

# **Rigidity Results of Lambda-Hypersurfaces**

by

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# Abstract

This dissertation introduces  $\lambda$ -hypersurfaces. These are hypersurfaces  $\Sigma^n \subset \mathbb{R}^{n+1}$  that satisfy the equation  $H = \frac{1}{2}\langle x, n \rangle + \lambda$ . Such hypersurfaces generalize the notion of a self-shrinking soliton of mean curvature flow. They also are stationary solutions to an isoperimetric-type problem on a Gaussian measure. We will motivate the study of  $\lambda$ -hypersurfaces, and then give several rigidity results that can help to classify such surfaces. These results include a stability result (that the only stable hypersurfaces, suitably defined, are hyperplanes) with versions that apply to both complete and incomplete hypersurfaces. They also include an eigenvalue and diameter estimate for compact hypersurfaces. Finally, we state a classification result about compact surfaces with small curvature.

Advisor: William P. Minicozzi II

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# Chapter 1

## Introduction

The main subject of study in this dissertation is the notion of a  $\lambda$ -hypersurface. This is an  $n$ -dimensional hypersurface immersed in  $\mathbb{R}^{n+1}$  that satisfies the equation

$$H = \frac{1}{2}\langle x, n \rangle + \lambda \tag{1.1}$$

Here  $x$  is the position vector of the immersed hypersurface,  $n$  is the normal vector,  $H$  is the mean curvature, and  $\lambda$  is a fixed constant. Such hypersurfaces are relevant to two different topics of interest in geometric analysis. First, they may be viewed as a generalization of self-shrinking solitons for the mean curvature flow, and one might study which theorems regarding self-shrinkers can be generalized. Second, they may be viewed as solutions to an isoperimetric problem in a suitable weighted ambient space (much as self-shrinkers may be viewed as solutions to the minimal surface equation). These hypersurfaces were first introduced by the author and Mcgonagle

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in [MR13], and have subsequently been studied by [CW14], [Gua14], and [Cha14].

In this dissertation, I will give several results on  $\lambda$ -hypersurfaces. The first two, which are joint work with Matt McGonagle and recorded in [MR13], discuss which complete or incomplete  $\lambda$ -hypersurfaces can be stable. The third result gives an estimate on the first non-zero eigenvalue of a compact  $\lambda$ -hypersurface, and uses that to obtain an estimate on the diameter of the hypersurface. The final result, which complements theorems found in [CW14] and [Gua14], discusses which compact  $\lambda$ -hypersurfaces can have small curvature.

### 1.1 Overview of results

To properly motivate  $\lambda$ -hypersurfaces, we will introduce a variational problem.

Let  $\mathcal{A}$  be  $n$ -dimensional Hausdorff measure and  $\mathcal{V}$  be  $n+1$ -dimensional Hausdorff measure (both on  $\mathbb{R}^{n+1}$ ). Then, consider area and volume measures with Gaussian weights attached:

$$d\mathcal{A}_\mu := e^{-|x|^2/4} d\mathcal{A} \tag{1.2}$$

$$d\mathcal{V}_\mu := e^{-|x|^2/4} d\mathcal{V} \tag{1.3}$$

Given a hypersurface  $\Sigma$  immersed in  $\mathbb{R}^{n+1}$ , a compact variation of  $\Sigma$  is a map  $F : \Sigma \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{n+1}$  such that  $F(\Sigma, 0) = \Sigma$  and  $F(x, t) = x$  outside a compact set. As  $\Sigma$  is varied, it is of general interest to see how the (weighted) area and the



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(weighted) enclosed volume of the hypersurface vary. In particular, in chapter 3, we will calculate the first and second derivatives of both the weighted area and the weighted volume (or, more accurately, change in volume).

It is well-known (and will be discussed in more detail in chapter 2) that self-shrinking solitons of mean curvature flow are critical points for this weighted area functional. Ie, if  $\Sigma$  is a self-shrinker, then its weighted area is a critical point for any compact variation. This turns out to be equivalent to saying that self-shrinkers are minimal hypersurfaces in a space with a conformally changed metric (see [CM12] for details on this).

In a related variational problem, one tries to find critical points of the (weighted or unweighted) area functional specifically for those variations that do not change the enclosed, (weighted or unweighted) volume. Without the Gaussian weight (ie, simply considering Euclidean area and volume), such surfaces are constant mean curvature hypersurfaces. Our first result, found in chapter 3 reveals that  $\lambda$ -hypersurfaces are solutions to this problem with weighted area and volume:

**Theorem 1.1.1.** *Hypersurfaces that satisfy equation 1.1 are precisely those hypersurfaces that are critical points for weighted area for every volume-preserving variation.*

In chapter 4, we study *stable*  $\lambda$ -hypersurfaces. Traditionally, a critical hypersurface for a variational problem is called *stable* if the second derivative of area is non-negative for every compact variation. In [CM12], Colding and Minicozzi show that no self-shrinkers are stable (and a more refined version of stability, known as  $\mathcal{F}$ -stability,

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is introduced to account for this). In our setting, however, we will only consider variations that leave the weighted volume unchanged. In this case, we get that there are stable hypersurfaces, namely planes:

**Corollary 1.1.2.** *Hyperplanes are the only stable  $\lambda$ -hypersurfaces.*

This result is actually a corollary of a stronger result, that any bound on the *index*,  $I$ , will force a non-planar  $\Sigma$  to split off a line. In our setting, we will define the index of a  $\lambda$ -hypersurface to be the number of linearly independent eigenfunctions (with respect to the stability operator) that are perpendicular to the constant functions (so that they are volume-preserving), and that have negative eigenvalues.

**Theorem 1.1.3.** *Let  $\Sigma$  be a  $\lambda$ -hypersurface with finite weighted area, and suppose that the index  $I$  satisfies  $I \leq n$ . Then there exists an  $i$  with  $(n + 1 - I) \leq i \leq n$  such that  $\Sigma = \Sigma_0 \times \mathbb{R}^i$ .*

In addition to 1.1.2, an immediate corollary of this result is

**Corollary 1.1.4.** *There are no  $\lambda$ -hypersurfaces of index 1.*

We then turn our attention stable  $\lambda$ -hypersurfaces that are incomplete (but defined to exist in a large ball). We are able to prove the following integral curvature estimate:

**Theorem 1.1.5.** *Let  $\Sigma \subset B_{2R}(0) \subset \mathbb{R}^{n+1}$  with  $\partial\Sigma \subset \partial B_{2R}(0)$  satisfy  $H = \frac{1}{2}\langle x, n \rangle + \lambda$  and be stable (ie,  $\Sigma$  satisfies (3.16)). If  $\mathcal{A}_\mu(\Sigma \cap B_R) \geq 2B_n R^{-2} \mathcal{A}_\mu(\Sigma \cap (B_{2R} \setminus B_R))$ ,*

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then we have that

$$\int_{B_R \cap \Sigma} |A|^2 d\mathcal{A}_\mu \leq 2B_n R^{-2} \mathcal{A}_\mu(\Sigma \cap (B_{2R} \setminus B_R)). \quad (1.4)$$

where  $B_n$  is a given constant.

Furthermore, in the case of  $n = 2$  we use a Choi-Shoen type argument (so called because it was first introduced in [CS85]) to prove a pointwise curvature estimate. The result is:

**Theorem 1.1.6.** *Given an  $M > 0$  and an  $R > 1$ , let  $\Sigma \subset B_{2R}(0)$  be an incomplete, stable, 2-dimensional  $\lambda$ -hypersurface with  $|\lambda| \leq M$  and  $\partial\Sigma \subset \partial B_{2R}(0)$ . Then there exists a  $\epsilon > 0$  such that, if  $2B_n A_\mu(\Sigma \cap (B_{2R} \setminus B_R)) \leq R^2 A_\mu(\Sigma \cap B_R)$  and if  $A_\mu(\Sigma \cap (B_{2R} \setminus B_R)) < (R^2/2B_n)e^{-\frac{1}{4}(\rho + \frac{1}{R})^2} \delta \epsilon$  for some  $0 < \delta \leq 1$  and some  $\rho \in (0, R - 1/R)$ , then*

$$\sup_{x \in B_\rho \cap \Sigma} |A|^2 \leq R^2 \delta. \quad (1.5)$$

We remark that these area conditions, while technical looking, are easily satisfied in the case of a complete hypersurface with  $A_\mu(\Sigma) < \infty$ . In this case, by looking at a fixed  $B_\rho(x_0) \cap \Sigma$  and taking  $R \rightarrow \infty$  and  $\delta \rightarrow 0$  appropriately, we recover our previous result: namely, that the only complete, smooth, two-sided, properly immersed  $\lambda$ -hypersurfaces are hyperplanes.

In chapter 5, we give some additional results about  $\lambda$ -hypersurfaces. The first result, applicable to compact hypersurfaces only, is an estimate on the smallest non-

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zero eigenvalue to the drift Laplacian (also called the Witten Laplacian),  $\mathcal{L}u = \Delta u - \frac{1}{2}\langle x, \nabla u \rangle$ . This estimate requires a bound on the curvature.

**Theorem 1.1.7.** *Suppose that  $\Sigma$  is a compact  $\lambda$ -hypersurface with  $|A|^2 \leq 2$  and  $\lambda(H - \lambda) \geq 0$ . Then the smallest non-zero eigenvalue,  $\gamma_1$ , satisfies  $\gamma_1 \leq \frac{n+2}{2}$*

Using this result, as well as an estimate created by [FLL13], we can get the following diameter lower bound:

**Corollary 1.1.8.** *Let  $\Sigma$  be as above. Then the diameter  $d$  of the hypersurface satisfies*

$$d \geq \sqrt{\frac{2\pi^2}{n+4+\lambda\sqrt{2}}} \quad (1.6)$$

Finally, we have a result about what happens when  $|A|^2$  is small:

**Theorem 1.1.9.** *Suppose  $\Sigma$  is a compact  $\lambda$ -hypersurface which satisfies  $H - \lambda \geq 0$  and  $|A|^2 \leq 1/2$ . Then  $\Sigma$  is a sphere.*

This partially generalizes a result of [CL]. We also compare this result to similar results in [CW14] and [Gua14]. In the first paper, a similar theorem is proved without the second condition but with an additional, technical condition. In the latter paper, the author proves a similar theorem without the first assumption but with stronger assumptions on  $|A|^2$ .

# Chapter 2

## Background material

We'll begin by providing the necessary background material on the isoperimetric problem, especially the special case of the isoperimetric problem in Gaussian space. We'll then redirect our attention to give some background remarks on mean curvature flow, especially self-shrinking solutions to mean curvature flow. This will leave us in a position where we'll be able to define our main topic of study,  $\lambda$ -hypersurfaces.

### 2.1 The isoperimetric problem

Throughout this section, we will be considering smooth, compact, properly immersed hypersurfaces  $\Sigma$  in  $\mathbb{R}^{n+1}$  (and by abuse of notation, we will identify the manifold  $\Sigma$  with its immersed image). Let  $\mathcal{A}$  and  $\mathcal{V}$  denote  $n$ -dimensional and  $(n + 1)$ -dimensional Hausdorff measure, respectively, defined on  $\mathbb{R}^{n+1}$ . By a slight abuse of

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notation and language,  $\mathcal{A}(\Sigma)$  will be called the “area” of  $\Sigma$ , and  $\mathcal{V}(\Sigma)$  will be used to denote the measure of the compact region whose boundary is  $\Sigma$  and will be called the “volume” of  $\Sigma$ . The classical isoperimetric problem is stated as follows:

**The Isoperimetric Problem 1** (classic). *Given a fixed constant  $V$ , find a hypersurface  $\Sigma$  of volume  $V$  and minimal area.*

The problem has been known since antiquity (for example, a version of it appears as *Dido’s problem* from Virgil’s *Aeneid*). It has long been believed that the solution is a circle (in dimension 2: more generally, an  $n$ -dimensional sphere in  $\mathbb{R}^{n+1}$ ), but a rigorous proof of this didn’t exist until the 1800’s due to work by Steiner [Ste38].

There are many generalizations on the isoperimetric problem. For example, one can try to find an isoperimetric region in an ambient space other than  $\mathbb{R}^{n+1}$ : this can be done, for example, by solving the isoperimetric problem in a region of Euclidean space (possibly with obstructions), solving the isoperimetric problem on a Riemannian manifold, or solving the isoperimetric problem subject to a weighted measure. Another way to generalize is to relax the requirements that the hypersurfaces enclose a finite amount of volume, or are absolute minimizers of area. We can do this using a variational approach.

### 2.1.1 Variational approaches to isoperimetry

We begin by defining a variation of a hypersurface:

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**Definition 2.1.1.** A *variation* of a hypersurface  $\Sigma$  is a smooth map  $F : \Sigma \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{n+1}$  such that  $F(\Sigma, 0) = \Sigma$  and  $F(x, t) = x$  outside of a compact set.

A local version of the isoperimetric problem can be construed using variations. Specifically, a hypersurface will locally solve the isoperimetric problem if every variation that keeps the enclosed volume fixed must initially increase area (which can be measured using  $\mathcal{A} \circ F(\Sigma, \cdot)$  and  $\mathcal{V} \circ F(\Sigma, \cdot)$ ). This gives us:

**The Isoperimetric Problem 2** (local). *Find a hypersurface in which no volume-preserving variations exist that locally decrease area.*

It can, of course, be generalized even further. For example, we do not need to find hypersurfaces that locally minimize area: we can find a hypersurface  $\Sigma$  for which, under any variation that preserves volume, the function  $\mathcal{A} \circ F(\Sigma, \cdot)$  is stationary:

**The Isoperimetric Problem 3** (critical points). *Find a hypersurface  $\Sigma$  which is a critical point for the area functional under every volume-preserving variation.*

A hypersurface will be called **volume-preserving stationary** (often shortened to stationary) if, under any volume-preserving variation  $F$ , we have  $\frac{d}{dt}|_{t=0} \mathcal{A} \circ F = 0$ . Additionally, a stationary hypersurface is called **volume-preserving stable** (or just stable) if  $\frac{d^2}{dt^2}|_{t=0} \mathcal{A} \circ F \geq 0$ .

Suppose  $F$  is a variation, and suppose  $f$  is a function defined on  $\Sigma$  such that

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$\langle \frac{d}{dt}|_{t=0}F, n \rangle = f$ . Some computations (see [BdC84]) yield

$$\frac{d}{dt}|_{t=0}\mathcal{A} \circ F = \int_{\Sigma} H f d\mathcal{A} \quad (2.1)$$

$$\frac{d}{dt}|_{t=0}\mathcal{V} \circ F = \int_{\Sigma} f d\mathcal{A} \quad (2.2)$$

**Theorem 2.1.2.** *From 2.1 and 2.2, one can derive the following well-known results:*

1. *A variation that preserves volume must necessarily satisfy  $\int_{\Sigma} f d\mathcal{A} = 0$ .*
2. *Hypersurfaces that are stationary for the area functional under all variations (ie, not just the volume-preserving ones) are called minimal hypersurfaces and satisfy  $H = 0$ .*
3. *Hypersurfaces that are stationary for the area functional under all volume-preserving variations are called constant mean curvature hypersurfaces and satisfy  $H = C$ .*

Constant mean curvature surfaces are natural generalizations of minimal surfaces. The most well-known constant mean curvature surface is the round sphere, but there are many other examples, including classic examples by Delaunay [Del41] and more recent constructions by many people (for example, the compact and immersed Wente Torus [Wen86] and the noncompact, complete glued examples of Kapouleas [Kap90]). These are all examples of stationary solutions to the isoperimetric problem. However, in 1984, J. Barbosa and M. do Carmo showed that the class of stable hypersurfaces



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is much more restrictive. Note that they require the hypersurface to be compact and two-sided, but do not require the hypersurface to be embedded.

