# Strichartz Estimates for Wave and Schrödinger Equations on Hyperbolic Trapped Domains

by

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# Abstract

In this thesis, I will establish the mixed norm Strichartz type estimates for the wave and Schrödinger equations on certain Riemannian manifold $(\Omega, g)$. Here $\Omega$ is the exterior domain of a smooth, normally hyperbolic trapped obstacle in $n$ dimensional Euclidean space. I studied the case when $n \geq 3$ is odd. As for the normally hyperbolic trapped obstacles, I got local Strichartz estimates for wave and Schrödinger equations with some loss of derivatives for the initial data. I also got a global Strichartz type estimates for the wave equation. In this case, I need two different $L_t^p L_x^q$ norms of the forcing term to bound the solution of the inhomogeneous equation.

Primary Reader: Christopher D. Sogge

Secondary Reader: Bernard Shiffman

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# Dedication

This thesis is dedicated to my family, who provide great support for my daily life and the PhD study. They are my dad Chengzhong Sun, my mum Lingjuan Chen, and my husband Dr. Lishui Cheng.

# Contents

# Chapter 1

# Introduction

## 1.1 Some Classical results for Strichartz Estimates

In this dissertation, we want to prove Strichartz-type estimates for wave and Shrödinger equations on the Riemannian manifold $(\Omega, g)$, where $\Omega$ is the region outside a normally hyperbolic trapped obstacle.

The Strichartz-type estimates was brought into interest by the study of the well-posedness for small data semilinear wave and Schrödinger equations. Based on the Strichartz-type estimates, a series of systematic techniques and theories has been developed to prove the global well-posedness or blow-up properties of wave and Schrödinger equations. Survey articles as [21] and [8] provide great introductions for the history and recent developments of such topics.

In particular, Strichartz estimates are well understood for Euclidean space. Here we mean $\Omega = \mathbb{R}^n$ and $g_{ij} = \delta_{ij}$. See for example Strichartz [18], Fritz John [9], Y. Zhou [24], Lindblad and Sogge [12], Georgiev and Lindblad and Sogge [5], D. Tararu [20], Ginibre and Velo [6], Keel and Tao [11], Smith and Sogge [16] and the references therein. In section 1.1.1, we will illustrate the basic idea using a classical example in three dimensional Euclidean space. For more details about this example, please refer to Christopher D. Sogge's book on wave equations [17].

Since 2008, the study of the problem on non-trapping manifolds also got many interesting results. For example Y. Du, J. Metcalfe, C. Sogge and Y. Zhou in [4]; K. Hidano, J. Metcalfe, H. Smith, C. Sogge and Y. Zhou in [7]; D. Tataru [19]; etc. In section 1.1.2, we will also illustrate the local-global idea in the proof of abstract Strichartz estimate in [7].

### 1.1.1 On Euclidean Space

Now let us start by considering the example of an wave equation on the Minkowski space $\mathbb{R}^{1+3}$. For readers that are interested in more details, please refer to Christopher D. Sogge's book [17] *Lectures on Nonlinear Wave Equations.*

Let $\Delta = \sum\limits_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ be the standard Laplacian on $\mathbb{R}^3$. Then consider the following wave equation

$$\begin{cases} (\partial_t^2 - \Delta)u(t,x) = F(t,x) \quad (t,x) \in \mathbb{R}^{1+3} \\ \\ u(0,x) = f(x),\ \partial_t u(0,x) = g(x) \quad x \in \mathbb{R}^3. \end{cases} \tag{1.1.1}$$

In 1977, Strichartz [18] proved that, if $u = u(t,x)$ is a solution for (1.1.1), we have

$$\|u\|_{L^4(\mathbb{R}^{1+3})} \leq C(\|f\|_{\dot{H}^{1/2}(\mathbb{R}^3)} + \|g\|_{\dot{H}^{-1/2}(\mathbb{R}^3)} + \|F\|_{L^{4/3}(\mathbb{R}^{1+3})}). \tag{1.1.2}$$

So if we consider the following semilinear Cauchy problem on $\mathbb{R}^{1+3}$,

$$\begin{cases} (\partial_t^2 - \Delta)u(t,x) = |u(t,x)|^3 \quad (t,x) \in \mathbb{R}^{1+3} \\ \\ u(0,x) = f(x),\ \partial_t u(0,x) = g(x) \quad x \in \mathbb{R}^3, \end{cases} \tag{1.1.3}$$

we have that

$$\|u\|_{L^4(\mathbb{R}^{1+3})} \leq C(\|f\|_{\dot{H}^{1/2}(\mathbb{R}^3)} + \|g\|_{\dot{H}^{-1/2}(\mathbb{R}^3)} + \|u\|^3_{L^4(\mathbb{R}^{1+3})}). \tag{1.1.4}$$

This suggests that, if the initial data $\|f\|_{\dot{H}^{1/2}(\mathbb{R}^3)}$ and $\|g\|_{\dot{H}^{-1/2}(\mathbb{R}^3)}$ are both small enough, estimate (1.1.4) provides sufficient criterion for the convergence of Picard iteration. In fact, this will deduce the global well-posedness for equation (1.1.3).

### 1.1.2 On Non-trapping Domain

In this section, we will give a brief review of the Local-Global Analysis Theory that will be used in later chapters.

The idea was originated from H. F. Smith and C. D. Sogge in [16] to deal with the non-trapping perturbation of Laplacians. J. Metcalfe and C. D. Sogge [13] used the Local-Global analysis and proved the long-time existence of quasilinear wave equations exterior to star-shaped obstacles in 2006.

In particular, the theory has been used to get the breakthrough results in the global well-posedness of 3 dimensional and 4 dimensional small data semilinear wave equations on non-trapping exterior domains by K. Hidano, J. Metcalfe, H. F. Smith, C. D. Sogge and Y. Zhou in [7]. Later in [23], Xin Yu generalized the theory so that it can deal with Strichartz estimates with derivative loss, as we will see later in Chapter 2.

Let $n \geq 2$ and $u = u(t,x)$ be the solution of the wave equation on the exterior domain $\Omega \in \mathbb{R}^n$ of a compact obstacle:

$$\begin{cases} (\partial_t^2 - \Delta_g)u = F(t,x) \qquad (t,x) \in \mathbb{R}^+ \times \Omega \\ u(0,x) = f(x),\, \partial_t u(0,x) = g(x) \qquad x \in \Omega \\ u(t,x)\mid_{\partial\Omega} = 0 \qquad (t,x) \in \mathbb{R}^+ \times \partial\Omega, \end{cases} \tag{1.1.5}$$

where $B$ is either the identity operator or the inward pointing normal derivative $\partial_\nu$. The operator $\Delta_g$ is the Laplace-Beltrami operator associated with a smooth, time-independent Riemannian metric $g(x)$ on $\Omega$. For some fixed $R > 0$, assume that $g_{ij}(x) = \delta_{ij}(x)$ when $|x| \geq 2R$ and the compact obstacle $K = \mathbb{R}^n \setminus \Omega$ is contained in $\{x \in \mathbb{R}^n : |x| < R\}$. That is, the metric coincides with the Euclidean metric near infinity.

The following hypothesis is a local energy decay estimate for the solution $u$:

**Hypothesis B.** [7] *Fix the boundary operator $B$ and the exterior domain $\Omega \in \mathbb{R}^n$ as above. We then assume that given $R_0 > 0$*

$$\int_0^\infty \left( \|u(t,\cdot)\|^2_{H^1(|x|<R_0)} + \|\partial_t u(t,\cdot)\|^2_{L^2(|x|<R_0)} \right) dt$$

$$\lesssim \|f\|^2_{H^1} + \|g\|^2_{L^2} \int_0^\infty \|F(s,\cdot)\|^2_{L^2} ds, \quad (1.1.6)$$

*whenever $u$ is a solution of* (1.1.5) *with data* $(f(x), g(x)$ *and forcing term* $F(t,x)$ *that both vanish for* $|x| > R_0$.

Let $X$ be an abstract norm, and here is the definition of abstract admissibility.

**Definition 1.1.1** (Definition 1.2 in [7])**.** *Let $q \geq 2$ and $\gamma \in \mathbb{R}$. We say that $(X, \gamma, q)$ is an admissible triple if the following two inequalities holds: (i) the Global Minkowski Abstract Strichartz Estimates*

$$\|v\|_{L^q_t X(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|v(0,\cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\partial_t v(0,\cdot)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}, \quad (1.1.7)$$

*assuming that*

$$(\partial_t^2 - \Delta)v = 0 \quad in \quad \mathbb{R}\times\mathbb{R}^n. \quad (1.1.8)$$

*(ii) Local Abstract Stichartz Estimates for $\Omega$*

$$\|u\|_{L^q_t X([0,1]\times\Omega)} \lesssim \|v(0,\cdot)\|_{\dot{H}^\gamma_B(\Omega)} + \|\partial_t v(0,\cdot)\|_{\dot{H}^{\gamma-1}_B(\Omega)}, \quad (1.1.9)$$

*provided that*

$$(\partial_t^2 - \Delta_g)u = 0 \quad in \quad [0,1]\times\Omega. \quad (1.1.10)$$

The result of Local-Global analysis for the abstract Strichartz estimates gives

**Theorem 1.1.1** (Corollary 1.5 in [7])**.** *Let $n \geq 2$. Assume that $(X, \gamma, q)$ and $(Y, 1-\gamma, r)$ are admissible triples and that Hypothesis B is valid. Also assume that*

$$q \geq 2 \quad \textit{and} \quad \gamma \in [0,1]. \tag{1.1.11}$$

*Then we have the following global abstract Strichartz estimates for the solution $u$ of* (1.1.5)

$$\|u\|_{L_t^q X(\mathbb{R}_+ \times \Omega)} \lesssim \|f\|_{\dot{H}_B^\gamma(\Omega)} + \|g\|_{\dot{H}_B^{\gamma-1}(\Omega)} + \|F\|_{L_t^{r'} Y'(\mathbb{R}^+ \times \Omega)}, \tag{1.1.12}$$

*where $r'$ denotes the Hölder conjugate exponent to $r$ and $\|\cdot\|_{Y'}$ is the dual norm to $\|\cdot\|_Y$*

Notice that, Theorem 1.1.1 derived Global Strichartz estimates from Local Strichartz estimates. The key idea is to bound the interaction terms generated when we decompose a global solutions into the sum of finite-time solutions.

And in fact, when $n = 3, 4$, if we take the following abstract norm $X_{\gamma,q}$, one can prove the global well-posedness for semilinear wave equations on $\Omega$ with supercritical exponent. [7]

$$\|h\|_{X_{\gamma,q}(\Omega)} = \|h\|_{L^{s_\gamma}(|x|<2R)} + \||x|^{n/2-(n+1)/q-\gamma} h\|_{L_r^q L_\omega^2(|x|>2R)}, \tag{1.1.13}$$

with $n(\frac{1}{2} - \frac{1}{s_\gamma}) = \gamma$.

## 1.2 The Hyperbolic Trapped Domain

In this section, we want to describe the geometric set-ups for Chapter 3 and Chapter 2. Throughout Chapter 3 and Chapter 2, we will restrict the dimension of the manifolds to be odd integers greater than 3.

$\Omega_0$ is assumed to be a smooth compact Riemannian manifold with boundary. For fixed $R > 0$, $B(0, R)$ is the ball of radius $R$ centred at the origin in $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus B(0, R)$ is the exterior domain in $\mathbb{R}^n$ outside the ball $B(0, R)$.

The ambient space $(\Omega, g)$ we will work on is taken to be $\Omega = \Omega_0 \sqcup (\mathbb{R}^n \setminus B(0, R))$ with Riemannian metric $g$. The metric $g$ equals to the Euclidean metric in the infinite end $\mathbb{R}^n \setminus B(0, R)$. Moreover, we want the trapped set $\mathcal{K}$ for the geodesic flow on the cosphere bundle $S^*\Omega$ to be normally hyperbolic, as it is defined in section 1.2 of [22]. We are now describing the geometric set up for the normally hyperbolic trapped domain $\Omega$.

