TOPICS IN TIGHT CLOSURE THEORY

by

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To my Family, Friends and Significant Others

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CHAPTER I

Introduction and Dissertation Outline

Many problems in Commutative Algebra have been successfully attacked with techniques using the Frobenius endomorphism of a ring of positive prime characteristic. Surprisingly, these methods often also yield results in characteristic 0. The tight closure theory created by Hochster and Huneke in the 1980s is now the main characteristic p method and one of the central tools in commutative algebra. It has provided simple proofs for a number of deep results that didn't seem to be particularly related before. For example, tight closure theory methods were used to prove the local homological conjectures for equicharacteristic cases (see, for instance, [13]), several extremely powerful vanishing results on maps of Tors (e.g., in [15]), the famous invariant theory result of Hochster-Roberts (e.g. in [14]and [19]), which asserts that the ring of invariants of a linearly reductive algebraic group acting on a regular ring is Cohen-Macaulay, and a great deal more. The proofs obtained via tight closure theory appear simpler and lead to much more general results (e.g., a series of "Briançon-Skoda theorems"; see, for instance, [14], [4], [25], [26]). There are lots of parallels and connections of tight closure with integral closure and multiplicities. Other strong results obtained or inspired by tight closure methods include the fact that for an excellent local domain of positive prime characteristic p the absolute integral closure R^+ is a big Cohen-Macaulay algebra ([17]), a uniform version of the Artin-Rees theorem (see e.g. [2] and references therein) and comparison theorems on ordinary and symbolic powers ([18]), just to name a few. There are also important connections between the ideas of tight closure in characterstic p and singularities of algebraic varieties in characteristic 0. Tight closure theory continues to be an area of very active research that has a deep influence on and is an immensely fruitful source for ideas in commutative algebra itself as well as in its neighboring fields such as Algebraic Geometry.

In its simplest form, tight closure is a closure operation for submodules of a module over a ring of finite prime characteristic (although a big part of the theory also works for rings containing the rationals).

Let R be a Noetherian ring of positive prime characteristic p and let $N \subseteq M$ be R-modules. For every integer $e \geq 0$ the eth power of Frobenius endomorphism maps R to itself by $x \mapsto x^q$ where $q = p^e$. Let S_e denote R considered as an R-algebra via the eth power of Frobenius. The Peskine-Szpiro functor, F^e is $S_e \otimes_R$, a covariant functor from R-modules to S_e-modules. Since $S_e = R$ this is actually a functor from R-modules to R-modules.

We say that $x \in M$ is in the *tight closure* of N (denoted by N^*_M), if there exists $c \in R$ that does not belong to any minimal prime of R and such that in $F^e(M)$ we have $cx^q \in \operatorname{im}(F^e(N) \to F^e(M))$ for all $q \gg 0$. Here x^q stands for the image of x under the natural map $M \to F^e(M)$ that takes $m \mapsto 1 \otimes m$.

This dissertation is centered around two important themes from tight closure theory: phantom homology and existence of test elements. In the next two sections we will outline the main results of this work and some related questions.

1.1 Results related to Phantom Homology

One of the most beautiful notions emerging from tight closure theory is that of phantom homology, introduced by Hochster and Huneke in [14]. Roughly speaking, the idea is to consider projective resolutions that are "almost acyclic": instead of requiring that the boundaries are the same as cycles, as is the case for the usual acyclic resolution, it is assumed that they are the same "up to tight closure".

Specifically, let

$$P_\bullet : \ldots \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to 0$$

be a complex of finitely generated projective R-modules, where R is a Noetherian ring of positive prime characteristic p. We say that $P_\bullet$ has *phantom homology* at the ith spot if the cycles at that spot lie inside the tight closure of the boundaries within the ambient module P_i: that is, $Z_i \subseteq (B_i)^*_{P_i}$. If $P_\bullet$ has a phantom homology at the ith spot for all $i \geq 1$ then we say that $P_\bullet$ is *phantom acyclic.* If not only $P_\bullet$, but also all of its Frobenius iterates $F^e(P_\bullet)$ are phantom acyclic for all $e \geq 0$, then $P_\bullet$ is called *stably phantom acyclic.* In this last case $P_\bullet$ is called a *phantom resolution* of the augmentation module $H^0(P_\bullet) = M$ (assuming that the augmentation module M is nonzero). The length of the shortest finite stably phantom projective resolution of M, if one exists, is called the *phantom projective dimension* of M over R and is denoted by $ppd_R(M)$. For the zero module $M = 0$ we define $ppd_R(M) = -1$.

It turns out that a module M having at least one phantom projective resolution will necessarily have a finite one as well (see Theorem 2.1.7 in [1]). Thus $ppd_R(M)$ is a well-defined natural number when M has any phantom projective resolution; otherwise, we define $ppd_R(M) = +\infty$.

Phantom resolutions are really not too exotic and arise very naturally. An important source of phantom resolutions is base change: if M is a module of finite projective dimension, then any of its resolutions tensored with a module-finite extension S of R produces a phantom resolution of $S \otimes_R M$ (under mild conditions on the ring R; e.g., it is enough to assume that R is excellent, equidimensional and reduced).

Phantom homology turned out to be a very fruitful idea and was developed in a number of papers (see, for instance, [1], [3], [15], [20]). The class of modules of finite phantom projective dimension over a fixed ring R includes those of finite projective dimension but is usually larger (these two coincide for Cohen-Macaulay rings). There are lots of results on modules of finite phantom projective dimension that parallel the results for the usual notion of projective dimension. Just to name a few, the Buchsbaum-Eisenbud Acyclicity Criterion for finite complexes of free modules (see [7]) has a phantom analogue with depth conditions replaced by the height conditions (see, for instance, Theorem 9.8 in [14]), Auslander-Buchsbaum's formula that expresses the projective dimension via depths has a very similar version for phantom projective dimension and phantom depth (see Theorem 3.2.7 in [1]), an extension of the regularity theorem by Auslander-Buchsbaum-Serre for a local ring holds with finite phantom projective dimension (see Theorem 2.4.1 in [1]), etc. These phantom homology ideas were used to give extremely powerful results on vanishing of maps of Tors (see Section 4 in [15]). Another reason to be interested in phantom projective resolutions stems from a result in [3], which gives an instance when tight closure commutes with localization: that is, under mild conditions on the ring R, for a pair of finitely generated modules $N \subseteq M$ for which $ppd_R(M/N) < \infty$ and for any multiplicative system W in R we have $W^{-1}(N^*_M) = (W^{-1}N)^*_{W^{-1}M}$ (see section 5 in [3]) Tight closure does not commute with localization in general, as shown recently by Brenner and Monsky (see [5]).

Nevertheless, the notions of phantom projective dimension and phantom depth do not enjoy all the good properties of the usual ones. One of the most important reasons for that is that when one is given a module of finite phantom projective dimension, there is no canonical "constructive" way to build up a phantom resolution. For a module of finite phantom projective dimension different resolutions can even fail to be chain-isomorphic. Also, the behavior of such modules in short exact sequences is more complicated (e.g., it can happen that for R-modules $N \subseteq M$ we have $ppd_R(M) < \infty$, $ppd_R(M/N) < \infty$ but N has no finite phantom resolution).

My results include the demonstration of further instances of "bad" behavior for modules of finite phantom projective dimension that are not parallel to what happens with modules of finite projective dimension. In [1], the following natural conjecture was posed:

Conjecture 1.1. *Let R be a Noetherian ring of prime characteristic p and let M be a finitely generated R-module. Then $ppd_R M < \infty$ if and only if $ppd_{R_\mathfrak{m}} M_\mathfrak{m} < \infty$ for* *all maximal ideals* $\mathfrak{m}$ *of* R*.*

The "only if" part of this conjecture is trivial, since the phantom acyclic (and stably phantom acyclic) resolutions are preserved by flat base change (in particular, by localization). One of the main results in my dissertation is the construction of a counterexample to the "if" part of this conjecture. Specifically, I have been able to show the following (essentially all of Chapter III is devoted to proving this; see section 3.1 for more details):

Theorem 1.2. *Fix an arbitrary field* k *of positive prime characteristic* p*. Let* z, v, a, x, y *be indeterminates over* k*. Consider the following subring* R *of the localized polynomial ring* $k[\sqrt{z}, v, a, x, y]_z$*:*

$$R = k[z, z^{-1}, v, a, vx, vy, ax, ay, x^2, y^2, (1-a)x\sqrt{z}, (1-a)y\sqrt{z}\,]$$

Take the R*-module* M *to be* $M = R/(vx, vy)$*. Then for every maximal ideal* $\mathfrak{m}$ *of* R *we have* $ppd_{R_\mathfrak{m}}(M_\mathfrak{m}) = 2$ *but* M *does* not *have a finite phantom projective dimension over* R*.*

This example came out from an attempt to obtain a ring in which, after localization either at a or at $1-a$, complexes similar to Koszul complex give phantom resolutions; however, they cannot be modified to "patch" together to give a global phantom resolution over R.

I will sketch the main points of the proof. Verification of the fact that $ppd_{R_\mathfrak{m}} M_\mathfrak{m} < \infty$ for every maximal ideal $\mathfrak{m}$ of R is relatively easy. We have a phantom resolution of M_a over R_a

$$(*) \qquad 0 \to R_a \xrightarrow{\begin{bmatrix} y \\ -x \end{bmatrix}} R_a^2 \xrightarrow{\begin{bmatrix} vx & vy \end{bmatrix}} R_a \to 0$$

and a phantom resolution of M_{1-a} over R_{1-a}

$$(**) \qquad 0 \to R_{1-a} \xrightarrow{\begin{bmatrix} y\sqrt{z} \\ -x\sqrt{z} \end{bmatrix}} R_{1-a}^2 \xrightarrow{\begin{bmatrix} vx & vy \end{bmatrix}} R_{1-a} \to 0$$

Each of $(*)$ and $(**)$ is a phantom resolution by the Phantom Acyclicity Criterion. Any maximal ideal $\mathfrak{m}$ of R necessarily misses at least one of the elements a and $b = 1 - a$. So localizing the corresponding resolution $(*)$ or $(**)$ at $\mathfrak{m}$ we obtain phantom resolution of $M_{\mathfrak{m}}$ over $R_{\mathfrak{m}}$.

The difficult part of the proof is to show that M does not have finite phantom projective dimension. In short, the proof goes as follows: assume that there is a phantom resolution $P_\bullet$ of M over R. First, by using various tricks we reduce to the case when the resolution has a specific form

$$(\dagger) \qquad 0 \to P \to R^2 \xrightarrow{(vx,vy)} R \to 0$$

This part of the proof actually does not depend on the specific ring R and R-module M. The techniques developed are useful also in more general cases, when one needs to reduce a phantom resolution to a simpler form.

Next, we translate the question of the existence of a projective module P that makes $(\dagger)$ a phantom resolution into a question about the existence of a very special projective ideal of R. Finally, we explicitly find all possible choices for P after localizing at the elements a and $1-a$, by computing certain Picard groups, and then show that these possible choices for P_a and P_{1-a} cannot patch together properly to give the projective module P required in resolution (†).

This counterexample gives rise to several other negative results. First, after some adjustments of R and M from above, I was also able to provide a counterexample to another conjecture posed in [1].

Conjecture 1.3. *Let R be a Noetherian ring of characteristic p and let M and N be finitely generated R-modules. Then $ppd_R(M \oplus N) < \infty$ if and only if $ppd_R(M) < \infty$ and $ppd_R(N) < \infty$*

The "if" part is trivial, since the direct sum of phantom resolutions is still a phantom resolution. However, a phantom resolution of $M \oplus N$ does not necessarily decompose into direct sum. Specifically, I have shown the following (see section 4.1 for further details):

Theorem 1.4. *Fix an arbitrary field k of positive prime characteristic p. Let $z, v, a, x, y, t_{11}, t_{12}, t_{21}, t_{22}$ be indeterminates over k. Consider the following subring R of the localized polynomial ring*

$k[\sqrt{z}, v, a, x, y, t_{11}, t_{12}, t_{21}, t_{22}]_{z(t_{11}t_{22}-t_{12}t_{21})^2}$:

$$R = k[z, z^{-1}, v, a, vx, vy, ax, ay, x^2, y^2, (1-a)x\sqrt{z}, (1-a)y\sqrt{z}\,,$$
$$t_{ij}x, t_{ij}y, t_{ij}^2, at_{ij}, \tfrac{1}{(t_{11}t_{22}-t_{12}t_{21})^2}, (1-a)t_{ij}\sqrt{z}\,]$$

where $1 \leq i, j \leq 2$. Take the R-module M to be $M = R/(vx, vy)$. Then we have $ppd_R(M \oplus M) < \infty$ but M does not *have a finite phantom projective resolution.*

This question is still open in the local case: I conjecture that it is false there as well.

Another corollary is related to the question of finding a homological characterization of the phantom projective dimension. There are classical homological criterions for determining the finite projective dimension and the depth for a finitely generated module M over a Noetherian ring R. The projective dimension of M is finite if and only if the modules $\mathrm{Ext}^i_R(M,N)$ vanish for all R-modules N for some $i \geq 1$ (or in the local case this can be checked by verifying the vanishing of $\mathrm{Tor}_i(M,k)$, where k is the residue class field of R), whereas the depth is determined as $\mathrm{depth}_I M = \inf\{j : \mathrm{Ext}^j_R(R/I,M) \neq 0\}$. So it is natural to try to find similar homological characterizations of finite phantom projective dimension and phantom depth. In [1], the following result was proved:

Theorem 1.5. *Let R be a Noetherian ring of positive prime characteristic p. Let $P_\bullet$ be any projective resolution of M. If M has finite phantom projective dimension then $F^e(P_\bullet)$ has phantom homology at the first spot for all $e \geq 0$.*

It was conjectured that the converse is also true, which would have given a nice homological characterization of finite phantom projective dimension. However, the counterexample above shows that this is not the case, since the phantomness of the homology can be checked locally.

1.2 Results related to Test Elements for Tight Closure

Another part of the dissertation concerns the existence of test elements. In the definition of the tight closure given in the introduction it is possible that the element c of R, that was used to establish that x is in the tight closure of N in M, can depend on the choice of N, M and x. It turns out that for large class of rings there exist elements c in R that can be used for all tight closure tests. For such an element c $cu^q \notin N^{[q]}$ for even one value of q shows that $u \notin N^*_M$. Such elements are known to exist in many cases: e.g., when R is excellent local (for some versions, see section 6 in [14]), when R is F-finite (i.e., when R considered as an R-algebra via the Frobenius endomorphism is a finite R-module), as well as in the case of algebras essentially of finite type over an excellent semilocal ring (see Theorem 6.20 in [16]) and for the case of excellent domains of dimension at most 2 (see Theorem 1.3 in [2]). In the cases above the test elements for the tight closure were arising as powers of non-zerodivisor elements of the defining ideal of the singular locus of R. It is still an open question whether test elements exist for excellent rings of finite Krull dimension ≥ 3 without any extra hypotheses.

In [2] (Theorem 1.2) it is shown that (under mild conditions on the domain R) the test elements for the tight closure can be obtained as powers of the test elements for the Frobenius closure. Recall that for the ideal I of the ring R of prime characteristic p its Frobenius closure consists of all x such that $x^q \in I^{[q]}$ for some q. Therefore one possible approach suggested in [2] is to show that every nonzero element of the defining ideal of the singular locus of an excellent domain has power that is test element for Frobenius closure (see the last paragraph in section 2.5 for the relevant definitions). Unfortunately, this seems as difficult as finding tight closure test elements outright.

