CONCERNING THE KLEIN-GORDON EQUATION ON ASYMPTOTICALLY EUCLIDEAN MANIFOLDS

by

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Abstract

The aim of the dissertation is to study the Klein-Gordon equation on asymptotically Euclidean manifolds. Using Mourre estimates, we obtain KSS estimates for the Klein-Gordon equation on $(\mathbb{R}^d, \mathfrak{g})$, $d \geq 3$, when metric $\mathfrak{g}$ is non-trapping and approaches the Euclidean metric like $\langle x \rangle^{-\rho}$ with $\rho > 0$. Together with local energy decay, we can obtain the Strichartz estimate for compact perturbation metric. Also, using stationary phase methods and parametrix constructions, we can obtain the dispersive estimate which implies the global Strichartz estimates for non-compact small perturbation.

Readers: Dr. Christopher Sogge (Advisor).

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Chapter 1

Introduction

1.1 Background

Our research interest lies within the field of Fourier Analysis and Partial Differential Equations. This is a wide area that naturally intersects with Riemannian geometry, quantum physics, and complex variables. We concern about the Klein-Gordon equation (1.1) on compact and non-compact asymptotically Euclidean non-trapping manifolds.

We focused on the asymptotic spectral behavior, local energy decay and global Strichartz estimate for Klein-Gordon equations on $(\mathbb{R}^d, \mathfrak{g}), d \geqslant 3$, when the metric $\mathfrak{g}$ is non-trapping and approaches the Euclidean metric like $\langle x\rangle^{-\rho}$ with $\rho > 0$. There has been important progress concerning these kinds of problems for Schrödinger equations and Wave equations. Sogge and Wang([2]) used the KSS (Keel-Smith-Sogge) estimate to prove almost global existence for quadratically semilinear wave equations with small initial data for $\rho > 1, d = 3$ and also established the Strauss conjecture when the metric is radial. Bony and Häfner applied Mourre estimate to obtain local energy decay for several evolution equations on asymptotically Euclidean manifolds(see [1] and [3]). Tataru used phase space methods to construct outgoing parametrices to prove global in time Strichartz type

estimates for Schrödinger equations(see [18]). And later on, Metcalfe and Tataru finished the local smoothing estimate and global Strichartz estimate for variable coefficient wave equation (see [17]). However, certain questions remain open for Klein-Gordon equations. Fortunately, Klein-Gordon equation is a relativistic version of the Schrödinger equation, and it is also a quantum type wave equation. Therefore, it's natural to apply similar techniques for Wave/Schrödinger equations to Klein-Gordon equations.

1.2 Preliminaries and Main Results

Let us now state our precise results. We consider asymptotically Euclidean manifolds $(\mathbb{R}^d, \mathfrak{g})$ with $d \geq 3$ and

$$\mathfrak{g} = \sum_{i,j=1}^{d} g_{ij}(x) dx^i dx^j.$$

We suppose $g_{ij}(x) \in C^\infty(\mathbb{R}^d)$ and, for some $\rho > 0$,

(H1) $$\forall \alpha \in \mathbb{N}^d \qquad \partial_x^\alpha (g_{ij} - \delta_{ij}) = O(\langle x \rangle^{-|\alpha|-\rho}),$$

with $\langle x \rangle = \sqrt{1+|x|^2}$ and $\delta_{ij} = \delta^{ij}$ being the Kronecker delta function.
We also assume that

(H2) $$\mathfrak{g} \text{ is non-trapping.}$$

where non-trapping means every geodesic on manifold reaches spatial infinity in forward and backward time.
Let $g(x) = (\det(\mathfrak{g}))^{1/4}$. The Laplace–Beltrami operator associated with $\mathfrak{g}$ is given by

$$\Delta_g = \sum_{i,j} \frac{1}{g^2} \partial_i (g^{ij} g^2 \partial_j),$$

where $g^{ij}(x)$ denotes the inverse metric. Note $-\Delta_{\mathfrak{g}}$ is self-adjoint non-negative on $L^2(\mathbb{R}^d, g^2dx)$, while $P = -g\Delta_{\mathfrak{g}}g^{-1}$ is self-adjoint non-negative on $L^2(\mathbb{R}^d, dx)$. Let $\Omega = \Omega_{k,\ell} := x_k\partial_\ell - x_\ell\partial_k$ be the rotational vector fields.