**Theorem 2.1.3** ([BdC84]). *The sphere is the only compact, two-sided, immersed constant mean curvature hypersurface that is stable under all volume-preserving variations.*

This proof was later simplified by Wente [Wen91], and generalized in several ways (see, for example, Morgan and Ritoré [MR02]). A rough outline of the proof is as follows: we consider two types of variations, namely variations that correspond to ambient homothetic shrinking in  $\mathbb{R}^n$  and variations that correspond to uniform movement in the normal direction. In the case of the sphere, these variations coincide; in the other case, a linear combination of these two types of variations will yield a variation whose second derivative is negative, proving that the hypersurface cannot be stable. This approach (of using linear combinations of well-known variations) will be used in our main argument for stability in the Gaussian space.

### 2.1.2 The isoperimetric problem in Gaussian Space

The Gaussian isoperimetric problem (ie, the problem of solving the isoperimetric problem in a weighted Gaussian space) has been studied since at least the 1970's. Besides probability theory (where a Gaussian weight is natural), there are many applications to such a problem (cf. [Led96] for a more general discussion of applications).

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To parallel our discussion from chapter 1, we define Gaussian weighted area and Gaussian weighted volume to be

$$d\mathcal{A}_\mu = e^{-|x|^2/4} d\mathcal{A} \quad (2.3)$$

$$d\mathcal{V}_\mu = e^{-|x|^2/4} d\mathcal{V} \quad (2.4)$$

We may define  $\mathcal{A}_\mu(\Sigma)$  and  $\mathcal{V}_\mu(\Sigma)$  as the area and (enclosed) volume of  $\Sigma$  in a way analogous to before (note that the weight allows a non-compact  $\Sigma$  to have a finite amount of area, and to enclose a finite amount of volume). We may then ask: for a fixed weighted volume, which hypersurfaces minimize weighted surface area? The main result in this field is that hyperplanes are global minimizers. This result is due to Borell [Bor75] and Sudakov and Tsirel'son [ST78], although more there are many recent proofs of this result (cf. [Bob97], [Led98], and [Ehr83]).

As before, one may relax the question to allow for local minimizers, or even stationary or stable solutions. These are the generalizations that we will study in the coming chapters.

## 2.2 Mean Curvature Flow

Let  $F_0 : M^n \rightarrow \mathbb{R}^{n+1}$  be a smooth immersion of a manifold  $M$ . Then the **mean curvature flow** of  $F_0$  (denoted MCF from now on) is a smooth map  $F : [0, T) \times M \rightarrow$

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$\mathbb{R}^{n+1}$  that satisfies the system of PDEs

$$\frac{dF}{dt}(t, x) = -H(t, x)n(t, x) \quad (2.5)$$

where  $H$  is the mean curvature of the hypersurface, and  $n$  is the normal vector to the hypersurface. MCF is the (negative) gradient flow for the area of the hypersurface, meaning that hypersurfaces flowing in this way have their area decreasing as quickly as possible. As a consequence, minimal surfaces (which already have minimal area, and have  $H \equiv 0$ ) are stationary under the MCF.

We will often use  $M_t$  to denote  $F(t, M)$ , and will identify the manifold  $M$  with its initial immersion  $M_0$ .

**Example 2.2.1. *Shrinking Spheres.*** *When the initial hypersurface  $S_0$  is the round  $n$ -sphere of radius  $R$ , the MCF equation simplifies to an ODE on the radius. This can be explicitly solved to get a solution  $S_t = S_0\sqrt{R^2 - 2nt}$ . This solution shrinks homothetically to a singular point at time  $T = \frac{R^2}{2n}$ .*

MCF is a nonlinear parabolic PDE, and so many standard results from parabolic PDEs apply (see, for example, [Eck04]). If the initial hypersurface is smooth and compact, the MCF will exist and be well-defined for (at least) a short amount of time, although the hypersurface can become singular at some positive time  $T$  (as the previous example indicates). Some other basic results include the following corollaries to a parabolic maximum principle:

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**Theorem 2.2.2.** *If a MCF solution is initially an embedding, then it remains embedded.*

**Theorem 2.2.3.** *If two compact solutions to MCF are initially not touching, then they will never touch.*

By the previous theorem, one can show that every compact hypersurface  $M$  evolving under MCF will eventually develop a singularity in a finite amount of time. Indeed,  $M$  can be enclosed by a sphere of large radius, and by Theorem 2.2.3  $M$  must develop a singularity before the sphere vanishes. This suggests that singularities appear readily in MCF, and that studying the nature of these singularities is of central importance to the theory.

**Example 2.2.4. *Dumbbell.*** *A dumbbell (two spheres connected by a thin neck in a smooth way) is a compact hypersurface that will develop a singularity in finite time. By making the spheres large enough and the neck thin enough, one can rigorously show that the neck “pinches off” into a singularity.*

### 2.2.1 Singularity Formation

With the importance of understanding singularities established, we turn towards understanding what types of singularities can occur. It can be shown [Hui84] that singularities will only occur at points in space-time where the curvature (ie the norm squared of the second fundamental form, denoted  $|A|^2$ ) blows up. Furthermore, the

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type of singularity that can occur depends largely on how fast the curvature is blowing up: we call a singularity a **type 1 singularity** if, near the singular point, the curvature is blowing up relatively slowly. To be more precise: consider a singularity that develops at time  $T$  and a point  $x \in \mathbb{R}^{n+1}$ . This singularity is type 1 if, for some constant  $C$ , every sequence of points  $(t_i, x_i)$  with  $t_i \rightarrow T$  and  $x_i \in M_{t_i}$ ,  $x_i \rightarrow x$  satisfies the curvature bound:

$$|A| \leq \frac{C}{\sqrt{2(T-t)}} \quad (2.6)$$

Otherwise, we call it a **type 2 singularity**.

The first fundamental result in this direction was found by Huisken, who distinguished the two types of singularities and who proved in [Hui90] the following monotonicity formula:

**Theorem 2.2.5.** *Let  $M_t$  be a solution for MCF on  $t \in [0, T)$ . Then we have*

$$\frac{d}{dt} \int_{M_t} (T-t)^{-n/2} \exp\left(\frac{-|x|^2}{4(T-t)}\right) d\mathcal{A} = \int_{M_t} (T-t)^{-n/2} \exp\left(\frac{-|x|^2}{4(T-t)}\right) \left| H - \frac{\langle x, n \rangle}{2(T-t)} \right|^2 d\mathcal{A} \quad (2.7)$$

This monotonicity formula allows us to create parabolic rescalings of the MCF in a controlled manner. Specifically, if we have a given MCF  $M_t$  with a singular point, we can rescale appropriately in space and time about that point to get a new MCF  $\tilde{M}_\tau$ . The new MCF will effectively look like the old one, only dilated about the singularity.

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We can take a sequence of such rescalings and, if the original singularity was type 1, a result by Brakke [Bra78] guarantees that we can pass to a limit. This limit is called a **tangent flow**, and is a generalization of the tangent cone from minimal surface theory. The tangent flow here will be a MCF, and the monotone quantity in Huisken's formula will be 0, ie at the initial time of the tangent flow the hypersurface will satisfy

$$H = \frac{\langle x, n \rangle}{2} \quad (2.8)$$

This equation is an elliptic PDE on the hypersurface. One can observe that a hypersurface that satisfies this will evolve by MCF by homothetically shrinking: ie, if  $M_0$  satisfies this equation, then  $M_t$  will satisfy

$$M_t = \sqrt{1-t} M_0 \quad (2.9)$$

which will homothetically shrink and develop a singularity after one unit of time. We call such a hypersurface a **self-shrinker** under mean curvature flow.

### 2.2.2 Self-shrinkers

**Example 2.2.6. *Shrinking Spheres again.*** *The shrinking  $n$ -sphere with initial radius  $\sqrt{2n}$  is a self-shrinker. It will disappear into a point.*

**Example 2.2.7. *Shrinking cylinders.*** *The noncompact cylindrical hypersurfaces*

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$S^k \times \mathbb{R}^{n-k}$  with radius  $\sqrt{2(n-k)}$  is a self-shrinker that will shrink to a singularity that looks like  $\mathbb{R}^{n-k}$

**Example 2.2.8. *Shrinking Donut.*** In [Ang92], Angenent proved the existence of a rotationally symmetric self-shrinking  $S^{n-1} \times S^1$  by creating a one-to-one correspondence between rotationally symmetric self-shrinkers and a “generating curve” that is rotated to create the surface. He then found the appropriate generating curve (an  $S^1$ ) using a shooting method. This was revisited by Møller in [Mol11], who found a torus using a double-shooting method.

As we said before, self-shrinkers are excellent models for singularities that can develop in a general MCF. Here are two examples of MCF singularities that can be modeled using a self-shrinker.

**Example 2.2.9. *Huisken, [Hui84].*** If a hypersurface  $M$  is initially convex, it will remain convex under MCF and will develop a singularity as it “shrinks to a round sphere”.

**Example 2.2.10. *Dumbbell again.*** If we parabolically rescale around the singular point in the previous dumbbell example, we see that the singularity is modeled by a shrinking cylinder. Or, put another way, the tangent flow to the dumbbell singularity is a shrinking cylinder.

Self-shrinkers, besides being models for singularities, can be viewed as minimal surfaces in a certain weighted sense. It is well-known (see Lemma 3.1 of [?]) that they

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are stationary solutions for a weighted area functional

$$F(M) = \int_M e^{-\frac{|x|^2}{4}} d\mathcal{A} \quad (2.10)$$

and are also minimal surfaces in Euclidean space with a conformally changed metric

$$g_{ij} = e^{-\frac{|x|^2}{2n}} \delta_{ij} \quad (2.11)$$

This means, to use our language from before, that any compact variation starting at a self-shrinker  $M$  will be stationary with respect the weighted area functional  $\mathcal{A}_\mu$ . This allows us to study self-shrinkers as minimal surfaces, and a great deal of minimal surface theory can be adapted to self-shrinkers. For example, Kapouleas, Kleene, and Moller [KKM12] recently used gluing and perturbation techniques from minimal surface theory to construct new self-shrinkers. See also [Mol11]. In another example, much has been done to study the **stability** of these self-shrinkers (that is, self-shrinkers for whom the second variation of weighted area is non-negative) (cf. [CM12], [Hus13]).



# Chapter 3

## Introduction to lambda-hypersurfaces

### 3.1 Volume preserving normal variations

As in section 1.1, we define weighted area and volume as

$$d\mathcal{A}_\mu = e^{-|x|^2/4} d\mathcal{A} \tag{3.1}$$

$$d\mathcal{V}_\mu = e^{-|x|^2/4} d\mathcal{V} \tag{3.2}$$

We will consider a, smooth, two-sided hypersurface  $\Sigma$  immersed in  $\mathbb{R}^{n+1}$  (and, by abuse of notation, will identify the manifold  $\Sigma$  with its image). We take a compact variation  $F(x, t) : \Sigma \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{n+1}$  as defined in chapter 2.1.1, and will define

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$\Sigma_t := F(\Sigma, t)$ . We will call  $F$  a normal variation if  $\frac{d}{dt}F$  is normal to  $\Sigma_t$  for all  $t$ , and we define the “normal variation function” as  $u(x, t) = u_t(x) := \langle \frac{d}{dt}F(x, t), n(x, t) \rangle$ .

It is important to define what we mean when we say that a variation “preserves volume.” Supposing momentarily that  $\Sigma$  is embedded, and remember that we define  $\mathcal{V}_\mu(\Sigma)$  as the (weighted) volume of the region that  $\Sigma$  encloses. Then  $F$  will be volume-preserving if  $\mathcal{V}_\mu(\Sigma_t) = \mathcal{V}_\mu(\Sigma)$ , ie, if  $\frac{d}{dt}\mathcal{V}_\mu(\Sigma_t) = 0$ .

At the same time, we know (see [BdC84]) that

$$\frac{d}{dt}\mathcal{V}_\mu(\Sigma_t) = \int_{\Sigma_t} \langle \frac{d}{dt}F(x, t), n(x, t) \rangle d\mathcal{A}_\mu \quad (3.3)$$

Therefore, if a normal variation  $F$  preserves  $\mathcal{V}_\mu$ , then we have that  $u_t(x)$  must satisfy  $\int_{\Sigma_t} u d\mathcal{A}_\mu = 0$ . An argument analogous to that made in [BdC84] shows that the converse is true as well:

**Lemma 3.1.1.** *Let  $\Sigma$  be an immersed, two-sided hypersurface, and let  $u$  be a function compactly supported on  $\Sigma$  such that  $\int_\Sigma u d\mathcal{A}_\mu = 0$ . Then there exists a compact normal variation  $F$  such that  $F$  is volume preserving and  $\langle \frac{d}{dt}F(x, 0), n(x) \rangle = u$ .*

Thus, we see that the volume preserving variations are exactly described (at time  $t = 0$ ) by the functions  $u \in C_0^\infty(\Sigma)$  satisfying  $\int_\Sigma u d\mathcal{A}_\mu = 0$ . By abuse of notation, we will talk about the “variation  $u$ ” as any normal variation with initial velocity determined by  $u$ .

## 3.2 The first variation formula

We now compute the first variation for area. For any compact normal variation  $F(x, t)$ , let  $u(x, t) = \langle \frac{d}{dt}|_{t=0} F(x, t), n(x, t) \rangle$ . It is standard (using the first variation of area) that  $\frac{d}{dt}|_{t=0}(d\mathcal{A}) = uH d\mathcal{A}$ , and it is clear that  $\frac{d}{dt}|_{t=0}(e^{-|x|^2/4}) = e^{-|x|^2/4}u\langle -\frac{x}{2}, n \rangle$  (see, for example, Lemma 3.1 of [CM12]). Then we have that

$$\frac{d}{dt}|_{t=0}\mathcal{A}_\mu(\Sigma_t) = \int_{\Sigma} u \left( H - \frac{1}{2}\langle x, n \rangle \right) d\mathcal{A}_\mu. \quad (3.4)$$

By using pairs of approximations to the identity with opposite weights and centered at different points for  $u$ , we find the following curvature condition must be satisfied by critical hypersurfaces of  $\mathcal{A}_\mu$  for all normal variations preserving  $V_\mu$ .

**Lemma 3.2.1.** *The hypersurface  $\Sigma$  will satisfy  $\frac{d}{dt}|_{t=0}\mathcal{A}_\mu(u) = 0$  for all  $\{u \in C_0^\infty(\Sigma) : \int_{\Sigma} u d\mathcal{A}_\mu = 0\}$  if and only if  $H - \frac{1}{2}\langle x, n \rangle$  is constant on  $\Sigma$ .*

We denote the constant  $\lambda$ , and use the name  $\lambda$ -hypersurfaces to denote those hypersurfaces that satisfy  $H = \frac{1}{2}\langle x, n \rangle + \lambda$ .