### 1.2.1 Trapped Set $\mathcal{K}$

First we have to specify the definition of trapped set of $\Omega$.

Let $\pi$ denote the projection from the cotangent bundles to $\Omega$. Let $U_\alpha$ for some indices $\alpha \in A$ form a covering to $\Omega$. Now let $\varphi^t$ denote the geodesic flow on $\Omega$. We fix a point $x_0 \in \Omega$, and let $r(x)$ be the distance from $x \in \Omega$ to $x_0$. Now for each $U_\alpha$, we can define its backward/forward trapped sets $\Gamma_+^\alpha/\Gamma_-^\alpha$ by:

$$\Gamma_{\pm}^{\alpha} = \{\rho \in \pi^{-1}(U_\alpha) : \lim_{t \to \mp\infty} r(\varphi^t(\rho)) \neq \infty\}.$$

Actually $\Gamma_{+}^{\alpha}$ consists of the cotangent vectors for $U_\alpha$, for which its geodesic flow was trapped in a bounded subset of $\Omega$ if we trace backward in time infinitely. And $\Gamma_{-}^{\alpha}$ consists of those cotangent vectors whose geodesic flows will be trapped in a bounded subset of $\Omega$ when being traced forward in time infinitely.

Then we can define the backward/forward trapped sets $\Gamma_+/\Gamma_-$ respectively by:

$$\Gamma_{\pm} = \bigcup_{\alpha \in A} \Gamma_{\pm}^{\alpha}.$$

These two sets consist of all the cotangent vectors that corresponds to geodesics on $\Omega$ that are trapped forward or backward respectively.

Now we can give the definition of the trapped set based on the above set-up.

**Definition 1.2.1.** *Let $\Gamma_+$ and $\Gamma_-$ be the forward and backward trapped set for the Riemannian manifold $(\Omega, g)$, as it is described above. Then $\mathcal{K}$ is called the trapped set for $\Omega$ if*

$$\mathcal{K} = \Gamma_+ \cap \Gamma_-.$$

In fact, the trapped set $\mathcal{K}$ is a subset of the cosphere bundle $S^*\Omega$. A cotangent vector is in $\mathcal{K}$ if and only if its corresponding geodesic flow is bounded in a compact subset of $\Omega$. If the wave propagate along a trapped geodesic, other than flow to the space infinity, it will stay inside a compact subset. This means the energy will not disperse for such waves. As a result, it is possible to cause accumulation of energy in a small region, and break the well posedness of a wave or Schrödinger equation.

### 1.2.2 Hyperbolic Dynamical Assumptions

Then definition in 1.2 characterizes the feature of a general trapped set. The trapped set of a normally hyperbolic trapped domain has some particular dynamical assumptions, which restrict the problem into relatively weaker cases. And we describe those dynamical assumptions here. In the following chapters, we will always assume these Dynamical assumptions are satisfied.

Let $\Delta_g$ be the Laplace-Beltrami operator associated with the metric $g$. And let $p$ be the principle symbol of $\Delta_g$. Let $\Gamma_\pm^\lambda = \Gamma\pm \cap p^{-1}(\lambda)$ be the level set of the backward/forward trapped set, and $\mathcal{K}_\pm^\lambda = \mathcal{K}\pm \cap p^{-1}(\lambda)$ be the level set of the trapped set. Then the dynamical hypotheses are

**Dynamical Hypothesis**

*1, There exists* $\delta > 0$ *such that* $dp \neq 0$ *on* $p^{-1}(\lambda)$ *for* $|\lambda| < \delta$.

*2,* $\Gamma_\pm$ *are codimension-one smooth manifolds intersecting transversely at* $\mathcal{K}$.

*3, The flow is hyperbolic in the normal direction to* $\mathcal{K}$ *within the energy surface. This means that there are subbundles* $E^+$ *(or* $E_-$*) of* $T_{\mathcal{K}_\lambda}\Gamma_+^\lambda$ *(or* $T_{\mathcal{K}_\lambda}\Gamma_-^\lambda$*), which is* $\cup_{q\in\mathcal{K}_\lambda} T_q\Gamma_+^\lambda$ *(or* $\cup_{q\in\mathcal{K}_\lambda} T_q\Gamma_-^\lambda$*), have the following three properties:*

*(i)* $T_{\mathcal{K}_\lambda}\Gamma_+^\lambda$ *(or* $T_{\mathcal{K}_\lambda}\Gamma_-^\lambda$*) can be decomposed into the direct sum of the tangent space*

*of $\mathcal{K}_\lambda$ and the subspace of normal directions $E_+$ (or $E_-$). That is,*

$$T_{K_\lambda}\Gamma^\lambda_\pm = T\mathcal{K}_\lambda \oplus E^\pm.$$

*(ii) For the gradient flow of $\varphi^t$, we have*

$$d\varphi^t : E^\pm \to E^\pm.$$

*(iii) There exists $\theta > 0$ such that for all $|\lambda| < \delta$,*

$$d\varphi^t(v) \leq Ce^{-\theta|t|}\|v\|$$

*for all $v \in E^\mp$, $\pm t \geq 0$.*

### 1.2.3 Sobolev Spaces on $\Omega$

Before stating the main theorems, we have to specify some function spaces over $(\Omega, g)$ that we shall consider. So in this section, we will give those definitions. Such Sobolev spaces were also used in [7] and [23].

Recall that the metric $g$ described in previous sections is a smooth, time independent Riemannian metric. And the operator $\Delta_g$ is the Laplace-Beltrami operator associated with $g$. In section 1.1.1, we used $\Delta$ to denote the standard Laplacian on the Euclidean space $\mathbb{R}^n$. As before, $\hat{f}$ denotes the Fourier transform of $f$, if $f$ is any function on $\mathbb{R}^n$. And the homogeneous Sobolev space $\dot{H}^\gamma(\mathbb{R}^n)$ on the Euclidean space has norm defined to be

$$\|f\|^2_{\dot{H}^\gamma(\mathbb{R}^n)} = \|(\sqrt{-\Delta})^\gamma f\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^{-n}\int_{\mathbb{R}^n} ||\xi|^\gamma \hat{f}(\xi)|^2 d\xi.$$

While the inhomogeneous Sobolev space $H^\gamma(\mathbb{R}^n)$ on the Euclidean space has norm defined by

$$\|f\|^2_{H^\gamma(\mathbb{R}^n)} = \|(1-\Delta)^{\gamma/2} f\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^{-n}\int_{\mathbb{R}^n} |(1+|\xi|^2)^{\gamma/2} \hat{f}(\xi)|^2 d\xi.$$

We can also define a Sobolev space on a compact manifold, with or without boundary.

Let $(\tilde{\Omega}, \tilde{g})$ be a compact manifold (with or without boundary), and $\Delta_{\tilde{g}}$ be the associated Laplace-Beltrami operator. Then $\Delta_{\tilde{g}}$ has a discrete spectrum. Assume $e_{i}{}_{i=0}^{\infty}$ are the eigenfunctions for $\Delta_{\tilde{g}}$, with eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$. Then for each function $f$ on $\tilde{\Omega}$ and each $i$, we can project it onto the $i$-th eigenspace corresponding to $e_i$. We denote the projection by $E_i f$. Then $f$ can be written as an spectral decomposition

$$f = \sum_{i=1}^{\infty} E_i f = \sum_{i=1}^{\infty} < f, e_i >_{\tilde{g}} e_i.$$

So the Sobolev spaces on $(\tilde{\Omega}, \tilde{g})$ is defined using this spectral decomposition.

In particular, the homogeneous Sobolev space $\dot{H}^\gamma(\tilde{\Omega})$ has a norm given by

$$\|f\|^2_{\dot{H}^\gamma(\tilde{\Omega})} = \|(\sqrt{-\Delta_{\tilde{g}}})^\gamma f\|^2_{L^2(\tilde{\Omega})} = \sum_{i=1}^{\infty} |\lambda_i^\gamma E_i f|^2.$$

And the inhomogeneous Sobolev space $H^\gamma(\tilde{\Omega})$ has a norm defined to be

$$\|f\|^2_{H^\gamma(\tilde{\Omega})} = \|(1-\Delta_{\tilde{g}})^{\gamma/2} f\|^2_{L^2(\tilde{\Omega})} = \sum_{i=1}^{\infty} |(1+\lambda_i^2)^{\gamma/2} E_i f|^2.$$

Now we can define the Sobolev spaces on $\Omega$ that we will use later.

Take a cut-off function $\beta$ on $\mathbb{R}^n$. That is, $\beta$ is a compactly supported smooth function on $\mathbb{R}^n$. Let $R$ be the same as we used in section 1.2.1. Assume $\beta$ is supported in $\{x \in \mathbb{R}^n : |x| < 2R\}$, and $\beta(x) = 1$ when $|x| < R$. Let $\Omega' = \Omega \cap \{x \in \mathbb{R}^n : |x| < 2R\}$. Notice that by our choice of $R$, $\Omega$ is contained in $\Omega'$. Moreover, the metric $g$ is the same as the Euclidean metric on $\Omega \setminus \Omega'$. Let $\tilde{\Omega}$ be the embedding of $\Omega'$ into a compact manifold with boundary, so that $\partial\tilde{\Omega} = \partial\Omega$.

Then for any function $f$ on $\Omega$, we can write it as $f = \beta f + (1-\beta)f$. Then $\beta f$ is supported on $\Omega'$, so it can be identified with its pull-back function on $\tilde{\Omega}$. And the second part $(1-\beta)f$ is supported outside the ball of radius $R$. By its support property and the definition of $g$, $(1-\beta)f$ can be think as a function on the Euclidean space $\mathbb{R}^n$.

So we can define a homogeneous Sobolev norm for $f$ by

$$\|f\|_{\dot{H}^\gamma(\Omega)} = \|\beta f\|_{\dot{H}^\gamma(\tilde{\Omega})} + \|(1-\beta)f\|_{\dot{H}^\gamma(\mathbb{R}^n)}, \tag{1.2.1}$$

and an inhomogeneous Sobolev norm by

$$\|f\|_{H^\gamma(\Omega)} = \|\beta f\|_{H^\gamma(\tilde{\Omega})} + \|(1-\beta)f\|_{H^\gamma(\mathbb{R}^n)}, \tag{1.2.2}$$

The function spaces on $\Omega$ equipped with the norms given by (1.2.1) and (1.2.2) are the homogeneous Sobolev space $\dot{H}^\gamma(\Omega)$ and inhomogeneous Sobolev space $H^\gamma(\Omega)$ respectively.

Back to the geometric property of $\Omega$, we need to point out that when $\Omega$ has non-trivial trapped set, our estimates will contain some extra derivative loss. In order to deal with those loss, we introduce a Sobolev-type norm, as it described in [23].

