

ESSAYS ON GOVERNMENT DEBT AND DEFAULT

by

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Abstract

This dissertation proposes theories of government debt and default in the context of external sovereign debt as well as domestic public debt. The essential contribution of this dissertation is to model limited commitment of repaying the debt from the perspective of the borrowing government.

Chapter 1 provides a theory of external sovereign debt default. It introduces limited commitment into a dynamic optimization model of sovereign debt and strategic default. The idea of limited commitment contrasts with the existing literature, which assumes that the sovereign cannot commit to repay. Estimating the model for Argentina shows that limited commitment improves the model's ability to match the data in many ways, in particular the level of debt, the level of spreads, and the countercyclicality of the trade balance. Welfare is increasing with the degree of commitment only for low levels of debt, and most of the gains only accrue at relatively high levels of commitment.

Chapter 2 extends the model in Chapter 1 to domestic public debt markets. In the model, the government borrows from domestic households to smooth its expenditure. A quantitative analysis calibrated to the Greece economy shows that the model is able to sustain an equilibrium with both recurrent defaults and a high level of debt as observed

in the data. The model predicts a default and interest rate spike in the period when the Greek government defaulted, and the dynamics of the model are consistent with stylized facts of the Greek economy such as countercyclical interest rate spreads and countercyclical primary balance. An alternative model specification is also considered where tax policy is countercyclical. Compared to the baseline model where tax rate is constant, countercyclical tax policy significantly reduces the likelihood of default and lowers interest rate spreads, but does not have any significant effect on the level of debt.

Chapter 3 adapts the endogenous gridpoints method to solve a simplified version of the model in Chapter 1. The paper shows that the endogenous gridpoints method yields model solution that is very close to the discrete-state-space method widely used in the literature. However, there does not seem to be any gain in computation time.

Keywords: Sovereign Debt, Domestic Debt, Strategic Default,
Limited Commitment, Dynamic Stochastic General Equilibrium Models,
Business Cycle, Dynamic Optimization

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Chapter 1

Introduction

1.1 Introduction

Existing dynamic-stochastic-general-equilibrium (DSGE) models of sovereign debt and default fail to match the data in several ways.¹ First, these models are unable to predict both the level of debt and the frequency of default that are observed in the data. Second, in order to explain the observed frequency of default in emerging market economies, it is necessary to assume unrealistically low discount factors in these models. Third, these models are unable to match key business cycle features of emerging market economies such as the strong countercyclical trade balance.²

¹These papers include, but are not limited to, (Aguiar and Gopinath 2006), (Arellano 2008), (Yue 2010), (D'Erasmus 2011), (Asonuma 2012). (Aguiar and Amador 2013) provide an overview of the economics of sovereign debt and review some of the papers in the literature.

²Some of these empirical failures have been recognized in the literature, for example, (Dias, Richmond and Wright 2011) argue that the levels of debt that the existing models produce are much smaller than the levels reported in traditional sovereign debt statistics.

This paper relaxes the unrealistic assumption in the existing literature that the government is never committed to repaying the debt. In conformance with the terminology used in other literature, we use the concept of *limited commitment*. A government with limited commitment is essentially equivalent to a government that sometimes does not have the option to default, and commitment is especially meaningful if the government does not default although the benefit of a default is greater than the cost.

Introducing the notion of limited commitment improves the empirical performance of the model in several ways, in particular, the level of debt, the level of interest rate spreads, and the countercyclicality of trade balance. Intuitively, when the government can commit to some degree, it will be able to borrow at lower rates conditional on issuing the same amount of debt. This encourages borrowing, supports higher debt levels and may generate more defaults in equilibrium.

Limited commitment not only improves the quantitative performance of the model, it is also a more reasonable assumption than no commitment. In the real world there are ways in which a country can commit itself to repaying its external debt to some extent, even when making repayments means making sacrifices. For example, the decision to repay could be delegated to a conservative policymaker with a large personal disutility from defaulting, or to one who will be sanctioned for defaulting in terms of reputation or loss of power. The fact that the preferences of the policymakers affect a country's willingness and commitment to repay has been recognized in the literature.³

Following other literature, such as (Schaumburg and Tambalotti 2007), we model limited commitment as a probability λ that in each period the government repays even

³For example, (Santiso 2003), (Rijckeghem and Weder 2009), and (Hatchondo and Martinez 2010).

if repaying is more costly than defaulting. Limited commitment corresponds to the case where the commitment probability is strictly between zero and one. The commitment probability can be rationalized in a framework of political turnovers. In this framework, each policymaker is obliged to repay the debt that she issues, but has no such obligation on debt issued by her predecessor. The incumbent policymaker is replaced by a newly elected policymaker with probability $1 - \lambda$ in every period. The political turnover could result from elections, or could be accompanied by revolutions or military coups.⁴ Then, default can happen only if a political turnover takes place.

This specification is simple yet powerful, and has many advantages. It encompasses the no-commitment and full-commitment assumptions in the existing models as two extreme cases. Setting the probability of commitment to zero corresponds to the no-commitment case, while setting the probability at one is the full-commitment case. By varying the probability of commitment, we can study the welfare effects of increasing commitment.

This paper is related to three categories of papers. The first category includes papers that model the borrowing and default decisions of a government as an optimization problem that the government is trying to solve. (Eaton and Gersovitz 1981) first model the case where a country defaults and is permanently excluded from financial markets. (Bulow and Rogoff 1989) state that sanctions are necessary in sustaining sovereign debt in equilibrium. Recently, the literature has expanded to include the DSGE models of a small open economy that account for recurring debt crises, countercyclical interest rates and other

⁴Arguably it would be more realistic to assume that the date of the election is deterministic rather than stochastic. However, this would complicate the model as the time remaining before an election, if modelled, would become another state variable and increase the computational burden.

key patterns in business cycles for emerging markets. In these models, the government solves the optimization problem in each period, and the net benefit of default is modelled as a function of a number of economic determinants. These models have shown their abilities in explaining certain empirical regularities. Important contributions such as (Arellano 2008) predict that interest rates respond to output fluctuations through endogenous time-varying default probabilities. Defaults are associated with recessions because a risk-averse borrower will find it more costly to repay non-contingent debt and is hence more likely to default. In contemporaneous works, (Aguiar and Gopinath 2006) highlight the role of a stochastic trend in output process in improving the model's ability to match the data. (Yue 2010) explicitly models post-default negotiations and endogenous periods of exclusion. (Alfaro and Kanczuk 2005) and (D'Erasmus 2011) develop models of sovereign default with information asymmetry and heterogeneous governments whose types change over time. (Asonuma 2012) models renegotiations between a defaulting country and its creditors. (Phan 2013) models the borrowing government with private information on the stochastic domestic economy. However, all these papers fail to explain the data in one way or another, including the level of debt, the level of interest rate spreads, and the countercyclicity of the trade balance.

The second category of papers consists of those discussing how government commitment, political stability and institutional quality affect sovereign borrowing. The idea that emerging economies tend to have lower government stability and higher risks of political turnover has been discussed in the literature. Studies such as (Alesina, Ozler, Roubini and Swagel 1996) define political instability as the tendency of a government to collapse. (Annett 2001) shows how political instability in emerging markets is linked to racial and religion divisions. (Jeanne 2009) suggests that a realistic assumption for many develop-

ing and emerging economies is limited commitment, meaning that the credibility of the commitment of repayment is more limited than for advanced economies. (Hatchondo and Martinez 2010) discuss how political factors may influence sovereign default risk and find that the Argentina default in 2001 is most likely to have been triggered by political turnover. (Qian 2010) shows that a country with a unified government is less likely to default than one with a polarized government.

The third category of relevant literature includes several recent papers that study limited commitment in the context of monetary policy and fiscal policy. (Schaumburg and Tambalotti 2007) assumes that monetary policy is delegated to a central banker with a policy plan to which she is committed until she is replaced, which occurs with a certain constant probability. They find that in a simple model of the monetary transmission mechanism, most of the gains from commitment accrue at relatively low levels of credibility. (Debortoli and Nunes 2010) discuss limited commitment in fiscal policy. They assume that the government in power always fulfills its promises, but with some probability it will be replaced by other governments. The new government may make a new promise based on the economic environment, while the old government's promises are no longer considered. They find that the properties of labor and capital income taxes are significantly different under the full-commitment and no-commitment assumptions. (Bauducco and Caprioli 2008) analyze how the tax-smoothing result obtained in models of optimal fiscal policy is altered in a context of international risk sharing with limited commitment.

The rest of the paper is structured as follows. Section 1.2 identifies some existing problems in the literature. Section 1.3 presents the model with limited commitment. Section 1.4 defines its recursive equilibrium. Section 1.5 estimates the model using data

for Argentina. The results suggest that limited commitment improves the model fit in every aspect, and resolves the problems in the existing literature. Section 1.6 explores the quantitative role of commitment, and explains why the problems are resolved once limited commitment is introduced. Section 1.7 studies the welfare effects of commitment. Section 1.8 concludes the paper.

1.2 Problems with the Existing Literature

The DSGE literature on sovereign debt explains countercyclical spreads and procyclical capital inflows, but it has a hard time explaining several empirical facts present in the data. This section presents four problems that are unresolved in the literature.

Problem 1: Existing DSGE models of sovereign debt explain only the lower measures of a country's indebtedness.

Table A.1 shows different measures of indebtedness for Argentina before its 2001 debt crisis. Two of those five measures are flow (debt service), while the rest are stocks. Within each category, some only measures those debts that are public and publicly guaranteed, while some includes debts issued by all identities in the economy. The debt ratios range from 12% to as high as 172%.⁵ Table A.2 summarizes the measures of debt that papers in the literature are trying to match, and the levels of debt they succeed to explain. As one of the important contributions to the literature, (Arellano 2008) matches a debt service to GDP ratio at 5.95% with an observed annual default frequency at 3%. Some

⁵The ratios are computed using quarterly GDP data to allow a fair comparison with the statistics generated by the existing models, as all those models are calibrated on a quarterly basis.

other papers in the sovereign debt literature, such as (Aguiar and Gopinath 2006)(referred to in the tables as AG) and (Yue 2010), propose variants of the (Arellano 2008) model and are only able to explain debt measures that are less than 20% of output. (D’Erasmus 2011) targets the public and public guaranteed debt stock to output ratio at 81.2%, but is only able to generate half of the level observed in the data. (Asonuma 2012) targets short-term debt stocks, which constitute only one-fifth of the total debt stocks.

We think that the best data analog of the debt ratio in the model is the public and publicly guaranteed external debt stock. The reasons are the following. First, when the government defaults, it does so on all debt obligations. Sovereign debt contracts often contain an acceleration clause and a cross-default clause. These two clauses imply that once a government defaults on one debt issue, the other debt issues are accelerated and become current. In this sense, the total outstanding stock of debt can be renegotiated in a default. In December 2001, Argentina defaulted on 93 billion USD of its external government debt. Given the 2001 GDP at 325.5 billion USD, the debt defaulted represented 28.6% of its GDP. Multiplying this number by four gives us an estimate of the debt in quarterly GDP of about 114%. Of all the measures in we have discussed, the public and publicly guaranteed external debt has the closest magnitude. Second, the public and publicly guaranteed debt is backed by the government, consistent with the assumption in the models that the decisions regarding repayment and default are made by the government rather than private debtors.⁶

Problem 2: Existing DSGE models of sovereign debt assume low levels of discount factors.

In order to match the observed frequency of default, papers in the sovereign debt literature assume very low discount factors. The authors argue that high impatience reflects

⁶In fact, in these models there is no private debt.

the high political instability in the borrowing country. This is consistent with the view that the policymakers are short-sighted in the sense that they maximize domestic welfare at the horizon at which they are in power. In that case, we can model the policymakers' optimization problem assuming that they have a discount factor that is the product of the households' discount factor and the probability that the policymakers stay in power. If we believe that the households are as patient as the foreign creditors, or not more impatient by much, the very low discount factors must imply that policymakers have very low probability of staying in power in each period. Table A.3 summarizes the quarterly discount factors calibrated or estimated by the existing models. Except (Arellano 2008), all the papers have very low quarterly discount factors around 0.75, implying that the policymakers expect themselves to be in power for about one year, or even less. This horizon seems too short compared to their real life counterparts.⁷

Problem 3: Existing DSGE models of sovereign debt are unable to predict the strong countercyclicality observed in the trade balance.

Emerging market economies often have countercyclical trade balance. One explanation, offered by the sovereign debt literature, is that the availability of international capital varies with the business cycle. With persistent output shocks, the terms of bond contracts are much more stringent in recessions than in booms because of endogenous default risk. Although the DSGE models of sovereign debt are able to predict a negative correlation between the trade balance and output, the correlation is much smaller than the data. Table A.4 shows the correlation between the trade balance and output in the data

⁷Even for a country like Argentina which is not politically very stable, there has been 52 presidents in power since 1801. On average each president was in place for four years.

and the existing models. In the data for Argentina before its debt crisis in 2001, the trade balance is negatively correlated with output with a correlation of -0.658.⁸ However, the existing models predict a negative correlation less than -0.25.

Problem 4: Authors of the existing DSGE models of sovereign debt do not estimate their models, instead, they calibrate the parameters to match certain moments from the data.

Although there are many similarities in the methodology, for example both require finding the roots of some functions, there is substantial difference between statistical estimation and calibration. First, on the data part, in the calibration exercise of the model, we do not assume any distribution of the data. We take data from the history, calculate some moments of the time series, and pick parameter values for the model so that the model can reproduce those moments. We do not talk about whether the data could be representative enough for a longer horizon or they just come from rare events, nor are we explicit about why we choose to match these moments as opposed to others. Second, calibration exercise do not give standard errors for the parameter estimates, so if we care about the parameter values and would like to draw some policy conclusion, we do not know if the estimates are significant or not. Estimation resolves those problems by assuming that the data comes from some known distribution, and using the distribution to determine the relative importance of the moments in the moment-matching exercise. We can also obtain the standard errors of the parameter estimates to get a better idea if the parameter estimates are signif-

⁸Literature has shown that this large negative correlation is robust across time. (Kaminsky, Reinhart and Vegh 2004) document that Argentina has the largest negative correlation between the current account and output among most OECD and developing countries, relying on data for 104 countries for the period 1960-2003.

icant or not. Although statistical estimation requires sufficiently large amount of data, and arguably default is a rare event in history meaning that we do not have much data to play with, it is still worth estimating the model using the data that we have.

1.3 Model

1.3.1 Preferences and Endowment

This model is set up in a small open endowment economy similar to (Arellano 2008). The economy is composed of the households and the government. All the agents in the economy discount the future with a discount factor $\beta < \frac{1}{1+r}$, where r is the risk-free interest rate. The economy receives a flow of stochastic income y in every period. The households cannot lend or borrow abroad and the government does it on their behalf. The government is benevolent and maximizes the household's lifetime utility defined as

$$E_t \sum_{t=0}^{\infty} \beta^t u(c_t).$$

1.3.2 Bond Contracts

Foreign risk-neutral competitive creditors supply credit to the government. In every period after output y_t is realized, the government repays its total outstanding debt or defaults on it, and makes lump-sum transfers to the households. If repayment is made, the government can choose its new borrowing level with the foreign creditors.

The financial market in this model features non-contingent contracts and incomplete insurance. The only contract that the government can enter is one-period zero-coupon bonds. Let b_t be the country's holding of bonds at time t , and b_t is negative. As long as the

government repays $-b_t$, it can enter a new contract with foreign creditors and issue a new bond with face value $-b_{t+1}$. Foreign creditors price each unit of the bond by q_t . This new contract provides the government with resource equal to $-q_t b_{t+1}$. Note that this contract is not contingent on future fundamentals, in other words, the amount that the government is obliged to repay (conditional on not declaring a default) does not depend on the realization of output.⁹ The government is subject to an external credit constraint \underline{b} .¹⁰ As a result, as long as the government does not default, debt moves according to

$$c_t - b_t \leq y_t - q_t b_{t+1},$$

and

$$b_{t+1} \geq \underline{b}.$$

1.3.3 Repayments and Defaults

So long as the government repays, it remains in the international financial market. If the government defaults, it defaults on all its outstanding debt. The country immediately

⁹In practice, the standard sovereign debt contract is typically non-contingent. As pointed out by (Aguiar and Amador 2013), the lack of contingency may reflect asymmetric information, for example, the government can manipulate the actual or reported behavior of microeconomic aggregates. Thus contracts with state-contingent payoffs may be prone to moral hazard. In reality, there do exist certain types of contingent sovereign bonds, including some of the Brady bond restructuring in the early 1990s. However, the prevalent form of sovereign bond is still the non-indexed bond.

¹⁰This constraint is useful because it guarantees that the level of debt is bounded from above. Papers in the literature do not explicitly discuss this constraint because under the no-commitment assumption, the level of debt will be bounded from above because of the endogenous borrowing constraint generated by the bond Laffer Curve. This will become clearer in the latter part of the paper.

experiences an output loss and is excluded from the international financial market for a stochastic number of periods. The output loss persists and consumption equals output as long as the country is in financial autarky.¹¹ The country leaves financial autarky and regains access to international financial markets with an exogenous probability each period. When the country is back to international financial markets, it repays the restructured debt with a haircut.¹²

The representative household delegates the decision to borrow and repay to a policymaker. The policymaker will be in power with an exogenous probability $\lambda \in [0, 1]$ in each period, and otherwise will be replaced by another policymaker. A newly-elected policymaker may renege on the promise made by the predecessor in her first incumbent period, but is committed to repay for the rest of her mandate. This could be the case, for example, because there is little reputational cost to defaulting when the policymaker can blame it on the predecessor. In the model, all policymakers are benevolent and intrinsically

¹¹Cost to default is necessary for a positive level of debt to be sustained in the model. The literature has proved that a positive level of external debt cannot exist if the government has access to a storage technology and effective punishment does not exist. In theory, debt can be sustained in equilibrium through financial exclusion alone; however, the level of debt will be small. This is because as Lucas (1987) points out, the welfare gains from consumption smoothing is relatively small. Therefore, a direct output cost is needed. Papers investigating post-default episodes typically find evidence for the existence of direct default costs, such as destruction in domestic collateral, contraction in domestic lending, reduction in trade credit, etc. This model takes a reduced-form approach regarding default cost and the resumption of market access. See (Mendoza and Yue 2012) for a model with endogenous output loss during default.

¹²This stochastic exclusion reflects the fact that many factors affect the post-default debt renegotiations and restructuring. (Yue 2010) models endogenous post-default renegotiations and haircuts in a similar model.

the same. No differentiation is made between honest repayers and frequent defaulters. $\lambda = 1$ represents the extreme case where the policymaker is never replaced and all the promises of the government are fulfilled. $\lambda = 0$ is the opposite extreme where there are frequent turnovers among policymakers.

To facilitate analysis, define Λ_t as the binary variable that determines whether the policymaker is replaced and a new one is elected. $\Lambda_t = 1$ indicates that the incumbent policymaker stays in power and repays the debt. $\Lambda_t = 0$ indicates that a new policymaker replaces the incumbent, and the new one has an option to declare a default on the debt. Λ_t is linked to λ in the sense that $\Lambda_t = 1$ happens with probability λ .

1.3.4 Timing of Actions

The timing of actions within each period is as follows. At the beginning of period t , output y_t and commitment Λ_t are realized. If the incumbent policymaker stays in power, she repays and makes new borrowing decisions, taking the price schedule q_t as given. If a new policymaker replaces the old one, the new policymaker decides whether to repay or not. If she repays, foreign creditors supply the funds to the government. If she defaults, the country is excluded from financial markets and enters financial autarky. In either case, the government makes lump-sum transfers to the households. The households then consume.

1.4 Recursive Equilibrium

1.4.1 Sovereign's Problem

The goal of the sovereign government is to maximize households' expected lifetime utility. Define $V(b, y, \Lambda)$ as the value of the objective function for the government before it knows whether it has the option to default or not. The value is

$$V(b, y, \Lambda) = \Lambda V^c(b, y) + (1 - \Lambda) V^o(b, y).$$

$V^c(b, y)$ is the value associated with not defaulting and continuing to borrow. $V^o(b, y)$ is the value associated with having the option to default. It is the maximum of the two choices

$$V^o(b, y) = \max_{c,d} \{V^c(b, y), V^d(b, y)\}.$$

The value of repaying and continuing to be in the financial contract is

$$\begin{aligned} V^c(b, y) &= \max_{b'} \{u(y - qb' + b) + \beta EV(b', y', \Lambda')\} \\ &= \max_{b'} \{u(y - qb' + b) + \beta \lambda EV^c(b', y') + \beta (1 - \lambda) EV^o(b', y')\}. \end{aligned}$$

The value given the government defaults is

$$V^d(b, y) = u(y^d) + \beta \theta EV^c((1 - h)b, y') + \beta (1 - \theta) EV^d(b, y'),$$

in which y^d is the reduced output after a default and θ is the probability that the economy will regain access to international credit markets. We assume that the government must repay the debt with a haircut in the period it regains access to the market.¹³

¹³If the country cannot repay all the debt from current output, it repays partially using all the output and consumption drops to zero.

The government's default policy is characterized by default and repayment sets. Let $\mathcal{D}(b, y)$ be the set in the two-dimensional space $B \times Y$ where default is preferred to repayment. By definition, $\mathcal{D}(b, y)$ is

$$\mathcal{D}(b, y) = \{b \in B, y \in Y | V^c(b, y) < V^d(b, y)\}.$$

The default set for any given b is

$$\mathcal{D}_b(y) = \{y \in Y | V^c(b, y) < V^d(b, y)\}.$$

The default frontier, which separates the default region with other states of the economy, is defined as

$$\mathcal{F}(b, y) = \{b \in B, y \in Y | V^c(b, y) = V^d(b, y)\}.$$

For the purpose of the analysis later, also define the vulnerability region as

$$\mathcal{V}(b, y) = \{b \in B, y \in Y | V^c(b, y) \geq V^d(b, y), \mathcal{D}_b(y) \neq \emptyset\}. \quad (1.1)$$

This is the set of the states where default is not preferred to repayment, but a different output realization could bring the state into the default region. Being in the vulnerability region is a prerequisite for defaults to happen in equilibrium.

1.4.2 Foreign Creditors' Problem

The foreign creditors choose the amount of debt b' to maximize their expected payoff. They are assumed to have perfect information about the country's output process, debt holdings and probability of commitment. The expected payoff is given by

$$G(b', y) = \begin{cases} q(b', y) - \frac{1}{1+r}b' & \text{if } b' \geq 0 \\ \frac{1}{1+r}\{\lambda + (1-\lambda)[(1-\pi) + \frac{\theta(1-h)\pi}{\theta+r}]\}(-b') - q(b', y)(-b') & \text{if } b' < 0 \end{cases},$$

where π is the expected probability that the sovereign prefers a default to repayment with an endowment y and debt b , which can be written as

$$\pi(b', y) = \Pr[V^c(b', y') < V^d(b', y')|y]. \quad (1.2)$$

The term $\frac{\theta(1-h)}{\theta+r}$ reflects the present value of the renegotiated bond recovery after a default.

