Potent Elements and Tight Closure in Artinian Modules

by

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CHAPTER I

Introduction

In the last twenty years, Hochster and Huneke's theory of tight closure has emerged as the state of the art in characteristic p methods. It has unified many deep results from commutative algebra that were not previously thought to be related while simultaneously providing vastly simplified proofs of far more general theorems. It has produced a wealth of entirely novel results, many of which are very powerful and seem to be unobtainable through other means. Moreover, in recent years the influences of tight closure theory have expanded beyond the borders of pure commutative algebra into neighboring fields such as algebraic geometry.

In its primary form, tight closure is a closure operation performed on a submodule of a module over reduced ring of characteristic p. For a reduced ring R of prime characteristic and an inclusion of R-modules $N \subseteq M$, the tight closure of N in M, denoted N^*_M, is a potentially larger submodule of M containing the original module N. When $N^*_M = N$ we say N is tightly closed (in M). The definition of tight closure can be extended to rings containing a field of characteristic zero by reduction to characteristic p. In this way, one is often able to prove theorems about rings containing the rational numbers using tight closure.

A very important property of tight closure is that over a regular ring all ideals are tightly closed. This relatively easy fact is the key ingredient in the proof that direct summands of regular rings are Cohen-Macaulay, as well as many other important results. Observing that they share many of the nice properties of regular rings, Hochster and Huneke defined a ring to be *weakly F-regular* if all of its ideals are tightly closed. The class of weakly F-regular rings includes the class of regular rings, but it is much larger.

One of the most important open problems in tight closure theory concerns its interaction with localization. In general, if $I \subseteq R$ is an ideal and $W \subseteq R$ a multiplicative system, then one easily verifies that $I^*(W^{-1}R) \subseteq (IW^{-1}R)^*$, but understanding exactly when equality holds is formidable. For many years, experts believed that equality may always hold, and over the years some special cases in this direction were obtained. However, due to very recent work of Brenner and Monsky it is now known that there exist rings, in fact three-dimensional normal hypersurface domains, in which tight closure does not commute with localization. Despite the troubling examples of Brenner and Monsky, it is still entirely possible that if R is weakly F-regular then $W^{-1}R$ is weakly F-regular for all multiplicative systems $W \subseteq R$. A ring such that $W^{-1}R$ is weakly F-regular for all multiplicative systems $W \subseteq R$ is called *F-regular*. The question of whether every weakly F-regular ring is F-regular remains one of the most important open questions in tight closure theory.

There is yet another kind of F-regularity known as strong F-regularity. One very natural but slightly non-standard characterization of these rings is that all submodules of *all* modules M are tightly closed, not just when M is finitely generated. It is well known that if R is strongly F-regular then $W^{-1}R$ is strongly F-regular for every multiplicative system $W \subseteq R$. Hochster and Huneke conjectured the equivalence of strong and weak F-regularity for excellent local rings and proved this in the case where the ring is Gorenstein. Since then, Lyubeznik and Smith have established the conjecture in the case of an $\mathbb{N}$-graded ring over a field and the case of a ring with an isolated non-Gorenstein point. Since every strongly F-regular ring is F-regular, we know that weak F-regularity is equivalent to F-regularity in these cases. It is interesting to note that in all cases where we know that weak F-regularity is equivalent to F-regularity, we actually know that it is equivalent to strong F-regularity.

This thesis addresses a related question involving Artinian modules. To describe it, let $N \subseteq M$ be an inclusion of R-modules which may not be finitely generated, and let $z \in M$. We say that z is in the *finitistic tight closure of N in M*, denoted N_M^{*fg}, if $z \in (N' \cap N)^*_{N'}$ for some finitely generated R-module $N' \subseteq M$. Clearly $N_M^{*fg} \subseteq N_M^*$, but the containment may be strict. The case $N = 0$ is central since, quite generally, $z \in N_M^*$ if and only if $\bar{z} + N \in 0^*_{M/N}$. Motivated by the fact that strong and weak F-regularity are equivalent if tight closure equals finitistic tight closure for the zero submodule of the injective hull of the residue class field, Lyubeznik and Smith conjectured that tight closure equals finitistic tight closure in all Artinian modules over an excellent local ring. In fact, their proofs that strong and weak F-regularity are equivalent in the cases described above work by establishing that tight closure is the same as finitistic tight closure in certain Artinian modules. Indeed, in article [LS99] they show that tight closure equals finitistic tight closure in every graded Artinian module over an $\mathbb{N}$-graded ring over a field. In [LS01] they show that tight closure equals finitistic tight closure for the zero submodule of the injective hull of the residue class field of a local ring with an isolated non-Gorenstein point, and in every Artinian module over a local ring with an isolated singularity. In [Abe02] using an argument of B. MacCrimmon, it is shown that tight closure is the same as finitsitic tight closure for the zero submodule of the injective hull of the residue class field when R has an isolated non $\mathbb{Q}$-Gorenstein point. At the time of this writing, there are no known examples where tight closure and finitistic tight closure differ.

In this thesis we develop the theory of potent ideals, a new method for studying the question of whether tight closure equals finitistic tight closure for in arbitrary Artinian module. Briefly, an ideal $I \subseteq R$ is called *potent* for an Artinian module M if $0^*_M = \bigcup_n 0^*_{\mathrm{Ann}_M(I^n)}$. One easily verifies that over a local ring R, tight closure is the same as finitistic tight closure in every Artinian R-module if and only if the maximal ideal of R is potent for every Artinian R-module. As a consequence of the theory of potent elements we prove results which simultaneously unify and generalize the results of Lyubeznik and Smith in [LS01], and which are strong enough to establish Lyubeznik and Smith's conjecture in several new cases. See, for example, V.7, VI.10, VI.11.

An Overview of the Thesis

In Chapter 2 we establish the basic notation and definitions used throughout the rest of the thesis, including the definition of a potent element and a potent ideal. We also record standard results about tight closure, Matlis duality, and other theories which we make frequent use of.

In Chapter 3 we introduce and develop the technique of u-split complexes. Using this technique we give our first proof that the defining ideal of the singular locus is potent (for the formal definition of potent ideal, see II.12). The proof we give is inspired by Lyubeznik and Smith's original proof that tight closure and finitistic tight closure agree in every Artinian module over an isolated singularity. The theory of potent elements allows us to simplify some arguments. It should be noted that the results of Chapter 3 are indeed special cases of the main results of Chapter 4. However, the arguments in Chapter 4 are much more difficult, and the results of Chapter 3 already provide substantially more information than the original results of Lyubeznik and Smith. We also expect the theory of u-split complexes to have other applications, and there is reason to believe that the arguments of Chapter 3 may be generalized in a different direction from those of Chapter 4.

The most significant results of this thesis are contained in Chapter 4 where we ultimately prove that, over a local ring, the defining ideal of the non-finite injective dimension locus of a finitely generated R-module W is potent for the Matlis dual of W. This result unifies and extends the results of Lyubeznik and Smith in [LS01] as well as the results of Chapter 3 of this thesis, but the techniques we employ in Chapter 4 are quite different from those in the other treatments. The first goal of the chapter is a uniform annihilator theorem for certain Ext modules (see IV.5 and IV.41), a result interesting in and of itself. The setup is that for all q, the modules $\mathrm{Ext}^i_R(R^{1/q}, W)$ vanish when we localize at a fixed element $c \in R$ and we want to show that a fixed power of c annihilates all of the modules. The basic idea is to show that the Ext modules in question are related to certain local cohomology modules "up to a bounded power of c". This enables us to prove the result since uniform ahhihilators of the local cohomology modules are known to exist by results of Hochster and Huneke. The transition from Ext modules to local cohomology modules is a lengthy process involving several new ideas. The arguments are most natural and transparent in the case where the ring is assumed to be equidimensional, and that case is presented in its entirety before the more general result is proved.

After obtaining the uniform annihilator result for Ext modules, Chapter 4 concludes by applying the theorem to the theory of potent elements. The main result obtained is strong enough to imply that the defining ideal of the singular locus is potent for every Artinian module, that the defining ideal of the non-Gorenstein locus is potent for the injective hull of the residue class field, and that the defining ideal of the non-Cohen-Macaulay locus is potent for the top local cohomology module of the ring with support in the maximal ideal. The main result therefore unifies several results in the literature while simultaneously imparting substantially more information.

In Chapter 5 we study potent elements for local cohomology modules. We establish that the Cohen-Macaulay locus and the finite projective dimension locus behave well with respect to completion. Given a finitely generated module W over a local ring R, let $I \subseteq R$ be the radical ideal such that $c \in I$ if and only if W_c is Cohen-Macaulay and has finite projective dimension over R_c. Using our work in Chapter 4 we prove that I is potent for the local cohomology modules $H^j_m(W)$. See Theorem V.1. This result enables us to establish Lyubeznik and Smith's conjecture for certain local cohomology modules in Corollary V.7.

In Chapter 6 we develop the theory of potent elements for graded rings. We observe that the methods of [LS99] show that the ideal generated by elements of positive degree in any $\mathbb{N}$-graded ring is potent for any graded Artinian module. For completeness, a proof of this fact is included. We then combine this result with our earlier work to obtain new cases in which tight closure agrees with finitsitc tight closure for graded Artinain modules over $\mathbb{N}$-graded rings. For example, we show that if $R = A[x_1, \ldots, x_d]$ is a polynomial ring over an isolated singularity A, then tight closure equals finitistic tight closure in every graded Artinian R-module. We mention that in [Eli03], it is shown that for formal power series rings R over a local ring (A, m) with isolated singularity, mR is potent, though the "potent" terminology does not appear there. The results contained in this thesis constitute a substantial generalization of this result.

CHAPTER II

Background and Notation

In this chapter we adopt notation to be used throughout the rest of this work as well as review the standard notions of tight closure theory and other theories necessary for reading this thesis. We begin with a review of tight closure theory. While it is possible to define tight closure over any ring containing a field, we shall only be concerned with the characteristic $p > 0$ theory here. Some preliminary remarks about the characteristic of a ring are in order.

If R is a ring with multiplicative identity element 1, then we say R has (finite) characteristic p if p is the smallest positive integer such that $p \cdot 1 = 0$ where $p \cdot 1 = (1 + 1 + \cdots + 1)$ (p times). If no such integer p exists we define the characteristic of the ring to be 0. It is clear that if R is an integral domain, then the characteristic is either 0 or a prime p. With few exceptions, the rings we consider in this thesis will have prime characteristic $p > 0$. While many "natural" rings such as the ring of polynomials over the real or complex numbers have characteristic 0, powerful techniques developed over the last thirty to forty years have shown that many theorems can be proven about rings of containing a field of characteristic 0 by first proving the corresponding theorem about rings of prime characteristic $p > 0$.

As is standard when working with tight closure, we shall make the following conventions when dealing with rings of positive characteristic: When R has positive prime characteristic, we will always denote the characteristic by p, and then $q = p^e$ will always denote a positive power of p for some $e \in \mathbb{N}$. Thus, the statement "for all $q \gg 0$" is synonymous with "for all $q = p^e \gg 0$". When R is reduced, we define the R-algebra

$$R^{1/q} := R[r^{1/q} \ : \ r \in R]$$

for every $q = p^e$. Note that given any map $N \to M$ is of R-modules there is an induced map $R^{1/q} \otimes_R N \to R^{1/q} \otimes_R M$.

Definition II.1. Let R be a reduced ring of characteristic $p > 0$. Let $N \subseteq M$ be an arbitrary inclusion of R-modules, and let $z \in M$. We say that $z \in N^*_M$, *the tight closure of* N *in* M, if there exists an element $c \in R$, not in any minimal prime of R, such that

$$c^{1/q} \otimes z \in \operatorname{Im}(R^{1/q} \otimes_R N) \subseteq R^{1/q} \otimes_R M$$

for all $q \gg 0$.

As is evident from the definition, the modules $R^{1/q}$ are of fundamental importance to tight closure. We often wish to impose a finiteness condition:

Definition II.2. Let R be a ring of characteristic $p > 0$. We say that R is *F-finite* if $R^{1/p}$ is finitely generated as an R-module.

If R is F-finite, then it is immediate that $R^{1/q}$ is a finitely generated R-module for all $q = p^e$. Moreover, it is easy to see that if R is F-finite, then so is every homomorphic image, localization, finitely generated algebra and formal power series ring over R. In the complete local case it follows that (R, m, K) is F-finite if and only if K is F-finite. More generally, if (R, m, K) is excellent then R is F-finite if and only if K is F-finite by a theorem of Kunz (see [Kun76], Theorem 2.5). This happens, for example, when K is perfect, so the F-finiteness condition is not very restrictive.

There is an alternative description of tight closure involving the Peskine-Szpiro functor (also called the Frobenius functor).

Definition II.3. Let R be a ring of characteristic p. The *Frobenius endomorphism* $F : R \to R$ is the map $F(r) = r^p$ for all $r \in R$. The e^{th} iteration of this map, denoted F^e, is such that $F^e(r) = r^q$.

For all $e \in \mathbb{N}$, we let $R^{(e)}$ denote the ring R considered as a module over itself via F^e. That is, for $x \in R^{(e)}$ and $r \in R$ we define the action $r \cdot x := r^q x$.

Definition II.4. The *Peskine-Szpiro functor* (or *Frobenius functor*) is the covariant functor from R-modules to R-modules which on an R-module M is given by $\mathcal{F}(M) := R^{(1)} \otimes_R M$. We let $\mathcal{F}^e$ denote the Frobenius functor iterated e times. Clearly, $\mathcal{F}^e(M) = R^{(e)} \otimes_R M$.

If $N \subseteq M$ are R-modules then there is a natural map $\mathcal{F}^e(N) \to \mathcal{F}^e(M)$. Note that this map need not be injective. We let $N_M^{[q]}$ denote the image of $\mathcal{F}^e(N)$ in $\mathcal{F}^e(M)$ under this natural map. It should be kept in mind that $N_M^{[q]}$ is a submodule of $\mathcal{F}^e(M)$, *not* of M itself. For $x \in M$ we let x^q denote the image of x in $\mathcal{F}^e(M)$. These notations are suggestive of the case of ideals: if $I \subseteq R$ is an ideal then the reader will easily verify that $I^{[q]}$ is the ideal generated by all q^{th} powers of elements in I.

Proposition II.5. ***(Alternative description of Tight closure).*** *Let R be a* *reduced ring. If $N \subseteq M$ is any inclusion of R-modules, then N^*_M may be identified with the set of elements $x \in M$ such that there exists an element $c \in R$, not in any minimal prime, such that for all $q \gg 0$, $cx^q \in N^{[q]}_M$.*

Proof: There is a commutative diagram:

$$\begin{array}{ccc} R & \hookrightarrow & R^{1/q} \\ \| & & F^{-e}\uparrow \\ R & \xrightarrow{F^e} & R \end{array}$$

where $F^{-e}(r) = r^{1/q}$. Both vertical arrows are isomorphisms and so for any R-module M we may readily identify $R^{(e)} \otimes_R M$ and $R^{1/q} \otimes_R M$. Under this identification, cx^q is identified with $c^{1/q} \otimes x$. This result now follows. □

Test elements are, roughly speaking, elements of R which may be used for all tight closure tests. The theory of test elements plays an important role in tight closure, in both computational and theoretical applications.

Definition II.6. Suppose that $c \in R$ is not in any minimal prime of R. We say that c is a *test element* if for all finitely generated R-modules $N \subseteq M$, if $z \in N^*_M$, then

$$c^{1/q} \otimes z \in \mathrm{Im}(R^{1/q} \otimes_R N) \subseteq R^{1/q} \otimes_R M$$

for all q.

We say that c is a *locally (respectively, completely) stable test element* if its image in (respectively, in the completion of) every local ring of R is a test element.

Note the finiteness conditions on $N \subseteq M$. Because of the emerging importance of tight closure in modules that are not finitely generated, we introduce the following terminology suggested by Mel Hochster.

Definition II.7. Suppose that $c \in R$ is not in any minimal prime of R. We say that c is a *big test element* if for all R-modules $N \subseteq M$, if $z \in N_M^*$, then

$$c^{1/q} \otimes z \in \mathrm{Im}(R^{1/q} \otimes_R N) \subseteq R^{1/q} \otimes_R M$$

for all q.

We say that c is a *locally (respectively, completely) stable big test element* if its image in (respectively, in the completion of) every local ring of R is a big test element.

The existence theorems currently available in the literature talk about test elements instead of big test elements; however, the assumption that the modules $N \subseteq M$ are finitely generated is usually not needed, and the proofs typically produce big test elements (or completely stable big test elements). For example, we make substantial use of the following theorem.

Theorem II.8. *Let (R, m, K) be an excellent, reduced local ring. Then R has a completely stable big test element. More precisely, if $c \in R$ is not in any minimal prime of R and is such that R_c is weakly F-regular and Gorenstein (for example, if R_c is regular), then c has a power, c^n, which is a completely stable big test element. In particular, the element c^n is a test element for R and its completion, $\widehat{R}$.*

Remark. This result is known to the experts, but since a proof in this generality is lacking from the literature we include a short sketch.

Sketch of Proof: By standard results we may assume that R is complete. The idea is then to use the gamma construction to reduce to the case where R is F-finite. This construction can be found in Section 6 of [HH94] and is reviewed briefly in Chapter 4 of this thesis. When we pass to the gamma construction, we may loose completeness, but we may then complete again. Without loss of generality we may assume R is F-finite and complete. The result then follows by arguing precisely as in the proof of Theorem 5.10 of [HH94], noting that the proof produces big test elements. $\square$

When the modules $N \subseteq M$ are not finitely generated, there is another important notion of tight closure, the finitistic tight closure of N in M.

Definition II.9. If $N \subseteq M$ is an inclusion of R-modules and $z \in M$, then we say z is in the *finitistic tight closure of N in M*, denoted N_M^{*fg} if there is a finitely generated R-module $N' \subseteq M$ such that $z \in (N' \cap N)^*_{N'}$.

It is clear from the definition that $N_M^{*fg} \subseteq N_M^*$. The Lyubeznik-Smith conjecture discussed in the introduction is the following

Conjecture II.10. *If (R, m, K) is an excellent local ring and $N \subseteq M$ is an inclusion of Artinian R-modules then $N_M^* = N_M^{*fg}$.*

The conjecture is easily reduced to the case $N = 0$. The main contribution of this thesis is the notion of a potent element and its consequences. First we set up some convenient notation.

Notation II.11. For any ideal $I \subseteq R$, for every integer $v \in \mathbb{N}$ and every R-module M, let $M_{(-v,I)} = \mathrm{Ann}_M(I^v)$. We sometimes write M_{-v} for $M_{(-v,I)}$ when I is clear from the context.

For $v' > v$, we note the following easy identities:

$$\Big(\frac{M}{M_{-v}}\Big)_{-v'} = \frac{M_{-v'}}{M_{-v}} = \Big(\frac{M}{M_{-v}}\Big)_{-(v'-v)}.$$

We now give the definition of a potent element.

Definition II.12. If $x \in R$ is any element, and M is an Artinian R-module, we say that x is *potent for M* if we have

$$0^*_M = \bigcup_{n \in \mathbb{N}} 0^*_{M_{(-n,x)}}.$$

Similarly, we say that an ideal $I \subseteq R$ is *potent for M* if we have

$$0^*_M = \bigcup_{n \in \mathbb{N}} 0^*_{M_{(-n,I)}}.$$

If x (respectively, I) is potent for every Artinian R-module we simply say x (respectively, I) is *potent*.

The next proposition collects some basic results about potent elements and potent ideals.

Proposition II.13. *Let (R, m, K) be a local ring and let M be an Artinian R-module.*

(a) If $I \subseteq R$ is an ideal, then I is potent (respectively, potent for M) if and only if $Rad(I)$ is potent (respectively, potent for M).

(b) Let $\widehat{R}$ be the completion of R at m. If $I \subseteq R$ and if R and $\widehat{R}$ have a common test element for Artinian modules, then I is potent for M if and only if $I\widehat{R}$ is potent for M. For example, if R is reduced and excellent, then I is potent for M if and only if $I\widehat{R}$ is potent for M.

(c) If $I, J \subseteq R$ are potent ideals then $I + J$ is potent. It follows that the set of all potent elements forms an ideal called the potent ideal, and it is a radical ideal.

Proof: (a). Since $\mathrm{Rad}(I)^t \subseteq I$ for $t >> 0$, it follows that $\mathrm{Rad}(I)^{tn} \subseteq I^n$ for all n and all $t >> 0$. Therefore, $\bigcup_{n\in\mathbb{N}} 0^*_{M_{(-n,Rad(I))}} \subseteq \bigcup_{n\in\mathbb{N}} 0^*_{M_{(-n,I)}}$ which proves the result.

(b). The second statement follows from the first since every reduced excellent local ring has a completely stable test element by Theorem II.8. We prove the first statement. Since M has DCC, $M \cong M \otimes_R \widehat{R}$, and is thus an $\widehat{R}$-module. The same is true for $F^e_R(M)$ since it also has DCC. Also, for any map $R \to S$ of algebras, $F^e_R(M) \otimes_R S \cong F^e_S(M \otimes_R S)$. It follows that $F^e_R(M) \cong F^e_R(M) \otimes_R \widehat{R} \cong F^e_{\widehat{R}}(\widehat{M}) = F^e_{\widehat{R}}(M)$. Similarly, for any submodule $N \subseteq M$, $F^e_R(N) = F^e_{\widehat{R}}(N)$.

Let $c \in R$ be a test element for R and $\widehat{R}$. Then for $z \in N \subseteq M$, $cz^q = 0$ in $F^e_R(N)$ if and only if $cz^q = 0$ in $F^e_{\widehat{R}}(N)$ by our comments above. So for a submodule $N \subseteq M$, 0^*_N is the same whether computed over R or $\widehat{R}$. Therefore, the result will follow if we show that $\mathrm{Ann}_M(I^n) = \mathrm{Ann}_M(I\widehat{R}^n)$.

Let $I^n = (f_1, \ldots, f_h)$. Then there is an exact sequence $0 \to \mathrm{Ann}_M(I^n) \to M \to M^{\oplus h}$. Since $\widehat{R}$ is faithfully flat, $0 \to \mathrm{Ann}_M(I^n) \otimes_R \widehat{R} \to \widehat{M} \to \widehat{M}^{\oplus h}$ is still exact and we find that

$$\mathrm{Ann}_M(I^n) = \mathrm{Ann}_M(I^n) \otimes_R \widehat{R} = \mathrm{Ann}_{\widehat{M}}(I\widehat{R}^n) = \mathrm{Ann}_M(I\widehat{R}^n).$$

This proves (b).

(c). Let M be any Artinian R-module. We have a decomposition $0^*_M = \bigcup_{n\in\mathbb{N}} 0^*_{M_{(-n,I)}}$. For each $n \in \mathbb{N}$, to simplify notation let $W_n := M_{(-n,I)}$. Then W_n is an Artinian

R-module, so using the potent property for J, we get

$$0^*_{W_n} = \bigcup_{v \in \mathbb{N}} 0^*_{(W_n)_{(-v,J)}}.$$

Notice that $(W_n)_{(-v,J)} = \mathrm{Ann}_M(I^n + J^v)$. It follows that

$$0^*_M = \bigcup_{n \in \mathbb{N}} \bigcup_{v \in \mathbb{N}} 0^*_{\mathrm{Ann}_M(I^n + J^v)}$$

and so it suffices to show

$$\bigcup_{n \in \mathbb{N}} \bigcup_{v \in \mathbb{N}} 0^*_{\mathrm{Ann}_M(I^n + J^v)} \subseteq \bigcup_{t} 0^*_{M_{(-t,I+J)}}.$$

Therefore, we will be done if we can prove that for all $n, v \in \mathbb{N}$, $\mathrm{Ann}_M(I^n + J^v) \subseteq \mathrm{Ann}_M((I+J)^t)$ for some t. But for this it is enough to show that $I^n + J^v \supseteq (I+J)^t$ for some t. This follows, for example, for $t = 2nv$. $\square$

Note that the modules $\mathrm{Ann}_M(m^t)$ are finitely generated for all t. Moreover, since M is Artinian, if $N \subseteq M$ is any finitely generated module then $N \subseteq \mathrm{Ann}_M(m^t)$ for all $t \gg 0$. Therefore, the conjecture of Lyubeznik and Smith has a nice translation into the language of potent ideals:

Conjecture II.14. *If (R, m, K) is an excellent local ring then m, the maximal ideal of R, is a potent ideal.*

In Chapter 5 we study potent elements for local cohomology modules. Recall that if R is a Noetherian ring, $I \subseteq R$ is an ideal and M an R-module, then for each $j \in \mathbb{N}$, the j^{th} local cohomology module of M with support in I is defined to be

$$H^j_I(M) := \lim_{\to t} \mathrm{Ext}^j_R(R/I^t, M).$$

For an introduction to the theory of local cohomology we refer the reader to section 3.5 of [BH93].

If (R, m, K) is a local ring we will let $E_R(K)$ denote the injective hull of the residue class field. The functor $\operatorname{Hom}_R(-, E_R(K))$ is exact since $E_R(K)$ is injective. The following result, known as Matlis Duality, will be used throughout the thesis:

Theorem II.15. *Let (R, m, K) be a complete local ring, let M be an Artinian R-module and let W be a finitely generated R-module. Let $E_R(K)$ denote the injective hull of the residue class field and let $(-)^\vee$ denote the functor $\operatorname{Hom}_R(-, E_R(K))$.*

(a) $R^\vee \cong E_R(K)$ while $E_R(K)^\vee \cong R$.

(b) $W^\vee$ is Artinian while $M^\vee$ is finitely generated.

(c) There are natural isomorphisms $(W^\vee)^\vee \cong W$ and $(M^\vee)^\vee \cong M$.

(d) The functor $(-)^\vee$ establishes an anti-equivalence between categories of modules with ACC and modules with DCC.

Proof: See 3.2.13 in [BH93]. □

In the following chapters we will make frequent use of canonical modules. We review the notion and some standard results.

Definition II.16. Let (R, m, K) be a local ring of dimension d. We say that a finitely generated R-module ω is a canonical module for R if $\operatorname{Hom}_R(\omega, E_R(K)) \cong H^d_m(R)$ where $E_R(K)$ is the injective hull of the residue class field, and $H^d_m(R)$ is the d^{th} local cohomology module with supports in m.

When R is locally equidimensional, we define ω to be a canonical module if ω_m is a canonical module for R_m for every maximal ideal $m \subseteq R$.

The next theorem contains the standard results about canonical modules which we will need.

Theorem II.17. *Let (R, m, K) be a local ring of dimension d. Let $E_R(K)$ be the injective hull of the residue class field, and let $(-)^\vee$ denote the functor $\operatorname{Hom}_R(-, E_R(K))$.*

(a) If R is complete then R has a canonical module, and any canonical module is isomorphic with $H^d_m(R)^\vee$.

(b) Any two canonical modules for R are (non-canonically) isomorphic.

(c) If R is a homomorphic image of a Gorenstein ring S, then R has a canonical module. If $R = S/J$ then $\operatorname{Ext}^h_S(R, S)$ is a canonical module for R where $h = \dim(S) - \dim(R)$.

(d) If R is equidimensional and a homorphic image of a Gorenstein ring then let ω denote a canonical module for R. For every prime P of R, ω_P is a canonical module for R_P.

Proof: Part (a) follows at once from (c) and Matlis duality. Part (b) is immediate from part (c), and part (d) follows from (c) and standard facts about equidimensional rings. Part (c) is easily deduced from local duality. □

We shall also need the following celebrated result in characteristic $p > 0$:

Theorem II.18. *Let (R, m, K) be a local ring and let W be a finitely generated R-module. If W has finite injective dimension then R is a Cohen-Macaulay ring.*

This theorem was first raised as a question by Bass in the seminal work [Bas63]. It is now a theorem in all characteristics. The original proof in characteristic $p > 0$ (which is the only case we shall use) is due to Peskine and Szpiro ([PS72]). In equal characteristic the result is due to Hochster (see [Hoc75]) while the general case follows from the new intersection theorem proved by Roberts in [Rob87].

CHAPTER III

u-Split Complexes and the Defining Ideal of the Singular Locus

In this chapter we introduce the notion of a u-split complex, see Definition III.3. This technique will allow us to prove Theorem III.1 just below. The idea is to first use Matlis duality to translate the problem to a question about Ext modules which are finitely generated. Then using the technique of u-split complexes, we are able, in effect, to bound the relevant Artin-Rees numbers. See the proof of Proposition III.8 where the case of a principal ideal is obtained.

After proving Theorem III.1, we give a corollary which generalizes a result of Lyubeznik and Smith in [LS01]. Section 2 is devoted to using Theorem III.1 to give a proof that the defining ideal of the singular locus is potent for every Artinian module.

3.1 u-Split Complexes

The main goal of this section is the following result.

Theorem III.1. *Let R be a Noetherian ring and let $J \subseteq R$ be an ideal. Let M be an Artinian R-module. There exists an integer $k = k(J, M) \in \mathbb{N}$ depending only on J and M, such that for all $i \geq 0$ and for all R-modules N such that $JN = 0$, the* *natural map*

$$\operatorname{Tor}_i^R(N, M_{(-k,J)}) \to \operatorname{Tor}_i^R(N, M)$$

is surjective.

Before giving the proof we need several preliminary results. In this first lemma, we do not require any finiteness conditions on the ring.

Lemma III.2. *Let R be a commutative ring, let E be an injective module, and let M, N, W be arbitrary R-modules.*

(a) If $^\vee$ is the functor $\operatorname{Hom}_R(-, E)$ then $\operatorname{Ext}_R^i(N, M^\vee) \cong \operatorname{Tor}_i^R(N, M)^\vee$ for every $i \in \mathbb{N}$.

(b) If $I \subseteq R$ is a finitely generated ideal, then $\operatorname{Ann}_W(I)^\vee \cong \dfrac{W^\vee}{IW^\vee}$.

(c) If the natural map $\operatorname{Ext}_R^i(N, M^\vee) \to \operatorname{Ext}_R^i(N, M^\vee/I^k M^\vee)$ is injective then the natural map $\operatorname{Tor}_i^R(N, M_{(-k,I)}) \to \operatorname{Tor}_i^R(N, M)$ is surjective.

Proof: (a). Let $\mathcal{P}_\bullet$ be a projective resolution of N. Then

$$\operatorname{Tor}_i^R(N, M) := H_i(\mathcal{P}_\bullet \otimes_R M).$$

Applying $\operatorname{Hom}_R(-, E)$ shows that

$$\operatorname{Tor}_i^R(N, M)^\vee \cong \operatorname{Hom}_R(H_i(\mathcal{P}_\bullet \otimes M), E) \cong H^i(\operatorname{Hom}_R(\mathcal{P}_\bullet \otimes M, E)).$$

By the adjointness of tensor and Hom this is

$$H^i(\operatorname{Hom}(\mathcal{P}_\bullet, \operatorname{Hom}(M, E))) \cong H^i(\operatorname{Hom}(\mathcal{P}_\bullet, M^\vee)) =: \operatorname{Ext}_R^i(N, M^\vee).$$

This proves (a).

(b). Suppose $I = (u_1, \ldots, u_h)R$. There is an exact sequence

$$0 \to \mathrm{Ann}_W(I) \to W \xrightarrow{f} W^{\oplus h}$$

where the map f is given by the matrix $[u_1, \ldots, u_h]$. That is, for $x \in W$, $f(x) = (u_1 x, \ldots, u_h x)$. When we apply $\mathrm{Hom}_R(-, E)$ we get an exact sequence

$$(W^{\oplus h})^\vee \xrightarrow{f^\vee} W^\vee \to (\mathrm{Ann}_W(I))^\vee \to 0$$

where $f^\vee$ is given by the matrix $[u_1, \ldots, u_h]$ as well. Since $(\mathrm{Ann}_W(I))^\vee$ is the cokernel of $f^\vee$, it follows that $(\mathrm{Ann}_W(I))^\vee = W^\vee / \mathrm{Im}(f^\vee) = W^\vee / IW^\vee$.