Let us start with the Euclidean space $\mathbb{R}^n$ with the standard Laplacian $\Delta$. For all $\varepsilon$ and $\gamma$ in $\mathbb{R}$, we define $\tilde{H}^\gamma_\varepsilon(\mathbb{R}^n)$ to be the space equipped with norm

$$\begin{aligned}\|h\|_{\tilde{H}^\gamma_\varepsilon(\mathbb{R}^n)} &= \||\sqrt{-\Delta}|^\gamma(1-\Delta)^{\varepsilon/2}h\|_{L^2(\mathbb{R}^n)} \\ &= \left(\int_{\mathbb{R}^n} ||\xi|^\gamma(1+|\xi|^2)^{\varepsilon/2}\hat{h}(\xi)|^2 d\xi\right)^{1/2}.\end{aligned}$$

Notice that, for all $\varepsilon \in \mathbb{R}$, the norm defined above has the following equivalent form:

$$\|h\|_{\tilde{H}^\gamma_\varepsilon(\mathbb{R}^n)} \approx \|h\|_{\dot{H}^\gamma(|\xi|<1)} + \|h\|_{\dot{H}^{\gamma+\varepsilon}(|\xi|>1)}. \tag{1.2.3}$$

And for $\varepsilon \geq 0$,

$$\|h\|_{\tilde{H}^\gamma_\varepsilon(\mathbb{R}^n)} \approx \|h\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|h\|_{\dot{H}^{\gamma+\varepsilon}(\mathbb{R}^n)}. \tag{1.2.4}$$

For compact manifolds $(\tilde{\Omega}, \tilde{g})$, we can define the Sobolev-type norm $\tilde{H}^\gamma_\varepsilon(\tilde{\Omega})$ using spectral decomposition similarly. That is, for function $f$ on $\tilde{\Omega}$, its $\tilde{H}^\gamma_\varepsilon$ norm is given by

$$\|f\|_{\tilde{H}^\gamma_\varepsilon(\tilde{\Omega})} = \left(\sum_{i=0}^{\infty}(|\lambda_i|^\gamma(1+|\lambda_i|^2)^{\varepsilon/2}|E_i f|)^2\right)^{1/2}. \tag{1.2.5}$$

And for a normally hyperbolic trapped domain $(\Omega, g)$ we described in previous sections, the Sobolev-type norm $\|\cdot\|_{\tilde{H}^\gamma_\varepsilon(\Omega)}$ is defined using a cut-off function $\beta$, as we did earlier,

$$\|f\|_{\tilde{H}^\gamma_\varepsilon(\Omega)} = \|\beta f\|_{\tilde{H}^\gamma_\varepsilon(\tilde{\Omega})} + \|(1-\beta)f\|_{\tilde{H}^\gamma_\varepsilon(\mathbb{R}^n)}. \tag{1.2.6}$$

Notice that (1.2.3) and (1.2.4) also hold for $\|\cdot\|_{\tilde{H}^\gamma_\varepsilon(\Omega)}$ and $\|\cdot\|_{\tilde{H}^\gamma_\varepsilon(\tilde{\Omega})}$ when $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$ respectively.

## 1.3 Main Theorems

### 1.3.1 Main Theorems: Wave Equations

Now we can state our main result for the wave equations on the hyperbolic trapped domain $(\Omega, g)$ we described above.

Consider the following wave equation on $(\Omega, g)$.

$$\begin{cases} (\partial_t^2 - \Delta_g)u(t,x) = F(t,x), \quad (t,x) \in \mathbb{R}_+ \times \Omega \\ u(0,x) = f(x), \quad x \in \Omega \\ \partial_t u(0,x) = g(x), \quad x \in \Omega \\ u(t,x) = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{1.3.1}$$

If the forcing term $F(t,x) = 0$ for all $(t,x) \in \mathbb{R}_+ \times \Omega$, we get a homogeneous wave equation on the ambient space. Our first result here is for the homogeneous wave equation.

**Theorem 1.3.1.** *Let $n \geq 3$ be odd and $\Omega$ be a normally hyperbolic trapped domain described above. Let $u = u(t,x)$ be the solution of* (1.3.1) *with $F(t,x) = 0$. If $p > 2$, $\gamma \in (-\frac{n-3}{2}, \frac{n-1}{2})$, and $(p,q,\gamma)$ satisfies*

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma, \quad \begin{cases} \frac{3}{p} + \frac{2}{q} \leq 1, \quad n = 3 \\ \frac{2}{p} + \frac{2}{q} \leq 1, \quad n > 3. \end{cases} \tag{1.3.2}$$

*Then for all $\varepsilon > 0$,*

$$\|u\|_{L_t^p L_x^q(\mathbb{R}_+ \times \Omega)} \lesssim_\varepsilon \|f\|_{\tilde{H}_\varepsilon^\gamma(\Omega)} + \|g\|_{\tilde{H}_\varepsilon^{\gamma-1}(\Omega)}. \tag{1.3.3}$$

In section 1.2, we see that the wave propagating along trapped geodesics will stay in a bounded subset. This could possibly cause energy concentration in a small region. Above result shows that, in hyperbolic trapped case, the $L_p^t L_x^q$ mixed norm Strichartz estimate only needs an arbitrarily small derivative loss. This is understandable if we think about the dynamics for hyperbolic trapped set. From section 1.2.2, we see that the trapped set of $\Omega$ is highly unstable. An arbitrary small perturbation with a normal direction summand will result in an escaped geodesic.

Furthermore, we can get an estimate for the inhomogeneous problem. That is for equation (1.3.1) when $F(t,x)$ is not constantly zero.

**Theorem 1.3.2.** *Let $n \geq 3$ be odd, and $\Omega$ be the same as above. Assume $p > 2$, $\gamma \in (-\frac{n-3}{2}, \frac{n-1}{2})$, and $(p,q,\gamma)$ and $(r,s,1-\gamma)$ both satisfy* (1.3.3)*. Let $u = u(t,x)$ be the solution of* (1.3.1)*, then for any $\varepsilon, \delta > 0$, we have,*

$$\begin{aligned} \|u\|_{L_t^p L_x^q(\mathbb{R}_+\times\Omega)} \lesssim \|f\|_{\tilde{H}_\varepsilon^\gamma(\Omega)} + \|g\|_{\tilde{H}_\varepsilon^{\gamma-1}(\Omega)} \\ + \|F\|_{L_t^{r'} L_x^{s'}(\mathbb{R}_+\times\Omega)} + \|F\|_{L_t^{r'} L_x^{s'-\delta}(\mathbb{R}_+\times\Omega)}. \end{aligned} \tag{1.3.4}$$

*Here $r'$ and $s'$ are the Hölder's conjugate of $r$ and $s$ respectively.*

Compare our results with the classical mixed norm estimates for non-trapping domain, it turns out that the forcing term $F(t,x)$ needs to have better decay rates when $x$ approaches infinity.

We will prove theorem 1.3.1 and theorem 1.3.2 in Chapter 2.

### 1.3.2 Main Theorems: Schrödinger Equations

Now we can state our main results for the Schrödinger equations on the hyperbolic trapped domain $(\Omega, g)$ we described above.

Our goal here is to study the local well-posedness for the following nonlinear Schrödinger Equation on $\Omega$,

$$\begin{cases} (i\partial_t + \Delta_g)u(t,x) = F(u(t,x)), \quad (t,x) \in \mathbb{R}_+ \times \Omega \\ u(0,\cdot) = f \\ u(t,x) = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{1.3.5}$$

The nonlinear interaction $F$ is assumed to behave like $|u|^p$ when $|u|$ is small. By which we mean, there is a $\varepsilon_0 > 0$ such that,

$$\sum_{0 \le j \le 2} |u|^j |\partial_u^j F_p(u)| \lesssim |u|^p \tag{1.3.6}$$

When $|u| < \varepsilon_0$.

The main estimate we get for (1.3.5) is

**Theorem 1.3.3.** *For every finite $T > 0$, assume $(p,q) \in \mathbb{R}^2$, $p > 2$ satisfies*

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} \tag{1.3.7}$$

*Then there esists $C > 0$ such that*

$$\|u\|_{L_T^p W^{s,q}(\Omega)} \le CT^{1/(2p)} \|f\|_{H^{s+\varepsilon}(\Omega)} \tag{1.3.8}$$

*Where $s \in [0,\infty)$, $u(t) = e^{it\Delta_g f}$. Moreover,*

$$\|u\|_{L_T^p W^{s,q}(\Omega)} \le CT^{1/2p} \|F\|_{L_T^1 H^{s+\varepsilon}(\Omega)} \tag{1.3.9}$$

*Where $s \in [0,\infty)$, $u(t) = \int_0^t e^{i(t-\tau)\Delta_g} F(\tau) d\tau$.*

(1.3.8) is the Strichartz estimate for homogeneous Schrödinger equations. It is the estimate for (1.2.1) when $F = 0$. (1.3.9) is the Stichartz estimate for the inhomogeneous linear Schrödinger equations. It is the estimate for (1.2.1) when $F$ is independent of $u$.

We may notice that, the right hand side of both (1.3.8) and (1.3.9) depends on the ending time $T$.

Using the estimates in Theorem 1.3.3, we can show the following local well-posedness for (1.2.1).

**Theorem 1.3.4.** *For $n = 2$, $T > 0$, and $(p, q) \in \mathbb{R}^2$, $p > 2$, such that $1/p + 1/q = 1/2$. If*

$$X_T = L_T^\infty H^1(\Omega) \cap L_T^p W^{1-1/p-\varepsilon,q}(\Omega),$$

*and $R < CT^{-1/p(p-1)}$, then (1.3.5) has a solution in $B_R \cap X_T$.*

*For $n \geq 3$, $T > 0$ and $(p, q) \in \mathbb{R}^2$, $p > 2$ satisfies $1/p + n/q = n/2$. We also assume that $s > n/q$. If*

$$X_T^s = L_T^\infty H^{s+\varepsilon}(\Omega) \cap L_T^p W^{s,q}(\Omega)$$

*and $R < CT^{-1/p(p-1)}(1 + T^{1/(2p)})^{-1/(p-1)}$ is a positive radius. Then (1.3.5) has a solution in $B_R \cap X_T^s$.*

We will prove Theorem 1.3.3 and Theorem 1.3.4 in Chapter 3

# Chapter 2

# Global Strichartz Estimates for Wave Equations

## 2.1 Introduction

In this chapter, we will focus on the wave equation 1.3.1. Same as before, $(\Omega, g)$ is the normally hyperbolic trapped Rimannian manifold. $\Delta_g$ is the Laplace-Beltrami operator associated with the metric $g$. Our goal is to prove theorem 1.3.1 and theorem 1.3.2 for $\Omega$.

This work is based on X. Yu's Generalized Strichartz estimates theory. In [23], X. Yu generalized the abstract Strichartz estimate theory from [7] and developed the Generalized Strichartz estimates theory that takes the derivative loss into account. The idea is, if there are strong enough controls to the energy momentum of the solutions local in time, as the wave flows have finite speed of propagation, we can get a global estimate of the solution.

In particular, in order to get the desired global mixed norm estimate for wave equations, we have to first prove a good enough time global bounds for the energy momentum. So in section 2.2, we will derive a local energy estimate. It is important in the sense that, such estimates provides the control for the solution near the trapped set. It turns out that, no matter how long does the wave propagate, the bad behaviour generated from the trapped set will be controlled by the initial values and the forcing term. However, instead of using the classical Sobolev data, the data we need here has to be in the Sobolev-type space we defined in Chapter 1.

The second essential estimate we need is the Strichartz estimate for finite time. And we will prove it in section 2.3. The problem here is that, as the wave propagate to the space infinity, the bad behaviours generated near the trapped set will also propagate and cause bad effect to the part of the solution near the infinity end of the manifold. That is the reason we start with the Strichartz estimate for finite time. Because of the superposition property of wave equations, we can decompose the solution of our (1.3.1) into two parts: the part near the trapped set, and the part near the infinity end. Since the wave has finite propagation speed, the bad effects only propagated for finite distances in finite time. It is interesting to see that, in this case, we do not need the data to have any extra derivative loss.

Then in section 2.4, we will prove two of our main theorems. That is the theorem 1.3.1 and the theorem 1.3.2. After proving the local energy decay and the local Strichartz estimates, the global Strichartz estimates for homogeneous wave equation follows by applying the theory of generalized Strinchartz estimates in [23]. The inhomogeneous estimates is more interesting if we want to avoid loss of derivatives for the forcing term. As you will see in section 2.4.2, we can hedge the derivative loss using two parallel $TT^*$ argument.

## 2.2 Local Energy Estimates

Consider $\Omega = \Omega_0 \bigsqcup(\mathbb{R}^n \setminus B(0,R))$ be a normally hyperbolic trapped domain, as it is described in section 1.2 of Chapter 1.