### 3.2.1 Examples of $\lambda$ -hypersurfaces

**Example 3.2.2.** *Any hyperplane in  $\mathbb{R}^{n+1}$  (not necessarily passing through the origin) is a  $\lambda$ -hypersurface as  $\langle x, n \rangle$  is constant and  $H = 0$ .*

**Example 3.2.3.** *Any sphere with center  $x_0$  and radius  $R$  satisfies the equation  $|x -$*

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$|x_0| = R$ . Therefore, the normal vector satisfies  $n = \frac{x-x_0}{R}$  and the mean curvature satisfies  $H = \frac{n}{R}$ . This implies that  $\langle x, n \rangle = R + (1/R)\langle x_0, x - x_0 \rangle$ . Hence, in order for the sphere to be a  $\lambda$ -hypersurface, we need  $\frac{n}{R} - \frac{R}{2} - \frac{1}{2R}\langle x_0, x - x_0 \rangle$  to be a constant. Therefore, the only spheres that satisfy the equation are the spheres centered at the origin.

**Example 3.2.4.** For any cylinder  $\{(x, y) \in S^k \times \mathbb{R}^{n-k} : |x - x_0| = R\}$ , we have the normal vector satisfies  $n = \frac{1}{R}(x - x_0)$  and the mean curvature satisfies  $H = k/R$ . Hence, to be a  $\lambda$ -hypersurface, the quantity  $\frac{k}{R} - \frac{R}{2} - \frac{1}{2R}\langle x_0, x - x_0 \rangle = \lambda$  must be constant. Thus, the only cylinders that are also  $\lambda$ -hypersurfaces are those that are cylinders over spheres  $S^k$  in some  $(k+1)$ -plane and centered at the origin.

**Example 3.2.5.** As noted before, if  $\lambda = 0$ , then the hypersurfaces are the self-shrinkers of the mean curvature flow (see section 2.2). There are many examples of self-shrinkers, including Angenent's self-shrinking torus [Ang92] and the noncompact examples of Kapouleas, Kleene, and Møller [KKM12].

### 3.3 The second variation formula

Now we discuss the second variation  $\frac{d^2}{dt^2}\mathcal{A}_\mu(u)$ , where  $u$  represents a normal variation of  $\Sigma$  that is compact and satisfies  $u \in \{v \in C_0^\infty(\Sigma) : \int_\Sigma v d\mathcal{A}_\mu = 0\}$  (ie,  $u$  is  $\mathcal{V}_\mu$  preserving).

**Lemma 3.3.1.** *Let  $\Sigma$  be a  $\lambda$ -hypersurface. Then for any  $u$  representing a compact*

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normal variation that preserves  $\mathcal{V}_\mu$ , we get

$$\frac{d^2}{dt^2}|_{t=0}\mathcal{A}_\mu(\Sigma)(u) = - \int_\Sigma u \left( \Delta u + |A|^2 u + \frac{1}{2}u - \frac{1}{2}\langle x, \nabla u \rangle \right) d\mathcal{A}_\mu. \quad (3.5)$$

*Proof.* Given  $u \in \{v \in C_0^\infty(\Sigma) : \int_\Sigma v d\mathcal{A}_\mu = 0\}$ , let  $F(x, t)$  be a normal variation preserving  $\mathcal{V}_\mu$  such that  $u(x) = \langle \partial_t F(x, 0) \rangle$ . Extend  $u(x)$  to be  $u(x, t) = \langle \partial_t F(x, t), n(x, t) \rangle$ . Since the variation is volume-preserving, we have that  $\int_{\Sigma_t} u d\mathcal{A}_\mu = 0$  for all  $t$ .

Let  $H = \frac{1}{2}\langle x, n \rangle + \lambda$  on  $\Sigma$ . From 3.4, we have that

$$\frac{d^2}{dt^2}|_{t=0}\mathcal{A}_\mu(\Sigma_t)(u) = \frac{d}{dt}|_{t=0} \left( \int_{\Sigma_t} \left( H - \frac{1}{2}\langle x, n \rangle \right) u d\mathcal{A}_\mu \right) \quad (3.6)$$

$$= \int_\Sigma \frac{d}{dt}|_{t=0} \left( H - \frac{1}{2}\langle x, n \rangle \right) (u d\mathcal{A}_\mu) + \int_\Sigma \left( H - \frac{1}{2}\langle x, n \rangle \right) \frac{d}{dt}|_{t=0}(u d\mathcal{A}_\mu) \quad (3.7)$$

$$= \int_\Sigma \frac{d}{dt}|_{t=0} \left( H - \frac{1}{2}\langle x, n \rangle \right) (u d\mathcal{A}_\mu) + \lambda \frac{d}{dt}|_{t=0} \left( \int_{\Sigma_t} u d\mathcal{A}_\mu \right) \quad (3.8)$$

$$= \int_\Sigma \frac{d}{dt}|_{t=0} \left( H - \frac{1}{2}\langle x, n \rangle \right) (u d\mathcal{A}_\mu) \quad (3.9)$$

It well-known that, in Euclidean space, we have that  $\frac{d}{dt}|_{t=0}n = -\nabla u$  and that  $\frac{d}{dt}|_{t=0}H = -\Delta u - |A|^2 u$ . For an explanation of these formulas, see Colding-Minicozzi [CM12]. Also, it is clear that  $\langle \frac{d}{dt}x, n \rangle = u$  at time  $t = 0$  from our definition of  $u$ . This

gives us

$$\frac{d^2}{dt^2}|_{t=0}\mathcal{A}_\mu(\Sigma_t)(u) = - \int_\Sigma u \left( \Delta u + |A|^2 u + \frac{1}{2}u - \frac{1}{2}\langle x, \nabla u \rangle \right) d\mathcal{A}_\mu \quad (3.10)$$

which completes the proof. □

### 3.3.1 The $\mathcal{L}$ and $L$ operators, and the quadratic form $\mathcal{Q}$

Here we define two operators that will be of central importance. The first is called the drift Laplacian (also called the Witten-Laplacian in certain settings, see chapter 5.1). It is denoted by  $\mathcal{L}$ :

$$\mathcal{L}u := \Delta u - \frac{1}{2}\langle x, \nabla u \rangle. \quad (3.11)$$

We remark that the  $\mathcal{L}$  operator is self-adjoint in the weighted  $L^2$  space: in other words, if the functions  $u, v$  are “in the weighted  $L^2$  space”, then the following integral identities hold:

$$\int_\Sigma u \mathcal{L}v d\mathcal{A}_\mu = \int_\Sigma v \mathcal{L}u d\mathcal{A}_\mu = - \int_\Sigma \langle \nabla u, \nabla v \rangle d\mathcal{A}_\mu \quad (3.12)$$

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When we say that  $u, v$  are in the weighted  $L^2$  space, we mean that

$$\int_{\Sigma} (|u|^2 + |\nabla u|^2 + |\mathcal{L}u|^2) d\mathcal{A}_{\mu} < \infty \quad (3.13)$$

This general requirement will guarantee that the integrals defined in 3.12 are well-defined, and will be satisfied (for example) on smooth, compactly supported  $u$ , or on smooth  $u$  when  $\Sigma$  is compact.

We also introduce the stability operator, denoted  $L$ :

$$Lu := \Delta u + |A|^2 u + \frac{1}{2}u - \frac{1}{2}\langle x, \nabla u \rangle \quad (3.14)$$

This operator is equivalent to  $\mathcal{L}$  up to the lowest order terms. Thus, it is also self-adjoint in the weighted space. It is called the stability operator because the second variation of weighted area can be written as

$$\frac{d^2}{dt^2}|_{t=0}\mathcal{A}_{\mu}(\Sigma)(u) = - \int_{\Sigma} uLu d\mathcal{A}_{\mu} \quad (3.15)$$

For convenience, we define the related quadratic form  $\mathcal{Q}$  on  $C_0^{\infty}(\Sigma)$  by  $\mathcal{Q}(u, v) = - \int_{\Sigma} uLv d\mathcal{A}_{\mu}$ . This motivates the following definition:

**Definition 3.3.2.**  $\Sigma$  is called *stable* (more precisely,  $\mathcal{V}_{\mu}$ -preserving stable) if

$$\mathcal{Q}(u, u) \geq 0 \quad (3.16)$$

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for every  $u \in C_0^\infty(\Sigma)$  that preserves the weighted volume (ie, that satisfies  $\int_\Sigma u d\mathcal{A}_\mu = 0$ ).

**Definition 3.3.3.** We define the index  $I$  of  $\Sigma$  (more precisely, the  $\mathcal{V}_\mu$ -preserving index) to be the number of linearly independent functions  $u \in C_0^\infty(\Sigma)$  which preserve weighted volume and have  $\mathcal{Q}(u, u) < 0$ .

These definitions make sense since each normal variation can be represented at the initial time-slice by a function  $u$  satisfying  $\int_\Sigma u d\mathcal{A}_\mu = 0$ .



# Chapter 4

## Stability of lambda-hypersurfaces

The main results of this thesis are a series of results detailing necessary and sufficient conditions for a hypersurface to be (volume-preserving) stable. We will begin by showing that hyperplanes are stable solutions. This is well-known since, by [ST78] and [Bor75], and more recently by [Bob97], hyperplanes are actually global minimizers to the isoperimetric problem and thus clearly stable. We present a new proof using a comparison between the stability operator  $L$  and the quantum harmonic oscillator, which is a trick that has been used in similar problems (see, for example, [KKM12]). We then discuss the case where  $\Sigma$  is complete, and prove a general result about the index of  $\Sigma$ . Finally, we will turn to the incomplete case, and derive a series of curvature estimates that a stable solution must satisfy. The curvature estimates in the incomplete case will recover the results from the complete case when passing to a limit on the size of the ball of definition.

## 4.1 Hyperplanes are stable

**Theorem 4.1.1.** *Hyperplanes are  $\mathcal{V}_\mu$ -preserving stable hypersurfaces.*

*Proof.* We begin by observing that if a hyperplane does not pass through the origin, then after a change of coordinates it may be considered to be the plane  $x_{n+1} = c$ . A change of variables  $x \rightarrow x - (0, 0, \dots, c)$  then shifts this plane to pass through the origin and changes the quadratic functional  $\mathcal{Q}$  by the constant factor  $e^{-|c|^2/4}$ . As this will not change the sign of  $\mathcal{Q}$ , it suffices to consider the stability of a hyperplane through the origin.

For such a hyperplane, the second fundamental form satisfies  $A \equiv 0$  and the stability operator  $L$  takes the form

$$Lu = \Delta u - \frac{1}{2} \langle x, \nabla u \rangle + \frac{1}{2} u. \quad (4.1)$$

Such an operator is well-known to be comparable to the harmonic oscillator: for example, such a comparison is done in Kapouleas-Kleene-Møller [KKM12]. Indeed, the operator  $L$  may be factored as

$$Lu = e^{|x|^2/8} \left( \Delta - \frac{|x|^2}{16} + \frac{n+2}{4} \right) e^{-|x|^2/8} u \quad (4.2)$$

where the new operator

$$H_x := \left( \Delta - \frac{|x|^2}{16} + \frac{n+2}{4} \right) \quad (4.3)$$

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is being applied to  $e^{-|x|^2/8}u$ .  $H_x$  may be viewed as a shifted version of the harmonic oscillator,

$$\tilde{H} := \Delta - |x|^2 \tag{4.4}$$

as a change of variables  $x = 2y$  gives us the operator

$$H_y := \frac{1}{4}\Delta - \frac{|x|^2}{4} + \frac{n+2}{4}, \tag{4.5}$$

and  $\tilde{H} = 4H_y - (n+2)$ .

The eigenvalues of  $\tilde{H}$  (i.e, the values  $\gamma$  such that  $\tilde{H}u = -\gamma u$ ) are well-known to be  $n + 2k$  for  $k = 0, 1, 2, \dots$ , and the eigenfunctions are products of  $e^{-|y|^2/2}$  and Hermite polynomials. So the eigenvalues of  $H_y$  (which are equivalent to the eigenvalues of  $L$ ) take the form  $(k-1)/2$ . Except for the first eigenvalue, these are all positive.

Observe that  $n$  is the lowest eigenvalue of  $\tilde{H}$  and has an eigenspace spanned by  $e^{-|y|^2/2}$ . Undoing the change of variables, the lowest eigenvalue of  $L$  is  $-1/2$ . Furthermore, the lowest eigenspace of  $L$  is spanned by the constant functions. Such functions do not correspond to  $u$  that are volume preserving, so we discount them.

Note that if  $u \in C_0^\infty(\Sigma)$  such that  $\int_\Sigma u d\mathcal{A}_\mu = 0$ , then  $u$  is orthogonal to the constant functions under the weighted  $d\mathcal{A}_\mu$  measure. Since the other eigenvalues of  $L$  are non-negative, we then have that the hyperplanes are  $\mathcal{V}_\mu$ -preserving stable hypersurfaces.

□

## 4.2 Stability of complete $\lambda$ -hypersurfaces

We now want to examine the the index of complete, nonplanar hypersurfaces. The argument may be broken into two cases, for compact and non-compact hypersurfaces. The compact case is much simpler because we do not need any cutoff functions, and will motivate the non-compact case. First, however, we remark on an important identity.

**Lemma 4.2.1.** *Let  $\Sigma \subset \mathbb{R}^{n+1}$  satisfy  $H = \frac{1}{2}\langle x, n \rangle + \lambda$ , and let  $v \in \mathbb{R}^{n+1}$  be a constant vector. Then*

$$L\langle v, n \rangle = \frac{1}{2}\langle v, n \rangle \quad (4.6)$$

*Proof.* The proof is identical to the corresponding proof for self-shrinkers in Colding-Minicozzi [CM12]. In particular, the main computation on  $\nabla H$  is the same for self-shrinkers and for our hypersurfaces.  $\square$

**Lemma 4.2.2.** *Let  $\Sigma$  be a compact hypersurface that satisfies  $H = \frac{1}{2}\langle x, n \rangle + \lambda$ . Then  $\mathcal{Q}$  is negative definite on  $\text{Span}\{1, \langle v, n \rangle : v \in \mathbb{R}^{n+1}\}$ .*

*Proof.* From (4.6) we have that  $\mathcal{Q}$  is negative definite on  $\text{Span}\{\langle v, n \rangle : v \in \mathbb{R}^{n+1}\}$ . So it is sufficient to check that  $\mathcal{Q}(1 + u, 1 + u) < 0$  for any  $u = \langle v, n \rangle$ .

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We use the divergence theorem and lemma 4.2.1 to get

$$\frac{1}{2} \int_{\Sigma} u d\mathcal{A}_{\mu} = \int_{\Sigma} Lu d\mathcal{A}_{\mu} \quad (4.7)$$

$$= \int_{\Sigma} \mathcal{L}u d\mathcal{A}_{\mu} + \int_{\Sigma} \left( |A|^2 + \frac{1}{2} \right) u d\mathcal{A}_{\mu} \quad (4.8)$$

$$= \int_{\Sigma} \left( |A|^2 + \frac{1}{2} \right) u d\mathcal{A}_{\mu}. \quad (4.9)$$

Therefore,  $\int_{\Sigma} |A|^2 u d\mathcal{A}_{\mu} = 0$ , and so  $u$  is orthogonal to  $|A|^2$  in the weighted  $L^2$  space.