(c). Let $\mathrm{Tor}_i^R(N, M_{(-k,I)}) \to \mathrm{Tor}_i^R(N, M) \to C \to 0$ be exact so that the relevant map is surjective if and only if $C = 0$. Then, applying $^\vee$, we get an exact sequence

$$\mathrm{Tor}_i^R(N, M)^\vee \leftarrow \mathrm{Tor}_i^R(N, M_{(-k,I)})^\vee \leftarrow C^\vee \leftarrow 0$$

and by parts (a) and (b) this is

$$\mathrm{Ext}_R^i(N, M^\vee) \leftarrow \mathrm{Ext}_R^i(N, M^\vee / I^k M^\vee) \leftarrow C^\vee \leftarrow 0.$$

The result now follows. $\square$

The next several results deal with u-split complexes, a notion we introduce here. This notion generalizes the notion of a split-exact complex.

Definition III.3. If $\mathcal{F}_\bullet : \ \cdots \to F_{i+1} \xrightarrow{\alpha_{i+1}} F_i \xrightarrow{\alpha_i} F_{i-1} \to \cdots$ is any complex of R-modules, and if $u \in R$ is any element, then we say that $\mathcal{F}_\bullet$ is *u-split at the i^{th} spot* if there exist R-linear maps $\beta_i : F_i \to F_{i+1}$ and $\beta_{i-1} : F_{i-1} \to F_i$ such that

$$(*) \quad u \cdot \mathbb{1}_{F_i} = \beta_{i-1} \circ \alpha_i + \alpha_{i+1} \circ \beta_i.$$

If $\mathcal{F}_\bullet$ is u-split at every i then we say that $\mathcal{F}_\bullet$ is *u-split.*

Similarly, if $\mathcal{F}^\bullet : \cdots \to F_{i-1} \stackrel{\alpha_i}{\to} F_i \stackrel{\alpha_{i+1}}{\to} F_{i+1} \to \cdots$ is any cohomological complex of R-modules, and if $u \in R$ is any element, then we say that $\mathcal{F}^\bullet$ is *u-split at the i^{th} spot* if there exist R-linear maps $\beta_i : F_i \to F_{i-1}$ and $\beta_{i+1} : F_{i+1} \to F_i$ such that

$$(**) \quad u \cdot \mathbb{1}_{F_i} = \alpha_i \circ \beta_i + \beta_{i+1} \circ \alpha_{i+1}.$$

If $\mathcal{F}^\bullet$ is u-split at every i then we say that $\mathcal{F}^\bullet$ is *u-split.*

The following result justifies the use of the terminology u-split.

Proposition III.4. *Let*

$$\mathcal{F}_\bullet : \cdots \to F_{k+1} \stackrel{\alpha_{k+1}}{\to} F_k \stackrel{\alpha_k}{\to} F_{k-1} \to \cdots \to F_1 \stackrel{\alpha_1}{\to} F_0 \stackrel{0}{\to} 0$$

be any (left) u-split complex. Then for all $k \geq 0$:

(a) $uF_k = \mathrm{Im}(\alpha_{k+1}) + \mathrm{Im}(\beta_{k-1}|_{\mathrm{Im}(\alpha_k)})$.

(b) $\alpha_k \beta_{k-1}|_{\mathrm{Im}(\alpha_k)} = u \cdot \mathbb{1}_{\mathrm{Im}(\alpha_k)}$.

(c) $\mathrm{Im}(\alpha_{k+1}) \cap \mathrm{Im}(\beta_{k-1}|_{\mathrm{Im}(\alpha_k)})$ *is killed by u.*

Proof: Property (a) follows immediately from Definition III.3.

We establish property (b). Suppose $f \in F_k$ is any element. Then equation $(*)$ in (III.3) for $i = k-1$ gives:

$$\alpha_k \beta_{k-1} = u\mathbb{1}_{F_{k-1}} - \beta_{k-2}\alpha_{k-1}$$

Hence,

$$\alpha_k \beta_{k-1} \alpha_k(f) = (u\mathbb{1}_{F_{k-1}} - \beta_{k-2}\alpha_{k-1})\alpha_k(f) = u\alpha_k(f) - \beta_{k-2}\alpha_{k-1}\alpha_k(f).$$

As $\alpha_{k-1}\alpha_k = 0$, we conclude $\alpha_k\beta_{k-1}\alpha_k(f) = u\alpha_k(f)$, and since $\alpha_k(f) \in \mathrm{Im}(\alpha_k)$ was arbitrary, this proves part (b).

Finally, to prove (c), let $y = \alpha_{k+1}(g) = \beta_{k-1}(f)$ for some $f \in \mathrm{Im}(\alpha_k)$. We want to show $uy = 0$. Applying α_k we find $\alpha_k(y) = \alpha_k\alpha_{k+1}(g) = 0 = \alpha_k\beta_{k-1}(f)$. By what we have just shown in (b),

$$\alpha_k\beta_{k-1}(f) = uf$$

since $f \in \mathrm{Im}(\alpha_k)$. Hence, $uf = 0$ and so

$$uy = u\beta_{k-1}(f) = \beta_{k-1}(uf) = 0$$

as required. $\square$

We have an analogous statement for right cohomological complexes. The proof is very similar.

Proposition III.5. *Let*

$$\mathcal{F}^\bullet : \; 0 \to F_0 \overset{\alpha_1}{\to} F_1 \overset{\alpha_2}{\to} \cdots \to F_{i-1} \overset{\alpha_i}{\to} F_i \overset{\alpha_{i+1}}{\to} F_{i+1} \to \cdots$$

be any (right) u-split cohomological complex. Then for all $i \geq 0$:

(a) $uF_i = \mathrm{Im}(\alpha_i) + \mathrm{Im}(\beta_{i+1}|_{\mathrm{Im}(\alpha_{i+1})})$.

(b) $\alpha_i\beta_i|_{\mathrm{Im}(\alpha_i)} = u \cdot \mathbb{1}_{\mathrm{Im}(\alpha_i)}$.

(c) $\mathrm{Im}(\alpha_i) \cap \mathrm{Im}(\beta_{i+1}|_{\mathrm{Im}(\alpha_{i+1})})$ *is killed by u.*

Proof: Property (a) follows immediately from Definition (III.3).

We establish property (b). Suppose $f \in F_{i-1}$ is any element. Then equation $(**)$ in (III.3) gives:

$$\alpha_i\beta_i = u\mathbb{1}_{F_i} - \beta_{i+1}\alpha_{i+1}$$

Hence,

$$\alpha_i\beta_i\alpha_i(f) = (u\mathbb{1}_{F_i} - \beta_{i+1}\alpha_{i+1})\alpha_i(f) = u\alpha_i(f) - \beta_{i+1}\alpha_{i+1}\alpha_i(f).$$

As $\alpha_{i+1}\alpha_i = 0$, we conclude $\alpha_i\beta_i\alpha_i(f) = u\alpha_i(f)$, and since $\alpha_i(f) \in \mathrm{Im}(\alpha_i)$ was arbitrary, this proves part (b).

Finally, to prove (c), let $y = \alpha_i(g) = \beta_{i+1}(f)$ for some $f \in \mathrm{Im}(\alpha_{i+1})$. We want to show $uy = 0$. Applying α_{i+1} we find $\alpha_{i+1}(y) = \alpha_{i+1}\alpha_i(g) = 0 = \alpha_{i+1}\beta_{i+1}(f)$. By what we have just shown in (b),

$$\alpha_{i+1}\beta_{k+1}(f) = uf$$

since $f \in \mathrm{Im}(\alpha_{i+1})$. Hence, $uf = 0$ and so

$$uy = u\beta_{i+1}(f) = \beta_{i+1}(uf) = 0$$

as required. $\square$

The u-split property is preserved by additive functors:

Proposition III.6. *Let $u \in R$ and suppose Γ is any additive functor from R-modules to R-modules (covariant or contravariant). If*

$$\mathcal{F}_\bullet : \cdots \longrightarrow F_{k+1} \overset{\alpha_{k+1}}{\to} F_k \overset{\alpha_k}{\to} F_{k-1} \longrightarrow \cdots$$

is any complex of R-modules that is u-split at the k^{th} spot then $\Gamma(\mathcal{F}_\bullet)$ is u-split at the k^{th} spot as well. Moreover, if $\mathcal{F}_\bullet$ is u-split then $\Gamma(\mathcal{F}_\bullet)$ is u-split.

Proof: The second statement follows immediately from the first. For the first statement, the point is that additive functors preserve the addition, composition and scalar multiplication of morphisms, and they take the identity morphism to the identity morphism. For example, suppose Γ is contravariant (the covariant case being similar) and look at the k^{th} spot of the complex $\mathcal{F}_\bullet$:

$$F_{k+1} \stackrel{\alpha_{k+1}}{\rightarrow} F_k \stackrel{\alpha_k}{\rightarrow} F_{k-1}.$$

If $\mathcal{F}_\bullet$ is u-split at k, then we have maps $\beta_k : F_k \to F_{k+1}$ and $\beta_{k-1} : F_{k-1} \to F_k$ satisfying $(*)$ of (III.3). When we apply Γ we get a complex

$$\Gamma(F_{k+1}) \stackrel{\Gamma(\alpha_{k+1})}{\leftarrow} \Gamma(F_k) \stackrel{\Gamma(\alpha_k)}{\leftarrow} \Gamma(F_{k-1})$$

and maps $\Gamma(\beta_k) : \Gamma(F_{k+1}) \to \Gamma(F_k)$ and $\Gamma(\beta_{k-1}) : \Gamma(F_k) \to \Gamma(F_{k-1})$. Furthermore we have the identity:

$$\Gamma(u \cdot \mathbb{1}_{F_k}) = \Gamma(\alpha_{k+1}\beta_k + \beta_{k-1}\alpha_k)$$

By our introductory remarks, we can write

$$u \cdot \mathbb{1}_{\Gamma(F_k)} = \Gamma(\alpha_{k+1}\beta_k) + \Gamma(\beta_{k-1}\alpha_k)$$

and therefore,

$$u \cdot \mathbb{1}_{\Gamma(F_k)} = \Gamma(\beta_k) \circ \Gamma(\alpha_{k+1}) + \Gamma(\alpha_k) \circ \Gamma(\beta_{k-1}).$$

Hence, $\Gamma(\beta_k), \Gamma(\beta_{k-1})$ have the right property. This completes the proof. $\square$

The next result provides us with examples of u-split complexes. It will be of great utility in proving Theorem III.1.

Proposition III.7. *Let N be any R-module and let $u \in \mathrm{Ann}_R(N)$. Let*

$$\mathcal{F}_\bullet : \cdots \to F_k \stackrel{\alpha_k}{\rightarrow} F_{k-1} \to \cdots \to F_1 \stackrel{\alpha_1}{\rightarrow} F_0 \to 0$$

be a free resolution of N. Then $\mathcal{F}_\bullet$ is u-split.

Proof: For simplicity, we isolate and prove the following two statements: for all $i \geq 0$, there exist R-linear maps $\beta_i : F_i \to F_{i+1}$ such that:

(1) $\alpha_{i+1} \circ \beta_i + \beta_{i-1} \circ \alpha_i = u \cdot \mathbb{1}_{F_i}$

(2) $\alpha_{i+1} \circ \beta_i = u \cdot \mathbb{1}_{\mathrm{Im}(\alpha_{i+1})}$

where $F_{-1} = 0$ and $\alpha_0, \beta_{-1} = 0$. Of course, the result follows from (1).

We use induction on i. If $i = 0$ then $\beta_{-1} \circ \alpha_0 = 0$ so the left side of equation (1) is just $\alpha_1 \circ \beta_0$. To define β_0 we only have to specify the values on a set of generators. Moreover, for each element e in a set of free generators for F_0, we can choose $\beta_0(e)$ to be any element of F_1. Since $u \in J$, $uF_0 \subseteq \mathrm{Im}(\alpha_1)$, so suppose $\alpha_1(y) = ue$. We define $\beta_0(e) := y$ and then clearly $\alpha_1 \circ \beta_0 = u\mathbb{1}(F_0)$ since this is true on a set of (free) generators. Note that when $i = 0$, equation (2) just says $\alpha_1 \circ \beta_0 = u\mathbb{1}_{\mathrm{Im}\,\alpha_1}$ which we have already verified.

Now assume β_{n-1} has been defined and we want to define β_n, $n \geq 1$. We will define β_n on a set of free generators for F_n and then extend linearly. Let e be a free generator of F_n. We first prove:

Claim: $ue - \beta_{n-1}\alpha_n(e) \in \mathrm{Im}(\alpha_{n+1})$.

To see this, note that since $n \geq 1$, $\mathrm{Im}(\alpha_{n+1}) = \mathrm{Ker}(\alpha_n)$ so we only must see that $\alpha_n(ue - \beta_{n-1}\alpha_n(e)) = 0$. Equivalently, $\alpha_n(ue) = \alpha_n(\beta_{n-1}\alpha_n(e))$. By the induction hypothesis, equation (2), we know $\alpha_n\beta_{n-1} = u\mathbb{1}_{\mathrm{Im}(\alpha_n)}$ so it follows that $\alpha_n(\beta_{n-1}\alpha_n(e)) = u\alpha_n(e)$. Therefore, the statement is equivalent to $\alpha_n(ue) = u\alpha_n(e)$

which is true by the R-linearity of α_n. This proves the claim.

Now, by the claim, choose $y \in F_{n+1}$ such that $\alpha_{n+1}(y) = ue - \beta_{n-1}\alpha_n(e)$ and define $\beta_n(e) := y$. Then

$$\alpha_{n+1}\beta_n(e) + \beta_{n-1}\alpha_n(e) = ue - \beta_{n-1}\alpha_n(e) + \beta_{n-1}\alpha_n(e) = ue$$

establishing equation (1). Notice that equation (2) now follows immediately: if $y \in \operatorname{Im}(\alpha_{i+1})$ then $\alpha_i(y) = 0$ since $\alpha_i\alpha_{i+1}$ is the zero map. Equation (1) then simplifies to equation (2). □

Proposition III.8. *Let $u \in R$ be any element, and let W be a finitely generated R-module. Then there exists an integer $k \in \mathbb{N}$ depending only on u and W such that for all finitely generated R-modules N such that $uN = 0$, and for every $i \geq 0$, the natural map*

$$\phi_{i,k} : \operatorname{Ext}^i_R(N, W) \to \operatorname{Ext}^i_R(N, \frac{W}{u^k W})$$

is injective.

Proof: By the Artin-Rees Lemma there exists $c \in \mathbb{N}$ such that for all $N > c$,

$$\operatorname{Ann}_W(u) \cap u^N W = u^{N-c}(u^c W \cap \operatorname{Ann}_W(u)) = 0.$$

Set $k := c + 2$ (so that $k - 1 > c$).

Suppose N is any finitely generated R-module with $uN = 0$ and let $\mathcal{F}_\bullet$ be a free resolution of N with boundary maps $\alpha_i : F_i \to F_{i-1}$. Let $\mathcal{W}_\bullet := \operatorname{Hom}_R(\mathcal{F}_\bullet, W)$ with induced maps $\alpha_i^\vee : W_{i-1} \to W_i$. Notice that if $F_i = R^{\oplus b_i}$ then $W_i = \operatorname{Hom}_R(F_i, W) \cong$

$W^{\oplus b_i}$. It follows that for c as above,

$$\mathrm{Ann}_{W_i}(u) \cap u^N W_i = 0$$

for all $N > c$ and for all $i \geq 0$. Furthermore, it is easy to see that $\phi_{i,k}$ is injective $\iff u^k W_i \cap \mathrm{Ker}(\alpha_{i+1}^\vee) \subseteq \mathrm{Im}(\alpha_i^\vee)$.

Suppose $z \in u^k W_i \cap \mathrm{Ker}(\alpha_{i+1}^\vee)$. We will show $z \in \mathrm{Im}(\alpha_i^\vee)$. Since $uN = 0$, $\mathcal{F}_\bullet$ is u-split by Proposition III.7. Therefore, since $\mathrm{Hom}_R(-, W)$ is an additive functor, it follows from Proposition III.6 that $\mathcal{W}_\bullet$ is u-split as well. For each i, let $\beta_i^\vee : W_i \to W_{i-1}$ be the R-linear map guaranteed by the u-split property. If $z = u^k w$, then by definition of u-split,

$$(**) \;\; z = \alpha_i^\vee \beta_i^\vee (u^{k-1}w) + \beta_{i+1}^\vee \alpha_{i+1}^\vee (u^{k-1}w).$$

Since $z \in \mathrm{Ker}(\alpha_{i+1}^\vee)$ we see that

$$0 = \alpha_{i+1}^\vee \alpha_i^\vee \beta_i^\vee (u^{k-1}w) + \alpha_{i+1}^\vee \beta_{i+1}^\vee \alpha_{i+1}^\vee (u^{k-1}w) = \alpha_{i+1}^\vee \beta_{i+1}^\vee \alpha_{i+1}^\vee (u^{k-1}w).$$

By III.5(b), $\alpha_{i+1}^\vee \beta_{i+1}^\vee = u \mathbb{1}_{\mathrm{Im}(\alpha_{i+1}^\vee)}$ and so we find that $u\alpha_{i+1}^\vee(u^{k-1}w) = 0$. In other words, $\alpha_{i+1}^\vee(u^{k-1}w) \in \mathrm{Ann}_{W_{i+1}}(u)$. But on the other hand, $\alpha_{i+1}^\vee(u^{k-1}w) = u^{k-1}\alpha_{i+1}^\vee(w)$ so we conclude that $\alpha_{i+1}^\vee(u^{k-1}w) \in u^{k-1}W_{i+1}$. Hence, $\alpha_{i+1}^\vee(u^{k-1}w) \in u^{k-1}W_{i+1} \cap \mathrm{Ann}_{W_{i+1}}(u) = 0$ since $k - 1 > c$. By $(**)$ above we conclude $z = \alpha_i^\vee \beta_i^\vee (u^c w) \in \mathrm{Im}(\alpha_i^\vee)$ as desired. $\square$

Now are now ready to prove Theorem III.1.

Proof of Theorem III.1: We can immediately reduce to proving the result for all finitely generated R-modules N such that $JN = 0$: any R-module N is a direct limit of its finitely generated submodules, and a direct limit of surjective maps is surjective.

We next reduce to the local case and then the complete local case. By standard results, a map of R-modules $\theta : A \to B$ is surjective if and only if $\theta_m : A_m \to B_m$ is surjective for every (maximal) ideal $m \in \mathrm{Spec}(R)$. Furthermore, $\mathrm{Tor}_i^R(N, M)_m \cong \mathrm{Tor}_i^{R_m}(N_m, M_m)$ for all R-modules N, M. Next we claim that $\mathrm{Ann}_M(I^t) \otimes_R R_m = \mathrm{Ann}_{M \otimes R_m}(IR_m^t)$. To see this, suppose $I^t = (f_1, \ldots, f_h)R$. Then we have an exact sequence

$$0 \to \mathrm{Ann}_M(I^t) \to M \stackrel{[f_1,\ldots,f_h]}{\to} M^{\oplus h}$$

and applying $- \otimes_R R_m$ we obtain an exact sequence:

$$0 \to \mathrm{Ann}_M(I^t) \otimes_R R_m \to M \otimes_R R_m \stackrel{[f_1,\ldots,f_h]}{\to} (M \otimes_R R_m)^{\oplus h}.$$

This justifies the claim. Finally, the hypotheses that $IN = 0$ and M has DCC are preserved when we pass to R_m. Therefore, without loss of generality we may assume R is local. Then $R \to \widehat{R}$ is faithfully flat and since the module M has DCC, it is already a module over $\widehat{R}$. It follows that we may replace R, N, M with $\widehat{R}$, $N \otimes \widehat{R}$, and $M \otimes \widehat{R} \cong M$ and assume that R is complete.

Let $W := \mathrm{Hom}_R(M, E)$ where $E = E_R(K)$ is the injective hull of the residue class field. Since M has DCC, W has ACC by Matlis duality. Therefore, by Lemma III.2 (c), it suffices to show that for a finitely generated R-module W, there exists $k = k(J, W) \in \mathbb{N}$ (depending on J and W) such that for all finitely generated R-modules N such that $JN = 0$, the natural map

$$\mathrm{Ext}^i_R(N, W) \to \mathrm{Ext}^i_R(N, \frac{W}{J^k W})$$

is injective.

Let $J = (u_1, \ldots, u_h)R$. By Proposition III.8, there exists k_1 such that for all finitely generated R-modules N such that $JN = 0$, the natural map

$$\operatorname{Ext}^i_R(N, W) \to \operatorname{Ext}^i_R(N, W/u_1^{k_1}W)$$

is injective. Let $W_1 = W/u_1^{k_1}W$. Then by Proposition III.8 again, there exists k_2 such that for all finitely generated R-modules N such that $JN = 0$, the map

$$\operatorname{Ext}^i_R(N, W_1) \to \operatorname{Ext}^i_R(N, W_1/u_2^{k_2}W_1)$$

is injective. It follows that for all finitely generated R-modules N such that $JN = 0$, the composition map

$$\operatorname{Ext}^i_R(N, W) \to \operatorname{Ext}^i(N, W_1/u_2^{k_2}W_1) \cong \operatorname{Ext}^i_R(N, W/(u_1^{k_1}, u_2^{k_2})W)$$

is injective as well. Proceeding in this way, for each $1 \leq i \leq h$ we get $k_i \in \mathbb{N}$ such that for all finitely generated R-modules N such that $JN = 0$, the natural map

$$\operatorname{Ext}^i_R(N, W) \to \operatorname{Ext}^i_R(N, \frac{W}{(u_1^{k_1}, \ldots, u_h^{k_h})W})$$

is injective. Now pick k so large that $J^k \subseteq (u_1^{k_1}, \ldots, u_h^{k_h})R$ (for example, take $k := k_1 + \cdots + k_h - h + 1$). Since the map $\operatorname{Ext}^i_R(N, W) \to \operatorname{Ext}^i_R(N, W/(u_1^{k_1}, \ldots, u_h^{k_h})W)$ factors as

$$\operatorname{Ext}^i_R(N, W) \to \operatorname{Ext}^i_R(N, \frac{W}{J^kW}) \to \operatorname{Ext}^i_R(N, W/(u_1^{k_1}, \ldots, u_h^{k_h})W)$$

it follows that the natural map

$$\operatorname{Ext}^i_R(N, W) \to \operatorname{Ext}^i_R(N, \frac{W}{J^kW})$$

is injective for all finitely generated R-modules N such that $JN = 0$. This completes the proof. $\square$

Theorem III.1 allows us to give a simplified proof of the following result, an improvement of [LS01], Proposition 8.4:

Corollary III.9. *Let $I \subseteq R$ be an ideal. For every Artinian R-module M and integers $t, v, i \geq 0$ there exist integers $\psi(M,t)$ and $\phi(M,t,v) > v$ such that:*

(a) For all R-modules N annihilated by I^t and all $v' > \psi(M,t)$, the natural map

$$\mathrm{Tor}_i^R(N, M_{(-v',I)}) \to \mathrm{Tor}_i^R(N, M)$$

is surjective.

(b) For all R-modules N annihilated by I^t and all $v' > \phi(M,t,v)$, the map

$$\mathrm{Tor}_i^R(N, M_{(-v',I)}) \to \mathrm{Tor}_i^R(N, M)$$

induced by the inclusion $M_{(-v',I)} \to M$ induces an isomorphism on the images of $\mathrm{Tor}_i^R(N, M_{(-v,I)})$ in both modules.

Proof: By Theorem III.1, with $J = I^t$, there exists $k \in \mathbb{N}$ such that for all R-modules N annihilated by I^t and for all $i \geq 0$, the map $\mathrm{Tor}_i^R(N, M_{(-k,I^t)}) \to \mathrm{Tor}_i^R(N, M)$ is surjective. Let $\psi(M,t) := kt$. Then since $M_{(-k,I^t)} = M_{(-kt,I)}$ we see that $\mathrm{Tor}_i^R(N, M_{(-kt,I)}) \to \mathrm{Tor}_i^R(N, M)$ is surjective. Now, for all $v' > kt = \psi(M,t)$, the map $\mathrm{Tor}_i^R(N, M_{(-kt,M)}) \to \mathrm{Tor}_i^R(N, M)$ factors as

$$\mathrm{Tor}_i^R(R, M_{(-kt,I)}) \to \mathrm{Tor}_i^R(N, M_{(-v',I)}) \to \mathrm{Tor}_i^R(N, M).$$

It follows that for all $v' > \psi(M,t)$, the map $\mathrm{Tor}_i^R(N, M_{(-v',I)}) \to \mathrm{Tor}_i^R(N, M)$ is surjective as well. This proves part (a).

To prove part (b), we follow [LS01], Proposition 8.4. It is enough to prove the following statement:

Claim: For a fixed t, if $\psi(M,t)$ exists for all M and i, then $\phi(M,t,v)$ also exists for all M, v and i.

Proof of Claim: Set $\phi(M,t,v) := v + \psi(M/M_{-v}, t)$. The commutative diagram with short exact rows:

$$\begin{array}{ccccccccc} 0 & \to & M_{-v} & \to & M & \to & M/M_{-v} & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & M_{-v} & \to & M_{-v'} & \to & \left(\frac{M}{M_{-v}}\right)_{-(v'-v)} & \to & 0 \end{array}$$

produces, for each i, the following commutative diagram with exact rows

$$\begin{array}{ccccc} \mathrm{Tor}^R_{i+1}(N, M/M_{-v}) & \longrightarrow & \mathrm{Tor}^R_i(N, M_{-v}) & \longrightarrow & \mathrm{Tor}^R_i(N, M) \\ f\uparrow & & \mathbb{1}\uparrow & & g\uparrow \\ \mathrm{Tor}^R_{i+1}(N, (M/M_{-v})_{-(v'-v)}) & \longrightarrow & \mathrm{Tor}^R_i(N, M_{-v}) & \longrightarrow & \mathrm{Tor}^R_i(N, M_{-v'}) \end{array}$$

By our choice of $\phi(M,t,v)$, when $v' > \phi(M,t,v)$ it follows that $v'-v > \psi(M/M_{-v}, t)$. Therefore, when $v' > \phi(M,t,v)$, f is surjective. An easy diagram chase then shows that g restricts to an isomorphism between $\mathrm{Im}\left(\mathrm{Tor}^R_i(N, M_{-v}) \to \mathrm{Tor}^R_i(N, M_{-v'})\right)$ and $\mathrm{Im}\left(\mathrm{Tor}^R_i(N, M_{-v}) \to \mathrm{Tor}^R_i(N, M)\right)$. This proves the claim and completes the proof of the theorem. $\square$

3.2 The Defining Ideal of the Singular Locus is Potent

Our goal for the rest of this chapter is to prove that the defining ideal of the singular locus is potent for every Artinian R-module.

Theorem III.10. *Let (R, m, K) be a reduced, excellent local ring, let $J \subseteq R$ be a defining ideal of the singular locus, and let M be an Artinian R-module. Then J is potent for M. That is,*

$$0^*_M = \bigcup_{v' \in \mathbb{N}} 0^*_{M_{(-v',J)}}.$$

First we need to know that a defining ideal of the the singular locus is generated by nonzerodivisors in the reduced case.

Lemma III.11. *Let R be a Noetherian ring.*

(a) If $J \subseteq R$ is any ideal of depth at least one then J is generated by nonzerodivisors.

(b) If R is reduced and if there is a radical ideal $J \subseteq R$ defining the singular locus (for example, if R is excellent there will exist such a J) then J is generated by nonzerodivisors.

Proof: (a). Let $I \subseteq J$ be the ideal generated by the nonzerodivisors of J. Then $J \subseteq I \cup \bigcup_{P \in Ass(R)} P$. By prime avoidance J is contained in one of them, and since the depth of J is at least one, $J \subseteq I$.

(b). By part (a), it is enough to show that if R is reduced and contains a radical ideal $J \subseteq R$ defining the singular locus then the depth of J is at least one. If not then J is contained in an associated prime, say P, of R. Since R is reduced, the associated primes of R are all minimal primes, so that $J \subseteq P$ a minimal prime. But then R_P is zero-dimensional and singular which means that R_P is not reduced (the only zero-dimensional regular rings are fields). This contradicts the fact that R is reduced. $\square$

Before giving the proof of Theorem III.10 we need to generalize several results of section 8 of [LS01]. This first lemma generalizes [LS01], Lemma 8.3. The proof we give is identical to the original.

Lemma III.12. *Let M be an Artinian R-module, let $I \subseteq R$ be a proper ideal, and write M_{-v} for $M_{(-v,I)}$. Assume for every v there exists v' such that for all $q = p^e$, the natural map $\operatorname{Tor}_1^R(R^{1/q}, (M/M_{-v})_{-(v'-v)}) \to \operatorname{Tor}_1^R(R^{1/q}, M/M_{-v})$ is surjective. Then*

$$0^*_M = \bigcup_{v' \in \mathbb{N}} 0^*_{M_{-v'}}$$

Proof: The short exact sequences $0 \to M_{-v} \to M \to M/M_{-v} \to 0$ and $0 \to M_{-v} \to M_{-v'} \to (M/M_{-v})_{-(v'-v)} \to 0$ induce the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
\operatorname{Tor}_1^R(R^{1/q}, (M/M_{-v})_{-(v'-v)}) & \longrightarrow & R^{1/q} \otimes_R M_{-v} & \xrightarrow{a} & R^{1/q} \otimes_R M_{-v'} \\
f \downarrow & & \mathbb{1} \downarrow & & g \downarrow \\
\operatorname{Tor}_1^R(R^{1/q}, (M/M_{-v})) & \longrightarrow & R^{1/q} \otimes_R M_{-v} & \xrightarrow{b} & R^{1/q} \otimes_R M
\end{array}$$

The surjectivity of f guarantees that g restricts to an isomorphism between $\operatorname{Im}(a)$ and $\operatorname{Im}(b)$. Therefore, if $x \in 0^*_M$ and $x \in M_{-v}$ then $x \in 0^*_{M_{-v'}}$. $\square$

We recall the following discussion from [LS01].

Discussion. Let $T \subseteq R$ be a free $R^{1/p}$-module. We use T to construct, for all $q = p^e$, a free $R^{1/q}$-module, $T_e \subseteq R$ recursively as follows. First, set $T_1 := T$. Now suppose

$T_1, \ldots, T_{e-1}$ have been constructed and we want to construct T_e. Let $\{t_{e,i}\}$ be a free $R^{1/p^{e-1}}$-basis of T_{e-1}. By identifying R with $R^{1/p^{e-1}}$, $R^{1/p}$ with $R^{1/q}$, we can view T_1 as a free $R^{1/q}$-submodule of $R^{1/p^{e-1}}$. That is, $T_1 = \oplus_i R^{1/q} t_{1,i} \subseteq R^{1/p^{e-1}}$. Now, identify $R^{1/p^{e-1}}$ with $R^{1/p^{e-1}} t_{e,j} \subseteq R$ and let $T_{1,e,i,j}$ be the submodule of $R^{1/p^{e-1}} t_{e,j}$ corresponding to $R^{1/q} t_{1,i}$. Let T_e be the sum of the $T_{1,e,i,j}$ over all i and j. It follows that T_e is a free $R^{1/q}$-submodule of R.

Hence, if $T \subseteq R^{1/p}$ is a free R-submodule, then the construction above produces, for each $q = p^e$, a free R-submodule $T_e \subseteq R^{1/q}$.