Consider a distribution $u = u(t,x)$ that solves (1.3.1) with $f$, $g$, $F$ compactly supported near $\Omega_0$.

Then a special case of Theorem 3 in Section 5 of [22] is,

**Corollary 2.2.1.** *Supposed that $\Omega$ is a normally hyperbolic trapped domain as above. Assume that, if for a fixed $R > 0$, $u$ solves* (1.3.1)*. Assume that $F = 0$ and $f$,$g$ compactly supported in $\{|x| < R\}$. Then there exists $\alpha > 0$, $K \in \mathbb{Z}^+$ such that,*

$$\int_{|x|<R} (|u'(t,x)|^2 + |\partial_t u(t,x)|^2)dx \le Ce^{-\alpha t}(||f||^2_{H^{K+1}(|x|<R)} + ||g||^2_{H^K(|x|<R)}), \quad (2.2.1)$$

*for some $C = C(\alpha, K, R)$.*

Interpolate (2.2.1) with the energy estimate, we get the following lemma, which suggests the exponential decay for the local energy of the solutions with respect to arbitrarily small derivative loss of the initial data.

**Lemma 2.2.1.** *Fix $R > 0$. If $u = u(t,x)$ solves (1.3.1) with $f$, $g$, $F$ compactly supported near $\partial\Omega$, $\exists c > 0$, so that*

$$\|u'(t,x)\|_{L^2_x(|x|<R)} \lesssim e^{-ct}(\|f\|_{\dot{H}^{1+\varepsilon}(|x|<R)} + \|g\|_{\dot{H}^{\varepsilon}(|x|<R)}) \tag{2.2.2}$$

*For any $\varepsilon > 0$*

Using this estimate, we can get the local energy decay estimate, as it is stated in the following proposition.

**Proposition 2.2.1.** *Assume $u = u(t,x)$ solves (1.3.1), and $f$, $g$, $F$ vanishes for $|x| > R$. Then we have*

$$\int_0^\infty (\|u(t,x)\|^2_{\dot{H}^1(|x|<R)} + \|\partial_t u(t,x)\|^2_{L^2(|x|<R)})dt$$
$$\lesssim \|f\|^2_{\dot{H}^{1+\varepsilon}(\Omega)} + \|g\|^2_{\dot{H}^{\varepsilon}(\Omega)} + \int_0^\infty \|F(s,x)\|^2_{\dot{H}^{\varepsilon}(\Omega)}ds \tag{2.2.3}$$

**Proof:**

Now let us prove Proposition 2.2.1.

Notice that the functions of both sides of the equation are compactly supported, the left hand side of (2.2.3) is equivalent to,

$$A = \|u'(t,x)\|^2_{L^2_t L^2_x(\mathbb{R}_+\times\{|x|<R\})} \tag{2.2.4}$$

When $F = 0$, square and integrate both sides of (2.2.2), we can get that

$$A^2 \lesssim \|f\|^2_{\dot{H}^{1+\varepsilon}(\Omega)} + \|g\|^2_{\dot{H}^{\varepsilon}(\Omega)} \tag{2.2.5}$$

Assume that $D_g = \sqrt{-\Delta_g}$. Take $f = 0$ in (2.2.2), we can see that

$$\|\sin(t|D_g|)h\|_{L^2_x(|x|<R)} \lesssim e^{-ct}\|h\|_{\dot{H}^{\varepsilon}(\Omega)}. \tag{2.2.6}$$

for $h \in \dot{H}^{\varepsilon}(\Omega)$ Supported in $|x| < R$.

And take $f = 0$ in (2.2.2), we can get

$$\|\cos(t|D_g|)h\|_{L^2_x(|x|<R)} \lesssim e^{-ct}\|h\|_{\dot{H}^{\varepsilon}(\Omega)}. \tag{2.2.7}$$

So by Euler's equation,

$$\|e^{it|D_g|}h\|_{L^2_x(|x|<R)} \lesssim e^{-ct}\|h\|_{\dot{H}^{\varepsilon}(\Omega)}. \tag{2.2.8}$$

By Duhamel's Principle, if we apply Minkowski inequality to

$$\|e^{it|D_g|}h\|_{L^2_x(|x|<R)} \lesssim e^{-ct}\|h\|_{\dot{H}^{\varepsilon/2}(\Omega)},$$

we get the desired estimate for $F \neq 0$.

$\square$

## 2.3 Local Strichartz Estimates

In order to prove the case global Strichartz estimates, we need to first prove a Local Strichartz Estimate in this section. Because we are considering finite time estimate, the problem is relatively simple, as the trapped effect would not propagate to infinity.

The main result for this section is the following estimate for homogeneous wave equations.

**Proposition 2.3.1.** *If $u = u(t,x)$ solves (1.3.1) with $F = 0$. Assume that $p > 2$ and $\gamma \in (-\frac{n-3}{2}, \frac{n-1}{2})$. Then for $(p,q,\gamma)$ satisfying (1.3.2), we have,*

$$\|u\|_{L_t^p L_x^q([0,1]\times\Omega)} \lesssim \|f(x)\|_{\dot{H}^\gamma(\Omega)} + \|g(x)\|_{\dot{H}^{\gamma-1}(\Omega)} \tag{2.3.1}$$

The idea of the proof is to decompose the solution $u(t,x)$ into the sum of two parts, because of the principle of superposition. The first part is supported near $\Omega_0$. We will see that it is equivalent to a solution of a wave equation on a compact manifold with boundary. The second part is supported near the infinity end $\mathbb{R}^n \setminus B(0,R)$. This part is, in fact, equivalent to a solution of a wave equation on the Minkowski space.

Now let us see the proof explicitly.

**Proof of Proposition 2.3.1:**

Consider a smooth cutoff function $\varphi \in C_0^\infty(\mathbb{R}^n)$. We assume that $\varphi(x) = 0$ on $\{x \in \mathbb{R}^n : |x| > 2R\}$ and $\varphi(x) = 1$ when $|x| < R$. Recall that in Section 1.2 of Chapter 1, we assumed $\Omega \in \{x \in \mathbb{R}^n : |x| > 2R\}$.

For the $u$ solving (1.3.1), let us take $v = \varphi u$ and $w = (1-\varphi)u$. Then we can write $u$ as

$$u = v + w = \varphi u + (1-\varphi)u \tag{2.3.2}$$

**(i) Estimate** $w = (1-\varphi)u$

As we assumed, $F(t,x) = 0$ for all $(t,x) \in \mathbb{R}_+ \times \Omega$. So

$$\begin{aligned}(\partial_t^2 - \Delta_g)w &= (1-\varphi)\partial_t^2 u - \varphi\Delta_g u - [\varphi, \Delta_g]u \\ &= -[\varphi, \Delta_g]u.\end{aligned}$$

Notice that $g_{ij} = \delta_{ij}$ when $|x| > R$, and $w = (1-\varphi)u$ is supported away from $\{x \in \mathbb{R}^n : |x| > 2R\}$. This means that taking $\Delta_g w$ is the same as taking $\Delta w$, where $\Delta$ is the standard Laplacian on the Euclidean space. So $w$ solves the following wave equation:

$$\begin{cases}(\partial_t^2 - \Delta)w = -[\varphi, \Delta_g]u \\ w(0,x) = (1-\varphi(x))f(x) \\ \partial_t w(0,x) = (1-\varphi(x))g(x)\end{cases} \tag{2.3.3}$$

In fact, as $G(t,x) = -[\varphi, \Delta_g]u$ is supported on the annulus $\{x \in \mathbb{R}^n : R < |x| < 2R\}$, (2.3.3) is a wave equation on the Minkowski Space $\mathbb{R}_+ \times \mathbb{R}^n$.

Now we can write $w$ as the sum of $w_0$ and $w_1$, the solution for the homogeneous equation and a particular solution for the inhomogeneous equation.

So we want $w_0$ to solve the corresponding homogeneous equation of (2.3.3), that is,

$$\begin{cases} (\partial_t^2 - \Delta)w_0 = 0 \\ w_0(0,x) = (1-\varphi(x))f(x) \\ \partial_t w_0(0,x) = (1-\varphi(x))g(x). \end{cases} \tag{2.3.4}$$

By Corollary 1.2 in chapter 4 of [17], we have,

$$\|w_0\|_{L_t^p L_x^q([0,1]\times\mathbb{R}^n)} \lesssim \|(1-\varphi(x))f(x)\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|(1-\varphi(x))g(x)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}. \tag{2.3.5}$$

for $(p,q,\gamma)$ satisfying (1.3.2).

And for $w_1$, it solves the inhomogeneous equation with zero initial data, that is,

$$\begin{cases} (\partial_t^2 - \Delta)w_1 = -[\varphi, \Delta_g]u \\ w_1(0,x) = 0 \\ \partial_t w_1(0,x) = 0 \end{cases} \tag{2.3.6}$$

By (3.2.6) and Minkowski inequality, we have that

$$\begin{aligned} \| \int_0^t e^{i(t-s)|D|}|D|^{-1}G(t,\cdot)ds\|_{L_t^p L_x^q([0,1]\times\mathbb{R}^n)} & \\ \lesssim \int_0^t \|e^{i(t-s)|D|}|D|^{-1}G(t,\cdot)\|_{L_t^p L_x^q([0,1]\times\mathbb{R}^n)}ds & \\ \lesssim \|G(t,x)\|_{L_t^2 \dot{H}^{\gamma-1}([0,1]\times\mathbb{R}^n)} & \end{aligned} \tag{2.3.7}$$

Where $|D| = \sqrt{-\Delta}$.

So combine (3.2.6) and (2.3.7), we get that

$$\|w\|_{L_t^p L_x^q([0,1]\times\Omega)} \lesssim \|(1-\varphi)f\|_{\dot H^\gamma(\mathbb{R}^n)} + \|(1-\varphi)g\|_{\dot H^{\gamma-1}(\mathbb{R}^n)} + \|[\varphi,\Delta_g]u\|_{L_t^2 \dot H^{\gamma-1}([0,1]\times\mathbb{R}^n)} \quad (2.3.8)$$

As $[\varphi,\Delta_g]u = -(\Delta_g\varphi)u + \nabla_g\varphi\cdot\nabla_g u$ is supported on $\{x\in\mathbb{R}^n : R<|x|<2R\}$, and $g_{ij}=\delta_{ij}$ on this annulus, we know that

$$\|[\varphi,\Delta_g]u\|_{L_t^2 \dot H^{\gamma-1}([0,1]\times\mathbb{R}^n)} = \|[\varphi,\Delta_g]u\|_{L_t^2 \dot H^{\gamma-1}([0,1]\times\Omega)}.$$

So by energy inequality, we can get

$$\|[\varphi,\Delta_g]u\|_{L_t^2 \dot H^{\gamma-1}([0,1]\times\mathbb{R}^n)} \lesssim \|f\|_{\dot H^\gamma(\Omega)} + \|g\|_{\dot H^{\gamma-1}(\Omega)} \quad (2.3.9)$$

So plug (2.3.9) into (2.3.8), we can see that $w$ is bounded by

$$\|(1-\varphi)f\|_{\dot H^\gamma(\mathbb{R}^n)} + \|(1-\varphi)g\|_{\dot H^{\gamma-1}(\mathbb{R}^n)} + \|f\|_{\dot H^\gamma(\Omega)} + \|g\|_{\dot H^{\gamma-1}(\Omega)},$$

which is controlled by a constant times

$$\|f\|_{\dot H^\gamma(\Omega)} + \|g\|_{\dot H^{\gamma-1}(\Omega)}.$$

**(ii) Estimate $v=\varphi u$**

Now let us consider $v=\varphi u$.

