T)) = \sum_{F \in \mathcal{F}_J(N_T)} \mathrm{wt}(F),$$

where

- N_T is the Γ-network corresponding to the Le-tableau T, and its boundary source set is labeled by I,
- $\mathcal{F}_J(N_T)$ is the collection of non-intersecting directed path families $F = \{F_i\}_{i \in I}$ in N_T from the boundary sources I to the boundary destinations J, and
- $\mathrm{wt}(F) = \prod_{i \in I} \mathrm{wt}(F_i)$.

We note that the destination set J may contain both sources and sinks, so that I and J may overlap, in which case some of the paths in the collection will have zero edges.

CHAPTER 4

Formulas for the boundary measurement map

This chapter contains the statements and proofs of explicit combinatorial formulas for the Plücker coordinates of the boundary measurement matrix of a planar circular network.

4.1 Statements of the formulas

We will first present the case of planar circular networks which are *perfectly oriented*. Perfectly oriented networks form a particularly nice class of examples, and every planar circular network can be transformed into a perfectly oriented one using a few simple local moves.

The perfectly oriented case

Definition 4.1 ([Pos07])**.** A planar circular directed graph (or network) is said to be *perfectly oriented* if every interior vertex either has exactly one outgoing edge (with all other edges incoming) or exactly one incoming edge (with all other edges outgoing).

For example, let G be a circular directed graph in which all interior vertices are trivalent, with no interior sources or sinks. Then G is perfectly oriented. Such a graph is shown in Figure 3.1; this will serve as our running example in the perfectly oriented case.

Note that in a perfectly oriented circular graph, any self-intersecting walk between boundary vertices must repeat at least one edge at every point of self-intersection.

Recall that $I = \{i_1 < \cdots < i_k\}$ indexes the set of boundary sources of G.

Definition 4.2. A subset F of (distinct) edges in a graph G is called a *flow* if, for each interior vertex v in G, the number of edges of F that arrive at v is equal to the number of edges of F that leave from v.

For our perfectly oriented planar circular directed graphs, this means that each flow is a union of k non-intersecting self-avoiding walks, each connecting a boundary source b_i $(i \in I)$ to a distinct boundary vertex b_j $(j \in J)$, together with a (possibly empty) collection of pairwise disjoint cycles, none of which intersect any of the walks.

A flow C is *conservative* if it contains no edges incident to the boundary. We denote by $\mathcal{C}(G)$ the set of all conservative flows in G.

Let J be a k-element subset of $[n]$. We say that a flow F is a *flow from I to J* if each boundary source b_i is connected by a walk in F to a (necessarily unique) boundary vertex b_j with $j \in J$. If G is perfectly oriented, we denote by $\mathcal{F}_J(G)$ the set of all flows from I to J.

The *weight* of a flow F, denoted $\mathrm{wt}(F)$, is by definition the product of the weights of all edges in F. A flow with no edges has weight 1.

We note that each flow of a perfectly oriented planar circular directed graph G lies in precisely one of the sets $\mathcal{F}_J(G)$. In particular, for a conservative flow, each of the k walks between boundary vertices is trivial, and $\mathcal{C}(G) = \mathcal{F}_I(G)$.

We can now precisely state the key theorem in the perfectly oriented case.

Theorem 4.3. *Let $N = (G, x)$ be a perfectly oriented planar circular network. Then* *the maximal minors of the boundary measurement matrix* $A(N)$ *are given by*

$$\Delta_J(A(N)) = \frac{\displaystyle\sum_{F \in \mathcal{F}_J(G)} \mathrm{wt}(F)}{\displaystyle\sum_{C \in \mathcal{C}(G)} \mathrm{wt}(C)}. \tag{4.1}$$

This is the subtraction-free rational expression we will use when evaluating at a positive specialization of the edge weights. Recall Corollary 3.11, which states that each minor will be nonnegative if we choose positive edge weights. Viewing the $\Delta_J(A(N))$ as Plücker coordinates, we then obtain a point in the totally nonnegative Grassmannian $(\mathrm{Gr}_{kn})_{\geq 0}$.

Example 4.4. Consider the planar circular network N in Figure 3.1, in which $I = \{1,4\}$. For $J = \{1,5\}$, let us describe the set of flows $\mathcal{F}_J(G)$. The boundary vertex b_1 must be connected to itself by the trivial walk $b_1 \rightsquigarrow b_1$. Together with the unique self-avoiding walk $P : b_4 \rightsquigarrow b_5$ of weight $a_4 w_3 f y_3 a_5$, this gives a flow from $\{1,4\}$ to $\{1,5\}$. There is one additional flow, consisting of P and the cycle of weight $Z = z_1 z_2 z_3 z_4$ (along with the trivial walk $b_1 \rightsquigarrow b_1$). See Figure 4.1.

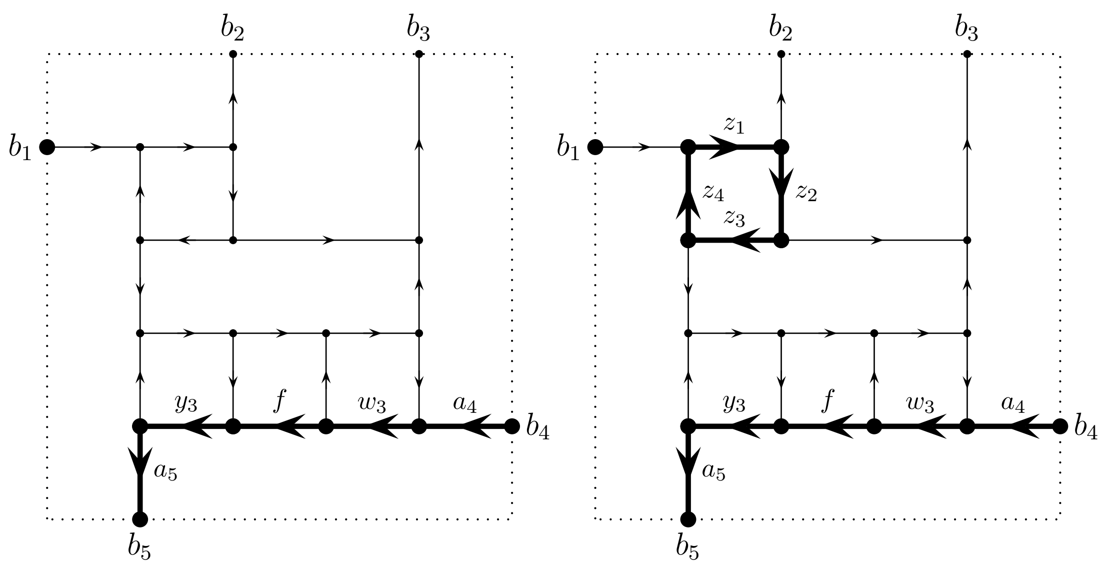


Figure 4.1: The two flows in $\mathcal{F}_{1,5}(G)$, shown in bold.

Thus, for this particular network, the numerator of (4.1) is

$$\sum_{F\in\mathcal{F}_{\{1,5\}}(G)} \mathrm{wt}(F) = a_4 w_3 f y_3 a_5 (1+Z).$$

The only cycles in the network N are those of weights W, Y, Z, and T. Since conservative flows in G are unions of disjoint cycles, we have

$$\begin{aligned}\sum_{C\in\mathcal{C}(G)} \mathrm{wt}(C) &= 1+W+Y+Z+T+WZ+WY+YZ+WYZ+ZT \\ &= (1+Z)[(1+W)(1+Y)+T].\end{aligned}$$

Consequently,

$$\Delta_{\{1,5\}}(A(N)) = \frac{a_4 w_3 f y_3 a_5 (1+Z)}{(1+Z)[(1+W)(1+Y)+T]} = \frac{a_4 w_3 f y_3 a_5}{(1+W)(1+Y)+T}.$$

The general case

In this section, we provide an extension of Theorem 4.3 for arbitrarily oriented planar circular networks.

When G is a perfectly oriented graph, the following definition is equivalent to Definition 4.2. This extension provides the appropriate setup for working in arbitrarily oriented planar circular directed graphs.

Definition 4.5. A subset F of (distinct) edges in a planar circular directed graph G (not necessarily perfectly oriented) is called an *alternating flow* if, for each interior vertex v in G, the edges $e_1, \ldots, e_d$ of F which are incident to v, listed in clockwise order around v, alternate in orientation (that is, directed towards v or directed away from v).

In an alternating flow F, define the walks W_i (with $i \in I$) as follows. If b_i is isolated in F, set W_i to be the trivial walk from b_i to itself. Otherwise, let W_i be the unique path leaving b_i which, at each subsequent vertex, takes the first left turn in F, until it arrives at another boundary vertex.

For a k-element subset J of $[n]$, we say that an alternating flow F is a *flow from I to J* if each boundary source b_i is connected by W_i to a boundary vertex b_j with $j \in J$. (The vertices b_j are necessarily distinct.) Let $\mathcal{A}_J$ denote the set of alternating flows from I to J. In particular, $\mathcal{A}_I$ is precisely the set of conservative alternating flows.

Note that in the special case of perfectly oriented networks, every flow is an alternating flow.

Definition 4.6. Suppose F is an alternating flow. For each vertex v in G, let $\tau(v, F)$ denote the number of edges of F coming into v. Set

$$\theta(F) = \sum_v \max\{\tau(v, F) - 1, 0\}.$$

Theorem 4.7. *Let $N = (G, x)$ be a planar circular network with source set indexed by I. Then the maximal minors of the boundary measurement matrix $A(N)$ are given by the formula*

$$\Delta_J(A(N)) = \frac{\displaystyle\sum_{F \in \mathcal{A}_J(G)} 2^{\theta(F)} \operatorname{wt}(F)}{\displaystyle\sum_{C \in \mathcal{A}_I(G)} 2^{\theta(C)} \operatorname{wt}(C)}. \tag{4.2}$$

Corollary 3.11 applies here as well; if we choose positive edge weights, then each minor is nonnegative, and we obtain a point in the totally nonnegative Grassmannian $(\mathrm{Gr}_{kn})_{\geq 0}$.