Lemma III.13. *If $r \in R$ annihilates $R^{1/p}/T_1$ then r^2 annihilates $R^{1/q}/T_e$ for all e.*

Proof: This is [LS01], Lemma 8.9. □

Proposition III.14. *Let $J \subseteq R$ be the defining ideal of the singular locus. Then there exists $t \in \mathbb{N}$ such that for all $i, e > 0$ and all R-modules M, J^t annihilates the R-modules $\operatorname{Tor}^i_R(R^{1/q}, M) = 0$.*

Proof: Using Lemma III.11 we may assume J is generated by nonzerodivisors. The rest of the proof is the same as in [LS01], Proposition 8.10. □

We now come to the proof of the main theorem:

Proof of Theorem III.10: By Lemma III.11 we may assume $J = (x_1, \ldots, x_h)$ is generated by nonzerodivisors. Since the potent elements form an ideal, it suffices to show that

$$0^*_M = \bigcup_{v \in \mathbb{N}} 0^*_{M_{(-v, x_i)}}$$

for $1 \leq i \leq h$. To simplify notation, let $x = x_i$. Let $v \in \mathbb{N}$ be given, and set

$M' := M/M_{(-v,x)}$. By Lemma III.12 it is enough to show that there exists $v' \in \mathbb{N}$ such that for all q, the natural map $\mathrm{Tor}_1^R(R^{1/q}, M'_{-v'}) \to \mathrm{Tor}_1^R(R^{1/q}, M')$ is surjective. For this, we first prove the following claim:

Claim: There exists an integer $T \in \mathbb{N}$ such that for all $q = p^e$, for all R-modules N', and for all $i \geq 1$, x^T annihilates the R-modules $\mathrm{Tor}_i^R(R^{1/q}, N')$ and $\mathrm{Tor}_i^R(R^{1/q}/x^T R^{1/q}, N')$.

Proof of Claim: Since $x \in J$, by Proposition III.14 there exists a $T \in \mathbb{N}$ such that x^T annihilates $\mathrm{Tor}_i^R(R^{1/q}, N')$ for all $i \geq 1$ and all q. Obviously x^T annihilates all of the modules $\mathrm{Tor}_i^R(R^{1/q}/x^T R^{1/q}, N')$ since it annihilates $R^{1/q}/x^T R^{1/q}$. $\square$

The second part of the following statement will complete the proof:

Claim: There exists $v' \in \mathbb{N}$ such that:

(1) For all $i \geq 1$ and for all q, the map $\mathrm{Tor}_i^R(R^{1/q}/x^T R^{1/q}, M'_{-v'}) \to \mathrm{Tor}_i^R(R^{1/q}/x^T R^{1/q}, M')$ is surjective.

(2) For all $i \geq 1$ and for all q, the map $\mathrm{Tor}_i^R(R^{1/q}, M'_{-v'}) \to \mathrm{Tor}_i^R(R^{1/q}, M')$ is surjective.

Proof: For (1), $R^{1/q}/x^T R^{1/q}$ is killed by x^T so there exists such a v' by Corollary 3.9, part (a). We claim this v' works for (2) as well. Since x is a nonzerodivisor we have the following exact sequence

$$(*) \quad 0 \to R^{1/q} \xrightarrow{\cdot x^T} R^{1/q} \to R^{1/q}/x^T R^{1/q} \to 0$$

Since x^T annihilates $\mathrm{Tor}_i^R(R^{1/q}, -)$, by exact sequence $(*)$ there is a commutative diagram with exact rows:

$$\begin{array}{ccccc}
\operatorname{Tor}_{i+1}^R(R^{1/q}/x^T R^{1/q}, M'_{-v'}) & \longrightarrow & \operatorname{Tor}_i^R(R^{1/q}, M'_{-v'}) & \xrightarrow{\cdot 0} & \operatorname{Tor}_i^R(R^{1/q}, M'_{-v'}) \\
f \downarrow & & g \downarrow & & \\
\operatorname{Tor}_{i+1}^R(R^{1/q}/x^T R^{1/q}, M') & \longrightarrow & \operatorname{Tor}_i^R(R^{1/q}, M') & \xrightarrow{\cdot 0} & \operatorname{Tor}_i^R(R^{1/q}, M')
\end{array}$$

Therefore, we actually have the following commutative diagram with exact rows

$$\begin{array}{ccccc}
\operatorname{Tor}_{i+1}^R(R^{1/q}/x^T R^{1/q}, M'_{-v'}) & \longrightarrow & \operatorname{Tor}_i^R(R^{1/q}, M'_{-v'}) & \longrightarrow & 0 \\
f \downarrow & & g \downarrow & & \\
\operatorname{Tor}_{i+1}^R(R^{1/q}/x^T R^{1/q}, M') & \longrightarrow & \operatorname{Tor}_i^R(R^{1/q}, M') & \longrightarrow & 0
\end{array}$$

We have already proven that the map f is surjective. It follows that the map g is surjective as well. This completes the proof. $\square$

CHAPTER IV

The Defining Ideal for the Non-finite Injective Dimension Locus is Potent for the Matlis Dual

In this chapter we obtain a result on potent elements that substantially generalizes the result of the previous chapter. The main result may be stated as follows:

Theorem IV.1. *Let (R, m, K) be a reduced, excellent local ring. Let W be a finitely generated R-module, let $E_R(K)$ be the injective hull of the residue class field, and put $M := \operatorname{Hom}_R(W, E_R(K))$ so that M is an Artinian R-module. Let $I \subseteq R$ be the defining ideal of the non-finite injective dimension locus of W (that is, $c \in I$ if and only if the injective dimension of W_c over R_c is finite). If $z \in 0^*_M$, then z is in the tight closure of 0 in $\operatorname{Ann}_M(I^t)$ for some $t \in \mathbb{N}$.*

This theorem is given in Corollary IV.58 in Section 3. As first corollaries we get:

Corollary IV.2. *Let (R, m, K) be a reduced excellent local ring, let W be a finitely generated R-module, and let $E_R(K)$ be the injective hull of the residue class field. Set $M := \operatorname{Hom}_R(W, E_R(K))$, the Matlis dual of W.*

(a) Assume that W has finite injective dimension on the punctured spectrum of R. Then $0^*_M = 0^{*fg}_M$.

(b) Let $J \subseteq R$ be the defining ideal of the singular locus of R. If $z \in 0^*_M$ then there exists $t \in \mathbb{N}$ such that z is in the tight closure of 0 in $\operatorname{Ann}_M(J^t)$.

Part (b) of the above is the main result of the previous chapter. As additional consequences we immediately recover three important results already in the literature:

Corollary IV.3. *Let (R, m, K) be a reduced, excellent local ring.*

(a) If R is an isolated singularity, then tight closure equals finitistic tight closure in every Artinian R-module.

(b) If R is has an isolated non-Gorenstein point then tight closure equals finitstic tight closure in the injective hull of the residue class field.

(c) If (R, m, K) is local ring of dimension d which is Cohen-Macaulay on the punctured spectrum, then tight closure equals finitisitic tight closure in $H^d_m(R)$, the top local cohomology module with supports in the maximal ideal.

Parts (a) and (b) are due to Lyubeznik and Smith and are proved in [LS01]. It should be noted that their techniques require the ring to be equidimensional so that parts (a) and (b) are already an improvement. Part (c) is due to Smith, originally proved in [Smi94], where she obtains the result for all equidimensional local rings.

The theory of potent elements also allows one to combine these results with the main result of [LS99] to get results for graded Artinian modules over polynomial rings (and slightly more generally). We shall deduce these consequences in Chapter 6 after developing the theory of potent elements for graded rings.

We prove the main theorem of this chapter by first proving a uniform annihilator result for certain Ext modules which is interesting in its own right. The idea of the proof is to make a transition from Ext modules to local cohomology modules where we know the uniform annihilator result. This transition requires a substantial effort and is most transparent in the case where the ring is equidimensional. The remainder of the chapter is organized as follows: in Section 1 we prove our uniform annihilator result when the ring is equidimensional (see Theorem IV.5 for the precise statement) and in Section 2 we obtain our uniform annihilator result without the equidimensional hypothesis (see Theorem IV.41). In Section 3 we develop the notions of potent elements and potent ideals. Using Theorem IV.41 we then prove Theorem IV.1.

4.1 Uniform Annihilators When R is Equidimensional

In this section, we prove an existence theorem on uniform annihilators of certain Ext modules under the assumption that our ring is equidimensional. To state our main result we first make a definition.

Definition IV.4. Suppose W is any R-module and $c \in R$. We say that W is *(c, F)-injective (as a module)* if there exists $k = k(c, W) \in \mathbb{N}$ such that

$$c^k \cdot \operatorname{Ext}^j_R(R^{1/q}, W) = 0$$

for all $q > 0$ and all $j > 0$.

Our main theorem on uniform annihilators of Ext modules may be stated as follows.

Theorem IV.5. *Let (R, m, K) be a reduced, F-finite, equidimensional, excellent local ring of characteristic $p > 0$. Let $c \in R$ and let W be a finitely generated R-module such that W_c has finite injective dimension. Then W is (c, F)-injective.*

In Section 2 we obtain this result without the assumption that R is equidimensional (see Theorem IV.41), but the proof in the equidimensional case is substantially more transparent. As a consequence we obtain results on our theory of potent elements in Section 3. The reader may wish to consult Theorems IV.47, IV.53, and Corollary IV.58 at this time.

We would like to give an overview of the proof of Theorem IV.5. We begin with some basic statements about (c, F)-injective modules. Next, the theorem reduces to the complete case where we may assume that R is a homomorphic image of a Gorenstein ring. Then R has a canonical module which we denote by ω. Note that

R_c is forced to be Cohen-Macaulay since it has a finitely generated module of finite injective dimension, and the first major step is to reduce to proving the theorem in the case $W = \omega$. This reduction requires the use of (c, ω)-resolutions, defined below.

Once we are in the case $W = \omega$, we make use of the dualizing complex as well as results on colon-killers from [HH92] to prove the theorem. We point out that if R itself is assumed to be Cohen-Macaulay the proof may be simplified considerably (for example, the theorem is quite easy in the case $W = \omega$, and so the use of dualizing complexes and results on colon-killers can be avoided), and we can obtain a uniform annihilator result for Ext's against all R-modules, not just the modules $R^{1/q}$.

To begin, we collect some basic results about (c, F)-injective modules.

Proposition IV.6. *Let (R, m, K) be a local ring of dimension d, and let*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence.

(a) If M is any R-module such that $c \in \operatorname{Rad}(\operatorname{Ann}_R(M))$ then M is (c, F)-injective.

(b) If A and C are (c, F)-injective then so is B.

(c) If A and B are (c, F)-injective then so is C.

(d) If the exact sequence above splits, then B is (c, F)-injective $\iff$ A and C are both (c, F)-injective.

Proof: Part (a) follows from the fact that $c^t \in \operatorname{Ann}_R(M) \Longrightarrow c^t \in \operatorname{Ann}_R(\operatorname{Ext}^i_R(-, M))$ for all i.

From the long exact sequence for Ext we have an exact sequence

$$\operatorname{Ext}^j_R(N, A) \to \operatorname{Ext}^j_R(N, B) \to \operatorname{Ext}^j_R(N, C).$$

If $j \geq 1$ then c^{t_1} kills the first term and c^{t_2} kills the third term for some $t_1, t_2 \in \mathbb{N}$. It follows that $c^{t_1+t_2}$ kills the middles term. This proves part (b).

For part (c) one uses the exact sequence

$$\operatorname{Ext}^j_R(N, B) \to \operatorname{Ext}^j_R(N, C) \to Ext^{j+1}_R(N, A)$$

and proceeds as in (b).

To prove (d), if $B \cong A \oplus C$, then

$$\operatorname{Ext}^j_R(N, B) \cong \operatorname{Ext}^j_R(N, A) \oplus \operatorname{Ext}^j_R(N, C).$$

Therefore, c^t annihilates $\operatorname{Ext}^j_R(N, B) \iff c^t$ annihilates $\operatorname{Ext}^j_R(N, A)$ and $\operatorname{Ext}^j_R(N, C)$.

□

Before proceeding further we introduce the following:

Definition IV.7. Assume R has a canonical module ω and let $c \in R$. We say that a module W has a *finite* (c, ω)*-resolution (of length s)* if there is a complex

$$0 \to \omega^{\oplus b_s} \xrightarrow{\phi_p} \cdots \to \omega^{\oplus b_0} \xrightarrow{\phi_0} W \xrightarrow{\phi_{-1}} 0$$

and an integer $t \in \mathbb{N}$ such that

(a) For all $-1 \leq j \leq s-1$, c^t kills $\operatorname{Ker}(\phi_j)/\operatorname{Im}(\phi_{j+1})$.

(b) If $Z := \operatorname{Ker}(\phi_p)$, then Z_c is a direct summand of $\omega_c^{\oplus n}$ for some $n \in \mathbb{N}$.

We shall eventually see that the finitely generated modules W such that W_c has finite injective dimension over R_c are exactly the finitely generated modules possessing a (c, ω)-resolution (see Proposition IV.11 for the precise statement; when R is Cohen-Macaulay the case $c = 1$ is a classical result recorded in Lemma IV.9). But first we want to point out that the module Z in the above definition will be forced to be (c, F)-injective once we know ω is. We prove this in the next proposition by making use of ideas from Chapter 3.

Proposition IV.8. *Suppose A, B are finitely generated R-modules, assume B is (c, F)-injective, and assume that A_c is a direct summand of B_c. Then A is (c, F)-injective as well.*

Proof: There is a complex $A \xrightarrow{\alpha} B \xrightarrow{\gamma} C$ and a split exact sequence $0 \to A_c \to B_c \to C_c \to 0$. Since the modules are finitely generated, it follows that the original complex is c^t-split for some $t \in \mathbb{N}$; i.e., there exist R-linear maps $\beta : B \to A$ and $\psi : C \to B$ such that for some $t \in \mathbb{N}$,

$$c^t \cdot \mathbb{1}_B = \alpha\beta + \psi\gamma.$$

Note that since $\gamma \circ \alpha = 0$,

$$(*) \qquad \alpha \circ \beta = c^t \cdot \mathbb{1}_{\mathrm{Im}(\alpha)}.$$

By enlarging t if necessary we may assume

$$(**) \qquad c^t \cdot \mathrm{Ker}(\alpha) = 0.$$

Since B is (c, F)-injective, there exists $h \in \mathbb{N}$ such that, for all q, for all $j \geq 1$

$$c^h \cdot \mathrm{Ext}^j_R(R^{1/q}, B) = 0.$$

Let $\mathcal{F}_\bullet$ be a free resolution of $R^{1/q}$. We have a diagram

$$\begin{array}{ccccc} A_{j+1} & \overset{\delta_j^A}{\longleftarrow} & A_j & \overset{\delta_{j-1}^A}{\longleftarrow} & A_{j-1} \\ \beta \uparrow\downarrow \alpha & & \beta \uparrow\downarrow \alpha & & \beta \uparrow\downarrow \alpha \\ B_{j+1} & \overset{\delta_j^B}{\longleftarrow} & B_j & \overset{\delta_{j-1}^B}{\longleftarrow} & B_{j-1} \end{array}$$

with the following commutativity properties:

(1) $\delta_i^A \circ \beta = \beta \circ \delta_i^B$, for $i = j, j-1$ and

(2) $\delta_i^B \circ \alpha = \alpha \circ \delta_i^A$, for $i = j, j-1$.

Here, $A_i = \mathrm{Hom}_R(F_i, A) \cong A^{m_i}$, $B_i = \mathrm{Hom}_R(F_i, B) \cong B^{m_i}$, and we have written α and β for the maps they induce on the Hom modules. It is easy to verify that equations $(*)$ and $(**)$ hold for the induced maps. Note that the homology of the top row (respectively, the bottom row) is $\mathrm{Ext}^j_R(R^{1/q}, A)$ (respectively $\mathrm{Ext}^j_R(R^{1/q}, B)$). We have

$$(***) \qquad c^h \operatorname{Ker}(\delta_j^B) \subseteq \operatorname{Im}(\delta_{j-1}^B)$$

and we claim $c^{h+2t} \operatorname{Ker}(\delta_j^A) \subseteq \operatorname{Im}(\delta_{j-1}^A)$. This will complete the proof, since h, t did not depend on q or j.

To see this, suppose $z \in \operatorname{Ker}(\delta_j^A)$. By commutativity property (2), $\alpha(z) \in \operatorname{Ker}(\delta_j^B)$ so by $(***)$ there exists y such that $\delta_{j-1}^B(y) = c^h\alpha(z) = \alpha(c^h z)$. It follows that $\delta_{j-1}^B(y) \in \operatorname{Im}(\alpha)$, so by equation $(*)$, we have

$$\alpha\beta\delta_{j-1}^B(y) = c^t\delta_{j-1}^B(y) = \alpha(c^{t+h}z).$$

Now, let $z' := \delta_{j-1}^A(\beta(y))$ which is $\beta(\delta_{j-1}^B(y))$ by commutativity property (1). By what we have just shown, $\alpha(z') = \alpha(c^{t+h}z)$, and it follows that $\alpha(c^{t+h}z - z') = 0$. Therefore, $c^{t+h}z - z' \in \operatorname{Ker}(\alpha)$, so by $(**)$,

$$c^{h+2t}z - c^t z' = 0 \Longrightarrow c^{h+2t}z = c^t z'.$$

Since $z' \in \mathrm{Im}(\delta^A_{j-1})$ it follows that $c^t z' \in \mathrm{Im}(\delta^A_{j-1})$ and hence $c^{h+2t} z \in \mathrm{Im}(\delta^A_{j-1})$. This proves the claim and completes the proof of the proposition. □

We prepare two lemmas before clarifying which finitely generated modules posses finite (c, ω)- resolutions.

Lemma IV.9. *Let S be a (not necessarily local) Cohen-Macaulay ring of finite Krull dimension with canonical module ω, and let W be a finitely generated S-module.*

(a) The natural map

$$S \to \mathrm{Hom}_S(\omega, \omega)$$

is an isomorphism.

(b) If W has finite injective dimension, then $\mathrm{Hom}_S(\omega, W)$ has finite projective dimension.

(c) If $0 \to M_n \to M_{n-1} \to \cdots \to M_0 \to 0$ is an exact sequence of S-modules such that each M_i is finitely generated and has finite injective dimension, then $0 \to \mathrm{Hom}_S(\omega, M_n) \to \cdots \to \mathrm{Hom}_S(\omega, M_0) \to 0$ is exact also.

(d) If S is a local ring, then W has finite injective dimension if and only if there is an exact sequence

$$(*) \qquad 0 \to \omega^{\oplus b_d} \to \cdots \to \omega^{\oplus b_0} \to W \to 0,$$

where $d = \dim S - \mathrm{depth}_m(W)$ and where $b_j := \dim_L(\mathrm{Ext}_S^{\dim S - j}(L, W))$. In particular, if S is a local ring and W is a finitely generated, maximal Cohen-Macaulay S-module of finite injective dimension, then $W \cong \omega^{\oplus h}$ for some h.

Proof: Part (a) reduces to the local case where the result is given in 3.3.4(d) of

[BH93] while part (b) follows from 3.11(i) of [Sha72].

We prove (c). Since Hom commutes with localization when the first variable is finitely generated, the statement reduces at once to the local case so we may assume that (S, m) is Cohen-Macaulay, local. We use induction on n. We first consider the case of a short exact sequence $0 \to M_2 \to M_1 \to M_0 \to 0$. The key point is that when a finitely generated module M has finite injective dimension, for any finitely generated module N, $\mathrm{Ext}^j_S(N, M) = 0$ for all $j > \dim(S) - \mathrm{depth}(N)$: one proves this by induction on $\mathrm{depth}(N)$. When $\mathrm{depth}(N) = 0$ this follows at once from the fact that $\mathrm{id}(M) = \dim(S)$. If $\mathrm{depth}(N) > 0$ then for some $x \in R$ we have a short exact sequence

$$0 \to N \xrightarrow{\cdot x} N \to N/xN$$

which produces

$$\mathrm{Ext}^i(N, M) \xrightarrow{\cdot x} \mathrm{Ext}^i_R(N, M) \to 0$$

since, by the induction hypothesis, $\mathrm{Ext}^{i+1}_R(N/xN, M) = 0$. The result then follows from Nakayama's Lemma. Now, returning to the short exact sequence

$$0 \to M_2 \to M_1 \to M_0 \to 0,$$

since ω is a maximal Cohen-Macaulay module, by the result just proved, $\mathrm{Ext}^1_R(\omega, M_2) = 0$, so it follows that

$$0 \to \mathrm{Hom}_R(\omega, M_2) \to \mathrm{Hom}_R(\omega, M_1) \to \mathrm{Hom}_R(\omega, M_0) \to 0$$

is exact.

Finally, given $0 \to M_n \to \cdots \to M_0 \to 0$, we may from two shorter exact sequences:

$$(1)\;\; 0 \to M_n \to M_{n-1} \to \mathrm{Im}\, M_{n-1} \to 0$$

and

$$(2)\ \ 0 \to \operatorname{Im} M_{n-1} \to M_{n-2} \to \cdots \to M_0 \to 0.$$

Note that by (1), it follows that $\operatorname{id}(\operatorname{Im} M_{n-1}) < \infty$. Applying the induction hypothesis we find that

$$0 \to \operatorname{Hom}_R(\omega, M_n) \to \operatorname{Hom}_R(\omega, M_{n-1}) \to \operatorname{Hom}_R(\omega, \operatorname{Im} M_{n-1}) \to 0$$

and

$$0 \to \operatorname{Hom}_R(\omega, \operatorname{Im} M_{n-1}) \to \operatorname{Hom}_R(\omega, M_{n-2}) \to \cdots \to \operatorname{Hom}_R(\omega, M_0) \to 0$$

are both exact. The result then follows easily.

Part (d) is contained in 3.3.28 of [BH93]. □

Lemma IV.10. *Let (R, m, K) be an equidimensional local ring with a canonical module, ω and let $c \in R$. Suppose W is a finitely generated R-module and that W_c has finite injective dimension over R_c. Then, for some $b \in \mathbb{N}$, there exists an exact sequence*

$$\omega^{\oplus b} \to W \to C \to 0$$

such that $c^t \cdot C = 0$ for some $t \in \mathbb{N}$.

Remark: By Theorem II.18, since W is finitely generated, if W_c has finite injective dimension then R_c must be a Cohen-Macaulay ring.

Proof: Consider the family $\mathcal{F}$ of submodules of W defined as follows: a submodule $U \subseteq W$ is in $\mathcal{F}$ if there exists an R-linear map $f : \omega^{\oplus b} \to W$ with $\operatorname{Im}(f) = U$. If $U, V \subseteq W$ are in $\mathcal{F}$ then we claim $U + V$ is in $\mathcal{F}$ as well. To see this, suppose there are R-linear maps $f : \omega^{\oplus b_1} \to W$ and $g : \omega^{\oplus b_2} \to W$ with $\operatorname{Im}(f) = U$ and $\operatorname{Im}(g) = V$. Then the direct sum map $f \oplus g : \omega^{\oplus b_1 + b_2} \to W$ has image $\operatorname{Im}(f \oplus g) = U + V$ and the claim is proved. Since W is Noetherian, it follows that there is a submodule $B \in \mathcal{F}$ such that if $U \in \mathcal{F}$ then $U \subseteq B$.

We first prove that $(W/B)_c = 0$. If not, then since W/B is finitely generated there is a prime $P \in \mathrm{Supp}(W/B)$, with $c \notin P$. Since W_c has finite injective dimension over R_c, W_P has finite injective dimension over R_P. Therefore, by the proceeding lemma (since R_c and hence R_P are Cohen-Macaulay), there is a surjective map $\omega_{R_P}^{\oplus n} \twoheadrightarrow W_P$. Now, $\omega_{R_P} \cong \omega_P$, and since Hom commutes with localization, we find an R-linear map $\theta : \omega^{\oplus n} \to W$ such that $\mathrm{Coker}(\theta)_P = 0$. But $\mathrm{Im}(\theta) \subseteq B$ and so $\mathrm{Coker}(\theta) \twoheadrightarrow W/B$. Since $\mathrm{Coker}(\theta)_P = 0$ it follows that $(W/B)_P = 0$, a contradiction.

We have shown that $(W/B)_c = 0$ and since W/B is finitely generated, there exists $t \in \mathbb{N}$ such that $c^t \cdot (W/B) = 0$. With $C := W/B$, it follows that there is an exact sequence

$$\omega^{\oplus b} \to W \to C \to 0$$

such that $c^t \cdot C = 0$. $\square$

Proposition IV.11. *Let (R, m, K) be an equidimensional local ring with canonical module ω and let $c \in R$. Let W be a finitely generated R-module. Then W_c has finite injective dimension over R_c if and only if W has a finite (c, ω)-resolution.*

Proof: Let W be a finitely generated R-module. Since R is local it has finite Krull dimension. We first prove the implication ($\Longrightarrow$). Assume W_c has finite injective dimension. Then R_c must be Cohen-Macaulay and also R_c has finite Krull dimension since R does. By the proceeding lemma, there is an exact sequence $0 \to Z_0 \to \omega^{\oplus b_0} \to W \to C_{-1} \to 0$ and an integer t_{-1} such that $c^{t_{-1}} C_{-1} = 0$. Since localization is flat, there is a short exact sequence $0 \to (Z_0)_c \to \omega_c^{\oplus b_0} \to W_c \to 0$.

It follows that $(Z_0)_c$ has finite injective dimension. Moreover, since ω_c is a maximal Cohen-Macaulay module, it follows that either $(Z_0)_c$ is a maximal Cohen-Macaulay module or $\operatorname{depth}_n(Z_0)_c \geq \operatorname{depth}_n W_c + 1$ for all maximal ideals $n \subseteq R_c$.

Applying the proceeding lemma to Z_0, we obtain an exact sequence

$$0 \to Z_1 \to \omega^{\oplus b_1} \to Z_0 \to C_0 \to 0$$

and an integer t_0 such that $c^{t_0} C_0 = 0$. Again, we have an exact sequence

$$0 \to (Z_1)_c \to \omega_c^{\oplus b_1} \to (Z_0)_c \to 0$$

and it follows that the injective dimension of $(Z_1)_c$ is finite and either $(Z_1)_c$ is a maximal Cohen-Macaulay module, or $\operatorname{depth}_n(Z_1)_c \geq \operatorname{depth}_n(Z_0)_c + 1 \geq \operatorname{depth}_n(W_c) + 2$ for all maximal ideals $n \subseteq R_c$.

Proceeding in this way, for each $0 \leq j \leq d = \dim(R)$, we obtain modules Z_j, C_j, integers b_j, t_j and exact sequences $0 \to Z_{j+1} \to w^{\oplus b_j} \to Z_j \to C_j \to 0$ such that $c^{t_j} \cdot C_j = 0$. Putting all of this together, we obtain a complex:

$$(*) \qquad 0 \to \omega^{\oplus b_d} \to \omega^{\oplus b_{d-1}} \to \cdots \to \omega^{\oplus b_0} \to W \to 0$$

with the following properties:

(a) the homology at $\omega^{\oplus b_d}$ is Z_{d+1} (i.e., $\operatorname{Ker}(\omega^{\oplus b_d} \to \omega^{\oplus b_{d-1}}) = Z_{d+1}$), and $(Z_{d+1})_c$ is a maximal Cohen-Macaulay module,

(b) for $0 \leq j \leq d-1$, the homology at $\omega^{\oplus b_j}$ is C_j and is therefore annihilated by c^{t_j}, and

(c) the homology at W is C_{-1} and is therefore annihilated by $c^{t_{-1}}$.

Letting $t := \max\{t_{-1}, \ldots, t_d\}$ we see that this complex satisfies property (a) of Definition IV.7. Therefore, the proof will be complete as soon as we know that $(Z_{d+1})_c$ is a direct summand of $\omega_c^{\oplus n}$ for some n.

To simplify notation, let $Z := Z_{d+1}$. Notice that since Z_c is a maximal Cohen-Macaulay module, we have that for all maximal ideals $n \in \operatorname{Spec}(R_c)$, Z_n is a maximal Cohen-Macaulay module over R_n. Therefore, $Z_n \cong \omega_n^{\oplus h}$ for some h where the h may depend on n. This is just the second statement of (IV.9d). Since the canonical module localizes, it follows that for all $P \in \operatorname{Spec}(R_c)$ there exists h such that $Z_P \cong \omega_P^{\oplus h}$.

Now, localizing the complex $(*)$ at c produces the following exact sequence

$$0 \to Z_c \to \omega_c^{\oplus b_d} \to \cdots \to \omega_c^{\oplus b_0} \to W_c \to 0.$$

Set $N := \operatorname{Hom}_{R_c}(\omega_c, W_c)$. Since W_c has finite injective dimension, N has finite projective dimension by (IV.9b). Further, by (IV.9a) and (IV.9c), applying $\operatorname{Hom}_{R_c}(\omega_c, -)$ to the above exact sequence produces the exact sequence

$$0 \to \operatorname{Hom}_{R_c}(\omega_c, Z_c) \to R_c^{\oplus b_d} \to \cdots \to R_c^{\oplus b_0} \to N \to 0.$$

Since $\dim(R) = d$, it follows at once that $\operatorname{Hom}_{R_c}(\omega_c, Z_c)$ is projective as it is a $(d+1)^{st}$ module of syzygies of N which has finite projective dimension. Therefore, there is an R_c-module Q such that $\operatorname{Hom}_{R_c}(\omega_c, Z_c) \oplus Q = R_c^{\oplus n}$ for some n. Applying $- \otimes_{R_c} \omega_c$, we find that

$$(\operatorname{Hom}_{R_c}(\omega_c, Z_c) \otimes_{R_c} \omega_c) \oplus (Q \otimes_{R_c} \omega_c) \cong R_c^{\oplus n} \otimes_{R_c} \omega_c \cong \omega_c^{\oplus n}.$$

Hence, $\operatorname{Hom}_{R_c}(\omega_c, Z_c) \otimes_{R_c} \omega_c$ is a direct summand of $\omega_c^{\oplus n}$. We claim we have an isomorphism

$$\operatorname{Hom}_{R_c}(\omega_c, Z_c) \otimes_{R_c} \omega_c \cong Z_c$$

and it is enough to check this locally. Indeed, if $P \in \operatorname{Spec}(R_c)$, then since Hom commutes with localization, the left hand side is $\operatorname{Hom}_{R_P}(\omega_P, Z_P) \otimes_{R_P} \omega_P$. Since $Z_P \cong \omega_P^{\oplus h}$, this is

$$\operatorname{Hom}_{R_P}(\omega_P, \omega_P^{\oplus h}) \otimes_{R_P} \omega_P \cong (R_P)^{\oplus h} \otimes_{R_P} \omega_P \cong \omega_P^{\oplus h} \cong Z_P$$

yielding the isomorphism. Therefore Z_c is a direct summand of $\omega_c^{\oplus n}$, as required.

It remains to prove the ($\Longleftarrow$) implication. Assume that W has a finite (c, ω)-resolution

$$0 \to \omega^{\oplus b_d} \to \cdots \to \omega^{\oplus b_0} \to W \to 0.$$

We must show that W_c has finite injective dimension. Localizing at c produces the exact sequence

$$0 \to Z_c \to \omega_c^{\oplus b_d} \to \cdots \to \omega_c^{\oplus b_0} \to W_c \to 0.$$

If $d = 0$ then we have an exact sequence $0 \to Z_c \to \omega_c^{\oplus b_0} \to W_c \to 0$. Since ω_c has finite injective dimension so does Z_c being a direct summand. But then Z_c and ω_c have finite injective dimension, so W_c does as well. This handles the case $d = 0$. The result now follows from a straightforward induction on d. □

Proposition IV.12. ***(Reduction to the case*** $W = \omega$***.)*** *Suppose that (R, m, K) is an equidimensional, reduced, excellent local ring with canonical module ω. Suppose that W is a finitely generated R-module such that the injective dimension of W_c is finite. If ω is (c, F)-injective then W is (c, F)-injective.*

Proof: By Proposition IV.11, there exists a complex

$$0 \to \omega^{\oplus b_d} \to \cdots \to \omega^{\oplus b_0} \to W \to 0$$

where $Z = \mathrm{Ker}(\phi_d)$ is such that Z_c is a direct summand of $\omega_c^{\oplus n}$ for some n. By Proposition IV.8, Z is (c, F)-injective since ω is (c, F)-injective. Furthermore, there exists $t \in \mathbb{N}$ such that the other homology modules are killed by c^t. We use induction on d.