Because of the perfect competitive assumption in the bond market, the foreign creditors' expected profit is zero. Using the zero profit condition, the pricing function of the bond is

$$q(b', y) = \begin{cases} \frac{1}{1+r} & \text{if } b' \geq 0 \\ \frac{1}{1+r} \{ \lambda + (1-\lambda)[(1-\pi) + \frac{\theta(1-h)\pi}{\theta+r}] \} & \text{if } b' < 0 \end{cases},$$

where π is defined in Equation (1.2).

The pricing equation models how the price of the debt is determined. The probability of preferring a default π is endogenous to the model. By definition, $q \in [0, \frac{1}{1+r}]$. At one extreme, as λ goes to one, q approaches the inverse of risk-free rate $\frac{1}{1+r}$. In this world, the government is fully committed to repaying its debt. The bond bears no risk premium. At the other extreme, as λ goes to zero, the government is fully opportunistic. The interest rate is generally higher than the risk-free rate and bears a time-varying positive risk premium.

Using the bond-pricing function, the vulnerability set defined in Equation (1.1) is equivalent to the set where bond carries a positive default premium. Thus the vulnerability region can also be defined as

$$\mathcal{V}(b, y) = \{y \in Y | q(b, y) < \frac{1}{1+r}\}.$$

Note that when $\lambda = 1$, the vulnerability region will disappear.

1.4.3 Recursive Equilibrium

The equilibrium of the model is defined.

Definition. *The model's recursive equilibrium is a set of functions for (i) bond-pricing function $q^*(b', y)$, (ii) the sovereign's value functions $V(b, y, \Lambda)$, $V^o(b, y)$, $V^c(b, y)$, $V^d(b, y)$, (iii) bond holdings $b'^*(b, y)$, (iv) default set $\mathcal{D}(b, y)$, (v) consumption $c^*(b, y)$, such that*

1. *Given the bond pricing function $q^*(b', y)$, the value functions $V(b, y, \Lambda)$, $V^o(b, y)$, $V^c(b, y)$, $V^d(b, y)$, bond holdings $b'^*(b, y)$, default set $\mathcal{D}(b, y)$, and consumption $c^*(b, y)$ satisfy the government's optimization problem;*
2. *The bond-pricing function $q^*(b', y)$ reflects the government's limited commitment and is consistent with foreign creditors' break-even condition.*

Condition 1 requires that the sovereign's default and repayment decisions must be governed by the commitment probability. Depending on the realizations of the commitment shock, when the sovereign is given the option to choose between defaulting and repaying, it does so taking the interest rates on the debt as given. Condition 2 requires that equilibrium bond prices be consistent with the optimal lender behavior.

1.5 Quantitative Analysis

We estimate the model using data for Argentina before its 2001 debt crisis. Argentina has been the focus of all the quantitative models of sovereign default because it is a typical emerging market economy that has repeatedly defaulted over the course of its history, and whose economy features strongly countercyclical interest rates and current account. Argentina defaulted in December 2001 in the midst of an economic turmoil that

continued through 2002 as the economy virtually collapsed. It defaulted on \$93 billion of external debt, representing around 28.6% of its annual GDP in 2001. After the default, a large-scale debt restructuring was implemented, and only 27% of the debt's face value was recovered by creditors.

1.5.1 Model Specification

Define one period as a quarter. The utility function is assumed to be:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

where σ is the coefficient of relative risk aversion. The endowment is assumed to follow a log-normal AR(1) process

$$\log(y_t) = \rho \log(y_{t-1}) + \epsilon_t,$$

where ϵ has a normal distribution with mean 0 and variance η^2 . The mean endowment is normalized to 1. This output process features shocks around a long-run level of mean output.¹⁴ The shock is discretized into a finite-state Markov chain with 21 states, using the method of (Tauchen and Hussey 1991).

The literature has diverged in the way in which output loss in default is modeled. Most of the studies assume that output loss is a constant fraction of output. (Arellano 2008), on the other hand, adopts an asymmetric cost of default, an assumption that I will follow

¹⁴(Aguiar and Gopinath 2006) highlight the role of shocks to the growth rate instead of shocks to the level, and claimed that the ability of the model to fit the data improved significantly when they assume trend shocks in the GDP process. However, (Hatchondo, Martinez and Sapriza 2010) cast doubt on the above claims. Using their improved numerical method with finer grids over the state space, these authors show that the model fitness does not necessarily improve when assuming trend shocks.

here. Output in default is assumed to be

$$y^d = \begin{cases} \alpha E(y) & \text{if } y > \alpha E(y) \\ y & \text{if } y \leq \alpha E(y) \end{cases}, \quad (1.3)$$

where $E(y)$ is the mean level of output.¹⁵ The asymmetric default output cost makes the default cost a more sensitive function of output, which is key for extending the range of b' that carry positive but finite default premiums. The important role that the asymmetric default output cost plays in generating endogenous defaults has been recognized in the literature.

1.5.2 Estimation

The parameters can be classified into two groups. The first group includes parameters that are calibrated independently of other parameters, whose values are reported in Table A.5. The risk aversion coefficient σ is set to 2, which is standard in the macroeconomic literature. The risk-free interest rate is set to 1%. The stochastic process for output is calibrated to Argentina's quarterly real GDP. The probability of redemption after a default is assumed to be 0.25. It implies an average exclusion period of one year, consistent with the findings in (Gelos, Sahay and Sandleris 2011).¹⁶ The haircut is set to 73%, consistent with various estimates of the debt recovery rate in the Argentine default episode. The debt limit

¹⁵One possible interpretation of this assumption is related to the argument by (Grossman and Huyck 1988) about excusable and non-excusable defaults. In their argument, when the state of the economy gets worse (i.e. the income level decreases) for a given level of debt, the penalty that would follow is likely to be smaller as the default would be "more excusable". (English 1996) discuss this idea in the case of U.S. state bonds.

¹⁶(Gelos et al. 2011) find that the average years until resumption was 5.4 in the 1980's and 0.9 in the 1990's.

is set at -3, representing 300% of mean output. The second group contains parameters that will be jointly estimated by matching target statistics of the country. These parameters include the discount factor, the probability of commitment, and output in default.

The estimation is done by the Simulated Method of Moments. This method aims to minimize the distance between moments from the data and from the model. Let Ω be the set of parameters. This method is intended to choose Ω to minimize the loss function defined as

$$\text{Loss}(\Omega) = [M(\text{Data}) - M(\Omega)]W^*[M(\text{Data}) - M(\Omega)]',$$

where $M(\text{Data})$ are the moments from the data, $M(\Omega)$ are the moments generated by the model, and W^* is the optimally derived weighting matrix. Due to the potential kinks in the loss function, we use the simulated annealing algorithm to conduct the optimization. The weighting matrix is obtained using the bootstrap method. We bootstrap data to get bootstrap samples, calculate the moments for each bootstrap sample, and the weighting matrix is the inverse of the variance-covariance matrix of those moments. Standard errors of the parameter estimates are computed from the derivative of the simulated moments with respect to the parameters evaluated at the point estimates. Appendix A.2 provides a more detailed explanation of the estimation procedure.

The moments to match are the average interest rate spreads, the average debt-to-output ratio, the ratio of the standard deviation of consumption to the standard deviation of output, and the correlation between trade balance and output. The time series to construct the data moments starts in 1993q1, and ends in 2001q4 when the default took place. The interest rate spread is defined as the difference of the Emerging Markets Bond Index (EMBI) yield and the yield of a 5-year U.S. bond. During that period, the average spreads is

6.6%. As discussed in Section 3.2, we use the public and publicly guaranteed debt stock to quarterly GDP ratio as the data analog of the measure of indebtedness for this model. During the above time period, the average debt-to-output ratio is 99.70%. The ratio of the standard deviation of consumption to the standard deviation of output is 1.1002. The correlation between trade balance and output is -0.6580. The data on output, consumption and trade balance are taken from the Ministry of Finance in Argentina for the period of 1993q1 to 2001q4. The variables are real, seasonally adjusted and detrended using the HP-filter with parameter 1600. The trade balance is expressed as a percentage of GDP.

These moments are key characteristics of the Argentina economy business cycle, and also contain enough information to identify the parameters of interest. Spreads are high and volatile. Consumption is more volatile than output and trade balance is negatively correlated with output, reflecting the fact that borrowing plays an imperfect role in smoothing consumption in the country. These moments are also chosen as target statistics in the calibration in other sovereign debt papers.¹⁷ Having similar moments allows us to make a comparison of our results with the literature. The frequency of default is not chosen as one of the target statistics simply because the risk-neutrality of the lenders establishes a direct link from the frequency of default to bond spreads.

The model is solved numerically and simulated to obtain moments. The model

¹⁷(Arellano 2008) calibrates the model to match the frequency of default, trade balance volatility and debt-service-to-output ratio. (Yue 2010) calibrates the model to match the frequency of default and debt recovery rate. (D’Erasmus 2011) estimates the model to match the frequency of default, the standard deviation of current account to output ratio, the average period in the state of default and the ratio of the standard deviation of consumption to the standard deviation of output. (Asonuma 2012) calibrates the model to match the frequency of default and the average recovery rate.

is solved by value function iterations using the discrete-state-space (DSS) technique, which is also used in other default studies. Appendix A.3 provides a detailed description of the computational algorithm. We conduct 100 simulations of the model economy, each with 10500 periods. The first 500 periods are discarded to eliminate the effects of initial conditions. To construct the model analog of the default statistics, we first extract time series that satisfy the following criteria: i) the sample has 36 periods, ii) a default is declared immediately after each sample, iii) the last exclusion period is observed at least two periods before the beginning of the sample. These three criteria are also used by other default studies when extracting simulated samples. We impose an additional criterion that the sample must have a negative correlation between trade balance and output for reasons that will be explained later. For each of the simulated default samples, we calculate their moments. We then take an average of all the moments from default samples we have to obtain the model simulated statistics. Note that for some parameter values it is possible that no sample can be extracted simply because there is no default. In that case, we use all the paths in the simulation as a sample and calculate the moments based on that. This method is the same as in the literature.

1.5.3 Results

Table A.6 displays parameter values and moments generated by the baseline model. The discount factor is 0.9756, higher than the values in other default studies. The probability of commitment is 0.9554. By definition, it is the quarterly probability that the incumbent policymaker stays in place and commits to repay. The value implies that this event takes place around once every 4 years on average. The value for output in default

is 0.8508, implying that the country loses around 15% of output during a default. The value seems larger than the corresponding values calibrated in other default studies, but is plausible with the Argentine estimates in the empirical literature.¹⁸ All the parameter estimates have very small standard errors, indicating that the model statistically reject the hypothesis that $\lambda = 0$.

The model matches well the business cycle statistics. Table A.7 compares business cycle moments from the data, from the model and from other papers in the sovereign debt literature. Our model delivers an average spreads of 5.02% and debt-to-output ratio of 100.83%, both much higher than their counterparts from the existing literature. Note that our model do not need to assume risk-averse pricing to get the high spreads.¹⁹ The standard deviation of consumption to the standard deviation of output is 1.1673, and the correlation between trade balance and output is -0.5460. Both the volatile consumption and the countercyclical current account balances are salient features of emerging market economies. The bond price depends on probability for repayment. Since output has some

¹⁸The default costs in (Aguiar and Gopinath 2006), (Yue 2010), (D’Erasmus 2011) and (Asonuma 2012) are respectively 2%, 2%, 6% and 2% of output. Although the empirical literature does not give a precise estimate for the cost of a default, various sources suggest that the cost is large. For example, (Borensztein and Panizza 2009) empirically evaluates four types of cost that may result from an international sovereign default. They find that growth falls by 2.6% in the first year of a default episode. Also, they do not find statistically significant evidence that output catches up after a default, suggesting that the cumulative cost of default is much higher than the cost observed in the first year.

¹⁹As an attempt to increase the spreads, (Arellano 2008) extends the baseline model with risk-averse creditors and maintains an average spread of 10.4. However, with risk-averse pricing, the calibrated value for the quarterly discount factor drops to 0.882. Some other authors give a more complex maturity structure to the bond, to increase the average spreads.

persistent components, the default probability is higher in recessions and lower in booms. As a result, interest rates are countercyclical, meaning that they are low in good times and high in bad times. This constrains those countries' borrowing capacities, and limits the role of insurance that financial integration and foreign credit are intended to provide. Consumption and output are highly correlated, and both experience large drops at the time of default. Spreads are volatile and countercyclical. The model predicts a 3.24% frequency of default, in line with the historical evidence that Argentina defaulted three times in a hundred years.

1.6 Resolution of the Problems

This section is devoted to explaining why introducing limited commitment resolves the problems present in the literature. Section 1.6.1 revisits the problems discussed in Section 3.2 by estimating the “no-commitment model”, namely a constrained version of the benchmark model where the probability of commitment is set to zero. Note that by restricting the probability of commitment to zero, our baseline model boils down to a model that is very close to (Arellano 2008).²⁰ In Section 1.6.2 we discuss the quantitative role of commitment and why allowing λ to be greater than zero resolves the problems.

²⁰There are a few differences, though. For example, (Arellano 2008) assumes a haircut of 100% while we use 73% which comes from data. Her model fits the time series using a linear trend while we use the HP filter with coefficient 1600.

1.6.1 Problems in the No-Commitment Model

We estimate the no-commitment model and report the results in Table A.8. The same parameters β and α are estimated following the same procedure to match the same four target statistics as in the baseline estimation. The parameter values that we obtain are 0.9811 and 0.7189 for β and α respectively. Not surprisingly, the no-commitment model does badly in reproducing the data moments. The model delivers an average interest rate spread of 0.06% and frequency of default of 0.02%, which are only one-hundredth of those in the data. The average debt-to-output ratio is 62.90%, higher than the literature but still by far lower than the data, and bond spreads exhibit much lower volatility. Comparing the results with the baseline estimation, it is worth noting that allowing the probability of commitment to deviate from zero improves the model fit for virtually every model moment.

The no-commitment model fails to match the four target moments at the same time because it cannot explain a high level of debt and recurrent defaults (or equivalently, high spreads) without generating excessive consumption volatility. To see this, the last column in Table A.8 reports simulated statistics when we set λ to zero and calibrate the two parameters β and α to match only the first two moments, i.e., the average spreads and average debt-to-output ratio. The results show that while the two target moments are roughly in line with the data, the model exhibits too much volatility. Consumption, trade balance and spreads all have much higher standard deviations compared to the data. In addition, consumption is less correlated with output than in the data.

This raises the question as to why consumption volatility is much closer to the data than the other three moments in the estimation results of the no-commitment model? The reason is that, the estimation procedure aims to minimize the distance between the moments

from the data and from the model. When the method exhausts all possible parameter values but still fails to match all the moments at the same time, it uses the weighting matrix to determine the relative importance of the moments in the moment matching exercise. The weighting matrix is obtained by the bootstrap method. We bootstrap the original time series to get a number of bootstrap samples, calculate bootstrap moments for each of the bootstrap samples, and use the inverse of the variance-covariance matrix of the bootstrap moments as the weighting matrix. This method implies that moments with lower standard errors have larger weights. In the bootstrap samples, consumption volatility has the smallest variance compared to other moments while spreads have the largest variance. This seems plausible because in the original data, consumption is more stable over time, while spreads are more subject to large movements caused by various factors, especially when the data comes from the periods before a crisis. As a result, the estimation procedure puts the highest weight on consumption volatility and we get a better match of the ratio of the standard deviation of consumption to the standard deviation of output compared to other moments.

Besides the obvious failure in the estimation, another drawback of the no-commitment model is that it needs a very low β to bring both spreads and level of debt closer to the data. The parameter values for β and α that produce the stimulation statistics in the last column of Table A.8 are 0.8742 and 0.7899 respectively. As discussed in Section 1.2, a discount factor as low as 0.8742 seems implausible. Indeed, a low discount factor is needed in the no-commitment model to generate recurrent defaults on a large amount of debt. With less debt, the discount factor could be higher and more reasonable. For example, (Arellano 2008) explains a debt-to-output ratio around 6% using a discount factor at 0.953.

Having said that, in some of the existing papers the discount factors are low although they explain low levels of debt, for example, (Yue 2010) and (Asonuma 2012) predict levels of debt around 10% of output using discount factors around 0.75. This is due to the post-default debt restructure they impose on the model which makes default less preferable for a patient borrower.

1.6.2 Quantitative Role of Commitment

One way to understand the role of commitment is to compute the equilibrium of the baseline model assuming all parameter values are held at their baseline values but the probability of commitment is set to zero. Under the no-commitment assumption, the sovereign defaults as long as the value of defaulting is strictly greater than the value of repaying. With limited commitment, default does not necessarily happen even when the value of repaying falls below the value of defaulting, simply because the sovereign is not given the option to default. We discuss the quantitative role of commitment, first in the solution of the model, and then in the results in simulation.

First, commitment shrinks the possible default region, as is shown in Figure A.2. We call this region “possible default region” instead of “default region” (a term used in the literature) because in this region the government prefers default to repayment, but whether the it is able to engineer a default still hinges on the commitment shock that gives the government the option to do so. However, the change is small even if the probability of commitment increases from 0 to 0.95. It is only when the probability of commitment gets close to 1 that there is significant movement in the default threshold. The default threshold moves towards higher levels of debt with commitment because lower spreads make default

a less attractive option for the sovereign, and such benefit is greater when the sovereign is more likely to have to rollover its debt, i.e., when the probability of commitment is higher. Note that even in the extreme case where $\lambda = 1$ and the sovereign is never allowed to default, it would still prefer default to repayment in some states of the economy where debt is high and output is low.

The two default thresholds for $\lambda = 0$ and $\lambda = 1$ partition the whole state space into three regions. In Figure A.2, the region above the $\lambda = 0$ line (in the direction of less debt) is where the sovereign prefers not to default even if creditors believe that the sovereign lacks commitment and they price the bond taking into account default risks. The region below the $\lambda = 1$ line (in the direction of more debt) is where the sovereign prefers a default even if it is borrowing with no default risk premium. The region in-between the two lines is where the sovereign is willing to repay provided it can roll over all future debt with no default premium, but is unwilling to do so if the creditors believe that it has a zero probability of commitment. This region captures the idea of self-reinforcing expectation. As can be seen from the graph, a permanent increase in the probability of commitment from zero to one moves the default threshold by almost 30% of output.

Second, commitment shifts up the bond price schedule, and bond price becomes much less steep as a function of debt. This is because, conditional on borrowing the same level of new debt, a higher probability of commitment reduces the likelihood of default, which will be priced in to new debt by risk-neutral creditors. Figure A.3 compares the bond price schedules for the benchmark model and the no-commitment model. In the no-commitment case, bond price drops from almost one to 0.25 when the level of debt increases from 50% to 100% of output. By contrast, in the benchmark model where $\lambda = 0.95$, the

bond price drops by no more than 5%, which implies an annual spread of at most 20%. Note that although the no-commitment model implies higher spreads for the same level of new borrowings, in the simulation it does much worse in reproducing the average spreads comparable to the data, for reasons we will discuss later.

Given the difference in the solution of the model for different values of the commitment probability, it is not surprising that the model simulation statistics would exhibit stark difference with and without commitment. This is shown in Table A.9, which displays the simulation statistics for the baseline model and the no-commitment model, where the two models differ only in the probability of commitment. In the no-commitment model, the model produces lower average spreads, much lower level of debt and smaller frequency of default.

To understand the difference in simulated statistics which are basically an average of moments from all simulations, we need to look into how the variables move and interact in the simulation. With commitment, because bond price function is less steep in the level of debt, the sovereign is more willing to extend its borrowing into the region where the probability of default is significantly positive. Graphically, that is the region close to the default frontier. Once the economy is in that region, a negative output shock could push the economy into the default region, and the closer the economy is to the frontier, the higher likelihood that such situation will happen. Once it happens, instead of declaring a default immediately, the sovereign may not have the option to default and therefore has to stay in the default region.

To visualize this idea, we track how the state of the economy moves in the whole state space in the simulation. Figure A.4 shows the frequency of the states being visited

in the simulation, with darker color indicating higher frequency. The simulation statistics that the two models produce are in Table A.9. In the no-commitment case, the economy stays far from the default threshold and mainly hovers in the range where spreads are low. The economy never enters the default region because a default would immediately follow, resulting in debt reduced and the state of the economy out of the default region. With commitment, the economy moves closer to the default frontier, which implies a higher likelihood that a negative output shock brings the sovereign into the default region. Once this bad event happens and the sovereign is inside the default region, default is solely determined by the realization of the commitment shock that gives the sovereign the option to default. Before default becomes an option, the sovereign could go further deep into the default region because of another bad output shock, or it could move out of the default region because of a positive output shock. In the benchmark model simulation, the economy on average spends 17% of the time in the default region, and once it is in there, there is a probability of 5% in each period that it defaults. 85% of the defaults happen when the economy is already in the default region. Note that in a bad but unlikely scenario where a sequence of bad shocks hit the economy, debt could continue accumulating until it reaches an exogenous limit that we set to the model. In our baseline simulation, such cases are rare and are not going to affect the simulation results in any significant way. In the simulation, debt is at the exogenous limit only 1.3% percent of the time and is above 250% of output only 3.32% of the time.

As explained in the estimation methodology, the simulated moments reported in the tables are the averages of the corresponding moments of many default samples extracted from the simulation. Although the estimation procedure only aims to match the averages

of those moments to their data counterparts, it is still worth looking at the distribution of those simulated moments in more detail as they exhibit very different patterns with and without commitment. Figure A.5 shows the distribution of the simulated moments for both the baseline model and the no-commitment model. To allow a fair comparison, again we use the same parameter values for the two models except for the probability of commitment. Note that in the no-commitment case, the number of effective samples is smaller because we get much fewer defaults (the frequency of defaults shown in Table A.9).

In the baseline model where the probability of commitment is at 0.95, the average spreads are much higher because the sovereign is willing to borrow more which implies higher spreads despite the fact that commitment lowers interest rates conditional on borrowing the same level of debt. This is also the reason why in the baseline model we do not need a very low discount factor to generate frequent defaults. Recall that in the no-commitment case, in order to have high spreads and frequent defaults, the sovereign must be willing to borrow debt with high default premium. The benefit of doing so comes in the current period, in the sense that the sovereign enjoys more consumption today which increases utility. The cost, on the contrary, is in the future, as either a default does not happen in the next period and the sovereign pays high default premium, or, a default happens which brings in output loss and exclusion. It is especially more costly when the level of debt is high, as the total interest cost of borrowing risky debt is proportional to the level of debt. Consequently, only an impatient borrower who values current consumption much more than future consumption is willing to extend its borrowing into the risky region. With commitment, the cost of being close to the default frontier is much lower, therefore one does not need to assume a high impatience level to induce frequent defaults.