4.2 Some technical lemmas

Definition 4.8 ([Pos07])**.** Let $\pi : I \to J$ be a bijection such that $\pi(i) = i$ for all $i \in I \cap J$. A pair of indices (i_1, i_2), where $\{i_1 < i_2\} \subset I \setminus J$, is called a *crossing*, an *alignment*, or a *misalignment* of π, if the two directed chords $[b_{i_1}, b_{\pi(i_1)}]$ and $[b_{i_2}, b_{\pi(i_2)}]$

are arranged with respect to each other as shown in Figure 4.2. Define the *crossing number* $\operatorname{xing}(\pi)$ of π as the number of crossings of π.

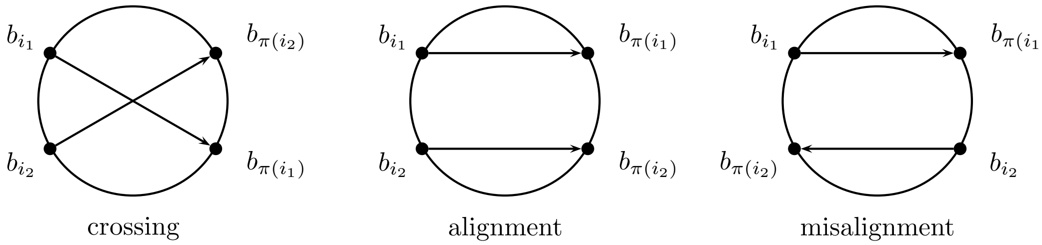


Figure 4.2: Crossings, alignments, and misalignments

Lemma 4.9. *For distinct $i_1, i_2, j_1, j_2 \in [n]$, the chords $[b_{i_1}, b_{j_1}]$ and $[b_{i_2}, b_{j_2}]$ cross if and only if*

$$(i_1 - j_2)(j_2 - j_1)(j_1 - i_2)(i_2 - i_1) < 0. \tag{4.3}$$

Proof. The proof is a straightforward verification. Without loss of generality, we may assume that $i_1 < i_2$, so that $i_2 - i_1$ is always positive. The following table summarizes the possibilities in this case.

$i_1 - j_2$	$j_2 - j_1$	$j_1 - i_2$		
+	+	+	$i_2 > i_1 > j_2 > j_1 > i_2$	impossible
+	+	-	$i_2 > i_1 > j_2 > j_1$	crossing
+	-	+	$j_1 > i_2 > i_1 > j_2$	crossing
+	-	-	$i_2 > i_1 > j_2$ and $i_2 > j_1 > j_2$	(mis)alignment
-	+	+	$j_2 > j_1 > i_2 > i_1$	crossing
-	+	-	$j_2 > i_1, j_1$ and $i_2 > i_1, j_1$	(mis)alignment
-	-	+	$j_1 > i_2 > i_1$ and $j_1 > j_2 > i_1$	(mis)alignment
-	-	-	$i_2 > j_1 > j_2 > i_1$	crossing

For each sign pattern, the consequent inequalities ensure that we have crossings precisely when the inequality (4.3) holds. $\square$

Definition 4.8 and Lemma 4.9 immediately yield the following corollary.

Corollary 4.10. *Let $\pi : I \to J$ be a bijection such that $\pi(i) = i$ for all $i \in I \cap J$. For $\{i_1 < i_2\} \subset I \setminus J$, the following are equivalent:*

1. *(i_1, i_2) is a misalignment;*
2. *the chords $[b_{i_1}, b_{i_2}]$ and $[b_{\pi(i_1)}, b_{\pi(i_2)}]$ cross;*
3. *$(i_1 - \pi(i_2))(\pi(i_2) - i_2))(i_2 - \pi(i_1))(\pi(i_1) - i_1) < 0$.*

We provide a new proof of the following result.

Proposition 4.11 ([Pos07]). *Let I index the boundary sources of a planar circular network N and let $J \subseteq [n]$, with $|J| = |I|$. Then*

$$\Delta_J(A(N)) = \sum_{\pi : I \to J} (-1)^{\mathrm{xing}(\pi)} \prod_{i \in I} M_{i,\pi(i)}, \tag{4.4}$$

the sum over all bijections π from I to J.

Proof. Taking the appropriate determinant, we see that

$$\Delta_J(A(N)) = \sum_{\pi : I \to J} (-1)^{\mathrm{inv}(\pi)} \prod_{i \in I} (-1)^{s(i,\pi(i))} M_{i,\pi(i)},$$

where $s(i, \pi(i))$ is the number of elements of I strictly between i and $\pi(i)$, as in Definition 3.8, and $\mathrm{inv}(\pi)$ is the number of inversions of π. Here, an *inversion* of π is a pair (i_1, i_2) with $i_1 < i_2$ and $\pi(i_1) > \pi(i_2)$. Note that $\prod_{i \in I} M_{i,\pi(i)} = 0$ unless $\pi(i) = i$ for all $i \in I \cap J$. Thus, we wish to show that if π fixes the elements in $I \cap J$, then

$$(-1)^{\mathrm{xing}(\pi)} = (-1)^{\mathrm{inv}(\pi)} \prod_{i \in I} (-1)^{s(i,\pi(i))}. \tag{4.5}$$

Consider the right-hand side of (4.5). Each pair (i_1, i_2) with $i_1 < i_2$ contributes a factor of $\operatorname{sgn}((\pi(i_2) - \pi(i_1))$ to $(-1)^{\operatorname{inv}(\pi)}$. Furthermore, i_1 contributes a factor of $\operatorname{sgn}((i_1 - i_2)(i_1 - \pi(i_2)) = -\operatorname{sgn}(i_1 - \pi(i_2))$ to $(-1)^{s(i_2,\pi(i_2))}$, since this product is negative if and only if $\pi(i_2) < i_1 < i_2$. Similarly, i_2 contributes a factor of $\operatorname{sgn}((i_2 - i_1)(i_2 - \pi(i_1))) = -\operatorname{sgn}((i_2 - i_1)(\pi(i_1) - i_2))$ to $(-1)^{s(i_1,\pi(i_1))}$. Thus, the total contribution by the pair (i_1, i_2) is

$$\operatorname{sgn}[(i_2 - i_1)(i_1 - \pi(i_2))(\pi(i_2) - \pi(i_1))(\pi(i_1) - i_2)].$$

Taking the product over all pairs $\{i_1 < i_2\}$, we get $(-1)^{\operatorname{xing}(\pi)}$, by Lemma 4.9. □

Lemma 4.12. *Let $\pi : I \to J$ be a bijection such that $\pi(i) = i$ for all $i \in I \cap J$. For $l, m \in I \setminus J$ with $l < m$, let $s_{\pi(l),\pi(m)}$ denote the transposition of the boundary vertices $b_{\pi(l)}$ and $b_{\pi(m)}$, and let $\pi^* = s_{\pi(l),\pi(m)} \circ \pi$. Then*

$$(-1)^{\operatorname{xing}(\pi^*)} = \begin{cases} (-1)^{\operatorname{xing}(\pi)+1} & \text{if } (l,m) \text{ is a crossing or an alignment;} \\ (-1)^{\operatorname{xing}(\pi)} & \text{if } (l,m) \text{ is a misalignment.} \end{cases}$$

Proof. Applying Lemma 4.9 and simplifying, we obtain:

$$\begin{aligned} (-1)^{\operatorname{xing}(\pi)}(-1)^{\operatorname{xing}(\pi^*)} &= \operatorname{sgn}\left[\prod_{i_1<i_2}(i_1 - \pi(i_2))(\pi(i_2) - \pi(i_1))(\pi(i_1) - i_2)\right] \\ &\quad \cdot \operatorname{sgn}\left[\prod_{i_1<i_2}(i_1 - \pi^*(i_2))(\pi^*(i_2) - \pi^*(i_1))(\pi^*(i_1) - i_2)\right] \\ &= \operatorname{sgn}\left[(l - \pi(m))(\pi(m) - m)(m - \pi(l))(\pi(l) - l)\right], \end{aligned}$$

and the lemma follows from Corollary 4.10. □

4.3 Proofs of the main formulas

Proof of Theorem 4.3

Proof of Theorem 4.3. For a bijection $\pi : I \to J$, let $\mathcal{P}_\pi$ denote the set of all (possibly intersecting) collections of walks $\mathbf{P} = (P_i)_{i \in I}$ connecting I and J in accordance with π:

$$\mathcal{P}_\pi = \{\mathbf{P} = (P_i : b_i \rightsquigarrow b_{\pi(i)})_{i \in I}\}.$$

In view of (3.1) and (4.4), we can rewrite the claim (4.1) as

$$\sum_{C \in \mathcal{C}(G)} \sum_{\pi : I \to J} \sum_{\mathbf{P} \in \mathcal{P}_\pi} \operatorname{wt}(C, \mathbf{P}) = \sum_{F \in \mathcal{F}_J(G)} \operatorname{wt}(F), \tag{4.6}$$

where $\operatorname{wt}(C, \mathbf{P})$, for $\mathbf{P} \in \mathcal{P}_\pi$, is defined by

$$\operatorname{wt}(C, \mathbf{P}) = \operatorname{wt}(C)(-1)^{\operatorname{xing}(\pi)} \prod_{i \in I} (-1)^{\operatorname{wind}(P_i)} \operatorname{wt}(P_i).$$

Note that if C and $\mathbf{P}$ form a flow F from I to J, then $\operatorname{xing}(\pi) = 0$ and $\operatorname{wind}(P_i) = 0$ for all i, so that $\operatorname{wt}(C, \mathbf{P}) = \operatorname{wt}(F)$. Hence (4.6) can be restated as saying that all terms on its left-hand side cancel except for the ones for which C and $\mathbf{P}$ form a flow from I to J. It remains to construct a sign-reversing involution proving this claim. More precisely, we need an involution φ on the set of pairs $(C, \mathbf{P})$ such that

(i) $C \in \mathcal{C}(G)$ is a conservative flow,

(ii) $\mathbf{P}$ is a collection of $k = |I|$ walks connecting I and J, and

(iii) C and $\mathbf{P}$ do *not* form a non-conservative flow.

Furthermore, φ must satisfy $\operatorname{wt}(\varphi(C, \mathbf{P})) = -\operatorname{wt}(C, \mathbf{P})$.