If $d = 0$, then for some modules N, C we have two short exact sequences:

$$(1) \qquad 0 \to Z \to \omega^{\oplus b_0} \to N \to 0$$

and

$$(2) \qquad 0 \to N \to W \to C \to 0.$$

By hypothesis, ω is (c, F)-injective and so by Proposition IV.6(d), $\omega^{\oplus b_0}$ is (c, F)-injective. Therefore, by exact sequence (1) and Proposition IV.6(c), N is (c, F)-injective as well. Now, C is the homology of the complex at W and thus is killed by c^t. Therefore, C is (c, F)-injective by Proposition IV.6(a), so using exact sequence (2) and Proposition IV.6(b), we conclude W is (c, F)-injective.

For the inductive step, let $T := \mathrm{Ker}(\omega^{\oplus b_0} \to W)$. The complex

$$0 \to \omega^{\oplus b_d} \to \cdots \to \omega^{\oplus b_1} \to T \to 0$$

is a (c, ω)-resolution for T of length $d - 1 < d$, as is easily verified. Therefore by the induction hypothesis, T is (c, F)-injective. We have an exact sequence $0 \to T \to \omega^{\oplus b_0} \to W \to C \to 0$ which breaks up into two short exact sequences. As before, since T, ω, and C are (c, F)-injective so is W. $\square$

In handling the case $W = \omega$ we will require the technique of the dualizing complex. The next two results are well known but are included for the convenience of the reader and to allow us to set up some notation that will be used throughout the rest of this section. See [Har66].

Lemma IV.13. *Let (R, m, K) be an equidimensional local ring of dimension d and let (T, n, L) be a Gorenstein local ring of dimension $d + h$ such that $T \twoheadrightarrow R$. Write $R = T/I$. Then:*

(i) The injective dimension of T (as a T-module) is $d + h$, and T has an injective resolution:

$$0 \to \bigoplus_{ht(P)=0} E(T/P) \to \bigoplus_{ht(P)=1} E(T/P) \to \cdots \to E(L) \to 0$$

where $E(T/P)$ is the injective hull of T/P.

(ii) $\mathrm{Hom}_T(R, E(T/P)) = 0$ if $I \not\subseteq P$ while $\mathrm{Hom}_T(R, E(T/P)) \cong E(R/P)$ if $I \subseteq P$. In particular, $\mathrm{Hom}_T(R, E(T/P)) = 0$ if $\mathrm{height}(P) < h$.

(iii) The module $\mathrm{Ext}^h_T(R, T)$, thought of as an R-module, is a canonical module for R.

Proof: Part (i) is Theorem 3.3.10(b) of [BH93], since T is a canonical module for itself. The first statement of (ii) is obvious, and the second is proved by simply noting that $\mathrm{Hom}_T(R, E(T/P)) = \mathrm{Ann}_{E(T/P)}(I) = E(T/P) = E(R/P)$. Part (iii) follows from local duality. □

Proposition IV.14. ***(The Dualizing Complex)*** *Assume all of the notations and conventions of the previous lemma; let $c \in R$ and assume R_c is Cohen-Macaulay. Let ω be a canonical module for R. Then there is a complex:*

$$0 \to \omega \to D^0 \to \cdots \to D^d \to 0$$

and an integer $k_1 \in \mathbb{N}$ such that *(i) D^j is an injective R-module for all $0 \leq j \leq d$.*

(ii) The map $\omega \to D^0$ is injective (i.e., the homology at ω is 0).

(iii) c^{k_1} annihilates the homology at D^j for all $0 \leq j \leq d$.

Proof: Applying $\mathrm{Hom}_T(R, -)$ to the injective resolution of T described in the previous lemma produces

$$0 \to 0 \to \cdots \to 0 \to \bigoplus_{ht(P)=h, I \subseteq P} \mathrm{Hom}_T(R, E(T/P)) \to \cdots \to \mathrm{Hom}_T(R, E(T/n)) \to 0.$$

Notice that the homology of this complex is finitely generated since it is $\mathrm{Ext}_T^{\bullet}(R, T)$. By part (iii) of the lemma,

$$\mathrm{Ker}\Big(\bigoplus_{ht(P)=h, I \subseteq P} \mathrm{Hom}_T(R, E(T/P)) \to \bigoplus_{ht(P)=h+1, I \subseteq P} \mathrm{Hom}_T(R, E(T/P))\Big)$$
$$\cong \mathrm{Ext}_T^h(R, T) \cong \omega_R.$$

By part (ii) we may rewrite the complex as:

$$\mathcal{D}^{\bullet}: \quad 0 \to \bigoplus_{ht(P)=0, I \subseteq P} E(R/P) \to \cdots \to E(R/m) \to 0,$$

and so, with these notations, D^j is an injective R-module for $0 \leq j \leq d$ and, $H^0(\mathcal{D}^{\bullet}) = \omega_R$. We have already seen that $H^j(\mathcal{D}^{\bullet})$ is finitely generated. We claim that $H^j(\mathcal{D}^{\bullet})_c = 0$ for all $j \geq 1$. To see this, first note that ω_c is the canonical module for R_c since R is equidimensional. Now, since R_c is Cohen-Macaulay, ω_c has finite injective dimension and $(\mathcal{D}^{\bullet})_c$ is the injective resolution of ω_c. In particular, $H^j(\mathcal{D}^{\bullet})_c = 0$ for all $j \geq 1$. Since each $H^j(\mathcal{D}^{\bullet})$ is finitely generated, it follows that there exists $k_1 \in \mathbb{N}$ such that $c^{k_1} \cdot H^j(\mathcal{D}^{\bullet}) = 0$ for all $j \geq 1$. $\square$

Remark. It is often convenient in applications to replace T with T/J where $J = (x_1, \ldots, x_k)$ is generated by a maximal regular sequence in $I = \mathrm{Ker}(T \to R)$. Then

T/J is still Gorenstein, and one has $\dim(T) = \dim(R)$.

Using the dualizing complex, we are already able to produce uniform annihilators of the higher Ext modules.

Proposition IV.15. *Let (R, m, K) be a local ring of dimension d, let $c \in R$, and let W be any R-module. Suppose there is a complex*

$$(*) \qquad 0 \to W \to D^0 \to \cdots \to D^d \to 0$$

and an integer $k_1 \in \mathbb{N}$ such that:

(1) The complex is exact at W, and

(2) D^j is injective and the homology at D^j is killed by c^{k_1} for all $0 \leq j \leq d$.

Then, with $k := (d+1)k_1$,

$$c^k \cdot \operatorname{Ext}_R^{d+l}(N, W) = 0$$

for all R-modules N and all integers $l > 0$.

Proof: We use induction on d. If $d = 0$ then we have a short exact sequence

$$0 \to W \to D^0 \to C \to 0$$

and $c^{k_1} \cdot C = 0$ since C is the homology of the complex $(*)$ at D^0. The long exact sequence for Ext produces the exact sequences

$$\operatorname{Ext}_R^j(N, C) \to \operatorname{Ext}_R^{j+1}(N, W) \to 0$$

for all $j \geq 0$ since $\operatorname{Ext}_R^{j+1}(-, D^0) = 0$. Now, $c^{k_1} \cdot \operatorname{Ext}_R^j(N, C) = 0$, and it follows from the surjectivity of the above map that $c^{k_1} \cdot \operatorname{Ext}_R^{j+1}(N, W) = 0$. This proves the case $d = 0$.

For the induction step, assume $d \geq 1$, let $Z := \operatorname{Ker}(D^1 \to D^2)$, and let $C := \operatorname{Im}(D^0 \to D^1)$. Then we have a short exact sequences

$$(**) \qquad 0 \to W \to D^0 \to C \to 0$$

and

$$(***) \qquad 0 \to C \to Z \to M \to 0$$

and $c^{k_1} \cdot M = 0$ since M is the homology of the original complex $(*)$ at the D^1 spot. Notice also that

$$0 \to Z \to D^1 \to \cdots \to D^d \to 0$$

is a complex of shorter length satisfying conditions (1) and (2) of the proposition. Therefore, by the induction hypothesis, $c^{dk_1} \cdot \operatorname{Ext}_R^{d+l-1}(N, Z) = 0$ for all $l > 0$. The long exact sequence for Ext induced by $(***)$ produces the following exact sequence:

$$\operatorname{Ext}_R^{d+l-2}(N, M) \to \operatorname{Ext}_R^{d+l-1}(N, C) \to \operatorname{Ext}_R^{d+l-1}(N, Z).$$

Since c^{k_1} kills $\operatorname{Ext}_R^{d+l-2}(N, M)$ and c^{dk_1} kills $\operatorname{Ext}_R^{d+l-1}(N, Z)$, it follows from exactness that the product, $c^{(d+1)k_1}$ kills $\operatorname{Ext}_R^{d+l-1}(N, C)$. But by $(**)$ the long exact sequence for Ext produces an isomorphism $\operatorname{Ext}_R^{d+l-1}(N, C) \cong \operatorname{Ext}_R^{d+l}(N, W)$. This completes the proof. □

Corollary IV.16. *Let (R, m, K) be an equidimensional local ring of dimension d with canonical module ω, and let $c \in R$ such that R_c is Cohen-Macaulay. Let k_1 be as in IV.14. Then, with $k := (d+1)k_1$,*

$$c^k \cdot \operatorname{Ext}_R^{d+l}(N, \omega) = 0$$

for all $l > 0$ and all R-modules N.

Proof: This follows at once from (IV.14) and (IV.15). □

We focus on proving the existence of uniform annihilators for the modules $\mathrm{Ext}_R^j(N, \omega)$ for $1 \leq j \leq d$. We will use the dualizing complex to relate these modules to local cohomology modules. The first step is Proposition IV.18 below; however, it is convenient notationally to first prove the following lemma.

Lemma IV.17. *Let (R, m, K) be a local ring of dimension d with canonical module ω, and let $c \in R$. Let $\mathcal{D}^\bullet$ be the dualizing complex, and let*

$$\mathcal{I}^\bullet : \quad 0 \to I^0 \to \cdots \to I^d \to I^{d+1} \to \cdots$$

be an injective resolution of ω. Let k_1 be as in (IV.14). Then, with $h := k_1 \cdot d$, there exist maps $\psi_j : D^j \to I^j$ such that the following diagram commutes:

$$\begin{array}{ccccccccccccc} 0 & \to & \omega & \to & D^0 & \xrightarrow{\Delta_0} & D^1 & \xrightarrow{\Delta_1} & \cdots & \xrightarrow{\Delta_{d-1}} & D^d & & \\ & & \downarrow \cdot c^h & & \downarrow \psi_0 & & \downarrow \psi_1 & & & & \downarrow \psi_d & & \\ 0 & \to & \omega & \to & I^0 & \xrightarrow{\delta_0} & I^1 & \xrightarrow{\delta_1} & \cdots & \xrightarrow{\delta_{d-1}} & I^d & \to & I^{d+1} \end{array}$$

Proof: We will construct the ψ_j inductively. If $d = 0$, then $h = 0$ and we are considering the identity mapping $\mathbb{1} : \omega \to \omega$. So by part (ii) of (IV.14), we have an injective map $\omega \hookrightarrow D^0$, and a map $\omega \to I^0$. Since I^0 is injective, by the universal mapping property of injective modules, there exists $\psi_0 : D^0 \to I^0$ making the diagram commute.

If $d = 1$, then let $C := D^0/\omega$. The composite map $\delta^0 \circ \psi_0 : D^0 \to I^1$ kills ω and so induces a map $\gamma : C \to I^1$. Notice that $\mathrm{Im}(\Delta_0) \cong C/H$, where $H = \mathrm{Ker}(\Delta_0)/\omega$. By part (iii) of (IV.14), $c^h \cdot H = 0$, and it follows that $c^h \cdot \gamma : \mathrm{Im}(\Delta_0) \to I^1$. Since $\mathrm{Im}(\Delta_0) \hookrightarrow D^1$ and I^1 is injective, the universal mapping property of injectives gives a map $\psi_1 : D^1 \to I^1$. After replacing ψ_0 with $c^h \cdot \psi_0$, the diagram commutes.

For the induction step, we may assume that the maps $\psi_j : D^j \to I^j$ have been constructed for $0 \leq j \leq n-1$ making the following diagram commute:

$$\begin{array}{ccccccccccccc}
0 & \to & \omega & \to & D^0 & \overset{\Delta_0}{\to} & \cdots & \overset{\Delta_{n-3}}{\to} & D^{n-2} & \overset{\Delta_{n-2}}{\to} & D^{n-1} & \overset{\Delta_{n-1}}{\to} & D^n \\
 & & \downarrow \cdot c^{(n-1)k_1} & & \downarrow \psi_0 & & & & \downarrow \psi_{n-2} & & \downarrow \psi_{n-1} & & \\
0 & \to & \omega & \to & I^0 & \overset{\delta_0}{\to} & \cdots & \overset{\delta_{n-3}}{\to} & I^{n-2} & \overset{\delta_{n-2}}{\to} & I^{n-1} & \overset{\delta_{n-1}}{\to} & I^n
\end{array}$$

We must construct $\psi_n : D^n \to I^n$. Let $C := D^{n-1}/\operatorname{Im}(\Delta_{n-2})$. Notice that $\delta_{n-1} \circ \psi_{n-1} \circ \Delta_{n-2}$ is the zero map since the diagram commutes, and $\delta_{n-2} \circ \delta_{n-1}$ is the zero map. It follows that the map $\delta_{n-1} \circ \psi_{n-1} : D^{n-1} \to I^n$ induces a map $\gamma : C \to I^n$. Now put $H := \operatorname{Ker}(\Delta_{n-1})/\operatorname{Im}(\Delta_{n-2})$. Then $\operatorname{Im}(\Delta_{n-1}) \cong D^{n-1}/\operatorname{Ker}(\Delta_{n-1}) \cong C/H$, and $c^{k_1} \cdot H = 0$. Therefore, $c^{k_1} \cdot \gamma : \operatorname{Im}(\Delta_{n-1}) \to I^n$ is well defined. By the universal mapping property of injective modules, there exists $\psi_n : D^n \to I^n$. Finally, if we replace the original maps $\cdot c^{(n-1)k_1}, \psi_0, \ldots, \psi_{n-1}$ with the maps $\cdot c^{nk_1}, c^{k_1}\psi_0, \ldots, c^{k_1}\psi_{n-1}$, respectively, then the new diagram (with ψ_n included) commutes. This proves the lemma. $\square$

Proposition IV.18. *Assume all the notations and conventions of the previous result. Then there exist maps $\psi_j : D^j \to I^j$ and $\psi : 0 \to I^{d+1}$ such that the following diagram commutes:*

$$\begin{array}{ccccccccccccc}
0 & \to & \omega & \to & D^0 & \overset{\Delta_0}{\to} & D^1 & \overset{\Delta_1}{\to} & \cdots & \overset{\Delta_{d-1}}{\to} & D^d & \to & 0 \\
 & & \downarrow \cdot c^h & & \downarrow \psi_0 & & \downarrow \psi_1 & & & & \downarrow \psi_d & & \downarrow \psi \\
0 & \to & \omega & \to & I^0 & \overset{\delta_0}{\to} & I^1 & \overset{\delta_1}{\to} & \cdots & \overset{\delta_{d-1}}{\to} & I^d & \to & I^{d+1}
\end{array}$$

Proof: Given the previous lemma, all we have to do is construct the map ψ and show that the diagram commutes. We have the following commutative diagram:

$$\begin{array}{ccccc} X & \overset{\Delta_{d-1}}{\rightarrow} & D^d & \rightarrow & 0 \\ \downarrow f & & \downarrow \psi_d & & \\ Y & \overset{\delta_{d-1}}{\rightarrow} & I^d & \overset{\delta_d}{\rightarrow} & I^{d+1} \end{array}$$

where $X = D^{d-1}$ and $Y = I^{d-1}$ if $d > 0$ and $X = Y = \omega$ if $d = 0$. Now, define $C := D^d/\Delta_{d-1}(X)$. By the commutativity of the diagram, the map $\delta_d \circ \psi_d : D^d \rightarrow I^{d+1}$ kills $\Delta_{d-1}(X)$ and therefore induces a map $\gamma : C \rightarrow I^{d+1}$. But C is the homology of the complex at D^d, so $c^{k_1} \cdot C = 0$. It follows that $c^{k_1} \cdot \gamma : 0 \rightarrow I^{d+1}$, so we let $\psi := c^{k_1} \cdot \gamma$. Replacing the previous ψ_j with $c^{k_1}\psi_j$ ensures that the new diagram commutes. □

Proposition IV.19. *Let (R, m, K) be a local ring of dimension d with canonical module ω, and let $c \in R$. Let $\mathcal{D}^\bullet$ be the dualizing complex and let $\{N_\lambda\}$ be a family of R-modules. Suppose there exists an integer $t \in \mathbb{N}$ such that, for all R-modules N_λ in the family,*

$$c^t \cdot H^i(\mathrm{Hom}_R(N_\lambda, \mathcal{D}^\bullet)) = 0$$

for all $0 < i \leq d$. Then, with h as in the previous proposition,

$$c^{h+t} \cdot \mathrm{Ext}^i_R(N_\lambda, \omega) = 0$$

for all $0 < i \leq d$ and all R-modules N_λ in the family.

Proof: We first construct two endomorphisms of the complex

$$0 \rightarrow \omega \rightarrow I^0 \rightarrow \cdots \rightarrow I^{d+1}$$

and show that they are homotopic. The map $\omega \xrightarrow{\cdot c^h} \omega$ given by multiplication by c^h induces an endomorphism $f^\bullet : I^\bullet \rightarrow I^\bullet$. On the other hand, since $I^\bullet$ is exact and $\mathcal{D}^\bullet$ is injective, the identity mapping, $\mathbb{1} : \omega \rightarrow \omega$ can be lifted to a map of complexes $\gamma^\bullet : I^\bullet \to \mathcal{D}^\bullet$. Recall the map $\psi^\bullet : D^\bullet \to I^\bullet$ constructed in (IV.18). The composition,

$$g^\bullet := \psi^\bullet \circ \gamma^\bullet$$

produces a second enodmorphism of $I^\bullet$ lifting the map $\omega \xrightarrow{\cdot c^h} \omega$.

Since both $f^\bullet$ and $g^\bullet$ lift the map $\cdot c^h$, by standard results these two maps of complexes are homotopic. Further, the homotopy is preserved after applying $\mathrm{Hom}_R(N, -)$. Therefore, the two maps induce the same map on homology. Obviously, $f^\bullet$ simply induces the map $\mathrm{Ext}^\bullet(N, W) \xrightarrow{\cdot c^h} \mathrm{Ext}^\bullet(N, W)$. Therefore, the composition:

$$\mathrm{Ext}^\bullet(N, W) \xrightarrow{\gamma^\bullet} H^\bullet(\mathrm{Hom}(N, \mathcal{D}^\bullet)) \xrightarrow{\psi^\bullet} \mathrm{Ext}^\bullet(N, W)$$

is simply $\cdot c^h$. Here, we have written $\gamma^\bullet$ and $\psi^\bullet$ for the maps they induce. Suppose $z \in \mathrm{Ext}^\bullet_R(N, W)$ and we would like to show that $c^{h+t}z = 0$. Then $(\psi^\bullet \circ \gamma^\bullet)(c^t z) = c^{h+t}z$, and yet,

$$(\psi^\bullet \circ \gamma^\bullet)(c^t z) = \psi^\bullet(c^t \gamma^\bullet(z)) = 0$$

since c^t kills the middle module. □

The dualizing complex derives its power in part from the fact that its homology is dual to local cohomology: this is well known (see [Har66]), but we include a short proof for completeness.

Proposition IV.20. *Let (R, m, K) be a local ring of dimension d such that $R = T/I$ for a Gorenstein ring T of dimension $d + h$. Let $\mathcal{D}^\bullet$ be the dualizing complex. Let ${}^{\vee_R}$ be the functor $\mathrm{Hom}_R(-, E_R(K))$ where $E_R(K)$ is the injective hull of the residue class field, and similarly for ${}^{\vee_T}$. Let N be a finitely generated R-module. Then there* *is a natural isomorphism:*

$$H^j_m(N) \cong (H^{d-j}(\mathrm{Hom}_R(N, \mathcal{D}^\bullet)))^{\vee_R}$$

for all $0 \leq j \leq d$.

Proof: By local duality over the Gorenstein ring T (see Corollary 3.5.9 of [BH93]), we have an isomorphism

$$H^j_n(N) \cong \mathrm{Ext}^{d+h-j}_T(N, T)^{\vee_T}.$$

However, as N is an R-module, $H^j_n(N) \cong H^j_m(N)$, and

$$\mathrm{Ext}^{d+h-j}_T(N, T)^{\vee_T} \cong \mathrm{Ext}^{d+h-j}_T(N, T)^{\vee_R},$$

so we conclude

$$H^j_m(N) \cong \mathrm{Ext}^{d+h-j}_T(N, T)^{\vee_R}.$$

Now,

$$\mathrm{Ext}^{d+h-j}_T(N, T) \cong H^{d+h-j}(\mathrm{Hom}_T(N, E^\bullet)) \cong H^{d+h-j}(\mathrm{Hom}_T(N, \mathrm{Hom}_T(R, E^\bullet))),$$

the last isomorphism holding because N is an R-module; *Proof:* N is killed by I, so $\mathrm{Hom}_T(N, E) = \mathrm{Hom}_T(N, \mathrm{Ann}_E(I))$ for any T-module E. But,

$$\mathrm{Hom}_T(R, E) = \mathrm{Hom}_T(T/I, E) = \mathrm{Ann}_E(I)$$

and so

$$\mathrm{Hom}_T(N, E) = \mathrm{Hom}_T(N, \mathrm{Ann}_E(I)) = \mathrm{Hom}_T(N, \mathrm{Hom}_T(R, E))$$

for any T-module E.

But $\mathrm{Hom}_T(R, E^\bullet) = \mathcal{D}^\bullet$ with a shift by h in degree, and so

$$H^{d+h-j}(\mathrm{Hom}_T(N, \mathrm{Hom}_T(R, E^\bullet))) \cong H^{d-j}(\mathrm{Hom}_T(N, \mathcal{D}^\bullet))$$

$$\cong H^{d-j}(\mathrm{Hom}_R(N, \mathcal{D}^\bullet))).$$

In conclusion,

$$H^j_m(N) \cong \mathrm{Ext}_T^{d+h-j}(N,T)^{\vee_R} \cong (H^{d-j}(\mathrm{Hom}_R(N, \mathcal{D}^\bullet)))^{\vee_R}$$

as desired. □

Corollary IV.21. *Let (R, m, K) be a local ring of dimension d with canonical module ω, and let $c \in R$. Let $\mathcal{D}^\bullet$ be a dualizing complex. Suppose there exists an integer $t \in \mathbb{N}$ such that, for a family of R-modules $\{N_\lambda\}$,*

$$c^t \cdot H^{d-i}_m(N_\lambda) = 0$$

for all R-modules N_λ in the family and all $1 \le i \le d$. Then, with h as in (IV.19),

$$c^{h+t} \cdot \mathrm{Ext}^i_R(N_\lambda, \omega) = 0$$

for all R-modules N_λ in the family and all $1 \le i \le d$.

Proof: This is immediate from (IV.19) and the isomorphism of (IV.20). □

Remark: Of course, the corollary is also valid with $i = 0$ included, but we have stated precisely the result we will need below.

To complete the proof of Theorem IV.5, we need results on colon-killers from [HH92].

Proposition IV.22. *Assume R is a reduced, equidimensional local ring that is a homomorphic image of a Gorenstein ring. Let $c \in R$ such that R_c is Cohen-Macaulay. There exists $t \in \mathbb{N}$ such that for all systems of parameters $x_1, \ldots, x_d$, c^t annihilates*

$$\frac{(x_1, \ldots, x_k) : x_{k+1}}{(x_1, \ldots, x_k)}$$

for all $1 \leq k \leq d-1$.

Proof: See Lemma 3.2 of [HH92]. $\square$

Corollary IV.23. *Let* R, c *be as above. There exists an integer* $t \in \mathbb{N}$ *such that, for all systems of parameters* $x_1, \ldots, x_d$, *for all* $q = p^e$ *and for all* $1 \leq k \leq d-1$,

$$c^t \cdot \frac{(x_1, \ldots, x_k)R^{1/q} : x_{k+1}}{(x_1, \ldots, x_k)R^{1/q}} = 0.$$

Proof: By the previous result, there exists t such that c^t kills

$$(x_1^q, \ldots, x_k^q) : x_{k+1}^q / (x_1^q, \ldots, x_k^q)$$

for all k, q. Taking q^{th} roots we find that $c^{t/q}$ kills

$$(x_1, \ldots, x_k)R^{1/q} : x_{k+1} / (x_1, \ldots, x_k)R^{1/q}$$

for all k, q. Since c^t is an $R^{1/q}$ multiple of $c^{t/q}$, it follows that c^t has this property as well. This proves the result. $\square$

Corollary IV.24. *Let* R, c *be as above. There exists an integer* $t \in \mathbb{N}$ *such that*

$$c^t \cdot H^j_m(R^{1/q}) = 0$$

for all $j < d$. *Consequently if* R *is* F*-finite, then with* h *as in Proposition IV.19,*

$$c^{h+t} \cdot \mathrm{Ext}^i(R^{1/q}, \omega) = 0$$

for all $1 \leq i \leq d$ *and for all* q.

Proof: The second statement follows from the first and Corollary IV.21. It suffices to prove the first statement. By the previous corollary, there exits $t \in \mathbb{N}$ annihilating all Koszul homology on $R^{1/q}$. If $(x) = x_1, \ldots, x_d$ is a system of parameters, then

$H^j_m(R^{1/q}) = H^j_{(x)}(R^{1/q})$ which can be computed as a direct limit of Koszul cohomology, which is Koszul homology with the numbering reversed. □

We need one last lemma before we can complete the proof of Theorem IV.5.

Lemma IV.25. *Let $R \to S$ be a flat map of (not necessarily local) Noetherian rings, let W be a finitely generated R-module, and let $c \in R$.*

(a) The injective dimension of W (whether finite or not) is the same as

$$\sup\{i(m) \;:\; \operatorname{Ext}^{i(m)}_{R_m}(R_m/mR_m, W_m) \neq 0, m \in \operatorname{MaxSpec}(R)\}.$$

In particular, if (R, m, K) is local, then the injective dimension of W is

$$\sup\{i \;:\; \operatorname{Ext}^i_R(K, W) \neq 0\}.$$

(b) If $\operatorname{id}_S(W \otimes_R S) < \infty$ then $\operatorname{id}_R(W) < \infty$.

(c) If additionally, S has finite Krull dimension and the map $R \to S$ has Gorenstein fibers then

$$\operatorname{id}_S(W \otimes_R S) < \infty \Longleftrightarrow \operatorname{id}_R(W) < \infty$$

(d) Assume (R, m) is an excellent local ring. If R is equidimensional (repectively, reduced) then $\widehat{R}$, the completion of R with respect to the maximal ideal, is equidimensional (respectively, reduced).

Proof: (a). By definition, the i^{th} Bass number of W with respect to m is the number $\mu_i(m, W) := \dim_{\kappa(m)}\left(\operatorname{Ext}^i_{R_m}(\kappa(m), W_m)\right)$ where $\kappa(m) = R_m/mR_m$. Since the minimal injective resolution of W over R has $\mu_i(m, W)$ copies of $E_R(R/m)$ in the i^{th} spot, the result follows. The second statement is a special case of the first.

We establish part (b). Set $b := \mathrm{id}_S(W \otimes_R S)$. Let $P \in \operatorname{Spec}(R)$ and let Q be a prime of S lying over P. Since injective resolutions localize, $\mathrm{id}_{S_Q}(W \otimes_R S_Q) \leq b$ for all $Q \in \operatorname{Spec}(S)$. It follows that for all $i > b$,

$$\operatorname{Ext}^i_{S_Q}(\kappa_R(P) \otimes_R S_Q, W \otimes_R S_Q) = 0$$

where $\kappa_R(P) = R_P/PR_P$. Since $R_P \to S_Q$ is faithfully flat, we therefore have

$$S_Q \otimes_{R_P} \operatorname{Ext}^i_{R_P}(\kappa_R(P), W_P) = 0,$$

for all $i > b$. But since $R_P \to S_Q$ is faithfully flat we must have $\operatorname{Ext}^i_{R_P}(\kappa_R(P), W_P) = 0$ for all $i > b$. By [BH93], 3.1.14, it follows that $\mathrm{id}_{R_P}(W_P) \leq b$. We claim $\mathrm{id}_R(W) \leq b$: if the minimal injective resolution of W over R has length greater than b, we may localize at some prime and preserve this, a contradiction since the minimal injective resolution of W_P has length less than or equal to b.

For part (c), it suffices to prove the ($\Longleftarrow$) implication. Assume $\mathrm{id}_R W < \infty$. We need to show that $\mathrm{id}_S(W \otimes S) < \infty$. Let d be the Krull dimension of S. We first claim that this issue is local on the maximal ideals of S. Note first that for any maximal ideal $n \subseteq S$, $\dim(S_n) \leq d$ and therefore any finitely generated module of finite injective dimension over S_n has injective dimension $\leq d$. Therefore, if we prove that $\mathrm{id}_{S_n}(W \otimes S_n) < \infty$ for all maximal ideals n, then we will have that $\mathrm{id}_{S_n}(W \otimes S_n) \leq d$ for all n, and then $\mathrm{id}_S(W \otimes S) \leq d$ (if not, we could preserve this by localizing the minimal injective resolution of $W \otimes_R S$ at a maximal ideal of S). Therefore, we are free to replace S by S_n, and then we can replace R by R_m where $m = n \cap R$: we still have that $R_m \to S_n$ is faithfully flat with Gorenstein fibers. Thus, without loss of generality we may assume that R and S are local.

We next want to replace R, S by their completions $\widehat{R}, \widehat{S}$. We begin with R. If $k = R/m$ is the injective hull of the residue filed of R, then by 3.1.14 of [BH93], $\mathrm{Ext}^i_R(k, W) = 0$ for all $i > \mathrm{id}_R(W)$. Since $R \to \widehat{R}$ is faithfully flat, $\mathrm{Ext}^i_{\widehat{R}}(k, W \otimes \widehat{R}) = 0$ for all $i > \mathrm{id}(W)$ from which is follows that $\widehat{W} = W \otimes_R \widehat{R}$ has finite injective dimension over $\widehat{R}$. Observe that

$$W \otimes_R \widehat{R} \otimes_{\widehat{R}} \widehat{S} \cong W \otimes_R S \otimes_S \widehat{S}.$$

If we show that $\mathrm{id}_{\widehat{S}}(\widehat{W} \otimes_{\widehat{R}} \widehat{S})$ is finite, then by part (b) just proved, it will follow that $\mathrm{id}_S(W \otimes_R S)$ is finite, since $S \to \widehat{S}$ is faithfully flat. So, we may replace $R \to S$ by $\widehat{R} \to \widehat{S}$: the map is still faithfully flat by the local criterion for flatness and the closed fiber is $\widehat{S}/m\widehat{S}$ which is still Gorenstein since it is just the completion of the Gorestein ring S/mS. Hence, without loss of generality we may assume that R and S are complete. R therefore has a canonical module which we denote by ω, and by [BH93], Theorem 3.3.14(a), $\omega \otimes_R S$ is a canonical module for S. Now, the result is trivial if $W = 0$, so we may assume that $W \neq 0$. It follows that R is a Cohen-Macaulay ring, since it possesses a finitely generated, non-zero module of finite injective dimension, and it follows that S is Cohen-Macaulay as well, since $R \to S$ is a flat local map with Gorenstein fibers (in fact, only a Cohen-Macaulay closed fiber is needed; cf., 2.1.7 of [BH93]). In this case, W has a finite ω-resolution

$$0 \to \omega^{\oplus b_h} \to \cdots \to \omega^{\oplus b_0} \to W \to 0$$

and applying $- \otimes_R S$ yields the exact sequence

$$0 \to \omega^{\oplus b_h} \otimes S \to \cdots \to \omega^{\oplus b_0} \otimes S \to W \otimes S \to 0.$$

It follows that $W \otimes S$ has finite injective dimension as well, completing the proof.