We then compute

$$\mathcal{Q}(1+u, 1+u)$$

$$= - \int_{\Sigma} (1+u)L(1+u) d\mathcal{A}_{\mu} \quad (4.10)$$

$$= - \int_{\Sigma} \left( \frac{1}{2} + |A|^2 \right) d\mathcal{A}_{\mu} - \int_{\Sigma} u \left( |A|^2 + \frac{1}{2} \right) d\mathcal{A}_{\mu} - \frac{1}{2} \int_{\Sigma} u^2 d\mathcal{A}_{\mu} \quad (4.11)$$

$$= - \int_{\Sigma} |A|^2 d\mathcal{A}_{\mu} - \frac{1}{2} \int_{\Sigma} (u+1)^2 d\mathcal{A}_{\mu} \quad (4.12)$$

where we have used the self-adjointness of  $L$  and the fact that  $u$  is orthogonal to  $|A|^2$ .

This shows that  $\mathcal{Q}$  is negative definite on  $\text{Span}\{1, \langle v, n \rangle : v \in \mathbb{R}^{n+1}\}$ .  $\square$

The argument for the non-compact case is morally similar to the compact case, but we need to multiply our normal variation function  $u$  by a cutoff function  $\phi \in C_0^{\infty}(\Sigma)$  to make it compactly supported. For the proof of Lemma 4.2.2, the orthogonality of  $|A|^2$  and  $\langle v, n \rangle$  was critical. For the non-compact case, once we introduce the cutoff functions we lose orthogonality: in particular, we don't have that  $|A|^2$  and  $\phi^2 \langle v, n \rangle$

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are orthogonal. We'll get around this by using (4.14), which gives us control on the product of  $|A|^2$  and  $\phi^2 \langle v, n \rangle$ .

**Lemma 4.2.3.** *For any functions  $\phi \in C_0^\infty(\Sigma)$  and  $f \in C^\infty(\Sigma)$  we have that*

$$\int_{\Sigma} \phi f L(\phi f) d\mathcal{A}_{\mu} = \int_{\Sigma} \phi^2 f Lf d\mathcal{A}_{\mu} - \int_{\Sigma} |\nabla \phi|^2 f^2 d\mathcal{A}_{\mu}. \quad (4.13)$$

Also, for any  $\Sigma$  satisfying  $H = \frac{1}{2} \langle x, n \rangle + \lambda$  and constant vector  $v \in \mathbb{R}^{n+1}$  we have that

$$\int_{\Sigma} \phi^2 |A|^2 \langle v, n \rangle d\mathcal{A}_{\mu} = 2 \int_{\Sigma} \phi A(\nabla \phi, v^T) d\mathcal{A}_{\mu}, \quad (4.14)$$

*Proof.*

$$\begin{aligned} \int_{\Sigma} \phi f L(\phi f) d\mathcal{A}_{\mu} &= \int_{\Sigma} \left( f^2 \phi \mathcal{L} \phi + \frac{1}{2} \langle \nabla \phi^2, \nabla f^2 \rangle + \phi^2 f Lf \right) d\mathcal{A}_{\mu} \\ &= \int_{\Sigma} (f^2 \phi \mathcal{L} \phi - f^2 \phi \mathcal{L} \phi - |\nabla \phi|^2 f^2 + \phi^2 f Lf) d\mathcal{A}_{\mu} \\ &= \int_{\Sigma} (\phi^2 f Lf - |\nabla \phi|^2 f^2) d\mathcal{A}_{\mu}. \end{aligned} \quad (4.15)$$

This shows (4.13). To prove (4.14), consider  $\Sigma$  satisfying  $H = \frac{1}{2} \langle x, n \rangle + \lambda$ . Using Lemma 4.2.1 and (4.13) we have that

$$\begin{aligned} \frac{1}{2} \int_{\Sigma} \phi \langle v, n \rangle d\mathcal{A}_{\mu} &= \int_{\Sigma} \phi L \langle v, n \rangle d\mathcal{A}_{\mu} \\ &= \int_{\Sigma} \left( \frac{1}{2} + |A|^2 \right) \phi \langle v, n \rangle d\mathcal{A}_{\mu} - \int_{\Sigma} \langle \nabla \phi, \nabla \langle v, n \rangle \rangle d\mathcal{A}_{\mu}. \end{aligned} \quad (4.16)$$

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Therefore, we get that

$$\int_{\Sigma} |A|^2 \phi \langle v, n \rangle d\mathcal{A}_{\mu} = \int_{\Sigma} A(\nabla \phi, v^T) d\mathcal{A}_{\mu}. \quad (4.17)$$

Replacing  $\phi$  with  $\phi^2$  gives us (4.14).  $\square$

Define  $V$  to be a subspace of  $C^{\infty}(\Sigma)$  spanned by the constant functions and those corresponding to translation variations, ie  $V := \text{Span}\{1, \langle v, n \rangle : v \in \mathbb{R}^{n+1}\}$ . Our next lemma shows that there exists a compactly supported function  $\phi$  that allows  $\mathcal{Q}$  to be negative definite on  $\phi V$ .

**Lemma 4.2.4.** *Let  $\Sigma$  be a hypersurface satisfying  $H = \frac{1}{2}\langle x, n \rangle + \lambda$  and  $\mathcal{A}_{\mu}(\Sigma) < \infty$ . Then there exists  $\phi \in C_0^{\infty}(\Sigma)$  such that  $\mathcal{Q}$  is negative definite on  $\phi V$  and  $\text{Dim}(\phi V) = \text{Dim} V$ .*

**Remark 4.2.5.** *The vector space  $V$  consists of all functions that are spanned by the constant function and those functions that correspond to translating the hypersurface. It is not true that all functions in  $V$  represent variations preserving  $\mathcal{V}_{\mu}$ .*

**Remark 4.2.6.** *We are assuming  $\Sigma$  is non-compact here (or else Lemma 4.2.2 would suffice). Nevertheless, it is not a stretch to assume that  $\mathcal{A}_{\mu}(\Sigma) < \infty$ , as the weighted area includes an exponential decay (any hypersurface with polynomial volume growth, for example, will satisfy this).*

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*Proof.* Consider  $u := c_0 + \langle v, n \rangle$ , and consider  $\mathcal{Q}(\phi u, \phi u)$ . From (4.13) we have that

$$\mathcal{Q}(\phi u, \phi u) = - \int_{\Sigma} \phi^2 u L u \, d\mathcal{A}_{\mu} + \int_{\Sigma} |\nabla \phi|^2 u^2 \, d\mathcal{A}_{\mu} \quad (4.18)$$

$$= - \int_{\Sigma} \phi^2 u c_0 (1/2 + |A|^2) \, d\mathcal{A}_{\mu} - \int_{\Sigma} \phi^2 u \frac{1}{2} \langle v, n \rangle \, d\mathcal{A}_{\mu} + \int_{\Sigma} |\nabla \phi|^2 u^2 \, d\mathcal{A}_{\mu}. \quad (4.19)$$

Using that  $u = c_0 + \langle v, n \rangle$ , we get

$$\mathcal{Q}(\phi u, \phi u) = -\frac{1}{2} \int_{\Sigma} \phi^2 u^2 \, d\mathcal{A}_{\mu} - \int_{\Sigma} \phi^2 |A|^2 c_0^2 \, d\mathcal{A}_{\mu} - \int_{\Sigma} \phi^2 |A|^2 c_0 \langle v, n \rangle \, d\mathcal{A}_{\mu} + \int_{\Sigma} |\nabla \phi|^2 u^2 \, d\mathcal{A}_{\mu}. \quad (4.20)$$

Now, using (4.14) and a Cauchy-Schwarz inequality of the form  $2ab \leq a^2 + b^2$ , we get

$$\begin{aligned} \left| \int_{\Sigma} \phi^2 |A|^2 c_0 \langle v, n \rangle \, d\mathcal{A}_{\mu} \right| &= 2 \left| \int_{\Sigma} \phi A (\nabla \phi, v^T) c_0 \, d\mathcal{A}_{\mu} \right| \\ &\leq \int_{\Sigma} \phi^2 |A|^2 c_0^2 \, d\mathcal{A}_{\mu} + \int_{\Sigma} |\nabla \phi|^2 |v^T|^2 \, d\mathcal{A}_{\mu}. \end{aligned} \quad (4.21)$$

Therefore,

$$\mathcal{Q}(\phi u, \phi u) \leq -\frac{1}{2} \int_{\Sigma} \phi^2 u^2 \, d\mathcal{A}_{\mu} + \int_{\Sigma} |\nabla \phi|^2 (u^2 + |v^T|^2) \, d\mathcal{A}_{\mu} \quad (4.22)$$

Now, fix a point  $p \in \Sigma$  and let  $r$  be the Euclidean distance from the origin. For each large  $R > 0$ , define the cut-off function  $\phi_R$  such that

- $\phi_R(r) \in C_0^{\infty}(\Sigma)$



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- $\phi_R(r) = 1$  whenever  $r \leq R$
- $\phi_R(r) = 0$  whenever  $r \geq 2R$
- $|\nabla \phi_R| \leq \frac{2}{R}$

With this, (4.22) becomes

$$\mathcal{Q}(\phi_R u, \phi_R u) \leq -\frac{1}{2} \int_{\Sigma} \phi_R^2 u^2 d\mathcal{A}_{\mu} + \frac{4}{R^2} \int_{\Sigma \cap (B_{2R}(0) \setminus B_R(0))} (u^2 + |v^T|^2) d\mathcal{A}_{\mu}. \quad (4.23)$$

Since  $\mathcal{A}_{\mu}(\Sigma) < \infty$ , we have that  $\int_{\Sigma} (u^2 + |v^T|^2) d\mathcal{A}_{\mu} < \infty$  for any  $u \in V$ . For fixed  $u \neq 0$  and taking  $R \rightarrow \infty$ , we see that there exists  $R_u$  such that  $\mathcal{Q}(\phi_{R_u} u, \phi_{R_u} u) < 0$ .

We wish to show that we can find such an  $R$  that is independent of  $u \in V$ .

Note that, while the dimension of  $V$  is not necessarily  $n+1$ , it is certainly finite. Let  $\{c_i + \langle v_i, n \rangle\}$  be a basis for  $V$  with  $|c_i|^2 + |v_i|^2 = 1$ . Define  $S \equiv \{d^i(c_i + \langle v_i, n \rangle) : \sum d_i^2 = 1\}$  to be the “unit sphere in  $V$ ”. Since  $\dim V < \infty$ , we have that  $S$  is compact.

This implies there exists  $R_0$  such that for all  $u \in S$  we have that  $B_{R_0} \cap \{u \neq 0\} \neq \emptyset$ ; otherwise, there would exist a sequence of  $u_j = d_j^i(c_i + \langle v_i, n \rangle) \in S$  such that  $u_j \equiv 0$  on  $B_j$ . By the compactness of  $S$ , after passing to a subsequence we would have a limit  $d_j^i \rightarrow d_{\infty}^i \in S$  such that  $d_{\infty}^i(c_i + \langle v_i, n \rangle) \equiv 0$ : a contradiction, since  $0 \notin S$ .

Thus, for  $R \geq R_0$  and all  $u \in S$  there exists a constant  $M_R > 0$  such that

$$\int_{\Sigma} \phi_R^2 u^2 d\mathcal{A}_{\mu} \geq M_R > 0. \quad (4.24)$$

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Note that  $M_R$  is increasing in  $R$ , and that  $\text{Dim}(\phi_R V) = \text{Dim} V$ . Since  $S$  is compact, we may find  $D_S$  such that  $\int_{\Sigma} (u^2 + |v^T|^2) d\mathcal{A}_{\mu} < D_S$  for all  $u \in S$ . Therefore, (4.23) becomes

$$\mathcal{Q}(\phi_R u, \phi_R u) \leq -\frac{M_R}{2} + \frac{4D_S}{R^2}, \quad (4.25)$$

for all  $u \in S$  and  $R \geq R_0$ . Taking  $R \rightarrow \infty$ , and remembering that  $M_R$  is increasing in  $R$ , we may find  $R$  independent of  $u$  such that  $\mathcal{Q}(\phi_R u, \phi_R u) < 0$  for all  $u \in S$ . So we have that  $\text{Dim} V = \text{Dim}(\phi_R V)$  and that  $\mathcal{Q}$  is negative definite on  $\phi_R V$ .  $\square$

Now, we will prove the main theorem of this section. We use the space  $\phi V$  from Lemma 4.2.4 and dimension counting to show that bounds on the index of  $\mathcal{Q}$  will force  $\Sigma$  to split off a linear space.

**Theorem 4.2.7.** *Consider any two-sided, smooth, properly immersed, non-planar hypersurface  $\Sigma \subset \mathbb{R}^{n+1}$  such that  $\mathcal{A}_{\mu}(\Sigma) < \infty$ ,  $\Sigma$  satisfies the mean curvature condition  $H = \frac{1}{2}\langle x, n \rangle + \lambda$ , and  $\text{Index } \mathcal{Q} \leq n$ . Then there exists an  $i$  such that  $n + 1 - (\text{Index } \mathcal{Q}) \leq i \leq n$ , and we have that*

$$\Sigma = \Sigma_0 \times \mathbb{R}^i. \quad (4.26)$$

Furthermore, for such non-planar  $\Sigma$  it is impossible that  $\text{Index } \mathcal{Q} = 0$  or  $\text{Index } \mathcal{Q} = 1$ .

*Proof.* Let  $V := \text{Span}\{1, \langle v, n \rangle\}_{v \in \mathbb{R}^{n+1}}$ . First, we comment on  $\text{Dim} V$ . Consider the case that the constant function  $c_0 \in \text{Span}\{\langle v, n \rangle\}_{v \in \mathbb{R}^{n+1}}$ . We have that  $Lc_0 = \frac{1}{2}c_0$ ,

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but  $c_0$  is also constant, so  $Lc_0 = (\frac{1}{2} + |A|^2)c_0$ . Therefore,  $|A|^2c_0 \equiv 0$ , and since  $\Sigma$  is non-planar, we have that  $c_0 = 0$ . Hence,

$$\text{Dim}V = 1 + \text{DimSpan}\{\langle v, n \rangle\}_{v \in \mathbb{R}^{n+1}}. \quad (4.27)$$

By Lemma 4.2.4, we have for some  $\phi \in C_0^\infty(\Sigma)$  that  $\text{Dim}\phi V = \text{Dim}V$  and that  $Q$  is negative definite on  $\phi V$ .