For a pair $(C, \mathbf{P})$ satisfying (i)-(iii), we define $\varphi(C, \mathbf{P}) = (C^*, \mathbf{P}^*)$ as follows. Let $\mathbf{P} = (P_i)_{i \in I} \in \mathcal{P}_\pi$, with $\pi : I \to J$ a bijection. Choose the smallest $i \in I$ such that P_i is not self-avoiding or has a common vertex with C or with some $P_{i'}$ with $i' > i$. (Such an i exists by the assumptions we made regarding $(C, \mathbf{P})$.)

Let $P_i = (e_1, \dots, e_m)$. Choose the smallest q such that the edge e_q lies in C or in some $P_{i'}$ with $i' > i$, or $e_q = e_r$ for some $r > q$.

- If e_q lies in some $P_{i'}$ with $i' > i$, choose the smallest such i'. (This case allows for the possibility that P_i intersects itself or C at e_q.) We will then swap the tails of P_i and $P_{i'}$ as follows. Let $P_{i'} = (h_1, \dots, h_{m'})$, and choose the smallest q' such that $h_{q'} = e_q$. Set $P_i^* = (e_1, \dots, e_{q-1}, e_q = h_{q'}, h_{q'+1}, \dots, h_{m'})$ and $P_{i'}^* = (h_1, \dots, h_{q'-1}, h_{q'} = e_q, e_{q+1}, \dots, e_m)$. Set $\mathbf{P}^* = \mathbf{P} \setminus \{P_i, P_{i'}\} \cup \{P_i^*, P_{i'}^*\}$ and set $C^* = C$. (Note that $q < \min(m, m')$ in this case, so $\mathbf{P}^* \neq \mathbf{P}$.)
- Otherwise we will find the first point along P_i where we can move a cycle from C to P_i or vice versa, as follows. If P_i is not self-avoiding, let ℓ_P be the first cycle that P_i completes. That is, choose the smallest s such that $e_r = e_s$ for some $r < s$; then $\ell_P = (e_r, e_{r+1}, \dots, e_{s-1})$. If P_i is self-avoiding, then set $s = \infty$. If C intersects P_i, choose the smallest t such that e_t occurs in a (necessarily unique) cycle $\ell_C = (l_1, l_2 \dots, l_w)$ in C, where $l_1 = e_t$. If $C \cap P_i = \emptyset$, then set $t = \infty$. Note that at least one of t or s must be finite, and $t \neq s$, since $e_s = e_r$ and $r < s$.
  - If $t < s$, we move ℓ_C from C to P_i, as follows. Set $C^* = C \setminus \{\ell_C\}$, $P_i^* = (e_1, \dots, e_{t-1}, e_t = l_1, \dots, l_w, e_t, \dots, e_m)$, and $\mathbf{P}^* = \mathbf{P} \setminus \{P_i\} \cup \{P_i^*\}$.
  - If $t > s$, we move ℓ_P from P_i to C, as follows. Set $C^* = C \cup \{\ell_P\}$, $P_i^* = (e_1, \dots, e_{r-1}, e_s, \dots, e_m)$, and $\mathbf{P}^* = \mathbf{P} \setminus \{P_i\} \cup \{P_i^*\}$.

It is easy to see that, with this definition, the image $(C^*, \mathbf{P}^*)$ is again a pair of the required kind, i.e., it satisfies the conditions (i)-(iii) above.

Let us verify that φ is an involution. First, we check that φ does not change the value of i. That is, among all walks in $\mathbf{P}^*$ which intersect themselves, another walk, or a cycle in C^*, the walk with the smallest index (of its starting point) is P_i^*. Indeed, our moves only affect P_i, $P_{i'}$, and C, keeping their combined set of edges intact, so the involution will not introduce a new self-intersection in any P_a with $a < i$, nor will it introduce an intersection between P_a and any path or cycle.

Consider $\varphi(C, \mathbf{P}) = (C, \mathbf{P}^*)$ in the first case. After swapping tails, P_i^* still has no intersections with C or any of the other paths before the edge e_q. Further, P_i^* does not have any self-intersections before e_q (though it may have self-intersections at e_q), since P_i did not have any self-intersections before e_q and the tail of $P_{i'}$ did not intersect P_i before e_q. Thus, e_q remains the first edge along P_i^* with an intersection. Now, P_i^* and $P_{i'}^*$ intersect at this edge, and no path with smaller index intersects P_i^* at e_q, so we will swap the same tails again.

Consider the second case, with $\varphi(C, \mathbf{P}) = (C \setminus \{\ell_C\}, \mathbf{P}^*)$ or $\varphi(C, \mathbf{P}) = (C \cup \{\ell_P\}, \mathbf{P}^*)$. Here, P_i intersects itself or C at e_q, but does not intersect any other path at e_q. After moving a cycle, the same is true for P_i^*. (If the cycle moved starts at e_q, then either a self-intersection becomes an intersection with C, or an intersection with C becomes a self-intersection. If the cycle moved starts later, then the intersections at e_q remain as they are.) If P_i intersects C before completing its first cycle, then P_i^* will complete its first cycle before intersecting $C \setminus \{\ell_C\}$. If P_i completes its first cycle ℓ_P before intersecting C, then P_i^* will intersect $C \cup \{\ell_P\}$ before completing its first cycle. Thus, the same cycle is moved both times. We have now shown that φ is an involution.

Finally, we verify that φ is sign-reversing. In the case of tail swapping, we need to show that $\mathrm{wind}(P_i) + \mathrm{wind}(P_{i'}) + \mathrm{xing}(\pi) + \mathrm{wind}(P_i^*) + \mathrm{wind}(P_{i'}^*) + \mathrm{xing}(\pi^*)$ is odd, where π^* is the bijection such that $\mathbf{P}^* \in \mathcal{P}_{\pi^*}$. By Lemma 4.12, $\mathrm{xing}(\pi) + \mathrm{xing}(\pi^*)$ is even if and only if (i, i') is a misalignment. Thus, we need to show that (i, i') is a misalignment if and only if

$$\mathrm{wind}(P_i) + \mathrm{wind}(P_{i'}) + \mathrm{wind}(P_i^*) + \mathrm{wind}(P_{i'}^*) \equiv 1 (\mathrm{mod}\ 2). \tag{4.7}$$

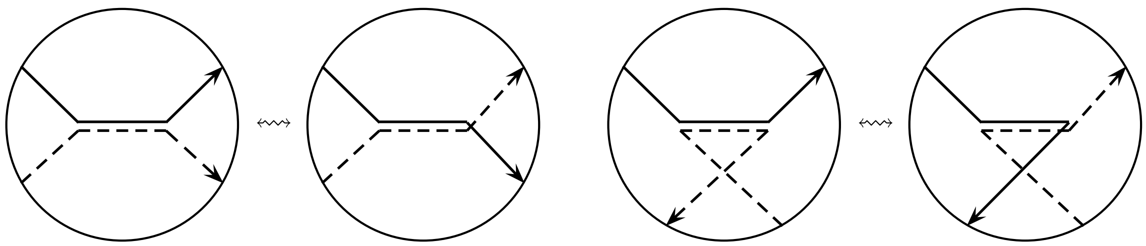


Figure 4.3: Winding index and tail swapping

This statement is in fact true for *any* instance of tail swapping, i.e., it does not rely on our particular choice of the walks P_i and $P_{i'}$ sharing an edge e_q. Viewing (4.7) as a purely topological condition, we can "unwind" each of the 4 subwalks from which our walks P_i and $P_{i'}$ are built, keeping e_q fixed. This will not change the parity in (4.7) since each loop contained entirely in one of the initial or terminal subwalks will contribute twice, once to $\mathrm{wind}(P_i) + \mathrm{wind}(P_{i'})$ and once to $\mathrm{wind}(P_i^*) + \mathrm{wind}(P_{i'}^*)$. Deforming the walks as necessary, we then obtain one of the four pictures shown in Figure 4.3. The last two of the four pictures represent misalignments, and indeed, these are precisely the two cases in which (4.7) holds.

In the remaining case (moving a cycle from C to $\mathbf{P}$ or vice versa), $\mathrm{wind}(P_i)$ changes parity, but $\mathrm{xing}(\pi)$ and all other winding numbers remain the same. Hence $\mathrm{wt}(\varphi((C, \mathbf{P})) = -\mathrm{wt}(C, \mathbf{P})$, as desired. $\square$

Proof of Theorem 4.7

We begin by explaining the process given in [Pos07] for transforming an arbitrary planar circular network into a partially specialized perfectly oriented planar circular network.

For boundary measurements, there is no loss of generality in assuming that G has no internal sources or sinks. Further, we may assume that there are no vertices of degree 2. Indeed, if there is a vertex v with exactly one incoming edge e_1 and exactly one outgoing edge e_2, we may remove v and glue e_1 and e_2 into a single edge e of weight $x_e = x_{e_1}x_{e_2}$.

Let $N = (G, x)$ be a planar network with boundary sources indexed by I and positive weight function $\alpha : \{x_e\}_{e\in E(G)} \to \mathbb{R}$, with $x_e \mapsto \alpha_e$. Let $N' = (G', x')$ and $\alpha' : \{x_e\}_{e\in E(G')} \to \mathbb{R}$ (with $x'_e \mapsto \alpha'_e$) denote a perfectly oriented planar network and corresponding positive weight function obtained from N and α by the process described below.

In general, N' will not be unique. That is, different choices made during the process below may yield different trivalent planar networks, though all of them will have the same boundary measurements.

To obtain a trivalent planar network N' from N, we perform the following operations in stages (1)-(3).

1. First, suppose that N has an internal vertex v of degree greater than 3; let $e_1, \ldots, e_d$ denote the edges incident to v, listed in clockwise order. If two adjacent edges e_i and e_{i+1} (modulo d) have the same orientation, either both towards v or both away from v, we choose such a pair, pull these edges away from v, insert a new vertex v' and a new edge e (directed from v' to v when e_i and e_{i+1}

are edges entering v and from v to v' when e_i and e_{i+1} are edges leaving v), and attach the edges e_i and e_{i+1} to v'. (See Figure 4.4.) We set $\alpha'_e = 1$. Repeat until the resulting network has no vertices v of this form.