Part (d) follows from the fact that R is excellent. $\square$

Proof of Theorem IV.5: By the previous lemma, the hypothesis are preserved after we complete: $R_c \to \widehat{R}_c$ is faithfully flat and the fibers are regular since R is excellent, so by part (c) of the previous lemma, $\widehat{W_c}$ has finite injective dimension over $\widehat{R}_c$. We may therefore assume R is a homomorphic image of a Gorenstein ring with canonical module ω. By Proposition IV.12 it is enough to show ω is (c, F)-injective. This follows from Corollaries IV.16 and IV.24. $\square$

4.2 When R is Not Equidimensional

In this section we obtain Theorem IV.5 without assuming the ring R is equidimensional. See Theorem IV.41. The proof in the equidimensional case involves most of the ideas; however, when the ring is not assumed to be equidimensional, there is a technical difficulty that must be resolved. What we are going to know is that R_c is a product of Cohen-Macaulay, equidimensional rings of potentially different dimensions. Roughly speaking, we are able work on each component separately using the ideas from the previous section.

Definition IV.26. Let R be any Noetherian ring and let $c \in R$. Then an R-module W is called a *c-canonical module* if W is a finitely generated R-module, and for all primes $P \in \operatorname{Spec}(R)$ such that $c \notin P$, W_P is a canonical module for R_P.

The following construction ensures that c-canonical modules exist in our situation.

Proposition IV.27. *Let (R, m, K) be a local ring and let $c \in R$ such that R_c is Cohen-Macaulay. Assume R is a homomorphic image of a Gorenstein ring, T. Then R has a c-canonical module, W. Moreover, in this case, $R_c = R_1 \times \cdots \times R_k$ where each R_i is a Cohen-Macaulay, equidimensional ring and W is such that $W_c = \omega_1 \times \cdots \times \omega_k$ where ω_i is a canonical module for R_i.*

Proof: It suffices to construct a module $W(c)$ over R_c with the right properties: if $W(c)$ has been constructed over R_c, say with presentation $R_c^{\oplus m} \to R_c^{\oplus n} \to W(c) \to 0$, then we may clear denominators in the presentation by multiplying by a high power of c to get $R^{\oplus m} \xrightarrow{\theta} R^{\oplus n} \to \operatorname{Coker}(\theta) \to 0$. Letting $W = \operatorname{Coker}(\theta)$ we have that $W_c \cong W(c)$ and so W will have the desired properties. We work over the Cohen-Macaulay (though not local) ring R_c.

The surjection $\pi : T \twoheadrightarrow R$ induces a surjection $T_{\pi^{-1}(c)} \twoheadrightarrow R_c$. It suffices to prove the statement that $R_c = R_1 \times \cdots \times R_k$ where each R_i is Cohen-Macaulay and equidimensional. For then, each R_i is still a homomorphic image of a Gorenstein ring (each is a homomorphic image of R_c), and so by standard results, each R_i has a canonical module ω_i. The fact that each R_i is equidimensional ensures that ω_i localizes properly. We may then set $W(c) := \omega_1 \times \cdots \times \omega_k$ and it follows that $W(c)$ localizes properly.

It remains to prove the statement that if R_c is Cohen-Macaulay and a homomorphic image of a Gorenstein ring, then R_c is a product of equidimensional rings. If $\mathrm{Spec}(R_c)$ is not connected, then $R_c = R_1 \times \cdots \times R_k$ with $\mathrm{Spec}(R_i)$ connected. Therefore, replacing R_c with $S = R_i$, it suffices to prove that if a Cohen-Macaulay ring S is a homomorphic image or a Gorenstein ring T, and $\mathrm{Spec}(S)$ is connected then S is equidimensional. Write $S = T/J$. We claim that all minimal primes of J have the same height which will prove that S is equidimensional. This is clear if the two minimal primes are contained in the same maximal ideal m of T: replacing T with T_m is harmless, and then S is Cohen-Macaulay local and therefore equidimensional, so the primes have the same height. Finally, given two primes P, Q not contained in the same maximal ideal, since $\mathrm{Spec}(S)$ is connected we can construct a sequence $P = P_1, m_1, P_2, m_2, \ldots, P_{h-1}, m_h, P_h = Q$ such that all P_j are minimal, all m_j are maximal, and each m_j contains both P_{j-1} and P_j. The result now follows. $\square$

There is a sort of uniqueness that c-canonical modules enjoy when R_c is Cohen-Macaulay.

Proposition IV.28. *Let $c \in R$ such that R_c is Cohen-Macaulay and let V, W be* *two c-canonical modules. Then V_c is a direct summand of $W_c^{\oplus n}$ for some n.*

Proof: For any $P \in Spec(R_c)$ we have $V_P \cong R_P$ since R_P is Cohen-Macaulay and local. Therefore, $\mathrm{Hom}_{R_P}(V_P, W_P) \cong \mathrm{Hom}_{R_P}(V_P, V_P) \cong R_P$ since V_P is a canonical module for R_P. It follows that $\mathrm{Hom}_{R_c}(V_c, W_c)$ is projective as it is locally free and finitely generated. Say $\mathrm{Hom}_{R_c}(V_c, W_c) \oplus Q = R_c^{\oplus n}$. Applying $- \otimes_{R_c} W_c$ we find

$$\big(\mathrm{Hom}_{R_c}(V_c, W_c) \otimes W_c\big) \oplus (Q \otimes W_c) = W_c^{\oplus n}.$$

So, the result will be proved if we can show that $\mathrm{Hom}_{R_c}(V_c, W_c) \otimes W_c \cong V_c$. But it suffices to prove the isomorphism locally, and, for each $P \in \mathrm{Spec}(R_c)$, we have $\mathrm{Hom}_{R_P}(V_P, W_P) \cong R_P$. Hence,

$$\mathrm{Hom}_{R_P}(V_P, W_P) \otimes W_P \cong R_P \otimes W_P \cong W_P \cong V_P.$$

This completes the proof. □

Corollary IV.29. *Let $c \in R$ such that R_c is Cohen-Macaulay. Suppose there exists a c-canonical module W for R such that W is (c, F)-injective. Then every c-canonical module is (c, F)-injective.*

Proof: This is immediate from the previous result and Proposition IV.8. □

When R is not equidimensional, the dualizing complex does not suffice for all of our purposes, but a modification of a certain subcomplex does. We begin with a few observations about the dualizing complex.

Lemma IV.30. *Let R be a homomorphic image of a Gorenstein ring $T \twoheadrightarrow R$. Assume that R_c is Cohen-Macaulay for some $c \in R$ so that, as we have seen above, $R_c = R_1 \times \cdots \times R_n$ where each R_i is Cohen-Macaulay and equidimensional. Let*

$d = \dim(R)$. Then there is a complex:

$$\mathcal{D}^\bullet: \quad 0 \to D^0 \to \cdots \to D^d \to 0$$

such that:

(a) Each module D^j is an injective R-module.

(b) For some $\mathcal{H}^\bullet(c)$, there is a short exact sequence of complexes

$$0 \to \mathcal{H}(c)^\bullet \to \mathcal{D}^\bullet \to \mathcal{D}^\bullet_c \to 0$$

where $\mathcal{D}^\bullet_c$ is the complex obtained by localizing $\mathcal{D}^\bullet$ at c. Moreover, this exact sequence splits so that $\mathcal{D}^\bullet = \mathcal{H}(c)^\bullet \oplus \mathcal{D}^\bullet_c$.

(c) The complex $\mathcal{D}^\bullet_c$ splits (even over R) as

$$\mathcal{D}^\bullet_c = \mathcal{D}^\bullet_{(c,1)} \oplus \cdots \oplus \mathcal{D}^\bullet_{(c,k)}.$$

We therefore have

$$\mathcal{D}^\bullet = \mathcal{H}(c)^\bullet \oplus \mathcal{D}^\bullet_{(c,1)} \oplus \cdots \oplus \mathcal{D}^\bullet_{(c,k)}.$$

Proof: By replacing T with $T/(x_1, \ldots, x_h)$ where $x_1, \ldots, x_h$ is a maximal regular sequence in $\mathrm{Ker}(T \to R)$ we may assume that $\dim(T) = \dim(R) = d$. We start with a minimal injective resolution $\mathcal{E}^\bullet$ of T over itself. We set $\mathcal{D}^\bullet := \mathrm{Hom}_T(R, \mathcal{E}^\bullet)$ and it follows that

$$\mathcal{D}^\bullet = 0 \to D^0 \to \cdots \to D^d \to 0$$

where D^j is injective over R just as in the equidimensional case. Now, each $D_j = \oplus E(R/P)$ is a direct sum of injective hulls of R/P for certain primes P. The key point is that if $c \in P$ then $E_R(R/P)_c = 0$ since every element of $E_R(R/P)$ is annihilated by a power of P and hence by a power of c. On the other hand, if $c \notin P$, then $E_R(R/P)_c \cong E_{R_c}(R_c/PR_c)$. We therefore define

$$[\mathcal{H}(c)]^j := \bigoplus_{E(R/P)\subseteq D^j,\, c\in P} E_R(R/P),$$

and it follows that $(*)\quad 0 \to \mathcal{H}(c)^\bullet \to \mathcal{D}^\bullet \to \mathcal{D}_c^\bullet \to 0$ is exact. The complex $\mathcal{D}_c^\bullet$ is such that

$$[\mathcal{D}_c]^j := \bigoplus_{E(R/P)\subseteq D^j,\, c\notin P} E_R(R/P),$$

and it is clear that the exact sequence $(*)$ splits.

Since every $E_R(R/P)$ occurring in $\mathcal{D}_c^\bullet$ is such that $c \notin P$, the prime P corresponds to a prime of R_c. But $R_c = R_1 \times \cdots \times R_n$, so each such P corresponds to a prime of some R_j. Thus, we may divide the $E_R(R/P)$ occurring in $D_c^\bullet$ into n different subcomplexes corresponding to the R_i. This establishes the decomposition in part (c). $\square$

We will make extensive use of a complex closely related to $\mathcal{D}_c^\bullet$. Before we can give the construction we need to prove a lemma.

Lemma IV.31. *Let $T \twoheadrightarrow R$ where T is Gorenstein of the same dimension as R, and let $c \in R$ such that R_c is Cohen-Macaulay. Write $R_c = R_1 \times \cdots \times R_n$ and let $d_j = \dim(R_j)$. After renumbering if necessary we may assume that $d_1 \geq d_2 \geq \cdots \geq d_n$. Then for each $1 \leq j \leq n$,*

(a) $[\mathcal{D}_{(c,j)}]^i = 0$ for all $0 \leq i < \dim(T) - d_j$.

(b) $H^{\dim(T)-d_j}(\mathcal{D}_{(c,j)})_c \cong \omega_j$ where ω_j is a canonical module for R_j.

Proof: By construction, $\mathcal{D}_c^\bullet$ consists of all $E_R(R/P)$ in $\mathcal{D}^\bullet$ such that $c \notin P$, and $\mathcal{D}_{(c,j)}^\bullet$ consists of all $E_R(R/P)$ such that P is a prime of R_j. Now by construction,

$[\mathcal{D}]^i = \oplus_{htP=i} \operatorname{Hom}_T(R, E_T(T/P))$ and so $[\mathcal{D}_{(c,j)}]^i = \oplus_{htP=i} \operatorname{Hom}_R(R_j, E_T(T/P))$. But these will all be zero for all $0 \leq i < \dim(T) - d_j$. This proves part (a).

For part (b) the point is that $H^j(\mathcal{D}^\bullet) = \operatorname{Ext}^j_T(R,T)$ and also $H^j(\mathcal{D}^\bullet)_c \cong H^j(\mathcal{D}^\bullet_c)$, hence $\operatorname{Ext}^j_{T_c}(R_c, T_c) \cong H^j(\mathcal{D}^\bullet_c)$ follows. Now, $\operatorname{Ext}^j_{T_c}(R_c, T_c) \cong \prod_i \operatorname{Ext}^j_{T_c}(R_i, T_c)$. To complete the proof, we may localize at an arbitrary prime P such that $c \notin P$, and then the result follows from (IV.13) part (iii). $\square$

Construction. Let $R, c, \mathcal{D}^\bullet_{(c,j)}$ and $\mathcal{D}^\bullet_c$ be as above. We define a new complex, $\mathcal{A}^\bullet$, as the sum of all $\mathcal{D}^\bullet_{(c,j)}$ shifted by $\dim(T) - d_j$. Formally,

$$\mathcal{A}^\bullet := \mathcal{D}^\bullet_{(c,1)}(\dim(T) - d_1) \oplus \mathcal{D}^\bullet_{(c,2)}(\dim(T) - d_2) \oplus \cdots \oplus \mathcal{D}^\bullet_{(c,n)}(\dim(T) - d_n)$$

where for an integer m and a complex $\mathcal{B}^\bullet$, $\mathcal{B}^\bullet(m)$ is the complex such that $[\mathcal{B}(m)]^j := [\mathcal{B}]^{j+m}$. We have

Proposition IV.32. *Let $R, c,$ and $\mathcal{A}^\bullet$ be as above. Then the complex $\mathcal{A}^\bullet$ has the following properties:*

(i) $\mathcal{A}^\bullet :\ 0 \to A^0 \to \cdots \to A^{d_1} \to 0$ *where each A^i is an injective R-module.*

(ii) $H^0(\mathcal{A}^\bullet)_c \cong \omega_1 \times \cdots \times \omega_n$ *where ω_j is a canonical module for R_j. Moreover, there exists a c-canonical module W for R and an integer $k' \in \mathbb{N}$ such that*

$$0 \to W \overset{\Delta_{-1}}{\to} A^0 \overset{\Delta_0}{\to} \cdots \to A^d \to 0$$

is a complex and $c^{k'} \cdot \operatorname{Ker}(\Delta_{-1}) = 0$.

(iii) $H^j(\mathcal{A}^\bullet)_c = 0$ *for $j \geq 0$ and $H^j(\mathcal{A}^\bullet)$ is finitely generated for all j. It follows that there exists $k_1 \in \mathbb{N}$ such that c^{k_1} annihilates $H^j(\mathcal{A}^\bullet)$ for all $j \geq 0$ and c^{k_1} annihilates* $\operatorname{Ker}(\Delta_{-1})$.

Proof: Part (i) is obvious from the construction, and the first statement of part (ii) follows from (b) of the previous lemma. Let $W' := \omega_1 \times \cdots \times \omega_n$, a finitely generated R_c-module. Then W' has a presentation, $R_c^{\oplus a} \xrightarrow{\theta_c} R_c^{\oplus b} \to W' \to 0$. After multiplying θ_c by a high power of c, we clear the denominators of the entries in a matrix for θ_c and we obtain $R^{\oplus a} \to R^{\oplus b} \to W \to 0$. It follows that $W_c \cong W'$, and so W is a c-canonical module for R. Also, after clearing denominators appearing in the isomorphism $W_c \cong W'$ we obtain an R-linear map $W_1 \to H^0(\mathcal{A}^\bullet)$, hence a map $\Delta_{-1} : W_1 \to A^0$. Now $\mathrm{Ker}(\Delta_{-1}) \subseteq W_1$, and so $\mathrm{Ker}(\Delta_{-1})$ is finitely generated. Since $(\Delta_{-1})_c$ is injective, it follows that $c^{k'} \cdot \mathrm{Ker}(\Delta_{-1}) = 0$ for k' sufficiently large. This proves (ii).

For (iii), first note that $H^j(\mathcal{A}^\bullet) = \bigoplus_{j=1}^n H^j(\mathcal{D}^\bullet_{(c,j)}(d-d_j))$ where $d = \dim(T)$. It suffices to prove the result about $H^j(\mathcal{D}^\bullet_{(c,j)})$. We have

$$\mathrm{Ext}^j_T(R,T) \cong H^j(\mathcal{D}^\bullet) = \bigoplus_{i=1}^n H^j(\mathcal{D}^\bullet_{(c,i)}) \oplus H^j(\mathcal{H}(c)^\bullet)$$

and so each $H^j(\mathcal{D}^\bullet_{(c,i)})$ is finitely generated. It suffices to show that $H^j(\mathcal{D}^\bullet_{(c,i)})_c = 0$, for then a fixed power of c, say c^{k_2}, will annihilate $H^j(\mathcal{D}^\bullet_{(c,i)})$ for all $i \geq 1$, and then taking $k_1 := \max\{k_2, k'\}$ will work. For this purpose, it is enough to show that the homology vanishes at every $P \in \mathrm{Spec}(R)$ such that $c \notin P$. But for such a P, $R_P \cong (R_i)_P$ for some i while $(R_j)_P = 0$ for all other $j \neq i$. Then $(\mathcal{D}^\bullet_{(c,i)})_P$ is an injective resolution of $(\omega_i)_P$ over the Cohen-Macaulay local ring $(R_i)_P$, and therefore the homology is zero while $(\mathcal{D}^\bullet_{(c,j)})_P = 0$ for all $j \neq i$. It follows that the homology vanishes in this case too, and this completes the proof. $\square$

For the rest of this section, W will denote the specific c-canonical module with complex $0 \to W \to \mathcal{A}^\bullet$ as in the previous result. The following is an easy corollary of Proposition IV.15.

Proposition IV.33. *Let c, R, W, k_1 and $\mathcal{A}^\bullet$ be as above. Then with $k = (d+2)k_1$,*

$$c^k \cdot \mathrm{Ext}_R^{d+l}(N, W) = 0$$

for all R-modules N and all integers $l > 0$.

Proof: Let $H_0 = \mathrm{Ker}(W \to A^0)$. Then $c^{k_1} \cdot H_0 = 0$, we have a short exact sequence $(*)$ $0 \to H_0 \to W \to W/H_0 \to 0$, and a complex $0 \to W/H_0 \to A^0 \to \cdots \to A^d \to 0$ such that the map $W/H_0 \to A^0$ is injective. By the previous result and Proposition IV.15, $c^{(d+1)k_1} \cdot \mathrm{Ext}_R^{d+l}(N, W/H_0) = 0$ for all $l > 0$. But $(*)$ gives rise to exact sequences:

$$\mathrm{Ext}^j(N, H_0) \to \mathrm{Ext}_R^j(N, W) \to \mathrm{Ext}^j(N, W/H_0)$$

and $c^{k_1} \cdot \mathrm{Ext}_R^j(N, H_0) = 0$ so $c^{(d+2)k_1} \cdot \mathrm{Ext}_R^{d+l}(N, W) = 0$. $\square$

Our next goal is to prove that a fixed power of c annihilates the modules $\mathrm{Ext}_R^j(R^{1/q}, W)$ for $1 \leq j \leq d$. Together with (IV.33) this will show that W is (c, F)-injective, and then by (IV.29) every c-canonical module will be (c, F)-injective. As in Section 1, the first step it to construct a map $\psi^\bullet : \mathcal{A}^\bullet \to \mathcal{I}^\bullet$ lifting the map $W \xrightarrow{\cdot c^h} W$, where $\mathcal{I}^\bullet$ is an injective resolution of W.

Lemma IV.34. *Let $c, R, W,$ and $\mathcal{A}^\bullet$ be as above and let k_1 be as in (IV.32)(iii). Let*

$$\mathcal{I}^\bullet : \quad 0 \to I^0 \to \cdots \to I^d \to I^{d+1} \to \cdots$$

be an injective resolution of W. Then, with $h := k_1(d+1)$, there exist maps $\psi_j : D^j \to I^j$ such that the following diagram commutes:

$$\begin{array}{ccccccccccccc} 0 & \to & W & \to & A^0 & \xrightarrow{\Delta_0} & A^1 & \xrightarrow{\Delta_1} & \cdots & \xrightarrow{\Delta_{d-1}} & A^d & & \\ & & \downarrow \cdot c^h & & \downarrow \psi_0 & & \downarrow \psi_1 & & & & \downarrow \psi_d & & \\ 0 & \to & W & \to & I^0 & \xrightarrow{\delta_0} & I^1 & \xrightarrow{\delta_1} & \cdots & \xrightarrow{\delta_{d-1}} & I^d & \to & I^{d+1} \end{array}$$

Proof: We will construct the ψ_j inductively. If $d = 0$, then $h = k_1$ and we are considering the mapping $W \overset{\cdot c^{k_1}}{\to} W$. Let $H = \mathrm{Ker}(\Delta_{-1})$. By part (ii) of (IV.32), $c^h \cdot H = 0$. Then Δ_{-1} induces an injective map $W/H \to A^0$ and we also have a map $\delta_{-1} \circ (\cdot c^h) : W/H \to I^0$. Since I^0 is injective, by the universal mapping property of injective modules, there exists $\gamma_0 : A^0 \to I^0$. We claim that the diagram $(*)$

$$\begin{array}{ccccc} 0 & \to & W & \xrightarrow{\Delta_{-1}} & A^0 \\ & & \downarrow \cdot c^h & & \downarrow \gamma_0 \\ 0 & \to & W & \xrightarrow{\delta_{-1}} & I^0 \end{array}$$

commutes. To see this, first note that the map Δ_{-1} factors as $W \to W/H \to A^0$. Therefore, we have a commutative diagram:

$$\begin{array}{ccccc} A^0 & \xrightarrow{\mathbb{1}} & A^0 & \xrightarrow{\gamma_0} & I^0 \\ \Delta_{-1} \uparrow & & \uparrow & & \uparrow \mathbb{1} \\ W & \to & W/H & \xrightarrow{\delta_{-1} \circ (\cdot c^h)} & I^0 \end{array}$$

(the right square commutes by the universal property). Moreover, since $c^h \cdot H = 0$, we also know that

$$\begin{array}{ccc} W & \to & W/H \\ \downarrow \cdot c^h & & \downarrow \\ W & \xrightarrow{\delta_{-1}} & I^0 \end{array}$$

is commutative. Since the diagram $(*)$ in question is (contained in) the concatenation of the these two diagrams, it also commutes.

For the induction step, we may assume that the maps $\psi_j : A^j \to I^j$ have been constructed for $0 \leq j \leq n-1$ making the following diagram commute:

$$\begin{array}{ccccccccccccc}
0 & \to & \omega & \to & A^0 & \overset{\Delta_0}{\to} & \cdots & \overset{\Delta_{n-3}}{\to} & A^{n-2} & \overset{\Delta_{n-2}}{\to} & A^{n-1} & \overset{\Delta_{n-1}}{\to} & A^n \\
 & & \downarrow \cdot c^{nk_1} & & \downarrow \psi_0 & & & & \downarrow \psi_{n-2} & & \downarrow \psi_{n-1} & & \\
0 & \to & \omega & \to & I^0 & \overset{\delta_0}{\to} & \cdots & \overset{\delta_{n-3}}{\to} & I^{n-2} & \overset{\delta_{n-2}}{\to} & I^{n-1} & \overset{\delta_{n-1}}{\to} & I^n
\end{array}$$

We must construct $\psi_n : A^n \to I^n$. Let $C := A^{n-1}/\operatorname{Im}(\Delta_{n-2})$. Notice that $\delta_{n-1} \circ \psi_{n-1} \circ \Delta_{n-2}$ is the zero map, since the diagram commutes and since $\delta_{n-2} \circ \delta_{n-1}$ is the zero map. It follows that the map $\delta_{n-1} \circ \psi_{n-1} : A^{n-1} \to I^n$ induces a map $\gamma : C \to I^n$. Now put $H := \operatorname{Ker}(\Delta_{n-1})/\operatorname{Im}(\Delta_{n-2})$. Then $\operatorname{Im}(\Delta_{n-1}) \cong A^{n-1}/\operatorname{Ker}(\Delta_{n-1}) \cong C/H$, and $c^{k_1} \cdot H = 0$. Therefore, $c^{k_1} \cdot \gamma : \operatorname{Im}(\Delta_{n-1}) \to I^n$ is well defined. By the universal mapping property of injective modules, there exists $\psi_n : A^n \to I^n$. Finally, if we replace the original maps $\cdot c^{nk_1}, \psi_0, \ldots, \psi_{n-1}$ with the maps $\cdot c^{(n+1)k_1}, c^{k_1}\psi_0, \ldots, c^{k_1}\psi_{n-1}$, respectively, then the new diagram (with ψ_n included) commutes. The argument is very similar to that given above. This proves the lemma. □

Proposition IV.35. *Assume all the notations and conventions of the previous result. Then, with $h := k_1(d+2)$, there exist maps $\psi_j : A^j \to I^j$ and $\psi : 0 \to I^{d+1}$ such that the following diagram commutes:*

$$\begin{array}{ccccccccccccc}
0 & \to & \omega & \to & A^0 & \overset{\Delta_0}{\to} & A^1 & \overset{\Delta_1}{\to} & \cdots & \overset{\Delta_{d-1}}{\to} & A^d & \to & 0 \\
 & & \downarrow \cdot c^h & & \downarrow \psi_0 & & \downarrow \psi_1 & & & & \downarrow \psi_d & & \downarrow \psi \\
0 & \to & \omega & \to & I^0 & \overset{\delta_0}{\to} & I^1 & \overset{\delta_1}{\to} & \cdots & \overset{\delta_{d-1}}{\to} & I^d & \to & I^{d+1}
\end{array}$$

Proof: Given the previous lemma, all we have to do is construct the map ψ and show that the diagram commutes. We have the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\Delta_{d-1}} & A^d & \rightarrow & 0 \\ \downarrow f & & \downarrow \psi_d & & \\ Y & \xrightarrow{\delta_{d-1}} & I^d & \xrightarrow{\delta_d} & I^{d+1} \end{array}$$

where $X = A^{d-1}$ and $Y = I^{d-1}$ if $d > 0$ and $X = Y = W$ if $d = 0$. Now, define $C := A^d/\Delta_{d-1}(X)$. By the commutativity of the diagram, the map $\delta_d \circ \psi_d : A^d \to I^{d+1}$ kills $\Delta_{d-1}(X)$ and therefore induces a map $\gamma : C \to I^{d+1}$. But C is the homology of the complex at A^d, so $c^{k_1} \cdot C = 0$. It follows that $c^{k_1} \cdot \gamma : 0 \to I^{d+1}$, so we let $\psi := c^{k_1} \cdot \gamma$. Replacing the previous ψ_j with $c^{k_1}\psi_j$ ensures that the new diagram commutes. □

The next result is the same as Proposition IV.19 for the complex $\mathcal{A}^\bullet$. The proofs are identical.

Proposition IV.36. *Let $c, W, \mathcal{I}^\bullet, \mathcal{A}^\bullet$ be as above. Let $\{N_\lambda\}$ be a family of R-modules, and suppose there exists an integer $t \in \mathbb{N}$ such that, for all R-modules N_λ in the family,*

$$c^t \cdot H^i(\mathrm{Hom}_R(N_\lambda, \mathcal{A}^\bullet)) = 0$$

for all $0 < i \leq d$. Then, with h as in the previous proposition,

$$c^{h+t} \cdot \mathrm{Ext}^i_R(N_\lambda, W) = 0$$

for all $0 < i \leq d$ and all R-modules N_λ in the family.

Proof: We first construct two endomorphisms of the complex

$$0 \to W \to I^0 \to \cdots \to I^{d+1}$$

and show that they are homotopic. The map $W \xrightarrow{\cdot c^h} W$ given by multiplication by c^h induces an endomorphism $f^\bullet : \mathcal{I}^\bullet \to \mathcal{I}^\bullet$. On the other hand, since $\mathcal{I}^\bullet$ is exact and $\mathcal{A}^\bullet$ is injective, the identity mapping, $\mathbb{1} : W \to W$ can be lifted to a map of complexes $\gamma^\bullet : \mathcal{I}^\bullet \to \mathcal{A}^\bullet$. Recall the map $\psi^\bullet : A^\bullet \to \mathcal{I}^\bullet$ constructed in (IV.35). The composition,

$$g^\bullet := \psi^\bullet \circ \gamma^\bullet$$

produces a second endomorphism of $\mathcal{I}^\bullet$ lifting the map $W \xrightarrow{\cdot c^h} W$.

Since both $f^\bullet$ and $g^\bullet$ lift the map $\cdot c^h$, by standard results these two maps of complexes are homotopic. Further, the homotopy is preserved after applying $\mathrm{Hom}_R(N, -)$. Therefore, the two maps induce the same map on homology. But $f^\bullet$ simply induces the map $\mathrm{Ext}^\bullet(N, W) \xrightarrow{\cdot c^h} \mathrm{Ext}^\bullet(N, W)$. Therefore, the composition:

$$\mathrm{Ext}^\bullet(N, W) \xrightarrow{\gamma^\bullet} H^\bullet(\mathrm{Hom}(N, \mathcal{D}^\bullet)) \xrightarrow{\psi^\bullet} \mathrm{Ext}^\bullet(N, W)$$

is simply $\cdot c^h$. Here, we have written $\gamma^\bullet$ and $\psi^\bullet$ for the maps they induce. Suppose $z \in \mathrm{Ext}^\bullet_R(N, W)$ and we would like to show that $c^{h+t}z = 0$. Then $(\psi^\bullet \circ \gamma^\bullet)(c^t z) = c^{h+t}z$, and yet,

$$(\psi^\bullet \circ \gamma^\bullet)(c^t z) = \psi^\bullet(c^t \gamma^\bullet(z)) = 0$$

since c^t kills the middle module. $\square$

To finish off the proof that W is (c, F)-injective, we want to use the fact that the homology of $\mathcal{D}^\bullet$ is dual to certain local cohomology modules that are "controlable" in the sense that there is a uniform bound on the power of c needed to annihilate them. This was one of the key facts in Section 1. Since $\mathcal{A}^\bullet$ splits off from $\mathcal{D}^\bullet$ (up to a shift in degree), the homology of $\mathcal{A}^\bullet$ is dual to a submodule of these controllable local cohomology modules. We need the following generalization of (IV.22) to the case where R is not necessarily equidimensional.

Proposition IV.37. *Suppose (R, m, K) is a local ring of dimension d and a homomorphic image of a Gorenstein ring. Let $c \in R$ such that R_c is Cohen-Macaulay. There exists $t \in \mathbb{N}$ such that for every system of parameters $x_1, \ldots, x_d$,*

$$c^t \cdot \frac{(x_1, \ldots, x_k) : x_{k+1}}{(x_1, \ldots, x_k)} = 0$$

for all $1 \leq j \leq d - 1$.

Proof: This follows from Proposition 2.16 and Theorem 2.13 of [HH93]. □

Corollary IV.38. *Let (R, m, K) be a reduced, local ring of dimension d and a homomorphic image of a Gorenstein ring. Suppose $c \in R$ is such that R_c is Cohen-Macaulay. Then there exists $t \in \mathbb{N}$ such that for all $q = p^e$,*

$$c^t \cdot H^j_m(R^{1/q}) = 0$$

for all $j < d$.

Proof: This follows from the previous result in the exact same way that (IV.24) followed from (IV.22). □

We can now complete the proof that W is (c, F)-injective.

Proposition IV.39. *Suppose (R, m, K) is local, F-finite, reduced, and a homomorphic image of a Gorenstein ring T. Let $c \in R$ such that R_c is Cohen-Macaulay. Let $\mathcal{D}^\bullet$ and $\mathcal{A}^\bullet$ be the complexes constructed above and let W be the c-canonical module as in (IV.32)(ii). Then W is (c, F)-injective, i.e., there exists $t \in \mathbb{N}$ such that for all $q = p^e$ and all $i > 0$,*

$$c^t \cdot \mathrm{Ext}^i_R(R^{1/q}, W) = 0.$$

Proof: Let $d = \dim(R)$. We have seen in Proposition IV.33 that there exists $k \in \mathbb{N}$ such that c^k annihilates $\mathrm{Ext}^{d+l}_R(R^{1/q}, W)$ for all $l > 0$ and all q. It suffices to find a fixed power of c annihilating $\operatorname{Ext}^j_R(R^{1/q}, W)$ for all $1 \leq j \leq d$ and all q. By Proposition IV.36, it suffices to find $t \in \mathbb{N}$ such that $c^t \cdot H^j(\operatorname{Hom}_R(R^{1/q}, \mathcal{A}^\bullet)) = 0$ for all q.