Let us therefore consider the corresponding linear equation. Let u be solution of

$$\begin{cases} (\partial_t^2 - \Delta_{\mathfrak{g}} + 1)u(t,x) = F(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

With the notation

$$A_\mu^\epsilon(T) = \begin{cases} 1 & \mu > \frac{1}{2} \\ \ln(2+T) & \mu = \frac{1}{2} \\ T^{1-2\mu+2\epsilon} & \mu < \frac{1}{2} \end{cases}$$

and

$$B_\mu^\epsilon(T) = \begin{cases} 1 & \mu > \frac{1}{2} \\ T^{1-2\mu+2\epsilon} & \mu \leq \frac{1}{2} \end{cases}$$

we have the following KSS estimate. Throughout this paper, $L_T^2 = L^2_{t\in[0,T]}$.

Theorem 1.2.1. *Assume that (H1) and (H2) hold with $\rho > 0$ and let $d \geq 3$. For all $\Psi \in C_0^\infty(\mathbb{R})$, $C > 0$ small, and $\varepsilon > 0$, the solution of (1.1) satisfies*

(1)For Low frequency, that is, if $0 < \mu \leq 1$,

$$\|\langle x\rangle^{-\mu} e^{it\sqrt{1+P}}\Psi(P \leq C)u\|^2_{L^2_{T,x}} \lesssim A_\mu^\epsilon(T)\|\langle x\rangle^{\mu} u\|^2_{L^2_x} \tag{1.2}$$

if $\mu > 1$, we have

$$\|\langle x\rangle^{-\mu} e^{it\sqrt{1+P}}\Psi(P \leq C)u\|^2_{L^2_{T,x}} \lesssim \|u\|^2_{L^2_x} \tag{1.3}$$

(2)For intermediate frequency,

$$\|\langle x\rangle^{-\mu}e^{it\sqrt{1+P}}\Psi(C\le P\le\frac{1}{C})u\|^2_{L^2_{T,x}}\lesssim B^{\epsilon}_{\mu}(T)\|u\|^2_{L^2_x} \tag{1.4}$$

(3)For high frequency,

$$\|\langle x\rangle^{-\mu}e^{it\sqrt{1+P}}\Psi(P\ge\frac{1}{C})u\|^2_{L^2_{T,x}}\lesssim B^{\epsilon}_{\mu}(T)\|u\|^2_{L^2_x} \tag{1.5}$$

Theorem 1.2.2. *Assume that* (H1) *and* (H2) *hold with* $\rho>0$, $\mu\ge\frac{1}{2}$ *and let* $d\ge 3$. *For all* $\varepsilon>0$, *the solution of* (1.1) *satisfies*

(1.6)

$$\big\|\langle x\rangle^{-\mu}\nabla_x u\big\|_{L^2_TL^2_x}+\big\|\langle x\rangle^{-\mu-1}u\big\|_{L^2_TL^2_x}\lesssim(A^{\epsilon}_{\mu}(T))^{\frac{1}{2}}(\|u'(0,\cdot)\|_{L^2_x}+\|u(0,\cdot)\|_{L^2_x}+\|\langle x\rangle^{1+\epsilon}F\|_{L^2_{T,x}})$$

where we use L^2_T *to denote* $L^2_{t\in[0,T]}$.

We also have the higher-order KSS estimates. Here, we put $Z=\{\partial_t,\partial_x,\Omega\}$, $Y=\{\partial_x,\Omega\}$, $X=\{\partial_x\}$. Then following the argument of Theorem 1.3 in [2] we have

Theorem 1.2.3. *Assume that* (H1) *and* (H2) *hold with* $\rho>\frac{3}{2}$. *Let* $N\ge 0$ *and* $\mu\ge\frac{1}{2}$. *The solution of* (1.1) *satisfies*

$$\sup_{0\le t\le T}\sum_{1\le k+j\le N+1}\|\partial_t^k P^{\frac{j}{2}}gu(t,\cdot)\|_{L^2_x}+\sum_{|\alpha|\le N}(A^{\epsilon}_{\mu}(T))^{-\frac{1}{2}}\|\langle x\rangle^{-\mu}(|\nabla_x(Z^\alpha u)|+|\frac{|Z^\alpha u|}{\langle x\rangle})\|_{L^2_TL^2_x}$$

$$\lesssim\sum_{|\alpha|\le N}\|(Y^\alpha u)'(0,\cdot)\|_{L^2_x}+\sum_{|\alpha|\le N}\int_0^T\|Z^\alpha F(s,\cdot)\|_{L^2_x}ds$$

Note that the estimate (1.2)-(1.5) can be viewed as the local energy decay estimate for $\Box_{\mathfrak{g}}+1=\partial_t^2-\Delta_{\mathfrak{g}}+1$. By combining these with local energy estimates we shall prove global Strichartz estimates for compact perturbation.