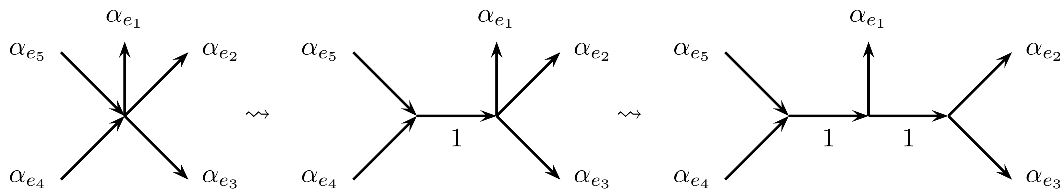


Figure 4.4: Pulling out adjacent edges with the same orientation (first e_4 and e_5, then e_2 and e_3).

2. If a vertex v of degree greater than 3 remains, its incident edges must alternate orientation in clockwise order. In this case we blow up[1] the vertex v into a cycle whose edges are all oriented clockwise, as in Figure 4.5. If e is an edge coming into v, we set $\alpha'_e = 2\alpha_e$, and if e is one of the new edges created to make the cycle, we set $\alpha'_e = 1$. Repeat until the resulting network has no vertices v of this form.

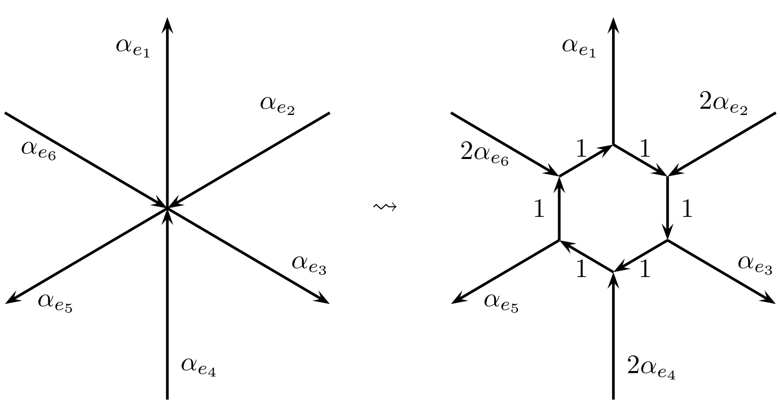


Figure 4.5: Blowing up a vertex with alternating edge directions.

[1]This terminology is taken from [Pos07] and has nothing to do with the notion of blowing up found in algebraic geometry.

3. Finally, for any remaining edge e unaffected by these steps (i.e. such that α'_e has not yet been specified), set $\alpha'_e = \alpha_e$. Let N' and α' denote the final result.

Proposition 4.13 ([Pos07]). *If $N' = (G', \alpha')$ is obtained from $N = (G, \alpha)$ via the process above, then for all $J \subset [n]$ with $|J| = |I|$, we have*

$$\Delta_J(A(N))|_\alpha = \Delta_J(A(N'))|_{\alpha'}.$$

By *contracting* an edge e, we mean removing the edge e and identifying its two endpoints. (If we contract all edges in a connected subset of edges, the image is a single vertex.) It is easy to see that by contracting all new edges created in Proposition 4.13, we obtain G from G'.

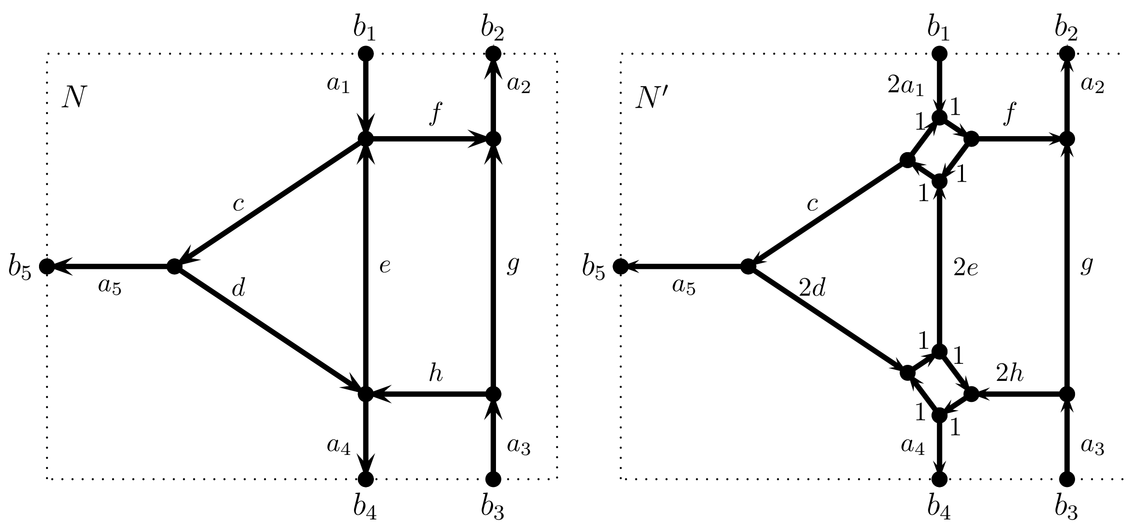


Figure 4.6: A network N and the corresponding perfectly oriented network N'.

Definition 4.14. Let $B(G)$ denote the set of vertices of G around which the orientations of edges switch at least four times. Call such a vertex v a *blowup vertex*. These are precisely the vertices which are blown up into cycles in the second stage of Proposition 4.13.

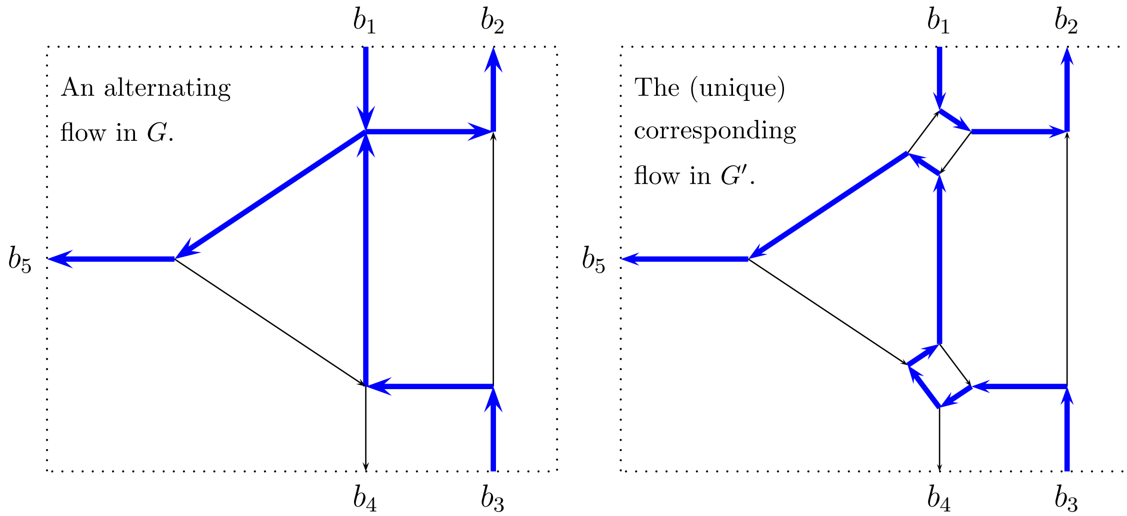


Figure 4.7: An alternating flow in G and the unique corresponding flow in G'.

Example 4.15. In Figure 4.6, we have an example of a network N and a corresponding perfectly oriented network N'. (In this example, N' happens to be the only possible network resulting from the process above.) Notice that the edges of the newly created cycles each have weight 1, and the weights of the edges entering those cycles have doubled.

Let us examine what happens to flows during the transformation from N to N'. We will find $\mathcal{A}_{\{2,5\}}(N)$. We can see that N has precisely two alternating flows from $\{1, 3\}$ to $\{2, 5\}$, namely those in Figures 4.7 and 4.8.

In Figure 4.7, there is only one flow in N' whose contraction is the given alternating flow in N. In Figure 4.8, there are two such flows in N' which contract to the given alternating flow in N. Note that they are identical except that one includes the cycle which contracts to the blowup vertex marked in G, and the other does not. Since this cycle has weight 1, these two flows both have the same weight.