Recall the notations of results (IV.31) and (IV.32). In particular, recall that $H^j(\mathcal{A}^\bullet) = \bigoplus_{j=1}^n H^j(\mathcal{D}^\bullet_{(c,j)}(d - d_j))$ where $d = \dim(T) = \dim(R)$. Since Hom commutes with direct sum, it follows that

$$H^j(\operatorname{Hom}_R(R^{1/q}, \mathcal{A}^\bullet)) = \bigoplus_{j=1}^n H^j(\operatorname{Hom}_R(R^{1/q}, \mathcal{D}^\bullet_{(c,j)}(d - d_j)).$$

Now, $\mathcal{D}^\bullet_{(c,j)}(d - d_j)$ splits off from $\mathcal{D}^\bullet(d - d_j)$, and therefore, $\operatorname{Hom}_R(R^{1/q}, \mathcal{D}^\bullet_{(c,j)}(d - d_j))$ splits off from $\operatorname{Hom}_R(R^{1/q}, \mathcal{D}^\bullet(d - d_j))$. So, $H^j(\operatorname{Hom}_R(R^{1/q}, \mathcal{D}^\bullet_{(c,j)}(d - d_j)))$ is a submodule of $H^j(\operatorname{Hom}_R(R^{1/q}, \mathcal{D}^\bullet(d - d_j)))$. But

$$H^j(\operatorname{Hom}_R(R^{1/q}, \mathcal{D}^\bullet(d - d_j)))^\vee \cong H_m^{d-d_j-j}(R^{1/q})$$

by (IV.20). By the previous result, there exists $t \in \mathbb{N}$ such that c^t annihilates $H^j_m(R^{1/q})$ for all q and for all $j < d$. For this t, it follows that

$$c^t \cdot \bigoplus_{j=1}^n H^j(\operatorname{Hom}_R(R^{1/q}, \mathcal{D}^\bullet_{(c,j)}(d - d_j)) = 0,$$

and this completes the proof since

$$H^j(\operatorname{Hom}_R(R^{1/q}, \mathcal{A}^\bullet)) = \bigoplus_{j=1}^n H^j(\operatorname{Hom}_R(R^{1/q}, \mathcal{D}^\bullet_{(c,j)}(d - d_j)).$$

□

Corollary IV.40. *Suppose (R, m, K) is local, reduced, F-finite, and a homomorphic image of a Gorenstein ring. If $c \in R$ such that R_c is Cohen-Macaulay, then every c-canonical module is (c, F)-injective.*

Proof: This is immediate from (IV.29) and the previous result. □

We now turn our attention to an arbitrary finitely generated module V such that $\mathrm{id}_{R_c}(V_c) < \infty$. The idea is to replace the theory of (c, ω)-resolutions developed in Section 1 with a theory of (c, W)-resolutions where W is c-canonical. Our main results are, just as in the equidimensional case, that for a finitely generated module V, V_c has finite injective dimension if and only if V has a (c, W)-resolution for one (equivalently every) c-canonical module W, and that, in this case, V is (c, F)-injective if and only if W is (c, F)-injective. Together with (IV.40), these results will ultimately complete a proof of:

Theorem IV.41. *Let (R, mK) be a reduced, F-finite, excellent local ring. Let $c \in R$ and let V be a finitely generated R-module such that V_c has finite injective dimension. Then there exists $k = k(c, V)$ such that*

$$c^k \cdot \mathrm{Ext}^j_R(R^{1/q}, V) = 0$$

for all $j > 0$ and all q.

We begin with a result analogous to (IV.10).

Lemma IV.42. *Let (R, m, K) be a local ring, let $c \in R$. Suppose V is a finitely generated R-module and that V_c has finite injective dimension over R_c, and let W be any c-canonical module. Then, for some $b \in \mathbb{N}$, there exists an exact sequence*

$$W^{\oplus b} \to V \to C \to 0$$

such that $c^t \cdot C = 0$ for some $t \in \mathbb{N}$.

Proof: Consider the family $\mathcal{F}$ of submodules of V defined as follows: a submodule $U \subseteq V$ is in $\mathcal{F}$ if for some $b \in \mathbb{N}$, there exists an R-linear map $f : W^{\oplus b} \to V$ with

$\mathrm{Im}(f) = U$. If $U_1, U_2 \subseteq V$ are in $\mathcal{F}$ then we claim $U_1 + U_2$ is in $\mathcal{F}$ as well. To see this, suppose there are R-linear maps $f : W^{\oplus b_1} \to V$ and $g : W^{\oplus b_2} \to V$ with $\mathrm{Im}(f) = U_1$ and $\mathrm{Im}(g) = U_2$. Then the direct sum map $f \oplus g : W^{\oplus b_1 + b_2} \to V$ has image $\mathrm{Im}(f \oplus g) = U_1 + U_2$ and the claim is proved. It follows that there is a submodule $B \in \mathcal{F}$ such that if $U \in \mathcal{F}$ then $U \subseteq B$.

We first prove that $(V/B)_c = 0$. If not, then since V/B is finitely generated there is a prime $P \in \mathrm{Supp}(V/B)$ with $c \notin P$. Since V_c has finite injective dimension over R_c, V_P has finite injective dimension over R_P. Note that W_P is the canonical module for the Cohen-Macaulay ring R_P for all $c \notin P$. Therefore, by Lemma IV.9 (since R_c and hence R_P are Cohen-Macaulay), there is a surjective map $W_P^{\oplus n} \twoheadrightarrow V_P$. Since Hom commutes with localization when the left variable is finitely presented, we find an R-linear map $\theta : W^{\oplus n} \to V$ such that $\mathrm{Coker}(\theta)_P = 0$. But $\mathrm{Im}(\theta) \subseteq B$ and so $\mathrm{Coker}(\theta) \twoheadrightarrow V/B$. Since $\mathrm{Coker}(\theta)_P = 0$ it follows that $(V/B)_P = 0$, a contradiction.

We have shown that $(V/B)_c = 0$ and since V/B is finitely generated, there exists $t \in \mathbb{N}$ such that $c^t \cdot (V/B) = 0$. With $C := V/B$, it follows that there is an exact sequence

$$W^{\oplus b} \to V \to C \to 0$$

such that $c^t \cdot C = 0$. $\square$

Proposition IV.43. *Let (R, m, K) be a local ring, let $c \in R$ and let W be a c-canonical module. Let V be a finitely generated R-module. Then V_c has finite injective dimension over R_c if and only if V has a finite (c, W)-resolution.*

Proof: Let V be a finitely generated R-module. Since R is local it has finite krull dimension. We first prove the implication ($\Longrightarrow$). Assume V_c has finite injective dimension. Then R_c must be Cohen-Macaulay and also R_c has finite krull dimension since R does. By the proceeding lemma, there is an exact sequence

$$0 \to Z_0 \to W^{\oplus b_0} \to W \to C_{-1} \to 0$$

and an integer t_{-1} such that $c^{t_{-1}} C_{-1} = 0$. Since localization is flat, there is a short exact sequence

$$0 \to (Z_0)_c \to W_c^{\oplus b_0} \to V_c \to 0.$$

It follows that $(Z_0)_c$ has finite injective dimension. Moreover, since W_c is a maximal Cohen-Macaulay module, it follows that either $(Z_0)_c$ is a maximal Cohen-Macaulay module or $\operatorname{depth}_n (Z_0)_c \geq \operatorname{depth}_n V_c + 1$ for all maximal ideals $n \subseteq R_c$.

Applying the proceeding lemma to Z_0, we obtain an exact sequence

$$0 \to Z_1 \to W^{\oplus b_1} \to Z_0 \to C_0 \to 0$$

and an integer t_0 such that $c^{t_0} C_0 = 0$. Again, we have an exact sequence

$$0 \to (Z_1)_c \to W_c^{\oplus b_1} \to (Z_0)_c \to 0$$

and it follows that the injective dimension of $(Z_1)_c$ is finite and either $(Z_1)_c$ is a maximal Cohen-Macaulay module, or $\operatorname{depth}_n (Z_1)_c \geq \operatorname{depth}_n (Z_0)_c + 1 \geq \operatorname{depth}_n (V_c) + 2$ for all maximal ideals $n \subseteq R_c$.

Proceeding in this way, for each $0 \leq j \leq d = dim(R)$, we obtain modules Z_j, C_j, integers b_j, t_j and exact sequences

$$0 \to Z_{j+1} \to W^{\oplus b_j} \to Z_j \to C_j \to 0$$

such that $c^{t_j} \cdot C_j = 0$. Putting all of this together, we obtain a complex:

$$(*) \qquad 0 \to W^{\oplus b_d} \to W^{\oplus b_{d-1}} \to \cdots \to W^{\oplus b_0} \to V \to 0$$

with the following properties:

(a) the homology at $W^{\oplus b_d}$ is Z_{d+1} (i.e., $\mathrm{Ker}(W^{\oplus b_d} \to \omega^{\oplus b_{d-1}}) = Z_{d+1}$), and $(Z_{d+1})_c$ is a maximal Cohen-Macaulay module,

(b) for $0 \le j \le d-1$, the homology at $W^{\oplus b_j}$ is C_j and is therefore annihilated by c^{t_j}, and

(c) the homology at V is C_{-1} and is therefore annihilated by $c^{t_{-1}}$.

Letting $t := \max\{t_{-1}, \ldots, t_d\}$ we see that this complex satisfies property (a) of the definition of (c, W)-resolution. Therefore, the proof will be complete as soon as we know that $(Z_{d+1})_c$ is a direct summand of $W_c^{\oplus n}$ for some n.

To simplify notation, let $Z := Z_{d+1}$. Notice that since Z_c is a maximal Cohen-Macaulay module, we have that for all maximal ideals $n \in \mathrm{Spec}(R_c)$, Z_n is a maximal Cohen-Macaulay module over R_n. Therefore, $Z_n \cong W_n^{\oplus h}$ for some h where the h may depend on n. This is just the second statement of (IV.9d). Since the canonical module localizes over the Cohen-Macaulay ring R_c, it follows that for all $P \in Spec(R_c)$ there exists h such that $Z_P \cong W_P^{\oplus h}$.

Now, localizing the complex $(*)$ at c produces the following exact sequence

$$0 \to Z_c \to W_c^{\oplus b_d} \to \cdots \to W_c^{\oplus b_0} \to V_c \to 0.$$

Set $N := \mathrm{Hom}_{R_c}(W, V_c)$. Since V_c has finite injective dimension, N has finite projective dimension by (IV.9b). Further, by (IV.9a) and (IV.9c), applying $\mathrm{Hom}_{R_c}(W_c, -)$ to the above exact sequence produces the exact sequence

$$0 \to \mathrm{Hom}_{R_c}(W_c, Z_c) \to R_c^{\oplus b_d} \to \cdots \to R_c^{\oplus b_0} \to N \to 0.$$

Since $\dim(R) = d$, it follows at once that $\mathrm{Hom}_{R_c}(W_c, Z_c)$ is projective as it is a $(d+1)^{st}$ module of syzygies of N which has finite projective dimension. Therefore, there is an R_c-module Q such that $\mathrm{Hom}_{R_c}(W_c, Z_c) \oplus Q = R_c^{\oplus n}$ for some n. Applying $- \otimes_{R_c} W_c$, we find that

$$(\mathrm{Hom}_{R_c}(W_c, Z_c) \otimes_{R_c} W_c) \oplus (Q \otimes_{R_c} W_c) \cong R_c^{\oplus n} \otimes_{R_c} W_c \cong W_c^{\oplus n}.$$

Hence, $\mathrm{Hom}_{R_c}(W_c, Z_c) \otimes_{R_c} W_c$ is a direct summand of $W_c^{\oplus n}$. We claim we have an isomorphism

$$\mathrm{Hom}_{R_c}(W_c, Z_c) \otimes_{R_c} W_c \cong Z_c$$

and it is enough to check this locally. Indeed, if $P \in \mathrm{Spec}(R_c)$, then since Hom commutes with localization, the left hand side is $\mathrm{Hom}_{R_P}(W_P, Z_P) \otimes_{R_P} W_P$. Since $Z_P \cong W_P^{\oplus h}$, this is

$$\mathrm{Hom}_{R_P}(W_P, W_P^{\oplus h}) \otimes_{R_P} W_P \cong (R_P)^{\oplus h} \otimes_{R_P} W_P \cong W_P^{\oplus h} \cong Z_P$$

yielding the isomorphism. Therefore Z_c is a direct summand of $W_c^{\oplus n}$, as required.

It remains to prove the ($\Longleftarrow$) implication. Assume V has a finite (c, W)-resolution

$$0 \to W^{\oplus b_d} \to \cdots \to W^{\oplus b_0} \to V \to 0.$$

We must show W_c has finite injective dimension. Localizing at c produces the exact sequence

$$0 \to Z_c \to W_c^{\oplus b_d} \to \cdots \to W_c^{\oplus b_0} \to V_c \to 0.$$

If $d = 0$ then we have an exact sequence $0 \to Z_c \to W_c^{\oplus b_0} \to V_c \to 0$. Since W_c has finite injective dimension so does Z_c being a direct summand. But then Z_c and W_c have finite injective dimension, so V_c does as well. This handles the case $d = 0$. The result now follows from a straightforward induction on d. □

The next result is analogous to (IV.12). The proof is very similar.

Proposition IV.44. *Suppose (R, m, K) is a reduced, excellent local ring, let $c \in R$ and assume W is a c-canonical module. Suppose that V is a finitely generated R-module such that the injective dimension of V_c is finite. If W is (c, F)-injective then V is (c, F)-injective.*

Proof: By Proposition IV.43, there exists a complex

$$0 \to W^{\oplus b_d} \to \cdots \to W^{\oplus b_0} \to V \to 0$$

where $Z = \ker(\phi_d)$ is such that Z_c is a direct summand of $W_c^{\oplus n}$ for some n. By Proposition IV.8, Z is (c, F)-injective since W is. Furthermore, there exists $t \in \mathbb{N}$ such that the other homology modules are killed by c^t. We use induction on d.

If $d = 0$, then for some modules N, C we have two short exact sequences:

$$(1) \quad 0 \to Z \to W^{\oplus b_0} \to N \to 0$$

and

$$(2) \quad 0 \to N \to V \to C \to 0.$$

By hypothesis, W is (c, F)-injective, and so by Proposition IV.6(d), $W^{\oplus b_0}$ is (c, F)-injective. Therefore, by exact sequence (1) and Proposition IV.6(c), N is (c, F)-injective as well. Now, C is the homology of the complex at V and thus is killed by c^t. Therefore, C is (c, F)-injective by Proposition IV.6(a), so using exact sequence (2) and Proposition IV.6(b), we conclude V is (c, F)-injective.

For the inductive step, let $T := \mathrm{Ker}(W^{\oplus b_0} \to V)$. The complex

$$0 \to W^{\oplus b_d} \to \cdots \to W^{\oplus b_1} \to T \to 0$$

is a (c, W)-resolution for T of length $d - 1 < d$, as is easily verified. Therefore by the induction hypothesis, T is (c, F)-injective. We have an exact sequence $0 \to T \to W^{\oplus b_0} \to V \to C \to 0$ which breaks up into two short exact sequences. As before, since T, W, and C are (c, F)-injective so is V. $\square$

Proof of Theorem IV.41: By Lemma IV.25 we may assume that R is complete and therefore a homomorphic image of a Gorenstein ring. By Proposition IV.44 and Corollary IV.29 it is enough to show that the c-canonical module W constructed in (IV.32) is (c, F)-injective. This is proved in Proposition IV.39. $\square$

4.3 Potent Elements

In this section we use our uniform annihilator result for Ext modules to prove that the defining ideal of the non-finite injective dimension locus of a finitely generated R-modules W is potent for the Matlis dual of W. See Theorems IV.47 and IV.53 as well as Corollary IV.58. We recall the following notation from Chapter 2:

Notation IV.45. For any ideal $I \subseteq R$, for every integer $v \in \mathbb{N}$ and every R-module M, let $M_{(-v,I)} = \mathrm{Ann}_M(I^v)$. We sometimes write M_{-v} for $M_{(-v,I)}$ when I is clear from the context.

For $v' > v$, we note the following easy identities:

$$\left(\frac{M}{M_{-v}}\right)_{-v'} = \frac{M_{-v'}}{M_{-v}} = \left(\frac{M}{M_{-v}}\right)_{-(v'-v)}.$$

We also recall the definition of potent elements and potent ideals from Chapter 2:

Definition IV.46. If $x \in R$ is any element, and M is an (Artinian) R-module, we say that x is *potent for M* if we have

$$0^*_M = \bigcup_{n \in \mathbb{N}} 0^*_{M_{(-n,x)}}.$$

Similarly, we say that an ideal $I \subseteq R$ is **potent for M** if we have

$$0^*_M = \bigcup_{n \in \mathbb{N}} 0^*_{M_{(-n,I)}}.$$

If x (respectively, I) is potent for every Artinian R-module we simply say x (respectively, I) is *potent.*

We now state our first main result on the existence of potent elements.

Theorem IV.47. *Let (R, m, K) be a reduced, F-finite, excellent local ring, and let W be a finitely generated R-module. Set $E := E_R(K)$, the injective hull of the residue class field, and let $M := Hom_R(W, E)$. For any $c \in R$, if W_c has finite injective dimension over R_c, then c is potent for M.*

Before giving the proof, we need several preliminary results. The first is well known.

Lemma IV.48. *Let (R, m, K) be a local ring, let $E = E_R(K)$ be the injective hull of the residue class field, let M, N, W be R-modules, and let $^\vee$ be the functor* $\operatorname{Hom}_R(-, E)$.

(a) $\operatorname{Ext}^i_R(N, M^\vee) \cong \operatorname{Tor}_i^R(N, M)^\vee$ *for every* $i \in \mathbb{N}$.

(b) If $I \subseteq R$ is a finitely generated ideal, then $\operatorname{Ann}_M(I)^\vee \cong \frac{M^\vee}{IM^\vee}$.

(c) If $I \subseteq R$ is a finitely generated ideal, then $(M/M_{(-b,I)})^\vee \cong I^b M^\vee$.

(d) Suppose $b' > b$ are integers and assume that R is complete. Let W be an R-module and let $M := W^\vee$. The natural map

$$\operatorname{Ext}^i_R(N, I^b W) \to \operatorname{Ext}^i_R(N, I^b W/I^{b'} W)$$

is injective if and only if the natural map

$$\operatorname{Tor}_i^R(N, (M/M_{(-b,I)})_{(b'-b,I)}) \to \operatorname{Tor}_i^R(N, M/M_{(-b,I)})$$

is surjective.

Proof: (a). Let $\mathcal{P}_\bullet$ be a projective resolution of N. Then

$$\operatorname{Tor}_i^R(N, M) := H_i(\mathcal{P}_\bullet \otimes_R M).$$

Applying $\mathrm{Hom}_R(-,E)$ shows that

$$\mathrm{Tor}_i^R(N,M)^\vee \cong \mathrm{Hom}_R(H_i(\mathcal{P}_\bullet \otimes M),E) \cong H^i(\mathrm{Hom}_R(\mathcal{P}_\bullet \otimes M,E)),$$

the last isomorphism holding because E is injective. By the adjointness of tensor and Hom this is

$$\cong H^i(\mathrm{Hom}(\mathcal{P}_\bullet,\mathrm{Hom}(M,E))) \cong H^i(\mathrm{Hom}(\mathcal{P}_\bullet,M^\vee)) =: \mathrm{Ext}_R^i(N,M^\vee).$$

This proves (a).

(b). Suppose $I=(u_1,\ldots,u_h)R$. There is an exact sequence

$$0 \to \mathrm{Ann}_M(I) \to M \xrightarrow{f} M^{\oplus h}$$

where the map f is given by the matrix $[u_1,\ldots,u_h]$. That is, for $x \in M$, $f(x)=(u_1x,\ldots,u_hx)$. When we apply $\mathrm{Hom}_R(-,E)$ we get an exact sequence

$$(M^{\oplus h})^\vee \xrightarrow{f^\vee} M^\vee \to (\mathrm{Ann}_M(I))^\vee \to 0$$

where $f^\vee$ is given by the matrix $[u_1,\ldots,u_h]$ as well. Since $(\mathrm{Ann}_M(I))^\vee$ is the cokernel of $f^\vee$, it follows that $(\mathrm{Ann}_M(I))^\vee = M^\vee/\operatorname{Im}(f^\vee) = M^\vee/IM^\vee$.

(c). Consider the short exact sequence

$$0 \to M_{(-b,I)} \to M \to \frac{M}{M_{(-b,I)}} \to 0.$$

Applying $^\vee$, we get a short exact sequence

$$0 \to \left(\frac{M}{M_{(-b,I)}}\right)^\vee \to M^\vee \to M^\vee_{(-b,I)} \to 0,$$

which, by the previous result we can rewrite as

$$0 \to \left(\frac{M}{M_{(-b,I)}}\right)^\vee \to W \to W/I^bW \to 0$$

where $W = M^\vee$. The result follows.

(d). By Matlis duality, $M^\vee = W$. Let

$$\operatorname{Tor}_i^R(N, (M/M_{(-b,I)})_{(b'-b,I)}) \to \operatorname{Tor}_i^R(N, M/M_{(-b,I)}) \to C \to 0$$

be exact so that the relevant map is surjective if and only if $C = 0$. Then, applying $^\vee$, we get an exact sequence

$$\operatorname{Tor}_i^R(N, (M/M_{(-b,I)})_{(b'-b,I)})^\vee \leftarrow \operatorname{Tor}_i^R(N, M/M_{(-b,I)})^\vee \leftarrow C^\vee \leftarrow 0$$

and by part (a) this is

$$(*) \qquad \operatorname{Ext}^i_R(N, ((M/M_{(-b,I)})_{(b'-b,I)})^\vee) \leftarrow \operatorname{Ext}^i_R(N, (M/M_{(-b,I)})^\vee) \leftarrow C^\vee \leftarrow 0.$$

Now, $(M/M_{(-b,I)})^\vee = W/I^bW$ by part (c), so by part (b) applied to $M/M_{(-b,I)}$ we get that $((M/M_{(-b,I)})_{(b'-b,I)})^\vee = I^bW/I^{b'}W$. Therefore, $(*)$ may be rewritten as

$$\operatorname{Ext}^i_R(N, I^bW/I^{b'}W) \leftarrow \operatorname{Ext}^i_R(N, W/I^bW) \leftarrow C^\vee \leftarrow 0.$$

Since $C = 0$ if and only if $C^\vee = 0$, the result follows. $\square$

The next result is a dual form of [LS01], Lemma 8.3, in the language of potent elements.

Lemma IV.49. *Let (R, m, K) be a complete, reduced local ring and put $E := E_R(K)$, the injective hull of the residue class field. Let W be a finitely generated R-module, let $M := \operatorname{Hom}_R(W, E)$, and let $c \in R$. If for all $b \gg 0$ there exists $b' > b$ such that for all q, the natural map*

$$\operatorname{Ext}^1_R(R^{1/q}, c^bW) \to \operatorname{Ext}^1_R(R^{1/q}, \frac{c^bW}{c^{b'}W})$$

is injective, then c is potent for M.

Proof: The short exact sequences

$$0 \to M_{-b} \to M \to M/M_{-b} \to 0$$

and

$$0 \to M_{-b} \to M_{-b'} \to (M/M_{-b})_{-(b'-b)} \to 0$$

induce the following commutative diagram with exact rows:

$$\begin{array}{ccccc} \mathrm{Tor}_1^R(R^{1/q}, (M/M_{-b})_{-(b'-b)}) & \longrightarrow & R^{1/q} \otimes_R M_{-b} & \xrightarrow{a_q} & R^{1/q} \otimes_R M_{-b'} \\ f_q \downarrow & & \mathbb{1} \downarrow & & g_q \downarrow \\ \mathrm{Tor}_1^R(R^{1/q}, (M/M_{-b})) & \longrightarrow & R^{1/q} \otimes_R M_{-b} & \xrightarrow{d_q} & R^{1/q} \otimes_R M \end{array}$$

If the map f_q is surjective for all q, then g_q restricts to an isomorphism between $\mathrm{Im}(a_q)$ and $\mathrm{Im}(d_q)$ for all q as the reader will easily verify. Then, if $x \in 0^*_M$ and $x \in M_{-b}$ then $x \in 0^*_{M_{-b'}}$. Therefore, it is enough to show that for all $b \gg 0$ there exists b' such that for all q the maps f_q are surjective. But this follows from the hypothesis and part (d) of the previous lemma. $\square$

Lemma IV.50. *Let $c \in R$ and let M, W be two R-modules. If $c^t \cdot \mathrm{Ext}^i_R(M, W) = 0$ then $c^t \cdot \mathrm{Ext}^i_R(M, c^n W) = 0$ for all $n \geq 1$.*

Proof: Let $\mathcal{F}_\bullet$ be a free resolution of M. We can compute the two Ext modules in question by applying $\mathrm{Hom}_R(-, W)$ and $\mathrm{Hom}_R(-, c^n W)$. The result (at the i^{th} spot) is the following commutative diagram:

$$\begin{array}{ccccc} W_{i+1} & \overset{\alpha_i}{\leftarrow} & W_i & \overset{\alpha_{i-1}}{\leftarrow} & W_{i-1} \\ \uparrow & & \uparrow & & \uparrow \\ c^n W_{i+1} & \leftarrow & c^n W_i & \leftarrow & c^n W_{i-1} \end{array}$$

If we let $Z = \ker(\alpha_i)$ and $B = \mathrm{Im}(\alpha_{i-1})$, then $\mathrm{Ext}^i_R(M, W) = Z/B$ and, by the commutativity of the diagram, $\mathrm{Ext}^i_R(M, c^n W) = c^n Z / c^n B$. The hypothesis therefore tells us that $c^t Z \subseteq B$. But then $c^t(c^n Z) = c^n(c^t Z) \subseteq c^n B$ and so c^t annihilates $\mathrm{Ext}^i_R(M, c^n W)$, as desired. $\square$

Theorem IV.51. *Let R be a (not necessarily local) Noetherian ring and let W be a finitely generated R-module.*

(a) Suppose R is equidimensional and a homomorphic image of a regular ring and let $P \in \mathrm{Spec}(R)$ such that $\mathrm{id}_{R_P}(W_P) < \infty$. Then there exists $c \in R - P$ such that $\mathrm{id}_{R_c}(W_c) < \infty$. It follows that the set $V := \{P \in \mathrm{Spec}(R) \mid \mathrm{id}_{R_P}(W_P) < \infty\}$ is an open set in this case. Therefore, there is a radical ideal $I \subseteq R$ such that $c \in I$ if and only if $\mathrm{id}_{R_c}(W_c) < \infty$.

(b) Let (R, m, K) be an excellent, local ring. The set

$$U := \{P \in \mathrm{Spec}(R) \mid \mathrm{id}_{R_P}(W_P) < \infty\}$$

is an open set and hence, there is a radical ideal $J \subseteq R$ such that $c \in J$ if and only if $\mathrm{id}_{R_c}(W_c) < \infty$.

(c) If (R, m, K) is an excellent local ring and $J \subseteq R$ is the defining ideal of the non-finite injective dimension locus of W (as in (b)), then $J\widehat{R}$ is the defining ideal of the non-finite injective dimension locus of $\widehat{W} = W \otimes_R \widehat{R}$, where $\widehat{R}$ is the completion of R with respect to the maximal ideal.

Proof: Let ω denote a canonical module for R, and suppose $\mathrm{id}_{R_P}(W_P)$ is finite. If $W_P = 0$ then because W is finitely generated there exists $c \in R - P$ such that $W_c = 0$ and we are done. We may assume $W_P \neq 0$. Then since R is equidimensional, ω_P is a canonical module for R_P which is Cohen-Macaulay since it possess a finitely generated, nonzero module of finite injective dimension. Therefore, there exists an exact sequence

$$(*)\ \ 0 \to \omega_P^{\oplus b_h} \xrightarrow{\alpha_h} \cdots \xrightarrow{\alpha_1} \omega_P^{\oplus b_0} \to W_P \to 0$$

The maps α_j are given by matrices with entries in R_P since $\operatorname{Hom}_{R_P}(\omega_P, \omega_P) \cong R_P$. Let c_1 be the product of all elements appearing in denominators of the matrices for the maps α_j, $1 \le j \le h$. Then $c_1 \in R - P$ and we may form

$$0 \to \omega_{c_1}^{\oplus b_h} \xrightarrow{\alpha_h} \cdots \xrightarrow{\alpha_1} \omega_{c_1}^{\oplus b_0} \to W_{c_1} \to 0.$$

The maps $\alpha_j : \omega_{c_1}^{\oplus b_j} \to \omega_{c_1}^{\oplus b_{j-1}}$ are given by the exact same matrices as the original α_j and when we localize at P, we obtain the exact sequence $(*)$. Because the homology is finitely generated, it follows that all the homology modules, including the homology at W_{c_1}, are all killed by an element $c_2 \in R - P$. If we let $c := c_1 c_2$ it then follows that

$$0 \to \omega_c^{\oplus b_h} \xrightarrow{\alpha_h} \cdots \xrightarrow{\alpha_1} \omega_c^{\oplus b_0} \to W_c \to 0$$

is exact and so W_c has finite injective dimension over R_c. This shows that the set V is open, and it follows at once that $\{Q \in \operatorname{Spec}(R) \mid \operatorname{id}(W_Q) = \infty\}$ is a closed set and therefore defined by a (radical) ideal.

(b). Let $P \in \operatorname{Spec}(R)$ such that $\operatorname{id}_{R_P}(W_P) < \infty$, let $\widehat{R}$ denote the completion of R at the maximal ideal, and consider the multiplicative system $U := R - P \subseteq \widehat{R}$. Then $\widehat{R}$ and thus $U^{-1}\widehat{R}$ are homomorphic images of regular rings and the map $R_P \to U^{-1}\widehat{R}$ is faithfully flat with regular fibers. Furthermore, $U^{-1}\widehat{R}$ has finite Krull dimension, so from part (c) of Lemma IV.25, $\operatorname{id}_{U^{-1}\widehat{R}}(W \otimes_R U^{-1}\widehat{R}) < \infty$. It follows that $U^{-1}\widehat{R}$ is Cohen-Macaulay and since the Cohen-Macaulay locus is open, there exists $c_1 \in R - P$ such that $\widehat{R}_{c_1}$ is Cohen-Macaulay. By part (a) of this result applied to $\widehat{R}_{c_1}$, there exists $c_2 \in R-P$ such that, with $c = c_1c_2$, $\mathrm{id}_{\widehat{R}_c}(W \otimes \widehat{R}_c) < \infty$. But the map $R_c \to \widehat{R}_c$ is faithfully flat with regular fibers, so by part (c) of Lemma IV.25, $\mathrm{id}_{R_c}(W_c) < \infty$. Since $c \in R-P$, this proves that U is an open set. The other statements then follow.

(c). If I is defining the non-finite injective dimension locus of $\widehat{W}$, then we have to show that $J\widehat{R} = I$. First note that for a prime $P \in \mathrm{Spec}(R)$,

$$\mathrm{id}_{R_P}(W_P) < \infty \Longleftrightarrow \mathrm{id}_{\widehat{R}_Q}(\widehat{W_Q}) < \infty$$

for some (equivalently, all) minimal primes Q of $P\widehat{R}$: this is immediate from Lemma IV.25(c), since $R_P \to \widehat{R}_Q$ is faithfully flat with Gorenstein fibers.

Since J is a radical ideal, $J = P_1 \cap \cdots \cap P_h$ for some primes $P_i \in \mathrm{Spec}(R)$. By the fact that R is excellent, $P_i\widehat{R}$ is a radical ideal. Let $P_i\widehat{R} = \bigcap Q_{ij}$. Then since $R \to \widehat{R}$ is flat, we have that $J\widehat{R} = \bigcap Q_{ij}$. Now, by our remark above, $\widehat{W}_{Q_{ij}}$ does not have finite injective dimension and it follows that $I \subseteq \bigcap Q_{ij} = J\widehat{R}$. On the other hand, if Q is a minimal prime of I, then $\widehat{W}_Q$ does not have finite injective dimension. But the contraction $Q \cap R$ is a prime of R, and so again by the remark above, the injective dimension of $W_{Q\cap R}$ is infinite. Therefore, $J \subseteq Q \cap R$, whence $J\widehat{R} \subseteq (Q \cap R)\widehat{R} \subseteq Q$. We have just showed that every minimal prime of I contains $J\widehat{R}$. Since I is radical, it follows that $J\widehat{R} \subseteq I$, so $J\widehat{R} = I$. $\square$

.