Recall that we are interested in variations that preserve  $\mathcal{V}_\mu$ , and therefore we must only consider functions  $u \in C_0^\infty(\Sigma)$  that satisfy  $\int_\Sigma u d\mathcal{A}_\mu = 0$ . So, we need to consider the space  $\phi V \cap 1^\perp$  (ie, those functions perpendicular to the constant functions with respect to the weighted metric). By counting dimensions, we have  $\text{Dim}(\phi V \cap 1^\perp) \geq \text{DimSpan}\{\langle v, n \rangle\}_{v \in \mathbb{R}^{n+1}}$ . Hence, we know  $\text{DimSpan}\{\langle v, n \rangle\}_{v \in \mathbb{R}^{n+1}} \leq \text{Index } \mathcal{Q}$ . Considering the kernel of the linear transformation  $\mathbb{R}^{n+1} \rightarrow C^\infty(\Sigma)$  given by  $v \rightarrow \langle v, n \rangle$ , we have that

$$\text{Dim}\{v : \langle v, n \rangle \equiv 0\} = n + 1 - \text{DimSpan}\{\langle v, n \rangle\}_{v \in \mathbb{R}^{n+1}} \quad (4.28)$$

$$\geq n + 1 - (\text{Index } \mathcal{Q}). \quad (4.29)$$

Finally, note that

$$\Sigma = \Sigma_0 \times \{v : \langle v, n \rangle \equiv 0\}. \quad (4.30)$$

□

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**Remark 4.2.8.** *We will show that the index bound in Theorem 4.2.7 is sharp by considering the case of  $\Sigma = S_R^n \subset \mathbb{R}^{n+1}$ . Here, one has that the eigenvalues of  $\Delta_{S_R^n}$  are given by  $k(k+n-1)/R^2$  for  $k = 0, 1, 2, \dots$ , and each eigenspace is given by the restriction of harmonic polynomials in  $x_1, \dots, x_{n+1}$  that are homogeneous of degree  $k$ .*

*Therefore, for  $\Sigma = S_R^n$ , we have that  $L$  has eigenvalues*

$$\gamma_k = \frac{1}{R^2} (k(k+n-1) - n) - \frac{1}{2}. \quad (4.31)$$

*The lowest eigenspace is given by the constant functions, so all other eigenspaces represent variations preserving  $\mathcal{V}_\mu$ .*

*The next eigenspace, for  $\gamma_1 = -1/2$ , is given by the functions  $\{\langle v, n \rangle : v \in \mathbb{R}^{n+1}\}$ .*

*Note that its dimension is  $n+1$ .*

*The next eigenvalue is  $\gamma_2 = (n+2)/R^2 - (1/2)$ . So we see that, for  $R^2 < 2n+4$ ,  $\Sigma = S_R^n$  is an example of a hypersurface satisfying  $H = \frac{1}{2}\langle x, n \rangle + \lambda$  for some  $\lambda$ , Index  $\mathcal{Q} = n+1$ , and  $\Sigma$  does not split off a linear space. Therefore, the index bound in Theorem 4.2.7 is sharp.*

**Corollary 4.2.9.** *The hyperplanes are the only two-sided, smooth, complete, properly immersed  $\lambda$ -hypersurfaces  $\Sigma \subset \mathbb{R}^{n+1}$  such that  $\mathcal{A}_\mu(\Sigma) < \infty$  and  $\Sigma$  satisfies the locally stable condition (3.16).*

*Furthermore, there are no two-sided, smooth, complete, properly immersed  $\Sigma$  such that  $\mathcal{A}_\mu(\Sigma) < \infty$ ,  $\Sigma$  satisfies  $H = \frac{1}{2}\langle x, n \rangle + \lambda$ , and Index  $\mathcal{Q} = 1$ .*

## 4.3 The incomplete case

### 4.3.1 An integral curvature estimate

Using stability inequalities to obtain integral estimates for the norm of the second fundamental form,  $|A|$ , and then turning these estimates into pointwise estimates for  $|A|$  is a long established technique in geometric analysis, dating back to at least Schoen-Simon-Yau [SSY75]. Over the next two sections, we use this approach to get estimates on  $|A|$ . These estimates will require  $\Sigma$  to be defined over a suitably large ball: however,  $\Sigma$  will not need to be complete. One complication that we must overcome is that, in its current form, our stability condition (3.16) can only be applied to test functions  $u \in C_0^\infty(\Sigma)$  that are volume preserving.

Since the functions  $\langle v, n \rangle$  for  $v \in \mathbb{R}^{n+1}$  play a key role in the proof of theorem 4.2.7, it is not surprising that they play a key role in creating an integral estimate for the incomplete case. We will use these functions with appropriate cut-off functions to prove our estimate.

For two-sided  $\Sigma$  and any  $\phi \in C_0^\infty(\Sigma)$  such that  $\phi \not\equiv 0$ , let  $n_\phi$  be the constant vector defined by

$$n_\phi := \frac{\int_\Sigma \phi n d\mathcal{A}_\mu}{\int_\Sigma \phi d\mathcal{A}_\mu}. \quad (4.32)$$

In the case that  $\phi \equiv 0$ , we may define  $n_\phi := 0$ .

We find a modified version of the stability condition (3.16) that is valid for any

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$\phi \in C_0^\infty(\Sigma)$ : in particular, such  $\phi$  will not necessarily satisfy  $\int_\Sigma \phi d\mathcal{A}_\mu = 0$ . One should compare (4.33) to the stability inequality for minimal hypersurfaces in Euclidean space, which states that  $\int_\Sigma \phi^2 |A|^2 d\mathcal{A} \leq \int_\Sigma |\nabla \phi|^2 d\mathcal{A}$  (see [CM11] for more details).

**Lemma 4.3.1.** *Let  $\Sigma \subset \mathbb{R}^{n+1}$  be a two-sided, smooth, immersed hypersurface satisfying the mean curvature condition  $H = \frac{1}{2}\langle x, n \rangle + \lambda$  and satisfying the stability condition (3.16). Let  $\phi \in C_0^\infty(\Sigma)$  such that  $\phi \geq 0$  and  $\phi \not\equiv 0$ . Then, we have that*

$$\int_\Sigma \phi^2 |n - n_\phi|^2 d\mathcal{A}_\mu + |n_\phi|^2 \int_\Sigma \phi^2 |A|^2 d\mathcal{A}_\mu \leq B_n \int_\Sigma |\nabla \phi|^2 d\mathcal{A}_\mu \quad (4.33)$$

where  $B_n$  is a constant depending only on  $n$ .

*Proof.* Let  $v \in \mathbb{R}^{n+1}$  such that  $|v| = 1$ , and observe that  $\int_\Sigma \phi \langle v, n - n_\phi \rangle d\mathcal{A}_\mu = 0$ :

$$\begin{aligned} \int_\Sigma \phi \langle v, n - n_\phi \rangle d\mathcal{A}_\mu &= \int_\Sigma \phi \langle v, n \rangle d\mathcal{A}_\mu - \int_\Sigma \phi \langle v, n_\phi \rangle d\mathcal{A}_\mu \\ &= \int_\Sigma \phi \langle v, n \rangle d\mathcal{A}_\mu - \int_\Sigma \phi \langle v, \frac{\int_\Sigma \phi n d\mathcal{A}_\mu}{\int_\Sigma \phi d\mathcal{A}_\mu} \rangle d\mathcal{A}_\mu \\ &= \int_\Sigma \phi \langle v, n \rangle d\mathcal{A}_\mu - \frac{1}{\int_\Sigma \phi d\mathcal{A}_\mu} \int_\Sigma \phi \langle v, \int_\Sigma \phi n d\mathcal{A}_\mu \rangle d\mathcal{A}_\mu \\ &= \int_\Sigma \phi \langle v, n \rangle d\mathcal{A}_\mu - \frac{1}{\int_\Sigma \phi d\mathcal{A}_\mu} \int_\Sigma \phi d\mathcal{A}_\mu \int_\Sigma \phi \langle v, n \rangle d\mathcal{A}_\mu \\ &= 0 \end{aligned}$$

where the second-to-last equality comes from writing  $\int_\Sigma \phi n d\mathcal{A}_\mu$  as a vector and taking the inner product. Therefore we may plug  $u = \phi \langle v, n - n_\phi \rangle$  into the stability condition

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(3.16). A computation analogous to that in (4.20) gives us

$$\frac{1}{2} \int_{\Sigma} \phi^2 \langle v, n - n_{\phi} \rangle^2 d\mathcal{A}_{\mu} \leq \int_{\Sigma} \phi^2 |A|^2 \langle v, n_{\phi} \rangle \langle v, n - n_{\phi} \rangle d\mathcal{A}_{\mu} + \int_{\Sigma} 4|\nabla \phi|^2 d\mathcal{A}_{\mu}. \quad (4.34)$$

Applying a Cauchy inequality of the form  $2ab \leq (1/2)a^2 + 2b^2$  to (4.14), we get

$$\langle v, n_{\phi} \rangle \int_{\Sigma} |A|^2 \phi^2 \langle v, n \rangle d\mathcal{A}_{\mu} \leq \frac{1}{2} \int_{\Sigma} \phi^2 \langle v, n_{\phi} \rangle^2 |A|^2 d\mathcal{A}_{\mu} + 2 \int_{\Sigma} |\nabla \phi|^2 |v^T|^2 d\mathcal{A}_{\mu}. \quad (4.35)$$

Combining (4.34) and (4.35) gives us

$$\int_{\Sigma} \phi^2 \langle v, n - n_{\phi} \rangle^2 d\mathcal{A}_{\mu} + \langle v, n_{\phi} \rangle^2 \int_{\Sigma} \phi^2 |A|^2 d\mathcal{A}_{\mu} \leq 12 \int_{\Sigma} |\nabla \phi|^2 d\mathcal{A}_{\mu}. \quad (4.36)$$

Since  $v$  was chosen to be arbitrary and  $|v| = 1$ , we can sum this over a constant orthonormal frame for  $\mathbb{R}^{n+1}$  to prove the lemma.  $\square$

Now, we use Lemma 4.3.1 to obtain an integral estimate for  $|A|^2$ .

**Theorem 4.3.2.** *Let  $\Sigma \subset B_{2R}(0) \subset \mathbb{R}^{n+1}$  with  $\partial\Sigma \subset \partial B_{2R}(0)$  satisfy  $H = \frac{1}{2}\langle x, n \rangle + \lambda$  and be stable (ie,  $\Sigma$  satisfies (3.16)). If  $\mathcal{A}_{\mu}(\Sigma \cap B_R) \geq 2B_n R^{-2} \mathcal{A}_{\mu}(\Sigma \cap (B_{2R} \setminus B_R))$ , then we have that*

$$\int_{B_R \cap \Sigma} |A|^2 d\mathcal{A}_{\mu} \leq 2B_n R^{-2} \mathcal{A}_{\mu}(\Sigma \cap (B_{2R} \setminus B_R)). \quad (4.37)$$

Here,  $B_n$  is the constant from Lemma 4.3.1.

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*Proof.* We construct a cut off function  $\phi = \phi(|x|)$  depending on Euclidean distance  $r = |x|$  such that

$$\phi(r) = \begin{cases} 1 & r \leq R \\ \text{linear} & R \leq r \leq 2R \\ 0 & 2R \leq r \end{cases} \quad (4.38)$$

Our modified stability inequality lemma gives us

$$\int_{\Sigma} \phi^2 d\mathcal{A}_{\mu} - 2|n_{\phi}| \int_{\Sigma} \phi^2 d\mathcal{A}_{\mu} + |n_{\phi}|^2 \int_{\Sigma} \phi^2 (|A|^2 + 1) d\mathcal{A}_{\mu} \leq B_n R^{-2} \mathcal{A}_{\mu}(\Sigma \cap (B_{2R} \setminus B_R)). \quad (4.39)$$

Note that the left hand side of this inequality is quadratic in  $|n_{\phi}|$ , and since any quadratic with  $a > 0$  satisfies  $au^2 + bu + c \geq c - \frac{b^2}{4a}$ , we get that

$$\int_{\Sigma} \phi^2 d\mathcal{A}_{\mu} - \frac{(\int_{\Sigma} \phi^2 d\mathcal{A}_{\mu})^2}{\int_{\Sigma} \phi^2 (|A|^2 + 1) d\mathcal{A}_{\mu}} \leq B_n R^{-2} \mathcal{A}_{\mu}(\Sigma \cap (B_{2R} \setminus B_R)). \quad (4.40)$$

So, we have

$$\frac{\int_{\Sigma} \phi^2 d\mathcal{A}_{\mu} \int_{\Sigma} \phi^2 |A|^2 d\mathcal{A}_{\mu}}{\int_{\Sigma} \phi^2 (1 + |A|^2) d\mathcal{A}_{\mu}} \leq B_n R^{-2} \mathcal{A}_{\mu}(\Sigma \cap (B_{2R} \setminus B_R)). \quad (4.41)$$

This inequality is of the form  $\frac{ab}{a+b} \leq c$  where  $a = \int_{\Sigma} \phi^2 d\mathcal{A}_{\mu}$  and  $b = \int_{\Sigma} \phi^2 |A|^2 d\mathcal{A}_{\mu}$ . This can be put into the form  $(a-c)b \leq ca$ . From our assumption that  $\mathcal{A}_{\mu}(\Sigma \cap B_R) \geq 2B_n R^{-2} \mathcal{A}_{\mu}(\Sigma \cap (B_{2R} \setminus B_R))$  we get that  $a \geq 2c$  and  $a - c \geq \frac{a}{2}$ . Therefore, we have



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that  $b \leq 2c$ , which gives us that

$$\int_{\Sigma} \phi^2 |A|^2 d\mathcal{A}_{\mu} \leq 2B_n R^{-2} \mathcal{A}_{\mu}(\Sigma \cap (B_{2R} \setminus B_R)). \quad (4.42)$$

So, we get that

$$\int_{B_R(0)} |A|^2 d\mathcal{A}_{\mu} \leq 2B_n R^{-2} \mathcal{A}_{\mu}(\Sigma \cap (B_{2R} \setminus B_R)). \quad (4.43)$$

□

**Remark 4.3.3.** *When considering incomplete  $\Sigma$ , there are conditions that are sufficient to guarantee that  $\mathcal{A}_{\mu}(\Sigma \cap B_R) \geq 2B_n R^{-2} \mathcal{A}_{\mu}(\Sigma \cap (B_{2R} \setminus B_R))$ .*

*For example, let  $H = \frac{1}{2}\langle x, n \rangle + \lambda$  with  $|\lambda| \leq M$ . We need a lower bound on  $\mathcal{A}_{\mu}(\Sigma \cap B_R)$ . In order to accomplish this, we look at getting some control over  $\min |x|$  and the Euclidean mean curvature  $H$  around some point realizing  $\min |x|$ . Let  $\Sigma$  achieve  $\min |x|$  at the point  $p \in \Sigma$ . At  $p$ , we have that*

$$2n - |x|^2 + 2M|x| \geq 2n - |x|^2 - 2\lambda\langle x, n \rangle = \mathcal{L}|x|^2 \geq 0. \quad (4.44)$$

*So, there exists a large positive constant  $D = D(M, n)$  such that  $\min |x| \leq D$ , and that  $|H| \leq 2D$  on  $B_{2D}$ . Using an adaptation of the monotonicity formula (A.0.6 in the appendix) and renaming  $D$ , we can turn these bounds into a lower bound on the Euclidean area  $\mathcal{A}(\Sigma \cap B_{2D}) \geq D^{-1}$ . Again renaming  $D$ , we get that  $\mathcal{A}_{\mu}(\Sigma \cap B_{2D}) \geq$*

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$e^{-D^2}D^{-1}$ . Upon renaming constants, if  $R > 2D$ , then  $R^2\mathcal{A}_\mu(\Sigma \cap B_R) \geq R^2D^{-1}$ .