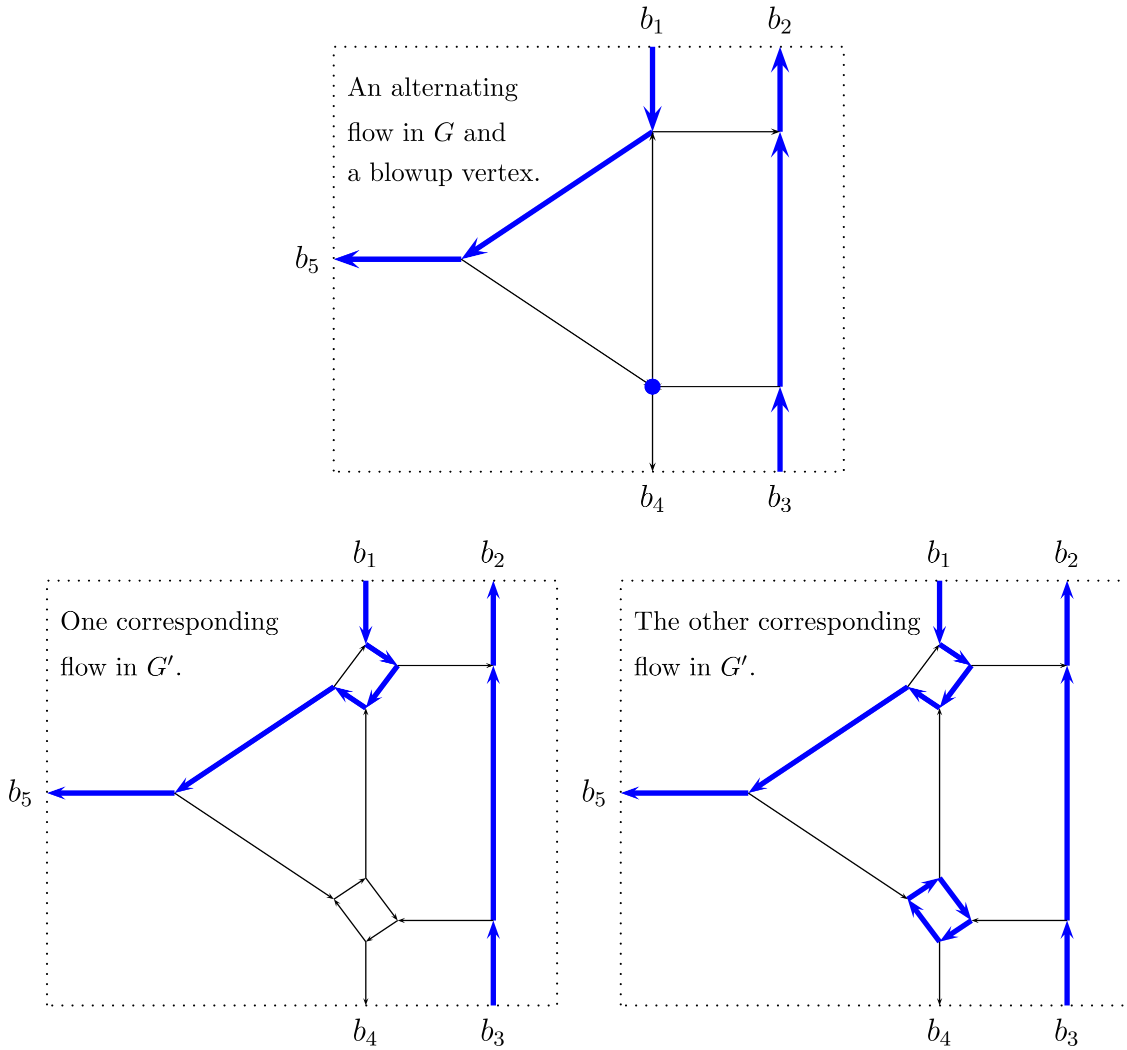


Figure 4.8: An alternating flow in G and the two corresponding flows in G'.

For an alternating flow F in a planar network N, we define $\epsilon(F)$ to be the number of edges of F which enter a blowup vertex of G, $\beta(F)$ to be the number of blowup vertices of G which occur as the endpoint of some edge in F, and $\eta(F)$ to be the number of blowup vertices of G which are not endpoints of any edge in F. Recalling Definition 4.6, note that $\theta(F) = \epsilon(F) - \beta(F)$.

Proof of Theorem 4.7. Fix an image N' of N under the transformation in Proposition 4.13, and let F' be a flow in N'. It is easily verified that contracting all edges in $E(G') - E(G)$ gives a bijection between flows F' in the perfectly oriented graph

G' and pairs (F, A), where F is an alternating flow in the original graph G and A is a subset of the $\eta(F)$ vertices in $B(G)$ which are not endpoints of any edges in F.

Further, extending α and α' linearly, we have

$$\alpha'(\mathrm{wt}(F')) = 2^{\epsilon(F)}\alpha(\mathrm{wt}(F)),$$

and there are $2^{\eta(F)}$ flows F' in G' corresponding to a given alternating flow F in G, all with the same weight.

Since this relationship holds for every positive specialization α, Theorem 4.3 and Proposition 4.13 then imply that

$$\Delta_J(A(N)) = \frac{\displaystyle\sum_{F\in\mathcal{A}_J(G)} 2^{\epsilon(F)+\eta(F)}\,\mathrm{wt}(F)}{\displaystyle\sum_{C\in\mathcal{A}_I(G)} 2^{\epsilon(C)+\eta(C)}\,\mathrm{wt}(C)}.$$

Cancelling a factor of $|B(G)| = \eta(F) + \beta(F)$ from each term in the numerator and denominator, we obtain

$$\Delta_J(A(N)) = \frac{\displaystyle\sum_{F\in\mathcal{A}_J(G)} 2^{\epsilon(F)-\beta(F)}\,\mathrm{wt}(F)}{\displaystyle\sum_{C\in\mathcal{A}_I(G)} 2^{\epsilon(C)-\beta(C)}\,\mathrm{wt}(C)},$$

which is equivalent to the desired formula (4.2), since $\theta(F) = \epsilon(F) - \beta(F)$. □

4.4 Plücker coordinates for perfectly oriented non-planar networks

It is natural to ask to what extent we can develop these constructions in the non-planar setting; this section is a slight detour which addresses this question. While the notion of the topological winding index only makes sense for planar graphs, Lawler's notion of loop-erasure in [Law80] allows us to give a non-planar analogue of the winding index if G is perfectly oriented. In this non-planar setting, we no longer have the positivity results, but we can describe those Plücker coordinates which are individual boundary measurements.

We begin by extending the definition of circular directed graphs and networks (Definition 3.4) to suit the non-planar setting. For a general *circular directed graph*, we no longer require that G has a planar embedding in a disk, but we still ask for the boundary vertices to be labeled in cyclic order and for each boundary vertex to be adjacent to at most one edge.

Definition 4.16 ([Fom01, Law91])**.** The *loop-erased part* of a walk $P : b_i \rightsquigarrow b_j$, denoted $\mathrm{LE}(P)$, is defined recursively as follows. If $P = (e_1, \ldots, e_m)$ does not have any self-intersections, then $\mathrm{LE}(P) = P$. Otherwise, we set $\mathrm{LE}(P) = \mathrm{LE}(P')$, where P' is obtained from P by removing the first cycle it completes. More precisely, when G is perfectly oriented, find the smallest value of s such that there exists $r < s$ with $e_r = e_s$, and remove the segment $e_r, e_{r+1}, \ldots, e_{s-1}$ from P to obtain P'. The *loop-erasure number* $\mathrm{loop}(P)$ is defined as the number of cycles erased during the calculation of $\mathrm{LE}(P)$. With the notation as above, we have $\mathrm{loop}(P) = \mathrm{loop}(P') + 1$, and $\mathrm{loop}(P) = 0$ when P is a self-avoiding walk.

Proposition 4.17 ([Pos07])**.** *Suppose that G is a perfectly oriented planar circular directed graph. If P is a walk from a boundary vertex b_i to a boundary vertex b_j, then $(-1)^{\mathrm{loop}(P)} = (-1)^{\mathrm{wind}(P)}$.*

Proof. Each boundary vertex is incident to at most one edge, so P has no self-intersections at its endpoints. Since G is perfectly oriented, P repeats at least one edge at every self-intersection. The claim then follows by induction on $\mathrm{loop}(P)$, as an erasure of a cycle changes the winding index by ± 1. $\square$

Proposition 4.17 allows us to view $\mathrm{loop}(P)$ as a natural generalization of $\mathrm{wind}(P)$

for perfectly oriented graphs, allowing us to work with non-planar graphs. This observation leads to an extension of Postnikov's construction (which applies to planar networks and employs the winding index) to arbitrary perfectly oriented graphs. Definitions 4.1, 3.5, 3.6, 3.8, 4.8, and 4.2 then extend to perfectly oriented non-planar networks in the obvious way, replacing wind(P) with loop(P) wherever appropriate.

Corollary 4.18. *Suppose $N = (G, x)$ is a perfectly oriented circular network with boundary source set indexed by I. Then, for $i \in I$ and $j \in [n]$, we have*

$$M_{ij} = \Delta_{(I\setminus\{i\})\cup\{j\}}(A(N)) = \frac{\displaystyle\sum_{F\in\mathcal{F}_{(I\setminus\{i\})\cup\{j\}}(G)} \mathrm{wt}(F)}{\displaystyle\sum_{C\in\mathcal{C}(G)} \mathrm{wt}(C)}.$$

Proof. This follows directly from the proof of Theorem 4.3, since for these special Plücker coordinates, the tail swapping process of the proof is never called upon. □

Although the result holds for those minors Δ_J which are boundary measurements M_{ij}, it is generally not valid for the remaining Plücker coordinates. For non-planar networks, tail swapping does not always yield the sign change in $(-1)^{\mathrm{xing}}$ that we obtain in the planar case. As a result, the numerator and denominator of a minor Δ_J do not necessarily simplify to linear combinations of flow weights after cancellation of common factors.

Example 4.19. Consider the network N in Figure 4.9, with boundary measurement matrix $A(N)$ below. The minors Δ_{12}, Δ_{13}, Δ_{14}, Δ_{24}, and Δ_{34} all satisfy Theorem 4.3.

$$A(N) = \begin{pmatrix} 1 & \frac{a_1 f a_2}{1+cdef} & \frac{a_1 feda_3}{1+cdef} & 0 \\ 0 & \frac{a_4 dcfa_2}{1+cdef} & \frac{a_4 da_3}{1+cdef} & 1 \end{pmatrix}$$

However, for Δ_{23}, we do not get the desired cancellation; in the simplified rational expression, both the numerator and denominator are quadratic in flow weights. We have

$$\Delta_{23}(A(N)) = \frac{a_1 a_2 a_3 a_4 \cdot df(1 - cdef)}{(1 + cdef)^2} \ .$$

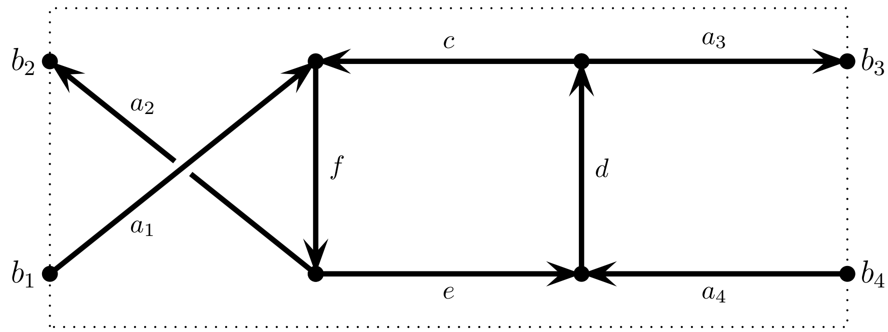


Figure 4.9: Boundary measurements in a non-planar network.

Remark 4.20. If we consider flow weights as polynomials with coefficients in the finite field of two elements, $\mathbb{F}_2$, then equation (4.1) holds for all Δ_J in the perfectly oriented non-planar case; this also follows directly from the proof of Theorem 4.3.

CHAPTER 5

Formulas for the inverse boundary measurement map

In Postnikov's work [Pos07], Theorem 3.14 is proved by giving a recursive algorithm for finding the Le-tableau T corresponding to a given point in $(\mathrm{Gr}_{kn})_{\geq 0}$. In this paper, we obtain explicit combinatorial formulas solving the same problem. This is done in two stages. In Section 5.1, we give an explicit rule for determining which positroid cell contains a given point. In Sections 5.2 and 5.3, we give two combinatorial formulas for the inverse of each particular map Meas_L (i.e., formulas for the corresponding Le-coordinates) in terms of the relevant Plücker coordinates.