Proof of Theorem IV.47: Since R is excellent and reduced, $\widehat{R}$ is also reduced. $\widehat{R}$ is F-finite since R is (R is F-finite $\Longrightarrow$ $R/m = \widehat{R}/\widehat{m}$ is F-finite since it is a homomorphic image, and this implies $\widehat{R}$ is F-finite since $\widehat{R}$ is complete). The hypothesis that W_c has finite injective dimension is preserved when we replace W with $\widehat{W}$ by

Proposition IV.51. Therefore all of the hypothesis are preserved when we pass to $\widehat{R}$. So by (II.13b), we may assume that R is complete.

By Lemma IV.49 it is enough to show that for all $b \gg 0$ there exists b' such that the natural map

$$i_q : \mathrm{Ext}^1_R(R^{1/q}, c^b W) \to \mathrm{Ext}^1_R(R^{1/q}, \frac{c^b W}{c^{b'} W})$$

is injective for all q. Notice that since W is finitely generated, there exists $b_1 \in \mathbb{N}$ such that $H^0_{(c)}(W) = \mathrm{Ann}_W(c^{b_1})$. Then c is not a zerodivisor on $c^b W$ for all $b > b_1$ since $c \cdot c^b z = 0 \implies z \in H^0_{(c)}(W) \implies c^b z = 0$. Without loss of generality, assume $b > b_1$.

By Theorem IV.41, there exists k such that for all q,

$$c^k \cdot \mathrm{Ext}^1_R(R^{1/q}, W) = 0$$

and so by Lemma IV.50,

$$c^k \cdot \mathrm{Ext}^1_R(R^{1/q}, c^b W) = 0$$

for all q. Set $b' := b + k$. Since c is a nonzerodivisor on $c^b W$, there is a short exact sequence

$$0 \to c^b W \xrightarrow{\cdot c^k} c^b W \to \frac{c^b W}{c^{b'} W} \to 0.$$

The long exact sequence for Ext produces the exact sequence

$$\mathrm{Ext}^1_R(R^{1/q}, c^b W) \xrightarrow{\cdot c^k} \mathrm{Ext}^1_R(R^{1/q}, c^b W) \xrightarrow{i_q} \mathrm{Ext}^1_R(R^{1/q}, \frac{c^b W}{c^{b'} W})$$

for all q. Since c^k kills $\mathrm{Ext}^1_R(R^{1/q}, c^b W)$ for all q, it follows that i_q is injective for all q. $\square$

Lemma IV.52. *Let $I \subseteq R$ be an ideal and W a finitely generated R-module. Put $\bar{W} := W/H_I^0(W)$. Then either $\bar{W} = 0$ or there exists $c \in I$ not a zerodivisor on $\bar{W}$.*

Proof: We claim it is enough to see that if $I \subseteq \bigcup_{P \in \mathrm{Ass}(\bar{W})} P$ then $\bar{W} = 0$. Indeed, if every element of I is a zerodivisor on $\bar{W}$, then $I \subseteq \bigcup_{P \in \mathrm{Ass}(\bar{W})} P$, and then by the claim $\bar{W} = 0$.

Assume for contradiction that $\bar{W} \neq 0$ but $I \subseteq \bigcup_{P \in \mathrm{Ass}(\bar{W})} P$. By Proposition 3.13 of [Eis95], $\mathrm{Ass}(\bar{W}) = \mathrm{Ass}(W) - A$ where $A = \{P \in \mathrm{Ass}(W) \ : \ I \subseteq P\}$. In particular, no associated prime of $\bar{W}$ contains I. Then $\mathrm{Ass}(\bar{W})$ is a finite, non-empty set or primes, none of which contain I. By prime avoidance, I is not contained in the union, a contradiction. $\square$

Theorem IV.53. *Let (R, m, K) be a reduced, F-finite, excellent local ring, and let W be a finitely generated R-module. Set $E := E_R(K)$, the injective hull of the residue class field, and let I be an ideal contained in the defining ideal of the non-finite injective dimension locus of W. Let $M := \mathrm{Hom}_R(W, E)$. Then I is potent for M.*

Proof: By (II.17b), (IV.51c) and the excellence hypothesis on R, we may assume R is complete. Let $z \in 0^*_M$. We will show $z \in 0^*_{M_{(-n,I)}}$ for some n. Let $\bar{W} := W/H_I^0(W)$.

We use induction on $\mathrm{depth}_I(\bar{W})$. If $\mathrm{depth}_I(\bar{W}) = 0$ then $\bar{W} = 0$ and then $I^n W = 0$ for some n. It follows that $I^n M = 0$ and so I is potent for M. Now assume $\mathrm{depth}_I(\bar{W}) > 0$ so that there exists $x \in I$ such that x is a nonzerodivisor on $\bar{W}$. Since $x \in I$, W_x has finite injective dimension over R_x, and by Theorem IV.47, x is potent for M. So $z \in 0^*_{M_{(-n_1,x)}}$ for some n_1.

Let P be a prime of R not containing the ideal I. Then $W_P = \bar{W}_P$ and we see that both have finite injective dimension over R_P. Since x is a nonzerodivisor on $\bar{W}$, there is a short exact sequence

$$0 \to \bar{W} \xrightarrow{\cdot x^{n_1}} \bar{W} \to \bar{W}/x^{n_1}\bar{W} \to 0$$

and hence

$$0 \to \bar{W}_P \xrightarrow{\cdot x^{n_1}} \bar{W}_P \to (\bar{W}/x^{n_1}\bar{W})_P \to 0.$$

Therefore, $(\bar{W}/x^{n_1}\bar{W})_P$ has finite injective dimension over R_P. Again, since $I \not\subseteq P$, $W_P = \bar{W}_P$ and hence $W_P/x^{n_1}W_P = \bar{W}_P/x^{n_1}\bar{W}_P$. It follows that $W_P/x^{n_1}W_P$ has finite injective dimension over R_P and so I is contained in a defining ideal of the non-finite injective dimension locus for $W/x^{n_1}W$. By the induction hypothesis, I is potent for $\mathrm{Hom}_R(W/x^{n_1}W, E)$.

Note that $\mathrm{Hom}_R(M, E) = W$ since R is complete. Therefore by Lemma IV.48(b),

$$\mathrm{Hom}_R(M_{(-n_1,x)}, E) = W/x^{n_1}W,$$

and hence, $\mathrm{Hom}_R(W/x^{n_1}W, E) = M_{(-n_1,x)}$. It follows that I is potent for $M_{(-n_1,x)}$. Therefore:

$$z \in 0^*_{M_{(-n_1,x)}} = \bigcup_{t \in \mathbb{N}} 0^*_{Ann_M(x^{n_1}+I^t)}$$

which completes the proof of the theorem. □

Using the technique of the gamma construction developed in Section 6 of [HH94] we are able to remove the F-finiteness assumption from Theorem IV.53. Below we collect some of the salient features of this construction. But first we prove a base change result for potent ideals. The base change result requires the following lemma:

Lemma IV.54. *Suppose $(R, m, K) \to (S, n, L)$ is a flat map of local rings.*

(a) If M is any R-module and $I \subseteq R$ is an ideal, then $\operatorname{Ann}_{M \otimes_R S}(IS) = \operatorname{Ann}_M(I) \otimes_R S$. (We are tacitly identifying $\operatorname{Ann}_M(I) \otimes_R S$ with its image in $M \otimes S$).

(b) If in addition $mS = n$, then $E_S(L) = E_R(K) \otimes_R S$.

Proof: (a). Let $I = (f_1, \dots, f_h)$ and consider the map $M \to M^{\oplus h}$ where for $x \in M$, $x \mapsto (f_1 x, \dots, f_h x)$. Then we have a short exact sequence

$$0 \to \operatorname{Ann}_M(I) \to M \to M^{\oplus h} \to 0$$

and since $R \to S$ is flat, tensoring with S produces

$$0 \to \operatorname{Ann}_M(I) \otimes S \to M \otimes S \to (M \otimes S)^{\oplus h} \to 0$$

exact. The result follows.

(b). This is a special case of Lemma 7.10(d) from [HH94]. □

Proposition IV.55. *Let $(R, m, K) \to (S, n, L)$ be a flat local map, and assume R and S have a common test element. Let $I \subseteq R$ be an ideal and let M be any R-module.*

(a) If IS is potent for $M \otimes_R S$, then I is potent for M.

(b) If additionally $mS = n$, W is a finitely generated R-module, and IS is potent for $\operatorname{Hom}_S(W \otimes_R S, E_S(L))$ then I is potent for $\operatorname{Hom}_R(W, E_R(K))$.

Proof: (a). Suppose $z \in 0^*_M$ and we want to show $z \in 0^*_{\operatorname{Ann}_M(I^t)}$ for some t. Let c be a common test element for R and S. Then $cz^q = 0$ in $F^e(M)$, so $cz^q = 0$ in $F^e_S(S \otimes_R M) = S \otimes_R F^e(M)$. (Note that in complete generality, without any assumptions on the map $R \to S$, we have that $F^e_S(S \otimes_R M) \cong S \otimes_R F^e_R(M)$ - this is just the commutative diagram:

$$\begin{array}{ccc} F^e(R) & \to & F^e(S) \\ \uparrow & & \uparrow \\ R & \to & S \end{array}$$

together with the associativity of base change.) It follows that $z \in 0^*_{M\otimes S}$ so, since IS is potent for $M \otimes S$ there exists t such that $z \in 0^*_{\mathrm{Ann}_{M\otimes S}(IS^t)}$. However, since $I^tS \subseteq IS^t$, it follows that $z \in 0^*_{\mathrm{Ann}_{M\otimes S}(I^tS)}$. Therefore, $cz^q = 0$ in $F^e_S(\mathrm{Ann}_{M\otimes S}(I^tS))$. But $R \to S$ is flat, so $\mathrm{Ann}_{M\otimes S}(I^tS) = \mathrm{Ann}_M(I^t) \otimes S$, so $cz^q = 0$ in $F^e_S(\mathrm{Ann}_M(I^t) \otimes S) \cong S \otimes F^e_R(\mathrm{Ann}_M(I^t))$. Finally, since $R \to S$ is flat and local, it is pure, so the map $F^e_R(\mathrm{Ann}_M(I^t)) \to F^e_R(\mathrm{Ann}_M(I^t)) \otimes_R S$ is injective. Therefore, $cz^q = 0$ in $F^e_R(\mathrm{Ann}_M(I^t))$ and so $z \in 0^*_{\mathrm{Ann}_M(I^t)}$ as required.

(b). By (IV.54b), $\mathrm{Hom}_S(W \otimes_R S, E_S(L)) \cong \mathrm{Hom}_S(W \otimes_R S, E_R(K) \otimes_R S)$ and since $R \to S$ is flat and W is finitely presented, this is $\cong S \otimes_R \mathrm{Hom}_R(W, E_R(K))$. The result now follows from part (a). $\square$

In the next result we collect some of the important properties of the gamma construction from Section 6 of [HH94]. For simplicity, we assume that R is a complete local ring. The reader is referred to [HH94] for details and generalizations. First we adopt some notation from that paper.

Let (R, m, K) be a complete local ring of characteristic $p > 0$. Fix a coefficient field $K \subseteq R$ and fix a p-base $\Lambda \subseteq K$ for K. Then for each cofinite subset $\Gamma \subseteq \Lambda$, the gamma construction produces a local R-algebra $(R^\Gamma, m^\Gamma, K^\Gamma)$ and we have the following:

Proposition IV.56. *Let (R, m, K) and $(R^\Gamma, m^\Gamma, K^\Gamma)$ be as above. Then:*

(a) For all Γ, R^Γ is an F-finite (and therefore, excellent), Henselian ring.

(b) For all Γ, R^Γ is faithfully flat and purely inseparable over R (the latter meaning that every element of R^Γ has a q^{th} power in R) with Gorenstein fibers, and $mR^\Gamma = m^\Gamma$.

(c) For all Γ, R^Γ is a homomorphic image of a regular ring.

(d) For all choices of Γ sufficiently small, if R is reduced (respectively, a domain) then R^Γ is reduced (respectively, a domain). Furthermore, for all Γ sufficiently small, if R is equidimensional then R^Γ is equidimensional.

Proof: Parts (a) through (c) follow from 6.6, 6.8 and the discussion in 6.11 of [HH94]. Part (d) follows from 6.13(b) of [HH94] (apply 6.13(b) to every minimal prime of R). □

Lemma IV.57. *Let $(R, m, K) \to (S, n, L)$ be a flat map of local rings such that $mS = n$. Assume R has a canonical module ω. Then $\omega \otimes_R S$ is a canonical module for S.*

Proof: This is a special case of [BH93], Theorem 3.3.14(a). □

Corollary IV.58. *Let (R, m, K) be a reduced, excellent local ring, and let W be a finitely generated R-module. Set $E := E_R(K)$, the injective hull of the residue class field, and let I be an ideal contained in the defining ideal of the non-finite injective dimension locus of W. Let $M := \mathrm{Hom}_R(W, E)$. Then I is potent for M.*

Proof: Without loss of generality we may assume that R is complete. We want to replace R, W with $R^\Gamma, W \otimes_R R^\Gamma$, and by (IV.55) and (IV.56) we can do this as soon as we know that IR^Γ is contained in the defining ideal of the non-finite injective dimension locus for $W \otimes R^\Gamma$. We show this now.

By (IV.51b) there is a radical ideal defining the non-finite injective dimension locus for $W \otimes R^\Gamma$. Since R is complete, R has a canonical module, say ω, and then $\omega_{R^\Gamma} = \omega_R \otimes_R R^\Gamma$ is a canonical module for R^Γ by (IV.57). Since IS is the smallest ideal of S containing I, it is enough to show that for all $c \in I$, if $\mathrm{id}_{R_c}(W_c) < \infty$, then $\mathrm{id}_{R_c^\Gamma}(W \otimes R_c^\Gamma) < \infty$. But since $\mathrm{id}_{R_c}(W_c) < \infty$, W_c has a resolution:

$$0 \to \omega_c^{\oplus b_d} \to \cdots \to \omega_c^{\oplus b_0} \to W_c \to 0,$$

and applying $- \otimes R_c^\Gamma$ then produces:

$$0 \to \omega_{R_c^\Gamma}^{\oplus b_d} \to \cdots \to \omega_{R_c^\Gamma}^{\oplus b_0} \to W_c \otimes R_c^\Gamma \to 0.$$

It follows that $\mathrm{id}_{R_c^\Gamma}(W_c \otimes R_c^\Gamma) < \infty$. Therefore we may replace R, W with $R^\Gamma, W \otimes R^\Gamma$. Since R^Γ is F-finite, the result follows from Theorem IV.53. This completes the proof. $\square$

CHAPTER V

Potent Elements for Local Cohomology Modules

In this chapter we study potent elements for local cohomology modules. To describe out main result, let (R, m, K) be an excellent, local ring and let M be a finitely generated R-module. Then the locus of primes $P \in \operatorname{Spec}(R)$ such that M_P is Cohen-Macaulay and has finite projective dimension over R_P is known to be an open set (it is the intersection of two open sets; cf., Corollary 9.4.7 of [BS98] and section 24 of [Mat89]). The main result of this chapter is that the defining ideal for the closed complement of this open set is potent for certain local cohomology modules. More precisely, we have

Theorem V.1. *Let (R, m, K) be a reduced, excellent, equidimensional local ring of dimension d and let M be a finitely generated R-module. Let $I \subseteq R$ be the radical ideal such that $c \in I$ if and only if M_c is Cohen-Macaulay and has finite projective dimension over R_c. Then I is potent for $H^k_m(M)$, the k^{th} local cohomology module with support in m, for every $k \in \mathbb{N}$.*

One key ingredient is the following well known corollary of the Peskine-Szpiro intersection theorem.

Proposition V.2. *Let R be a Noetherian local ring containing a field and let $M \neq 0$ be a finitely generated R-module. If M is Cohen-Macaulay and has finite projective* *dimension then R is Cohen-Macaulay.*

Proof: By [BH93], Corollary 9.4.6, with $N := R$ we have

$$\dim(R) \leq \mathrm{pd}(M) + \dim(M).$$

Since M is Cohen-Macaulay, by the Auslander-Buchsbaum formula, $\dim(R) \leq \mathrm{depth}(R)$. The other inequality is always satisfied. $\square$

The proof of Theorem V.1 relies on the main result of Chapter 4, which appears as Corollary IV.58 there:

Theorem V.3. *Let (R, m, K) be a reduced, excellent local ring, and let W be a finitely generated R-module. Set $E := E_R(K)$, the injective hull of the residue class field, and let $I \subseteq R$ be an ideal contained in the defining ideal of the non-finite injective dimension locus of W. Let $H := \mathrm{Hom}_R(W, E)$. Then I is potent for H.*

We will make use of the dualizing complex. The next proposition recalls some of its salient features.

Proposition V.4. *Let (R, m, K) be an equidimensional local ring of dimension d and let (T, n, L) be a Gorenstein local ring such that $T \twoheadrightarrow R$. By killing a maximal regular sequence in $\mathrm{Ker}(T \to R)$ we may assume $\dim(T) = d$. Let $P \in \mathrm{Spec}(R)$ and assume R_P is Cohen-Macaulay. Let ω be a canonical module for R. Then there is a complex:*

$$\mathcal{D}^\bullet \ : \ 0 \to \omega \to D^0 \to \cdots \to D^d \to 0$$

such that

(i) D^j is an injective R-module for all $0 \leq j \leq d$.

(ii) The result of localizing $\mathcal{D}^\bullet$ at P is the exact sequence

$$0 \to \omega_P \to D_P^0 \to \cdots \to D_P^{\dim(R_P)} \to 0,$$

and this is the minimal injective resolution of the canonical module, ω_P, for R_P.

(iii) For any finitely generated R-module N, we have an isomorphism

$$H_m^j(N) \cong H^{d-j}(\mathrm{Hom}_R(N, \mathcal{D}^\bullet)^\vee$$

for all $0 \leq j \leq d$, where $-^\vee$ is the functor $\mathrm{Hom}_R(-, E_R(K))$.

Proof: See for example, IV.14 and IV.20. □

We will require a lemma on the behavior of Ext.

Lemma V.5. *Let (R, m, K) be a Cohen-Macaulay local ring of dimension d with canonical module ω. Let M be a finitely generated Cohen-Macaulay R-module of dimension k. Then $\mathrm{Ext}_R^i(M, \omega) = 0$ for $i \neq d-k$ while $\mathrm{Ext}_R^{d-k}(M, \omega) \neq 0$.*

If in addition M has finite projective dimension, then $\mathrm{pd}(M) = d - k$ and so $\mathrm{Ext}_R^{\mathrm{pd}(M)}(M, \omega)$ is the unique non-vanshing $\mathrm{Ext}_R^i(M, \omega)$. Moreover, $\mathrm{Ext}_R^{\mathrm{pd}(M)}(M, \omega)$ has finite injective dimension over R.

Proof: The first statement is given in 3.3.3(b) of [BH93]. We prove the second statement. By the Auslander-Buchsbaum formula, the projective dimension of M is $d - k$. Let

$$0 \to R^{\oplus b_{d-k}} \to \cdots \to R^{\oplus b_0} \to M \to 0$$

give a free resolution of M. Dropping M and applying $\mathrm{Hom}(-, \omega)$ yields the complex

$$0 \leftarrow \omega^{\oplus b_{d-k}} \leftarrow \cdots \leftarrow \omega^{\oplus b_0} \leftarrow 0,$$

whose homology at the i^{th} spot is $\operatorname{Ext}^i_R(M,\omega)$. However, by the first part of this lemma, these modules vanish except when $i = d-k$. It follows that the complex

$$0 \leftarrow \operatorname{Ext}^{d-k}_R(M,\omega) \leftarrow \omega^{\oplus b_{d-k}} \leftarrow \cdots \leftarrow \omega^{\oplus b_0} \leftarrow 0$$

is exact. By 3.11(ii) of [Sha72], $\operatorname{Ext}^{d-k}_R(M,\omega)$ has finite injective dimension. $\square$

To prove (V.1) we first reduce to the complete case to guarantee the existence of a canonical module. We collect the necessary results in a proposition.

Proposition V.6. *Let (R,m,K) be an excellent, local ring and M a finitely generated R-module. Let $\widehat{R}$ denote the completion of R at the maximal ideal.*

(a) If I is the defining ideal of the non-Cohen-Macaulay locus of M then $I\widehat{R}$ is the defining ideal of the non-Cohen-Macaulay locus of $\widehat{M}$.

(b) Similarly, if J is the defining ideal of the non-finite projective dimension locus of M then $J\widehat{R}$ is the defining ideal of the non-finite projective dimension locus for $\widehat{M}$.

Proof: Let $Q \in \operatorname{Spec}(\widehat{R})$ and let $P := Q \cap R$ be the contraction of Q to R. Then P is a prime ideal of R and note that $P \in V(I) \Longleftrightarrow Q \in V(I\widehat{R})$. The map $R_P \to \widehat{R}_Q$ is flat and local, and since R is excellent, $\kappa_P \otimes_R \widehat{R}_Q = \widehat{\kappa}_Q$, where $\kappa_P = R_P/PR_P$ and $\widehat{\kappa}_Q = \widehat{R}_Q/Q\widehat{R}_Q$.

By [BH93], 2.1.7, M_P is Cohen-Macaulay $\Longleftrightarrow$ $\widehat{M}_Q$ is Cohen-Macaulay. Part (a) then follows since $\widehat{M}_Q$ is Cohen-Macaulay $\Longrightarrow$ M_P is Cohen-Macaulay $\Longrightarrow$ $P \notin V(I)$ $\Longrightarrow$ $Q \notin V(I\widehat{R})$. Conversely, if $\widehat{M}_Q$ is not Cohen-Macaulay, then M_P is not Cohen-Macaulay, so $P \in V(I)$ and thus $Q \in V(I\widehat{R})$.

Part (b) follows similarly since $\mathrm{pd}(M_P) < \infty \iff \mathrm{Tor}_i^{R_P}(M_P, \kappa_P) = 0$ for all $i \gg 0 \iff \mathrm{Tor}^i_{\widehat{R}_Q}(\widehat{M}_Q, \widehat{\kappa}_Q) = 0$ for all $i \gg 0 \iff \mathrm{pd}(\widehat{M}_Q) < \infty$. □

We can now give the proof of our main result.

Proof of Theorem V.1: By the previous result and (II.13b), we may assume R is complete and therefore is a homomorphic image of a Gorenstein ring T. Let ω denote the canonical module of R. Fix $k \in \mathbb{N}$. Let $\mathcal{D}^\bullet$ be the dualizing complex as in (V.4). Then by (V.4 iii),

$$H^{d-k}(\mathrm{Hom}_R(M, \mathcal{D}^\bullet)^\vee \cong H^k_m(M).$$

Let I be the radical ideal given by $c \in I$ if and only if M_c is Cohen-Macaulay and has finite projective dimension. Then by (V.3) it suffices to show that I is contained in the defining ideal of the non-finite injective dimension locus of $W := H^{d-k}(\mathrm{Hom}_R(M, \mathcal{D}^\bullet))$. So, it suffices to show that for $c \in I$, $\mathrm{id}_{R_c}(W_c) < \infty$. The issue is local on the maximal ideals of R_c.

Let P be a maximal ideal of R_c. We have to show that W_P has finite injective dimension over R_P. Since M is finitely generated,

$$H^{d-k}(\mathrm{Hom}_R(M, \mathcal{D}^\bullet))_P \cong H^{d-k}(\mathrm{Hom}_{R_P}(M_P, \mathcal{D}^\bullet_P)) = W_P.$$

If $M_P = 0$ then $W_P = 0$ and we are done. We may assume $M_P \neq 0$. Then M_P is a finitely generated, Cohen-Macaulay R_P-module of finite projective dimension. It follows from (V.2) that R_P is Cohen-Macaulay. By (V.4 ii), $\mathcal{D}^\bullet_P$ is the minimal injective resolution of ω_P and so we find that

$$W_P \cong H^{d-k}(\mathrm{Hom}_{R_P}(M_P, \mathcal{D}^\bullet_P)) \cong \mathrm{Ext}^{d-k}_{R_P}(M_P, \omega_P).$$

Now, if $d-k = \mathrm{pd}(M_P)$, then $\mathrm{Ext}^{d-k}_{R_P}(M_P, \omega_P)$ has finite injective dimension by (V.5). Otherwise, $d-k \neq \mathrm{pd}(M_P)$ and then $\mathrm{Ext}^{d-k}_{R_P}(M_P, \omega_P) = 0$ by (V.5). $\Box$

As a consequence we have the following special case of the Lyubeznik-Smith conjecture.

Corollary V.7. *Let (R, m, K) be a reduced, excellent, equidimensional local ring and let M be a finitely generated R-module. Suppose that for all $P \in \mathrm{Spec}(R)$ such that $P \neq m$, M_P is Cohen-Macaulay and has finite projective dimension over R_P. Then tight closure equals finitistic tight closure for the zero submodule in $H^j_m(M)$ for all $j \in \mathbb{N}$.*

Proof: This follows immediately since, by Theorem V.1 the maximal ideal is potent for $H^j_m(M)$ for all $j \in \mathbb{N}$. $\Box$

CHAPTER VI

Potent Elements for Graded Modules Over Graded Rings

In this chapter we develop the notion of a potent element for graded rings. We recast the main result of [LS99] in the language of potent elements. After developing the theory of potent elements for graded rings, we then combine this result with previous work to obtain new cases where we know tight closure agrees with finitistic tight closure.

Throughout this section we let $R = \bigoplus_{j\in\mathbb{N}} R_j$ denote an $\mathbb{N}$-graded ring and $R_+ = \bigoplus_{j=1}^{\infty} R_j$, the ideal generated by homogeneous elements of positive degree. We extend the notion of a potent ideal to the graded setting.

Definition VI.1. If R is an $\mathbb{N}$-graded ring and $I \subseteq R$ is a homogeneous ideal, then we say I is a *graded potent ideal* if for all $\mathbb{Z}$-graded Artinian modules M,

$$0^*_M = \bigcup_{n\in\mathbb{N}} 0^*_{M_{(-n,I)}}.$$

The main result of [LS99] in this terminology is that when R is reduced and F-finite, R_+ is a graded potent ideal:

Theorem VI.2. *Let $R = \bigoplus_{j\in\mathbb{N}} R_j$ be an $\mathbb{N}$-graded, Noetherian, F-finite ring, and let $R_+ = \bigoplus_{j=1}^{\infty} R_j$, the ideal generated by homogeneous elements of positive degree. Then R_+ is a graded potent ideal.*

Indeed the methods of [LS99] produce Theorem VI.2, but the results appearing there do not mention potent elements and are stated for $\mathbb{N}$-graded rings over a field. For completeness we present a proof of Theorem VI.2, the same in spirit as the original.

Discussion. When R is any reduced, $\mathbb{N}$-graded ring we can define an $\mathbb{N}[1/q]$-grading on $R^{1/q}$ by defining $deg(r^{1/q}) := \frac{deg(r)}{q}$ for homogeneous elements. More generally for any $\mathbb{Z}$-graded R-module, M, we can give $R^{1/q} \otimes_R M$ a $\mathbb{Z}[1/q]$-grading by defining

$$deg(r^{1/q} \otimes x) := deg(r^{1/q}) + deg(x).$$

Note that this grading is compatible with the $\mathbb{N}$-grading from R in the sense that

$$deg(r' \cdot (r^{1/q} \otimes x)) = deg(r') + deg(r^{1/q}) + deg(x).$$

As in [LS99], the key point is the following lemma.

Lemma VI.3. ***(Main Lemma)*** *Let $R = \bigoplus_{j \in \mathbb{N}} R_j$ be an F-finite Noetherian R_0-algebra. There exists an integer t, depending only on R, such that whenever*

$$\phi : M \to N$$

is a degree-preserving map of graded R-modules that is bijective in degrees at least s, then for every $q = p^e$ the induced map

$$\phi^{1/q} : R^{1/q} \otimes M \to R^{1/q} \otimes N$$

is bijective in degrees at least $s + t$.

Proof of Main Lemma: We can place an $\mathbb{N}[1/p^e]$-grading on R by taking the pieces of non-integer degree to be 0. More formally, we let R' be the ring R graded as follows:

$$[R']_{a+b/q} = 0 \textit{ if } b \textit{ does not equal } 0$$

$$[R']_{a+b/q} = R_a \textit{ if } b \textit{ equals } 0$$

where $a, b \in \mathbb{N}$ and $0 < b < q$. Then $R^{1/p}$ is module-finite and graded over R' and therefore has a graded finite presentation over R'. The integer t above will eventually be defined in terms of this presentation.

Let

$$\bigoplus_j R'(-a_j - b_j/p) \to \bigoplus_i R'(-c_i - d_i/p) \to R^{1/p} \to 0$$

be a finite graded presentation of $R^{1/p}$, where $a_j, b_j, c_i, d_i \in \mathbb{N}$ and $0 \leq b_j, d_i < p$ for all i and j. Here, $R'(-m - n/p)$ denotes the ring R' which has been graded so that

$$[R'(-m - n/p)]_{v+w/p} = [R']_{v-m+(w-n)/p}.$$

Let $t' := max_{i,j}\{a_j + 1, c_i + 1\}$. The first step is the following:

Claim 1. For any q, there is a presentation of $R^{1/q}$ given by

$$\bigoplus_j R'(-a_{j,q} - b_{j,q}/q) \to \bigoplus_i R'(-c_{i,q} - d_{i,q}/q) \to R^{1/q} \to 0$$

such that if $t_q := max_{i,j}\{a_{j,q} + 1, c_{i,q} + 1\}$ then $t_q < 2t' + 1$. Again, we are assuming $a_{j,q}, b_{j,q}, c_{i,q}, d_{i,q} \in \mathbb{N}$ and $0 < b_{j,q}, d_{j,q} < q$.

Proof of Claim 1: Since R is F-finite, $R^{1/p}$ is finitely generated over R. By taking the forms of the generators, we may assume the generators are homogeneous. Let $F_1, \ldots, F_s$ be a set of such generators. We first show that $R^{1/q}$ is generated over R by the elements

$$\{F_{k_0} F_{k_1}^{1/p} \cdots F_{k_{e-1}}^{1/p^{e-1}} \mid 1 \leq k_0, \ldots, k_{e-1} \leq s\}.$$

Let $q = p^e$. We use induction on e, the case $e = 1$ being trivial. We assume the result for $e - 1$. For any $x \in R$, using the induction hypothesis we can write

$$x^{p/q} = \sum_i r_i F_{i,1} \cdots F_{i,e-1}^{1/p^{e-2}}$$

We have indexed the F_i somewhat unusually to make the notation work out. Taking p^{th} roots we get

$$x^{1/q} = \sum_i r_i^{1/p} F_{i,1}^{1/p} \cdots F_{i,e-1}^{1/p^{e-1}},$$

Write $r_i^{1/p} = \sum_j s_{i,j} F_j$. When we plug this in we get

$$x^{1/q} = \sum_i \sum_j s_{i,j} F_j F_{i,1}^{1/p} \cdots F_{i,e-1}^{1/p^{e-1}}.$$

which after regrouping we can express as

$$x^{1/q} = \sum_k s_k F_{k,0} F_{k,1}^{1/p} \cdots F_{k,e-1}^{1/p^{e-1}}.$$

This shows that the elements $\{F_{k_0} F_{k_1}^{1/p} \cdots F_{k_{e-1}}^{1/p^{e-1}}\}$ do indeed generate $R^{1/q}$ over R.