For the Minkowski case, it is known ([34]) that we have Strichartz estimates if (s,q,r) is admissible, i.e., $\frac{1}{q} \leq \min(\frac{1}{2}, \frac{d}{2}\left(\frac{1}{2}-\frac{1}{r}\right))$, $\quad (q,r) \neq (2,\infty),(\infty,\infty), \quad s = d\left(\frac{1}{2}-\frac{1}{r}\right) - \frac{1}{q}$. When $\Delta_{\mathfrak{g}}$ is the Laplace-Beltrami operator associated with a smooth, time independent Riemannian metric $g_{jk}(x)$ which we assume equals the Euclidean metric δ_{jk} for $|x| \geq R$, some R. Our global Strichartz estimates are the following.

Theorem 1.2.4 (Global Strichartz estimate for compact perturbation metric)**.** *Assume that* (H1) *and* (H2) *hold with* $\rho > 0$, $d \geq 3$ *and* $s \in [0,1]$ *(*$s \in (0,1)$ *if* $d = 3$*). The solution of* (1.1) *satisfies*

$$\left\|u\right\|_{L_t^q L_x^r} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \|F\|_{L_t^1 H^{s-1}}, \tag{1.7}$$

for any admissible (s,q,r) *with* $q > 2$, $r < \infty$.

Chapter 2

Strichartz estimate for compact metric perturbations

2.1 Mourre Estimate

2.1.1 The general setting

In this section, we'll present some estimate which will be used to prove our theorem. We begin this section with the notion of regularity with respect to an operator. A full presentation of this theory can be found in the book of Amrein, Boutet de Monvel, and Georgescu [31].

Definition 2.1.1. Let $(A, D(A))$and $(H, D(H))$ be self-adjoint operators on a separable Hilbert space $\mathbb{H}$. The operator H is of class $C^k(A)$ for $k > 0$, if there is $z \in \mathbb{C}\backslash\sigma(H)$ such that

$$\mathbb{R} \ni t \to e^{itA}(H - z)^{-1}e^{-itA},$$

is C^k for the strong topology of $L(\mathbb{H})$.

Note: The spectrum of a given operator H is often denoted $\sigma(H)$, and its complement,

the resolvent set, is denoted by $\mathbb{C}\backslash\sigma(H)$.

Let $H \in C^1(A)$ and $I \subset \sigma(H)$ be an open interval. We assume that A and H satisfy a Mourre estimate on I:

$$\mathbf{1}_I(H)i[H,A]\mathbf{1}_I(H) \geq \delta \mathbf{1}_I(H) \tag{2.1}$$

for some $\delta > 0$. As usual, we define the multi-commutators $\mathsf{ad}_A^j B$ inductively by $\mathsf{ad}_A^0 B = B$ and $\mathsf{ad}_A^{j+1} B = [A, \mathsf{ad}_A^j B] = A \cdot \mathsf{ad}_A^j B - \mathsf{ad}_A^j B \cdot A$.

Theorem 2.1.2. *Let $H \in C^2(A)$ be such that $\mathsf{ad}_A^j H$, $j = 1, 2$, are bounded on $\mathbb{H}$. Assume furthermore (2.1). Then for all closed intervals $J \subset I$ and $\mu > \frac{1}{2}$, there exists $C_{J,\mu} > 0$ such that*

$$\sup_{Rez \in J, Imz \neq 0} \|\langle A\rangle^{-\mu}(H-z)^{-1}\langle A\rangle^{-\mu}\| \leq C_{J,\mu} \tag{2.2}$$

Note: Throughout this dissertation, $\langle A\rangle = (1+A^2)^{\frac{1}{2}}$.

We now state a result of Kato [32] which is under the conclusions of Theorem 2.1.2.