The average debt-to-output ratios are also much higher as the sovereign spends time in the default region which is associated with more debt and less output. The average consumption volatility is smaller in the baseline model. In the no-commitment model, consumption is more volatile than output because bond price moves negatively with output, implying net capital outflows and reduction in consumption in bad times and the opposite in good times. A problem with the no-commitment model, as we have discussed, is that when matching both a high level of debt and frequent defaults, consumption becomes too volatile. This is not surprising as the size of the capital outflows and the resulting reduction in consumption are proportional to the level of debt. With commitment, since the sovereign can enter the default region where default is solely determined by the commitment shock and bond price is constant, the benevolent government will smooth consumption of the risk-averse households, which will lead to a more smoothed consumption and less volatility.

For the same reason, the correlation between the trade balance and output is more dispersed in the baseline model. In the no-commitment model, because the bond price moves negatively with the level of output, the endogenous borrowing constraint is binding in bad times and lax in good times, which implies that consumption drops more than output in bad times and increases more than output in good times. Therefore, the trade balance, defined as output less consumption, moves negatively with output. With commitment, the correlation between trade balance and output can become positive if a sample includes enough periods when the sovereign is in the default region. Since the correlation exhibits both a positive and a negative mode once commitment is introduced, simply taking an average of all the correlations from default samples and trying to match it with the data seems problematic. As a compromise, we only keep default samples that have

a negative correlation between trade balance and output, and calculate all the simulated moments based on those samples.²¹ In the baseline simulation, one third of the default samples have a negative correlation while the rest have a correlation greater than zero.

To further illustrate the point that commitment brings the model closer to the data because it allows the sovereign to go into the default region where spreads and debt are higher and consumption volatility is lower, Figure A.6 shows how the simulated moments vary with the number of periods when the economy is in the default region. In each of the charts, the horizontal axis is the total number of periods that the sovereign spends in the default region in one default sample. The maximum number of periods is 36 which is the length of the default samples. The data points on the chart include all the default samples we get in the baseline simulation. Using different colors, we distinguish the samples that are selected, i.e., those that have a negative correlation between trade balance and output, from those that are not selected, meaning those that have a positive correlation.

The figure shows clear patterns that the average spreads and the average debt-to-output ratio both increase with the duration of stay in the default region. The relative volatility of consumption to output decreases from above one to below one as the sovereign spends more time in the default region because of the consumption smoothing motive. The correlation between the trade balance and output moves increases when the sample includes more periods in the default region. Note that the correlation can be positive when the economy spends no time in the default region, and negative when it spends all the

²¹Another way of getting around this problem is to calibrate the three parameters in the baseline model using the first three moments but without the correlation between trade balance and output. The results will not change by much.

time, which both may seem counterintuitive at first glance. In fact, the correlation could be positive when the economy is never in the default region before a default because these are the samples where the economy moves in the region where there is no default premium, bond price is fixed and the consumption smoothing motive dominates. On the other hand, the correlation could be negative when the economy is always in the default region because that is when the economy is at the exogenous borrowing limit, borrowing is constrained, and consumption moves in the same direction as output does.

1.7 Welfare Gains from Commitment

In this section, we examine the welfare effects in the baseline model of having higher probability of commitment. Do higher levels of commitment probability increase welfare? The answer is not *a priori* obvious. On the one hand, higher probability of commitment reduces interest rates which increases utility, but on the other hand, having to repay the debt in bad times reduces the insurance associated with defaults, and moreover, the sovereign may end up with more debt in the long run.

We measure welfare using the discounted future expected utility of the households before the realization of the commitment shock, defined as

$$V(b, y) = \lambda V^c(b, y) + (1 - \lambda) V^o(b, y).$$

Figure A.7 plots the value against λ for different debt-to-output ratios, assuming that output is at its mean.

The figure shows that welfare measured by the discounted future expected utility of the households is increasing and convex in the probability of commitment when the level

of debt is low. Contrary to the findings in (Schaumburg and Tambalotti 2007), in our case most of the gains from commitment accrue at higher levels of commitment. Welfare almost does not change when the probability of commitment increases from 0 to 0.95, but then shoots up as the probability of commitment gets closer to one. This is intuitive because the benefit of being committed to repay comes from the reduction in the spreads, and such benefit is larger the more likely the sovereign has to repay its debt. For higher levels of debt, increasing commitment does not necessarily increase welfare. For a debt-to-output ratio of 125%, the sovereign is better off with no commitment rather than having a medium level of commitment probability, and welfare is the lowest when the probability of commitment is at 0.5. This is because the economy starts with a level of debt that is so high that it will be better off if part of its debt could be written off by default.

1.8 Conclusion

This paper introduces limited commitment in a DSGE model of sovereign debt. The commitment is rationalized by assuming that the households can delegate the decision to default to a policymaker who has a large personal cost of default. It is because of this large personal cost that the policymaker sometimes acts against the good of the households, in the sense that it postpones a default even if it is more beneficial for the households to default. The commitment is limited rather than full because the costs are affected by the environment in which the policymakers act: for instance, the proximity of elections, or the possibility of government breakdowns caused by revolutions or military coups. The limited commitment is modeled as a probability that determines whether the government has the option to default or not in each period. This specification incorporates the (Arellano 2008)

model as a special case of our model.

By estimating the model, we show that limited commitment significantly improves the model's ability to match the data in all ways. The model is able to account for recurrent defaults, high interest rate spreads, a high debt-to-output ratio, volatile consumption, strongly countercyclical current accounts and other empirical regularities in the Argentinean data before the country's debt crisis in 2001. All these improvement are achieved without assuming unrealistic parameter values, in particular the discount factor. The estimated value for the probability of commitment is around 0.95, implying that the political turnover happens once every four years, in line with the history of Argentina. All the parameter estimates have very small standard errors pointing to the robustness of our results.

Whether commitment is welfare enhancing or welfare decreasing depends on the relative sizes of the benefit and cost of commitment. The benefit is the reduction in the interest rates that the borrower pays, while the cost comes from the fact that default is less likely to be used as an insurance against capital outflows in bad times when the borrower has to rollover its debt. Using the parameter values in our baseline model, we show that the benefit of default dominates the cost for low levels of debt, and the welfare gains accrue at higher levels of debt. For higher levels of debt, the welfare exhibits a U-shape pattern with the probability of commitment, meaning that the sovereign is most worse off if it has a probability of commitment in the middle.

One possible extension of the model is to endogenize the probability of commitment. It will be interesting to see that when the government can make costly investment in the probability of commitment, how the choice depends on the current situation and how it affects debt accumulation and defaults. Another possible direction is to bring asymmetric

information into this model. Assume that the creditors do not have perfect information about the true probability of commitment of the government, and they act according to their expectations about the probability. The discrepancy between the true and the perceived probability of commitment could cause a vicious cycle of self-fulfilling expectations and recurrent defaults.

Chapter 2

Domestic Public Debt with Endogenous Default

2.1 Introduction

This paper offers a theoretical model of domestic public debt default in a dynamic stochastic general equilibrium framework. Instead of employing the common assumption that the government repays whenever possible, we allow the possibility that the government can strategically default on its domestic debt obligations. Strategic default has been modeled in papers that focus on external sovereign debt in small-open economies by authors such as (Eaton and Gersovitz 1981), (Aguilar and Gopinath 2006), (Arellano 2008) and (Yue 2010). However, few attempts have been made to introduce strategic default to models where the defaultable debt is held domestically.

There has been an increasing emphasis both in the academic literature and in the policy world on the importance of studying domestic debt accumulation and default crises.

A comprehensive framework is needed to help understand the interactions among domestic debt, default, tax policy and welfare. The reliance on internal financing as opposed to external has been rising over time, and this trend is particularly pronounced in medium- and low-income countries. The potential consequences of the development of domestic debt market in those countries are still not fully understood. At the same time, recent domestic default episodes, among them the ongoing European debt crisis, have also highlighted the relevance of having new models to study those crises and of developing strategies to avoid them. Although domestic defaults occur less frequently than external defaults over history, they are equally or even more important because of their impact on the macroeconomy, and also because many of them have triggered external defaults.

We model a government that levies income tax from domestic households and creates government consumption in a closed economy. The production side of the economy is simplified and the households receive a flow of stochastic taxable endowment. Volatile income flow implies volatile tax revenue, and therefore the government borrows from the households to smooth its expenditure. In bad times, where tax revenue could fall short of spending, the government issues new debt to cover its fiscal gap; in good times, the government may use tax revenue to retire part of its outstanding debt. For simplicity, the only bond that the government is able to issue is a one-period zero-coupon non-contingent bond, which cannot be traded outside the country. The households get utility from both private and government consumption.

The government has only limited commitment to repaying its debt. It fulfills its commitment to repay most of the time, but it can renege on its promise occasionally if it finds it is optimal to do so. This idea of sporadic reoptimization is meant to capture the

political frictions in the real world. One example of such frictions is that the households could delegate the decision to default to a conservative policymaker who would never default as long as he or she is in place, but the policymaker is subject to political turnovers that take place once in a while. A new policymaker who comes in may not feel the obligation to honor the promise made by his or her predecessor, in which case default could occur. In modeling the idea of limited commitment, we assume that whether or not the government has the option to deviate from its commitment is determined by a probabilistic event.¹ Note that when reoptimization becomes available to the government, default is an option, not an obligation — a government that has the option to default does not necessarily do so in equilibrium.

Default can be the optimal choice of a benevolent government under some circumstances. The government maximizes domestic welfare from private and government consumption, subject to its budget constraint. We allow the possibility that the government has a different (in particular, lower) discount factor than the households when discounting future utility flows, reflecting additional political frictions that make the decision horizon of the policymakers shorter than that of the households. The government is benevolent, in the sense that it defaults on its debt (when it has the option to do so) only if this implies a higher value for the households. Note that as opposed to sovereign debt models, the

¹The idea of limited commitment in monetary policy rule or fiscal rule has been studied by (Schaumburg and Tambalotti 2007) and (Debortoli and Nunes 2010), and both model the breaking of commitment as a probabilistic event. Wang (2015) uses this approach to model limited commitment in the sovereign debt market. The standard assumption in the sovereign debt literature, is that the sovereign cannot commit to repay, which is to say that the probability of reoptimization is one in any given period.

temptation to default does not come from the expropriation of foreign creditors, but rather from the reduction in debt and the interest payment associated with that debt. The costs of default, which we follow the literature on sovereign debt in modeling them, are output loss and temporary suspension from issuing new debt.

Households have rational expectations, and price debt according to the probability of repayment. Bond price decreases with the level of debt, reflecting a higher default premium. The model gives rise endogenously to a borrowing Laffer curve due to the endogenous bond price response, which puts an upper limit on the resources that the government can pledge. Bond price also depends on current level of endowment. Because endowment shocks have some persistency, periods with positive endowment shocks are likely to be followed by subsequent periods of high endowment. This shifts the borrowing Laffer curve in a favorable way and increases the bond price. The opposite is true for negative endowment shocks. This endogenous bond price movement allows the model to explain the empirical observation that default events usually coincide with periods when output is low.

We use this model to explain the recent Greek debt crisis in the same way as papers in the sovereign debt literature (for example, (Arellano 2008) and (Yue 2010)) explain the Argentinean sovereign debt crisis of 2001. For this purpose, we calibrate the model to the Greek economy prior to the crisis. In fact, our model is an extension of Arellano (2008) to the domestic debt market. One of the key quantitative properties of the model in Arellano (2008) is the countercyclicality of the trade balance – in particular, the fact that when output goes down, consumption goes down by even more because of capital outflow. The analog of the trade balance in our model is the primary balance, which also exhibits strong countercyclical behavior. Historically, Greece’s primary balance is strongly

negatively correlated with output, in contrast to most other advanced economies. The recent data developments are shown in Figure B.1. This empirical evidence makes the application of our model to Greece plausible. One potential caveat in our application, though, is that in our model, domestic debt corresponds to debt held by domestic residents. However, as noted in papers such as (Reinhart and Rogoff 2009) and (D’Erasmus and Mendoza 2014), domestic public debt data are hard to obtain. In particular, the breakdown of public debt in terms of the residence of holders is not always available or reliable. As a short-cut, we assume that all public debts in the data are held domestically. As we will show in the numerical exercise, our model is able to explain even this relatively high estimate of domestic debt.

The model performs well in the quantitative exercise. It supports equilibria with significant levels of debt with non-zero default risks and default events are frequently observed. Moreover, it matches key moments from the data – in particular, the correlations between trade balance and output, between interest rate spreads and output, and between private consumption and output. In the data, Greece’s GDP collapsed shortly after the global financial crisis and interest rate spreads jumped from almost zero to about 6% in the year of default. The model predicts a default and an interest rate spreads spike at the end of the sample, where the Greek government received the first bailout loan and was considered by the public to be in default. The parameter values needed to generate those quantitative predictions are plausible. The government discount factor is lower than that of the households, the commitment probability implies a moderate level of commitment, and the output cost of default is also in line with other default studies and empirical estimates for Greece.

The baseline model lends itself to several extensions. One illustrative extension shows that countercyclical tax policy is successful in countering the effects of business cycles fluctuations on government revenues. Compared to the baseline case, where tax rate is kept constant, the average interest rate spreads and the volatility of the economy are both much lower. However, the level of debt does not fall in any significant manner compared to the baseline case. This finding suggests that countries faced with less volatile income and revenue are more likely to sustain higher levels of domestic debt with less default and lower spreads. It also suggests that hedging against income shocks is likely to decrease default risks, but has less effect on the level of debt.

The paper is organized as follows. Section 2.2 discusses related literature. Section 2.3 sets out the model. Section 2.4 describes the equilibrium, and rewrites the optimization problem in a recursive manner. Section 2.5 examines the model's ability to explain the recent Greek crisis. Section 2.6 looks at an extension of the baseline model, where countercyclical tax policy is used to smooth the cycle of government revenue. Section 2.7 concludes the paper and discusses potential future research that may stem from this paper.

2.2 Related Literature

The main contribution of this paper is to propose a model of government strategic default on domestic public debt. It is thus related to the two strands of papers that model default in different settings: models on external sovereign debt default and those on domestic public debt default.

Modeling framework for explaining strategic defaults first appeared in models of external sovereign debt, in such papers as (Eaton and Gersovitz 1981), (Aguiar and

Gopinath 2006), (Arellano 2008) and (Yue 2010). In those papers, sovereign defaults arise endogenously as the optimal choice of a benevolent government which borrows from foreign creditors and uses lump-sum transfers to smooth consumption of the domestic households. Our model differs from those models in three major ways. First and most important, the creditors in our model are domestic households, not foreigners. This means that in terms of wealth, default is simply a transfer of wealth from some households to others, and the benefits of default no longer come from expropriation of foreigners. We assume that the government prefers to default because defaulting moves government consumption towards its first-best level, and this is particularly true when output is low and debt interest payments are high. Second, we introduce limited commitment of repaying the debt by the government into the model. All other papers in the sovereign debt literature either assume full commitment or no commitment, while we allow the degree of commitment to fall between the two extremes. This is not only a more realistic assumption, it also improves the quantitative performance of the model. Third, our model allows the government to have a different level of patience than the households, while in the sovereign debt models, the government and the domestic households are equally patient. As we will show, the level of patience has a great impact on the equilibrium level of debt and the likelihood of default.

This paper is also linked to papers that study domestic default in a close-economy context, such as (Bi 2012) and (Bi and Traum 2012). The major difference between our model and theirs is the focus on “unwillingness to repay” versus “inability to repay”. In their models, the focus is on the “inability to repay” problem. They introduce the concept of fiscal limit, defined as the maximum level of debt that the government is able to service. It exists because the tax rate may eventually climb up to the peak of the tax Laffer curve

where the tax revenue reaches its upper limit. This prompts forward-looking households to demand a higher default risk premium on debt, and forces the government to default today. In our model, even if the government has the capacity to service its debt, it could be unwilling to repay in some states of the economy, as default may lead to a higher value for the households. The inability to repay situation is also a possibility in our model, if the price of new debt is too low and the government cannot rollover the outstanding debt. However, the optimal choice of the government would be to default strategically before such a situation occurred.

The paper is also closely related to a recent strand of papers that proposes alternative theories of domestic sovereign default, among which the most recent are (D’Erasmus and Mendoza 2014) and (Pouzo and Presno 2014). The former paper presents a heterogeneous-agent model where the government defaults if the distributional benefits of default outweigh costs. Our paper, on the contrary, assumes a representative agent, and focuses on the role of public debt to smooth and front-load government consumption. Their paper also relies on a fiscal reaction function of the government that drives the supply of public debt, while in our model, the government optimally chooses the level of debt in each period. The paper by (Pouzo and Presno 2014) has similar settings as our paper. The difference lies in that they assume distortionary taxes and are interested in the joint determination of tax policy and default. However, their paper is not able to deliver quantitative features observed in the data for emerging economies but only qualitative. For example, the equilibrium level of debt in their paper is only around 6% of output.

Our model assumes that the benevolent government promises to repay but cannot fully commit to it, and it has the option to default in each period with some exoge-

nous probability. This idea of occasional reoptimization is related to (Schaumburg and Tambalotti 2007) and (Debortoli and Nunes 2010). Those papers study imperfect commitment in the context of monetary policy and fiscal policy respectively. Wang (2015) introduces the idea of limited commitment into the quantitative sovereign debt models and shows that limited commitment improves the empirical performance of the model in many ways.

2.3 Model

In this section we set up the model. The country is in a closed endowment economy, composed of two sectors: households and the government. The economy receives an exogenous income flow y_t in each period. y_t could be calibrated to simple stochastic process such as an AR(1) or to more complex structures. This stochastic income flow is the main source of uncertainty in the model, and there is no way for the economy as a whole to hedge against those income shocks.

Households budget constraint is

$$c_t + q_t b_{t+1} = (1 - \tau_t)y_t + (1 - \delta_t)b_t,$$

where c_t is private consumption of the homogeneous good. b_{t+1} is the holdings of one-period government domestic real bond that promises to pay one unit of good in the beginning of period $t + 1$. q_t is the unit price of the bond, whose value will be endogenously determined in the model. τ_t is the income tax rate. δ_t is a dummy variable which takes the value of 1 if the government defaults and 0 otherwise.

The government taxes income, and issues domestic debt b_t in order to smooth

government expenditure. It maximizes households' discounted sum of utility from private consumption and government expenditure

$$\max_{\{b_{t+1}, g_t\}} \sum_{t=0}^{\infty} (\beta_g)^t E_t[c_t + zv(g_t)],$$

subject to the government budget constraint

$$\tau_t y_t + q_t b_{t+1} = b_t + g_t,$$

and an exogenous borrowing limit

$$b_t \leq \bar{b}.$$

β_g is the government discount factor, which could have a different value than that of the households. $v(x) = \frac{x^{1-\sigma}}{1-\sigma}$ is a constant relative risk aversion (CRRA) utility function for government expenditure. z is a parameter that determines the relative importance of public consumption to private consumption in the utility function. We assume that the households are risk-neutral in private consumption for reasons that will be stated later. \bar{b} is exogenous and constant. In the numerical exercise, it is calibrated so that the constraint is rarely binding. The households welfare, can be expressed as

$$\sum_{t=0}^{\infty} (\beta)^t E_t[c_t + zv(g_t)].$$

If the government were perfectly committed to repaying its debt, there would be no default crisis in equilibrium. Only two equilibria would exist under the perfect commitment assumption: debt fluctuates around a steady-state level less than \bar{b} if the government is patient enough, or, debt is constant at \bar{b} if the government is less patient. In order to have equilibria with recurrent default crises, we relax the perfect commitment assumption. To avoid going to the other extreme assumption where the government is totally opportunistic,

we assume limited commitment. This means that, most of the time the government is constrained by its commitment to repay and does not have the option to default, but once in a while the constraint is relaxed and the government has the option to default. The government will default on its debt (when it has the option to do so) if the payoff from default is greater than that from repayment. When the government defaults, the outstanding amount of debt is reduced by the haircut, which is assumed to be exogenous and constant.

The idea of limited commitment is modeled in a simple but powerful way. In the model, whether the government has the option to default or not is determined by a probabilistic event. Define Λ_t as the binary variable that determines whether the government has the option to default. $\Lambda_t = 1$ indicates that the government is bounded by its commitment and repays the debt. $\Lambda_t = 0$ indicates that the government has the option to reoptimize and could renege on its promise to repay if needed. We use λ for the probability that $\Lambda_t = 1$.

The temptation to default comes from the fact that when debt is high, default moves government consumption towards the first-best level, defined as the level of government consumption that the social planner would choose in order to maximize households' utility. In order to have positive levels of government debt in the equilibrium, we also impose the condition that the first-best level of government consumption is higher than the maximum tax revenue that the government can pledge. The intuition behind is straight-forward. In a closed economy, domestic debt is simply a transfer of wealth from some households to others. Since the households are assumed to be identical in this paper, there is no wealth redistribution effect from default. If the first-best level of government consumption is less than the tax receipts, meaning that such government consumption is always feasible, the

government can simply make such government consumption without borrowing from the residents and debt will not exist. On the contrary, if the first-best level of government consumption is less than the tax revenues, the government would have an incentive to borrow and increase its public consumption.

Default is costly. During the period of default, output falls to y_t^{def} . We follow (Arellano 2008) in assuming that output in default takes the following form,

$$y_t^{def} = \begin{cases} \bar{y} & \text{if } y_t > \bar{y} \\ y_t & \text{if } y_t \leq \bar{y} \end{cases}.$$

The loss in output captures the negative impact of the financial disruption on the productive sector induced by government default, a reduced-form approach taken by the quantitative models on sovereign default. It is plausible that the pure output cost of default is likely to be larger in a domestic debt default event than an external one, possibly due to the macro-financial linkages in the economy.² Note that by employing a constant \bar{y} across all states, we implicitly assume an asymmetric default cost in the sense that default is more costly in good times compared to bad ones. It has been recognized in the literature that asymmetric default cost is key to generate realistic default frequency in equilibrium.³ Besides output loss, the government temporarily loses its ability to borrow after a default, representing the disruption in the domestic debt market associated with default. The government budget

²See (Bolton and Jeanne 2011) for a micro-founded model in which government debt is used as a collateral in private financial contracts and default reduces the efficiency of resource allocation in the private sector. Also see (Broner, Martin and Ventura 2010) for discussion of the consequences of domestic default and the role of secondary markets.

³See (Chatterjee and Eyigungor 2012) for a discussion of the role of asymmetric default cost and their more general specification that allows for a variety of cost functions.

constraint in default is

$$\tau y_t^{def} = g_t^{def}.$$

There is an exogenous probability θ that the government resumes borrowing and output loss discontinues after a default. The government must repay the domestic creditors what it has defaulted with a haircut in the period when it regains access to the domestic debt market.