Since $v=\varphi u$,

$$
\begin{aligned}
(\partial_t^2 - \Delta_g)v &= \varphi\partial_t^2 u - \varphi\Delta_g u + [\varphi, \Delta_g]u \\
&= [\varphi, \Delta_g]u.
\end{aligned}
$$

So $v$ is a solution to the following wave equation,

$$
\begin{cases}
(\partial_t^2 - \Delta_g)v = [\varphi, \Delta_g]u \\
v(0, x) = \varphi(x)f(x) \\
\partial_t v(0, x) = \varphi(x)g(x) \\
v(t, x)|_{\partial\Omega} = 0
\end{cases}
\tag{2.3.10}
$$

Since $v$ itself, the forcing term $[\varphi, \Delta_g]u$ and the initial conditions are all compactly supported in $\{x \in \Omega : |x| < 2R\}$, (2.3.10) can be considered as a wave equation on the compact manifold $(\tilde{\Omega}, \tilde{g})$ with boundary. In particular, we can take $\tilde{\Omega}$ so that $\partial\tilde{\Omega} = \partial\Omega$, and $\tilde{g}$ coincide with $g$.

Now we can write $v = v_0 + v_1$, where $v_0$ solves the corresponding homogeneous equation and $v_1$ solves the inhomogeneous equation.

More precisely, for $v_0$ solves the homogeneous wave equation

$$\begin{cases} (\partial_t^2 - \Delta_{\tilde{g}})v_0 = 0 \\ v_0(0,x) = \varphi(x)f(x) \\ \partial_t v_0(0,x) = \varphi(x)g(x) \\ v_0(t,x)|_{\partial\tilde{\Omega}} = 0. \end{cases} \tag{2.3.11}$$

Then by Theorem 1.1 in [1], if $(p, q, \gamma)$ satisfies (1.3.2), we have that

$$\|v_0\|_{L_t^p L_x^q([0,1]\times\Omega)} \approx \|v_0\|_{L_t^p L_x^q([0,1]\times\tilde{\Omega})} \tag{2.3.12}$$

$$\lesssim \|\varphi f\|_{\dot{H}^\gamma(\tilde{\Omega})} + \|\varphi g\|_{\dot{H}^{\gamma-1}(\tilde{\Omega})} \tag{2.3.13}$$

$$\approx \|\varphi f\|_{\dot{H}^\gamma(\Omega)} + \|\varphi g\|_{\dot{H}^{\gamma-1}(\Omega)}. \tag{2.3.14}$$

Notice that, by Duhammel's principle and (2.3.13) for $f = 0$, we get that

$$\|h\|_{L_t^p L_x^q([0,1]\times\Omega)} \lesssim \|G(t,x)\|_{L_t^2 H^{\gamma-1}([0,1]\times\tilde{\Omega})} \tag{2.3.15}$$

If $h$ solves $(\partial_t^2 - \Delta_g)h = G(t,x)$ for $(t,x) \in [0,1]\times\tilde{\Omega}$ with zero initial data and zero boundary value.

In fact, $v_1$ solves the inhomogeneous wave equation below:

$$\begin{cases} (\partial_t^2 - \Delta_{\tilde{g}})v_1 = [\varphi, \Delta_g]u \\ v_0(0,x) = 0 \\ \partial_t v_0(0,x) = 0 \\ v_0(t,x)|_{\partial\tilde{\Omega}} = 0. \end{cases} \tag{2.3.16}$$

So by (2.3.15),

$$\|v_1\|_{L_t^p L_x^q([0,1]\times\Omega)} \lesssim \|[\varphi, \Delta_g]u\|_{L_t^2 H^{\gamma-1}([0,1]\times\tilde{\Omega})}. \tag{2.3.17}$$

So by (2.3.13) and (2.3.17), we have

$$\|v\|_{L_t^p L_x^q([0,1]\times\Omega)} \lesssim \|\varphi f\|_{\dot{H}^\gamma(\tilde{\Omega})} + \|\varphi g\|_{\dot{H}^{\gamma-1}(\tilde{\Omega})} + \|[\varphi, \Delta_g]u\|_{L_t^2 H^{\gamma-1}([0,1]\times\tilde{\Omega})} \tag{2.3.18}$$

where the right hand side of the inequality is bounded by $\|f\|_{\dot{H}^\gamma(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}(\Omega)}$, as $\|[\varphi, \Delta_g]u\|_{L_t^2 H^{\gamma-1}([0,1]\times\tilde{\Omega})} \approx \|[\varphi, \Delta_g]u\|_{L_t^2 H^{\gamma-1}([0,1]\times\Omega)}$.

Therefore, by Minkowski's inequality, we got the desired local Strichartz estimate for $u$ solving (1.3.1).

$\square$

# 2.4 Global Strichartz Estimates

## 2.4.1 Homogeneous Wave Equation

Let us first prove Theorem 1.3.1, the global mixed norm estimates when $F(t,x) = 0$ in (1.3.1).

There is a long history of establishing the global Minkowski Strichartz estimate, beginning with the original work by Strichartz [18]. Some subsequent works are done by Genibre-Velo [6], Pecher [15], Kapitanski [10], Lindblad-Sogge [12], Mockenhaupt-Seeger-Sogge [14], Keel-Tao [11], etc. One of the result is the following estimate, which is Corollary 2.1 in chapter 4 of [17], as we used several times above.

**Global Minkowski Strichartz estimates.**

Let $u = u(t,x)$ be a solution to (1.3.1) with $F = 0$, in the case of $\Omega = \mathbb{R}^n$ and $g_{ij} = \delta_{ij}$. Assume that $p > 2$, and $(p, q, \gamma)$ satisfies (1.3.2). Then we have the following mixed norm Strichartz estimate hold for $u$:

$$\|u\|_{L_t^p L_x^q(\mathbb{R}^{1+n})} \lesssim \|f\|_{\dot{H}^{\gamma}(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} \tag{2.4.1}$$

**Almost admissibility**

As we mentioned in the introduction of this Chapter, in [23] Xin Yu developed generalized Strichartz estimate theory to deal with Strichartz estimates with derivative loss on the initial data. A key definition in her theory is the **almost admissibility** of function spaces. Let us quote the definition as follows.

**Definition 2.4.1** (Definition 1.4 in [23]). *Let $X$ be some Strichartz norm, and $\gamma$, $\eta$, $p$ are some real numbers. We say that $(X, \gamma, \eta, p)$ is* **almost admissible** *if it satisfies*

*i), Minkowski almost Strichartz estimates*

$$\|u\|_{L^p_t X([0,S]\times\mathbb{R}^n)} \lesssim \|u(0,\cdot)\|_{\dot{H}^{\gamma}(\mathbb{R}^n)} + \|\partial_t u(0,\cdot)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} \tag{2.4.2}$$

*ii), Local almost Strichartz estimates for $\Omega$*

$$\|u\|_{L^p_t X([0,1]\times\Omega)} \lesssim \|u(0,\cdot)\|_{\tilde{H}^{\gamma}_{\eta}(\Omega)} + \|\partial_t u(0,\cdot)\|_{\tilde{H}^{\gamma-1}_{\eta}(\Omega)} \tag{2.4.3}$$

According to the generalized Strichartz estimate theory in [23], for almost admissible spaces $(X, \gamma, \eta, p)$, we have the following Local-global theorem for abstract Strichartz estimates.

**Theorem 2.4.1** (Theorem 1.5 in [23] for Dirichlet-wave equation). *Let $n > 2$ and assume that $(X, \gamma, \eta, p)$ is almost admissible with*

$$p > 2 \quad \text{and} \quad \gamma \in (-\frac{n-3}{2}, \frac{n-1}{2}).$$

*Then if there exists some $\varepsilon$ such that the local smoothing estimate* (2.2.3) *is valid and if $u$ solves* (1.3.1) *with forcing term $F = 0$, we have the abstract Strichartz estimates*

$$\|u\|_{L^p_t X([0,\infty)\times\Omega} \lesssim \|f\|_{\tilde{H}^{\gamma}_{\varepsilon+\eta}(\Omega)} + \|g\|_{\tilde{H}^{\gamma-1}_{\varepsilon+\eta}(\Omega)} \tag{2.4.4}$$

**End of proof for Theorem 1.3.1:**

Now let us consider $u = u(t, x)$ solves the (1.3.1) with $F(t, x) = 0$. From the result of section 2.2, we can see that (2.2.3) is valid for $u$, $f$, and $g$. By proposition 2.3.1

and (2.4.1), it follows that $(L^q(\Omega), \gamma, 0, p)$ is almost admissible, if $\gamma \in (-\frac{n-3}{2}, \frac{n-1}{2})$ and $(p, q, \gamma)$ satisfies (1.3.2). So our Theorem 1.3.1 follows directly from Theorem 2.4.1 above. $\square$

### 2.4.2 Inhomogeneous Wave Equation

The inhomogeneous estimate is interesting in the hyperbolic trapped case, because the loss of derivative grows up when there are non-trivial forcing terms. In particular, if the initial data $f$ and $g$ have $\varepsilon$-loss of derivative, the forcing term $F$ will have $2\varepsilon$-loss of derivative. It is obvious to see that if we iterate for a non-linear equation, such growing effect could accumulate and blow up and the number of iterations tends to infinity. So we want to get rid of such effect.

In Theorem 1.3.2, we in fact get rid of the derivative loss by using two mixed norms for the forcing term $F(t, x)$. This means, the control for the inhomogeneous solutions can be depend on $F$ itself, which is independent of its derivatives. As you can see in the following proof, it turns out that we can hedge the derivative loss by using the two different mixed norms. You may also notice that, our main idea is still the classical $TT^*$ duality argument.

Another important ingredient is the Christ-Kiselev lemma [3]. We quote the lemma here for completeness. A proof can be found in [16].

**Lemma 2.4.1** (Christ-Kiselev Lemma [3]). *Let $X$ and $Y$ be Banach spaces and assume that $K(t, s)$ is a continuous function taking its values in $B(X, Y)$, the space* *of bounded linear mappings from $X$ to $Y$ that $-\infty \leq a < b \leq \infty$, and set*

$$Tf(t) = \int_a^b K(t,s)f(s)ds. \tag{2.4.5}$$

*Assume that*

$$\|Tf\|_{L^q[a,b],Y} \leq C\|f\|_{L^p([a,b],X)}. \tag{2.4.6}$$

*Set*

$$Tf(t) = \int_a^t K(t,s)f(s)ds. \tag{2.4.7}$$

*Then, if $1 \leq p < q < \infty$,*

$$\|Wf\|_{L^q([a,b],Y)} \leq \frac{2^{-2(1/p-1/q)} \cdot 2C}{1-2^{-(1/p-1/q)}}\|f\|_{L^p([a,b],X)}. \tag{2.4.8}$$

Now let us prove the inhomogeneous estimate in Theorem 1.3.2.

**Proof of Theorem 1.2:**

Comparing Theorem 1.3.1 and Theorem 1.3.2, we can see that it suffices to show the case in which $f = g = 0$.

By Duhammel's Principle, we have to show,

$$\|\int_0^t e^{i(t-s)|D|}|D|^{-1}F(s,\cdot)ds\|_{L^p_t L^q_x(\mathbb{R}_+\times\Omega)}$$
$$\lesssim \|F(t,x)\|_{L^{r'}_t L^{s'}_x(\mathbb{R}_+\times\Omega)} + \|F(t,x)\|_{L^{r'}_t L^{s'-\delta}_x(\mathbb{R}_+\times\Omega)}, \tag{2.4.9}$$

for $(p,q,\gamma)$ and $(r,s,1-\gamma)$ satisfying (1.3.2) respectively, and any $\delta > 0$

Taking $T$ to be the operator

$$TF(t,\cdot) = \int_0^t e^{i(t-s)|D|}|D|^{-1}F(s,\cdot)ds \tag{2.4.10}$$

in Lemma 2.4.1, we can see that it suffices to show

$$\begin{aligned}&\|\int_0^\infty e^{i(t-s)|D|}|D|^{-1}F(s,\cdot)ds\|_{L_t^pL_x^q(\mathbb{R}_+\times\Omega)} \\ &\qquad\lesssim \|F(t,x)\|_{L_t^{r'}L_x^{s'}(\mathbb{R}_+\times\Omega)} + \|F(t,x)\|_{L_t^{r'}L_x^{s'-\delta}(\mathbb{R}_+\times\Omega)}\end{aligned} \tag{2.4.11}$$

By theorem 1.3.1, the left hand side of (2.4.11) is bounded by

$$\|\int_0^\infty e^{-is|D|}|D|^{-1}F(s,\cdot)ds\|_{\tilde{H}^\gamma_{2\varepsilon}(\Omega)}, \tag{2.4.12}$$

for any $\varepsilon > 0$.