5.1 Determining the positroid cell of a point in $(\mathrm{Gr}_{kn})_{\geq 0}$

In this section, we give an explicit formula for the Le-diagram $L(x)$ that determines which positroid cell $(S_{\mathcal{M}_L})_{\geq 0}$ a given point $x \in (\mathrm{Gr}_{kn})_{\geq 0}$ belongs to.

Let $x \in (\mathrm{Gr}_{kn})_{\geq 0}$ be given by its Plücker coordinates $(P_J(x) : J \in \binom{[n]}{k})$. Order the k-subsets of $[n]$ lexicographically. That is, a set $\{a_1 < a_2 < \cdots < a_k\}$ is less than or equal to a set $\{b_1 < b_2 < \cdots < b_k\}$ if at the smallest index m for which $a_m \neq b_m$, we have $a_m < b_m$.

For $\mathcal{M} \subseteq \binom{[n]}{k}$, let $I = \{i_1 < i_2 < \cdots < i_k\}$ be the lexicographically minimum set in $\mathcal{M}$. Let $[n] \setminus I = \{j_1 < j_2 < \cdots < j_{n-k}\}$ be the complement of I. The set I determines the shape $\lambda(\mathcal{M})$ of the Le-diagram $L(x)$.

Let $\lambda(\mathcal{M})$ be the partition in the $k \times (n-k)$ rectangle whose southeastern border is given by the path from the northeast corner of the $k \times (n-k)$ rectangle to its southwest corner which has vertical edges in positions I and horizontal edges in positions $[n] \setminus I$. More precisely, the length of the t^{th} row of λ is the number of elements of $[n] \setminus I$ which are greater than i_t, i.e., $\lambda_t = |j_s \in [n] \setminus I : j_s > i_t|$.

For a box $B = (r, c)$ in $\lambda(\mathcal{M})$, set

$$\begin{aligned} A_{r,c} &= [n] \setminus \{i_r + 1, i_r + 2, \ldots, j_c - 2, j_c - 1\} \\ &= \{1, 2, \ldots, i_r - 1, i_r\} \cup \{j_c, j_c + 1, \ldots, n - 1, n\} \end{aligned}$$

Set $M(B, \mathcal{M}) = (M'(B, \mathcal{M}) \setminus \{i_r\}) \cup \{j_c\}$, where

$$M'(B, \mathcal{M}) = \text{lexmax}\,\{J \in \mathcal{M} : J \cap A_{r,c} = I \cap A_{r,c}\}\,.$$

In plain language, this says that we are taking the maximum over sets J which contain all of the sources outside the open interval from i_r to j_c and none of the sinks, i.e., those sets whose interesting behavior happens *inside* the interval.

Recall that for a Le-diagram L, we have $J \in \mathcal{M}_L$ if and only if there exists a non-intersecting path collection in G_L with source set I and destination set J. Note that the lexicographically minimum set in $\mathcal{M}_L$ labels the sources of the appropriate Γ-graph, so that this set corresponds to the family of $|I|$ zero-edge paths, one from each source to itself.

Lemma 5.1. *Suppose that B is a box in a Le-diagram L of shape $\lambda(L)$. Then*

1. *$M'(B, \mathcal{M}_L)$ is the destination set of a unique non-intersecting path collection in the Γ-graph G_L, namely the northwest-most path collection whose edges lie strictly southeast of the B hook;*

2. *$M(B, \mathcal{M}_L) \in \mathcal{M}_L$ if and only if $L_B = +$; and*

3. *the vanishing pattern for the Plücker coordinates of $(S_{\mathcal{M}_L})_{\geq 0}$ is uniquely determined by the vanishing pattern of the subset $\{P_{M(B,\mathcal{M}_L)}\}$, ranging over all boxes B in $\lambda(L)$.*

Proof. The proof of the first claim is left as a straightforward exercise for the reader; the second and third then follow immediately from the definitions. □

Example 5.2. On the left in Figure 5.1, we have the Γ-graph of the example in Figure 3.2. We see that $M'((2,6),\mathcal{M}_L) = \{1,2,7,9,10\}$, corresponding to the solid path collection on the right in Figure 5.1. Adding in the potential (dotted) hook from $i_2 = 2$ to $j_6 = 11$, we have $M((2,6),\mathcal{M}_L) = \{1,7,9,10,11\}$. Since this hook does not occur in the Γ-graph, we must have $P_{M((2,6),\mathcal{M}_L)}(x) = 0$ for this point.

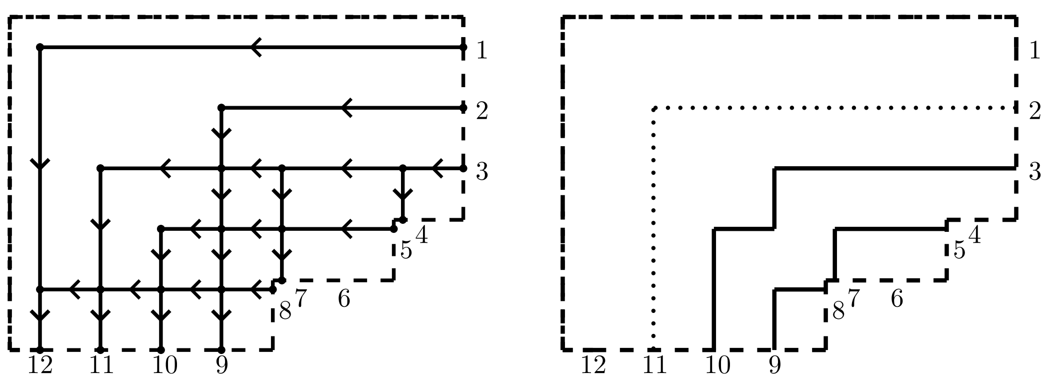


Figure 5.1: The Γ-graph of an example in $(\mathrm{Gr}_{5,12})_{\geq 0}$ and the path families corresponding to $M'((2,6),\mathcal{M}_L)$ and $M((2,6),\mathcal{M}_L)$.

Theorem 5.3. *For $x \in (\mathrm{Gr}_{kn})_{\geq 0}$, set $\mathcal{M}(x) = \{J \in \binom{[n]}{k} : P_J(x) \neq 0\}$. Then the filling of $\lambda(\mathcal{M}(x))$ given by*

$$L(x)_B = \begin{cases} 0 & \text{if } P_{M(B,\mathcal{M}(x))}(x) = 0; \\ + & \text{if } P_{M(B,\mathcal{M}(x))}(x) \neq 0. \end{cases} \tag{5.1}$$

is a Le-diagram, and x lies in the positroid cell $(S_{\mathcal{M}_{L(x)}})_{\geq 0}$.

Proof. Combining Theorem 3.14 and Lemma 5.1, each point $x \in (\mathrm{Gr}_{kn})_{\geq 0}$ lies in a unique positroid cell $(S_{\mathcal{M}_L})_{\geq 0}$ and therefore we must have a unique Le-diagram L such that $P_{M(B,\mathcal{M}(x))} = P_{M(B,\mathcal{M}_L)}$ for all boxes $B \in \lambda(L) = \lambda(\mathcal{M}(x))$. $\square$

5.2 The Le-tableau associated with a point in $(S_{\mathcal{M}_L})_{\geq 0}$

In Postnikov's original work, the map from $(\mathrm{Gr}_{kn})_{\geq 0}$ to $\bigcup_L \mathbf{T}_L$ is given recursively. In this section, we provide an explicit description of that map. More precisely, given a point $x \in (S_{\mathcal{M}_L})_{\geq 0}$, we give combinatorial formulas for the entries of the parametrizing Le-tableau, which we call *Le-coordinates* for x.

For each box B in λ, let $H(B)$ denote the collection of boxes lying weakly southeast of the B hook. For each box B with $L_B = +$, let $R(B)$ denote the face with northwest corner B, i.e., the collection of boxes which lie in the same face as B in the corresponding Γ-graph G. We may simply write R for $R(B)$ if there is no need to emphasize the northwest corner of R. The Le-property ensures that the northwest boundary of each face $R = R(B)$ is a portion of a single hook, namely the B hook; we may also refer to this hook as the R hook.

Definition 5.4. In a Γ-graph G, call a collection W of paths a *generalized path* if the paths in W are pairwise disjoint, and no path of W lies southeast of another path in W. (An example of a generalized path which is not a path can be seen on the right in Figure 5.1. The union of the hook from 5 to 7 and the hook from 8 to 9 is a generalized path.)

We say that a collection of paths lies (strictly or weakly) southeast of a given generalized path W if each of the edges in the path collection lies (strictly or weakly) southeast of some path of W.

For a generalized path W in a Γ-graph G, let $\mathcal{OC}(W)$ denote the set of *outer* *corners* of W, that is, those boxes B for which the northern and western boundaries of B are both edges of W. We order the outer corners from northeast to southwest. Let $\mathcal{IC}(W)$ denote the *inner corners* of W, that is, those boxes B such the northwest boundary of B is formed by portions of the hooks of two consecutive outer corners. Note that an inner corner need not be adjacent to the corresponding outer corners. The Le-property ensures that each outer or inner corner B satisfies $L_B = +$.

Consider the generalized paths which lie weakly southeast of the R hook and contain the entire southeast border of R; these generalized paths must all have the same edge set. That is, they are all identical up to addition or removal of paths with zero edges; take D_R to be the unique such generalized path whose paths each consist of at least one edge. Essentially, D_R traces out the southeast boundary of R, but it may be broken into several paths if R borders the boundary of our disk. We can see that $\mathcal{OC}(D_R)$ indexes the hooks which determine the southeast boundary of the face R, and $\mathcal{IC}(D_R)$ indexes the hooks which are intersections of two hooks corresponding to adjacent outer corners.