Households are risk-neutral in private consumption and are willing to hold the bonds as long as they break even. They price government bonds according to

$$q(b_{t+1}, y_t) = \beta \left\{ \lambda + (1 - \lambda) \left[(1 - \pi_t) + \frac{\theta(1 - h)\pi_t}{\theta + \frac{1}{\beta} - 1} \right] \right\},$$

where $\pi_t = \pi(b_{t+1}, y_t)$ is the probability of default. The assumption of risk-neutrality in private consumption of the households significantly simplified the determination of bond prices by implying risk-neutral pricing. Otherwise, the risk premium would have an additional component that comes from risk-aversion in private consumption, making the solution of the model more complicated.

In the baseline model, tax rates are constant and non state contingent. So we have

$$\tau_t = \bar{\tau}.$$

In an extension of the baseline model, we relax this constraint and instead assume that the government implements a countercyclical tax policy to counter the effects of business cycles fluctuations on tax revenues. We also assume that there is no feedback loop from tax rate to output. In other words, tax is not distortionary. In the real world, higher taxes represent more fiscal tightening and could have contractionary effects on output. But we abstract from adding more complexity here.

2.4 Equilibrium

In this section we characterize the equilibrium and define the value functions. The model has two state variables at the beginning of period t : the level of outstanding government debt b_t , and the realization of stochastic endowment y_t .

Combining the budget constraints of the government and the households gives the market-clearing condition that output is equal to the sum of private consumption and government consumption, both in normal times,

$$y_t = c_t + g_t,$$

and in default times,

$$y_t^{def} = c_t^{def} + g_t^{def}.$$

We now write the continuation values for different regimes. We denote by $V^c(b_t, y_t)$ the value associated with repaying the debt and staying in contract, by $V^d(b_t, y_t)$ the value associated with default, and by $V^o(b_t, y_t)$ the value before the government makes the decision. We also substitute the expression for consumption using the market-clearing conditions.

If the government repays, the continuation value is

$$V^c(b_t, y_t) = \max_{b_{t+1}, g_t} \left\{ y_t - g_t + zv(g_t) + \beta_g \lambda E_t V^c(b_{t+1}, y_{t+1}) + \beta_g (1 - \lambda) E_t V^o(b_{t+1}, y_{t+1}) \right\},$$

subject to

$$\tau_t y_t + q_t b_{t+1} = b_t + g_t.$$

If the government defaults, debt is reduced by h and output drops to y_t^{def} . So the continu-

ation value under default satisfies

$$V^d(b_t, y_t) = (1 - \tau_t)y_t^{def} + zv(\tau y_t^{def}) + \beta_g \theta E_t V^c((1 - h)b_t, y_{t+1}) + \beta_g (1 - \theta) E_t V^d(b_t, y_{t+1}).$$

Whenever the government has the option to default, it compares the values of repayment and default, and chooses the one that has a higher value. So the value before the government makes the decision is

$$V^o(b_t, y_t) = \max \left[V^c(b_t, y_t), V^d(b_t, y_t) \right],$$

and

$$\delta(b_t, y_t) = 1 \iff V^c(b_t, y_t) < V^d(b_t, y_t).$$

The value before the realization of the commitment probability is

$$V(b_t, y_t, \Lambda_t) = \Lambda_t V^c(b_t, y_t) + (1 - \Lambda_t) V^o(b_t, y_t),$$

where Λ_t takes the value of 0 or 1 depending on the realization of the commitment shock as we have specified in the previous section.

2.5 Quantitative Results

2.5.1 Calibration

The model is calibrated by reference to the Greek data for the period of 2001-2010 at the annual frequency. This period corresponds to the time after Greece's adaptation of the euro and up to its recent sovereign debt crisis. The purpose of the exercise is to test the model's quantitative performance in two respects. First, whether the model is able to predict a default and an interest rate spike in the period when the Greek government

actually defaulted. Second, whether the model is able to replicate observed macroeconomic relations by matching data moments in the Monte-Carlo simulation. A model with good quantitative properties should be able to survive both tests using plausible function forms and parameter values.

The parameters in the model can be classified into two groups. The first group includes parameters that are calibrated independently of other parameters, whose values are reported in Table B.1. The annual risk-free interest rate is set to 4%, in line with the average long-term interest rates for the German bonds during the time. The household discount factor is 0.96, equal to the inverse of the risk-free interest rate factor. Output is assumed to follow an AR(1) process with mean 1. We estimate the stochastic process of output using Greece's GDP at constant price for a longer period that starts in 1995 and ends in 2013. The autocorrelation coefficient is 0.4553 and the standard deviation of the error term is 1.31%. The data for GDP, private consumption, and government consumption are detrended using the HP filter with a parameter of 6.25.

The fiscal variables are calibrated to match their real-world counterparts. Income tax in Greece is progressive, which begins from 0% and up to 45% depending on the income level. Since the households in our model are homogeneous, we use an average income tax rate of 20% as our model analog of the income tax rate. Note from the government budget constraint that government expenditure should fluctuate around tax revenue in the equilibrium. A tax rate of 20% implies an average government expenditure to output ratio close to 20% in the simulation, in line with the data. In the data, the average central government net revenue (total revenue minus tax refunds) to GDP ratio for the period of 2003 to 2011 is 21.3%. Debt is assumed to have an upper limit of 2, representing 200% of

mean output. Haircut on outstanding debt is set to 50%, consistent with the recent Greek experience from various estimates. The probability of resuming borrowing after default is assumed to be 0.25. It implies an average of four-year period of suspension of borrowing and post-default renegotiation. The number is plausible as the Greece debt crisis began in 2009 and is still ongoing. The probability of commitment is set to 0.9. It implies that on average, the government is expected to serve its debt obligations for ten consecutive years. Given that Greece only had one domestic debt default episode in the past century (1932-1951) and another one recently (2011-present), the value seems plausible. The value also implies that the maximum level of spreads that the model can generate is consistent with the spreads observed in the end of the sample.⁴

The second group contains parameters that will be jointly calibrated to match target statistics of the Greek economy. These parameters include the discount factor of the government, the government consumption utility parameter, the risk aversion coefficient for government consumption, and output in default.

There are two conditions that we impose on the parameter values. First, the government consumption utility parameter z and the risk aversion coefficient for government consumption σ combined imply that the first-best level of government consumption is 22% of total consumption, consistent with the observation from data. In other words, the following

⁴In the data outside our sample, interest rate spreads are higher than what the model is able to generate given the above calibration. However, since the focus of the quantitative exercise is to predict default given the short time series prior to default, such calibration is still plausible. In a more generalized model, higher spreads can be the result of a time-varying probability of commitment.

condition must hold for g^* equal to 0.22,

$$z(g^*)^{-\sigma} = 1.$$

Second, the simulated moments generated by the model should be as close to the data moments as possible. Let Ω be the set of the four parameters to be jointly calibrated. Ω is chosen to minimize the loss function defined as

$$\text{Loss}(\Omega) = [(M(\Omega) - M(\text{Data}))/M(\text{Data})]W^*[(M(\Omega) - M(\text{Data}))/M(\text{Data})]',$$

where $M(\text{Data})$ are the moments from the data, $M(\Omega)$ are the moments generated by the model, and W^* is the weighting matrix. This is intended to minimize the sum of the percentage deviations of the simulated moments from their data counterparts. We use the identity matrix as the weighting matrix. Given the small sample size in our model, statistical methods are much limited in deriving the optimal weighting matrix.

The data moments we choose to match are the frequency of default, the correlation between spreads and output, the correlation between primary balance and output, and the correlation between private consumption and output. These moments contain useful information from the data. The frequency of default is calibrated to 2.2% given the two default episodes in the past one-hundred years. The interest rate spreads are constructed by taking the difference between the 10-year government bond yields for Germany and for Greece.⁵ Primary balance is represented as percentage of GDP. As discussed before,

⁵We look at long-term interest rates because the amount of debt securities we use to calculate the debt-to-output ratio covers long-term bonds and notes and money market instruments placed on domestic markets. Other quantitative papers that study the Greek debt crisis such as (Bi and Traum 2012) and (Grauwe and Ji 2012) also look at 10-year government bond yields. In the sovereign debt literature, Arellano (2008) also aims to match the average long-term (5 year) bond yields.

both the interest rate spreads and primary balance exhibit strong countercyclicality. The correlation between spreads and output is -0.6262 and that between primary balance and output is -0.2029.⁶ Private consumption is positive correlated with output with a relatively low correlation of 0.7402. The low correlation suggests that our assumption that households are risk-neutral in private consumption is plausible.

2.5.2 Solution and Simulation

The model is solved by value function iterations using the discrete-state-space (DSS) technique, which is also used in other default studies. The discontinuity in the choice variable of the optimization problem prohibits the use of the perturbation method often employed in other DSGE models. We use a one-loop algorithm that iterates on the value function and bond price function simultaneously. Appendix B.1 provides a detailed description of the computational algorithm in this paper.

The model economy is simulated 100 times, each with 10500 periods. The first 500 periods are discarded to eliminate the effects of initial conditions. To construct the model analog of the default statistics, we extract time series that satisfy the following criteria: 1) the sample has 10 periods, which has the same length as the times series in the data; 2) a default is declared at the end of each sample; 3) the last exclusion period is observed at

⁶In the data, the correlation between primary surplus and output for the period of 2001-2010 is 0.0070. However, this seemingly weak correlation does not capture the big picture. The correlation is close to zero purely because both primary balance and output dropped significantly the period before the crisis. If we exclude the last observation from the data sample, the correlation becomes -0.2118, and if we further exclude the last two observations, the correlation becomes -0.6910. A closer look at the Greek data over a longer horizon confirms that the primary balance and output are strongly negatively correlated in Greece.

least two periods before the beginning of the sample. We calculate business cycle moments for each simulated sample, and then take an average across all samples to obtain the model simulated moments.

Table B.2 displays parameter values and simulated moments generated by the model. The discount factor of the government is 0.6546, lower than the discount factor of the households. The low discount factor can be interpreted as taking account of the probability that policymakers will be replaced in each period. Note that although the value seems low, it implies a quarterly discount factor of 0.9, a value that is higher than in most of the sovereign debt default models.⁷ The government consumption utility parameter is 0.2034 and the risk aversion coefficient for government consumption is 1.0525. The output in default is 0.957, suggesting that the country loses about 4% of output during a default. It is in line with papers on sovereign debt default where default costs range from 2% to 6% of output. Some researchers suggest, however, that the cost of domestic debt default is greater than that of external debt default.⁸ As a reality check, the Greek economy is on average 12.9% below trend for the three years' period from 2012 to 2014.

The model does well in matching the data in that it simultaneously delivers the frequency of default and the correlation structures all close to the data. As we have dis-

⁷The quarterly discount factors in the quantitative sovereign debt models range from 0.73 to 0.953.

⁸Although the empirical literature does not give a precise estimate for the cost of a default, various sources suggest that the cost is large. For example, (Borensztein and Panizza 2009) empirically evaluates four types of cost that may result from an international sovereign default. They find that growth falls by 2.6% in the first year of a default episode. Also, they do not find statistically significant evidence that output catches up after a default, suggesting that the cumulative cost of default is much higher than the cost observed in the first year.

cussed, Greece’s primary balance exhibited strong countercyclical before the crisis, which makes the application of our model plausible. The average percentage deviation of the four simulated moments from their data counterparts is 11%. Other non-targeted moments are also in line with the data. We use the general government debt outstanding to GDP ratio as the data analog of the measure of domestic indebtedness. In the data, the average debt-to-output ratio is 126.02%, and the average interest rate spreads is 0.95%. The model generates an average debt-to-output ratio of 152.68% and an average interest rate spreads of 2.54%. Given the fact that in the data, the interest rate spreads were almost zero throughout the sample and jumped only at the end of the sample, and that the level of debt rose significantly over time, we do not target the average spreads and debt ratio, but will test the model by trying to reproduce the dynamics over the whole sample period. However, the model underestimates the standard deviation of bond spreads and that of the standard deviation of primary balance. Both are about half of their data counterparts.

We then examine the model’s predictive power. This is done by feeding into the model Greece’s GDP time series starting in 2001. The model predicts a default in 2010, the period when the Greek government defaulted. Figure B.2 compares the time series of output, interest rate spreads and debt-to-output ratio both in the data and in the model. The model generates an interest rate spreads hike of 6% with the exact timing correct. It also replicates the “tranquil” period before the crisis, where the interest rate spreads were almost constant at zero. In terms of the debt-to-output ratio, the model generates a gradual increase in the ratio during the period of interest, and predicts more than 96.6% of the data counterpart in the year of default.

To further understand the model properties, we examine the policy functions and

value functions of the model. Figure B.3 plots the bond price as a function of debt when output is at its mean. The bond price is constant at the inverse of the risk-free rate when debt is low because there is no default risk. As the level of debt goes up, bond price drops sharply. The price becomes a constant again when debt is above 150% of output, simply because default is solely determined by the commitment probability which is constant. Figure B.4 shows the value functions against the level of debt conditional on mean output. Both the value of repayment and the value of default are concave and decreasing in the level of debt. The value of default V^d is much less concave than the value of repayment V^c , because default incurs a haircut and delays repayment to a future period.

Figure B.5 shows the possible default region where the government prefers default to repayment. We call it “possible default region” instead of “default region” because when the economy is in this region, whether default will take place or not depends on the probabilistic event that gives the government the option to default. Other things equal, the government prefers default when debt is high and output is low. The default frontier is steep, allowing default to endogenously happen in equilibrium. The steepness in the default frontier is a result of the asymmetric default cost in our calibration. The literature has shown that an asymmetric default cost that makes default relatively more costly during good times generates a steep default frontier and realistic interest rate spreads.

2.5.3 Sensitivity Analysis

In this section, we examine how the model’s predictions change when the values of key parameters are altered. We look at the three parameters: the discount factor of the government, the probability of commitment, and the cost of default.

Table B.3 presents the results when the government is assumed to be more patient than in the baseline calibration. In the first case the β_g is assumed to be 0.8 and in the second one β_g is equal to 0.96. The latter value is almost identical to the discount factor of the households. As the table shows, when the government becomes more patient, the average spreads drop sharply. In the extreme case where the government is as patient as the households (not reported in the table), default risk vanishes and average spreads go to zero. The average debt-to-output ratio drops from 153% to 69% when the government becomes almost equally patient to the households. Not surprisingly, the volatility in the economy also decreases, as defaults occur less often.

We also consider the effects of varying the probability of commitment, the results of which are shown in Table B.4. In the first case λ is assumed to be 0, meaning that the government has no commitment in repaying its debt, an assumption that is often made in the sovereign debt literature. In this case, default frequency and average spreads are much lower. The level of debt is also lower but is still significantly above zero. In the second experiment, the government has a higher probability of commitment of 0.95. The average spreads and average debt-to-output ratio are both much higher than in the baseline case. Debt increases monotonically with the probability of commitment for two reasons. First, the possible default region is smaller when the probability of commitment is higher. So, conditional on being outside the possible default region, a higher probability of commitment means more debt is likely to be sustained. Second, because the government may not have the option to default at all times, the economy could stay in the possible default region which is associated with higher levels of debt. Average spreads are higher because there is an inverse U-shape relationship between the likelihood of default and the probability of

commitment, as explored in more detail in (Wang 2015).

In the third experiment we examine the role of default output cost by setting it to be between 1 and 2 percent higher or lower than in the baseline calibration. The results are summarized in Table B.5. A lower default cost (represented by higher \bar{y}) strengthens default incentives and yields higher spreads. Bond spreads and primary balance are both more volatile than in the baseline case. A higher default cost (lower \bar{y}) has the opposite effect on the incentive for default, therefore supporting lower spreads and lower volatility. In both the higher and lower default cost cases, the average debt-to-output ratio is higher than in the baseline. When default cost is high, the possible default region is smaller, implying higher debt levels in equilibrium. When default cost is low, the possible default region is larger. Given the fact that the probability of commitment is positive and debt can continue to grow even if the economy enters the possible default region, the average debt level that the model can sustain is higher.

The above results suggest that the combination of parameter values in the baseline calibration are important to ensure that the model's predictions are close to the data.

2.6 Extension

This section extends the baseline model by relaxing the assumption that income tax rate is constant and non state-contingent. Figure B.6 offers a scatterplot of the implied average tax rate (calculated as government tax revenue divided by GDP) against the cyclical component of real GDP for Greece during the period of 2001-2010. A simple linear regression suggests that the slope coefficient is -0.0431, negative but close to zero. The correlation between the two variables is -0.1320. This is consistent with empirical evidence in the

literature. For example, using a much longer and more comprehensive dataset, (Végh and Vuletin 2012) found that the government revenue-to-GDP ratio and real GDP for Greece are negatively correlated at -0.25, a number that is much lower than that of other countries.

The constant tax rate assumption in the baseline model is consistent with the near-zero slope coefficient from the simple regression, and it allows us to focus on the fundamental source of the volatility in the economy – output volatility. However, it implies that output volatility translates one-to-one to tax revenue volatility. Given the limited role of domestic borrowing in smoothing government consumption, volatile tax revenue necessitates taking on more debt and more defaults.

The question naturally arises whether reducing tax revenue volatility and fluctuations could reduce the frequency of default as well as the level of debt. In the following experiment, we assume that the government implements a countercyclical tax policy to counter the effects of business-cycle fluctuations on its revenues. In particular, we assume that tax rates move negatively with output in a linear fashion. We look at two different degrees of countercyclicality. In the first case, the tax rate is 1.45 percentage points higher (lower) than its medium if output is at its lowest (highest) possible value. In this case, the volatility in tax revenue is almost completely eliminated. In the second case, the tax rate is 2 percentage points higher (lower) than its medium if output is at its lowest (highest) possible value. In this case, the sign of the correlation between tax revenue and output is flipped compared to the baseline case. Figure B.7 shows a comparison of tax rates and tax revenues for the three cases.

We conduct Monte-Carlo simulations on the model with countercyclical tax rates. To allow a fair comparison, the parameter values are the same as in the baseline model.

Table B.6 compares the simulated model moments of the three cases. With countercyclical tax policy, average spreads and default risks drop quite significantly. Average spreads are close to zero, reflecting an almost zero default probability. Interestingly, the levels of debt are only slightly lower than the baseline case. Interest rate spreads and the primary balance are negatively correlated with output, similar to the baseline case. The standard deviations of bond spreads and primary balance are also much lower as default risks are low. Figure B.8 shows the predictions the model produces with the more aggressive countercyclical tax rates when we feed the GDP time series into the model before the crisis. The results for the less aggressive countercyclical tax rates are very similar. The model is unable to predict the jump in the spreads at the end of the sample. Figure B.9 compares possible default regions for the two models. They are almost identical.

The comparison of the three cases has the following implications. First, everything else being equal, countercyclical tax policy helps sustain high levels of domestic debt with less frequent defaults and low interest rate spreads. This suggests that key devices determining the external indebtedness capacity of economies could also explain their domestic default risks and the level of debt tolerance. “Debt intolerance” is a term coined by Carmen Reinhart, Kenneth Rogoff and others. It refers to the inability of emerging markets to manage levels of external debt that under the same circumstances, would be manageable for developed countries. Because they are more exposed to shocks, emerging economies tend to have more volatile income flow than advanced economies. The higher volatility increases the risk premium required when borrowing in external markets to insure against risks. The same mechanism could also be true in domestic debt markets.

Second, the government is unlikely to reduce its debt even when faced with less

volatile revenue. This is because an impatient government finds it optimal to front-load its consumption at the cost of future higher interest payment and default. In this case, debt can only be reduced through defaults. But such debt reduction is only temporary. The model predicts that, after a default, debt continues to grow until the government defaults again. As we have shown in the sensitivity analysis, a more patient government borrows much less and defaults much less often. However, these developments are usually slow in nature and difficult to implement. This suggests that there are many challenges to bringing down debt to sustainable levels in highly-indebted countries, and helping them stay permanently away from default.

2.7 Conclusion

This paper proposes a theoretical model of a government's strategic default on domestic debt in a closed economy. In the model, the government borrows from domestic residents to smooth and front-load its expenditures. The government is benevolent in the sense that it maximizes the discounted future utility of households from private and government consumption. The government has limited commitment to repaying its debt, and whether or not it has the option to default is determined by a probabilistic event. When given the option to default, the government defaults if the benefit of default outweighs its cost. The benefit comes from the fact that default reduces interest payment on outstanding debt, and moves government consumption up towards its first-best level. The cost of default is output loss and exclusion from issuing new debt. Domestic residents who lend to the government price the bond on the basis of the likelihood of repayment. Bond prices are endogenous to the model, which gives rise to a borrowing Laffer curve and endogenous

borrowing limit.

This numerical exercise shows that the model successfully explains Greece's recent default episode. It predicts not only a default but also a hike in interest rate spreads in the period when the Greek government defaulted. The model matches key moments in the data, such as the cyclicalities of interest spreads, the cyclicalities of primary balance, the correlation between private consumption and output, and the frequency of default. All of these are achieved with plausible parameter values. In the sensitivity analysis where the parameter values are altered, we find that the combination of parameter values in the baseline calibration is important in delivering realistic simulated moments.

We examine the role of countercyclical tax rates in reducing domestic debt default risks in an extension of the baseline model. This simple exercise assumes that tax rates move negatively with the level of output in a linear way, and can completely eliminate revenue fluctuations in some case. Our simulation exercise shows that the average interest rate spread is noticeably lower than in the baseline scenario. However, the level of debt remains roughly the same. This suggests that lower income volatility helps to reduce the frequency of default but not the level of debt.

There are important caveats to keep in mind when applying this model to a broader set of domestic default episodes, however. First, unlike external sovereign debt default, domestic default entails substantial redistribution across domestic agents. A representative agent model like ours is unable to explore the redistributive effects of domestic debt default. Second, to get a precise measure of the benefit and cost of default we need a good measure of how much debt is held by domestic residents. This is difficult because of various data constraints. In this paper we abstract from modeling debt that can be held partly

externally and partly domestically, and leave this generalization to future work. Third, in reality governments can resort to inflation as a means to reduce the real value of debt. Given that Greece is in the Eurozone and does not have control over its own monetary policy, there is small probability that the Greek government can resort to inflation to reduce its debt. However, when this model is applied to other countries with independent monetary policy, it would be more realistic to incorporate inflation as another way of reducing debt.

Our model can be extended in many other ways to cover a range of interesting topics. For example, the production side of the economy can be enriched to allow more tax instruments, including labor and capital taxes, to come into play. The feedback loop from tax to output could also be modeled. Future research could also delve further into the optimal tax policy under this environment to allow for more interesting interactions between debt, tax and default.

Chapter 3

Endogenous Gridpoints Method in Quantitative Sovereign Debt Models

3.1 Introduction

This paper adapts the endogenous gridpoints method to solve quantitative sovereign debt models. These models study sovereign debt and default in a dynamic stochastic small open-economy environment where a government optimally decides to borrow or default on behalf of the households. Previous important work in this literature includes (Arellano 2008), (Aguiar and Gopinath 2006) and (Yue 2010). These models have shown their abilities in explaining certain empirical regularities, in particular, recurring debt crises, countercyclical interest rate spreads and countercyclical trade balance.