The main result of Chapter V (Theorem 5.1)came out from an attempt to find Frobenius closure test element. It says that for obtaining these elements it is enough to deal only with module-finite ring extensions of R within $R^{\frac{1}{p}}$ vas opposed to dealing with much larger ring $R^{\frac{1}{p}}$. In this case the condition for being a test element for tight closure is equivalent to a splitting-type condition on such subrings of R.

Theorem 1.6. *Let R be an excellent approximately Gorenstein normal domain of prime characteristic p. Let $c \neq 0$ be an element from the defining ideal of the singular locus of R (so that R_c is regular). Consider the following properties of the element c:*

(i) *For any subring S of $R^{1/p}$ such that S is module-finite over R, there exists an R-module map $\theta : S \to R$ such that $\theta(1) = c$.*

(ii) *c is a test element for tight closure.*

(iii) *For any ideal $I \subseteq R$ and for any subring S of $R^{1/p}$ such that S is module-finite over R we have $cIS \cap R \subseteq I$.*

We have the following implications:

(A) c satisfies (i) $\Longrightarrow$ c^3 satisfies (ii)

(B) c satisfies (ii) $\Longrightarrow$ c satisfies (iii)

(C) c satisfies (iii) $\Longrightarrow$ c satisfies (i)

The heart of the proof is the following result which is interesting in its own right (see Chapter V, Theorem 5.2):

Theorem 1.7. *Let R be a reduced excellent approximately Gorenstein ring, S be a ring containing R as a subring which is finitely generated as an R-module and $c \in R$ be a nonzerodivisor on S. Then the following are equivalent:*

(i) *There exists an R-module map $\theta : S \to R$ such that $\theta(1) = c$.*

(ii) *For every ideal $I \subseteq R$ we have $c(IS \cap R) \subseteq I$.*

(iii) *For every ideal $I \subseteq R$ primary to a maximal ideal of R we have $c(IS \cap R) \subseteq I$.*

This result suggests that for constructing test elements in the excellent case it might be enough to work towards proving "milder" versions of F-finiteness that appear in part (i) of the Theorem 1.6.

CHAPTER II

Technical Background and Notation

2.1 Basic Conventions

In this chapter we will establish some notation to be used throughout this work, as well as outline the basic standard facts from tight closure theory and other theories needed for reading this thesis. Throughout, all rings are assumed to be commutative and with multiplicative identity 1, and all modules are assumed to be unitary. All rings are also assumed to be Noetherian.

By a *local* ring $(R, \mathfrak{m})$ (or $(R, \mathfrak{m}, k)$ we mean a Noetherian ring whose unique maximal ideal is $\mathfrak{m}$ that has residue class field $R/\mathfrak{m} = k$. If $(R, \mathfrak{m})$ is local we denote the $\mathfrak{m}$-adic completion of R by $\hat{R}$. For an arbitrary ring R we will denote by R_{red} the homomorphic image of R by its nilradical.

Additionally, we will assume that all rings under consideration have positive prime characteristic p. Recall that the ring R is said to have (finite) characteristic p if p is the smallest positive integer such that $p \cdot 1 = 0$ where $p \cdot 1 = (1 + 1 + \ldots + 1)$ (p times). If no such integer p exists, then characteristic of the ring is defined to be 0. For an integral domain R the characteristic is clearly either 0 or a prime number p.

While many "natural" rings such as the ring of polynomials over the real or complex numbers have characteristic 0, there are many powerful techniques that enable one to prove theorems about rings containing a field of characteristic 0 by first proving the corresponding theorem about rings of prime characteristic $p > 0$.

We shall use the notation for powers of the characteristic that by now has become quite standard in tight closure theory when dealing with rings of positive characteristic. Specifically, whenever R has positive prime characteristic, we will always denote this characteristic by p, e will always denote varying natural number and we will denote powers of characteristic by $q = p^e$ for $e \in \mathbb{N}$. Thus, the statement "for all $q \gg 0$" will always mean "for all $q = p^e \gg 0$".

If R is a domain then by adjoining p-th roots of all elements of R (say in algebraic closure of the fraction field of R) we obtain a ring which we denote by $R^{1/p}$ that contains R as a subring and is naturally isomorphic to R. This definition extends easily to the reduced case as well. By adjoining all p^e-th roots of all elements of R we can similarly define the ring R^{1/p^e} for all $e \geq 1$.

For any ring R and any ideal I of it we denote by $I^{[q]}$ the ideal of R generated by q-th powers of all elements of I. In general this "bracket" power is much smaller than the usual power I^q. The *Frobenius closure* of I, denoted by I^F, consists of all elements $x \in R$ such that $x^q \in I^{[q]}$ for at least one q (note that once it holds for some fixed q it will hold also for all larger values of q as well). It is clear that we have $I^F = \bigcup_e (IR^{1/p^e} \cap R)$. $IR^{1/p} \cap R$ is a part of the Frobenius closure I^F; following [2] we will call it *the p-th root closure of I*. Note that p-th root closure is not an honest closure operation in a sense that it is not idempotent: iterating it usually produces larger and larger ideals.

The *Krull dimension* (or just the *dimension* of the ring R, $dim(R)$, is defined to be the supremum of the lengths of strictly increasing chains of prime ideals of the ring: here the chain $P_0 \subset P_1 \ldots \subset P_n$ has length n. It is possible for the Noetherian ring to have infinite dimension (see Example 1 of Appendix 1 in [24]). A local ring always has a finite Krull dimension.

We shall say that the local ring $(R, \mathfrak{m})$ is *equidimensional* if for every minimal prime P of R we have $dim(R/P) = dim(R)$. A Noetherian ring is called *locally equidimensional* if its localization at every maximal ideal is equidimensional.

For a local ring $(R, \mathfrak{m})$ of dimension d elements $x_1, \ldots, x_d$ are called a *system of parameters for* R if $R/(x_1, \ldots, x_d)$ has dimension 0. Every local ring has a system of parameters.

We say that elements $x_1, \ldots, x_n$ in a Noetherian (not necessarily local) ring R are *parameters* if for every prime ideal P of R containing them ther images in R_P are part of system of parameters for R_P.

We recall that an element x of a ring R is called *integral* over ideal I of R if it satisfies an equation of the form

$$x^k + i_1 x^{k-1} + \ldots + i_k = 0$$

where $i_j \in I^j$ for all $1 \leq j \leq k$. The set of elements integral over I is an ideal in R

denoted by $\overline{I}$ and called *integral closure of I*.

2.2 Regular, Cohen-Macaulay, Gorenstein and Approximately Gorenstein Rings

We shall say that the local ring $(R, \mathfrak{m}, k)$ of Krull dimension d is *regular* if $dim_k(\mathfrak{m}/\mathfrak{m}^2) = n$. A Noetherian ring is *regular* if its localization at every maximal ideal is regular. Examples include polynomial and power series rings in finitely many variables and their localizations.

We shall say that the local ring $(R, \mathfrak{m}, k)$ of dimension d is *Cohen-Macaulay* if some (equivalently, every) system of parameters $x_1, \ldots, x_d$ is a regular sequence on R, i.e. x_1 is a nonzerodivisor on R and each x_i is a nonzerodivisor on $R/(x_1, \ldots, x_{i-1})$ for all $2 \leq i \leq d$. A Noetherian ring is *Cohen-Macaulay* if its localization at every maximal ideal is Cohen-Macaulay.

We shall say that the local ring $(R, \mathfrak{m}, k)$ of dimension d is *Gorenstein* if it is Cohen-Macaulay and for some (equivalently, every) system of parameters $x_1, \ldots, x_d$ we have $dim_k(Ann_{R/(x_1,\ldots,x_d)}(\mathfrak{m})) = 1$. A Noetherian ring is *Gorenstein* if its localizations at all maximal ideals are Gorenstein.

Properties of being regular, Cohen-Macaulay, Gorenstein pass to all localizations. We have the following implications: regular $\Longrightarrow$ Gorenstein $\Longrightarrow$ Cohen-Macaulay. All these implications are non-reversible.

A local ring $(R, \mathfrak{m})$ is called *approximately Gorenstein* if every power of $\mathfrak{m}$ contains an irreducible m-primary ideal. E.g. every complete Noetherian reduced ring is approximately Gorenstein (see Theorem 5.2 in [12]), so this class of rings is also not too restrictive.

2.3 Excellent Rings

We will sometimes impose the condition of *excellence* for the rings under consideration. An excellent ring is a Noetherian commutative ring with many of the good properties shared by finitely generated algebras over a field or over integers $\mathbb{Z}$ and by complete local rings. This class of rings was defined by Grothendieck and excellence of a ring is closely related to the resolution of singularities of the associated scheme (see, for instance, Chapter 13 in [22] for details). The formal definition is quite technical (we will present it below). Most Noetherian rings that occur in algebraic geometry or number theory are excellent, so it is also not too restrictive a condition.

A ring R containing a field k is called *geometrically regular over* k if for any finite extension K of k the ring $R \otimes_k K$ is regular.

A homomorphism of rings from R to S is called *regular* if it is flat and for every prime P of R the fiber $S \otimes_R R_P/(PR_P)$ is geometrically regular over the residue field $R_P/(PR_P)$ of R_P.

A ring R is called a *G-ring* (or *Grothendieck ring*) if it is Noetherian and for any prime P of R the map from the local ring R_P to its completion is regular in the sense above.

A ring R is *catenary*, or *has the saturated chain condition* if for any pair of prime ideals P, Q of R, the maximal strictly increasing chains of primes from P to Q all have the same length. A ring is called *universally catenary* if all finitely generated algebras over it are catenary.

A ring R is called *excellent* if it is a universally catenary G-ring and for every finitely generated R-algebra S, the singular points of $\mathrm{Spec}(S)$ form a closed subset.

As mentioned above, most naturally occurring commutative rings in number theory or algebraic geometry are excellent. Examples of excellent rings include complete local rings (in particular all fields), Dedekind domains of characteristic 0 (in particular the ring Z of integers is excellent). Any localization of an excellent ring is excellent and any finitely generated algebra over an excellent ring is excellent. Hence, all localizations of finitely generated algebras over a complete local ring or over a Dedekind domain of characteristic 0 are excellent.

2.4 Matlis Duality

If $(R, \mathfrak{m}, k)$ is a local ring we will let $E_R(k)$ denote the injective hull of the residue class field. The functor $Hom_R(-, E_R(k))$ is exact since the module $E_R(k)$ is injective. The following result, known as Matlis Duality, will be used in the last chapter of the thesis:

Theorem 2.1. *Let $(R, \mathfrak{m}, k)$ be a complete local ring, let M be an Artinian R-module* *and let W be a finitely generated R-module. Let $E_R(k)$ denote the injective hull of the residue class field k and let $(-)^\vee$ denote the functor $Hom_R(-, E_R(k))$.*

(a) $R^\vee = E_R(k)$ while $E_R(k)^\vee = R$.

(b) $W^\vee$ is Artinian while $M^\vee$ is finitely generated.

(c) There are natural isomorphisms $(W^\vee)^\vee \cong W$ and $(M^\vee)^\vee \cong M$.

(d) The functor $(-)^\vee$ establishes an anti-equivalence between categories of modules with ascending chain condition and modules with descending chain condition.

For the proof and necessary background see, for instance, 3.2.13 in [6].

2.5 Tight Closure and Test Elements

Tight closure was introduced in series of papers by Hochster and Huneke (see e.g. [14], [20] and references therein) and since then was developed by a very large number of people in numerous papers. We shall now briefly review some basic facts from the tight closure theory, especially those related to test elements. Although a big part of the theory also works for rings containing the rationals, we will be interested only in finite characteristic case here.

Tight closure in its simplest form is a closure operation for ideals of of a ring containing a field, but it easily extends to submodules of a module. The word "tight" refers to the fact that for ideals the tight closure is contained inside the integral closure but is often much smaller.

Specifically, let R be a Noetherian ring of positive prime characteristic p, let M be an R-module and let N be an R-submodule of M. For every integer $e \geq 0$ the eth power of Frobenius endomorphism maps R to itself by $x \mapsto x^q$ where $q = p^e$. Let S_e

denote R considered as an R-algebra via the eth power of Frobenius. The Peskine-Szpiro functor, F^e is $S_e \otimes_R$, a covariant functor from R-modules to S_e-modules. Since $S_e = R$ this is actually a functor from R-modules to R-modules. For any R-module M we have $a(b \otimes m) = (ab) \otimes m$ and $b \otimes am = (ba^q) \otimes m$ in $F^e(M)$. It is easy to see that explicitly the action of F^e on a finite presentation of an R-module is as follows: if $M = \operatorname{coker}(a_{ij})_{t \times s}$ then $F^e(M) = \operatorname{coker}(a_{ij}^q)_{t \times s}$.

We will denote by R° the complement of the union of all minimal prime ideals of R.

We say that $x \in M$ is in the *tight closure* of N, if there exists $c \in R^\circ$ such that in $F^e(M)$ we have $cx^q \in \operatorname{im}(F^e(N) \to F^e(M))$ for all $q \gg 0$. Here x^q stands for the image of x under the natural map $M \to F^e(M)$ that takes $m \mapsto 1 \otimes m$.

We say that N is *tightly closed* in M if $N_M^* = N$. We say that the ring R is *weakly F-regular* if every ideal of R is tightly closed. We say that R is *F-regular* if R_P is weakly F-regular for all prime ideals P of R. Quite recently Brenner and Monsky have shown that tight closure does *not* commute with localization; but it is still not known whether the notions of F-regularity and weak F-regularity coincide.

The tight closure of the R-submodule N of the module M is denoted by N_M^* or just by N^* when it is clear what M is; note that the tight closure *does* depend on which ambient module we are taking it in. It is customary to denote by $N_M^{[q]}$ the submodule of $F^e(M)$ generated by the image of $F^e(N)$, i.e. $\operatorname{im}(F^e(N) \to F^e(M))$. Clearly, $N_M^{[q]}$ is the R-submodule of $F^e(M)$ generated by all $\{x^q : x \in N\}$. In the special case $M = R$ and $N = I$ an ideal of R it agrees with the usual "bracket" power: $F^e(R) = R$ and I^q_R is the ideal of R generated by all $\{i^q : i \in I\}$.

The following proposition summarizes some of the basic properties of the tight closure:

Proposition 2.2. *Let R be a Noetherian ring of finite prime characteristic p, let M be an R-modules and let I be an ideal of R.*

(a) For any R submodule N of M, N^*_M is an R-submodule of M that contains N. For finitely generated M we have $(N^*_M)^*_M = N$. If $N_1 \subseteq N_2$ are R-submodules of M then $(N_1)^*_M \subseteq (N_2)^*_M$.

(b) If R is regular, then all ideals of R are tightly closed, i.e. R is F-regular.

(c) Let R be a weakly F-regular ring. Then every submodule N of a finitely generated module M is tightly closed in M.

(d) If $R \hookrightarrow S$ is an integral extension, then $(IS)^* \cap R \subseteq I^*$.

(e) Let R is local ring of dimension d which is a homomorphic image of a Cohen-Macaulay ring and let $x_1, \ldots, x_d$ be parameters in R. Then $(x_1, \ldots, x_i) : x_{i+1} \subseteq (x_1, \ldots, x_i)^*$ ("Colon Capturing").

(f) If l is the minimal number of generators of ideal I of positive height then we have $\overline{I^l} \subseteq I^* \subseteq \overline{I}$.

(g) Let $\phi : R \to S$ be a homomorphism of Noetherian rings of characteristic p and let I be an ideal of R. Assume that R is essentially of finite type over an excellent local ring. Then $\phi(I^*) \subseteq (IS)^*$ ("Persistence of Tight Closure").