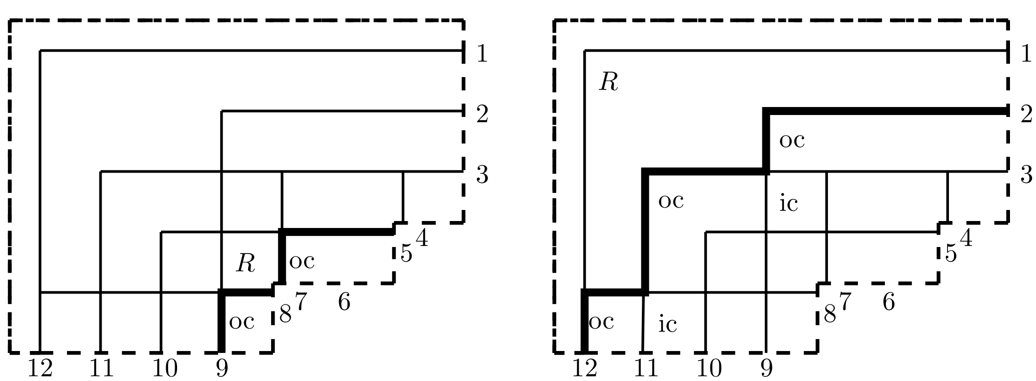


Figure 5.2: Finding the corners of $D_{R(4,4)}$ and $D_{R(1,7)}$.

Example 5.5. Consider the Γ-graph in Figure 5.2. We find the inner and outer corners of $D_{R(4,4)}$ and of $D_{R(1,7)}$. In each graph, the relevant face is labeled "R", the generalized path D_R is in bold, and outer and inner corners are labeled “oc” and “ic” respectively. Note that $D_{R(4,4)}$ consists of two paths, and it has no inner corners.

We recall that the Möbius function μ_S of a partially ordered set S is given recursively by the rules

$$\begin{aligned} \mu_S(x,x) &= 1, \text{ for all } x \in S, \text{ and} \\ \mu_S(x,y) &= -\sum_{x \leq z < y} \mu_S(x,z), \text{ for all } x < y \text{ in } S. \end{aligned}$$

For a Le-diagram L, let $\mathcal{R}_L$ denote the set of faces of the Γ-graph G_L which are indexed by + entries in L. We partially order $\mathcal{R}_L$ by the condition that $R_1 \leq_L R_2$ if the R_1 hook lies weakly northwest of the R_2 hook.

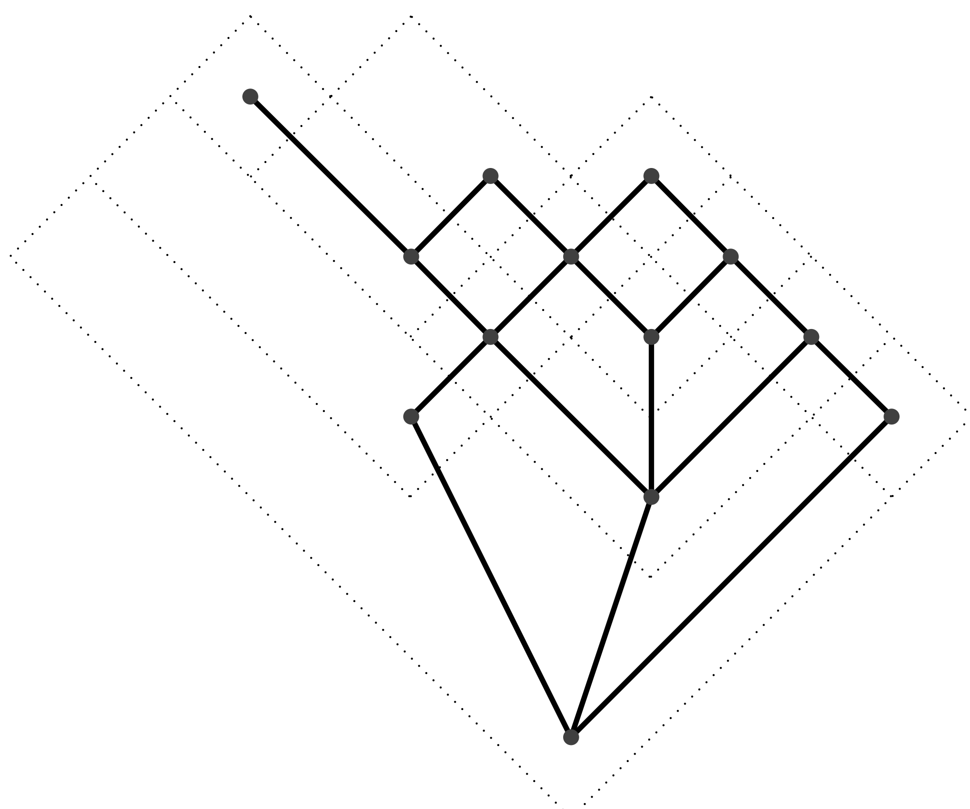


Figure 5.3: Here we have the poset for the example shown in Figures 3.2 and 5.1. Dotted lines indicate the boundaries of the original faces, rotated so that the northwest corner is at the lowest point of the poset.

Lemma 5.6. *Let* $\mu_L = \mu_{\mathcal{R}_L}$ *denote the Möbius function for* $\mathcal{R}_L$*, with the partial order* $\leq_L$*. Then for any two faces* $R_1 = R(B_1)$ *and* $R_2 = R(B_2)$ *of* G_L*, we have*

$$\mu_L(R_1, R_2) = \begin{cases} 1 & \textit{if } R_1 = R_2 \textit{ or } B_2 \in \mathcal{IC}(D_{R_1}) \\ -1 & \textit{if } B_2 \in \mathcal{OC}(D_{R_1}) \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We see that our Möbius function μ_L has the following interpretation. For a fixed face R_1, we assign to each face in the collection $H(R_2)$ the quantity $\mu_L(R_1, R_2)$. That is, we count the faces lying southeast of the R_2 hook with signed multiplicity $\mu_L(R_1, R_2)$. By the definition of a Möbius function, this means we want the total count for a face R to be exactly one if $R = R_1$ and zero if $R \neq R_1$. The proof is then completed by a simple inclusion-exclusion argument, which is left to the reader. □

To avoid unwieldy notation, we will write $M(B)$ and $M'(B)$ in place of $M(B, \mathcal{M}_L)$ and $M'(B, \mathcal{M}_L)$ when the appropriate Le-diagram L is clear from context.

Theorem 5.7. *Suppose* $x \in (S_{\mathcal{M}_L})_{\geq 0}$*. Then the Le-coordinates of* x *are the entries of the Le-tableau* $T(x) \in \mathbf{T}_L$ *defined below.*

$$T(x)_B = \begin{cases} 0 & \textit{if } P_{M(B)}(x) = 0; \\ \prod_{L_C=+} \left(\frac{P_{M(C)}(x)}{P_{M'(C)}(x)}\right)^{\mu_L(B,C)} & \textit{if } P_{M(B)}(x) \neq 0. \end{cases} \tag{5.2}$$

That is, $\mathrm{Meas}_L(T(x)) = x$*, and* $T(x)$ *is the unique Le-tableau whose image under* Meas_L *is* x*.*

Before proving Theorem 5.7, we note that the concrete description of μ_L given in Lemma 5.6 allows us to quickly write the expressions in equation (5.2) by simply inspecting the graph.

Proof. By Theorem 3.14, there exists a unique Le-tableau T satisfying $\mathrm{Meas}_L(T) = x$. Here we show that T must be the Le-tableau $T(x)$ defined above. Suppose that T satisfies $P_J(\mathrm{Meas}_L(T)) = P_J(x)$ for all $J \in \binom{[n]}{k}$. By Theorem 5.3, if $P_{M(B)}(x) = 0$, we must have $L_B = 0$, and therefore $T_B = 0$. Whenever $L_B = +$, we can easily see that the ratio

$$\frac{P_{M(B)}(\mathrm{Meas}_L(T))}{P_{M'(B)}(\mathrm{Meas}_L(T))}$$

is the product of the weights of all faces southeast of the B hook in the Γ-network N_T, each with multiplicity one. By assumption, we have

$$\frac{P_{M(B)}(\mathrm{Meas}_L(T))}{P_{M'(B)}(\mathrm{Meas}_L(T))} = \frac{P_{M(B)}(x)}{P_{M'(B)}(x)}.$$

Since the weight of a hook is simply the product of the weights of faces southeast of the hook, a multiplicative version of Möbius inversion implies that the weight of the face whose northwest corner is B is given by the ratio

$$\prod_{L_C=+} \left(\frac{P_{M(C)}(x)}{P_{M'(C)}(x)}\right)^{\mu_L(B,C)}.$$

Since the positive entries of T are simply the weights of the faces in N_T, the entries of T must be those of $T(x)$ given in equation (5.2). □

5.3 Le-coordinates in terms of a minimal set of Plücker coordinates

By Theorem 3.14, the dimension of a positroid cell $(S_{\mathcal{M}_L})_{\geq 0}$ is $|L|$, the number of "+" entries in the corresponding Le-diagram L. However, finding the Le-coordinates of a point $x \in (S_{\mathcal{M}_L})_{\geq 0}$ via equation (5.2) may require roughly twice this many Plücker variables. In this section, we give a formula for the map from $(S_{\mathcal{M}_L})_{\geq 0}$ to $\mathbf{T}_L$, using precisely $|L|$ Plücker variables. This formula is, of course, equivalent to our first formula modulo Plücker relations, but we now use exactly the desired number of parameters.

Suppose $x \in (S_{\mathcal{M}_L})_{\geq 0}$ and $\mathrm{Meas}_L(T) = x$. For a box B in λ with $L_B = +$, let $R = R(B)$ be the corresponding face in the Γ-network N_T. We have already defined U_R and D_R. Let U'_R and D'_R be the northwest-most generalized paths lying strictly southeast of U_R and D_R, respectively. See Figure 5.4 for an example.