Computation time is often a binding constraint that prevents expanding sovereign

debt models to study more interesting issues. Several key features make these models numerically challenging to solve. First, those models are based on iterative, infinite-horizon optimization with both discrete and continuous choices. In these models, the issuer of the debt in the model – the government – is conceived as a benevolent player who maximizes the discounted future utility of the households from consumption. The government has two choices to make, a discrete choice of whether to default or not, and a continuous choice of how much new debt to issue if it does not default. Both the government and the households live forever, so the horizon for the optimization problem is infinity. The second challenge comes from the fact that bond prices are endogenous to the model. Since debt is defaultable and foreign creditors who supply funds are rational, default risk will be reflected in lower bond prices. The stochastic nature of the model then requires calculating the probabilities of default and feeding them into bond prices.

Due to those characteristics, we cannot use the classical first-order perturbation method employed in many dynamic stochastic general equilibrium (DSGE) models. The perturbation method requires policy functions of the model to be smooth and differentiable, which is certainly not the case in sovereign debt models. In sovereign debt models, the decision to default or not is modeled as a discrete choice variable that takes the value of either 1 or 0. The jump in the choice variable prohibits the use of first-order Taylor expansion embedded in the perturbation method. Not only that, the perturbation method does not compute the value functions. In sovereign debt models, value functions are important because the decision to default or not is determined by comparing the value of default with the value of repayment.

In the literature, sovereign debt models are solved by value function iterations

on predetermined discrete gridpoints. This method is straightforward but computationally expensive. This is because the optimization problem needs to be solved on each gridpoint in the state space in every round of iteration before convergence. Obtaining precise solutions to the optimization problem requires a root-finding procedure to locate the level of new borrowing that equals the marginal benefit of borrowing today to the marginal cost of having to repay the debt tomorrow. As rootfinding procedures are generally computationally expensive, many authors use the so-called discrete-state-space (DSS) method to approximate the optimal policy. The idea is to discretize the choice space into gridpoints and do a gridsearch to find the maximizer. As shown in (Hatchondo et al. 2010), DSS is imprecise unless a great number of gridpoints is used, usually in a multiple of thousands, in which case the method becomes very slow and the computation time increases dramatically with the dimension of the model.

The endogenous gridpoints method (EGM), developed by (Carroll 2006), is a method for solving numerical dynamic stochastic optimization problems that avoids rootfinding operations. The strategy here is to begin with end-of-period assets and to use the end-of-period assets to back up the begin-of-period assets. The key distinction between the EGM and the standard method used in the literature is that in EGM the gridpoints for the begin-of-period assets are not predetermined; instead they are endogenously generated from a grid of values of end-of-period assets. The benefit of this approach, is that it is relatively easy to calculate the begin-of-period assets from the end-of-period assets without resorting to the rootfinding procedure. The EGM has succeeded in improving computation efficiency in precautionary savings models. Since sovereign debt models share many features with precautionary savings models, the EGM becomes a natural candidate for more

efficient solution methods.

Having said that, it is not simple to extend the endogenous gridpoints method to quantitative sovereign debt models. The first challenge comes from handling the mapping from end-of-period assets to begin-of-period assets. Since bond prices are endogenous to the model, some end-of-period asset levels are never the solution to the optimization problem because borrowing at those levels implies net resource outflow and the borrower would be better-off not to borrow at all. This creates the need for some selection mechanism to pick the levels of end-of-period assets that are the true optima, which we can then use to backup the begin-of-period assets. The second challenge is that the levels of end-of-period assets affect the prices that the borrower is paying for them; therefore the interest rate that connects one period to another cannot be taken as a given constant.

As a contemporaneous work, (Villemot 2012) addresses the first challenge mentioned above by introducing an additional step in the algorithm that iterates on the set of ergodic set as iterations run; and hence the suggested name of “doubly endogenous grid method” (2EGM). Our algorithm differs from his in the following ways. First, we assume an asymmetric output cost of default, which leads to a much wider ergodic set so that the EGM can be applied without iterating on the ergodic set. The assumption of asymmetric cost has become a common modeling technique in the sovereign debt models to induce realistic default frequencies. Second, we use much finer grids, so that we almost always have enough gridpoints in the ergodic set to conduct an interpolation.

The way that our algorithm addresses the second challenge is also different from (Villemot 2012). As explained before, in this model the interest rates (or bond prices) depend on the level of borrowing. Since the algorithm makes use of first-order conditions,

one key step is to determine the marginal effects on the price of debt of an additional new borrowing. While (Villemot 2012) takes numerical differentiations directly on the bond-price functions, we show through mathematical derivation that borrowing affects the bond price through its marginal effects on the values of repaying and default, and we use the partial derivatives of the value functions to find the marginal effects of new borrowing.

This paper demonstrates how the endogenous gridpoints method can be adapted to solve the quantitative sovereign debt models. We also improve our algorithm by using a one-loop algorithm that iterates simultaneously on the bond price function and the value functions. Using the canonical sovereign debt model in (Arellano 2008) as an example, we show that the solutions resulting from the two methods are very similar. However, there does not seem to be any gain in computation time.

The rest of the paper is structured as follows. Section 3.2 describes the model we are solving. Section 3.3 revisits the current solution method in the literature. Section 3.4 lays out the theory of the endogenous gridpoints method. Section 3.5 specifies the recursion from the initial condition and how to proceed from one iteration to another. Section 3.6 tests the EGM against the DSS method, and compares them both in terms of speed and accuracy. Section 3.7 concludes the paper.

3.2 The Problem

The problem we are solving is the canonical quantitative sovereign debt model in (Arellano 2008). Consider a benevolent government in an endowment economy that borrows from foreign creditors by issuing bonds and then makes lump-sum transfers to the households. The government's objective is to maximize discounted sum of future utility of

the households from consumption

$$E_t \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to the budget constraints

$$c_t + q_t b_{t+1} \leq y_t + b_t,$$

and

$$b_{t+1} \geq \underline{b}.$$

b is the government's asset holdings. Positive b means saving and negative b means debt.

Debt is assumed to be defaultable, one-period and non-contingent. We assume that $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$.

The government has the option to default in every period, and the decision is made after it observes the realization of endowment y_t . If the government defaults, it defaults on all its outstanding debt. The cost of default is modeled as an output loss and exclusion from the international financial market for a stochastic number of periods. If the government makes repayment, it remains in the financial market and can choose its new borrowing level b_{t+1} with the foreign creditors.

Foreign creditors are risk neutral and rational. The price of new bond b_{t+1} priced at time t is

$$q_t = \frac{1 - \pi_t}{1 + r},$$

where π_t is the probability of default at time $t + 1$ viewed at time t and r is the risk-free interest rate. This condition says that the price of the bond is equal to the expected return of the bond.

The decision to default or not is determined by comparing the value of default and not default. Define $V^o(b_t, y_t)$ as the value of the objective function of the government with the option to default. It is the maximum of the two choices

$$V^o(b_t, y_t) = \max_{c,d} \{V^c(b_t, y_t), V^d(y_t)\}.$$

The value of making repayment and not defaulting is given by

$$V^c(b_t, y_t) = \max_{b_{t+1}} \left\{ u(y_t + b_t - q_t b_{t+1}) + \beta E_t V^o(b_{t+1}, y_{t+1}) \right\}. \quad (3.1)$$

The value of default is

$$V^d(y_t) = u(y_t^{def}) + \beta E_t [\theta V^o(0, y_{t+1}) + (1 - \theta) V^d(y_{t+1})],$$

where y^{def} captures the output loss of default, modeled as a fraction $(1 - \alpha)$ of output absent of default. θ is the probability that the economy regains access to international financial markets after a default. The value of default does not depend on the current level of debt b_t because we assume a haircut of 100%.

Government's default policy is characterized by the default set. Let $\mathcal{D}(b)$ be the set of y for which default is preferred to repayment. $\mathcal{D}(b)$ is defined as

$$\mathcal{D}(b) = \{y \in Y | V^c(b, y) < V^d(y)\}. \quad (3.2)$$

By definition, for each b , the repayment set is the complement set of the default set. Using the default set, bond pricing function can be written as:

$$\begin{aligned} q(b_{t+1}, y_t) &= \frac{1}{1+r} \Pr[y_{t+1} \notin \mathcal{D}(b_{t+1}) | y_t] \\ &= \frac{1}{1+r} \Pr[V^c(b_{t+1}, y_{t+1}) \geq V^d(y_{t+1}) | y_t]. \end{aligned} \quad (3.3)$$

Finally, we assume that endowment follows a log-normal AR(1) process with mean 1,

$$\log y_{t+1} = \rho \log y_t + \varepsilon_{t+1}, \quad (3.4)$$

in which ε follows an i.i.d. normal distribution with mean 0 and variance η^2 .

3.3 Current Solution Method

The absence of a closed-form solution in the optimization problem presented in Equation (3.1) means that optimal policy functions must be constructed by calculating their values at a finite grid of possible values of the state variables. An interpolation or extrapolation is then used to cover the whole state space. Since interpolation or extrapolation is a computationally expensive operation, a majority of the papers in the literature, for example, (Aguiar and Gopinath 2006), (Arellano 2008), and (Yue 2010), use the discrete-state-space (DSS) method to solve their models.

The DSS method discretizes the state space along all dimensions, and restricts the choice and state variables to be on those pre-determined gridpoints. We use the model described in Section 3.2 to illustrate how the method works. In the model the two state variables are endowment y and debt b , and the choice variables are default and next period's debt holdings b' . We discretize endowment y into N_y gridpoints and debt b into N_b gridpoints, which gives us a total of $N_y N_b$ gridpoints on the state space. For each point on the state space, we solve the optimization problem by choosing the new borrowing level b' from the same N_b gridpoints for b such that the value in Equation (3.1) is the highest. After that we compare the value of default with the value of repayment and decide on the

discrete choice of default.

This method is straight-forward and easy to implement but computationally expensive. For each gridpoint on the state space in each iteration before convergence, we need to compute and loop over the N_b values to find the solution to the maximization problem. The difficulty increases with the dimension of the model and the number of gridpoints in each dimension.

The DSS method poses great computation cost to the quantitative sovereign debt models, and the problem is exacerbated by the fact that the authors use an algorithm with two loops: the inner loop iterates on value functions for given bond price, and the outer loop iterates on bond price. The model is not solved until two iterations both converge. In fact, as (Hatchondo et al. 2010) show, the algorithm can be further optimized by using a one-loop algorithm that iterates on the bond price function together with the value functions.

3.4 Theory

3.4.1 The Usual First-order Condition

The first-order condition for Equation (3.1) with respect to b_{t+1} is

$$u'(y_t + b_t - q_t b_{t+1}) \frac{\partial(q_t b_{t+1})}{\partial b_{t+1}} = \beta \frac{\partial E_t V^o(b_{t+1}, y_{t+1})}{\partial b_{t+1}}. \quad (3.5)$$

By definition,

$$\begin{aligned} E_t V^o(b_{t+1}, y_{t+1}) &= E_t[V^c(b_{t+1}, y_{t+1}) | V^c(b_{t+1}, y_{t+1}) \geq V^d(y_{t+1})] \\ &+ E_t[V^d(y_{t+1}) | V^c(b_{t+1}, y_{t+1}) < V^d(y_{t+1})]. \end{aligned}$$

For any given debt position b_{t+1} , we can find the level of endowment at time $t + 1$ that

makes the government indifferent between defaulting and not defaulting, and we call it the default threshold y_{t+1}^{thr} ,

$$V^c(b_{t+1}, y_{t+1}^{thr}) = V^d(y_{t+1}^{thr}). \quad (3.6)$$

Then the expected value of having the option to default can be written as

$$E_t V^o(b_{t+1}, y_{t+1}) = \int_{-\infty}^{y_{t+1}^{thr}(b_{t+1})} V^d(z) dF_{y_{t+1}|y_t}(z) + \int_{y_{t+1}^{thr}(b_{t+1})}^{+\infty} V^c(b_{t+1}, z) dF_{y_{t+1}|y_t}(z),$$

where $F_{y_{t+1}|y_t}(\cdot)$ is the cumulative density function for y_{t+1} conditional on y_t .

Note that the choice variable b_{t+1} appears in both the integrand and the limits of integration. Applying Leibniz integral rule to the above equation, we get

$$\begin{aligned} \frac{\partial E_t V^o(b_{t+1}, y_{t+1})}{\partial b_{t+1}} &= \frac{\partial y_{t+1}^{thr}(b_{t+1})}{\partial b_{t+1}} V^d(y_{t+1}^{thr}) - \frac{\partial y_{t+1}^{thr}(b_{t+1})}{\partial b_{t+1}} V^c(b_{t+1}, y_{t+1}^{thr}) \\ &\quad + \int_{y_{t+1}^{thr}(b_{t+1})}^{+\infty} \frac{\partial V^c(b_{t+1}, z)}{\partial b_{t+1}} dF_{y_{t+1}|y_t}(z). \end{aligned}$$

The first two terms cancel out each other from Equation (3.6) so we are left with

$$\frac{\partial E_t V^o(b_{t+1}, y_{t+1})}{\partial b_{t+1}} = \int_{y_{t+1}^{thr}(b_{t+1})}^{+\infty} \frac{\partial V^c(b_{t+1}, z)}{\partial b_{t+1}} dF_{y_{t+1}|y_t}(z).$$

Substituting it into the first-order condition in Equation (3.1) yields

$$u'(y_t + b_t - q_t b_{t+1}) \frac{\partial(q_t b_{t+1})}{\partial b_{t+1}} = \beta \int_{y_{t+1}^{thr}(b_{t+1})}^{+\infty} \frac{\partial V^c(b_{t+1}, z)}{\partial b_{t+1}} dF_{y_{t+1}|y_t}(z).$$

The economic intuition behind the above equation is straight-forward. The optimal level of new borrowing b_{t+1} should be such that the marginal utility today from issuing an additional unit of debt equals the expected marginal disutility tomorrow from having to repay it. Since the country has the option to default and debt is completely wiped out, the disutility comes only in the states where the government repays, which explains why we only integrate over those states where tomorrow's endowment is higher than the default

threshold. This equation will be used repeatedly in the iterations to solve the government's optimization problem.

To facilitate our analysis, it is convenient to define two functions. First, define $\mathbb{V}(b_{t+1}, y_t)$ as the discounted value of having the option to default in the next period.

$$\begin{aligned}\mathbb{V}(b_{t+1}, y_t) &\equiv \beta E_t V^o(b_{t+1}, y_{t+1}) \\ &= \beta \left(\int_{-\infty}^{y_{t+1}^{thr}(b_{t+1})} V^d(z) dF_{y_{t+1}|y_t}(z) + \int_{y_{t+1}^{thr}(b_{t+1})}^{+\infty} V^c(b_{t+1}, z) dF_{y_{t+1}|y_t}(z) \right).\end{aligned}$$

The partial derivatives are

$$\mathbb{V}'_b(b_{t+1}, y_t) = \beta \frac{\partial E_t V^o(b_{t+1}, y_{t+1})}{\partial b_{t+1}},$$

and

$$\mathbb{V}'_y(b_{t+1}, y_t) = \beta \frac{\partial E_t V^o(b_{t+1}, y_{t+1})}{\partial y_t}.$$

Second, define $\mathbb{V}^d(y_t)$ as the discounted value of default in the next period.

$$\begin{aligned}\mathbb{V}^d(y_t) &= \beta E_t V^d(y_{t+1}) \\ &= \beta \int_{-\infty}^{+\infty} V^d(z) dF_{y_{t+1}|y_t}(z).\end{aligned}$$

With our newly defined equations, we can rewrite the first-order condition as

$$u'(y_t + b_t - q_t b_{t+1}) \frac{\partial(q_t b_{t+1})}{\partial b_{t+1}} = \mathbb{V}'_b(b_{t+1}, y_t). \quad (3.7)$$

In the two subsections followed, we will discuss respectively how we get the two terms

$$\frac{\partial(q_t b_{t+1})}{\partial b_{t+1}} \text{ and } \mathbb{V}'_b(b_{t+1}, y_t).$$

3.4.2 Endogenous Bond Price

A salient feature of quantitative sovereign debt models is that the interest rates that the country is facing are endogenously determined by future probabilities of repaying.

Two things affect the likelihood of repaying in the next period: first, the level of debt that the country has to repay tomorrow, which is determined by how much new debt it chooses to issue today; and second, the level of endowment tomorrow. Because endowment follows an AR(1) process, we can form a distribution of tomorrow's endowment based on today's endowment. Bond price is thus a function of today's new borrowing b_{t+1} and endowment y_t .

$$\frac{\partial(q_t b_{t+1})}{\partial b_{t+1}} = \frac{\partial q(b_{t+1}, y_t)}{\partial b_{t+1}} b_{t+1} + q_t.$$

The above equation says that issuing an additional unit of debt has two opposite effects on the resource that the borrower can get. First, the borrower gets q_t because it borrows more. Second, an additional unit of debt increases the likelihood of default in the next period and decreases the bond price by $\frac{\partial q(b_{t+1}, y_t)}{\partial b_{t+1}}$. Because the drop in price applies to every unit of debt that the borrower issues, the total resource that the borrower can pledge is lowered by $\frac{\partial q(b_{t+1}, y_t)}{\partial b_{t+1}} b_{t+1}$. The sum of the two opposite terms can be positive or negative, depending on their magnitudes.

The bond price function can be written as

$$\begin{aligned} q(b_{t+1}, y_t) &= \frac{1 - \pi_t}{1 + r} \\ &= \frac{1 - \Pr[V^c(b_{t+1}, y_t) < V^d(y_t)]}{1 + r} \\ &= \frac{\Pr[V^c(b_{t+1}, y_t) \geq V^d(y_t)]}{1 + r} \\ &= \frac{1}{1 + r} \int_{y_{t+1}^{thr}(b_{t+1})}^{+\infty} dF_{y_{t+1}|y_t}(z). \end{aligned}$$

Taking the derivative with respect to b_{t+1} yields

$$\frac{\partial q(b_{t+1}, y_t)}{\partial b_{t+1}} = \frac{-1}{1 + r} \frac{\partial y_{t+1}^{thr}(b_{t+1})}{\partial b_{t+1}} f_{y_{t+1}|y_t}(y_{t+1}^{thr}), \quad (3.8)$$

where $f(\cdot)$ is the probability density function of tomorrow's endowment y_{t+1} conditional on the realization of today's endowment y_t .

Recall the definition of y_{t+1}^{thr} . It is the level of endowment that equals the value of defaulting to the value of not defaulting,

$$V^c(b_{t+1}, y_{t+1}^{thr}) = V^d(y_{t+1}^{thr}).$$

By implicit function theorem,

$$\frac{\partial y_{t+1}^{thr}(b_{t+1})}{\partial b_{t+1}} = -\frac{\partial V^c(b_{t+1}, y_{t+1})}{\partial b_{t+1}} / \left(\frac{\partial V^c(b_{t+1}, y_{t+1})}{\partial y_{t+1}} - \frac{\partial V^d(y_{t+1})}{\partial y_{t+1}} \right). \quad (3.9)$$

Equation (3.9) reveals that the steepness of the default frontier is tied to the relative responsiveness of the value functions to endowment. As is discussed in (Aguiar and Gopinath 2006), at the indifference point, it must be the case that the value function of repayment is more sensitive to an additional unit of endowment than the value function of default. This is because, if an agent is indifferent between defaulting or not, current consumption absent default must be weakly less than under default, implying an equal or higher marginal utility of consumption. The agent in default must consume this additional income. The agent not in default can consume the additional income and the utility increases more due to the higher marginal utility of consumption. She can also choose to save this additional income, and will only do so if it raises utility by more than consuming it immediately. Therefore, the derivative of the value under repayment with respect to endowment is greater than under default.

This observation is important because in order for those models to generate recurrent defaults, we need a steep not flat default frontier, steep and flat in the sense of Panel (a) and (b) in Figure C.1 respectively. A flat default frontier implies that the region of

risky borrowing – the region where debt carries positive default premium – is small, so that default is less likely to happen in the equilibrium. One way to make the default frontier steeper, as can be seen from Equation (3.9), is to decrease the responsiveness of the value of default to endowment. This can be achieved by employing an asymmetric cost of default, just as what (Arellano 2008) does in her calibration.

Substituting Equation (3.9) into Equation (3.8) we get

$$\frac{\partial q(b_{t+1}, y_t)}{\partial b_{t+1}} = \frac{1}{1+r} \frac{\partial V^c(b_{t+1}, y_{t+1})}{\partial b_{t+1}} f_{y_{t+1}|y_t}(y_{t+1}^{thr}) / \left(\frac{\partial V^c(b_{t+1}, y_{t+1})}{\partial y_{t+1}} - \frac{\partial V^d(y_{t+1})}{\partial y_{t+1}} \right) \quad (3.10)$$

3.4.3 Discretizing the Distribution

An important step to solve the country's optimization problem is to construct a discrete approximation to the log-normal distribution that can be used to conduct numerical integration. We use the method in (Tauchen and Hussey 1991) to construct a Markov chain on a discrete state space, whose probability distribution closely approximates the distribution of a given time series. The method gives us the vector of the endowment states

$$\mathbf{y} = [y_{(1)}, \dots, y_{(N_y)}],$$

and the transition matrix

$$\mathbf{P} = \begin{bmatrix} \Pr(y_{(1)}|y_{(1)}) & \dots & \Pr(y_{(1)}|y_{(N_y)}) \\ \vdots & & \vdots \\ \Pr(y_{(N_y)}|y_{(1)}) & \dots & \Pr(y_{(N_y)}|y_{(N_y)}) \end{bmatrix},$$

where the elements in each column add up to 1.

An approximation of the default threshold for given b is found by

$$y^{thr}(b) = y_{(K)}, \text{ if } \begin{cases} V^c(b, y_{(i)}) < V^d(y_{(i)}) & \text{for all } i < K \\ V^c(b, y_{(i)}) \geq V^d(y_{(i)}) & \text{for all } i \geq K \end{cases}.$$

In principle the “true” value of y^{thr} can fall between the gridpoints in \mathbf{y} . But using this discrete approximation is very convenient since the numerical integration needs to be done many times during the iterations. It is sufficient to show that as the number of gridpoints gets large, the error caused by the approximation is getting smaller.

Once we have the discrete state space and the transition matrix, we can write the following

$$\begin{aligned}\mathbb{V}'_b(b_{t+1}, y_t) &= \beta \frac{\partial E_t V^o(b_{t+1}, y_{t+1})}{\partial b_{t+1}} \\ &= \sum_{i=K}^{N_y} \frac{\partial V^c(b_{t+1}, y_{(i)})}{\partial b_{t+1}} \Pr(y_{(i)} | y_t),\end{aligned}$$

where K is the index for the approximated default threshold, or $y^{thr}(b) = y_{(K)}$.

We also set up the gridpoints for debt $\mathbf{b} = [b_{(1)}, \dots, b_{(N_b)}]$.