Proof. For (a)-(c) see Proposition 8.5, Theorem 4.6, Proposition 8.7 in [14]. For (d), (e) see Theorems 1.7 and 3.1 in [20]. For (f) see Theorems 5.2 and 5.4 in [14].

For (g) see Theorem 2.3 in [20].

In the definition of tight closure the element c can depend on x, N, M. It turns out that in many instances this element can be chosen so that it works for *all* tight closure tests. The existence of test elements in interesting in its own right but also makes it possible to prove many results that do not seem very related to tight closure. For example, existence of test elements enables one to prove a uniform version of Artin-Rees property for large class of rings (see, for instance, [20] and [2]).

We say that an element $c \in R^{\circ}$ is a *test element* for the reduced ring R if for any finitely generated R-modules $N \subseteq M$ we have $x \in N^*_M$ if and only if $cx^q \in N^{[q]}_M$ for all q. We say that c is *locally* (respectively, *completely*) *stable test element* if its image in (respectively, in the completion of) in R_P is a weak test element for R_P for every prime ideal P of R.

In [16] the following fundamental result was proved:

Theorem 2.3. *(a) Let $(R, \mathfrak{m}, k)$ be a reduced local ring of characteristic p such that $R \to \hat{R}$ has regular fibers and let $c \in R^{\circ}$ be any element such that R_c is regular. Then c has a power that is a completely stable test element for R.*

In particular, if R is a reduced excellent local ring, such elements c always exist, so R has a completely stable test element.

(b)Let R be a reduced algebra essentially of finite type over a local ring B such that $B \to \hat{B}$ has smooth fibers (e.g., in case when B is excellent). Let $c \in R^{\circ}$ be an element such that R_c is regular. Then c has a power that is a completely stable test element for R.

The proof is quite technical and is done essentially by reduction to F-finite case, for which existence of test elements is relatively easier to prove (F-finite means that R is a finitely generated R-module under the Frobenius map). It is quite natural to expect that test elements should exist for all sufficiently good rings. For excellent rings of finite Krull dimension less or equal than 2 the existence of completely stable test elements was proved in [2]. It is still an open question whether or not all excellent rings of dimension ≥ 3 have a test element.

It is clear that if c is a test element then for every ideal I of R we have $cI^* \subseteq I$. By analogy, we we will call an element $c \in R^\circ$ *a test element for the Frobenius closure* (respectively, *a test element for p-th root closure*) if for every ideal I of R we have $c(I^F) \subseteq I$ (respectively, $c(IR^{1/p} \cap R) \subseteq I$).

2.6 Phantom Homology and Phantom Projective Resolutions

In this section we will outline the basic definitions and results relevant to phantom homology. The notion of phantom homology was introduced by Hochster and Huneke in [14] and further developed in a number of papers (see, for instance, [1], [3], [15], [20], [9], [10]).

Let

$$P_\bullet : \ldots \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to 0$$

be a complex of finitely generated projective R-modules, where R is a Noetherian ring of positive prime characteristic p.

We shall say that $P_\bullet$ has a *phantom homology* at ith spot if the cycles at that spot lie inside the tight closure of the boundaries within the ambient module P_i: $Z_i \subseteq (B_i)^*_{P_i}$. If $P_\bullet$ has a phantom homology at ith spot for all $i \geq 1$ then we say that $P_\bullet$ is *phantom acyclic*. If not only $P_\bullet$, but also all of its Frobenius iterates $F^e(P_\bullet)$ are phantom acyclic for all $e \geq 0$ then $P_\bullet$ is called *stably phantom acyclic*. In this last case $P_\bullet$ is called a *phantom resolution* of the augmentation module $H^0(P_\bullet) = M$. The length of the shortest stably phantom projective resolution of M is called the *phantom projective dimension* of M over R and is denoted by $ppd_R(M)$.

If M is a module of finite projective dimension, then under mild conditions on the ring R (e.g., it is enough to assume that R is excellent, equidimensional and reduced) any of its resolutions tensored with a module-finite extension S of R produces a phantom resolution of $S \otimes_R M$ over S.

Roughly speaking, the phantom resolutions are "almost acyclic": at each spot the module of cycles lies between the module of boundaries and its tight closure. In other words, the modules of cycles and boundaries are the same "up to tight closure". The word "phantom" refers to the fact that the original homology vanishes after tensoring with weakly F-regular ring.

One of the beautiful results in this area that we will employ a great deal is the following criterion for the phantomness of a finite free resolution over a Noetherian ring of characteristic p (with mild conditions on the ring) which is analogous to the Buchsbaum-Eisenbud criterion (see [7]) that deals with "honest" acyclicity. In its simplest form the criterion says the following (see [14], [15], [3] for proof and generalizations):

Theorem 2.4. *(Phantom Acyclicity Criterion)*

Let R be a reduced Noetherian ring of characteristic p. Suppose that R is a homomorphic image of a Cohen-Macaulay ring and is locally equidimensional. Let

$$G_\bullet : 0 \to G_n \xrightarrow{d_n} \dots \xrightarrow{d_1} G_0 \to 0$$

be a complex of finitely generated free R-modules. Denote $b_i := rank(G_i)$ for $1 \leq i \leq n$. Suppose that $b_i = rank(d_i) + rank(d_{i+1})$ for $1 \leq i \leq n$ and suppose that the height of the ideal $I_i = I_{rank(d_i}$ is at least i for $1 \leq i \leq n$. Then $G_\bullet$ is stably phantom acyclic. Conversely, if $G_\bullet$ is stably phantom acyclic then the above-mentioned conditions on rank and height hold.

CHAPTER III

Finite Phantom Projective Dimension Locally Does Not Imply Finite Phantom Projective Dimension Globally

3.1 Statement of Conjecture, Counterexample and Sketch of Proof

We are going to construct a counterexample to the following conjecture posed in [1]:

Conjecture. *Let R be a Noetherian ring of prime characteristic p and let M be a finitely generated R-module. Then $ppd_R M < \infty$ iff $ppd_{R_{\mathfrak{m}}} M_{\mathfrak{m}} < \infty$ for all maximal ideals $\mathfrak{m}$ of R.*

The preservation of the phantom homology by flat base change (see (2.1.3.) in [1]) makes the "only if" part trivially true. We will construct a counterexample to the "if" part.

Fix an arbitrary field k of positive prime characteristic p. Let z, v, a, x, y be indeterminates over k. Consider the following ring:

$$R = k[z, z^{-1}, v, a, vx, vy, ax, ay, x^2, y^2, (1-a)x\sqrt{z}, (1-a)y\sqrt{z}\,]$$

which is a subring of the localized polynomial ring $k[\sqrt{z}, v, a, x, y]_z$. Take the R-module M to be $M = R/(vx, vy)$. We denote $b := 1 - a$.

Note that the normalization of R, i.e., the integral closure of R inside its field of fractions $\mathrm{frac}(R) = k(\sqrt{z}, x, y, a, v)$, is $S = R[\sqrt{z}\,] = k[\sqrt{z}, \sqrt{z}^{\,-1}, a, v, x, y]$. Indeed, the square of the element

$$\sqrt{z} = \frac{((1-a)x\sqrt{z})v}{(1-a)(vx)}$$

of $\mathrm{frac}(R)$ is in R and $R[\sqrt{z}\,]$ contains the elements $\sqrt{z}^{\,-1} = z^{-1} \cdot \sqrt{z}\,$, and

$$x = (ax) + \sqrt{z}^{\,-1}((1-a)x\sqrt{z})),\ y = (ay) + \sqrt{z}^{\,-1}((1-a)y\sqrt{z}))$$

Therefore $S = R[\sqrt{z}\,] = k[\sqrt{z}, \frac{1}{\sqrt{z}}, a, v, x, y]$ is a subring of $\mathrm{frac}(R)$ that is integral over R and is integrally closed (since it is the localization of the polynomial ring $k[\sqrt{z}, a, v, x, y]$ at the element $\sqrt{z}\,$), so it is indeed the integral closure of R inside $\mathrm{frac}(R)$.

First, let us verify that in fact $ppd_{R_{\mathfrak{m}}} M_{\mathfrak{m}} < \infty$ for every maximal ideal $\mathfrak{m}$ of R. We have a phantom resolution of M_a over R_a

$$(*) \qquad 0 \to R_a \xrightarrow{\begin{bmatrix} y \\ -x \end{bmatrix}} R_a^2 \xrightarrow{\begin{bmatrix} vx & vy \end{bmatrix}} R_a \to 0$$

and a phantom resolution of M_b over R_b

$$(**) \qquad 0 \to R_b \xrightarrow{\begin{bmatrix} y\sqrt{z} \\ -x\sqrt{z} \end{bmatrix}} R_b^2 \xrightarrow{\begin{bmatrix} vx & vy \end{bmatrix}} R_b \to 0$$

(note that $y = \frac{1}{a} \cdot (ay)$ and $x = \frac{1}{a} \cdot (ax)$ are in R_a and $y\sqrt{z} = \frac{1}{1-a} \cdot ((1-a)y\sqrt{z})$ and $x\sqrt{z} = \frac{1}{1-a} \cdot ((1-a)x\sqrt{z}\,)$ are in R_b). Each of $(*)$ and $(**)$ is a phantom resolution by the Phantom Acyclicity Criterion (see e.g., (3.21) of [15]): the ranks of all the relevant matrices are 1 and they add up correctly, $\mathrm{ht}(vx, vy) \geq 1$ in both domains R_a and

R_b, $\mathrm{ht}_{R_a}(x,y) = 2$ via the chain of primes $(0) \subset (x) \subset (x,y)$ in R_a, and for any prime $\mathfrak{p} \supseteq (x\sqrt{z}, y\sqrt{z})$ of R_b and any prime $\mathfrak{q}$ of S_b lying over it we have $\mathrm{ht}(\mathfrak{p}) \geq \mathrm{ht}(\mathfrak{q}) \geq 2$ (in S the same chain of primes works for $\mathfrak{q}$: $\mathfrak{q} \supseteq (x,y) \supset (x) \supset (0)$). Thus each of $(*)$ and $(**)$ is a phantom resolution of the corresponding augmentation module. Any maximal ideal $\mathfrak{m}$ of R necessarily misses at least one of the elements a and $b = 1-a$. So localizing the corresponding resolution $(*)$ or $(**)$ at $\mathfrak{m}$ we obtain phantom resolution of $M_{\mathfrak{m}}$ over $R_{\mathfrak{m}}$ (again by (2.1.3.) in [1]). Thus we have $ppd_{R_{\mathfrak{m}}} M_{\mathfrak{m}} = 2$ for all maximal ideals $\mathfrak{m}$ of R.

It remains to show that M does not have a finite phantom projective dimension over R. The proof is rather long and takes the rest of this chapter. Briefly, it goes as follows: assume that there is a phantom resolution $P_\bullet$ of M over R. First, by using various tricks we reduce to the case when the resolution has a specific form

$$(\dagger) \qquad 0 \to P \to R^2 \xrightarrow{(vx,vy)} R \to 0$$

Then we translate the question of existence of P that makes $(\dagger)$ a phantom resolution into the question of the existence of a very special projective ideal of R (this among other things requires determining the module of relations on vx, vy over R). Finally, we explicitly find all possible choices for P after localizing at a and $b = 1-a$ (by computing certain Picard groups) and show that these possible choices for P_a and P_b cannot patch together properly to give the projective module P required in the resolution $(\dagger)$.

3.2 Reduction to a Simpler Phantom Resolution

Now assume $ppd_R M < \infty$ and let

$$P_\bullet : 0 \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to 0$$

be a phantom projective resolution of M.

3.2.1 Reduction to resolutions of the form $0 \to P \xrightarrow{g} R^n \xrightarrow{f} R^m \oplus R \to 0$

First, we will make the resolution as short as possible with all but one of the modules in the resolution free.

Lemma 3.1. *Let R be a Noetherian ring of positive prime characteristic p and let M be an R-module such that for some fixed positive integer n we have $ppd_{R_\mathfrak{m}} M_\mathfrak{m} \leq n$ for every maximal ideal $\mathfrak{m}$ of R. If $ppd_R M < \infty$ then M has a phantom projective resolution of length $\leq n$.*

Proof. Fix any finite phantom resolution $P_\bullet$ of M. For every maximal ideal $\mathfrak{m}$ of R $(P_\bullet)_\mathfrak{m}$ is a phantom resolution of $M_\mathfrak{m}$ over $R_\mathfrak{m}$ (by (2.1.3) in [1]). $ppd_{R_\mathfrak{m}} M_\mathfrak{m} \leq n$ implies that $(P_\bullet)_\mathfrak{m}$ splits from nth spot on (by (2.1.5) and (3.1.1.) in [1]). A map of finitely generated R-modules that splits locally splits globally (see Lemma 1 in [11] and the remark after its proof) so $P_\bullet$ is a direct sum of a split exact complex and a phantom resolution of length $\leq n$. $\square$

Lemma 3.2. *Let R be a Noetherian ring of positive prime characteristic p and let $P_\bullet$ be a phantom projective resolution of M of length n. Then M has a phantom projective resolution of the same length in which every module except for the leftmost one is free.*

Proof. Let $P_\bullet$ be a phantom resolution of M:

$$P_\bullet : 0 \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to 0$$

where all P_i are finitely generated projective modules. Choose a finitely generated projective module Q_0 such that $P_0 \oplus Q_0$ is free of finite rank, and then for each $i = 1, \ldots, n-1$ choose a finitely generated projective module Q_i such that $P_i \oplus Q_{i-1} \oplus Q_i$ is free of finite rank. Taking direct sum with trivial complexes of the form $0 \to Q_i \xrightarrow{id} Q_i \to 0$ does not affect any of the homology modules. The resulting complex

$$\begin{array}{ccccccc}
0 \to P_n \to & \ldots \to & P_3 \to & P_2 & \to P_1 & \to P_0 & \to 0 \\
 & & & & \bigoplus & \bigoplus & \\
 & & & 0 & \to Q_0 & \xrightarrow{id} Q_0 & \to 0 \\
 & & & \bigoplus & \bigoplus & & \\
 & & 0 \to & Q_1 & \xrightarrow{id} Q_1 & \to 0 & \\
 & & \bigoplus & \bigoplus & & & \\
 & 0 \to & Q_2 \xrightarrow{id} & Q_2 & \to 0 & & \\
 & & \ldots & & & &
\end{array}$$

is still a phantom resolution of M of length n where every module except for the leftmost one is free. $\square$

We can say more about the last map in this phantom resolution for a *cyclic* module M.

Lemma 3.3. *Let R be a Noetherian ring of positive prime characteristic p and let I be an ideal of R such that $ppd_R R/I = d \geq 2$. Then R/I has a phantom projective* *resolution of length d*

$$0 \to P \to R^n \to \dots \xrightarrow{f} R^m \oplus R \to 0$$

where all modules except possibly the leftmost one are free and the image of the rightmost map is $im(f) = R^m \oplus I$.

Proof. By Lemma 3.2 we can assume that the augmented phantom resolution of R/I has the form

$$0 \to P \to \dots \to R^n \xrightarrow{f} R^m \xrightarrow{g} R/I \to 0$$

Take the direct sum of this complex with the trivial complex $0 \to R \xrightarrow{id_R} R \to 0$. As a result, we get a complex of the form

$$0 \to P \to \dots \to R^n \oplus R \xrightarrow{f \oplus id_R} R^m \oplus R \xrightarrow{g \oplus 0_R} R/I \to 0$$

It is clear that the homology at spots $i > 0$ is unaffected and the augmentation module is still R/I, so we again have a phantom resolution of R/I. Let $e_1, \dots, e_m, e_{m+1}$ be the standard basis in $R^m \oplus R$. The generators $e_1, \dots e_m$ map to elements $\bar{t}_1, \dots, \bar{t}_m$ of R/I such that there is an R-linear combination $r_1\bar{t}_1 + \dots + r_m\bar{t}_m = 1$ in R/I, whereas e_{m+1} maps to 0 in R/I. Let t_i be a lift of the element $\bar{t_i}$ to R for $i = 1, \dots, m$.