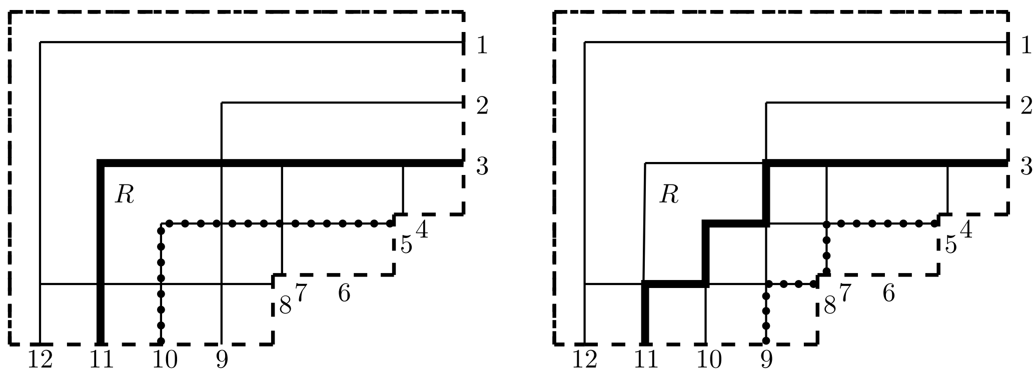


Figure 5.4: On the left, we have $U_{R(3,6)}$ in bold and $U'_{R(3,6)}$ dotted; on the right, we have $D_{R(3,6)}$ in bold and $D'_{R(3,6)}$ dotted.

For a generalized path W in a Γ-network N and a box B in λ, set

$$\varepsilon_W(B) = \begin{cases} 1 & \text{if } B \in \mathcal{OC}(W); \\ -1 & \text{if } B \in \mathcal{IC}(W); \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.8. *Suppose $x \in (S_{\mathcal{M}_L})_{\geq 0}$ and let $T(x)$ be the Le-tableau corresponding to x, so that $\mathrm{Meas}_L(T(x)) = x$. Then the Le-coordinates of x may be written in the alternate form*

$$T(x)_B = \begin{cases} 0 & \text{if } P_{M(B)}(x) = 0; \\ \prod_{L_C=+} (P_{M(C)}(x))^{\varepsilon(B,C)} & \text{if } P_{M(B)}(x) \neq 0, \end{cases} \tag{5.3}$$

where $\varepsilon(B,C) = [\varepsilon_{U_{R(B)}}(C) - \varepsilon_{U'_{R(B)}}(C)] - [\varepsilon_{D_{R(B)}}(C) - \varepsilon_{D'_{R(B)}}(C)]$.

Before proving Theorem 5.8, we first state one nearly immediate corollary using the notion of a *totally positive base* given in [FZ99].

Corollary 5.9. *The set of Plücker coordinates*

$$\mathbf{P}_L = \{P_{M(B)} : L_B = +\}$$

forms a totally positive base *for the collection* $\{P_J : J \in \mathcal{M}_L\}$ *of non-vanishing Plücker coordinates of the positroid cell* $(S_{\mathcal{M}_L})_{\geq 0}$*. That is, every Plücker coordinate* P_J *with* $J \in \mathcal{M}_L$ *can be written as a subtraction-free rational expression (i.e., a ratio of two polynomials with nonnegative integer coefficients) in the elements of* $\mathbf{P}_L$*, and* $\mathbf{P}_L$ *is a minimal set (with respect to inclusion) with this property. Further, each* P_J *with* $J \in \mathcal{M}_L$ *is a Laurent polynomial in the elements of* $\mathbf{P}_L$*, with nonnegative coefficients.*

Proof. Suppose $x \in (S_{\mathcal{M}_L})_{\geq 0}$, with $\mathrm{Meas}_L(T) = x$. By Theorem 5.8, every face weight of the Γ-network N_T can be written as a monomial rational expression in the elements of $\mathbf{P}_L$. Each Plücker coordinate $P_J(x)$ is a sum of products of face weights, by Definition 3.15. It is then clear that each P_J is a Laurent polynomial with nonnegative coefficients in elements of $\mathbf{P}_L$. Finally, $\mathbf{P}_L$ is minimal, as it is easily verified that the elements of $\mathbf{P}_L$ are algebraically independent. Indeed, the simple form of equation (5.3) shows that we can explicitly construct a network realizing any choice of positive values for the Plücker coordinates in $\mathbf{P}_L$, starting by choosing appropriate face weights for those faces at the top of the poset in Figure 5.2 (that is, in the northwest corner of the Le-tableau) and working towards the faces at the bottom of the poset. $\square$

To prove Theorem 5.8, we will need the following technical lemma, which gives the weights of certain nested path families. For a generalized path W, let $\mathrm{Nest}(W)$

denote the northwest-most non-intersecting path family lying weakly southeast of W. That is, $\mathrm{Nest}(W)$ consists of W, the northwest-most generalized path W' which lies strictly southeast of W, the northwest-most generalized path W'' which lies strictly southeast of W', and so on, until no more paths will fit.

Lemma 5.10. *Suppose T is a Le-tableau with corresponding Γ-network N_T. Let W be a generalized path in N_T. Then*

$$\mathrm{wt}(\mathrm{Nest}(W)) = \prod_{L_C=+} \left(P_{M(C)}(\mathrm{Meas}_L(T))\right)^{\varepsilon_W(C)}. \tag{5.4}$$

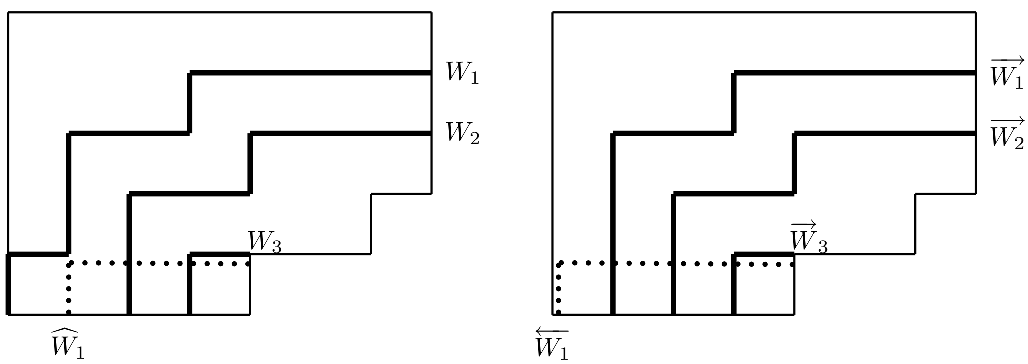


Figure 5.5: Finding the weight of a nested path family.

Proof. We proceed by induction on the number of outer corners of W. If W has a single outer corner, the result follows from the definition of $M(B)$. Otherwise, assume W has ℓ outer corners and split W as follows: let $\overrightarrow{W}$ be the path determined by the first $\ell-1$ outer corners of W (ordered from northeast to southwest) and let $\overleftarrow{W}$ be the hook determined by the last outer corner of W. If $\overrightarrow{W}$ and $\overleftarrow{W}$ do not intersect, the result clearly holds. (This can happen when λ is not the full $k \times n$ rectangle.) Otherwise, let $\widehat{W}$ be the hook determined by the inner corner of W which is between the last two outer corners of W. See Figure 5.5 for an example.

Now, $\mathrm{Nest}(W)$ is a disjoint union of paths in N_T. Write $\mathrm{Nest}(W)$ as the ordered collection of path families $(W_1, W_2, \ldots)$, where a path Y in $\mathrm{Nest}(W)$ lies in the block W_i if exactly $i-1$ paths of $\mathrm{Nest}(W)$ lie strictly northwest of Y. (For i large enough, W_i will be empty. Recall that the weight of an empty path collection is 1.) Write $\mathrm{Nest}(\overrightarrow{W})$, $\mathrm{Nest}(\overleftarrow{W})$, and $\mathrm{Nest}(\widehat{W})$ in the same manner.

We claim that for each i, $\mathrm{wt}(\overrightarrow{W}_i)\,\mathrm{wt}(\overleftarrow{W}_i) = \mathrm{wt}(W_i)\,\mathrm{wt}(\widehat{W}_i)$. More precisely, let $(v_1, \ldots, v_m)$ be the vertices at which $\overrightarrow{W}_i$ and $\overleftarrow{W}_i$ intersect. Then we claim that W_i is the path along edges of $\overrightarrow{W}_i$ or $\overleftarrow{W}_i$ which starts at the source of $\overrightarrow{W}_i$ and takes the northwest-most path between each v_m and v_{m+1} and $\widehat{W}_i$ is the path which starts at the source of $\overleftarrow{W}_i$ and takes the southeast-most path between each v_m and v_{m+1}. This is clearly true for $i = 1$. The remainder, which depends on the Le-property, is left as an exercise for the reader.

Since the weight of a path family is the product of the weights of the individual paths, we then have

$$\mathrm{wt}(\mathrm{Nest}(W)) = \frac{\prod_B \left(P_{M(B)}(\mathrm{Meas}_L(T))\right)^{\varepsilon_{\overrightarrow{W}}(B)} \cdot \prod_B \left(P_{M(B)}(\mathrm{Meas}_L(T))\right)^{\varepsilon_{\overleftarrow{W}}(B)}}{\prod_B \left(P_{M(B)}(\mathrm{Meas}_L(T))\right)^{\varepsilon_{\widehat{W}}(B)}},$$

which is precisely equation (5.4), since $\overleftarrow{W}$ has a single outer corner (which is an outer corner of W) and no inner corners, and $\widehat{W}$ has a single outer corner (which is an inner corner of W) and no inner corners. $\square$

Proof of Theorem 5.8: Suppose W is a generalized path in N_T. Let W' be the northwest-most generalized path lying strictly southeast of W. We can easily see that the ratio $\frac{\mathrm{wt}(\mathrm{Nest}(W))}{\mathrm{wt}(\mathrm{Nest}(W'))}$ is the product of the weights of the faces lying southeast of W, each with multiplicity one, since the weight of each face appearing in this ratio occurs exactly one more time in $\mathrm{wt}(\mathrm{Nest}(W))$ than it does in $\mathrm{wt}(\mathrm{Nest}(W'))$.

Then, since U_R and D_R bound precisely the face $R = R(B)$, the face weight T_B

must be given by the ratio

$$\left(\frac{\mathrm{wt}(\mathrm{Nest}(U_R))}{\mathrm{wt}(\mathrm{Nest}(U'_R))}\right) \Big/ \left(\frac{\mathrm{wt}(\mathrm{Nest}(D_R))}{\mathrm{wt}(\mathrm{Nest}(D'_R))}\right).$$

Combining this with equation (5.4) then yields equation (5.3), since we require that $P_J(\mathrm{Meas}_L(T)) = P_J(x)$ for all $J \in \binom{[n]}{k}$. $\square$

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