According to (1.2.3), (2.4.12) equals to

$$\|\int_0^\infty e^{-is|D|}F(s,\cdot)ds\|_{\dot{H}^{\gamma-1}(|\xi|\le 1)} + \|\int_0^\infty e^{-is|D|}F(s,\cdot)ds\|_{\dot{H}^{\gamma+2\varepsilon-1}(|\xi|\ge 1)} \tag{2.4.13}$$

So in order to prove the theorem, we need to show that

$$\begin{aligned}&\|\int_0^\infty e^{-is|D|}F(s,\cdot)ds\|_{\dot{H}^{\gamma-1}(|\xi|\le 1)} + \|\int_0^\infty e^{-is|D|}F(s,\cdot)ds\|_{\dot{H}^{\gamma+2\varepsilon-1}(|\xi|\ge 1)} \\ &\qquad\lesssim \|F(t,x)\|_{L_t^{r'}L_x^{s'}(\mathbb{R}_+\times\Omega)} + \|F(t,x)\|_{L_t^{r'}L_x^{s'-\delta}(\mathbb{R}_+\times\Omega)}\end{aligned} \tag{2.4.14}$$

In fact, if $(r, s, 1-\gamma)$ satisfies (1.3.2), by Theorem 1.3.1, for all $\varepsilon > 0$,

$$\|e^{it\sqrt{-\Delta_g}}f\|_{L_t^rL_x^s(\mathbb{R}_+\times\Omega)} \lesssim \|f\|_{\tilde{H}_\varepsilon^{1-\gamma}(\Omega)}. \tag{2.4.15}$$

By duality,

$$\|\int_0^\infty e^{-is|D|}F(s,\cdot)ds\|_{\tilde{H}_{-2\varepsilon}^{\gamma-1}(\Omega)} \lesssim \|F(t,x)\|_{L_t^{r'}L_x^{s'}(\mathbb{R}_+\times\Omega)} \tag{2.4.16}$$

According to the definition of the $\tilde{H}_\varepsilon^\gamma$ norm, the left hand side of (2.4.16) is equivalent to

$$\|\int_0^\infty e^{-is|D|}F(s,\cdot)ds\|_{\dot{H}^{\gamma-1}(|\xi|\le 1)} + \|\int_0^\infty e^{-is|D|}F(s,\cdot)ds\|_{\dot{H}^{\gamma-2\varepsilon-1}(|\xi|\ge 1)} \tag{2.4.17}$$

So the first term in (2.4.14) is bounded by $\|F(t,x)\|_{L_t^{r'}L_x^{s'}(\mathbb{R}_+\times\Omega)}$, which is exactly the first term on the left hand side of the inequality (2.4.14).

Then consider $(r,\tilde{s},1-(\gamma+4\varepsilon))$ satisfying (1.3.2). Similar as (2.4.15), we get

$$\|e^{it\sqrt{-\Delta_g}}f\|_{L_t^rL_x^{\tilde{s}}(\mathbb{R}_+\times\Omega)} \lesssim \|f\|_{\tilde{H}_{2\varepsilon}^{1-(\gamma+4\varepsilon)}(\Omega)}. \tag{2.4.18}$$

Then the duality of (2.4.18) gives

$$\|\int_0^\infty e^{-is|D|}F(s,\cdot)ds\|_{\tilde{H}_{-2\varepsilon}^{\gamma+4\varepsilon-1}(\Omega)} \lesssim \|F(t,x)\|_{L_t^{r'}L_x^{\tilde{s}'}(\mathbb{R}_+\times\Omega)} \tag{2.4.19}$$

And the left and side of (2.4.19) turns out to be,

$$\begin{aligned}\| \int_0^\infty e^{-is|D|} F(s,\cdot) ds \|_{\tilde{H}^{\gamma+4\varepsilon-1}_{-2\varepsilon}(\Omega)} \approx \| \int_0^\infty e^{-is|D|} F(s,\cdot) ds \|_{\dot{H}^{\gamma+2\varepsilon-1}(|\xi|\geq 1)} \\ + \| \int_0^\infty e^{-is|D|} F(s,\cdot) ds \|_{\dot{H}^{\gamma+4\varepsilon-1}(|\xi|\leq 1)}, \quad (2.4.20)\end{aligned}$$

which provides the bound for the second term in (2.4.14). Notice that $\varepsilon$ can be arbitrarily small, it follows that we can choose $\tilde{s}' = s' - \delta$, for arbitrarily small $\delta > 0$. And this completes the proof for Theorem 1.2.

$\square$

# Chapter 3

# Local Strichartz Estimates for Schrödinger Equations

## 3.1 Local Strichartz Estimates

Let us prove the Strichartz type estimate in Theorem 1.3.3 first.

Using the result in Corollary 2 of Wunsch and Zworski [22], we can prove the following lemma.

**Lemma 3.1.1.** *Let $u = u(t, x)$ solves (1.3.5) with forcing term $F = F(t, x)$ independent of $u$. Let $\rho(x) \in C_0^\infty(\mathbb{R}^n)$, and $\rho = 1$ on $\{x \in \mathbb{R}^n : |x| < 2R\}$. Then we have the following local energy decay estimates. If $f$, $F$ are supported in $|x| < 2R$,*

$$\|\rho u\|_{L_t^2 H^\gamma([0,T]\times\Omega)} \lesssim \|f\|_{H^{\gamma+\varepsilon-\frac{1}{2}}(\Omega)} + \|F\|_{L_t^2 H^{\gamma+2\varepsilon-1}([0,T]\times\Omega)}, \tag{3.1.1}$$

*and*

$$\| u \|_{L_t^\infty H^\gamma} \lesssim \| f \|_{H^\gamma} + \| F \|_{L_t^2 H^{\gamma+\varepsilon-\frac{1}{2}}} . \tag{3.1.2}$$

*For all* $\gamma \in \mathbb{R}$.

**Proof:**

(i).Firstly, let us consider (3.1.1).

Recall that, by Corollary 2 of [22], we have that for any $\varepsilon > 0$,

$$\int_0^T \|(1+|x|^2)^{-1/3} e^{it\Delta_g} f\|^2_{H^{1/2-\varepsilon}(\Omega)} \lesssim \|f\|^2_{L^2(\Omega)}. \tag{3.1.3}$$

So we can take $\rho(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\rho = 1$ on $\{x \in \mathbb{R}^n : |x| < 2R\}$. Since $f(x) = 0$ when $|x| \geq 2R$, it follows that

$$\begin{aligned} \|\rho e^{it\Delta_g} f\|^2_{L_t^2 H^{1/2-\varepsilon}([0,T]\times\Omega)} &\lesssim \int_0^T \|(1+|x|^2)^{-1/3} e^{it\Delta_g} f\|^2_{H^{1/2-\varepsilon}(\Omega)} \\ &\lesssim \|f\|^2_{L^2(\Omega)}. \end{aligned}$$

By Duhammel's principle and Minkowski inequality, if $F(t,x) = 0$ when $|x| \geq 2R$, we have that by $TT^*$ argument

$$\begin{aligned} \| \int_0^t \rho e^{i(t-s)\Delta_g} F(t,\cdot) ds\|^2_{L_t^2 H^{1/2-\varepsilon}([0,T]\times\Omega)} &\lesssim \| \int_0^t \|\rho e^{i(t-s)\Delta_g} F(t,\cdot)\|_{H^{1/2-\varepsilon}(\Omega)} ds\|^2_{L^2[0,T]} \\ &\lesssim \| \int_0^T \|\rho e^{i(t-s)\Delta_g} F(t,\cdot)\|_{H^{1/2-\varepsilon}(\Omega)} ds\|_{L^2[0,T]} \\ &\lesssim \int_0^T \|\rho e^{i(t-s)\Delta_g} F(t,\cdot)\|_{L_t^2 H^{1/2-\varepsilon}([0,T]\times\Omega)} ds \\ &\lesssim \|F\|_{L_t^2 H^{-1/2+\varepsilon}([0,T]\times\Omega)}. \end{aligned}$$

So we proved (3.1.1) for $\gamma = 1/2 - \varepsilon$.

Then by elliptic regularity of $\Delta_g$

$$\begin{aligned}\| \rho u \|_{L_t^2 H_x^{\frac{5}{2}-\varepsilon}} &\lesssim \| \triangle_g(\rho u) \|_{L_t^2 H_x^{\frac{1}{2}-\varepsilon}} + \| \rho u \|_{L_t^2 H_x^{\frac{1}{2}-\varepsilon}} \\ &\lesssim \| \rho \triangle_g u \|_{L_t^2 H_x^{\frac{1}{2}-\varepsilon}} + \| [\triangle_g, \rho] u \|_{L_t^2 H_x^{\frac{1}{2}-\varepsilon}} + \| \rho u \|_{L_t^2 H_x^{\frac{1}{2}-\varepsilon}}\end{aligned}$$

Since $\triangle_g u$ solves the equation with initial data $\triangle_g f$ and forcing term $\triangle_g F$, we get

$$\begin{aligned}\|\rho \Delta_g u\|_{L_t^2 H_x^{\frac{1}{2}-\varepsilon}} &\lesssim \|\Delta_g f\|_{L_x^2} + \|\Delta_g F\|_{L_t^2 H_x^{-\frac{1}{2}+\varepsilon}} \\ &\lesssim \|f\|_{H_x^2} + \|F\|_{L_t^2 \dot{H}_x^{\frac{3}{2}-\varepsilon}}\end{aligned}$$

Notice that $[\triangle_g, \rho] = \rho_1 \partial_x u + \rho_2 u$, where $\rho_1, \rho_2 \in C_0^\infty$ have support belonging to $supp(\rho)$. Thus,

$$\begin{aligned}\| [\triangle_g, \rho] u \|_{L_t^2 H_x^{\frac{1}{2}-\varepsilon}} &\lesssim \| \rho_3 u \|_{L_t^2 H_x^{\frac{3}{2}-\varepsilon}} \\ &\lesssim \| \rho_3 u \|^{\theta}_{L_t^2 H_x^{\frac{1}{2}-\varepsilon}} \| \rho_3 u \|^{1-\theta}_{L_t^2 H_x^{\frac{5}{2}-\varepsilon}}\end{aligned}$$

Where $\rho_3 \in C_0^\infty$ has support in $supp(\rho_1) \cap supp(\rho_2)$, and $\theta$ is any real number in $(0, 1)$.

Hence the $L^2$ part (3.1.1) is true for $\gamma = \frac{5}{2} - \varepsilon$. Similar arguments hold for $\gamma = \frac{9}{2} - \varepsilon, \frac{13}{2} - \varepsilon, \frac{17}{2} - \varepsilon, \ldots$. So by interpolation and duality, we proved (3.1.1) for all

$\gamma \in \mathbb{R}$.

(ii).To show (3.1.2), we use the result of part (i).