Therefore, there exists  $D(M, n)$  such that if  $\mathcal{A}_\mu(\Sigma \cap B_{2R}) \leq D^{-1}R^2$ , then we are guaranteed that  $\mathcal{A}_\mu(\Sigma \cap B_R) \geq 2B_nR^{-2}\mathcal{A}_\mu(\Sigma \cap (B_{2R} \setminus B_R))$ .

**Remark 4.3.4.** For the case of properly immersed, two-sided, smooth, complete  $\Sigma \subset \mathbb{R}^{n+1}$  satisfying the mean curvature condition  $H = \frac{1}{2}\langle x, n \rangle + \lambda$ , stability condition (3.16), and  $\mathcal{A}_\mu(\Sigma) < \infty$ , there exists an  $R_0$  large enough such that for  $R > R_0$ , we have that  $\mathcal{A}_\mu(\Sigma \cap B_R) \geq 2B_nR^{-2}\mathcal{A}_\mu(\Sigma \cap (B_{2R} \setminus B_R))$ . Sending  $R \rightarrow \infty$  in (4.37), we get that  $\int_\Sigma |A|^2 d\mathcal{A}_\mu = 0$ . So therefore, our estimate (4.37) also gives that the only such  $\Sigma$  are hyperplanes.

### 4.3.2 A pointwise curvature estimate

To achieve a pointwise curvature estimate from an integral estimate, we will need to make use of two inequalities: a Simons-type inequality and a Mean Value Inequality. These inequalities have well-known analogues in the theory of minimal surfaces, and we adjust them to fit our needs. We will prove our Simons-type inequality here, but will leave the proof of the Mean Value Inequality to the appendix.

A key element to the proof of the pointwise estimate is that it requires  $n = 2$ , that is  $\Sigma \subset \mathbb{R}^3$ . This is due to Theorem 4.3.7, which only holds for hypersurfaces of dimension 2.

**Lemma 4.3.5. *Simons Inequality:*** For a hypersurface  $\Sigma$  satisfying  $H = \frac{1}{2}\langle x, n \rangle +$

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$\lambda$ , we have that

$$\Delta|A|^2 \geq -(|x|^2/8)|A|^2 - (2 + \lambda^2)|A|^4. \quad (4.45)$$

*Proof.* First, Codazzi's equation tells us that  $\nabla A$  is symmetric. We fix a point  $p \in \Sigma$  and look at geodesic normal coordinates centered at  $p$ . Therefore, at  $p$  we have that

$$\Delta A_{jk} = \nabla_{jk}^2 H + H A_{jk}^2 - |A|^2 A_{jk}. \quad (4.46)$$

Now, from  $H = \frac{1}{2}\langle x, n \rangle + \lambda$  we have that

$$\nabla_{jk}^2 H = \frac{1}{2}\nabla_j A(k, x^T) + \frac{1}{2}A_{jk} - \frac{1}{2}\langle x, n \rangle A_{jk}^2. \quad (4.47)$$

Applying a Cauchy-Schwarz inequality of the form  $ab \leq 2a^2 + (1/8)b^2$  to  $\langle A, \nabla_{x^T} A \rangle$ ,

we get

$$\Delta|A|^2 \geq -(|x|^2/8)|A|^2 - (2 + \lambda^2)|A|^4. \quad (4.48)$$

□

**Lemma 4.3.6. Mean Value Inequality:** Suppose that, on a hypersurface with  $|H| \leq M$ , a function  $f$  satisfies  $f \geq 0$  and  $\Delta f \geq -Kt^{-2}f$  on  $B_t(x)$  for some  $K$ . Then, for  $s \leq t$  we have that

$$e^{(K/2t+M)s} s^{-n} \int_{B_s(x) \cap \Sigma} f d\mathcal{A} \geq \omega_n f(x). \quad (4.49)$$

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*Proof.* Left to the appendix □

We now use a standard tool due to Choi-Schoen [CS85] for turning our integral estimates for  $|A|$  into pointwise estimates.

**Theorem 4.3.7.** *Given any  $M > 0$ , there exists  $\epsilon_M > 0$  such that the following holds: Suppose  $\Sigma \subset \mathbb{R}^3$  is any hypersurface satisfying  $H = \frac{1}{2}\langle x, n \rangle + \lambda$  with  $|\lambda| \leq M$ , and suppose  $x_0 \in \Sigma$ . Also, suppose that for some  $R \geq 1$  and some  $r_0 < 1/R$ , we have  $B_{r_0}(x_0) \subset B_R(0)$  and  $\partial\Sigma \subset \partial B_R(0)$ . Finally, let  $0 < \delta \leq 1$  and suppose that*

$$\int_{B_{r_0}(x_0)} |A|^2 d\mathcal{A} < \delta\epsilon. \quad (4.50)$$

*Then for all  $0 < \sigma \leq r_0$  and  $y \in B_{r_0-\sigma}(x_0)$ ,  $|A|^2(y) \leq \delta/\sigma^2$ .*

**Remark 4.3.8.** *In Theorem 4.3.7, we need to require that  $r_0 < 1/R$ . This is to give more control of estimates coming from the Mean Value Inequality (4.49), which will give us (4.60). Also it is used to control the  $|x|^2$  term in inequality (4.45).*

*Proof.* On  $B_{r_0}(x_0)$ , define the function

$$F(y) = (r_0 - d(y, x_0))^2 |A(y)|^2 \quad (4.51)$$

where  $d(y, x_0)$  is the Euclidean distance between the two points. Observe that  $F \geq 0$  in  $B_{r_0}$ , and  $F = 0$  on  $\partial B_{r_0}$ . Set  $x_1$  to be the point where  $F$  achieves its maximum. Observe that if  $F(x_1) \leq \delta$  we will be done, since for  $y \in B_{r_0-\sigma}$ ,  $\sigma^2 |A|^2 \leq$

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$F(x_1) \leq \delta$ . We will now show that  $F(x_1) > \delta$  gives a contradiction for some  $\epsilon_M$  small enough and independent of  $\Sigma, \delta$ , and  $R \geq 1$ .

Suppose that  $F(x_1) > \delta$ , ie

$$(r_0 - d(x_1, x_0))^2 |A(x_1)|^2 > \delta, \quad (4.52)$$

and fix  $\sigma$  so that at  $x_1$  we have  $\sigma^2 |A|^2 = \delta/4$ . Observe that the following equations hold:

$$\sigma \leq \frac{1}{2} (r_0 - d(x_1, x_0)) \leq \frac{1}{2R} < 1, \quad (4.53)$$

$$\frac{1}{2} \leq \frac{r_0 - d(y, x_0)}{r_0 - d(x_1, x_0)} \leq 2, \forall y \in B_\sigma(x_1). \quad (4.54)$$

Using these, we compute

$$\begin{aligned} (r_0 - d(x_1, x_0))^2 \sup_{B_\sigma(x_1)} |A|^2 &\leq 4 \sup_{B_\sigma(x_1)} (r_0 - d(\cdot, x_0))^2 |A|^2, \\ &= 4 \sup_{B_\sigma(x_1)} F(\cdot) \leq 4F(x_1), \\ &= 4(r_0 - d(x_1, x_0))^2 |A|^2(x_1). \end{aligned}$$

Therefore,

$$\sup_{B_\sigma(x_1)} |A|^2 \leq 4|A|^2(x_1) = \frac{\delta}{\sigma^2} < \frac{1}{\sigma^2}. \quad (4.55)$$

Plugging (4.55) into Simons Inequality (4.45) gives us

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$$\Delta|A|^2 \geq -(R^2/8)|A|^2 - (2 + C^2)/\sigma^2|A|^2, \quad (4.56)$$

and using (4.53) (specifically, that  $R \leq 1/\sigma$ ) yields

$$\Delta|A|^2 \geq -\sigma^{-2}(3 + M^2)|A|^2. \quad (4.57)$$

Therefore,

$$\Delta|A|^2 \geq -K\sigma^{-2}|A|^2 \quad (4.58)$$

on  $B_\sigma(x_1)$ , where  $K = K(M) = 3 + M^2$ . Note that  $|H| \leq R + M$ . Then, by the Mean Value Inequality (4.49),

$$|A|^2(x_1) \leq \omega_n^{-1} e^{R\sigma + M\sigma + (3+M^2)/2} \sigma^{-2} \int_{B_\sigma(x_1) \cap \Sigma} |A|^2 d\mathcal{A} \quad (4.59)$$

$$\leq \omega_n^{-1} e^{M+2+M^2/2} \sigma^{-2} \int_{B_\sigma(x_1) \cap \Sigma} |A|^2 d\mathcal{A}. \quad (4.60)$$

Here for (4.60), we have used that  $\sigma R \leq 1$  which is a consequence of our hypothesis that  $r_0 < 1/R$ . Substituting back in our definition of  $\sigma$ , we get

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$$\delta/4 = \sigma^2 |A|^2(x_1) \leq \omega_n^{-1} e^{M+2+M^2/2} \int_{B_\sigma(x_1) \cap \Sigma} |A|^2 d\mathcal{A}, \quad (4.61)$$

$$\leq \omega_n^{-1} e^{M+2+M^2/2} \int_{B_{r_0}(x_1) \cap \Sigma} |A|^2 d\mathcal{A}, \quad (4.62)$$

$$\leq \omega_n^{-1} e^{M+2+M^2/2} \delta \epsilon. \quad (4.63)$$

We may choose  $\epsilon$  depending only on  $M$  such that there is a contradiction.  $\square$

Using Theorem 4.50 combined with Theorem 4.37 we get pointwise estimates for hypersurfaces that are stable (with respect to variations that preserve  $\mathcal{V}_\mu$ ).

**Theorem 4.3.9** (Pointwise for  $n = 2$ ). *Let  $M > 0$  be given and  $R > 1$ . Also, let  $\Sigma \subset B_{2R}(0) \subset \mathbb{R}^3$  with  $\partial\Sigma \subset \partial B_{2R}(0)$  be a hypersurface with  $H = \frac{1}{2}\langle x, n \rangle + \lambda$  and  $|\lambda| \leq M$  that satisfies the stability condition (3.16).*

*There exists  $\epsilon_M > 0$  such that if  $\mathcal{A}_\mu(\Sigma \cap B_R) \geq 2B_n R^{-2} \mathcal{A}_\mu(\Sigma \cap (B_{2R} \setminus B_R))$  and  $\mathcal{A}_\mu(\Sigma \cap (B_{2R} \setminus B_R)) < (R^2/2B_n) e^{-\frac{1}{4}(\rho + \frac{1}{R})^2} \delta \epsilon_M$  for some  $0 < \delta \leq 1$  and some  $\rho \in (0, R - 1/R)$ , then*

$$\sup_{x \in B_\rho \cap \Sigma} |A|^2 \leq R^2 \delta. \quad (4.64)$$

**Remark 4.3.10.** *Note that in the case that  $\mathcal{A}(\Sigma \cap B_R) \leq DR^N$  for some uniform  $D$  and some  $N$ , one does indeed get that for large enough  $R$  that  $\mathcal{A}_\mu(\Sigma \cap (B_{2R} \setminus B_R)) < (R^2/2B_n) e^{-\frac{1}{4}(\rho + \frac{1}{R})^2} \delta \epsilon_M$  for some  $0 < \delta \leq 1$  and some  $\rho \in (0, R - 1/R)$*

# Chapter 5

## Other rigidity results on lambda-hypersurfaces

### 5.1 An eigenvalue and diameter estimate

We begin with some standard notations. Given a triple  $(M, g, \phi)$  consisting of a manifold  $M$ , metric  $g$  and weight  $\phi$ , the Bakry-Emery Ricci Curvature is the quantity  $Ric + \nabla^2 \phi$ . With the same weight, we can define the *weighted volume* to be  $d\mu = e^{-\phi} dVol$ , where  $dVol$  represents the usual volume measure on the unweighted  $(M, g)$ . With this in mind, we define the Witten-Laplacian on  $(M, g)$  to be

$$\Delta_\phi = \Delta - \langle \nabla \phi, \nabla \cdot \rangle \tag{5.1}$$



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We observe that on a compact manifold the Witten-Laplacian is self-adjoint with respect to the weighted volume, ie for any smooth functions  $u, v$ , we have

$$\int_M u \Delta_\phi v \, d\mu = - \int_M \langle \nabla u, \nabla v \rangle \, d\mu = \int_M v \Delta_\phi u \, d\mu \quad (5.2)$$

Finally, if  $M$  is compact, we define the *diameter* of  $(M, g)$  to be the supremum of the shortest distance between points  $p, q \in M$ , where the supremum is taken over all pairs of points (we will always assume  $M$  is connected).

These objects were linked in the following theorem of Futaki, Li, and Li [FLL13]:

**Theorem 5.1.1.** *Let  $(M, g)$  be an compact Riemannian manifold with diameter  $d$ , and let  $\phi \in C^2(M)$ . Suppose that there exists a constant  $K \in \mathbb{R}$  such that the Bakry-Emery Ricci curvature satisfies*

$$\text{Ric} + \nabla^2 \phi \geq Kg \quad (5.3)$$

*Then the first non-zero eigenvalue  $\gamma_1$  of the Witten-Laplacian  $\Delta_\phi$  satisfies*

$$\gamma_1 \geq \sup_{s \in (0,1)} \left\{ 4s(1-s) \frac{\pi^2}{d^2} + sK \right\} \quad (5.4)$$

This result generalizes previous estimates on the first non-zero eigenvalue, specifically Zhong and Yang's result on manifolds with non-negative Ricci curvature [ZY84], as well as results by Shi and Zhang on manifolds with Ricci curvature bounded be-

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low [SZ07]. Furthermore, if we have an upper bound on the size of the first non-zero eigenvalue, this theorem allows us to draw conclusions about the size of the diameter. As examples, Futaki, Li, and Li prove in the following corollaries (about shrinking Ricci solitons and self-shrinkers of the MCF):

**Corollary 5.1.2.** *[FLL13] Let  $(M, g, \phi)$  be a non-trivial compact shrinking Ricci soliton with  $\text{Ric} + \nabla^2 \phi = Cg$ , where  $C > 0$  is a constant (nontrivial in this case simply means that  $\phi$  is not constant). Then the diameter of  $(M, g)$  satisfies*

$$d \geq \frac{(2\sqrt{2} - 2)\pi}{\sqrt{C}} \quad (5.5)$$

**Corollary 5.1.3.** *[FLL13] Let  $M$  be a compact self-shrinker with diameter  $d$  that is not a the self-shrinking sphere. Let  $\lambda_i$  be the principal curvatures of  $M$ , let  $h_{ij}$  represent the components of the second fundamental form of  $M$ , and let*

$$K_0 := \max_{1 \leq i \leq n} \left[ \sum_k h_{ik} h_{ki} \right]. \quad (5.6)$$

*Then*

$$d \geq \frac{2\pi}{\sqrt{2K_0 - 3}} \quad (5.7)$$

**Remark 5.1.4.** *In MCF, the Witten-Laplacian w.r.t. the weight  $\phi = -\frac{|x|^2}{4}$  is often*

*called the drift Laplacian and represented by the notation*

$$\Delta_\phi = \mathcal{L} = \Delta - \frac{1}{2}\langle x, \nabla(\cdot) \rangle \quad (5.8)$$

*This is the same drift Laplacian that was introduced in 3.3.1. We will use this  $\mathcal{L}$  notation.*

These corollaries are possible because of explicit estimates on the size of the first eigenvalue, as well as a lower bound on the Bakry-Emery Ricci curvature. In particular, for self-shrinkers, we have that  $\mathcal{L}(|x|^2 - 2n) = -(|x|^2 - 2n)$ , as shown in Colding-Minicozzi [CM12]. Since explicit eigenvalues of  $\mathcal{L}$  are not known for  $\lambda$ -hypersurfaces, we will need to be more clever in estimating our eigenvalue.