3.4.4 The Method of Endogenous Gridpoints

Solving the optimization problem is essentially to pin down the optimal choice b_{t+1}^* for each state of the economy (b_t, y_t) using Equation (3.7). This involves numerical root-finding procedure as the choice variable b_{t+1}^* appears on both sides of the equation. The method of endogenous gridpoints rearranges the equation so that it is much easier to solve for b_t from b_{t+1}^* and y_t . This can be seen from the following. Equation (3.7) is equivalent to

$$(y_t + b_t - q_t b_{t+1})^{-\sigma} \left(\frac{\partial q_t}{\partial b_{t+1}} b_{t+1} + q_t \right) = \mathbb{V}'_b(b_{t+1}, y_t).$$

We can rearrange it to get

$$b_t = \left\{ \frac{\mathbb{V}'_b(b_{t+1}, y_t)}{\frac{\partial q_t}{\partial b_{t+1}} b_{t+1} + q_t} \right\}^{-1/\sigma} - y_t + q_t b_{t+1}. \quad (3.11)$$

The above equation guarantees a unique solution of b_t for given b_{t+1} and y_t . These b_t gridpoints are “endogenous” to the model in contrast to the usual solution method of specifying ex-ante grid of values of b_t and then using a root-finding routine to locate the corresponding optimal b_{t+1}^* . This trick skips the most computational burdensome step in the algorithm.

3.4.5 Value Functions and Derivatives

The next step after solving for the optimizer is to compute the value functions which will be used in the next round of iteration. First, we calculate the expected value from next period onward at the optimal choice b_{t+1}^* obtained in the previous step.

$$\begin{aligned}\mathbb{V}(b_{t+1}^*, y_t) &= \beta E_t V^o(b_{t+1}^*, y_t) \\ &= \beta \sum_{i=1}^{K-1} V^d(y_{(i)}) \Pr(y_{(i)}|y_t) + \beta \sum_{i=K}^{N_y} V^c(b_{t+1}^*, y_{(i)}) \Pr(y_{(i)}|y_t).\end{aligned}$$

If the government defaults today, debt goes to zero and the country enters financial autarky immediately. Depending on the realization of a stochastic shock, the country could leave the financial autarky in the next period, or could stay. Therefore, the expected value from tomorrow is either

$$\begin{aligned}\mathbb{V}(0, y_t) &= \beta E_t V^o(0, y_{t+1}) \\ &= \beta E_t V^c(0, y_{t+1}) \\ &= \beta \sum_{i=1}^{N_y} V^c(0, y_{(i)}) \Pr(y_{(i)}|y_t),\end{aligned}$$

or

$$\begin{aligned}\mathbb{V}^d(y_t) &= \beta E_t V^d(y_{t+1}) \\ &= \beta \sum_{i=1}^{N_y} V^d(y_{(i)}) \Pr(y_{(i)}|y_t).\end{aligned}$$

Based on those, we get the value functions and their derivatives. The value functions are

$$V^c(b_t, y_t) = u(y_t + b_t - q(b_{t+1}^*, y_t)b_{t+1}^*) + \mathbb{V}(b_{t+1}^*, y_t), \quad (3.12)$$

and

$$V^d(y_t) = u(y_t^{def}) + \theta \mathbb{V}(0, y_t) + (1 - \theta) \mathbb{V}^d(y_t). \quad (3.13)$$

The derivative of the value of repaying with respect to b_t is

$$\frac{\partial V^c(b_t, y_t)}{\partial b_t} = u'(y_t + b_t - q(b_{t+1}^*, y_t)b_{t+1}^*), \quad (3.14)$$

which comes from the envelope theorem. The partial derivatives of the value functions with respect to endowment is a bit more complicated,

$$\begin{aligned}\frac{\partial V^c(b_t, y_t)}{\partial y_t} &= u'(y_t + b_t - q(b_{t+1}^*, y_t)b_{t+1}^*)(1 - \frac{\partial q(b_{t+1}^*, y_t)}{\partial y_t} b_{t+1}^*) \\ &+ \beta \frac{\rho}{y_t} E_t [\frac{\partial V^c(b_{t+1}, y_{t+1})}{\partial y_{t+1}} y_{t+1}] + \beta \frac{\rho}{y_t} E_t [\frac{\partial V^d(y_{t+1})}{\partial y_{t+1}} y_{t+1}],\end{aligned} \quad (3.15)$$

and

$$\begin{aligned}\frac{\partial V^d(y_t)}{\partial y_t} &= u'(y^{def}) \frac{\partial y^{def}}{\partial y_t} \\ &+ \beta \theta \frac{\rho}{y_t} E_t [\frac{\partial V^c(0, y_{t+1})}{\partial y_{t+1}} y_{t+1}] + \beta (1 - \theta) \frac{\rho}{y_t} E_t [\frac{\partial V^d(y_{t+1})}{\partial y_{t+1}} y_{t+1}].\end{aligned} \quad (3.16)$$

We need the partial derivatives of the value functions with respect to endowment to compute the derivative of the bond price with respect to debt as is shown in Equation (3.10).

3.4.6 Borrowing Laffer Curve

So far we have gone through the basic steps of the algorithm, most importantly, how to use the endogenous gridpoints method to solve the optimization problem and derive the value functions. Now we turn to the tricky part, namely, what makes the application of the EGM to sovereign debt models harder than the standard precautionary savings model.

The difficult part comes from the endogenous bond price. Because bond price q decreases with the level of new debt b_{t+1} and eventually goes to zero, qb_{t+1} , the amount of resource that the borrower can pledge by borrowing will first increase with the level of borrowing and then decrease, forming the so-called “borrowing Laffer curve”. On the curve, two different levels of new borrowing can give the borrower the same amount of resource, and the borrower always wants to incur the minimal level of borrowing. As a result, some levels of new borrowing are never chosen as the optimum. This means that if we mechanically plug those values of b_{t+1} into Equation (3.11), it will give us values of b_t that do not make sense. To avoid this problem, we impose the condition that

$$\frac{\partial(q_t b_{t+1})}{\partial b_{t+1}} \geq 0, \quad (3.17)$$

and we will be using only those b_{t+1} and their corresponding b_t given by Equation (3.11), if the above condition is satisfied.

This condition has also a numerical meaning. Note that Equation (3.11) is equivalent to

$$b_t = \left\{ \frac{\mathbb{V}'_{b,t}(b_{t+1}, y_t)}{\frac{\partial(q_t b_{t+1})}{\partial b_{t+1}}} \right\}^{-1/\sigma} - y_t + q_t b_{t+1}.$$

We assume that σ is 2, a common value in the literature. If $\frac{\partial(q_t b_{t+1})}{\partial b_{t+1}} < 0$ then we are taking the square root of a negative number, which will give us a b_t that is a complex number.

The algorithm simply discards those b_{t+1} as invalid. As we have discussed, by imposing this condition, we focus our attention only on those points that lie on the upward-sloping part of the borrowing Laffer curve.

3.4.7 An Interpolation

The model is solved by iterating on the value functions until convergence. Moving from one iteration to another requires knowing the values of the functions in Equations (3.12), (3.14) and (3.15) defined on $\mathbf{b} = [b_{(1)}, \dots, b_{(N_b)}]$ (as Equations (3.13) and (3.16) do not depend on b). This can be done by one of the following two ways. The first is to find the optimal choice b_{t+1}^* for each of the gridpoints in \mathbf{b} so that we can directly evaluate those equations. The second is to approximate those functions by interpolating them among the gridpoints where their values are known. Note that the EGM does not guarantee that b_t inferred by Equation (3.11) is on the grids for \mathbf{b} . Not only that, after we impose the condition in Equation (3.17), the method cannot guarantee the existence of valid points at all. It could be that all b_{t+1} do not satisfy the condition in Equation (3.17). As a result, we need to discuss case by case how to obtain the values of those functions on \mathbf{b} , conditional on how many valid points we get. Also note that in principle, an interpolation can be done along the \mathbf{y} dimension as well, but we refrain from doing that in this paper. The following discussions are conditional on a value for y .

Case 0: the number of valid points is zero.

When constructing the vector for debt, we always include zero debt as the last element in \mathbf{b} . If the number of valid points is zero, then borrowing any $b_{t+1} < 0$ is not optimal. This is because borrowing such b_{t+1} implies getting $q_t b_{t+1}$ today, and borrowing a

little less would give more resources and less debt to repay tomorrow. Note that $\frac{\partial(q_t b_{t+1})}{\partial b_{t+1}} = 0$ if $b_{t+1} = 0$. Therefore, the country is strictly better-off by not borrowing at all. Since we know exactly that the optimum is achieved at $b_{t+1}^* = 0$, we can simply plug in $b_{t+1}^* = 0$ in Equations (3.12), (3.14) and (3.15) to get the value functions.

Case 1: the number of valid points is one.

In this case, we identify one pair of state variable b_t and the optimal solution b_{t+1}^* . We cannot conduct an interpolation because in order to do so we need at least two data points. So we treat this case as in Case 0.

Case 2: the number of valid points is at least two.

In this case we conduct an interpolation to get the value functions defined on **b**. Let L and U be the indices for the first and last valid points on the predetermined b_{t+1} grids, $L, U \in \{1, \dots, N_b\}$ and $L < U$. We can write the vector for the next-period asset position as $[b_{t+1(L)}^*, \dots, b_{t+1(U)}^*]$ and their corresponding current-period asset position calculated by the endogenous gridpoints method as $[b(b_{t+1(L)}^*), \dots, b(b_{t+1(U)}^*)]$. Compute the values for Equations (3.12), (3.14) and (3.15) at $[b(b_{t+1(L)}^*), \dots, b(b_{t+1(U)}^*)]$ using the optimal solution $[b_{t+1(L)}^*, \dots, b_{t+1(U)}^*]$. On each of the $b_{(i)}$ gridpoints with $b_{(i)} \in \mathbf{b}$ and $b(b_{t+1(L)}^*) \leq b_{(i)} \leq b(b_{t+1(U)}^*)$, the values of the functions are inferred by an interpolation. For each of the $b_{(i)}$ gridpoints with $b_{(i)} < b(b_{t+1(L)}^*)$, we use $b_{t+1(L)}^*$ to approximate the optimal choice, and for each of the $b_{(i)}$ gridpoints with $b_{(i)} > b(b_{t+1(U)}^*)$, we use $b_{t+1(U)}^*$ as the optimal choice.

3.5 Recursion

Our algorithm involved period-by-period iteration from an initial guess. One natural candidate for the initial guess is the assumed last period of life as in standard life-cycle problems. It is sufficient to show that under certain conditions, as the horizon gets large, the equilibrium we get from the finite-horizon model will converge to the equilibrium of the infinite-horizon version of the model.

As a starting point, construct evenly-distributed gridpoints $\mathbf{b} = [b_{(1)}, \dots, b_{(N_b)}]$ for debt and $\mathbf{y} = [y_{(1)}, \dots, y_{(N_y)}]$ for endowment.

3.5.1 An Initial Guess

In the last period, we have

$$V_T^c(b_T, y_T) = u(y_T + b_T),$$

and

$$V_T^d(y_T) = u(y_T^{def}).$$

With known functions, we can analytically derive the derivatives as

$$\frac{\partial V_T^c(b_T, y_T)}{\partial b_T} = u'(y_T + b_T),$$

$$\frac{\partial V_T^c(b_T, y_T)}{\partial y_T} = u'(y_T + b_T),$$

and

$$\frac{\partial V_T^d(y_T)}{\partial y_T} = u'(y_T^{def}) \frac{\partial y_T^{def}}{\partial y_T}.$$

3.5.2 Backward Iterations

Assume that the problem has been solved up to period $t + 1$, meaning that we inherited from the previous iteration the following functions

$$V_{t+1}^c(b_{t+1}, y_{t+1}),$$

$$V_{t+1}^d(y_{t+1}),$$

$$\frac{\partial V_{t+1}^c(b_{t+1}, y_{t+1})}{\partial b_{t+1}},$$

$$\frac{\partial V_{t+1}^c(b_{t+1}, y_{t+1})}{\partial y_{t+1}},$$

and

$$\frac{\partial V_{t+1}^d(y_{t+1})}{\partial y_{t+1}}.$$

The steps of iteration t are as follows.

1. For each $b_{t+1(i)}$, find the approximated default threshold $y_{t+1(K)}$, $K \in \{1, \dots, N_y\}$.

This is done by finding K such that the following is true:

$$V_{t+1}^c(b_{t+1(i)}, y_{t+1(l)}) < V_{t+1}^d(y_{t+1(l)}) \quad \text{for all } l < K, \quad \text{and}$$

$$V_{t+1}^c(b_{t+1(i)}, y_{t+1(l)}) \geq V_{t+1}^d(y_{t+1(l)}) \quad \text{for all } l \geq K.$$

If for some $b_{t+1(i)}$, the value of defaulting is always greater than the value of repaying, we set $K = 1$ and if the value of defaulting is always less than the value of repaying, we set $K = N_y$.

2. For each gridpoint on the state space $(b_{t+1(i)}, y_{t(j)})$, calculate the values for:

$$\begin{aligned} q_t(b_{t+1(i)}, y_{t(j)}) &= \frac{1}{1+r} \int_{y_{t+1}^{thr}}^{+\infty} dF_{y_{t+1}|y_t}(z) \\ &= \frac{1}{1+r} \sum_{k=K}^{N_y} \Pr(y_{t+1(k)} | y_{t(j)}). \end{aligned}$$

$$\begin{aligned}
\frac{\partial q_t(b_{t+1(i)}, y_{t(j)})}{\partial b_{t+1}} &= \frac{1}{1+r} \frac{\partial V_{t+1}^c(b_{t+1(i)}, y_{t+1}^{thr})}{\partial b_{t+1}} f_{y_{t+1}|y_{t(j)}}(y_{t+1}^{thr}) / \left(\frac{\partial V_{t+1}^c(b_{t+1(i)}, y_{t+1}^{thr})}{\partial y_{t+1}} - \frac{\partial V_{t+1}^d(y_{t+1}^{thr})}{\partial y_{t+1}} \right) \\
&= \frac{1}{1+r} \frac{\partial V_{t+1}^c(b_{t+1(i)}, y_{t+1(K)})}{\partial b_{t+1}} f_{y_{t+1}|y_{t(j)}}(y_{t+1(K)}) \\
&\quad / \left(\frac{\partial V_{t+1}^c(b_{t+1(i)}, y_{t+1(K)})}{\partial y_{t+1}} - \frac{\partial V_{t+1}^d(y_{t+1(K)})}{\partial y_{t+1}} \right).
\end{aligned}$$

We can then calculate

$$\frac{\partial(q_t(b_{t+1(i)}, y_{t(j)})b_{t+1(i)})}{\partial b_{t+1}} = \frac{\partial q_t(b_{t+1(i)}, y_{t(j)})}{\partial b_{t+1}} b_{t+1(i)} + q_t(b_{t+1(i)}, y_{t(j)}).$$

If the value is negative, discard the $b_{t+1(i)}$ and move back to the next gridpoint for b_{t+1} .

Otherwise, move to the next step.

3. For a given $y_{t(j)}$, if there exists at least two valid points for $b_{t+1(i)}$, proceed with

Steps 3-6. Otherwise, jump to Step 7. Calculate

$$\begin{aligned}
\mathbb{V}_t(b_{t+1(i)}, y_{t(j)}) &= \beta E_t V_{t+1}^o(b_{t+1(i)}, y_{t(j)}) \\
&= \beta \sum_{k=1}^{K-1} V_{t+1}^d(y_{t+1(k)}) \Pr(y_{t+1(k)}|y_{t(j)}) \\
&\quad + \beta \sum_{k=K}^{N_y} V_{t+1}^c(b_{t+1(i)}, y_{t+1(k)}) \Pr(y_{t+1(k)}|y_{t(j)}). \\
\mathbb{V}_t(0, y_{t(j)}) &= \beta E_t V_{t+1}^o(0, y_{t+1}) \\
&= \beta E_t V_{t+1}^c(0, y_{t+1}) \\
&= \beta \sum_{k=1}^{N_y} V_{t+1}^c(0, y_{t+1(k)}) \Pr(y_{t+1(k)}|y_{t(j)}). \\
\mathbb{V}_t^d(y_{t(j)}) &= \beta E_t V_{t+1}^d(y_{t+1}) \\
&= \beta \sum_{k=1}^{N_y} V_{t+1}^d(y_{t+1(k)}) \Pr(y_{t+1(k)}|y_{t(j)}). \\
\mathbb{V}'_{b,t}(b_{t+1(i)}, y_{t(j)}) &= \beta \int_{y_{t+1}^{thr}}^{+\infty} \frac{\partial V_{t+1}^c(b_{t+1(i)}, y_{t+1})}{\partial b_{t+1}} dF(y_{t+1}|y_{t(j)}) \\
&= \beta \sum_{k=K}^{N_y} \frac{\partial V_{t+1}^c(b_{t+1(i)}, y_{t+1(k)})}{\partial b_{t+1}} \Pr(y_{t+1(k)}|y_{t(j)}).
\end{aligned}$$

4. Calculate $b_{t(i,j)}$ by

$$b_{t(i,j)} = \left\{ \frac{\mathbb{V}'_{b,t}(b_{t+1(i)}, y_{t(j)})}{\frac{\partial q_t}{\partial b_{t+1}} b_{t+1(i)} + q_t} \right\}^{-1/\sigma} - y_{t(j)} + q_t b_{t+1(i)}.$$

5. Calculate

$$V_t^c(b_{t(i,j)}, y_{t(j)}) = u(y_{t(j)} + b_{t(i,j)} - q_t b_{t+1(i)}) + \mathbb{V}_t(b_{t+1(i)}, y_{t(j)}),$$

$$V_t^d(y_{t(j)}) = u(y_{t(j)}^{def}) + \theta \mathbb{V}_t(0, y_{t(j)}) + (1 - \theta) \mathbb{V}_t^d(y_{t(j)}).$$

$$\frac{\partial V_t^c(b_{t(i,j)}, y_{t(j)})}{\partial b_t} = u'(y_{t(j)} + b_{t(i,j)} - q_t b_{t+1(i)}),$$

$$\begin{aligned} \frac{\partial V_t^c(b_{t(i,j)}, y_{t(j)})}{\partial y_t} &= u'(y_{t(j)} + b_{t(i,j)} - q_t b_{t+1(i)}) \left(1 - \frac{\partial q_t(b_{t+1(i)}, y_{t(j)})}{\partial y_t} b_{t+1(i)}\right) \\ &+ \beta \frac{\rho}{y_{t(j)}} \sum_{k=K}^{N_y} \frac{\partial V_{t+1}^c(b_{t+1(i)}, y_{t+1(k)})}{\partial y_{t+1}} y_{t+1(k)} \Pr(y_{t+1(k)} | y_{t(j)}) \\ &+ \beta \frac{\rho}{y_{t(j)}} \sum_{k=1}^{K-1} \frac{\partial V_{t+1}^d(y_{t+1(k)})}{\partial y_{t+1}} y_{t+1(k)} \Pr(y_{t+1(k)} | y_{t(j)}). \end{aligned}$$

$$\begin{aligned} \frac{\partial V_t^d(y_{t(j)})}{\partial y_t} &= u'(y_{t(j)}^{def}) \frac{\partial y_{t(j)}^{def}}{\partial y} \\ &+ \beta \theta \frac{\rho}{y_{t(j)}} \sum_{k=K}^{N_y} \frac{\partial V_{t+1}^c(0, y_{t+1(k)})}{\partial y_{t+1}} y_{t+1(k)} \Pr(y_{t+1(k)} | y_{t(j)}) \\ &+ \beta (1 - \theta) \frac{\rho}{y_{t(j)}} \sum_{k=1}^{K-1} \frac{\partial V_{t+1}^d(y_{t+1(k)})}{\partial y_{t+1}} y_{t+1(k)} \Pr(y_{t+1(k)} | y_{t(j)}). \end{aligned}$$

6. Construct interpolation approximations of

$$V_t^c(b_{t(i,j)}, y_{t(j)}),$$

$$\frac{\partial V_t^c(b_{t(i,j)}, y_{t(j)})}{\partial b_t},$$

and

$$\frac{\partial V_t^c(b_{t(i,j)}, y_{t(j)})}{\partial y_t},$$

over grids \mathbf{b} .

7. If the number of valid gridpoints is less than two, we approximate the optimal choice by $b_{t+1}^* = 0$. Calculate

$$V_t^c(b_{t(i)}, y_{t(j)}) = u(y_{t(j)} + b_{t(i)}) + \mathbb{V}_t(0, y_{t(j)}),$$

$$V_t^d(y_{t(j)}) = u(y_{t(j)}^{def}) + \theta \mathbb{V}_t(0, y_{t(j)}) + (1 - \theta) \mathbb{V}_t^d(y_{t(j)}),$$

$$\frac{\partial V_t^c(b_{t(i)}, y_{t(j)})}{\partial b_t} = u'(y_{t(j)} + b_{t(i,j)}),$$

$$\begin{aligned} \frac{\partial V_t^c(b_{t(i)}, y_{t(j)})}{\partial y_t} &= u'(y_{t(j)} + b_{t(i)} - q_t) \\ &+ \beta \frac{\rho}{y_{t(j)}} \sum_{k=K}^{N_y} \frac{\partial V_{t+1}^c(0, y_{t+1(k)})}{\partial y_{t+1}} y_{t+1(k)} \Pr(y_{t+1(k)} | y_{t(j)}) \\ &+ \beta \frac{\rho}{y_{t(j)}} \sum_{k=1}^{K-1} \frac{\partial V_{t+1}^d(y_{t+1(k)})}{\partial y_{t+1}} y_{t+1(k)} \Pr(y_{t+1(k)} | y_{t(j)}), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial V_t^d(y_{t(j)})}{\partial y_t} &= u'(y_{t(j)}^{def}) \frac{\partial y_{t(j)}^{def}}{\partial y} \\ &+ \beta \theta \frac{\rho}{y_{t(j)}} \sum_{k=K}^{N_y} \frac{\partial V_{t+1}^c(0, y_{t+1(k)})}{\partial y_{t+1}} y_{t+1(k)} \Pr(y_{t+1(k)} | y_{t(j)}) \\ &+ \beta (1 - \theta) \frac{\rho}{y_{t(j)}} \sum_{k=1}^{K-1} \frac{\partial V_{t+1}^d(y_{t+1(k)})}{\partial y_{t+1}} y_{t+1(k)} \Pr(y_{t+1(k)} | y_{t(j)}). \end{aligned}$$

8. With the new value functions at hand, we can move one-period back and work through the steps again. The solution of this period-by-period iteration will converge to a fixed rule as the horizon gets large. We exit the iteration until the distance between the value functions gets sufficiently small.