In fact, by (i) above, if $v$ solves the inhomogeneous equation with 0 initial data, $v$ satisfies

$$\|\rho v\|_{L_t^2 H_B^{\gamma}(\mathbb{R}_+\times\Omega)} \lesssim \|F\|_{L_t^1 H_B^{\gamma+\varepsilon-\frac{1}{2}}(\mathbb{R}_+\times\Omega)} \tag{3.1.4}$$

By duality, energy estimate and elliptic regularity, we get

$$\|u\|_{L_t^\infty H_B^{\gamma}(\mathbb{R}_+\times\Omega)} \lesssim \|f\|_{H^{\gamma}(\Omega)} + \|F\|_{L_t^2 H^{\gamma+\varepsilon-\frac{1}{2}}(\mathbb{R}_+\times\Omega)} \tag{3.1.5}$$

And we proved (3.1.2). $\square$

In the next proposition we show that away from the trapped set, the free evolution satisfies a Strichartz estimate with loss.

**Proposition 3.1.1.** *Let $n \geq 2$. Suppose $u = u(t,x)$ solves (1.3.5), with $F = 0$. Let $\chi \in C_0^\infty$ and $\chi(x) = 1$ when $|x| < R$.*

*Then there exists $C > 0$ such that*

$$\|(1-\chi)u\|_{L^p W^{s,q}(\mathbb{R}_+\times\Omega)} \leq C\|f\|_{H^{s+\varepsilon}} \tag{3.1.6}$$

*where $s \in \mathbb{R}_+$, $u(t) = e^{it\Delta_g}u_0$ and $(p,q)$, $p > 2$, is an Strichartz admissible pair, i.e.*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2} \tag{3.1.7}$$

*and $\varepsilon$ is the one in* (3.1.1) *and* (3.1.2).

**Proof:**

Set $v(t) = (1-\chi)e^{it\Delta_g}u_0$. Then $v$ satisfies the equation

$$\begin{cases} (i\partial_t + \Delta_g)v = [\Delta_g, -\chi]u, \\ v(0) = (1-\chi)u_0 \end{cases} \tag{3.1.8}$$

Since $\chi = 1$ when $|x| < R$, $1-\chi = 0$ when $|x| \geq R$. Remember that the metric $g$ is assumed to be the same as the Euclidean metric when $|x| > R$, $\Delta_g v = \Delta v$ on its support. So equation (3.1.8) can be regarded as a Schrödinger equation on the Euclidean space $\mathbb{R}^n$.

Hence

$$v(t) = e^{it\Delta}(1-\chi)u_0 + \int_0^t e^{i(t-\tau)\Delta}[\Delta_g, -\chi]u(\tau)d\tau \tag{3.1.9}$$

where $\Delta$ is the standard Laplacian on $\mathbb{R}^n$. Therefore, the first term on the right hand side satisfies the usual Strichartz estimate. And it is suffices to study the second term, i.e.

$$w(t) = \int_0^t e^{i(t-\tau)\Delta}[\Delta_g, -\chi]u(\tau)d\tau \tag{3.1.10}$$

Notice that, by lemma 3.1.1, we get

$$\|[\Delta_g, -\chi]u\|_{L^2_T H^{-\frac{1}{2}}} \lesssim \|f\|_{H^\varepsilon} \tag{3.1.11}$$

Let $Tf(t,x) = e^{it\Delta}f(x)$. We now investigate the smoothing effect for the operator $T$. Using (1.10) and (3.4) in [2], we have, for every cutoff function $\chi_0$ in $\mathbb{R}^n$,

$$\|(1-\Delta)^{\frac{1}{4}}(\chi_0 Tf)\|_{L^2(\mathbb{R}_+\times\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)} \tag{3.1.12}$$

The dual version is,

$$\|T^*(\chi_0(1-\Delta)^{1/4}g)\|_{L^2(\mathbb{R}^n)} \lesssim \|g\|_{L^2(\mathbb{R}_+\times\mathbb{R}^n)} \tag{3.1.13}$$

Applying Strichartz estimates on $\mathbb{R}^n$ for $T$, we can get

$$\|TT^*(\chi_0(1-\Delta)^{1/4}g)\|_{L^p_T L^q(\mathbb{R}^n)} \lesssim \|g\|_{L^2(\mathbb{R}_+\times\mathbb{R}^n)} \tag{3.1.14}$$

Notice that

$$TT^*(f)(t) = \int_0^T e^{i(t-\tau)\Delta} f(\tau)d\tau \tag{3.1.15}$$

Taking $H(t,x) = (1-\Delta)^{-1/4}[\Delta_g, -\chi]u$, we get

$$\begin{aligned}\| \int_0^T e^{i(t-\tau)\Delta}(1-\Delta)^{1/4}H\|_{L^p_T L^q(\mathbb{R}^n)} &\lesssim \|H\|_{L^2(\mathbb{R}_+\times\mathbb{R}^n)} \\ &\lesssim \|[\Delta_g, -\chi]u\|_{L^2_T H^{-1/2}(\mathbb{R}^n)} \lesssim \|u_0\|_{H^\varepsilon}\end{aligned} \tag{3.1.16}$$

Hence by Christ-Kiselev lemma, we completed the proof for $s = 0$.

The case $s = 1, 2, 3...$ can be treated similarly by differentiationg the first equation of (3.1.8), considered as an equation on the Euclidean space $\mathbb{R}^n$. Then interpolate between $s = k$ and $s = k + 1$, we can get the inequality for all $s \in \mathbb{R}_+$. □

Now we can deal with the Strichartz estimate for $e^{it\Delta_g}$.

**Proposition 3.1.2.** *There exists $C > 0$ such that*

$$\|u\|_{L_t^p W^{s,q}} \leq C\|f\|_{H^{s+\frac{1}{p}+\varepsilon}} \tag{3.1.17}$$

*where $\varepsilon$ satisfies* (3.1.1) *and* (3.1.2), $u(t) = e^{it\Delta_g} f$ *and* $(p, q)$, $p > 2$, *satisfies* (3.1.7).

**Proof:**

Take a cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$ which equals to 1 when $|x| < R$. We can write $u(t)$ as

$$u(t) = \chi e^{it\Delta_g} f + (1 - \chi)e^{it\Delta_g} f = v(t) + w(t) \tag{3.1.18}$$

Due to Proposition 3.1.1, we know that $w(t)$ satisfies the desired Strichartz estimate with loss. So by Minkowski inequality, it suffices to show the estimate for $v)(t)$.

Using lemma 3.1.1, we get

$$\|v\|_{L_t^2 H^1(\Omega)} \lesssim \|f\|_{H^{1/2+\varepsilon}(\Omega)} \tag{3.1.19}$$

Then using energy argument, we can deduce,

$$\|v\|_{L_t^\infty L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} \tag{3.1.20}$$

Interpolate between (3.1.19) and (3.1.20) gives

$$\|v\|_{L_t^p H^{2/p}(\Omega)} \lesssim \|f\|_{H^{\frac{1}{p}+\frac{2\varepsilon}{p}}} \tag{3.1.21}$$

By Sobolev embedding theorem, $H^{2/p}(\Omega) \subset L^q(\Omega)$ if $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$, we completed the proof for s=0.

Then we consider $s = 1$. By energy inequality, we get

$$\|v\|_{L_t^\infty H^{p/(p-2)}(\Omega)} \lesssim \|f\|_{H^{p/(p-2)}} \tag{3.1.22}$$

Interpolate between (3.1.19) and (3.1.22), we get

$$\|v\|_{L_t^p H^{1+2/p}(\Omega)} \lesssim \|f\|_{H^{1+1/p+\varepsilon}(\Omega)} \tag{3.1.23}$$

So by Sobolev embedding theorem, $H^{1+2/p(\Omega)} \subset W^{1,q}(\Omega)$, we get that (3.1.17) holds for $s = 1$. Interpolate between $s = 0$ and $s = 1$, we get the case $s \in [0, 1]$.

Similarly, we can get $s = 2, 3, 4...$, and interplation gives us for all $s \in \mathbb{R}_+$.

□

Now we can state the time-dependent Strichartz type estimate, which is key to the local existence theorem.

**Proposition 3.1.3.** *For every finite $T > 0$, there esists $C > 0$ such that*

$$\|u\|_{L^p_T W^{s,q}(\Omega)} \leq C T^{1/(2p)} \|f\|_{H^{s+\varepsilon}(\Omega)} \tag{3.1.24}$$

*Where $s \in [0, \infty)$, $u(t) = e^{it\Delta_g f}$, and $(p, q)$,$p \geq 2$ satisfies*

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} \tag{3.1.25}$$

*Moreover,*

$$\|u\|_{L^p_T W^{s,q}(\Omega)} \leq C T^{1/2p} \|F\|_{L^1_T H^{s+\varepsilon}(\Omega)} \tag{3.1.26}$$

*Where $s \in [0, \infty)$, $u(t) = \int_0^t e^{i(t-\tau)\Delta_g} F(\tau) d\tau$, and $(p, q)$, $p > 2$ satisfies* (3.1.25).

**Proof:**

Firstly, let us study $u(t) = e^{it\Delta_g f}$.

Let $\chi \in C^{\infty_0(\mathbb{R}^n)}$ be the cut-off function such that $\chi = 1$ when $|x| < R$. Similarly as it is in the proof of Theorem 1.3.1, we can write $u(t)$ as

$$u(t) = \chi u(t) + (1 - \chi) u(t) = v(t) + w(t) \tag{3.1.27}$$

Hence Minkowski's inequality implies

$$\|u\|_{L^p_T W^{s,q}(\Omega)} \leq \|v\|_{L^p_T W^{s,q}(\Omega)} + \|w\|_{L^p_T W^{s,q}(\Omega)} \tag{3.1.28}$$

So we evaluate each term on the right hand side separately.

By lemma 3.1.1, we have

$$\|v\|_{L^2_T H^{s+1/2}(\Omega)} \lesssim \|f\|_{H^{s+\varepsilon}(\Omega)} \tag{3.1.29}$$

Then an energy argument yields,

$$\|v\|_{L^\infty_T H^s(\Omega)} \lesssim \|f\|_{H^s(\Omega)} \tag{3.1.30}$$

Interpolate between (3.1.29) and (3.1.30), we can get,

$$\|v\|_{L^p_T H^{s+1/p}(\Omega)} \lesssim \|f\|_{H^{s+\varepsilon}(\Omega)} \tag{3.1.31}$$

By Sobolev embedding thoerem, if $(p, q)$ and $p > 2$ satisfies (3.1.25), we have $H^{s+1/p}(\Omega) \subset W^{s,q}(\Omega)$. Hence we obtain,

$$\|v\|_{L^p_T W^{s,q}(\Omega)} \lesssim \|f\|_{H^{s+\varepsilon}(\Omega)} \tag{3.1.32}$$

Then we investigate $w(t) = (1 - \chi)u(t)$.

Let $p^* = 2p$, then for $(p, q)$, $p > 2$ satisfying (3.1.25), we have $p^* > 2$ and $(p^*, q)$ satisfies (3.1.7). Hence by Hölder's inequality and Proposition 3.1.1, we get,

$$\|w\|_{L^p_T W^{s,q}([0,T]\times\Omega)} \lesssim T^{1/(2p)}\|w\|_{L^{p^*} W^{s,q}([0,T]\times\Omega)} \lesssim T^{1/(2p)}\|f\|_{H^{s+\varepsilon}(\Omega)} \tag{3.1.33}$$

This completes the proof of (3.1.24).