It is clear that the element $b_{m+1} = e_{m+1} + \sum_{i=1}^{m} r_i e_i$ maps to $\bar{1}$ in R/I, and so the elements $b_i = e_i - t_i b_{m+1}$ for $i = 1, \dots, m$ map to $\bar{0}$. Change the basis in $R^m \oplus R$ to $b_1, \dots, b_m, b_{m+1}$. In this new basis, the last map in the augmented phantom resolution has a kernel $R^m \oplus I$. $\square$

In the previous section we have shown that $ppd_{R_\mathfrak{m}} M_\mathfrak{m} = 2$ for every maximal ideal $\mathfrak{m}$ of R. By Lemma 3.1 and Lemma 3.3 we can assume that the phantom resolution of that specific cyclic module M has the form

$$0 \to P \xrightarrow{g} R^n \xrightarrow{f} R^m \oplus R \to 0$$

with $\mathrm{im}(f) = R^m \oplus (vx, vy)$.

3.2.2 Reduction to resolutions of the form $0 \to P \xrightarrow{g} R^n \xrightarrow{f} R \to 0$

Now we will need the following lemma:

Lemma 3.4. *Let R be Noetherian ring of prime characteristic p and let M be a finitely generated R-module with $ppd_R M < \infty$. Assume that we have fixed a stably phantom acyclic projective resolution of M:*

$$P_\bullet : 0 \to P_n \xrightarrow{d_n} \dots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$$

and a surjection $\theta : G \to Im(d_k)$, where $1 \le k \le n-1$ and G is a projective R-module. The projectivity of G implies that θ will lift to a map $\tilde{\theta} : G \to P_k$. We can modify $P_\bullet$ by changing the modules at k-th and $k+1$-th spots as following:

$$(P_\bullet)_{\theta,\tilde{\theta}} : 0 \to P_n \xrightarrow{d_n} \dots \xrightarrow{d_{k+2}} G \oplus P_{k+1} \xrightarrow{f} G \oplus P_k \xrightarrow{g} P_{k-1} \xrightarrow{d_{k-1}} \dots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$$

where $f|_{P_{k+1}} = d_{k+1}$, $f|G : m \mapsto m \oplus (-\tilde{\theta}(m))$ and $g|_{P_k} = d_k$, $g|_G = \theta$. Then the complex $(P_\bullet)_{\theta,\tilde{\theta}}$ is also a stably phantom acyclic resolution of M.

Proof. The fact that $(P_\bullet)_{\theta,\tilde{\theta}}$ is still a complex is immediate from the definitions of the maps f and g. Denote by G' the image of G under f. $f|_G$ is clearly injective so $G' \cong G$. Note that $G' \cap P_k = 0$: $m \oplus (-\tilde{\theta}(m)) \in P_k$ implies that $m \in P_k \cap G = 0$. Also every element in $G \subseteq G \oplus P_k$ can be rewritten as follows: $m \oplus 0_{P_k} = (m \oplus (-\tilde{\theta}(m))) + (0_G \oplus \tilde{\theta}(m)) \in G' + P_k$. Therefore $G \oplus P_k = G' \oplus P_k$.

Denote $B_i = \mathrm{im}(d_{i+1})$, $Z_i = \ker(d_i)$. Using the identification $G \oplus P_k = G' \oplus P_k$ we see that in $(P_\bullet)_{\theta,\tilde{\theta}}$ the boundaries at the kth spot are $\mathrm{im} f = f(G) + f(P_{k+1}) = G' + B_k = G' \oplus B_k$ (the last internal sum is direct because $B_k \subseteq P_k$ and $G' \cap P_k = 0$) and the cycles are $G' \oplus Z_k$; therefore we have phantom homology at that spot: $(G' \oplus B_k)^*_{G' \oplus P_k} \supseteq G' \oplus Z_k$ since, by the phantom acyclicity of $P_\bullet$, we have $Z_k \subseteq (B_k)^*_{P_k}$. θ is an onto mapping, so the image at $(k-1)$th spot is also unaffected. Thus at all spots of $(P_\bullet)_{\theta,\tilde{\theta}}$ we have phantom acyclicity.

It remains to show only that $(P_\bullet)_{\theta,\tilde{\theta}}$ is *stably* phantom acyclic. Applying F^e to $P_\bullet$ gives a stably phantom acyclic resolution of $F^e(M)$. Note that $F^e(\theta)$ is still a surjection of $F^e(G)$ onto $F^e(\mathrm{im}(d_k))$, $F^e(\tilde{\theta})$ still lifts $F^e(\theta)$ to $F^e(P_k)$ and we have $F^e(P_\bullet)_{F^e(\theta),F^e(\tilde{\theta})} = F^e((P_\bullet)_{\theta,\tilde{\theta}})$. Therefore the same argument as above applied to $F^e(P_\bullet)_{F^e(\theta),F^e(\tilde{\theta})}$ shows that the resolution $F^e((P_\bullet)_{\theta,\tilde{\theta}})$ is still phantom acyclic. $\square$

Take $\theta : R^m \oplus R^2 \to R^m \oplus (vx, vy)R = \mathrm{im}(f)$ to be the map that is the identity on R^m and maps the standard basis elements of R^2 to vx, vy. By Lemma 3.4 we get a phantom resolution of M of the form

$$0 \to R^m \oplus R^2 \oplus P \xrightarrow{\tilde{g}} R^m \oplus R^2 \oplus R^n \xrightarrow{\tilde{f}} R^m \oplus R \to 0$$

with $\tilde{f}|_{R^m \oplus R^2} = f$. Let $e_1, \ldots, e_m, e_{m+1}, e_{m+2}, u_1, \ldots, u_n$ be the standard basis of $R^m \oplus R^2 \oplus R^n$ and let $\tilde{f}(u_i) = a_i \oplus r_i$ where $a_i \in R^m$ and $r_i \in (vx, vy)R$. Replace the standard basis by $e_1, \ldots, e_m, e_{m+1}, e_{m+2}, (-a_1) \oplus 0_{R^2} \oplus u_1, \ldots, (-a_n) \oplus 0_{R^2} \oplus u_n$. Since $\tilde{f}(e_i) = e_i \oplus 0_R$ for $i = 1, \ldots, m$ and $\tilde{f}((-a_i) \oplus 0_{R^2} \oplus u_i) \in 0_{R^m} \oplus 0_{R^2} \oplus (vx, vy)$

the resolution in this new basis looks like

$$\begin{array}{ccccccc} & & & R^m & \xrightarrow{\mathrm{id}_{R^m}} & R^m & \\ & & & \bigoplus & & \bigoplus & \\ 0 \to & R^m \oplus R^2 \oplus P & \xrightarrow{\tilde{g}} & R^2 \oplus R^n & \xrightarrow{\alpha} & R & \to 0 \end{array}$$

It is clear that $\mathrm{im}(\alpha) = (vx, vy)$. Note that $\mathrm{im}(\tilde{g}) \subseteq \ker(\tilde{f}) = \ker(\alpha) \subseteq R^2 \oplus R^n$ so $R^m \oplus R^2 \oplus P$ maps into $R^2 \oplus R^n$ under $\tilde{g}$ and therefore we can remove the trivial complex $R^m \xrightarrow{id} R^m$ and we will still have a phantom resolution.

So without loss of generality the phantom resolution of M has the form

$$0 \to P_1 \xrightarrow{g} R^{n_1} \xrightarrow{f} R \to 0$$

where $\mathrm{im}(f) = (vx, vy)$. Here $n_1 = n + 2$ and $P_1 = R^m \oplus R^2 \oplus P$.

3.2.3 Reduction to resolutions of the form $0 \to P \xrightarrow{g} R^2 \xrightarrow{(vx,vy)} R \to 0$

Let $\theta : R^2 = Re_1 \oplus Re_2 \to R$ be the R-module map taking e_1, e_2 respectively to vx, vy. Again, applying Lemma 3.4 we get a phantom resolution of M of the form

$$0 \to R^2 \oplus P \xrightarrow{\tilde{g}} R^2 \oplus R^n \xrightarrow{\tilde{f}} R \to 0$$

with $\tilde{f}|_{R^n} = f$. By changing the standard basis $e_1, e_2, u_1, \ldots, u_n$ of $R^2 \oplus R^n$ to $e_1, e_2, u_1 - a_1e_1 - b_1e_2, \ldots, u_n - a_ne_1 - b_ne_2$ where $f(u_i) = a_i(vx) + b_i(vy)$ for $i = 1, \ldots, n$ with $a_i, b_i \in R$, we can assume that $\ker(\tilde{f}) = W \oplus R^n$ where $W = \mathrm{syz}^1(vx, vy) \subseteq R^2$ is the module of relations on vx, vy over R. The phantomness of the resolution means that $\mathrm{im}(\tilde{g}) \subseteq W \oplus R^n \subseteq (\mathrm{Im}(\tilde{g}))^*_{R^2 \oplus R^n}$. Now we want to see that in this case $\mathrm{im}(\tilde{g})$ has a very specific form. We will need the following lemma:

Lemma 3.5. *Let R be a Noetherian ring of prime characteristic p and let $W \subseteq R^h$, $V \subseteq R^n \oplus R^h$ be submodules such that $V \subseteq R^n \oplus W \subseteq V^*_{R^n \oplus R^h}$. Then there is a split* *exact sequence*

$$0 \to V \cap R^h \to V \to R^n \to 0$$

and $(V \cap R^h)^*_{R^h} \supseteq W$. *Also, if* V *is projective, then so is* $V \cap R^h$.

In order to prove the Lemma 3.5 we shall need to use the following

Lemma 3.6. *Let* R *be Noetherian ring of prime characteristic* p *and let* $N \subseteq M$ *be* R*-modules such that* $N^*_M = M$. *Then* $N = M$.

Proof. It is enough to check $M = N$ locally, since $(N_\mathfrak{m})^*_{M_\mathfrak{m}} \supseteq (N^*_M)_\mathfrak{m} = M_\mathfrak{m}$. So without loss of generality we can assume that $(R, \mathfrak{m}, K)$ is local. The canonical composition $p : M \twoheadrightarrow M/N \twoheadrightarrow K \otimes_R M/N \cong K^l$ maps N to 0. $N^*_M = M$ implies that $0^* = K^l$ (for every $m \in M$ $\exists c \in R^\circ$ such that $cx^q \in N^{[q]}$; now mapping by $p \otimes F^e$ we get $p(m) \in p(N)^* = 0^*$) which is possible only if $l = 0$, i.e. $M = N$.□

Proof of Lemma 3.5. Consider the projection map $\pi : R^n \oplus R^h \longrightarrow R^n$. For any $x \in R^n$, $x \oplus 0_{R^h} \in R^n \oplus W \subseteq V^*_{R^n \oplus R^h}$ so there exists $c \in R^\circ$ such that for all $q = p^e \gg 0$ we have $c(x \oplus 0_{R^h})^q \in V^{[q]}_{R^n \oplus R^h}$. Taking images under $\pi \otimes F^e$ we get $cx^q \in (\pi(V))^{[q]}_{R^n}$ for all $q \gg 0$ so that $x \in (\pi(V))^*_{R^n}$. Therefore we have $(\pi(V))^*_{R^n} = R^n$ which implies $\pi(V) = R^n$ by Lemma 3.6. Thus the projection map $V \subseteq R^n \oplus R^h \longrightarrow R^n$ is surjective with kernel $W_0 := V \cap \ker(\pi) = V \cap R^h$. This gives the required short exact sequence. R^n is projective, and so the sequence splits. In particular, $V \cap R^h$ will be a direct summand of V from which the very last statement of the lemma follows. It remains only to show that $(W_0)^*_{R^h} \supseteq W$.

Since $\pi(V) = R^n$ and $V \subseteq R^n \oplus W$ we have that for every $i = 1, \ldots, n$ there exist $w_i \in W$ such that $e_i \oplus w_i \in V$ (here $e_1, \ldots, e_n$ denotes the standard basis for R^n). Clearly we have $V = \sum R(e_i \oplus w_i) \oplus W_0$. Take an arbitrary $w \in W$. $0_{R_n} \oplus w \in V^*_{R^n \oplus R^h}$ so there exists $c \in R^\circ$ such that for every $q = p^e \gg 0$ there are $r_i \in R$ and $\delta \in (W_0)^{[q]}_{R^h}$ for which

$$c(0_{R^n} \oplus w)^q = \sum_{i=1}^{n} r_i(e_i \oplus w_i)^q + \delta$$

Taking the image of this equation under $\pi \otimes F^e$ we get $0 = \sum_{i=1}^n r_i e_i^q$, which necessarily gives $r_i = 0$. Hence the equation above is just $cw^q = \sum_{i=1}^n r_i w_i^q + \delta \in (W_0)^{[q]}_{R^h}$ so that $w \in (W_0)^*_{R^h}$. $\square$

The Lemma 3.5 and its proof show that $\mathrm{im}(\tilde{g}) = W_0 \oplus \sum_{i=1}^n R(u_i \oplus w_i)$ with $w_i \in W$ and W_0 such that $W_0 \subseteq W \subseteq W_0^*$. Replacing the basis elements u_i by $u_i \oplus w_i$ we can assume that in the resolution of M we still have $\ker(\tilde{g}) = W \oplus R^n$ and $\mathrm{im}(\tilde{f}) = W_0 \oplus R^n$ with $W_0 \subseteq W \subseteq (W_0)^*_{R^2}$.

We have now reduced to the case when the phantom resolution of M has the form

$$0 \to P \xrightarrow{f} R^n \oplus R^2 \xrightarrow{g} R \to 0$$

where $g(R^n) = 0$, $g|_{R^2} : R^2 \xrightarrow{(vx,vy)} R$, so that $\ker(g) = R^n \oplus W$, where W is the module of relations on elements vx, vy over R, and where $\mathrm{im}(g) = R^n \oplus W_0$ with $W_0 \subseteq W$. Now add a copy of R^n to P mapping identically to a copy of R^n in $R^n \oplus R^2$; clearly all images and kernels are unchanged and so we still have a phantom resolution of M:

$$0 \to P \oplus R^n \xrightarrow{f \oplus id_{R^n}} R^n \oplus R^2 \xrightarrow{g} R \to 0$$

Since P is a projective R-module, we can choose an R-module Q such that $P \oplus Q =$

R^m is free. Now, take the direct sum with the trivial complex $0 \to Q \stackrel{id_Q}{\to} Q \to 0$ (with the first Q appearing at spot 3) to get a phantom resolution of the form:

$$0 \to Q \stackrel{id}{\to} P \oplus R^n \oplus Q \stackrel{f \oplus id_{R^n} \oplus 0}{\longrightarrow} R^n \oplus R^2 \stackrel{g}{\to} R \to 0$$

where $P \oplus R^n \oplus Q = R^m \oplus R^n$ is mapped onto $R^n \oplus W$ with R^n mapping identically on $R^n \oplus 0_{R^2}$. Therefore, by changing the basis $u_1, \ldots, u_m, e_1, \ldots, e_n$ in $R^m \oplus R^n$ we can assume that R^m is mapped to R^2, so that the resolution has the form

$$\begin{array}{ccccc} & R^n & \stackrel{id}{\to} & R^n & \\ & \bigoplus & & \bigoplus & \\ 0 \to Q \to & R^m & \to & R^2 & \stackrel{(vx,vy)}{\to} R \to 0 \end{array}$$

After removing the trivial complex $R^n \stackrel{id}{\to} R^n$ we still have a phantom resolution of M. $ppd_{R_\mathfrak{m}} M_\mathfrak{m} = 2$ for every maximal ideal $\mathfrak{m}$ of R so the proof of Lemma 3.1 shows that the leftmost map of this resolution splits.