Now, by the definition of t' above, the degrees of the F_j are less than t'. Therefore each element in the generating set has degree less than

$$\sum_{n=0}^{e-1} \frac{t'}{p^n} = t' \sum_{n=0}^{e-1} \frac{1}{p^n} \leq t' \frac{p}{p-1} \leq 2t'.$$

The degree shifts required of the first free module in the presentation are thus less than $2t'$ and so the numbers $c_{j,q} + 1 < 2t' + 1$. To complete the proof of Claim 1 it remains to show that the numbers $a_{i,q} < 2t'$. That is, we must show that every element in a set of (homogeneous) generators for the kernel has degree less than $2t'$. This is done by showing that if $\{G_1, \ldots, G_n\}$ is a set of generators for the kernel of the map $\oplus R'(-c_j - d_j/p) \to R^{1/p}$ then the set of elements

$$\{F_{k_0} \cdot F_{k_1}^{1/p} \cdots F_{k_{e-2}}^{1/p^{e-2}} \cdot G_k^{1/p^{e-1}} \mid 1 \leq k_0, \ldots, k_{e-2} \leq s;\ 1 \leq k_{e-1} \leq n\}$$

generate the kernel of the map $\oplus R'(-c_{j_q} - d_{j,q}/p) \to R^{1/q}$. The proof is similar to the above, by induction on e, and is left to the reader.

Note that $deg(g_k) < t'$ and also $deg(F_k) < t'$. Then, as before, it is easy to verify that

$$deg(F_{k_0} \cdot F_{k_1}^{1/p} \cdots F_{k_{e-2}}^{1/p^{e-2}} \cdot g_k^{1/p^{e-1}}) < 2t'.$$

This completes the entire proof of Claim 1.

Claim 2. With $a, b \in \mathbb{N}$ and $0 < b < q$, the map $R'(-a-b/q) \otimes M \to R'(-a-b/q) \otimes N$ is bijective in degrees at least $s + a + 1$.

Proof of Claim 2: Since $R'(-a - b/q)$ is flat over R' we have an exact sequence

$$0 \to R'(-a - b/q) \otimes ker(\phi) \to R'(-a - b/q) \otimes M \to$$

$$R'(-a - b/q) \otimes N \to R'(-a - b/q) \otimes coker(\phi) \to 0$$

So, it suffices to check that for $K = coker(\phi), ker(\phi)$, if K vanishes in degrees at least s, then $K \otimes R'(-a - b/q)$ vanishes in degrees at least $s + a + 1$. Let e be a degree $a + b/q$ generator of $R'(-a - b/q)$, and note that $a + b/q < a + 1$. Consider an element $er \otimes x = e \otimes rx \in R'(-a - b/q) \otimes K$. Suppose $deg(e \otimes rx) > s + a + 1$. Then since

$$deg(e \otimes rx) = deg(e) + deg(rx) > s + a + 1$$

we find that $deg(rx) > s + a + 1 - deg(e) > s$. Hence, $rx \in K$ has degree greater than s, a contradiction. This proves Claim 2.

Now, take $t := 2t' + 1$. We are finally ready to finish the proof of the main lemma. We are given a map $\phi : M \to N$ bijective in degrees at least s and we want to show that $\phi^{1/q} : R^{1/q} \otimes M \to R^{1/q} \otimes N$ is bijective in degrees at least $s + t$.

By Claim 1, there is a presentation of $R^{1/q}$ of the form

$$\bigoplus_j R'(-a_{j,q} - b_{j,q}/q) \to \bigoplus_i R'(-c_{i,q} - d_{i,q}/q) \to R^{1/q} \to 0$$

such that $max_{i,j}\{a_{j,q} + 1, c_{i,q} + 1\} \leq t$. We have the following commutative diagram with exact rows and degree-preserving maps:

$$\begin{array}{ccccccc}
\oplus_j R'(-a_{j,q} - b_{j,q}/q) \otimes M & \to & \oplus_i R'(-c_{i,q} - d_{i,q}/q) \otimes M & \to & R^{1/q} \otimes M & \to & 0 \\
f_1 \downarrow & & f_2 \downarrow & & \phi^{1/q} \downarrow & & \\
\oplus_j R'(-a_{j,q} - b_{j,q}/q) \otimes N & \to & \oplus_i R'(-c_{i,q} - d_{i,q}/q) \otimes N & \to & R^{1/q} \otimes N & \to & 0
\end{array}$$

By Claim 2, the maps f_1 and f_2 are bijective in degrees at least $s + t$. A simple diagram chase establishes that $\phi^{1/q}$ is bijective in degrees at least $s + t$. $\square$

We now use Lemma VI.3 to prove the following result. Theorem VI.2 will then follow immediately.

Proposition VI.4. *Suppose $R = \bigoplus_{j \in \mathbb{N}} R_j$ is an $\mathbb{N}$-graded ring, and suppose $N \subseteq M$ is an inclusion of graded Artinian R-modules. For each $d \in \mathbb{Z}$, let $M_{\geq d}$ (respectively $N_{\geq d}$) be the R-submodule of M (respectively, N) spanned by all homogeneous element of degree greater than or equal to d. Then*

$$N^*_M = \bigcup_{d \in \mathbb{Z}} (N_{\geq d})^*_{M_{\geq d}}.$$

Proof: Suppose $z \in N^*_M$ and we want to show $z \in \bigcup_{d\in\mathbb{Z}}(N_{\geq d})^*_{M_{\geq d}}$. Since $N \subseteq M$ is graded, N^*_M is graded and we may assume $z \in M$ is a homogeneous element, of degree d say. Suppose $c^{1/q} \otimes z \in Im(R^{1/q} \otimes N \to R^{1/q} \otimes M)$ with $c \in R^o$. Pick t as in the Main Lemma; we will show $z \in (N_{\geq d-t})^*_{M_{\geq d-t}}$ which will complete the proof.

To simplify notation, set $M' := M_{\geq d-t}$ and $N' := N \cap M'$. Observe that $M' \to M$ is bijective in degrees at least $d-t$, so by the lemma, $R^{1/q} \otimes M' \to R^{1/q} \otimes M$ is bijective in degrees at least d. By the same reasoning, $R^{1/q} \otimes N' \to R^{1/q} \otimes N$ is bijective in degrees at least d. The element $c^{1/q} \otimes z$ has degree at least d and it follows that $c^{1/q} \otimes z$ is in $Im(R^{1/q} \otimes N \to R^{1/q} \otimes M)$ if and only if $c^{1/q} \otimes z$ is in $Im(R^{1/q} \otimes N' \to R^{1/q} \otimes M')$. That says $z \in N^*_M$ if and only if $z \in (N')^*_{M'}$ as required.
$\square$

Proof of Theorem VI.2: First note that for any Artinian module M, we have a chain of submodules $M_{\geq d} \supseteq M_{\geq d+1} \supseteq \cdots$ which must stabilize since M has DCC. It follows that there exists $n \in \mathbb{N}$ such that $M_t = 0$ for all $t \geq n$. Therefore, for every $w \in \mathbb{N}$, $M_{\geq n-w} \subseteq Ann_M((R_+)^w)$. Hence, by the previous proposition we have

$$0^*_M = \bigcup_{d\in\mathbb{Z}} 0^*_{M_{\geq d}} \subseteq \bigcup_{d\in\mathbb{Z}, n>d} 0^*_{Ann_M((R_+)^{n-d})}$$

as required. $\square$

Before proceeding we recall some standard notions about graded rings. We follow [BH93], sections 1.5 and 3.6.

Let M, N be graded modules over the graded ring R. We denote by $\mathrm{Hom}_i(M, N)$ the homogeneous homomorphisms of degree i and put ${}^*\mathrm{Hom}_R(M, N) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_i(M, N)$.

Note that ${}^*\operatorname{Hom}_R(M,N)$ is a $\mathbb{Z}$-graded module. In general, ${}^*\operatorname{Hom}_R(M,N)$ is a proper subgroup of $\operatorname{Hom}_R(M,N)$ but equality holds when M is finitely generated (see 1.5.19, [BH93]). If W is any finitely generated module over an arbitrary excellent ring, the locus of primes P such that W_P has finite injective dimension over R_P is open (see [Tak06]). It follows that there is a radical ideal $I \subseteq R$ such that $c \in I$ if and only if W_c has finite injective dimension over R_c. We require results on the behavior of the ideal I.

Lemma VI.5. *Let (R,m) be an $\mathbb{N}$-graded ring over an excellent local ring (R_0,m_0) where $m = m_0 + R_+$ is the unique homogeneous maximal ideal. Let W be a finitely generated, graded R-module, and let I be the radical ideal defining the non-finite injective dimension locus for W. Then*

(a) We have $\operatorname{id}_R(W) < \infty \Longleftrightarrow \operatorname{Ext}^i_R(R/m, W) = 0$ for all $i \gg 0$.

(b) Let $(R,m) \to (S,n)$ be a flat local map of graded rings where m, n are the homogeneous maximal ideals of R, S respectively. Assume that $mS = n$. Then $\operatorname{id}_R(W) < \infty$ if and only if $\operatorname{id}_S(W \otimes S) < \infty$.

(c) I is a homogeneous ideal.

(d) IR_m is the defining ideal of the non-finite injective dimension locus of $W \otimes R_m$.

Proof: (a). The implication $(\Longrightarrow)$ is obvious. Now assume $\operatorname{Ext}^i_R(R/m, W) = 0$ for all $i \gg 0$. By the assumption, $\operatorname{Ext}^i_{R_m}(R_m/mR_m, W_m) = 0$ for all $i \gg 0$, and it follows from [BH93], Proposition 3.1.14, that W_m has finite injective dimension over R_m. This means that there exists an integer $n \in \mathbb{N}$ such that for all graded primes $P \subseteq m$, $\mu_i(P, W) = 0$ for all $i > n$ where $\mu_i(P,W)$ is the i^{th} Bass number of W with respect to P. Now, for an arbitrary prime Q, let Q^* denote the ideal generated by homogeneous elements of Q. It is easy to see that Q^* is a homogeneous prime, and we have $\mu_0(Q, W) = 0$ while $\mu_{i+1}(Q, W) = \mu_i(Q^*, W)$ for all $i > 0$ (see [BH93], 3.6.6). It follows that for all $i > n+1$, $\mu_i(Q, W) = 0$ for all primes Q in R. This means that W has finite injective dimension.

(b). Since $R \to S$ is faithfully flat, $\mathrm{Ext}^i_R(R/m, W) = 0$ if and only if $\mathrm{Ext}^i_R(R/m, W) \otimes_R S = 0$. Furthermore, because W is finitely generated we have

$$\mathrm{Ext}^i_R(R/m, W) \otimes_R S \cong \mathrm{Ext}^i_S(R/m \otimes_R S, W \otimes_R S) \cong \mathrm{Ext}^i_S(S/mS, W \otimes S)$$

$$\cong \mathrm{Ext}^i_S(S/n, W \otimes S).$$

The result follows from (a).

(c). This follows from the standard Van der Monde matrix determinant trick. Let t be an indeterminate over R and let $R(t)$ denote the homogeneous localization of the polynomial ring $R[t]$ at $mR[t]$. By part (a) we may pass to $R(t)$: The ring $R(t)$ becomes a graded ring by letting $\deg(t) := 0$, and it is clear that $R \to R(t)$ is faithfully flat and the homogeneous maximal ideal of $R(t)$ is $mR(t)$. Without loss of generality we may assume that R has infinitely many unite u_i in degree 0 such that $u_i - u_j$ is also a unit.

The units in degree 0 act on R via the grading: if $u \in R_0$ is a unit and $x \in R$ is homogeneous of degree d, we define $\theta_u : R \to R$ by the rule $\theta_u(r_d) := u^d r_d$ for an element $r_d \in R_d$. It is clear that θ_u gives a ring automorphism of R. By standard results, an ideal $I \subseteq R$ is homogeneous if and only if for all $c \in I$ $\theta_u(c) \in I$ for all units $u \in R_0$. If I is the defining ideal of the non-finite injective dimension locus for W, then $c \in I$ if and only if W_c has finite injective dimension over R_c. For simplicity we write θ for θ_u. We must show that $\mathrm{id}_{R_c}(W_c) < \infty \Longrightarrow \mathrm{id}_{R_{\theta(c)}}(W_{\theta(c)}) < \infty$. We write θ_c for the induced automorphism of rings $\theta_c : R_c \to R_{\theta(c)}$.

There is an automorphism of abelian groups $\psi : M \to M$ given by $\psi(y_e) := u^e y^e$ for a homogeneous element $y_e \in M_e$. Note that the linearity of ψ is such that $\psi(ry) = \theta(r)\psi(y)$: indeed, it is enough to check this on homogeneous elements, and then

$$\psi(r_d y_e) = u^{d+e} r_d y_e = u^d r_d u^e y_e = \theta(r_d)\psi(y-e)$$

as required. Now define a map $\lambda : M_c \to M_{\theta(c)}$ by the rule $\lambda(m/c^k) := \psi(m)/\theta(c^k)$. We need to show that λ is well defined. First note that λ satisfies $\lambda(vy) = \theta_c(v)\lambda(y)$ for all $v \in R_c$ and all $y \in M$. Indeed,

$$\lambda(ry/c^k) = \lambda(ry)/\theta(c^k) = \theta(r)\psi(y)/\theta(c^k) = \theta(r/c^k)\psi(y).$$

It is easy to check that λ is well-defined: if $m/c^k = m'/c^{k'}$ then we must show that $\psi(m)/\theta(c^k) = \psi(m')/\theta(c^{k'})$. Equivalently, we need to find $n \in \mathbb{N}$ such that

$$\theta(c)^n\big(\psi(m)\theta(c^{k'}) - \theta(c^k)\psi(m')\big) = 0.$$

However, there exists n such that $c^n(mc^{k'} - m'c^k) = 0$. Applying ψ we find that

$$\psi\big(c^n(mc^{k'} - m'c^k)\big) = 0$$

and hence

$$\theta(c^n)\big(\psi(mc^{k'}) - \psi(m'c^k)\big) = 0 \Longrightarrow \theta(c)^n\big(\psi(m)\theta(c^{k'}) - \theta(c^k)\psi(m')\big) = 0$$

as required. We have established that $\lambda : M_c \to M_{\theta(c)}$ is an isomorphism of abelian groups compatible with the map $\theta_c : R_c \to R_{\theta(c)}$. It follows that M_c has finite injective dimension over R_c if and only if $M_{\theta(c)}$ has finite injective dimension over $R_{\theta(c)}$.

Therefore, $c \in I \Longrightarrow \theta(c) \in I$ and so I is homogeneous. This proves (c). Part (d) is obvious. $\square$

The main theorem of Chapter 4 is stated for local rings. In order to make use of it in the graded setting, we need to know that certain properties of graded modules may be checked after localizing at the homogeneous maximal ideal. We will need the following lemma.

Lemma VI.6. *Let $R = \bigoplus_{i \geq 0} R_i$ be an $\mathbb{N}$-graded ring over a local ring (R_0, m_0), and let M be a $\mathbb{Z}$-graded, Artinian R-module. Let $I \subseteq R$ be a homogeneous ideal, and let m denote the homogeneous maximal ideal of R. Set $S = R_m$.*

(a) For all $t \in \mathbb{N}$, $\operatorname{Ann}_{M \otimes S}(I^t S) = \operatorname{Ann}_M(I^t) \otimes S$.

(b) The modules $\operatorname{Ann}_M(I^t)$ and $F_R^e(\operatorname{Ann}_M(I^t))$ are graded Artinian R-modules.

(c) The natural map $M \to M \otimes S$ is injective.

Proof: Part (a) follows from the fact that $R \to S$ is flat. Let $I^t = (f_1, \ldots, f_h)R$. There is a short exact sequence

$$0 \to \operatorname{Ann}_M(I^t) \to M \xrightarrow{\theta} M^{\oplus h}$$

where $\theta(x) = (f_1 x, \ldots, f_h x)$ for all $x \in M$. Applying $- \otimes_R S$, we find an exact sequence

$$0 \to \operatorname{Ann}_M(I^t) \otimes_R S \to M \otimes S \xrightarrow{\theta} (M \otimes S)^{\oplus h}$$

and the result follows.

(b). Since I is homogeneous, I^t is homogeneous, so it suffices to show that $\operatorname{Ann}_M(J)$ is homogeneous when J and M are. Since J is homogeneous, $J = \oplus J_i$

where $J_i \subseteq R_i$. First we claim that $\mathrm{Ann}_M(J) = \cap_i \mathrm{Ann}_M(J_i)$. The containment $\subseteq$ is obvious. Conversely, let $x \in \cap_i \mathrm{Ann}_M(J_i)$. If $f \in J$, then $f = \sum_i f_i$ with $f_i \in J_i$. It follows that $fx = \sum_i f_i x = 0$ since $x \in \cap_i \mathrm{Ann}_M(J_i)$. This proves the claim. Now suppose $x = \sum_j x_j \in \mathrm{Ann}_M(J)$ where $\deg(x_1) < \deg(x_2) < \cdots$. Then $x \in \mathrm{Ann}_M(J_i)$ for all i, and it suffices to show that $x_j \in \mathrm{Ann}_M(J_i)$ for all i, j. If $f \in J_i$, then $fx = \sum_j fx_j = 0$. Clearly, $\deg(fx_j) \neq \deg(fx_i)$ for $i \neq j$, so it follows that $fx_j = 0$ for all j. Whence $x_j \in \mathrm{Ann}_M(J_i)$ for all i as needed. We have proved that $\mathrm{Ann}_M(J)$ is graded when M and J are. The last statement follows from the fact that $F^e(N)$ is graded when N is, as we saw at the beginning of this section. This completes the proof of (b).

For part (c), we have to show that if $u \in R - m$ and $ux = 0$ for $x \in M$ then $x = 0$. That $u \in R - m$ means that $u = u_0 + \cdots + u_h$ with $deg(u_i) = i$ and $u_0 \in A - n$. Therefore, u_0 is a unit in R and hence a nonzerodivisor. We may write $x = \sum_{j=0}^{n} x_j$ with $\deg(x_0) < \deg(x_1) < \cdots < \deg(x_n)$ and $x_0 \neq 0$. Observe that $u_0 x_0$ is the unique term of ux in degree $\deg(x_0)$ so if $ux = 0$ it follows that $u_0 x_0 = 0$. Since u_0 is a unit, we must have $x_0 = 0$, a contradiction. $\square$

The next result states that the DCC property may be checked after localizing at m.

Lemma VI.7. *Let R be an $\mathbb{N}$-graded ring over a local ring (R_0, m_0) and let $m = m_0 + R_+$ be the homogeneous maximal ideal. Note that m is a maximal ideal of R in this case. Suppose M is a graded R-module such that M_m is Artinian over R_m. Then M is Artinian over R.*

Proof: It suffices to see that the submodules of M are already R_m submodules for then a chain of submodules violating the descending chain condition over R would constitute a violation of the descending chain condition over R_m. First note that $M \to M_m$ is injective since M is graded. Therefore, since every element of M_m is killed by a power of m, every element of M is killed by a power of m. Let $N \subseteq M$ be a submodule. We need to show that $R_m N \subseteq N$. For this purpose, it suffices to show that $R_m xR \subseteq xR$ for $x \in N$. But $xR \cong R/\operatorname{Ann}_R(x)$ and $\operatorname{Ann}_R(x) \supseteq m^t$ for $t \gg 0$. Since m is a maximal ideal of R, it follows that $R/\operatorname{Ann}_R(x)$ is local, and that $R/\operatorname{Ann}_R(x)) = (R/\operatorname{Ann}_R(x))_m$. Therefore, $xR = (xR)_m$ and so $R_m(xR) = xR$. This proves the lemma. $\square$

We also have:

Proposition VI.8. *Let $R = \bigoplus_{i \geq 0} R_i$ be a reduced, excellent $\mathbb{N}$-graded ring over a local ring (R_0, m_0), and let M be a $\mathbb{Z}$-graded, Artinian R-module. Let $I \subseteq R$ be a homogeneous ideal, and let m denote the homogeneous maximal ideal of R. Set $S = R_m$. If IS is potent for $M \otimes S$ then I is potent for M.*

Proof: We may pick $c \in R$ homogeneous such that R_c is regular and then S_c is regular as well. After replacing c by a power, we have that c is a common big test element for R and S.

Suppose $z \in 0^*_M$ and we want to show that $z \in 0^*_{\operatorname{Ann}_M(I^t)}$ for some t. We have that $cz^q = 0$ in $F^e_R(M)$ so $cz^q = 0$ in $F^e_R(M) \otimes_R S = F^e_S(M \otimes S)$. Since IS is potent for $M \otimes S$, there exists t such that $cz^q = 0$ in $F^e(\operatorname{Ann}_{M \otimes S}(I^t S))$. By part (a) of the previous result, $cz^q = 0$ in $F^e_S(\operatorname{Ann}_M(I^t) \otimes S) = F^e_R(\operatorname{Ann}_M(I^t)) \otimes S$. By part (b) of that lemma, $F^e_R(\operatorname{Ann}_M(I^t))$ is graded since I and M are, so by part (c), the natural map $F^e_R(\operatorname{Ann}_M(I^t)) \to F^e_R(\operatorname{Ann}_M(I^t)) \otimes_R S$ is injective. It follows that $cz^g = 0$ even in $F_R^e(\mathrm{Ann}_M(I^t))$. This means $z \in 0^*_{\mathrm{Ann}_M(I^t)}$ as required. $\square$

We are now able to combine (VI.2) with the main result of Chapter 4 to obtain the following

Corollary VI.9. *Let $R = \bigoplus_{i=0}^{\infty} R_i$ be an $\mathbb{N}$-graded ring over a local ring (R_0, m_0) and let W be a finitely generated, graded R-module. Let m be the homogeneous maximal ideal of R and let $E = E_R(R/m)$ be the injective hull of the residue field of R at m. Let $M := \mathrm{Hom}_R(W, E)$. If I is the defining ideal of the non-finite injective dimension locus for W (that is, $c \in I$ if and only if $\mathrm{id}_{R_c}(W_c) < \infty$), then $I + R_+$ is potent for M.*

Proof: We first prove that I is potent for M. Since W is finitely generated, $M_m = \mathrm{Hom}_R(W, E_R(K))_m \cong \mathrm{Hom}_{R_m}(W_m, E_R(K))$. By Matlis duality over the local ring R_m, M_m has DCC. It follows from (VI.7) that M has DCC over R. By Lemma VI.5, I is a homogeneous ideal and IR_m is the defining ideal of the non-finite injective dimension locus for $M \otimes R_m$. Therefore, by (IV.58) IR_m is potent for $M \otimes R_m$. But then by (VI.8), I is potent for M.

We have a decomposition $0^*_M = \bigcup_t 0^*_{\mathrm{Ann}_M(I^t)}$. Therefore, the result will be proved if we can show that R_+ is potent for $\mathrm{Ann}_M(I^t)$ for all t. But as M and I are graded, $\mathrm{Ann}_M(I^t)$ is also graded so by (VI.2), R_+ is potent for $\mathrm{Ann}_M(I^t)$. This completes the proof. $\square$

We give two important consequences of this result.

Corollary VI.10. *Let R be an $\mathbb{N}$-graded ring over a local ring (R_0, m_0) and let W*

be a finitely generated, graded R-module. Let m be the homogeneous maximal ideal and set $M := \mathrm{Hom}_R(W, E_R(R/m))$. If m_0 is contained in the defining ideal of the non-finite injective dimension locus for W, then tight closure equals finitistic tight closure for the 0 submodule in M.

Proof: This follows at once from the previous result since $m_0 + R_+$ is potent for M. $\square$

Corollary VI.11. *Let $R = R_0[x_1, \ldots, x_d]$ be a polynomial ring over a complete, isolated singularity (R_0, m_0). Then tight closure is the same as finitistic tight closure for in every graded Artinian R-module.*

Proof: If $c \in m_0$ then $R_c = (R_0)_c[x_1, \ldots, x_d]$ is regular. It follows that m_0 is contained in the defining ideal of the singular locus for R. Since every finitely generated module over a regular ring has finite injective dimension, it follows that m_0 is contained in the defining ideal of the non-finite injective dimension locus of every finitely generated R-module W. Therefore, $m_0 + R_+$ is potent for every $\mathrm{Hom}_R(W, E_R(K))$ where W is a finitely generated, graded module. Hence, tight closure equals finitistic tight closure for the zero submodule in every $\mathrm{Hom}_R(W, E_R(K))$. By the graded version of Matlis duality (see [BH93], 3.6.17), every graded Artinian module is of this form. $\square$

CHAPTER VII

Questions

While the theory of potent elements has obtained new insights about tight closure in Artinian modules, many questions about the theory remain unsolved. In this chapter we collect the most important of these open problems and explain their connection to the question of whether strong and weak F-regularity are equivalent.

Unless stated otherwise, let (R, m, K) denote a reduced, excellent local ring of prime characteristic $p > 0$. As we noted in Chapter 2, the issue of whether the maximal ideal m is potent for an Artinian R-module M is equivalent to whether tight closure is the same as finitstic tight closure for the zero submodule of M. Perhaps the most ambitious open question is the following

Question VII.1. *If (R, m, K) is a reduced, excellent, local ring, is m potent for every Artinian R-module? Equivalently, if $c \in m$, is c potent for every Artinian R-module?*

We expect this question to be extraordinarily difficult to settle, even if one restricts to the case where R is a complete weakly F-regular ring (hence, a complete, Cohen-Macaulay, normal domain). This is because an answer to this question in the general case would settle the conjecture of Lyubeznik and Smith raised in [LS01], while even the restricted case an answer would settle the issue of whether every weakly F-regular ring is strongly F-regular.

Another open question that would have a major impact on the theory of tight closure is the following

Question VII.2. *If $c \in \tau(R)$, the test ideal of R, is c potent for every Artinian module M? Is $\tau(R)$ potent for $E_R(K)$, the injective hull of the residue class field?*

An affirmative answer to either of these questions would establish that strong and weak F-regularity are equivalent for excellent local rings. It would also establish that the test ideal behaves well with respect to localization and completion under mild conditions on the ring. See [LS01], section 2. For this reason, settling this question also seems formidable.

Question VII.3. *Suppose $c \in R$ is such that R_c is strongly F-regular. Is c potent for every Artinian module M? Is the defining ideal of the non-strongly F-regular locus potent for $E_R(K)$?*

An affirmative answer to this question would imply that F-regular rings are strongly F-regular. It would also imply that weakly F-regular rings that are strongly F-regular on the punctured spectrum are strongly F-regular. Note that we are free to replace c by a power, c^n. Using the gamma construction we will be able to assume that R is F-finite (or even complete and F-finite). In this case, if $c \in R$ is such that R_c is strongly F-regular, then for some n, c^n will be a big test element, hence a test element for any Artinian module M.

Part of the reason that the theory of test elements has been successful is that the set of elements potent for all Artinian modules forms a (radical) ideal. Unfortunately, we do not know whether the set of potent elements for a fixed Artinian module M forms an ideal, even if M is taken to be the injective hull of the residue class field. We do have the following result which was essentially proved in section IV.3, though it never appeared explicitly in this form:

Proposition VII.4. *Let (R, m, K) be a reduced, F-finite, excellent local ring of characteristic p, let W be a finitely generated R-module and let $c \in R$. Put $M :=$ $\mathrm{Hom}_R(W, E_R(K))$. If there exists $k \in \mathbb{N}$ such that for all $q \gg 0$*

$$c^k \cdot \mathrm{Ext}^1_R(R^{1/q}, W) = 0$$

then c is potent for M. Moreover the set $J = \{c \in R \mid \exists k, \forall q \gg 0, c^k \cdot \mathrm{Ext}^1_R(R^{1/q}, W) =$ $0\}$ forms a radical ideal of R.

Proof: First notice that since W is Noetherian, $\bigcup_{n\in\mathbb{N}} \mathrm{Ann}_W(c^n) = \mathrm{Ann}_W(c^h)$ for some $h \in \mathbb{N}$. It then follows that c is a nonzerodivisor on $c^h W$. Now, by Lemma IV.50, the condition that

$$c^k \cdot \mathrm{Ext}^1_R(R^{1/q}, W) = 0$$

implies that, for all $k_1 \geq 1$,

$$c^k \cdot \mathrm{Ext}^1_R(R^{1/q}, c^{k_1} W) = 0.$$

Assume $k_1 > h$. Then we have a short exact sequence

$$0 \to c^{k_1} W \xrightarrow{\cdot c^k} c^{k_1} W \to \frac{c^{k_1} W}{c^{k+k_1} W} \to 0$$

which yields the exact sequence

$$\mathrm{Ext}^1_R(R^{1/q}, c^{k_1} W) \xrightarrow{c^k} \mathrm{Ext}^1_R(R^{1/q}, c^{k_1} W) \to \mathrm{Ext}^1_R(R^{1/q}, \frac{c^{k_1} W}{c^{k+k_1} W})$$

and it follows that for all $k_1 > h$ the natural map

$$\mathrm{Ext}^1_R(R^{1/q}, c^{k_1}W) \to \mathrm{Ext}^1_R(R^{1/q}, \frac{c^{k_1}W}{c^{k+k_1}W})$$

is injective. By Lemma IV.48(a), after applying $\mathrm{Hom}_R(-, E_R(K))$ we find that for all $k_1 > h$ the natural map

$$\mathrm{Tor}^R_1(R^{1/q}, (\frac{c^{k_1}W}{c^{k+k_1}W})^\vee) \to \mathrm{Tor}^R_1(R^{1/q}, c^{k_1}W^\vee)$$

is surjective. It is routine to verify that $c^{k_1}W^\vee = M/\operatorname{Ann}_M(c^{k_1})$ and that

$$(\frac{c^{k_1}W}{c^{k+k_1}W})^\vee = \frac{\mathrm{Ann}_M(c^{k+k_1})}{\mathrm{Ann}_M(c^{k_1})}.$$

As in the proof of Lemma IV.49, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
\mathrm{Tor}^R_1(R^{1/q}, \frac{\mathrm{Ann}_M(c^{k+k_1})}{\mathrm{Ann}_M(c^{k_1})}) & \longrightarrow & R^{1/q} \otimes_R \mathrm{Ann}_M(c^{k_1}) & \xrightarrow{a_q} & R^{1/q} \otimes_R \mathrm{Ann}_M(c^{k+k_1}) \\
f_q \downarrow & & \mathbb{1} \downarrow & & g_q \downarrow \\
\mathrm{Tor}^R_1(R^{1/q}, \frac{M}{\mathrm{Ann}_M(c^{k_1})}) & \longrightarrow & R^{1/q} \otimes_R \mathrm{Ann}_M(c^{k_1}) & \xrightarrow{d_q} & R^{1/q} \otimes_R M
\end{array}$$

The map f_q is surjective for all q, so g_q restricts to an isomorphism between $\mathrm{Im}(a_q)$ and $\mathrm{Im}(d_q)$ for all q. Thus, if $x \in 0^*_M$ and $x \in \mathrm{Ann}_M(c^{k_1})$ then $x \in 0^*_{\mathrm{Ann}_M(c^{k+k_1})}$.

Finally, we want to see that J forms an ideal. Clearly, if $c \in J$ then $rc \in J$ for all $r \in R$. If $c_1, c_2 \in J$, say $c_1^{k_1} \cdot \mathrm{Ext}^1_R(R^{1/q}, W) = 0$ and $c_2^{k_2} \cdot \mathrm{Ext}^1_R(R^{1/q}, W) = 0$, then $(c_1 + c_2)^{k_1+k_2-1} \cdot \mathrm{Ext}^1_R(R^{1/q}, W) = 0$ so $c_1 + c_2 \in J$ as well. Therefore, J is an ideal and it follows at once that J is a radical ideal. □

This result leads to the following question:

Question VII.5. *Suppose (R, m, K) is a reduced, F-finite, excellent local ring, let W be a finitely generated R-module and let*

$$M := \mathrm{Hom}_R(W, E_R(K)).$$

Does the set of potent elements for M form an ideal of R?

One could hope for a theorem general enough to include both Theorem IV.1 and Theorem VI.2.

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