Estimate (3.1.26) comes from (3.1.24), Christ-Kiselev lemma and Minkowski inequality in time variable.

$$\begin{aligned}
\| \int_0^t e^{i(t-\tau)\Delta_g} F(\tau) d\tau \|_{L^p_T W^{s,p}(\Omega)} &\lesssim \| \int_0^T e^{i(t-\tau)\Delta_g} F(\tau) d\tau \|_{L^p_T W^{s,p}(\Omega)} \\
&\lesssim T^{1/(2p)} \| \int_0^t e^{-i\tau\Delta_g} F(\tau) d\tau \|_{H^{s+\varepsilon}(\Omega)} \\
&\lesssim T^{1/(2p)} \|F\|_{L^1_T H^{s+\varepsilon}(\Omega)}.
\end{aligned}$$

$\square$

**Proposition 3.1.4.** *For every $T > 0$, there exists $C > 0$ such that for all $s \in \mathbb{R}$,*

$$\|u\|_{L^p_T W^{s,q}} \leq C T^{1/(2p)+1/(2\tilde{p})} \|(1-\Delta_g)^{\frac{s+2\varepsilon}{2}} F\|_{L^{\tilde{p}}_T L^{\tilde{q}}(\Omega)}, \tag{3.1.34}$$

*If the right hand side is finite. Here $u(t) = \int_0^t e^{i(t-\tau)\Delta_g} F(\tau) d\tau$, $(p,q)$, $p > 2$ satisfies* (3.1.25)*, and $(\tilde{p}, \tilde{q})$, with $\tilde{p} \in [1,2)$, satisfies*

$$\frac{1}{\tilde{p}} + \frac{n}{\tilde{q}} = 1 + \frac{n}{2} \tag{3.1.35}$$

*Notice that $(p,q)$ and $(\tilde{p},\tilde{q})$ do not need to have any correspondence.*

**Proof:**

Due to Christ-Kiselev lemma, it suffices to evaluate

$$w(t) = \int_0^T e^{i(t-\tau)\Delta_g} F(\tau) d\tau \tag{3.1.36}$$

Using (3.1.24), we can get

$$\|w\|_{L_T^p W^{s,q}(\Omega)} \leq CT^{1/(2p)}\|h\|_{H^{s+\varepsilon}(\Omega)}, \quad s \in [0,\infty) \tag{3.1.37}$$

Where $h = \int_0^T e^{-i\tau\Delta_g F(\tau)d\tau}$.

The Dual of (3.1.24) gives

$$\|h\|_{H^s(\Omega)} \leq CT^{1/(2\tilde{p})}\|(1-\Delta_g)^{(s+\varepsilon)/2} f\|_{L_T^{\tilde{p}} L\tilde{q}(\Omega)}, \quad s \in (-\infty, 0) \tag{3.1.38}$$

Where $(\tilde{p}, \tilde{q})$, $\tilde{p} \in [1,2)$, satisfies (3.1.35). As $e^{it\Delta_g}$ commutes with $\Delta_g$, (3.1.38) is also true for $s \in [0,\infty)$. This completes the proof. □

## 3.2 Local Well-posedness of Schrödinger Equations

Now let us consider the initial value problem (1.3.5). In this section, we always assume $\varepsilon > 0$ be the small positive number in (3.1.1) and (3.1.2). And the initial value $f$ is small enough.

So by Duhamel's principle, the solution of (1.3.5) can be written as

$$u(t) = e^{it\Delta_g} f + \int_0^t e^{i(t-\tau)\Delta_g} F(u(\tau))d\tau, \tag{3.2.1}$$

Where $F$ and $\Omega$ are described as they are in the Chapter 1.

So $F(u)$ satisfies the following point-wise estimate:

$$|F(u)| \lesssim |u|^p \tag{3.2.2}$$

and

$$|\nabla F(u)| \lesssim |\nabla u||u|^{p-1} \tag{3.2.3}$$

Moreover, as $F(u) - F(v) = \int_0^1 F'(tu + (1-t)v)(u-v)dt$, it follows that

$$|F(u) - F(v)| \lesssim |u-v|(|u|^{p-1} + |v|^{p-1}) \tag{3.2.4}$$

$$|\nabla(F(u) - F(v)| \lesssim |\nabla(u-v)|(|u|^{p-1} + |v|^{p-1}) + |u-v|(|\nabla u| + |\nabla v|)(|u|^{p-2} + |v|^{p-1}) \tag{3.2.5}$$

Let $\Phi(u) = e^{it\Delta_g} f + \int_0^t e^{i(t-\tau)\Delta_g} F(u(\tau))d\tau$, it suffices to show that $\Phi$ is a contraction on some Hilbert Space $X$.

The next proposition is the result for 2 diminsion. The local well-posedness in $2d$ follow from it.

**Proposition 3.2.1.** *Let $n = 2$, and $(p,q) \in \mathbb{R}^2$, $p > 2$, such that $1/p + 1/q = 1/2$. If*

$$X_T = L_T^\infty H^1(\Omega) \cap L_T^p W^{1-1/p-\varepsilon,q}(\Omega),$$

*and*

$$\|u\|_{X_T} = \|u\|_{L_T^\infty H^1(\Omega)} + \|u\|_{L_T^p W^{1-1/p-\varepsilon,q}(\Omega)}$$

*Then $\Phi$ is a contraction from $X_T$ to $X_T$ in a ball of $X_T$*

**Proof:**

By Proposition 3.1.1, the free Schrödinger propagator $e^{it\Delta_g}$ is bounded. So it suffices to show that $\Lambda = \Phi - e^{it\Delta_g}$ is a contraction.

So $\Lambda G = \int_0^t e^{i(t-\tau)\Delta_g} G(\tau)d\tau$. By Christ-Kieslev lemma, energy argument, Proposition 3.1.1 and Minkowski inequality, we have

$$\|\Lambda G\|_{X_T} \leq C\|G\|_{L^1_T H^1(\Omega)} \tag{3.2.6}$$

On the other hand, we can bound $F(u(t))$, where $F$ is the forcing term in (1.1), satisfying the assumptions in the introduction. So using (3.2), we have

$$\|F(u(t))\|_{L^1_T H^1(\Omega)} \leq C\int_0^T \||u(\tau)|^p\|_{H^1(\Omega)} d\tau \tag{3.2.7}$$

$$\leq C\int_0^T \|u(\tau)\|_{H^1(\Omega)}\|u(\tau)\|^{p-1}_{L^\infty(\Omega)} d\tau \tag{3.2.8}$$

$$\leq CT^{1/p}\|u(\tau)\|_{L^\infty_T H^1\Omega}\|u(\tau)\|^{p-1}_{L^p_T L^\infty(\Omega)} \tag{3.2.9}$$

If we choose $1/p > \varepsilon > 0$, we will have $1 - 1/p - \varepsilon > 2/q$, hence $W^{1-1/p-\varepsilon,q}(\Omega) \subset L^\infty(\Omega)$. So we get

$$\|u\|^{p-1}_{L^p_T L^\infty(\Omega)} \leq C\|u\|^{p-1}_{X_T} \tag{3.2.10}$$

So the forcing term satisfies

$$\|F(u(t))\|_{L^1_T H^1(\Omega)} \leq CT^{1/p}\|u(t)\|^p_{X_T(\Omega)} \tag{3.2.11}$$

Substitute (3.2.11) into (3.2.6), we get

$$\|\Lambda F(u(t))\|_{X_T} \leq C\|F(u(t))\|_{L^1_T H^1(\Omega)} \leq CT^{1/p}\|u(t)\|^p_{X_T} \tag{3.2.12}$$

Similar to the above proof, we can show that

$$\|F(u) - F(v)\|_{L^1_T H^1(\Omega)} \leq CT^{1/p}\|u - v\|_{X_T}(\|u\|^{p-1}_{X_T} + \|v\|^{p-1}_{X_T}) \tag{3.2.13}$$

Which completes the proof. □

In conclusion, when $n = 2$, by Proposition 3.2.1, for any $T > 0$, in a ball $B_R$ of radius $R > 0$ of $X_T$, if $R < CT^{-1/p(p-1)}$, and the initial value $f$ is small, then $\Phi$ is a contraction on $B_R \cap X_T$. Hence there is a solution of (1.3.5) in $B_R \cap X_T$.

The higher dimensional case can be treated similarly.

**Proposition 3.2.2.** *Let $n \geq 3$, and $(p, q) \in \mathbb{R}^3$, $p > 2$ satisfies $1/p + n/q = n/2$. We also assume that $s > n/q$. If*

$$X^s_T = L^\infty_T H^{s+\varepsilon}(\Omega) \cap L^p_T W^{s,q}(\Omega)$$

*and*

$$\|u\|_{X^s_T} = \|u\|_{L^\infty_T H^{s+\varepsilon}(\Omega)} + \|u\|_{L^p_T W^{s,q}(\Omega)}$$

*We have*

$$\Phi(u) = e^{it\Delta_g} f + \int_0^T e^{i(t-\tau)\Delta_g} F(u(\tau))d\tau$$

*is a contraction in some ball of* $X_T^s$.

**Proof:**

By Proposition 3.1.4, the free Schrödinger propagator $e^{it\Delta_g}$ is bounded from $X_T^s$ to itself. So same as in Proposition 3.2.1, it suffices to show that $\Lambda = \Phi - e^{it\Delta_g}$ is a contraction.

By Christ-Kieslev lemma, Proposition 3.1.4 and energy argument,

$$\|\Lambda G\|_{X_T^s} \leq C(1 + T^{1/(2p)})\|G\|_{L_T^1 H^{s+\varepsilon}(\Omega)} \tag{3.2.14}$$

On the other hand, using (3.2.2) we can get,

$$\begin{aligned}
\|F(u(t))\|_{L_T^1 H^{s+\varepsilon}(\Omega)} &\leq C\int_0^T \||u(\tau)|^p\|_{H^{s+\varepsilon}} d\tau && (3.2.15)\\
&\leq C\int_0^T \|u(\tau)\|_{H^{s+\varepsilon}} \|u(\tau)\|^{p-1}_{L^\infty(\Omega)} d\tau && (3.2.16)\\
&\leq CT^{1/p}\|u\|_{L_T^\infty H^{s+\varepsilon}(\Omega)} \|u\|^{p-1}_{L_T^p L^\infty(\Omega)} && (3.2.17)
\end{aligned}$$

As $s > n/q$, we have $W^{s,q}(\Omega) \subset L^\infty(\Omega)$ is a continuous embedding. So

$$\|u\|^{p-1}_{L_T^p L^\infty(\Omega)} \leq C\|u\|^{p-1}_{X_T^s} \tag{3.2.18}$$

It follows that the forcing term $F(u)$ satisfies

$$\|F(u(t))\|_{L_T^1 H^{s+\varepsilon}(\Omega)} \leq CT^{1/p}\|u\|^p_{X_T^s} \tag{3.2.19}$$

Substitute (3.2.19) into (3.2.14), we get

$$\|\Lambda F(u(t))\|_{X_T^s} \leq C(1+T^{1/(2p)})\|F(u(t))\|_{L_T^1 H^{s+\varepsilon}(\Omega)} \leq CT^{1/p}(1+T^{1/(2p)})\|u\|_{X_T^s}^p \tag{3.2.20}$$

Similarly, using (3.2.4) we can get,

$$\|F(u)-F(v)\|_{L_T^1 H^{s+\varepsilon}} \leq CT^{1/p}\|u-v\|_{X_T^s}(\|u\|_{X_T^s}^{p-1}+\|v\|_{X_T^s}^{p-1}) \tag{3.2.21}$$

Hence

$$\|\Lambda(F(u)-F(v))\|_{X_T^s} \leq C(1+T^{1/(2p)})\|F(u)-F(v)\|_{L_T^1 H^{s+\varepsilon}(\Omega)} \tag{3.2.22}$$

$$\leq CT^{1/p}(1+T^{1/(2p)})\|u-v\|_{X_T^s}(\|u\|_{X_T^s}^{p-1}+\|v\|_{X_T^s}^{p-1}) \tag{3.2.23}$$

So for any $T>0$, if $R < CT^{-1/p(p-1)}(1+T^{1/(2p)})^{-1/(p-1)}$ is a positive radius, then $\Phi$ is a contraction on $B_R \cap X_T^s$, which completes the proof. □

By Proposition 3.2.2, when $n \geq 3$, (1.1) has a solution in $B_R \cap X_T^s$.

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# Vita

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