Thus, we have reduced to the case when the phantom resolution of M has the form

$$0 \to P \stackrel{g}{\to} R^2 \stackrel{(vx,vy)}{\longrightarrow} R \to 0$$

Note that g is actually injective: the phantomness of the resolution implies that $\ker(g) \subseteq 0^*_P$. But P is projective and the tight closure of 0 in a projective module over a domain (or even the tight closure of 0 in a bigger free module containing the projective as a direct summand) is 0. Hence, P is a projective submodule of $N = \mathrm{syz}^1(vx, vy) \subseteq R^2$ such that $P^*_{R^2} \supseteq N$. We will need to know the generators of N specifically, so we digress here for the computation of the module of relations on vx, vy over R.

3.3 Computing the module of relations on vx, vy

We will compute the module of relations $N \subseteq R^2$ on the elements vx, vy over the ring $R = k[z, z^{-1}, a, v, ax, ay, vx, vy, x^2, y^2, (1-a)x\sqrt{z}, (1-a)y\sqrt{z}\,]$. More specifically, we will show that N is generated as an R-module by the following 6 elements:

$$
\begin{array}{cc}
(-ay, ax) & (-(ax)y^2, (ay)x^2) \\
(-vy, vx) & (-(vx)y^2, (vy)x^2) \\
(-(1-a)y\sqrt{z}, (1-a)x\sqrt{z}) & (-(1-a)x\sqrt{z}\cdot y^2, (1-a)y\sqrt{z}\cdot x^2)
\end{array}
$$

The projection onto the second coordinate $pr_2 : R^2 \to R$ is injective when restricted to N since vx is a non-zerodivisor on R.The image under pr_2 of N is clearly $(vx :_R vy) = \{r \in R | \exists r_1 \in R \text{ such that } rvy = r_1 vx\}$.

Recall that the normalization of R is $S = R[\sqrt{z}\,] = k[\sqrt{z}, \sqrt{z}^{\,-1}, a, v, x, y]$; S is clearly a UFD so $rvy = r_1 vx$ implies that $x|r$ in S. Thus we have

$$(vx :_R vy) = \{r \in R | (\frac{r}{x})y = r_1 \in R\} \cong \{s \in S | xs \in R,\ ys \in R\} =: T$$

where the last R-module isomorphism takes $r \in pr_2(N)$ to $\frac{r}{x} \in S$. It is clear that for any element r in

$$\{a, v, (1-a)\sqrt{z}, axy, vxy, (1-a)\sqrt{z}xy\}$$

we have $rx, ry \in R$, so the R-submodule

$$aR + vR + (1-a)\sqrt{z}R + axyR + vxyR + (1-a)\sqrt{z}xyR$$

of S is contained in T. To show the other containment, note that there is a natural $\mathbb{N}^3 \times \mathbb{Z}$ grading on $S = k[a][x, y, v, \sqrt{z}\,]$ via degrees in $x, y, v, \sqrt{z}$ (we give a degree 0). Since $R \subseteq S$ is compatible with that grading, we have $T \cong \mathrm{Ann}_{S/R}(x) \cap \mathrm{Ann}_{S/R}(y)$ is also $\mathbb{N}^3 \times \mathbb{Z}$-graded. So it is enough to show that for the general homogeneous element $r = x^m y^n v^l (\sqrt{z})^t f(a)$ of S where $m, n, l \in \mathbb{N}$, $t \in \mathbb{Z}$ and $f(a) \in k[a]$, satisfying $rx, ry \in R$ we necessarily have

$$r \in aR + vR + (1-a)\sqrt{z}R + axyR + vxyR + (1-a)\sqrt{z}xyR$$

This last check is rather long but quite straightforward; we need to consider 8 cases of parity of t, m, n.

Case when t is even, m is even, n is even

We have in this case $r = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n}{2}} v^l z^{\frac{t}{2}} f(a)$.

If $l > 1$ then

$$rx = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n}{2}} v^{l-1} z^{\frac{t}{2}} f(a) vx \in R,$$

$$ry = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n}{2}} v^{l-1} z^{\frac{t}{2}} f(a) vy \in R$$

and

$$r = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n}{2}} v^{l-1} z^{\frac{t}{2}} f(a) \in Rv.$$

If $a | f(a)$ we have

$$rx = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n}{2}} v^{l} z^{\frac{t}{2}} (\frac{f(a)}{a}) ax \in R,$$

$$ry = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n}{2}} v^{l} z^{\frac{t}{2}} (\frac{f(a)}{a}) ay \in R$$

and

$$r = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n}{2}} v^{l} z^{\frac{t}{2}} (\frac{f(a)}{a}) a \in Ra.$$

Assume now that $l = 0$ and $a \nmid f(a)$. Then in the presentation

$$rx = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n}{2}}v^l z^{\frac{t}{2}} f(a)x \in R$$

as a product of homogeneous elements of R none of the elements ax, ay, vx, vy can appear; so rx can be rewritten as

$$rx = (x^2)^{N_1}(y^2)^{N_2}((1-a)x\sqrt{z})^{N_3}((1-a)y\sqrt{z})^{N_4} z^N g(a)$$

where $N_i \in \mathbb{N}$, $N \in \mathbb{Z}$ and $g(a) \in k[a]$ with $a \nmid g(a)$. Comparing these two presentations of rx we see that the power of x in rx $m+1 = 2N_1 + N_3$ should be odd so N_3 is odd. The power of y in rx $n = 2N_2 + N_4$ is even so N_4 is even. But the power with which z appears $t/2 = (N_3 + N_4)/2 + N$ should be integer for even t which is not the case with odd N_3 and even N_4.

Case when t is even, m is even, n is odd

Now we have $r = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n-1}{2}}v^l z^{\frac{t}{2}} f(a)y$. Note that in this case we always have $ry = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n-1}{2}}v^l z^{\frac{t}{2}} f(a)y^2 \in R$. Assume that $rx = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n-1}{2}}v^l z^{\frac{t}{2}} f(a)xy \in R$. rx has odd power in x, so in the representation of rx as a product of homogeneous elements of R at least one of $ax, vx, (1-a)x\sqrt{z}$ should appear with positive degree. Similarly, since rx has odd power in y at least one of $ay, vy, (1-a)y\sqrt{z}$ should also appear with positive degree. Thus the following cases arise:

If ax and ay appear, then $a^2|f(a)$ and we have that

$$r = ((x^2)^{\frac{m}{2}}(y^2)^{\frac{n-1}{2}}v^l z^{\frac{t}{2}}(\frac{f(a)}{a^2})ay)a \in Ra$$

If ax and vy appear or, if vx and ay appear, then we have $a|f(a)$ and $l \geq 1$ so that

$$r = ((x^2)^{\frac{m}{2}}(y^2)^{\frac{n-1}{2}}v^{l-1}z^{\frac{t}{2}}(\frac{f(a)}{a})vy)a \in Ra$$

If vx and vy appear, then $l \geq 2$ and we have

$$r = ((x^2)^{\frac{m}{2}}(y^2)^{\frac{n-1}{2}}v^{l-2}z^{\frac{t}{2}}f(a)vy)v \in Rv$$

If none of ax, vx, ay, vy appear then $rx \in R$ can be rewritten as

$$rx = (x^2)^{N_1}(y^2)^{N_2}((1-a)x\sqrt{z})^{N_3}((1-a)y\sqrt{z})^{N_4}z^N g(a)$$

where $N_i \in \mathbb{N}$, $N \in \mathbb{Z}$ and $g(a) \in k[a]$ with $a \nmid g(a)$; the considerations above show that in this remaining case $N_3 \geq 1$ and $N_4 \geq 1$ so $N_3 + N_4 \geq 2$. In particular $(1-a)^2 | f(a)$ so that

$$r = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n-1}{2}}v^l z^{\frac{t}{2}}(\frac{f(a)}{(1-a)^2})\frac{1}{z}((1-a)y\sqrt{z})((1-a)\sqrt{z}) \in R(1-a)\sqrt{z}$$

Case when t is even, m is odd, n is even

This case is done exactly as the previous one with roles of x and y interchanged.

Case when t is even, m is odd, n is odd

We have in this case $r = (x^2)^{\frac{m-1}{2}}(y^2)^{\frac{n-1}{2}}v^l z^{\frac{t}{2}}f(a)xy$. Assume that $rx, ry \in R$. If $a|f(a)$ we have

$$r = ((x^2)^{\frac{m-1}{2}}(y^2)^{\frac{n-1}{2}}v^l z^{\frac{t}{2}}(\frac{f(a)}{a}))axy \in Raxy.$$

Similarly, if $l \geq 1$ then we have

$$r = ((x^2)^{\frac{m-1}{2}}(y^2)^{\frac{n-1}{2}}v^{l-1}z^{\frac{t}{2}}f(a))vxy \in Rvxy$$

Assume now that $a \nmid f(a)$ and $l = 0$. Then in the representation of $rx \in R$ as a product of homogeneous elements of R neither of the elements ax, ay, vx, vy can appear. Thus we can write it as

$$rx = (x^2)^{N_1}(y^2)^{N_2}((1-a)x\sqrt{z})^{N_3}((1-a)y\sqrt{z})^{N_4}z^N g(a)$$

where $N_i \in \mathbb{N}$, $N \in \mathbb{Z}$ and $g(a) \in k[a]$ with $a \nmid g(a)$. rx has an even power $m+1 = 2N_1 + N_3$ in x so N_3 is even. y appears in an odd power $n = 2N_1 + N_4$ in rx so N_4 is odd. But the power of z appearing in rx $(N_3+N_4)/2+N$ should be integer $t/2$ which is not the case for even N_3 and odd N_4.

Case when t is odd, m is even, n is even

In this case $r = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n}{2}}v^l z^{\frac{t-1}{2}} f(a)\sqrt{z}$. If $rx \in R$ we must have $(1-a)|f(a)$ (because the homogeneous element of R can contain fractional power of z only if it is a multiple of either $(1-a)x\sqrt{z}$ or $(1-a)y\sqrt{z}$). Then we have

$$r = ((x^2)^{\frac{m}{2}}(y^2)^{\frac{n}{2}}v^l z^{\frac{t-1}{2}}(\frac{f(a)}{1-a}))(1-a)\sqrt{z} \in R(1-a)\sqrt{z}$$

Case when t is odd, m is even, n is odd

In this case we have $r = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n-1}{2}}v^l z^{\frac{t-1}{2}} f(a)\sqrt{z}y$. Since a fractional power appears in $rx \in R$ just as in the previous case we necessarily have $(1-a)|f(a)$.

If $l \geq 1$ we have

$$rx = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n-1}{2}}v^{l-1}z^{\frac{t-1}{2}}(\frac{f(a)}{1-a})((1-a)\sqrt{z}y)vx \in R,$$

$$ry = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n-1}{2}}v^{l-1}z^{\frac{t-1}{2}}(\frac{f(a)}{1-a})((1-a)\sqrt{z}y)vy \in R$$

and

$$r = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n-1}{2}}v^{l-1}z^{\frac{t-1}{2}}(\frac{f(a)}{1-a})((1-a)\sqrt{z}y)v \in Rv.$$

Similarly, if $a|f(a)$ we have

$$rx = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n-1}{2}}v^{l}z^{\frac{t-1}{2}}(\frac{f(a)}{a(1-a)})((1-a)\sqrt{z}y)ax \in R,$$

$$ry = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n-1}{2}}v^{l}z^{\frac{t-1}{2}}(\frac{f(a)}{a(1-a)})((1-a)\sqrt{z}y)ay \in R$$

and

$$r = (x^2)^{\frac{m}{2}}(y^2)^{\frac{n-1}{2}}v^{l}z^{\frac{t-1}{2}}(\frac{f(a)}{a(1-a)})((1-a)\sqrt{z}y)a \in Ra.$$

Assume now that $a \nmid f(a)$ and $l = 0$. Then in the representation of $rx \in R$ as a product of homogeneous elements of R none of the elements ax, ay, vx, vy can appear. Thus we can write

$$rx = (x^2)^{N_1}(y^2)^{N_2}((1-a)x\sqrt{z})^{N_3}((1-a)y\sqrt{z})^{N_4}z^N g(a)$$

where $N_i \in \mathbb{N}$, $N \in \mathbb{Z}$ and $g(a) \in k[a]$ with $a \nmid g(a)$. rx has an odd power $m+1 = 2N_1 + N_3$ in x so N_3 is odd. y appears in an odd power $n = 2N_1 + N_4$ in rx so N_4 is odd. But the power of z appearing in rx $(N_3+N_4)/2+N$ should be the non-integer $t/2$ which is not the case for odd N_3 and N_4.

Case when t is odd, m is odd, n is even

This case is again the same as the one immediately above with the roles of x, y interchanged.

Case when t is odd, m is odd, n is odd

We have $r = (x^2)^{\frac{m-1}{2}}(y^2)^{\frac{n-1}{2}}v^l z^{\frac{t-1}{2}}f(a)\sqrt{z}xy$. Again, as in the three previous cases, $(1-a)|f(a)$. Then we have

$$r = ((x^2)^{\frac{m-1}{2}}(y^2)^{\frac{n-1}{2}}v^l z^{\frac{t-1}{2}}(\frac{f(a)}{1-a}))(1-a)\sqrt{z}xy \in R((1-a)\sqrt{z}xy)$$

So in all cases whenever $r \in T$ we have

$$r \in aR + vR + (1-a)\sqrt{z}R + axyR + vxyR + (1-a)\sqrt{z}xyR$$

so we are done.

3.4 Continuation of the Proof

Now we continue with the proof that M is actually a counterexample for the "if" part of the Conjecture. Recall that we have reduced to the case when the phantom resolution of M has the form

$$0 \to P_0 \hookrightarrow R^2 \xrightarrow{(vx,vy)} R \to 0$$

where P_0 is a projective submodule of $N = \mathrm{syz}^1(vx, vy) \subseteq R^2$ such that $(P_0)^*_{R^2} \supseteq N$. We computed the R-module generators of N explicitly above. The projection onto the second coordinate $pr_2 : R^2 \to R$ is an injection on N since vx is a non-zerodivisor on R. So $P_0 \subseteq N$ are isomorphic as R-modules via pr_2 to some ideals $I \subseteq J$ of R.

$N \subseteq (P_0)^*_{R^2}$ means that for every $n \in N$ $\exists c \in R^\circ$ such that $cn^q \in (P_0)^{[q]}_{R^2}$ for all $q \gg 0$. Taking the images under $F^e(pr_2)$ we get $c(pr_2(n))^q \in I^{[q]}_R$ so that $J \subseteq I^*_R$. Recall that the normalization of R is $S = R[\sqrt{z}\,] = k[\sqrt{z}, \sqrt{z}^{\,-1}, a, v, x, y]$ which is regular (since it is a localization of the polynomial ring).