3.6 Test of the Method

In this section we test the endogenous gridpoints method algorithm and compare it with the discrete-state-space method algorithm widely used in the literature. The model we solve here has almost the same functional forms and parameter values as (Arellano 2008). The only difference is that the output cost of default is symmetric across all endowment states. Table C.1 summarizes the parameter values.

The two algorithms are both written in the MATLAB programming language and tested on the same computer with an Intel(R) Core(TM) i7-2600 CPU of 3.40GHz and 12GB Memory. The debt space is discretized using 201 fine gridpoints, so moving from one gridpoint to the adjacent one represents a change in the debt level of only 0.15% of GDP. For endowment, we choose a grid of 21 gridpoints. We use the method in (Tauchen and Hussey 1991) to generate a Markov approximation of the stochastic process on the endowment grid. We set the convergence criterion to 10^{-5} , same for both methods.

In terms of accuracy, the two methods yield very similar results. Figure C.2 shows the value of repayment and value of default as a function of outstanding debt when endowment is at its mean level. The value of repayment is a decreasing and concave function of the level of debt. The value of default is constant across all levels of debt, because haircut is assumed to be 100%. The point where the value of repayment and value of default equalize is the default threshold, which we will show later. The endogenous gridpoints method preserves the exact slope and concavity in the value functions, with slightly lower bias for both curves.

Figure C.3 shows the bond price functions when endowment is at its mean level. In

both cases, bond price function decreases with the level of debt, and goes to zero eventually, as default becomes a certainty when debt exceeds certain levels. Figure C.4 shows the optimal level of new borrowing given the current level of debt and endowment staying at its mean level. The results are also very close under the two algorithms.

Figure C.5 compares the default frontiers from the two methods. The choice of default is the most important policy function of the model, as it determines the properties of the simulated moments. The default frontier is defined as the level of debt after which the borrower prefers default to repayment. In the debt-endowment space, the default frontier is the line that separates the region where the government prefers to default and the region where it prefers to repay. The two methods produce almost identical default frontiers, except for very low levels of endowment. There, the default frontier from the endogenous gridpoints method is steeper, which implies a bigger default region and lower levels of debt in the equilibrium compared to the discrete-state-space method.

Table C.2 compares the simulated moments from the two methods. The moments are very close to each other. As we have discussed, the differences in the moments are mostly due to the differences in the default policy generated by the two algorithms.

In terms of speed, it takes longer for the the endogenous gridpoints method to find a converged solution. This is mostly because of the fact that the algorithm requires that linear interpolations for both the value function and the derivatives of the value function to be carried out on each gridpoint of endowment. However, future research can be done in this direction to speed up the linear interpolation in order to increase the speed of the method.

3.7 Conclusion

This paper shows how the endogenous gridpoints method can be adapted to solve quantitative sovereign debt models such as (Arellano 2008). Those models are successful in explaining certain empirical regularities of an emerging economy that issues sovereign debt and occasionally defaults on it, such as Argentina. However, applications of those models are always constrained by the computation time needed to solve them. This paper offers an algorithm that employs the endogenous gridpoints method as an alternative method to solve those models. We use the model in (Arellano 2008) as an example to illustrate how the method can be applied, and show that endogenous gridpoints method produces similar results compared to the discrete-state-space method widely used in the literature. Although the current version of our algorithm presented in this paper is slower in terms of speed, there can be potential gains in computation time if the linear interpolation can be optimized. Moreover, it is not difficult to extend the method to solve the model with more realistic features, such as, a positive haircut, or a more complex default output cost structure.

However, it is worth noting that the successful implementation of the algorithm offered in this paper hinges on the assumption that output is truly a continuous stochastic process. Although discretization is still a needed step when numerically solving the model, the algorithm makes heavy use of the derivatives of the value functions. If the true process of endowment is discrete, the discrete-state-space method might be a more efficient and accurate solution method.

Appendix A

Appendix to Chapter 1

A.1 Data Description

The frequency of default for Argentina is obtained by counting the number of actual default episodes and debt restructuring during the 213 years of 1800-2012, and divide it by 213. The numbers of default episodes are from (Reinhart and Rogoff 2009).

GDP and consumption data are from the Ministry of Finance (MECON), for the period 1980q1-2001q4. Trade balance data are from the Ministry of Finance (MECON), for the period 1993q1-2001q4. External debt data are from the World Bank World Development Indicators, for the period 1993-2001. The data on quarterly real GDP and consumption are seasonally adjusted and detrended using the HP-filter with parameter 1600. Trade balance is reported as percentage of GDP.

A.2 Estimation Procedure

The estimation is done by the Simulated Method of Moments. The method aims to minimize the distance between moments from the data and the model. The weighting matrix is obtained by the block bootstrap method. The block bootstrap is used when the data, or the errors in a model, are correlated. In this case, a simple case or residual resampling will fail, as it is not able to replicate the correlation in the data. The block bootstrap tries to replicate the correlation by resampling instead blocks of data.

The details of this method are as follows. Let D_t be the observations of all data at time t . In our case, D includes GDP, consumption, debt, trade balance, interest rate spreads, and $t \in \{1, 2, \dots, 36\}$. To start, we first draw one observation from the 36 observations at random with replacement. Suppose that we have picked observation D_τ . With some probability p , we pick the next observation $D_{\tau+1}$; with probability $1 - p$, we draw another observation at random (we could get D_τ again because this is random draw with replacement). If at any time we have reached the last observation of our data point D_{36} and there does not exist “the next one” available to be selected, we draw an observation at random just as how we draw the first observation. We stop when we have made 36 draws. By aligning these 36 observations in the order they were picked, we get one bootstrap sample.

For each of the bootstrap sample, we calculate its moments, including the average interest rate spreads, the average debt-to-output ratio, the relative volatility of consumption to output, and the correlation between trade balance and output. The weighting matrix is then the inverse of the variance-covariance matrix of the moments from all the bootstrap samples. We choose parameter values to minimize the weighted distance between data

moments and simulated moments. Standard errors of the parameter estimates are computed from the derivative of the simulated moments with respect to the parameters evaluated at the point estimates.

A.3 Numerical Resolution

The model is solved by value function iterations using the discrete-state-space (DSS) technique, which is also used in other default studies. The efficiency of the DSS technique is greatly improved by using a one-loop algorithm that iterates simultaneously on the value functions and the bond price functions, and finding the equilibrium as the limit of the finite-horizon version of the model. As has been shown in the literature, this one-loop algorithm significantly reduces the computation time required by the DSS method without loss of accuracy.¹

The one-loop algorithm works as follows. We start with a hypothetical last period of the model, solve for the optimal policy and value functions for that period given bond price functions. Using the value functions we are able to move one-period backward and update the bond price function for the second-to-last period. The optimal policy and value functions for the second-to-last period are then solved given the updated bond price function. This process goes on until the value functions converge. Since the bond price function is determined by value functions, the bond price functions should also converge.

¹(Hatchondo et al. 2010) discuss this algorithm and compare the speed of convergence with the two-loop algorithm that has been used in other default studies. They report that using the one-loop algorithm instead of the two-loop algorithm reduces the computation time by a factor of 6.

A.4 Graphs and Tables

Table A.1: Measures of Indebtedness

Argentina	
Measure	% of quarterly GDP
Debt service on external debt (PPG, TDS)	11.99
Debt service on external debt (total, TDS)	21.86
External debt stocks (short-term, DOD)	32.19
External debt stocks (PPG, DOD)	99.70
External debt stocks (total, DOD)	171.65

Note: Debt stock is the total value of debt that the country owes to all lenders. Debt service is the cash required to cover the payment of interest and principal over a time period. PPG, TDS, DOD stand for public and publicly guaranteed, total debt service ratio, and debt outstanding and disbursed. Short-term debt includes all debt having an original maturity of one year or less and interest in arrears on long-term debt. The ratios are computed using Argentina's quarterly GDP series from 1993q1 to 2001q4.

Table A.2: Targeted Measures of Indebtedness in Existing Models

	AG (2006)	Arellano (2008)	Yue (2010)	D'Erasmus (2011)	Asonuma (2012)
Targeted Measure (% of quarterly GDP)	–	debt service	debt service	PPG debt stocks	debt service or short-term debt stocks
Data	–	5.53	9.54	81.2	12.7 or 10.2
Model	19	5.95	10.13	45.1	9.5

Note: AG (2006) do not have a specific target debt ratio, instead, they refer the level of debt that their model explains as “debt payments”.

Table A.3: Discount Factors in Existing Models

	AG (2006)	Arellano (2008)	Yue (2010)	D'Erasmus (2011)	Asonuma (2012)
Discount factor (quarterly)	0.80	0.953	0.72	0.99/0.73	0.75
Implied probability of staying in power (quarterly)	0.792	0.944	0.713	0.98/0.723	0.743

Note: (D'Erasmus 2011) considers an environment with political uncertainty and two types of politicians ruling in turn following some stochastic process. The patient politician has a discount factor of 0.99 and the impatient one of 0.73. The risk-free rate used to calculate the implied probability of staying in power is 1%.

Table A.4: Correlation between the Trade Balance and Output in Existing Models

	Data	AG (2006)	Arellano (2008)	Yue (2010)	D’Erasmus (2011)	Asonuma (2012)
Corr(trade balance, output)	-0.658	-0.19	-0.25	-0.16	-0.10	-0.005

Note: The correlation is computed using data for Argentina from 1993q1 to 2001q4. The trade balance is expressed as a percentage of output.

Table A.5: Independent Targets

Parameter		Value	Reference
Risk aversion	σ	2	standard literature
Risk-free interest rate	r	0.01	standard literature
Endowment autocorrelation	ρ	0.8309	Argentina estimates
Endowment error std. (%)	η	2.57	Argentina estimates
Probability of redemption	θ	0.25	Gelos et al. (2011)
Haircut	h	0.73	27% debt recovery rate
Debt limit	\underline{b}	-3	300% of mean output

Table A.6: Parameter Values and Targeted Moments

Argentina

Parameter		Value	Std. Error
Discount factor	β	0.9756	3.1109e-06
Commitment probability	λ	0.9554	2.4450e-05
Output in default	α	0.8508	4.0013e-04
Moment		Data	Model
Average spreads (%)		6.60	5.02
Average debt/output ratio (%)		99.97	98.74
Consumption std./output std.		1.1002	1.1673
Corr (trade balance, output)		-0.6580	-0.5460

Figure A.1: Timeline

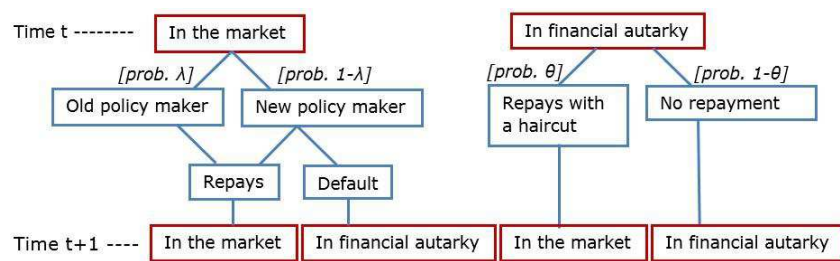


Table A.7: Simulation Statistics for the Baseline Model

Argentina							
Target Statistics	Data	Model	AG (2006)	Arellano (2008)	Yue (2010)	D’Erasmus (2011)	Asonum (2012)
Average spreads (%)	6.60	5.02	0.92	3.58	1.86	0.47*	3.1
Avg. debt/output ratio (%)	99.70	98.74	19	5.95	10.13	45.12	9.5
Consumption std./output std.	1.1002	1.1673	1.05	1.10	1.04	1.24	1.24
Corr(trade balance, output)	-0.6580	-0.5460	-0.19	-0.25	-0.16	-0.10	-0.005
Non-Target Statistics							
Business Cycle Variables							
Trade balance std. (%)	1.75	1.19	0.95	1.50	2.81	4.29	3.71
Corr(consumption, output)	0.9809	0.9630	0.98	0.97	–	0.85	–
Spreads							
Bond spreads std. (%)	3.08	2.49	0.32	6.36	1.58	0.21*	2.9
Corr(spreads, output)	-0.8081	-0.6111	-0.03	-0.29	-0.11	-0.14	-0.19
Corr(spreads, trade balance)	0.7039	0.5562	0.11	0.43	0.30		
Default Episodes							
Frequency of default (%)	2.80	3.24	0.92	3	2.67	0.63*	2.65
Output drop (%)	9.48	9.28	19	9.60	7.19	–	–
Consumption drop (%)	11.07	14.47	–	9.47	8.84	–	–

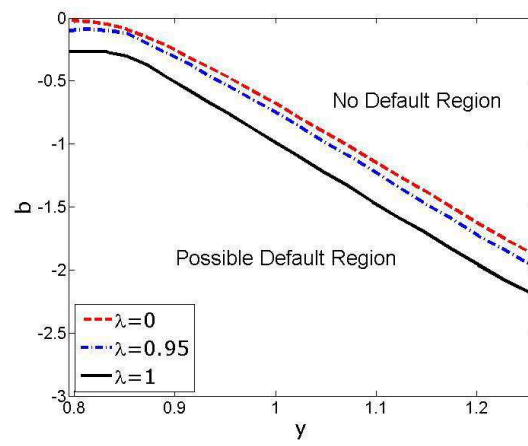
Note: Numbers with an asterisk are on a quarterly basis.

Table A.8: Simulation Statistics for the No-Commitment Model (Part I)

Argentina				
Target Statistics	Data	Baseline	No Commitment (I)	No Commitment (II)
Average spreads (%)	6.60	5.02*	0.06*	7.17*
Avg. debt/output ratio (%)	99.70	98.74*	62.90*	81.56*
Consumption std./output std.	1.1002	1.1673*	1.0709*	2.2592
Corr(trade balance, output)	-0.6580	-0.5460*	-0.3365*	-0.4050
Non-Target Statistics				
Business Cycle Variables				
Trade balance std. (%)	1.75	1.19	1.25	6.93
Corr(consumption, output)	0.9809	0.9630	0.9663	0.7523
Spreads				
Bond spreads std. (%)	3.08	2.49	0.25	5.28
Corr(spreads, output)	-0.8081	-0.6111	-0.4042	-0.5456
Corr(spreads, trade balance)	0.7039	0.5562	0.6035	0.5122
Default Episodes				
Frequency of default (%)	2.80	3.24	0.02	4.72
Output drop (%)	9.48	9.28	19.94	10.32
Consumption drop (%)	11.07	14.47	27.26	20.48

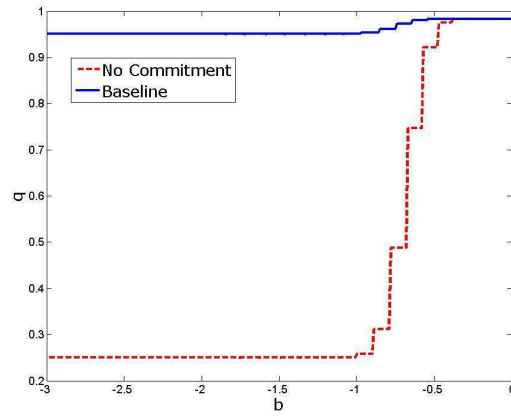
Note: Moments with an asterisk are target statistics.

Figure A.2: Default Thresholds



Note: The parameter values are from the baseline estimation except for λ .

Figure A.3: Bond Price Schedule



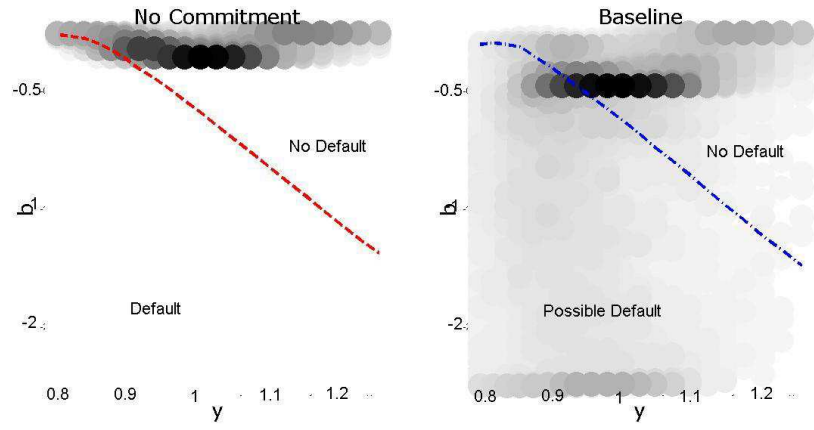
Note: The parameter values are from the baseline estimation except for λ . Output is assumed at its mean at 1.

Table A.9: Simulation Statistics for the No-Commitment Model (Part II)

Argentina			
			No
Target Statistics	Data	Baseline	Commitment (III)
Average spreads (%)	6.60	5.02*	2.17
Avg. debt/output ratio (%)	99.70	98.74*	16.44
Consumption std./output std.	1.1002	1.1673*	1.1769
Corr(trade balance, output)	-0.6580	-0.5460*	-0.4093
Non-Target Statistics			
Business Cycle Variables			
Trade balance std. (%)	1.75	1.19	1.63
Corr(consumption, output)	0.9809	0.9630	0.9638
Spreads			
Bond spreads std. (%)	3.08	2.49	4.55
Corr(spreads, output)	-0.8081	-0.6111	-0.6110
Corr(spreads, trade balance)	0.7039	0.5562	0.4970
Default Episodes			
Frequency of default (%)	2.80	3.24	1.17
Output drop (%)	9.48	9.28	17.23
Consumption drop (%)	11.07	14.47	16.09

Note: Moments with an asterisk are target statistics. For the no-commitment model there is no target statistics because we simply take the baseline model and set the probability of commitment to zero without trying to match any moment from the data.

Figure A.4: Frequency of the States



Note: Darker color indicates higher frequency. The lines are the default frontiers.

Figure A.5: Histograms of Model Simulated Moments

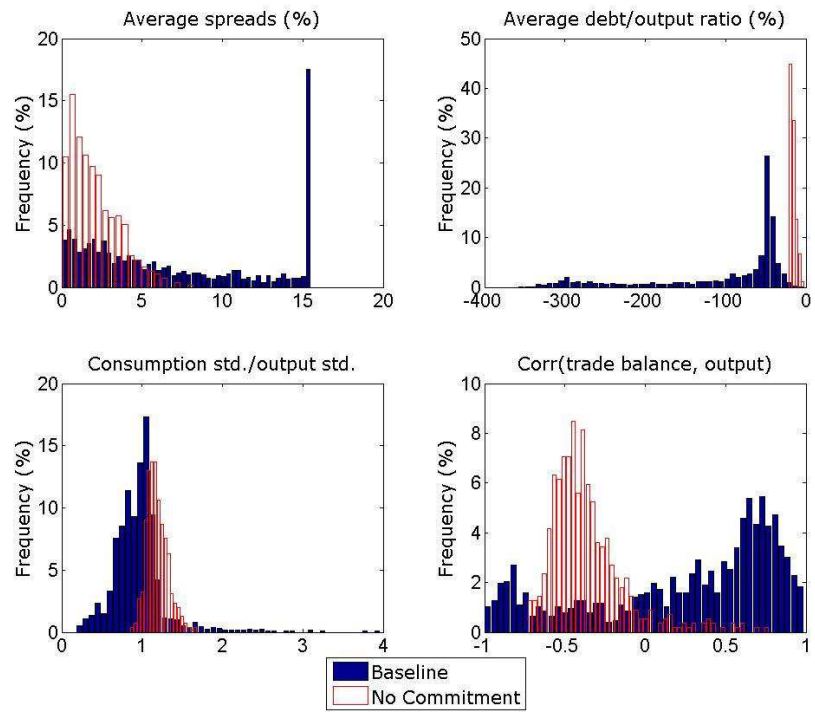


Figure A.6: Simulated Moments and Periods in the Default Region

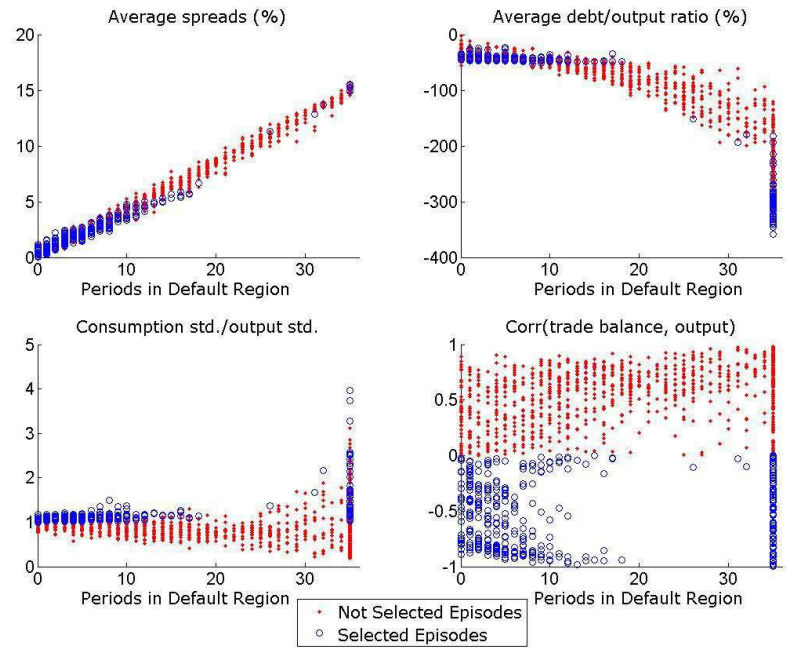
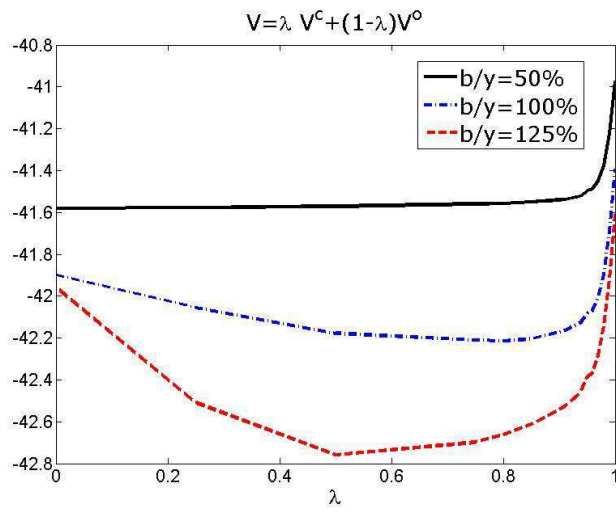


Figure A.7: Welfare of Commitment



Appendix B

Appendix to Chapter 2

B.1 Computational Algorithm

The model is solved using the discrete-state-space method. First we set fine grids on the spaces of state variables (b, y) . The limits of endowment space are large enough to incorporate large deviations around the mean. We use Tauchen and Hussey (1991)'s quadrature-based methods to obtain a Markov approximation to the endowment shocks. The debt space is discretized into 200 gridpoints and the endowment space is discretized into 11 gridpoints.