### 5.1.1 Computations using $\mathcal{L}$

Let  $\Sigma$  be a compact  $\lambda$ -hypersurface, and let  $\mathcal{L}u = \Delta u - \frac{1}{2}\langle x, \nabla u \rangle$ . As mentioned before, this operator is self-adjoint with respect to the weighted  $L^2$  inner product. Thus, standard spectral theory (see, for example, Corollary 5.15 of [CM12]) gives us the following:

1.  $\mathcal{L}$  has real eigenvalues  $\gamma_0 < \gamma_1 \leq \dots$ , with  $\gamma_k \rightarrow \infty$ .
2. There is an orthonormal basis of eigenfunctions  $u_k$  in the weighted  $L^2$  space.
3. The lowest eigenvalue  $\gamma_0$  will be 0, corresponding to the constant eigenfunctions.

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This can be seen by recalling that  $\gamma_0$  can be characterized as

$$\gamma_0 = \inf_f \frac{\int_M |\nabla f|^2 d\mathcal{A}_\mu}{\int_M f^2 d\mathcal{A}_\mu} \quad (5.9)$$

4. Since different eigenspaces are orthogonal w.r.t. the weighted inner product, we may characterize  $\gamma_1$  as

$$\gamma_1 = \inf_{g \perp 1} \frac{\int_M |\nabla g|^2 d\mathcal{A}_\mu}{\int_M g^2 d\mathcal{A}_\mu} \quad (5.10)$$

where the infimum is taken over all functions  $g$  that are perpendicular to constant functions (in the weighted  $L^2$  space), ie  $g$  such that  $\int_M g d\mathcal{A}_\mu = 0$ .

**Lemma 5.1.5.** *Let  $a$  be a constant vector. Then  $\mathcal{L}\langle x, a \rangle = -\frac{1}{2}\langle x, a \rangle - \lambda\langle n, a \rangle$*

*Proof.* We adopt the notation that  $u_i = \nabla_{e_i} u$  for a frame  $\{e_1, \dots, e_n, n\}$ . With this in mind, we know  $x_i = e_i$  and  $\Delta x = \sum_i x_{ii} = -Hn$ . Therefore, we get

$$\begin{aligned} \mathcal{L}\langle x, a \rangle &= \Delta\langle x, a \rangle - \frac{1}{2}\langle x, \nabla\langle x, a \rangle \rangle \\ &= -\langle Hn, a \rangle - \frac{1}{2} \sum_i \langle x, a \rangle_i \langle x, e_i \rangle \\ &= -H\langle n, a \rangle - \frac{1}{2}\langle x, a \rangle + \frac{1}{2}\langle x, n \rangle \langle n, a \rangle \\ &= \left( \frac{1}{2}\langle x, n \rangle - H \right) \langle n, a \rangle - \frac{1}{2}\langle x, a \rangle \\ &= -\frac{1}{2}\langle x, a \rangle - \lambda\langle n, a \rangle. \end{aligned} \quad (5.11)$$

□

**Lemma 5.1.6.**  $\mathcal{L}\langle n, a \rangle = -|A|^2 \langle n, a \rangle$

*Proof.* With the same notation as in the previous lemma, we know that  $\langle n, a \rangle_i = \sum_j -h_{ij} \langle e_j, a \rangle$  and  $\langle n, a \rangle_{ik} = -\sum_j h_{ijk} \langle e_j, a \rangle - \sum_j h_{ij} h_{kj} \langle n, a \rangle$ . Therefore, we get

$$\begin{aligned}
 \mathcal{L}\langle n, a \rangle &= \Delta \langle n, a \rangle - \frac{1}{2} \langle x, \nabla \langle n, a \rangle \rangle \\
 &= \sum_i \langle n, a \rangle_{ii} + \frac{1}{2} \sum_{ij} h_{ij} \langle a, e_j \rangle \langle x, e_i \rangle \\
 &= -\sum_{ij} h_{iij} \langle e_k, a \rangle + h_{ij} h_{ij} \langle n, a \rangle + \frac{1}{2} \sum_{ij} h_{ij} \langle a, e_j \rangle \langle x, e_i \rangle \\
 &= -\langle \nabla H, a \rangle - |A|^2 \langle n, a \rangle + \frac{1}{2} \sum_i \langle x, e_i \rangle \langle a, e_i \rangle h_{ii} \\
 &= -\sum_k \frac{1}{2} \langle x, n \rangle_k \langle e_k, a \rangle - |A|^2 \langle n, a \rangle + \frac{1}{2} \sum_i \langle x, e_i \rangle \langle a, e_i \rangle h_{ii} \\
 &= -\frac{1}{2} \sum_k h_{kk} \langle x, e_k \rangle \langle x, e_k \rangle - |A|^2 \langle n, a \rangle + \frac{1}{2} \sum_i \langle x, e_i \rangle \langle a, e_i \rangle h_{ii} \\
 &= -|A|^2 \langle n, a \rangle
 \end{aligned} \tag{5.12}$$

where the fourth line comes from choosing a frame at the point that diagonalizes  $h_{ij}$ , and the fifth line comes from the equation  $H = \frac{1}{2} \langle x, n \rangle + \lambda$ . □

**Lemma 5.1.7.**  $\mathcal{L}|x|^2 = 2n - |x|^2 - 2\lambda \langle x, n \rangle$

*Proof.* We know that  $\Delta x = -Hn$  and  $|\nabla x|^2 = n$ .

$$\begin{aligned}
 \mathcal{L}|x|^2 &= \Delta|x|^2 - \frac{1}{2}\langle x, \nabla|x|^2 \rangle \\
 &= 2\langle \Delta x, x \rangle + 2|\nabla x|^2 - \frac{1}{2}\langle x, \nabla|x|^2 \rangle \\
 &= -2H\langle x, n \rangle + 2n - \frac{1}{2}\langle x, \nabla|x|^2 \rangle \\
 &= -\langle x, n \rangle^2 - 2\langle x, n \rangle\lambda + 2n - \frac{1}{2}\langle x, \nabla|x|^2 \rangle \\
 &= -|x|^2 + 2n - 2\langle x, n \rangle\lambda
 \end{aligned} \tag{5.13}$$

where equality between the third and fourth line comes from the fact that  $H = \frac{1}{2}\langle x, n \rangle + \lambda$ , and equality between the last two lines comes from the fact that  $\langle x, \nabla|x|^2 \rangle = 2|x^T|^2$   $\square$

**Corollary 5.1.8.** *We have the following integral identities for any constant vector  $a \in \mathbb{R}^{n+1}$ :*

$$\int_{\Sigma} (\langle x, a \rangle + 2\lambda\langle n, a \rangle) d\mathcal{A}_{\mu} = 0 \tag{5.14}$$

$$\int_{\Sigma} |A|^2\langle n, a \rangle d\mathcal{A}_{\mu} = 0 \tag{5.15}$$

$$\int_{\Sigma} (2n - |x|^2 - 2\lambda\langle x, n \rangle) d\mathcal{A}_{\mu} = 0 \tag{5.16}$$

*Proof.* These identities follow immediately from the self-adjointness of  $\mathcal{L}$  (ie equation 3.12), with  $u = 1$  and  $v$  defined by the previous lemmas.  $\square$

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**Theorem 5.1.9.** *Suppose that  $\Sigma$  is a compact  $\lambda$ -hypersurface with  $|A|^2 \leq 2$  and  $\lambda(H - \lambda) \geq 0$ . Then  $\gamma_1 \leq \frac{n+2}{2}$*

*Proof.* Because the integrands in corollary 5.1.8 integrate to 0, they are perpendicular to the constant functions and therefore are good candidates to get an upper bound for  $\lambda_2$  as in equation 5.10. Fix a vector  $a \in \mathbb{R}^{n+1}$ . Then we have

$$\mathcal{L}(\langle x, a \rangle + 2\lambda \langle n, a \rangle) = -\frac{1}{2} \langle x, a \rangle - (|A|^2 \lambda + \lambda) \langle n, a \rangle \quad (5.17)$$

and therefore, for  $f = \langle x, a \rangle + 2\lambda \langle n, a \rangle$ , we have

$$\begin{aligned} -f \mathcal{L} f &= \frac{1}{2} (\langle x, n \rangle^2 + 4\lambda \langle x, a \rangle \langle n, a \rangle + 4\lambda^2 \langle n, a \rangle^2) \\ &\quad + |A|^2 \lambda \langle x, a \rangle \langle n, a \rangle + 2|A|^2 \lambda^2 \langle n, a \rangle^2 \end{aligned} \quad (5.18)$$

We can now estimate  $\lambda_2$  as in equation 5.10. We get

$$\frac{\int_{\Sigma} -f \mathcal{L} f \, d\mathcal{A}_{\mu}}{\int_{\Sigma} f^2 \, d\mathcal{A}_{\mu}} = \frac{1}{2} + \frac{\int_{\Sigma} |A|^2 \lambda \langle x, a \rangle \langle n, a \rangle + 2\lambda^2 \langle n, a \rangle^2 \, d\mathcal{A}_{\mu}}{\int_{\Sigma} \langle x, a \rangle^2 + 4\lambda^2 \langle n, a \rangle^2 + 4\lambda \langle x, a \rangle \langle n, a \rangle \, d\mathcal{A}_{\mu}} \quad (5.19)$$

Note that (5.19) is true for any vector  $a$ . We will choose a specific  $a$  as follows:

Compute the quantity

$$\int_M \langle x, v_i \rangle^2 + 4\lambda^2 \langle n, v_i \rangle^2 + 4\lambda \langle x, v_i \rangle \langle n, v_i \rangle \, d\mathcal{A}_{\mu} \quad (5.20)$$

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where  $v_i$ ,  $i = 1, \dots, n+1$  is the unit vector that points in the  $x_i$  direction. Averaging

5.20 over  $i$  gives us the quantity

$$\frac{1}{n+1} \int_{\Sigma} |x|^2 + 4\lambda^2 + 4\lambda \langle x, n \rangle d\mathcal{A}_{\mu} \quad (5.21)$$

Since (5.21) is an average, at least one of the  $v_i$  from before must result in the integral (5.20) being greater than or equal to (5.21). We choose  $a$  to be that  $v_i$ , and return to equation (5.19):

$$\begin{aligned} \frac{\int_{\Sigma} -f\mathcal{L}f d\mathcal{A}_{\mu}}{\int_{\Sigma} f^2 d\mathcal{A}_{\mu}} &\leq \frac{1}{2} + (n+1) \frac{\int_{\Sigma} |A|^2 \lambda \langle x, a \rangle \langle n, a \rangle + 2\lambda^2 \langle n, a \rangle^2 d\mathcal{A}_{\mu}}{\int_{\Sigma} |x|^2 + 4\lambda^2 + 4\lambda \langle x, n \rangle d\mathcal{A}_{\mu}} \\ &\leq \frac{1}{2} + (n+1) \frac{\int_{\Sigma} |A|^2 \lambda \langle x, n \rangle + 2\lambda^2 d\mathcal{A}_{\mu}}{\int_{\Sigma} |x|^2 + 4\lambda^2 + 4\lambda \langle x, n \rangle d\mathcal{A}_{\mu}} \end{aligned} \quad (5.22)$$

$$\leq \frac{1}{2} + (n+1) \frac{\int_{\Sigma} |A|^2 \lambda \langle x, n \rangle + 2\lambda^2 d\mathcal{A}_{\mu}}{\int_{\Sigma} 4\lambda^2 + 4\lambda \langle x, n \rangle d\mathcal{A}_{\mu}} \quad (5.23)$$

$$\leq \frac{1}{2} + (n+1) \frac{\int_{\Sigma} 2\lambda^2 + 2\lambda \langle x, n \rangle d\mathcal{A}_{\mu}}{\int_{\Sigma} 4\lambda^2 + 4\lambda \langle x, n \rangle d\mathcal{A}_{\mu}} \quad (5.24)$$

where the third inequality comes from dropping the  $|x|^2$  term in the denominator, the fourth inequality comes from our assumption that  $|A|^2 \leq 2$ , and we have used the assumption that  $\lambda(H - \lambda) \geq 0$  so that our integrals have the correct sign. Therefore, we get that

$$\gamma_1 \leq \frac{n+2}{2} \quad (5.25)$$



□

Next, we compute the Bakry-Emery Ricci curvature. By the Gauss equations, we have the Ricci curvature is

$$R_{ij} = R_{ikjk} = h_{ij}H - h_{ik}h_{jk} \quad (5.26)$$

and, for  $\phi = |x|^2/4$ ,

$$\phi_{ij} = \text{Hess}_\phi(e_i, e_j) = \frac{1}{2}(\langle x, e_i \rangle)_j = \frac{1}{2}g_{ij} + (H - \lambda)(-h_{ij}) \quad (5.27)$$

Taken together, this gives us

$$R_{ij} + \phi_{ij} = \frac{1}{2}g_{ij} - h_{ik}h_{kj} + \lambda h_{ij} \geq \left(\frac{1}{2} - 2 - \sqrt{2}\lambda\right) g_{ij} \quad (5.28)$$

where we have used that  $h_{ik}h_{kj} \leq (\max_i \sum_k h_{ik}h_{ik}) g_{ij} \leq 2g_{ij}$  and  $|h_{ij}| < |A|$  (see [FLL13]) Plugging this into Theorem 5.1.1, we have that

$$\frac{n+2}{2} \geq \sup_{s \in (0,1)} \left[ 4s(1-s) \frac{\pi^2}{d^2} + s \left( \frac{1}{2} - 2 - \sqrt{2}\lambda \right) \right] \quad (5.29)$$

Setting  $s = 1/2$  gives us

$$\frac{n+2}{2} \geq \left[ \frac{\pi^2}{d^2} + \frac{1}{2} \left( \frac{1}{2} - 2 - \sqrt{2\lambda} \right) \right] \quad (5.30)$$

from which we can derive

$$d \geq \frac{\sqrt{2}\pi}{\sqrt{n+4+\sqrt{2\lambda}}} \quad (5.31)$$

## 5.2 Rigidity result when $|A|^2$ is small

We end with a classification result of what can happen when  $\Sigma$  is compact and  $|A|^2$  is small. Specifically, we prove the following as an easy consequence of the maximum principle:

**Theorem 5.2.1.** *Suppose that  $\Sigma$  is a compact  $\lambda$ -hypersurface on which  $H - \lambda \geq 0$  and  $|A|^2 \leq 1/2$ . Then  $\Sigma$  is a sphere,  $\Sigma = S^n$*

This result should be compared to the following, similar results:

**Theorem 5.2.2.** *[Gua14] Suppose  $\Sigma^n \subset \mathbb{R}^{n+1}$  is a smooth, closed  $\lambda$ -hypersurface such that  $\lambda \geq 0$  and  $|A|^2 \leq \frac{1}{2} + \frac{\lambda(\lambda + \sqrt{\lambda^2 + 2n})}{2n}$ , then  $\Sigma$  is a round sphere.*