Lemma 3.7. *Let $R \subseteq S$ be Noetherian rings of prime characteristic p such that S is integral over R and S is F-regular. Let $I \subseteq J$ be ideals in R. Then $J \subseteq I^*$ if and only if $IS = JS$.*

Proof. First note that for any ideal $\mathfrak{P}$ of R we have $\mathfrak{P}^*_R \subseteq (\mathfrak{P}S)^*_S$ (persistence of tight closure) and $(\mathfrak{P}S)^*_S = \mathfrak{P}S$ (because S is F-regular). Thus we have $\mathfrak{P}^*_R \subseteq \mathfrak{P}S \cap R$ and the opposite inclusion holds since S is integral over R. Thus we have $\mathfrak{P}^*_R = \mathfrak{P}S \cap R$ for any ideal $\mathfrak{P}$ of R. Now the "if" part is immediate: $IS = JS$ implies $I^* = IS \cap R = JS \cap R = J^*$ so that $J \subseteq J^* = I^*$. Conversely, $J \subseteq I^* = IS \cap R$ implies that $JS \subseteq IS$ and the opposite inclusion is true by assumption $I \subseteq J$. $\square$

The explicit computation of N made in previous section gives

$$J = pr_2(N) = (ax, vx, (1-a)x\sqrt{z}, (ay)x^2, (vy)x^2, ((1-a)y\sqrt{z})x^2)R$$

We have $JS = xS$: clearly $JS \subseteq xS$ and $x = ax + (\sqrt{z}^{\,-1})((1-a)x\sqrt{z}) \in JS$. So by the lemma above, we will be done after proving the following

Proposition 3.8. *Let $R = k[z, \frac{1}{z}, a, v, ax, ay, vx, vy, x^2, y^2, (1-a)x\sqrt{z}, (1-a)y\sqrt{z}\,]$; the normalization of R is $S = R[\sqrt{z}\,] = k[\sqrt{z}, \frac{1}{\sqrt{z}}, a, v, x, y]$. Then there is* ***no*** *projective ideal P of R such that $P \subseteq (ax, vx, (1-a)x\sqrt{z}, (ay)x^2, (vy)x^2, ((1-a)y\sqrt{z})x^2)$ and $PS = (x)S$.*

Proof. To show the non-existence of P satisfying the given conditions we shall first do some reductions: namely, we shall show that it is enough to show the non-existence after killing y and v. We will need the following two lemmas first.

Lemma 3.9. *Let R be any domain and $P \subseteq R$ be an ideal. Then P is projective as an R-module iff there exists a* non-zero *ideal $Q \subseteq R$ ideal such that PQ is principal ideal of R.*

Proof. See e.g., (11.3) in [21].

Lemma 3.10. *Let R be a Noetherian domain such that the normalization S of R is module-finite over R and is a UFD. Let $g \in R - \{0\}$ be such that $R_g = S_g$. Let $t \in S - \{0\}$ be prime element in S such that $gcd(t, g) = 1$ in S. Let $I \subseteq R$ be an ideal in R and assume that there exists ideal $\mathfrak{A}$ of R such that $I\mathfrak{A}$ is principal. Then there exists an ideal $\mathfrak{B}$ of R such that $\mathfrak{B} \nsubseteq tS$ and $\mathfrak{A} = \mathfrak{B} \cdot f$ for some $f \in frac(S)$ (of course in that case $I\mathfrak{B}$ will also be principal).*

Proof. $R_g = S_g$ and S is module-finite over R so there exists an integer N such that $g^N S \subseteq R$. S is a UFD so there exists the largest integer k such that $\mathfrak{A} \subseteq t^k S$ (it can be 0 and in that case we are done by taking $\mathfrak{B} = \mathfrak{A}$). Then we have $\mathfrak{A} \subseteq t^k S$ and $\mathfrak{A} \nsubseteq t^{k+1}S$. So $R \supseteq \mathfrak{A} = t^k\mathfrak{B}_0$ where $\mathfrak{B}_0$ is an R-submodule of S with $\mathfrak{B}_0 \nsubseteq tS$.

Now, $g^N\mathfrak{A} = t^k(g^N\mathfrak{B}_0)$ and $g^N\mathfrak{B}_0 \subseteq R$ by choice of N. Take $\mathfrak{B} = g^N\mathfrak{B}_0 = \frac{g^N}{t^k}\mathfrak{A}$. $\mathfrak{B}$ is clearly subideal of R and $\mathfrak{B} \nsubseteq tS$ since $\mathfrak{B}_0 \nsubseteq tS$ and $\gcd(t, g) = 1$. □

Applying Lemma 3.9 and Lemma 3.10 for the given R, S with $g = vx \cdot (1 - a)$, $t = y$ and $I = P$ we get that for some ideal $\mathfrak{B}$ of R, $P\mathfrak{B}$ is principal and the image of $\mathfrak{B}$ in $R/(yS \cap R)$ is non-zero. Clearly $P \nsubseteq yS$ because of $PS = xS$. Therefore after killing yS in S and $yS \cap R$ in R the image $\hat{P}$ of P is still invertible ideal. So we get to the following setup: for $R_1 = k[z, \frac{1}{z}, a, v, ax, vx, x^2, (1-a)x\sqrt{z}\,]$ with normalization $S_1 = R_1[\sqrt{z}\,] = k[\sqrt{z}, \frac{1}{\sqrt{z}}, a, v, x]$ we have the ideal $P_1 \subseteq (ax, vx, (1-a)x\sqrt{z})$ which is still invertible (projective) and $P_1S_1 = xS_1$.

Now apply the same two Lemmas again for $R_1 \subseteq S_1$ now with $g = (1-a)(ax)$ and $t = v$. By the same reasoning as before, after killing vS_1 in S_1 and $vS_1 \cap R_1$ in R_1 we reduce to the case of the ring $R_2 = k[z, \frac{1}{z}, a, ax, x^2, (1-a)x\sqrt{z}\,]$ with normalization $S_2 = R_2[\sqrt{z}] = k[\sqrt{z}, \frac{1}{\sqrt{z}}, a, x]$ such that the image P_2 of P_1 in R_2 is still invertible and satisfies $P_2 \subseteq (ax, (1-a)x\sqrt{z}\,)R_2$ and $P_2S_2 = xS_2$.

So it remains only to prove the simpler proposition:

Proposition 3.11. *Let $R = k[z, z^{-1}, a, x^2, ax, (1-a)x\sqrt{z}\,]$; its normalization is $S = R[\sqrt{z}\,] = k[\sqrt{z}, \sqrt{z}^{\,-1}, a, x]$. Then there is* ***no*** *projective ideal P of R such that $P \subseteq (ax, (1-a)x\sqrt{z})R$ and $PS = (x)S$.*

Proof. The main idea of the proof is to find all such ideals P after localizing at the elements a and $b = 1-a$ separately and then to show that these "do not patch together" to give a projective ideal we need over R.

Assume for the moment that we have proved $\mathrm{Pic}(R_a)$ and $\mathrm{Pic}(R_b)$ are both trivial (these computations are rather long; they are done in the next section). We have $R_a = k[z, z^{-1}, a, a^{-1}, x, (1-a)x\sqrt{z}\,]$ and its normalization is $S_a = R_a[\sqrt{z}\,] = k[\sqrt{z}, \sqrt{z}^{\,-1}, a, a^{-1}, x]$. We want to find all projective subideals $P_a \subseteq (x, (1-a)x\sqrt{z})R_a$ such that $P_aS_a = xS_a$. Since $\mathrm{Pic}(R_a) = 0$ the projective rank 1 ideal P_a is actually free: $P_a = uR_a$ for some $u \in R_a$. The condition $P_aS_a = uS_a = xS_a$ implies that $u = x \cdot \tau$ where τ is a unit of S_a. The units of S_a are elements of the form $\alpha(\sqrt{z})^m a^n$ for $\alpha \in k - \{0\}$ and $m, n \in \mathbb{Z}$ so the only possible way for u to be in R_a is $u = x\alpha a^m$ so that $P_a = xR_a$.

On the other hand, localizing at the element b gives

$$R_b = k[z, z^{-1}, a, (1-a)^{-1}, ax, x\sqrt{z}\,]$$

($x^2 = \frac{1}{z}(x\sqrt{z})^2$ is automatically inside the ring) with normalization

$$S_b = k[\sqrt{z}, \sqrt{z}^{-1}, a, (1-a)^{-1}, x].$$

We want to find all the projective subideals $P_b \subseteq (ax, x\sqrt{z})R_b$ such that $P_bS_b = xS_b$. Again $\mathrm{Pic}(R_b) = 0$ gives that $P_b = wR_b$ for some $w \in R_b$ and the condition $P_bS_b = wS_b = xS_b$ implies that $w = x \cdot \tau_0$ for some unit τ_0 of S_b. The units of S_b are elements of the form $\alpha(\sqrt{z})^m(1-a)^n$ for $\alpha \in k - \{0\}$ and $m, n \in \mathbb{Z}$ so the only possible choice for w to get in R_b is $u = x\alpha(\sqrt{z})^{2m+1}$ so that $P_b = x\sqrt{z}R_b$.

Thus we have $P_a = xR_a$ and $P_b = x\sqrt{z}R_b$, so localizing further gives $P_{ab} = xR_{ab} = x\sqrt{z}R_{ab}$. Therefore $x = x\sqrt{z} \cdot \gamma$ where γ is a unit of R_{ab}. Units of R_{ab} are elements of the form $\alpha z^m a^n(1-a)^t$ for $\alpha \in k - \{0\}$ and $m, n, t \in \mathbb{Z}$. This gives $x = x\sqrt{z}\alpha z^m a^n (1-a)^t$ i.e. $\sqrt{z} = \alpha^{-1}z^{-m}a^{-n}(1-a)^{-t}$ which is not possible even in S_{ab}. Thus P_a and P_b "do not patch together" properly to give a projective ideal P of R. $\square$

3.5 Computing the Picard groups of R_a and R_{1-a}

3.5.1 The Technique Used for Computing the Picard Groups

We will make considerable use of the following fact:

Lemma 3.12. *Let S be any UFD. Then every invertible ideal I of S is principal. In particular, the Picard group of S is trivial.*

Proof. See e.g., (20.7) in [21]. □

For computing the Picard groups we shall use the Mayer-Vietoris exact sequences arising from the fiber squares (see [8] and [23] for details). Recall that a commutative diagram of rings and ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{\alpha_1} & R_1 \\ {\scriptstyle \alpha_2}\downarrow & & \downarrow{\scriptstyle \beta_1} \\ R_2 & \xrightarrow{\beta_2} & S \end{array}$$

is a *fiber square* if the map $(\alpha_1, \alpha_2) : R \to R_1 \oplus R_2$ identifies R with $\{(a_1, a_2) \in R_1 \times R_2 | \, \beta_1(a_1) = \beta_2(a_2)\}$ and at least one of the maps β_1, β_2 is surjective. Then there is an exact Mayer-Vietoris sequence of groups with natural maps

$$0 \to U(R) \to U(R_1) \oplus U(R_2) \to U(S) \to \mathrm{Pic}(R) \to \mathrm{Pic}(R_1) \oplus \mathrm{Pic}(R_2)$$

where $U(R)$ denotes the group of units for R .

We will use the Mayer-Vietoris sequences arising from the fiber squares of two particular forms:

1. For an arbitrary ring R and its two ideals I, J the commutative diagram with natural maps
$$\begin{array}{ccc} R/(I \cap J) & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ R/J & \longrightarrow & R/(I+J) \end{array}$$
is a fiber square.

2. Let R be any Noetherian domain and let S be its normalization. Let $\mathfrak{c}$ be the conductor of S into R. Then the diagram with natural maps
$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R/\mathfrak{c} & \longrightarrow & S/\mathfrak{c} \end{array}$$

is a fiber square, and the vertical map on the right is surjective.

3.5.2 Computation for R_a

We have $R_a = k[z, z^{-1}, a, a^{-1}, x, (1-a)x\sqrt{z}\,]$ and its normalization is $S_a = R_a[\sqrt{z}\,] = k[\sqrt{z}, \frac{1}{\sqrt{z}}, a, \frac{1}{a}, x]$. First note that the conductor $\mathfrak{c}$ of S_a into R_a is

$$\mathfrak{c} = ((1-a)x)S_a = ((1-a)x, (1-a)x\sqrt{z})R_a$$

Indeed, we have

$$\mathfrak{c} = (R_a :_{R_a} S_a) = (R_a :_{R_a} R_a[\sqrt{z}\,]) = (R_a :_{R_a} \sqrt{z})$$

so clearly $(1-a)x, (1-a)x\sqrt{z} \in \mathfrak{c}$. Every element of

$$R_a = k[z, z^{-1}, a, a^{-1}, x][(1-a)x\sqrt{z}\,]$$

can be written in the form $r = p + q(1-a)x\sqrt{z}$ with $p, q \in k[z, z^{-1}, a, a^{-1}, x]$; in particular a fractional power of z can appear only with a factor $(1-a)x$. Thus, if $r\sqrt{z} = q(1-a)xz + p\sqrt{z} \in R$ we will necessarily have $(1-a)x|p$ in S_a. Therefore $r = p + q(1-a)x\sqrt{z} \in ((1-a)x)S_a$.

Now let us compute the Picard group of

$$R_a/\mathfrak{c} = k[z, z^{-1}, a, a^{-1}, x, (1-a)x\sqrt{z}]/((1-a)x, (1-a)x\sqrt{z}) =$$

$$k[z, z^{-1}, a, a^{-1}, x]/((1-a)x)$$

Taking $T = k[z, z^{-1}, a, a^{-1}, x]$, $I = (1-a)T$ and $J = xT$, we get the fiber square:

$$\begin{array}{ccc}
T/(I \cap J) = T/(IJ) = R_a/\mathfrak{c} & \longrightarrow & T/I = k[z, z^{-1}, x] \\
\downarrow & & \downarrow \\
T/J = k[z, z^{-1}, a, a^{-1}] & \longrightarrow & T/(I+J) = k[z, z^{-1}]
\end{array}$$

As both T/I and T/J are UFDs, their Picard groups are trivial. So part of the corresponding Mayer-Vietoris sequence looks like

$$U(T/I) \oplus U(T/J) \to U(T/(I+J)) \to \mathrm{Pic}(T/(IJ)) \to \mathrm{Pic}(T/I) \oplus \mathrm{Pic}(T/J) = 0$$

and it follows that

$$\mathrm{Pic}(R_a/\mathfrak{c}) = U(T/(I+J))/\mathrm{im}(U(T/I) \oplus U(T/J))$$

But the units of the ring $T/(I+J) = k[z, z^{-1}]$ are all the elements of the form αz^k where $\alpha \in k - \{0\}$ and $n \in \mathbb{Z}$ and they all are images of units of T/I and T/J; thus $\mathrm{Pic}(R_a/\mathfrak{c}) = 0$.

Now in the fiber square coming from the conductor $\mathfrak{c}$ of S_a into R_a

$$\begin{array}{ccc} R_a & \longrightarrow & S_a \\ \big\downarrow & & \big\downarrow \\ R_a/\mathfrak{c} & \longrightarrow & S_a/\mathfrak{c} \end{array}$$

we have $\mathrm{Pic}(R_a/\mathfrak{c}) = 0$ by the previous computation and $\mathrm{Pic}(S_a) = 0$ since S_a is a UFD. Thus part of the corresponding Mayer-Vietoris sequence is

$$U(R_a/\mathfrak{c}) \oplus U(S_a) \to U(S_a/\mathfrak{c}) \to \mathrm{Pic}(R_a) \to \mathrm{Pic}(S_a) \oplus \mathrm{Pic}(R_a/\mathfrak{c}) = 0$$

and this gives that

$$\mathrm{Pic}(R_a) \cong U(S_a/\mathfrak{c})/\mathrm{im}(U(R_a/\mathfrak{c}) \oplus U(S_a))$$

But again all the units of $S_a/\mathfrak{c}$, namely $\{\alpha z^m a^n | \alpha \in k - \{0\}, m, n \in \mathbb{Z}\}$, are images of units of S_a, so that $\mathrm{Pic}(R_a) = 0$.