The equilibrium is obtained using a one-loop algorithm that iterates simultaneously on the value functions and the bond price function. We start with an initial guess $V_{[0]}^c$ and $V_{[0]}^d$ defined over the same grids. A natural candidate of the initial guess is the solution of the last period of the finite-horizon version of the model. It can be shown that the algorithm approximates the equilibrium as the limit of the equilibrium of the finite-horizon economy.

Given the initial guess, we calculate the default set. The government prefers default if $V_{[0]}^c$ is less than $V_{[0]}^d$. Using the default set derived and the zero-profit condition for bond holders, we compute the price of bonds $q_{[1]}$. Bond price also takes into consideration the probability that the government is constrained by its commitment and cannot default. After

we get the bond price, we solve the government's optimization problem one period ahead.

We continue the above steps until the value functions $V_{[1]}^c$ and $V_{[1]}^d$ are all sufficiently close to $V_{[0]}^c$ and $V_{[0]}^d$.

B.2 Graphs and Tables

Figure B.1: Time Series for Greece

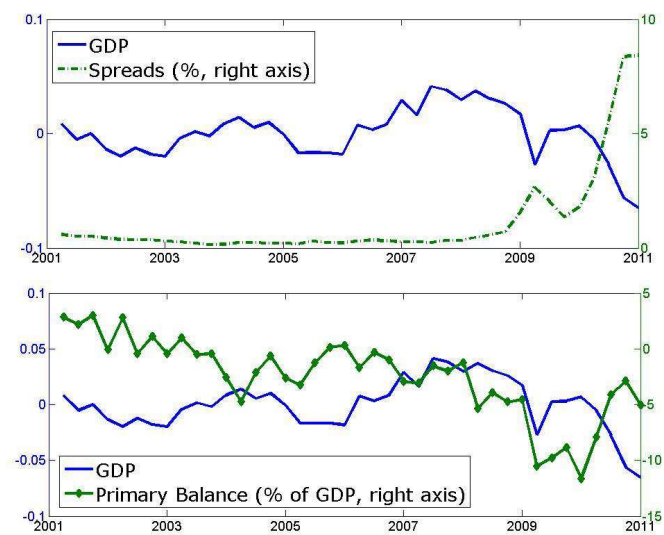


Table B.1: Independent Parameters

Parameter		Value
Risk-free interest rate (%)	r	4
Households discount factor	β	0.96
Output autocorrelation	ρ	0.4553
Output standard error (%)	η	1.31
Tax rate	$\bar{\tau}$	0.2
Debt upper limit	\bar{b}	2
Haircut	h	0.5
Probability of resuming borrowing after default	θ	0.25
Probability of commitment	λ	0.9

Table B.2: Parameter Values and Targeted Moments

Parameter		Value
Discount factor	β_g	0.6546
Govt. consumption utility parameter	z	0.2034
Risk aversion coefficient for govt. consumption	σ	1.0525
Output in default	\bar{y}	0.9570
Moment	Data	Model
Frequency of default* (%)	2.20	2.17
Corr(spreads, output)*	-0.6262	-0.5526
Corr(primary balance, output)*	-0.2029	-0.2419
Corr(consumption, output)*	0.7403	0.6503
Average spreads (%)	0.95	2.54
Average debt/output ratio (%)	126.02	152.68
Bond spreads std. (%)	1.93	1.00
Primary balance std. (%)	3.37	1.85

Note: Moments with an asterisk are targets.

Table B.3: Simulated Moments for Varying Discount Factor

Moment	Data	Baseline $\beta_g = 0.6546$	Alternative (I) $\beta_g = 0.80$	Alternative (II) $\beta_g = 0.96$
Frequency of default (%)	2.20	2.17	0.34	0.01
Corr(spreads, output)	-0.6262	-0.5526	-0.9251	-0.8182
Corr(primary balance, output)	-0.2029	-0.2419	-0.9324	0.1962
Corr(consumption, output)	0.7403	0.6503	0.9649	0.9907
Average spreads (%)	0.95	2.54	0.2	0.01
Average debt/output ratio (%)	126.02	152.68	122.99	68.86
Bond spreads std. (%)	1.93	1.00	0.26	0.01
Primary balance std. (%)	3.37	1.85	0.41	0.28

Figure B.2: Data and Model Times Series

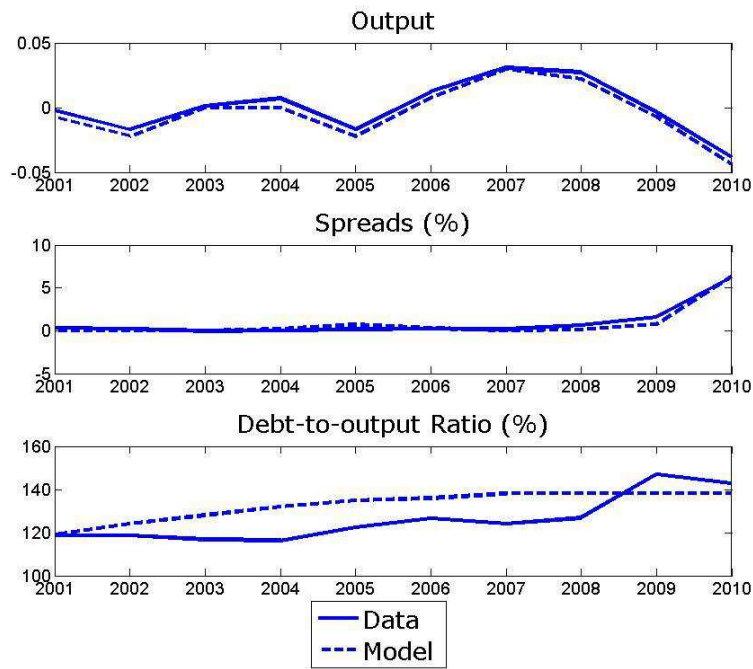


Figure B.3: Bond Price Function

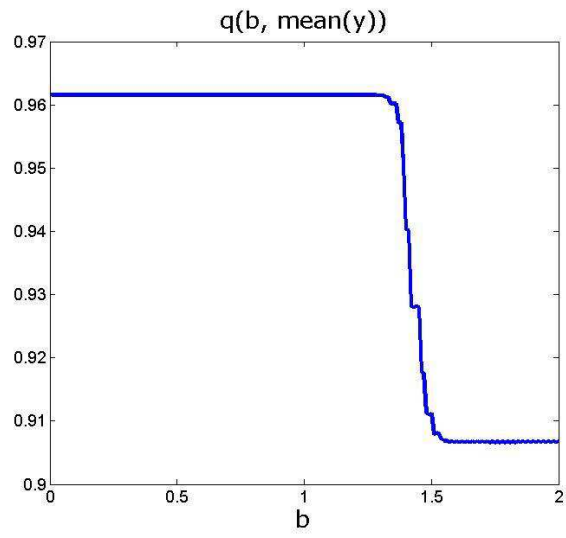


Figure B.4: Value Functions

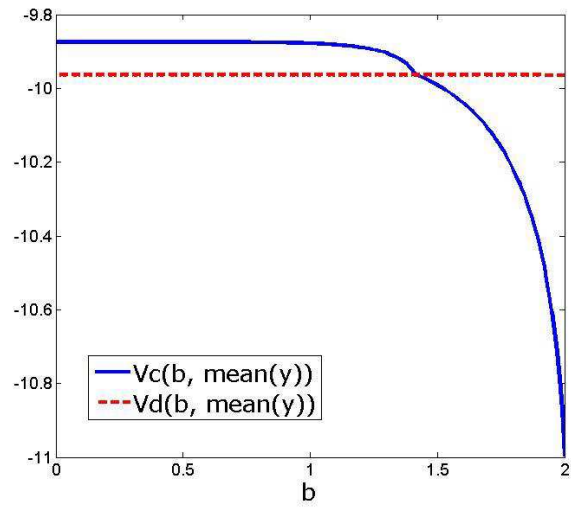


Figure B.5: Default Region

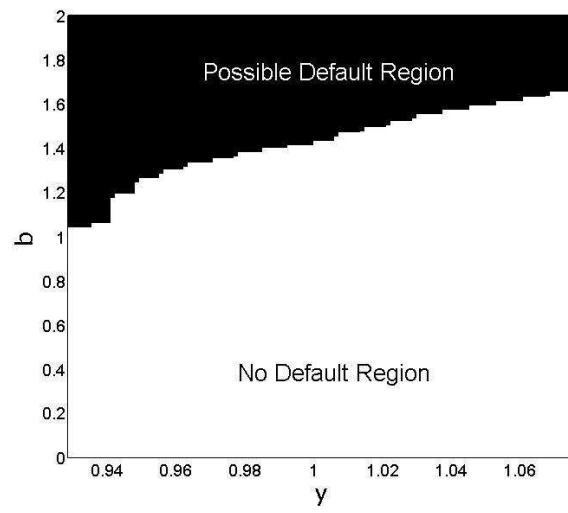


Table B.4: Simulated Moments for Varying Degree of Commitment

Moment	Data	Baseline $\lambda = 0.9$	Alternative (III) $\lambda = 0$	Alternative (IV) $\lambda = 0.95$
Frequency of default (%)	2.20	2.17	0.65	2.93
Corr(spreads, output)	-0.6262	-0.5526	-0.7851	-0.1368
Corr(primary balance, output)	-0.2029	-0.2419	-0.7440	-0.4823
Corr(consumption, output)	0.7403	0.6503	0.5352	0.6643
Average spreads (%)	0.95	2.54	0.45	2.45
Average debt/output ratio (%)	126.02	152.68	95.4624	187.92
Bond spreads std. (%)	1.93	1.00	0.37	0.43
Primary balance std. (%)	3.37	1.85	0.97	1.58

Table B.5: Simulated Moments for Varying Default Output Cost

Moment	Data	Baseline $\bar{y} = 0.957$	Alternative (V) $\bar{y} = 0.94$	Alternative (VI) $\bar{y} = 0.97$
Frequency of default (%)	2.20	2.17	0.22	5.10
Corr(spreads, output)	-0.6262	-0.5526	-0.7171	-0.1067
Corr(primary balance, output)	-0.2029	-0.2419	-0.7246	-0.1928
Corr(consumption, output)	0.7403	0.6503	0.7333	0.4393
Average spreads (%)	0.95	2.54	0.15	4.90
Average debt/output ratio (%)	126.02	152.68	173.77	165.48
Bond spreads std. (%)	1.93	1.00	0.09	1.25
Primary balance std. (%)	3.37	1.85	0.77	2.90

Table B.6: Simulated Moments for Model with Countercyclical Tax Rate

Moment	Data	Baseline	Countercyclical Tax (I)	Countercyclical Tax (II)
Frequency of default (%)	2.20	2.17	0.42	0.39
Corr(spreads, output)	-0.6262	-0.5526	-0.8158	-0.8238
Corr(primary balance, output)	-0.2029	-0.2419	-0.5581	-0.2628
Corr(consumption, output)	0.7403	0.6503	0.8661	0.8884
Average spreads (%)	0.95	2.54	0.34	0.29
Average debt/output ratio (%)	126.02	152.68	140.36	143.11
Bond spreads std. (%)	1.93	1.00	0.24	0.20
Primary balance std. (%)	3.37	1.85	0.60	0.52

Figure B.6: Tax Rates and Output

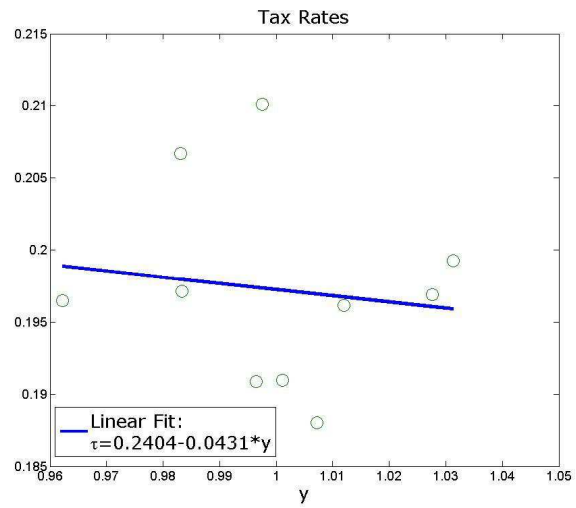


Figure B.7: Countercyclical Tax Rate and Tax Revenue

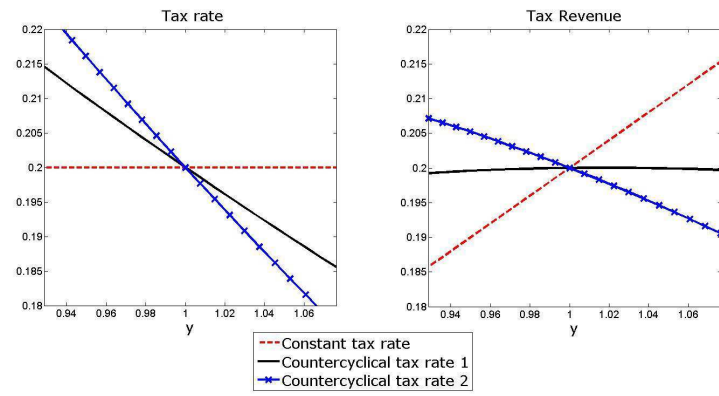


Figure B.8: Data and Model Times Series

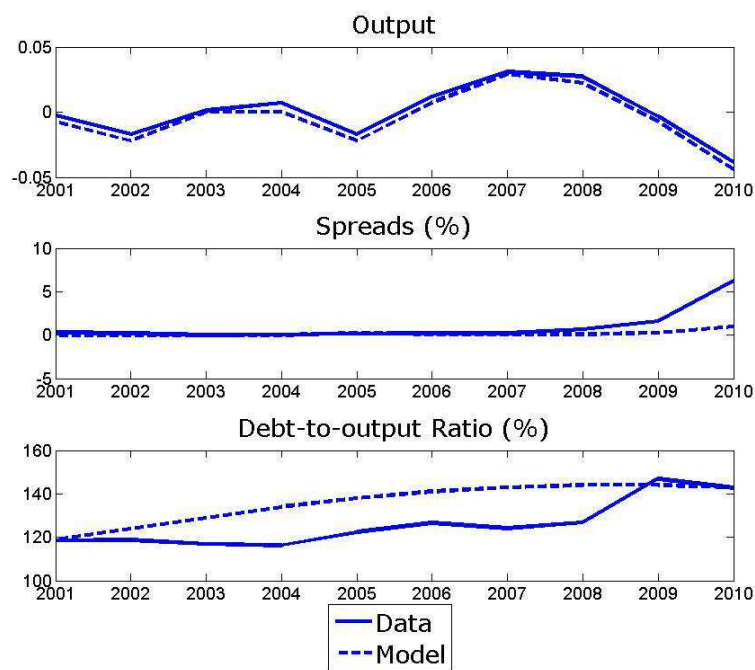
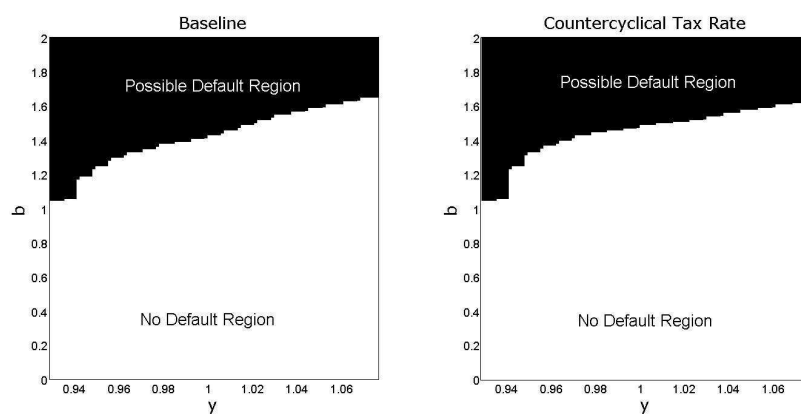


Figure B.9: Default Region



Appendix C

Appendix to Chapter 3

C.1 Graphs and Tables

Figure C.1: Default Frontier

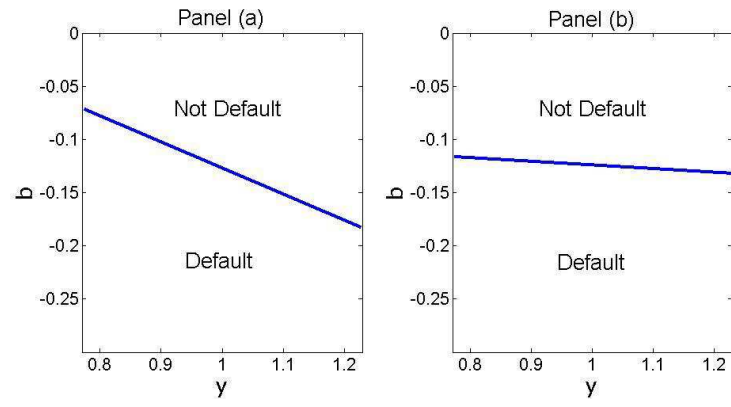


Table C.1: Parameter Values

Parameter		Value
Discount factor	β	0.95
Risk aversion	σ	2
Risk-free interest rate (%)	r	1.7
Output in default	α	0.969
Probability of Redemption	θ	0.282
Endowment autocorrelation	ρ	0.945
Endowment error std. (%)	η	2.5
Debt limit	\underline{b}	-0.3

Table C.2: Comparison of Simulation Statistics

Statistics	DSS	EGM
Average spreads (%)	0.10	0.09
Avg. debt/output ratio (%)	11.02	9.47
Corr(consumption, output)	0.9996	0.9977
Corr(trade balance, output)	-0.1677	-0.1513
Corr(spreads, output)	-0.3750	-0.2680
Corr(spreads, trade balance)	-0.0055	0.0522
Consumption std./output std.	1.0031	1.0098
Trade balance std. (%)	0.21	0.57
Bond spreads std. (%)	0.10	0.11

Table C.3: Comparison of the Methods

	DSS	EGM
Number of gridpoints for b	201	201
Number of gridpoints for y	21	21
Convergence criterion	10^{-5}	10^{-5}
Computation time (seconds)	6.16	70.35

Figure C.2: Value Functions

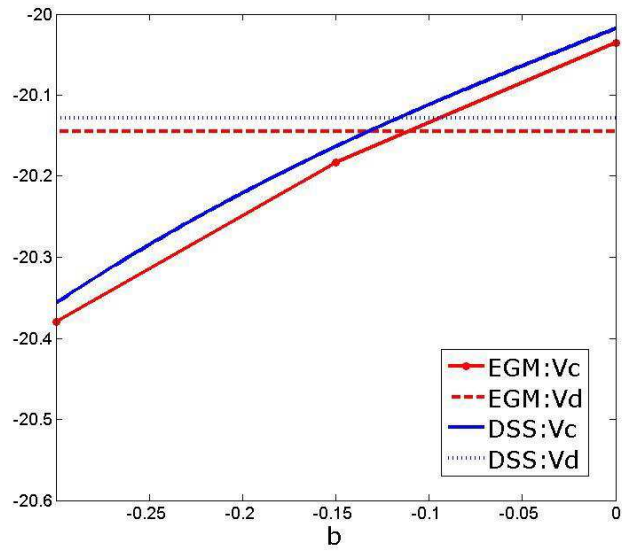


Figure C.3: Bond Price

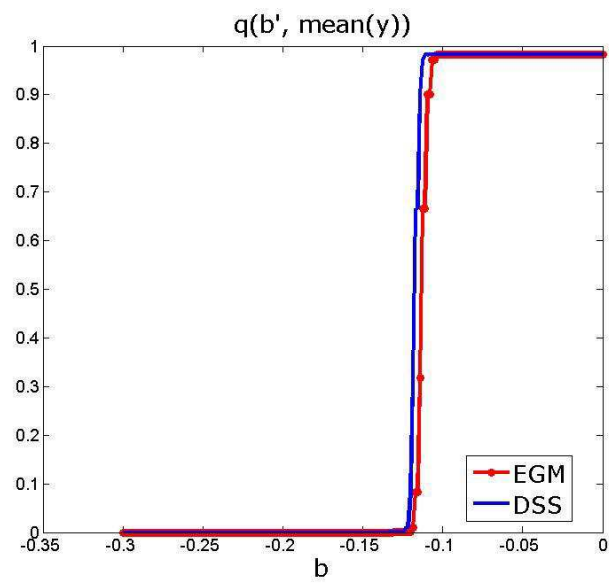


Figure C.4: Optimal New Borrowing

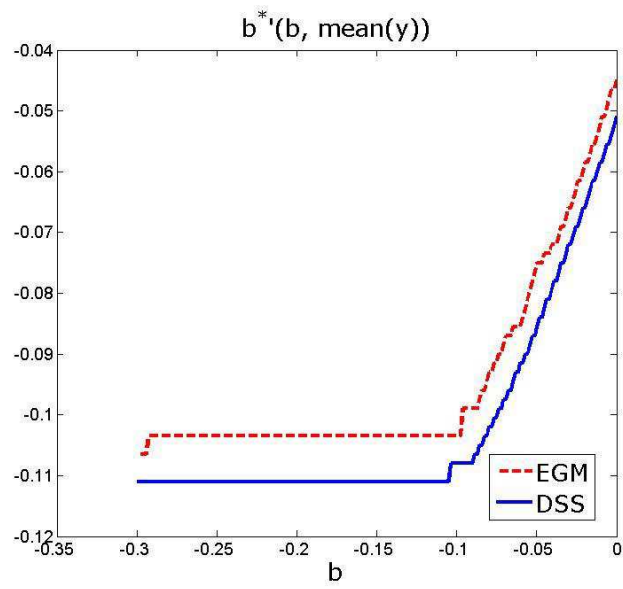
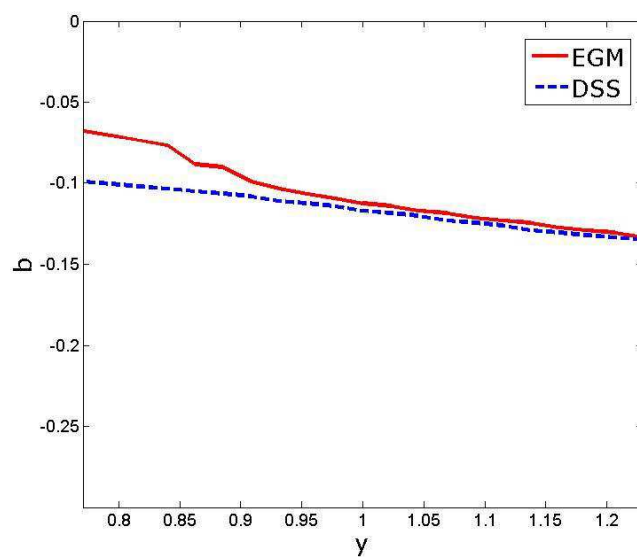


Figure C.5: Default Frontier



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Curriculum Vitae

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