**Theorem 5.2.3.** *[CW14] Suppose  $\Sigma^n \subset \mathbb{R}^{n+1}$  is a smooth, closed  $\lambda$ -hypersurface such that  $H - \lambda \geq 0$  and  $\lambda(f_3(H - \lambda) - |A|^2) \geq 0$ , where  $f_3 = \Sigma_{i,j,k} h_{ij} h_{jk} h_{ki}$ . Then  $\Sigma$  is a round sphere.*

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We begin by performing some computations using the  $\mathcal{L}$  operator:

**Lemma 5.2.4.**  $\mathcal{L}|A|^2 = |A|^2 - 2|A|^4 + 2\lambda \sum_{i,j,k} h_{ij}h_{jk}h_{ki} + 2 \sum_{i,j} |\nabla h_{ij}|^2$

*Proof.* Using the Ricci Formula and the Gauss equations, we calculate:

$$\begin{aligned} \Delta h_{ij} &= \sum_k h_{ijkk} \\ &= \sum_k h_{kkij} + \sum_{k,m} h_{im} R_{mkjk} + \sum_{k,m} h_{mk} R_{mijk} \\ &= H_{ij} + \sum_{k,m} h_{im} (h_{mj} h_{kk} - h_{mk} h_{jk}) + \sum_{k,m} h_{mk} (h_{mj} h_{ik} - h_{mk} h_{ij}) \end{aligned}$$

Since  $H = \frac{1}{2} \langle x, n \rangle + \lambda$ , we get

$$\begin{aligned} \Delta h_{ij} &= \sum_l \frac{1}{2} h_{ilj} \langle x, e_l \rangle + \frac{1}{2} h_{ij} + \sum_l h_{il} h_{lj} (\lambda - H) \\ &\quad + \sum_{k,m} h_{im} (h_{mj} h_{kk} - h_{mk} h_{jk}) + \sum_{k,m} h_{mk} (h_{mj} h_{ik} - h_{mk} h_{ij}) \\ &= \sum_l \frac{1}{2} h_{ijl} \langle x, e_l \rangle + \frac{1}{2} h_{ij} + \lambda \sum_l h_{il} h_{lj} - \sum_{k,m} h_{mk} h_{mk} h_{ij} \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{L}h_{ij} &= \Delta h_{ij} - \frac{1}{2} \sum_k h_{ijk} \langle x, e_k \rangle \\ &= \frac{1}{2} h_{ij} + \lambda \sum_l h_{il} h_{lj} - |A|^2 h_{ij} \end{aligned}$$

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Now, we use the fact that  $\mathcal{L}\alpha^2 = 2\alpha\mathcal{L}\alpha + 2|\nabla\alpha|^2$  to compute

$$\begin{aligned}
 \mathcal{L}|A|^2 &= \sum_{i,j} \mathcal{L}h_{ij}^2 \\
 &= \sum_{i,j} 2h_{ij}\mathcal{L}h_{ij} + 2|\nabla h_{ij}|^2 \\
 &= \sum_{i,j} 2h_{ij} \left( \frac{1}{2}h_{ij} + \lambda \sum_k h_{ik}h_{kj} - h_{ij}|A|^2 \right) + 2 \sum_{i,j} |\nabla h_{ij}|^2 \\
 &= |A|^2 + 2\lambda \sum_{i,j,k} h_{ij}h_{jk}h_{ki} - 2|A|^4 + 2 \sum_{i,j} |\nabla h_{ij}|^2
 \end{aligned}$$

□

**Lemma 5.2.5.**  $\mathcal{L}H = \frac{1}{2}H - |A|^2(H - \lambda)$

*Proof.* Since  $H = \frac{1}{2}\langle x, n \rangle + \lambda$ , we can calculate

$$\begin{aligned}
 H_{,i} &= \frac{1}{2}\langle x, n \rangle_{,i} = \frac{1}{2} \sum_j h_{ij} \langle x, e_j \rangle \\
 H_{,ik} &= \sum_j \frac{1}{2} h_{ijk} \langle x, e_j \rangle + \frac{1}{2} h_{ik} + \sum_j h_{ij} h_{jk} (\lambda - H) \\
 \Delta H &= \sum_i H_{,ii} = \frac{1}{2} \sum_i H_{,i} \langle x, e_i \rangle + \frac{1}{2} H - |A|^2 (H - \lambda) \\
 \mathcal{L}H &= \Delta H - \frac{1}{2} \sum_i \langle x, e_i \rangle H_{,i} = \frac{1}{2} H - |A|^2 (H - \lambda)
 \end{aligned}$$

□

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*proof of Theorem 3.1.* Let  $H - \lambda \geq 0$  and let  $|A|^2 \leq 1/2$ . We compute

$$\begin{aligned}\mathcal{L}(H - \lambda)^2 &= 2(H - \lambda) \left[ \frac{H}{2} - |A|^2(H - \lambda) \right] + 2|\nabla(H - \lambda)|^2 \\ &\geq 2 \left( \frac{1}{2} - |A|^2 \right) (H - \lambda)^2 + 2|\nabla(H - \lambda)|^2\end{aligned}\tag{5.32}$$

If  $\Sigma$  is compact, we can consider the point where  $(H - \lambda)^2$  attains its maximum (since  $\Sigma$  is a  $\lambda$ -hypersurface, this must happen at the point where  $|x|^2$  is maximized). But since  $|A|^2 \leq \frac{1}{2}$ , equation 5.32 will contradict the maximum principle unless either  $H - \lambda \equiv 0$  (implying  $\Sigma$  is a plane), or else  $H - \lambda \equiv C$  and  $|A|^2 \equiv 1/2$ . The only surfaces that satisfy these conditions are generalized cylinders, and the compactness condition forces  $\Sigma$  to be a sphere.  $\square$

# Appendix A

## Mean value inequality and monotonicity formula for $\lambda$ -hypersurfaces

Here we give a proof of the Mean Value Inequality for  $\lambda$ -hypersurfaces, Lemma 4.3.6. The techniques are well-known (see Colding-Minicozzi [CM11]), but we include a proof for completeness.

*Proof of Lemma 4.3.6.* Assume  $|H| \leq M$ . Lemma 4.3.6 is stated in terms of Euclidean quantities, so we are free to translate the surface to the origin in Euclidean

# APPENDIX A. MEAN VALUE INEQUALITY AND MONOTONICITY FORMULA FOR $\lambda$ -HYPERSURFACES

space and consider  $B_s(0)$ . Recall that  $\Delta|x|^2 = 2n - 2\langle x, n \rangle H$ . Then

$$2n \int_{B_s \cap \Sigma} f d\mathcal{A} = \int_{B_s \cap \Sigma} f \Delta|x|^2 d\mathcal{A} + 2 \int_{B_s \cap \Sigma} f \langle x, n \rangle H d\mathcal{A} \quad (\text{A.1})$$

$$\begin{aligned} &= \int_{B_s \cap \Sigma} |x|^2 \Delta f d\mathcal{A} + 2 \int_{\partial B_s \cap \Sigma} f |x^T| d\mathcal{A} \\ &\quad - s^2 \int_{B_s \cap \Sigma} \Delta f d\mathcal{A} + 2 \int_{B_s \cap \Sigma} \langle x, n \rangle H f d\mathcal{A}. \end{aligned} \quad (\text{A.2})$$

Let  $g(s) = s^{-n} \int_{B_s \cap \Sigma} f d\mathcal{A}$ . Using the coarea formula and (A.1), we get

$$g'(s) \geq \frac{1}{2} s^{-n+1} \int_{B_s \cap \Sigma} \Delta f d\mathcal{A} - s^{-n-1} \int_{B_s \cap \Sigma} \langle x, n \rangle f H d\mathcal{A}. \quad (\text{A.3})$$

Here we have used the positivity of  $f$ . Additionally, if we assume  $\Delta f \geq -Kt^{-2}f$  on  $B_t$ , our bound on  $|H|$  gives us

$$\begin{aligned} g'(s) &\geq \frac{-K}{2} s^{1-n} \int_{B_s \cap \Sigma} f t^{-2} d\mathcal{A} - M s^{-1-n} \int_{B_s \cap \Sigma} s f d\mathcal{A}, \\ &\geq - \left( \frac{K}{2t} + M \right) g(s) \end{aligned} \quad (\text{A.4})$$

for all  $s \leq t$ . Therefore,

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$$\frac{d}{ds} \left( g(s) e^{\left(\frac{K}{2t} + M\right)s} \right) \geq 0. \quad (\text{A.5})$$

Integrating (A.5) from  $s_0$  to  $s_1$  (both assumed to be less than  $t$ ) and letting  $s_0 \searrow 0$ , we get

$$e^{\left(\frac{K}{2t} + M\right)s_1} s_1^{-n} \int_{B_{s_1} \cap \Sigma} f d\mathcal{A} \geq \omega_n f(p). \quad (\text{A.6})$$

□

Note, that we get the following corollary (monotonicity):

**Corollary A.0.6.** *Let  $p \in \Sigma$ , and let  $|H| \leq M$  in  $B_t(p) \cap \Sigma$ . Then for  $s \leq t$ , we have  $\mathcal{A}(B_s \cap \Sigma) \geq \omega_n e^{-Ms} s^n$ , where  $\omega_n$  is the volume of the standard unit ball in  $\mathbb{R}^n$ .*

*Proof.* Use the Mean Value Inequality, Lemma 4.3.6 with  $f \equiv 1$  and  $K = 0$ .

□



# Bibliography

- [Ang92] S. Angenent. Shrinking doughnuts. *Progr. Nonlinear Differential Equations Appl.*, 7, 1992.
- [BdC84] J.L. Barbosa, and M. do Carmo. Stability of hypersurfaces of constant mean curvature. *Math. Zeit.*, 185:339–353, 1984.
- [Bob97] S. Bobkov. An isoperimetric inequality on the discrete cube, and an elementary proof of the isoperimetric inequality in gauss space. *The Annals of Probability*, 25:206–214, 1997.
- [Bor75] C. Borell. The brunn-minkowski inequality in gauss space. *Inventiones Mathematicae*, 30:207–216, 1975.
- [Bra78] K. Brakke. The motion of a surface by its mean curvature. *Princeton University Press*, 1978.
- [Cha14] J.E. Chang. One dimensional solutions of the lambda-self shrinkers, preprint, arxiv: 1410.1782. 2014.

## BIBLIOGRAPHY

- [CL] H. Cao and H. Li. A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension. *Calc. Var.*, page to appear.
- [CM11] T.H. Colding and W.P. Minicozzi. *A course in minimal surfaces*, volume 121 of *Graduate Studies in Mathematics*. AMS, 2011.
- [CM12] T.H. Colding and W.P. Minicozzi. Generic mean curvature flow i; generic singularities. *Annals of Math.*, (175):755–833, 2012.
- [CS85] H. Choi and R. Schoen. The space of minimal embeddings of a surface into a three-dimensional manifold of positive ricci curvature. *Invent. Math.*, 32:25–37, 1985.
- [CW14] Q.M. Cheng and G. Wei. Complete lambda-hypersurfaces of weighted volume-preserving mean curvature flow, preprint, arxiv:1403.3177. 2014.
- [Del41] C. Delaunay. Sur la surface de ré volution dont la courbure moyenne est constante. *J. Math. Pures Appl.*, 6:309–320, 1841.
- [Eck04] K. Ecker. Regularity theory for mean curvature flow. *Progress in Nonlinear Differential Equations and Their Applications*, 57, 2004.
- [Ehr83] A. Ehrhard. Symetrisation dans l’espace de gauss. *Math. Scand.*, 53:281–301, 1983.
- [FLL13] A. Futaki, H. Li, and X. Li. On the first eigenvalue of the witten-laplacian

## BIBLIOGRAPHY

- and the diameter of compact shrinking solitons. *Ann. Glob. Anal. Geom.*, 44:105–114, 2013.
- [Gua14] Q. Guang. Gap and rigidity theorems of lambda-hypersurfaces, preprint, arxiv: 1405.4871. 2014.
- [Hui84] G. Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Diff. Geom.*, 20(1):237–266, 1984.
- [Hui90] G. Huisken. Asymptotic behavior for singularities of the mean curvature flow. *J. Diff. Geom.*, 31:295–299, 1990.
- [Hus13] C. Hussey. Classification and analysis of low index mean curvature flow self-shrinkers, preprint, arxiv:1303.0354. 2013.
- [Kap90] N. Kapouleas. Complete constant mean curvature surfaces in euclidean three space. *Ann. of Math.*, 131:239–330, 1990.
- [KKM12] N. Kapouleas,, S. Kleene,, and N. Moller. Mean curvature self-shrinkers of high genus: non-compact examples. *J. Reine Angew. Math.*, to appear, 2012.
- [Led96] M. Ledoux. Isoperimetry and gaussian analysis. *Lectures on probability theory and statistics*, pages 165–294, 1996.
- [Led98] M. Ledoux. A short proof of the gaussian isoperimetric inequality. *Progress in Probability*, pages 229–232, 1998.

## BIBLIOGRAPHY

- [Mol11] N. Moller. Closed self-shrinking surfaces in  $\mathbb{R}^3$  via the torus, preprint, arxiv:1111.7318. 2011.
- [MR02] F. Morgan and M. Ritore. Isoperimetric regions in cones. *Trans. Amer. Math. Soc.*, 354:2327–2339, 2002.
- [MR13] M. McGonagle and J. Ross. The hyperplane is the only stable, smooth solution to the isoperimetric problem in gaussian space. 2013.
- [SSY75] R. Schoen, L. Simon, and S.T. Yau. Curvature estimates for minimal hypersurfaces. *Acta Math.*, 134:275–288, 1975.
- [ST78] V. Sudakov and B. Tsirel’son. Extremal properties of half-spaces for spherically invariant measures. *J. of Math. Sci.*, 9:9–18, 1978.
- [Ste38] J. Steiner. Einfacher beweis der isoperimetrischen hauptsatze. *J. reine angew Math.*, 18:281–296, 1838.
- [SZ07] Y.M. Shi and H.C. Zhang. Lower bounds for the first eigenvalue on compact manifolds. *Chinese Ann. Math. Ser. A.*, 28:863–866, 2007.
- [Wen86] H. Wente. Counterexample to a conjecture of h. hopf. *Pacific J. Math.*, 121:193–243, 1986.
- [Wen91] H. Wente. A note on the stability theorem of j.l. barbosa and m. do carmo for closed surfaces of constant mean curvature. *Pacific J. Math.*, 147:375–379, 1991.

## BIBLIOGRAPHY

- [ZY84] J.Q. Zhong and H.C. Yang. On the estimate of the first eigenvalue of a compact riemannian manifold. *Sci. Sinica Ser. A.*, 27:1265–1273, 1984.

# Vita



John Ross was born in Baltimore, Maryland on December 22, 1986. He attended Friends School of Baltimore for primary school, and St. Mary's College of Maryland for his undergraduate degree. It was at St. Mary's that he discovered his love of math, ultimately receiving the B. A. degree in mathematics in 2009 Summa Cum Laude. He returned to Baltimore to study math at Johns Hopkins, where he received an M. A. in 2011. His research focuses on various topics in Geometric Analysis, including Mean Curvature Flow; highly symmetric self-similar solitons under MCF; and the isoperimetric problem.