3.5.3 Computation for R_{1-a}

The computations for are very similar. The localization of R at $b = 1-a$ is $R_b = k[z, z^{-1}, a, (1-a)^{-1}, ax, x\sqrt{z}\,]$ and its normalization is

$$S_b = R_b[\sqrt{z}\,] = k[\sqrt{z}, (\sqrt{z})^{-1}, a, (1-a)^{-1}, x]$$

The conductor $\mathfrak{d}$ of S_b into R_b is $\mathfrak{d} = (ax)S_b = ((ax, ax\sqrt{z})R_a$. Indeed, we have

$$\mathfrak{d} = (R_b :_{R_b} S_b) = (R_b :_{R_b} R_b[\sqrt{z}\,]) = (R_b :_{R_b} \sqrt{z})$$

so clearly $ax, ax\sqrt{z} \in \mathfrak{d}$. Conversely, every element of

$$R_b = k[z, z^{-1}, a, (1-a)^{-1}][ax, x\sqrt{z}\,]$$

can be written as $r = \sum p_{m,n}(ax)^m(x\sqrt{z})^n$ where (m,n) varies over finite subset of $\mathbb{N} \times \mathbb{N}$ and $p_{m,n} \in k[z, z^{-1}, a, (1-a)^{-1}]$. If $r \in \mathfrak{d}$ then we also have that $r\sqrt{z} = \sum p_{m,n}(ax)^m(x\sqrt{z})^n\sqrt{z} \in R_b$. The terms of r involving ax and the terms $p_{0n}(x\sqrt{z})^n$ such that $a|p_{0n}$ in $k[z, z^{-1}, a, (1-a)^{-1}]$ are automatically in $(ax)S_b$. If there were a term $p_{0n}(x\sqrt{z})^n$ of r such that $a \nmid p_{0n}$ in $k[z, z^{-1}, a, (1-a)^{-1}]$ then the term $p_{0n}(x\sqrt{z})^n\sqrt{z}$ of $r\sqrt{z} \in R_b$ cannot arise from any polynomial in $x\sqrt{z}$ with coefficients in $k[z, z^{-1}, a, (1-a)^{-1}]$.

Now let us compute the Picard group of

$$R_b/\mathfrak{d} = k[z, z^{-1}, a, (1-a)^{-1}, ax, x\sqrt{z}]/(ax, ax\sqrt{z}) = k[z, z^{-1}, a, a^{-1}, x\sqrt{z}]/(a(x\sqrt{z}))$$

Taking $T = k[z, z^{-1}, a, (1-a)^{-1}, x\sqrt{z}]$, $I = (a)T$ and $J = (x\sqrt{z})T$ we get the fiber square:

$$\begin{array}{ccc} T/(I\cap J) = T/(IJ) = R_b/\mathfrak{d} & \longrightarrow & T/I = k[z, z^{-1}, x\sqrt{z}] \\ \downarrow & & \downarrow \\ T/J = k[z, z^{-1}, a, (1-a)^{-1}] & \longrightarrow & T/(I+J) = k[z, z^{-1}] \end{array}$$

As both T/I and T/J are UFDs, their Picard groups are trivial. So the same argument as for $R_a/\mathfrak{c}$ shows that $\mathrm{Pic}(R_b/\mathfrak{d}) = 0$.

Now in the fiber square coming from the conductor $\mathfrak{d}$ of S_b into R_b

$$\begin{array}{ccc} R_b & \longrightarrow & S_b \\ \downarrow & & \downarrow \\ R_b/\mathfrak{d} & \longrightarrow & S_b/\mathfrak{d} \end{array}$$

we have $\mathrm{Pic}(R_b/\mathfrak{d}) = 0$ by the previous computation and $\mathrm{Pic}(S_b) = 0$ since S_b is a UFD. Thus part of the corresponding Mayer-Vietoris sequence

$$U(R_b/\mathfrak{d}) \oplus U(S_b) \to U(S_b/\mathfrak{d}) \to \mathrm{Pic}(R_b) \to \mathrm{Pic}(S_b) \oplus \mathrm{Pic}(R_b/\mathfrak{d}) = 0$$

gives that

$$\mathrm{Pic}(R_b) \cong U(S_b/\mathfrak{d})/\mathrm{im}(U(R_b/\mathfrak{d}) \oplus U(S_b))$$

Again, all the units of $S_b/\mathfrak{d}$, namely $\{\alpha z^m(1-a)^n | \alpha \in k - \{0\}, m, n \in \mathbb{Z}\}$ are images of units of S_b so that $\mathrm{Pic}(R_b) = 0$ as well.

CHAPTER IV

Finite Phantom Projective Dimension Does Not Pass to Direct Summands

4.1 Statement of Conjecture, Counterexample and Sketch of Proof

The techniques developed in the previous chapter enable us to construct a counterexample to yet another conjecture posed in [1], namely:

Conjecture. *Let R be a Noetherian ring of characteristic p and let M and N be finitely generated R-modules. Then $ppd_R(M \oplus N) < \infty$ if and only if $ppd_R(M) < \infty$ and $ppd_R(N) < \infty$*

The "if" part is again trivial, since the direct sum of phantom resolutions is still a phantom resolution. However, a phantom resolution of $M \oplus N$ does not necessarily decompose into direct sum.

The main idea for constructing counterexample to this conjecture is roughly the following: take a direct sum of two copies of Koszul complexes of length 2, with a ring R very similar to the one in the previous chapter,

$$\begin{array}{ccccccc}
0 \to & R & \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} & R^2 & \xrightarrow{\begin{pmatrix} vx & vy \end{pmatrix}} & R & \to 0 \\
& \bigoplus & & \bigoplus & & \bigoplus & \\
0 \to & R & \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} & R^2 & \xrightarrow{\begin{pmatrix} vx & vy \end{pmatrix}} & R & \to 0
\end{array}$$

and make a "generic" change of basis in the leftmost copy of R^2 via an invertible matrix of indeterminates $(t_{ij})_{2\times 2}$. As a result of this basis change, the augmentation module will still have a finite phantom projective dimension, however the matrix $(t_{ij})_{2\times 2}$ "mixes" this phantom resolution enough so that it does not decompose. The proof heavily relies on the counterexample of the previous chapter and techniques developed there.

Specifically, fix an arbitrary field k of positive prime characteristic p and let $z, v, a, x, y, t_{11}, t_{12}, t_{21}, t_{22}$ be indeterminates over k. Consider the following ring

$$R = k[z, z^{-1}, v, a, vx, vy, ax, ay, x^2, y^2, (1-a)x\sqrt{z}, (1-a)y\sqrt{z}\,,$$
$$t_{ij}x, t_{ij}y, t_{ij}^2, at_{ij}, \tfrac{1}{(t_{11}t_{22}-t_{12}t_{21})^2}, (1-a)t_{ij}\sqrt{z}\,]$$

where $1 \le i, j \le 2$. It is a subring of the localized polynomial ring

$$S = k[\sqrt{z}, v, a, x, y, t_{11}, t_{12}, t_{21}, t_{22}]_{z(t_{11}t_{22}-t_{12}t_{21})^2}$$

Take the R-module M to be $M = R/(vx, vy)$. We claim that $ppd_R(M \oplus M) < \infty$ but M does *not* have a finite phantom projective resolution.

The proof we present will actually show that the module M also has finite phantom projective dimension locally but not globally, so it provides a counterexample for both conjectures. I was not yet able to construct a counterexample for the case of a local ring R.

4.2 $M \bigoplus M$ has Finite Phantom Projective Dimension

First, note that the normalization of R, i.e., the integral closure of R inside its field of fractions $\mathrm{frac}(R) = k(\sqrt{z}, x, y, a, v, t_{11}, t_{12}, t_{21}, t_{22})$, is

$$S = R[\sqrt{z}\,] = k[\sqrt{z}, \sqrt{z}^{\,-1}, a, v, x, y, t_{11}, t_{12}, t_{21}, t_{22}]_{z(t_{11}t_{22}-t_{12}t_{21})^2}$$

Indeed, the square of the element

$$\sqrt{z} = \frac{((1-a)x\sqrt{z})v}{(1-a)(vx)}$$

of $\mathrm{frac}(R)$ is in R and $R[\sqrt{z}\,]$ contains the elements $\sqrt{z}^{\,-1} = z^{-1} \cdot \sqrt{z}$, and

$$x = (ax) + \sqrt{z}^{\,-1}((1-a)x\sqrt{z}))$$
$$y = (ay) + \sqrt{z}^{\,-1}((1-a)y\sqrt{z}))$$
$$t_{ij} = (at_{ij}) + \sqrt{z}^{\,-1}((1-a)t_{ij}\sqrt{z}))$$

Therefore

$$S = R[\sqrt{z}\,] = k[\sqrt{z}, \frac{1}{\sqrt{z}}, a, v, x, y, t_{11}, t_{12}, t_{21}, t_{22}]_{z(t_{11}t_{22}-t_{12}t_{21})^2}$$

is a subring of $\mathrm{frac}(R)$ that is integral over R and is integrally closed (since it is the localization of the polynomial ring $k[\sqrt{z}, a, v, x, y]$ at the element $\sqrt{z}\,(t_{11}t_{22} - t_{12}t_{21})^2)$, so it is indeed the integral closure of R inside $\mathrm{frac}(R)$.

Now let us verify that $M \oplus M$ has finite phantom projective dimension. The complex mentioned above (direct sum of two Koszul complexes modified via "generic" change of basis) provides the appropriate phantom resolution:

$$(\dagger) \qquad 0 \longrightarrow R^2 \xrightarrow{A} R^4 \xrightarrow{B} R^2 \longrightarrow 0$$

where the matrices

$$A = \begin{pmatrix} y & 0 \\ -x & 0 \\ 0 & y \\ 0 & -x \end{pmatrix} \cdot \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} t_{11}y & t_{12}y \\ -t_{11}x & -t_{12}x \\ t_{21}y & t_{22}y \\ -t_{21}x & -t_{22}x \end{pmatrix}$$

and

$$B = \begin{pmatrix} vx & vy & 0 & 0 \\ 0 & 0 & vx & vy \end{pmatrix}$$

have entries in R.

It is clear that $(\dagger)$ is indeed a complex with augmentation module $M \bigoplus M$. Its phantomness follows again from the Phantom Acyclicity Criterion (see e.g., (3.21) of [15]). By computing determinants it is clear that the ranks of the relevant matrices A and B are both 2 and so they add up correctly. $\mathrm{ht} I_2(B) = \mathrm{ht}((v^2x^2, v^2xy, v^2y^2)) \geq 1$ in the domain R, and for any prime $\mathfrak{p}$ of R containing $I_2(A) = (x^2 \cdot \Delta, xy \cdot \Delta, y^2 \cdot \Delta)R$ (where $\Delta = \det(t_{ij})_{2\times 2}$) and any prime $\mathfrak{q}$ of $S = R[\sqrt{z}]$ lying over it we have $\mathrm{ht}(\mathfrak{p}) \geq \mathrm{ht}(\mathfrak{q}) \geq 2$ (in S Δ is a unit so the prime ideal $\mathfrak{q}$ contains the elements x^2, xy, y^2 and therefore $\mathfrak{q} \supseteq (x, y) \supset (x) \supset (0)$). Thus $(\dagger)$ is a phantom projective resolution of $M \oplus M$.

4.3 M Does Not Have Finite Phantom Projective Dimension

The ring R and the R-module M are very similar to ones in the previous chapter. So it is not at all surpsising that that the proof of M having no finite phantom projective dimension follows very closely the one given in the previous chapter. Specifically, the same argument as in section (2.1) shows that

$$(*) \qquad 0 \to R_a \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R_a^2 \xrightarrow{\begin{pmatrix} vx & vy \end{pmatrix}} R_a \to 0$$

and

$$(**) \qquad 0 \to R_b \xrightarrow{\begin{pmatrix} y\sqrt{z} \\ -x\sqrt{z} \end{pmatrix}} R_b^2 \xrightarrow{\begin{pmatrix} vx & vy \end{pmatrix}} R_b \to 0$$

are phantom resolutions of M_a over R_a and M_{1-a} over R_{1-a} respectively (here $b := 1 - a$), so that for every maximal ideal $\mathfrak{m}$ of R we have $ppd_{R_\mathfrak{m}} M_\mathfrak{m} = 2$.

Assume now that M is of a finite phantom projective dimension. Exactly the same argument as the one in section (2.2) shows that we can reduce to the case when the phantom resolution of M has the form:

$$0 \to P \xrightarrow{g} R^2 \xrightarrow{(vx,vy)} R \to 0$$

where g is an injection and P is a projective submodule of $N = \mathrm{syz}^1(vx, vy) \subseteq R^2$ such that $P^*_{R^2} \supseteq N$.

We need to know the generators of N explicitly: we claim that N as an R-module is generated by the following elements:

$$\begin{array}{cc} (-ay, ax) & (-(ax)y^2, (ay)x^2) \\ (-vy, vx) & (-(vx)y^2, (vy)x^2) \\ (-(1-a)y\sqrt{z}, (1-a)x\sqrt{z}) & (-(1-a)x\sqrt{z}\cdot y^2, (1-a)y\sqrt{z}\cdot x^2) \\ (t_{ij}y, -t_{ij}x) & ((t_{ij}x)y^2, -(t_{ij}y)x^2) \end{array}$$

where $1 \le i, j \le 2$.

The computation of the module of relations on vx, vy over R is almost verbatim the same as in section (2.3). First note that we can "un-localize" by forgetting that the element $(t_{11}t_{22}-t_{12}t_{21})^2$ is invertible in R: more precisely, the module of relations on vx, vy over our ring

$$R = k[z, z^{-1}, v, a, vx, vy, ax, ay, x^2, y^2, (1-a)x\sqrt{z}, (1-a)y\sqrt{z}\,,$$
$$t_{ij}x, t_{ij}y, t_{ij}^2, at_{ij}, \tfrac{1}{(t_{11}t_{22}-t_{12}t_{21})^2}, (1-a)t_{ij}\sqrt{z}\,]$$

is the localization at $(t_{11}t_{22}-t_{12}t_{21})^2$ of the module of relations on vx, vy over the ring

$$\check{R} = k[z, z^{-1}, v, a, vx, vy, ax, ay, x^2, y^2, (1-a)x\sqrt{z}, (1-a)y\sqrt{z}\,,$$
$$t_{ij}x, t_{ij}y, t_{ij}^2, at_{ij}, (1-a)t_{ij}\sqrt{z}\,]$$

Then the rest of the computation is similar to one in (2.4) yet a bit longer: we need to consider more cases according to not only the parity of the powers of $x, y, \sqrt{z}$ but also the parity of powers in which t_{ij}s appearing in the homogeneous elements.

Then, as in section (2.4), the projection of R^2 onto the second coordinate defines an isomorphism of $P_0 \subseteq N$ with some ideals $I \subseteq J$ of R. By Lemmas (2.7) and (2.9) we reduce to proving that there is no projective subideal $P \subseteq (ax, vx, (1-a)x\sqrt{z}, t_{ij}x, ayx^2, vyx^2, (1-a)y\sqrt{z}x^2, t_{ij}yx^2)R$ such that $PS = JS = (x)S$ (here

$S = R[\sqrt{z}\,]$ is the normalization of R). Argument after Lemma (2.10) works again, so that we can kill y and v.

Now it remains to prove only that there is no projective ideal $P \subseteq (t_{ij}x, ax, (1-a)\sqrt{z}x)$ in the ring

$$R = k[z, z^{-1}, a, ax, x^2, (1-a)x\sqrt{z}, t_{ij}x, t_{ij}^2, at_{ij}, \frac{1}{(t_{11}t_{22} - t_{12}t_{21})^2}, (1-a)t_{ij}\sqrt{z}\,]$$

such that $PS = (x)S$ where $S = R[\sqrt{z}\,]$ is the normalization or R. The proof is literally the same as the proof of Proposition (2.11) under assumptions $Pic(R_a) = Pic(R_b) = 0$. The latter assumptions are checked similarly to ones in section (2.5): the presence of t_{ij}s just adds extra elements to the conductors, but the computations stay virtually the same.

CHAPTER V

Characterization of Test Elements for Tight Closure and Frobenius Closure in Terms of Module-Finite Ring Extensions of R within $R^{\frac{1}{p}}$

Completely stable test elements for tight closure are known to exist in quite large classes of rings: e.g. for excellent local rings, for F-finite rings, and, by reduction to this case, for all reduced rings essentially of finite type over sufficiently nice semi-local rings (see Theorems 6.1, 6.2 in [16]; the semi-local case follows by combining these with (6.1a) in [14]). The main source of motivation for the results stated in this chapter comes from an attempt to show existence of test elements for tight closure in excellent domains of finite Krull dimension (in the *non-local* case).

In [2] it is shown that (under mild conditions on the domain R) the test elements for tight closure can be obtained as powers of the test elements for p-th root closure. Recall that for the ideal I of a reduced ring R of prime characteristic p, its *p-th root closure* is defined to be $IR^{\frac{1}{p}} \cap R$. We will call $c \in R^{\circ}$ *a test element for p-th root closure* if $c(IR^{1/p} \cap R) \subseteq I$ for all ideals I of R (following terminology in [2]). The main result of this chapter says that for obtaining these elements, it is enough to deal only with module-finite ring extensions of R within $R^{\frac{1}{p}}$, as opposed to dealing with the much larger ring $R^{\frac{1}{p}}$ itself. Moreover, in this case the condition for being a test element for tight closure is equivalent to a splitting-type condition on such subrings of $R^{\frac{1}{p}}$. The main result we have in this direction is as follows (see sections 2.2 and 2.3 for the definitions of excellent and approximately Gorenstein rings):

Theorem 5.1. Let R be an excellent approximately Gorenstein normal domain of prime characteristic p. Let $c \neq 0$ be an element from the defining ideal of the singular locus of R (so that R_c is regular). Consider the following properties of the element c:

(i) For any subring S of $R^{1/p}$ such that S is module-finite over R, there exists an R-module map $\theta : S \to R$ such that $\theta(1) = c$.

(ii) c is a test element.

(iii) For any ideal $I \subseteq R$ and for any subring S of $R^{1/p}$ such that S is module-finite over R we have $cIS \cap R \subseteq I$.

We have the following implications:

(A) c satisfies (i) $\Longrightarrow c^3$ satisfies (ii)

(B) c satisfies (ii) $\Longrightarrow c$ satisfies (iii)

(C) c satisfies (iii) $\Longrightarrow c$ satisfies (i)

Remark 1. If R is any domain of prime characteristic p, then for all ideals $I \subseteq R$, the Frobenius closure is inside the tight closure: $IR^{1/p^e} \cap R \subseteq I^*$ for all e. Indeed, if $x = \sum i_j r_j^{1/q_0}$ for some q_0, then for all $q = p^e \geq q_0$ we have $1x^q = \sum i_j^q r_j^{q/q_0} \in I^{[q]}$, so that $x \in I^*$. So in particular, test elements for tight closure are moreover test elements for Frobenius closure (and of course for even smaller p-th root closure). I. Aberbach has proven a partial converse to this in [2]: if c is a test element for the

p-th root closure then c^3 is a test element for tight closure. This sheds light on where condition (A) comes from.

Remark 2. The condition of normality of R is not too restrictive as the question of existence of test elements is easily reduced to this case (see e.g. [2] for details).

The heart of the proof is the following result, which is also interesting in its own right:

Theorem 5.2. Let R be a reduced excellent approximately Gorenstein ring, let S be a ring containing R as a subring which is finitely generated as an R-module and let $c \in R$ be a nonzerodivisor on S. Then the following are equivalent:

(i) There exists an R-module map $\theta : S \to R$ such that $\theta(1) = c$.

(ii) For every ideal $I \subseteq R$ we have $c(IS \cap R) \subseteq I$.

(iii) For every ideal $I \subseteq R$ primary to a maximal ideal of R we have $c(IS \cap R) \subseteq I$.

Remark. Note that c being a nonzerodivisor on S implies $c(IS \cap R) = cIS \cap cR$, so we can rewrite the conditions in (ii) and (iii) as $cIS \cap cR \subseteq I$.

Proof.

(i) $\Longrightarrow$ (ii) Pick an element $u \in IS \cap R$. $u \in IS$ implies $\theta(u) = \theta(\sum_j i_j s_j) = \sum_j i_j \theta(s_j) \in I$ for some $i_j \in I$ and $s_j \in S$. We also have $u \in R$ so that $\theta(u) = u\theta(1) = uc$. Thus $cu \in I$.

(ii) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (i)

Consider the R-linear map $\varphi : \mathrm{Hom}_R(S,R) \to R$ given by $f \mapsto f(1)$. Proving (i) is the same as proving that $c \in \mathrm{Im}(\varphi)$. Assume this is not true.

We will first reduce to the case when R is complete local. We can choose a maximal ideal $\mathfrak{m}$ from the support of the nonzero R-module $(cR + \mathrm{Im}\varphi)/(\mathrm{Im}\varphi)$. S is finitely presented over R so the flat base change $R \to R_{\mathfrak{m}}$ gives $(\mathrm{Hom}_R(S,R))_{\mathfrak{m}} = \mathrm{Hom}_{R_{\mathfrak{m}}}(S_{\mathfrak{m}}, R_{\mathfrak{m}})$ and also $(\mathrm{Im}\varphi)_{\mathfrak{m}} = \mathrm{Im}(\varphi_{\mathfrak{m}})$. Therefore $(cR_{\mathfrak{m}} + \mathrm{Im}(\varphi_{\mathfrak{m}}))/\mathrm{Im}(\varphi_{\mathfrak{m}}) = ((cR + \mathrm{Im}(\varphi))/\mathrm{Im}(\varphi))_{\mathfrak{m}} \neq 0$, so that $c \notin \mathrm{Im}(\varphi_{\mathfrak{m}})$. Let's check that the condition (iii) also holds for the rings R,S replaced by $R_{\mathfrak{m}}$,$S_{\mathfrak{m}}$. Since every ideal of $R_{\mathfrak{m}}$ is an extension from its contraction to R, it's enough to check the condition (iii) for the ideals $IR_{\mathfrak{m}}$, where $I \subseteq \mathfrak{m}$ is an $\mathfrak{m}$-primary ideal of R. We have $cIS \cap cR \subseteq I$ (by Remark above); flat base change commutes with finite intersections so localizing at $R - \mathfrak{m}$ gives $cIS_{\mathfrak{m}} \cap cR_{\mathfrak{m}} \subseteq IR_{\mathfrak{m}}$. Clearly c is nonzerodivisor on $S_{\mathfrak{m}}$, $R_{\mathfrak{m}} \subseteq S_{\mathfrak{m}}$ and $S_{\mathfrak{m}}$ is finitely generated as $R_{\mathfrak{m}}$ module. Also $R_{\mathfrak{m}}$ is reduced, excellent, approximately Gorenstein (because R is). Now, the map $R_{\mathfrak{m}} \to \widehat{R_{\mathfrak{m}}}$ is faithfully flat so exactly the same argument as above shows that after the base change $R_{\mathfrak{m}} \to \widehat{R_{\mathfrak{m}}}$ we have $(c\widehat{R_{\mathfrak{m}}} + \mathrm{Im}(\widehat{\varphi_{\mathfrak{m}}}))/\mathrm{Im}(\widehat{\varphi_{\mathfrak{m}}}) = ((cR + \mathrm{Im}(\varphi))/\mathrm{Im}(\varphi)) \otimes_R \widehat{R} \neq 0$, so that c is not in the image of $\widehat{\varphi_{\mathfrak{m}}}$ and the condition (iii) still holds for the rings R,S replaced by $\widehat{R_{\mathfrak{m}}}$, $\widehat{R_{\mathfrak{m}}} \otimes_{R_{\mathfrak{m}}} S$ (because every $\widehat{\mathfrak{m}}$-primary ideal of $\widehat{R}$ is extended from an $\mathfrak{m}$-primary ideal of R). $\widehat{R_{\mathfrak{m}}}$ is reduced, excellent, approximately Gorenstein (because $R_{\mathfrak{m}}$ is) and c is nonzerodivisor on $\widehat{R_{\mathfrak{m}}} \otimes_{R_{\mathfrak{m}}} S$. Therefore without loss of generality we can assume that R is complete local.

Let W be the cokernel of the map φ. We have the exact sequence

$$\mathrm{Hom}_R(S,R) \xrightarrow{\varphi} R \to W \to 0.$$

Denote by $E = E_R(K)$ the injective hull of the residue class field K of R. R is complete local, so applying the Matlis dual functor ${}^\vee = \mathrm{Hom}_R(—, E)$ to the exact sequence above produces the exact sequence

$$0 \to W^\vee \to E \xrightarrow{\lambda} S \otimes_R E \tag{5.1}$$

(to see that $(\mathrm{Hom}_R(S,R))^\vee = S \otimes_R E$, it is enough to see that their Matlis duals are isomorphic:we have $(S \otimes_R E)^\vee = \mathrm{Hom}_R(S \otimes_R E, E) = \mathrm{Hom}_R(S, \mathrm{Hom}_R(E,E)) = \mathrm{Hom}_R(S,R))$. Note that the map λ acts as $\lambda : e \mapsto 1 \otimes e$. We want to show that $c \in R$ kills W, or equivalently that c kills $\ker(\lambda) = W^\vee$ ($cW = 0 \iff cW^\vee = 0$; actually $\mathrm{Ann}_R W = \mathrm{Ann}_R W^\vee$).

R is approximately Gorenstein so there exists a sequence $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_t \supseteq \ldots$ of irreducible m-primary ideals that are cofinal with powers of m. The injective hull of the residue class field of R will be $E_R(K) = \varinjlim R/I_t$. Thus (5.1) becomes

$$0 \to \varinjlim W_t \to \varinjlim R/I_t \xrightarrow{\lambda} S \otimes_R \varinjlim R/I_t$$

where $W_t = \ker(R/I_t \to S \otimes_R R/I_t)$. If c kills W_t for all t then c will also kill all of $W^\vee$. So it is enough to show that c kills every W_t.

Note that R/I_t is Gorenstein of dim 0 so that R/I_t is injective as a module over itself. In particular, we have $\mathrm{Ann}_{E_R(K)} I_t \cong E_{R/I_t}(K) \cong R/I_t$. For every module M killed by I_t (in particular, for modules R/I_t, $S \otimes_R R/I_t$ and W_t) we have $\mathrm{Hom}(M, E_R(K)) \cong \mathrm{Hom}(M, R/I_t)$. Thus, by taking Matlis duals again, it suffices to show that c is in the image of $\mathrm{Hom}_R(S, R/I_t) \to R/I_t$ (where again $f \mapsto f(1)$) which in turn is equivalent to $\mathrm{Hom}_R(S/I_tS, R/I_t) \to R/I_t$ having c in the image. Since R/I_t is injective as a module over itself we need to show only that the R/I_t-

cyclic submodule of S/I_tS generated by 1 can be mapped to R/I_t so that $1 \mapsto c \iff R/(I_tS \cap R)$ can be mapped to R/I_t with $1 \mapsto c \iff c(I_tS \cap R) \subseteq I_t$, and we are done.

Proof of Theorem 5.1.

(A) It is enough to show that c kills the p-th root closure, i.e. that $c(IR^{1/p} \cap R) \subseteq I$ for any ideal $I = (f_1, \dots, f_k)$ of R (by Theorem 1.2 of [2] we will then have that c^3 is a completely stable test element for R). Pick any $u \in IR^{1/p} \cap R$ so that $u = \sum_{i=1}^k f_i r_i^{1/p}$ for some $r_i \in R$. Let $S = R[r_1^{1/p}, \dots, r_k^{1/p}]$. By the hypothesis (i) we have an R-module map $\theta : S \to R$ such that $\theta(1) = c$. Then we have $cu = \theta(1)u = \theta(u) = \theta(\sum_{i=1}^k f_i r_i^{1/p}) = \sum_{i=1}^k f_i \theta(r_i^{1/p}) \in I$.

(B) Assume that c is a test element for R. Fix an ideal $I \subseteq R$ and a subring S of $R^{1/p}$ that is finitely generated over R. We have $IS \cap R \subseteq IR^{1/p} \cap R \subseteq I^*$ (the last inclusion holds by the Remark right before the proof) so that $c(IS \cap R) \subseteq cI^* \subseteq I$ (last inclusion holds since c is a test element). Note that $c(IS \cap R) = cIS \cap cR$ (because c is nonzerodivisor in $R^{1/p}$), and that $cS \cap R \subseteq cR^{1/p} \cap R \subseteq (cR)^*$ (again by the Remark above). But $(cR)^* = \overline{cR}$ (by [14] Corollary 5.8]) and R being normal guarantees $\overline{cR} = cR$, so that $cS \cap R \subseteq (cR)^* = cR$. Therefore we get $cIS \cap R = (cIS \cap cS) \cap R = cIS \cap (cS \cap R) \subseteq cIS \cap cR \subseteq I$.

(C) Again fix an ideal $I \subseteq R$ and a subring S of $R^{1/p}$ that is finitely generated over R. $c(IS \cap R) = cIS \cap cR$ (because c is a nonzero divisor on S) so that $c(IS \cap R) = cIS \cap cR \subseteq cIS \cap R \subseteq I$. Now apply the part (ii) $\Longrightarrow$ (i) of the Lemma.

CHAPTER VI

Some Relevant Open Questions

The counterexample to Conjecture 4.1 that we constructed is "extremely non-local", so to speak. It heavily uses the fact that the module M has finite phantom projective dimension locally. So a natural question would be to prove or disprove this conjecture under the additional assumption of R being local. This is still an open question. I believe the answer in this case is also negative.

There is a uniqueness property for projective resolutions: any two projective resolutions are quasi-isomorphic, and in the local case it is possible to single out a canonical one - the minimal resolution. In the case of phantom resolution the situation is much more complicated because there is no "canonical" one to start with (even though the minimal resolutions in the local case are defined, they do not have to be chain isomorphic; the length of the minimal resolution and the Betti numbers are unique though -the proof of all these statements can be found in [2]). However, any two minimal phantom projective resolutions become chain isomorphic after tensoring with an appropriate module-finite extension of R. This is among the main difficulties in attacking the problem of finding good homological characterization of modules of finite phantom projective dimension.

Note that for any projective resolution $P_\bullet$, $H_i(F^e(P_\bullet))$ can be viewed as $\mathrm{Tor}_i^R(M, {}^e R)$, where ${}^eR$ is R viewed as an R algebra via eth iteration of the Frobenius endomorphism. Vanishing of these Tors unfortunately does not give a homological criterion for $ppd_R M < \infty$, as the remarks after Theorem (1.5) show. I believe that such a criterion might be obtained if we allow module-finite extensions in the definition of the phantom resolution in some neat way. The idea comes from the fact that tensoring a phantom resolution with a balanced big Cohen-Macaulay module R^+ makes the resolution exact.

Another natural direction to work in would be to modify the definition of finite phantom projective dimension, so that it becomes local and stable under taking direct summands, and generalize the existing theory to this case.

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