Theorem 2.1.3 (H-smoothness)**.** *Let A and H be two self-adjoint operators satisfying (2.2). Then for all closed intervals $J \subset I$ and $\mu > \frac{1}{2}$,*

$$\int_{\mathbb{R}} \|\langle A\rangle^{-\mu} e^{-itH} \mathbf{1}_J(H) u\|^2 dt \leq 8 C_{J,\mu} \|u\|^2, \tag{2.3}$$

Corollary 2.1.4 (Corollary 2.5 in [3])**.** *Let A and H satisfy the assumption in the H-smoothness theorem. Then for all closed intervals $J \subset I$ and $0 < \mu \leq \frac{1}{2}$,*

$$\int_0^T \|\langle A\rangle^{-\mu} e^{-itH} \mathbf{1}_J(H) u\|^2 dt \leq M_{J,\mu,\epsilon} T^{1-2\mu+\epsilon} \|u\|^2, \tag{2.4}$$

for all $0 < \epsilon < 2\mu$. *Here,*

$$M_{J,\mu,\epsilon} = (8C_{J,\mu/(2\mu-\epsilon)})^{2\mu-\epsilon}$$

Theorem 2.1.5. *(Hunziker, Sigal, Soffer [33]) Let* $\mu > 0$ *and* $\bar{\mu} = \min\{n \in \mathbb{N}, n > \mu+1\}$. *Let H, A be two self-adjoint operators satisfying* (2.2). *Then for all* $\chi \in C_0^\infty(I)$

$$\|\langle A\rangle^{-\mu} e^{itH}\chi(H)u\| \leq P_{\mu,\chi}(\|\mathsf{ad}_A^1 H\|, \dots, \|\mathsf{ad}_A^{\bar{\mu}} H\|)\langle t\rangle^{-\mu}\|\langle A\rangle^{\mu} u\| \tag{2.5}$$

where $P_{\mu,\chi}$ *is a polynomial.*

Then, we look at

$$\begin{cases} (i\partial_t - H)u(t) = \varphi(H)G(t) \\ u(0,x) = 0. \end{cases} \tag{2.6}$$

Corollary 2.1.6. *(Proposition 2.7 and Corollary 2.8 in [3] Let A and H be two self-adjoint operators satisfying* (2.2) *and* $\varphi \in C^1(\mathbb{R})$ *satisfying* $\|\varphi\|_\infty \leq 1$, $\|\varphi'\|_\infty \leq C_1$ *and* $suppG \in [0,\infty)$, *the solution of* (2.6) *satisfies*

$$\int_0^T \|\langle A\rangle^{-\mu} u(t)\|^2 dt \lesssim C_{J,\mu,C_1}(B_\mu^\epsilon(T))^2 \int_0^T \|\langle A\rangle^{\mu} G(t)\|^2 dt \tag{2.7}$$

for all $0 < \epsilon < 2\mu$.

2.1.2 The Klein-Gordon equation and the Mourre Estimates

In this part, we will split our operator into low frequency, high frequency and intermediate one respectively.

Part 1: Low Frequency Mourre Estimates

Let $\psi_\lambda \in C^\infty(\mathbb{R})$, for all $j \in \mathbb{N}$, we have uniformly in $\lambda \geq 1$

$$|\partial_x^j \psi_\lambda(x)| \lesssim 1$$

and that there exists an open interval $I \Subset [0, \infty)$ and constants $\delta, B, C > 0$, such that

$$\forall x \geq B, \psi_\lambda(x) = c, \quad and \quad \forall x \in I, \psi_\lambda'(x) \geq \delta \quad for \quad all \quad \lambda \geq 1.$$

And here, $U \Subset V$ means that the closure of U is a compact subset of V.
The aim of this section is to establish a Mourre estimate for the operators $\psi_\lambda(\lambda P)$ which holds uniformly in $\lambda \gg 1$. Then if $A = \frac{1}{2}(xD + Dx)$, $D(A) = \{u \in L^2(\mathbb{R}^d); Au \in L^2(\mathbb{R}^d)\}$, we have

Proposition 2.1.7. *(Proposition 9 in [1]) (i)For all $j \in \mathbb{N}$, and $\epsilon > 0$, we have $\psi_\lambda(\lambda P) \in C^j(A)$ and*

$$\|\mathsf{ad}_A^j \psi_\lambda(\lambda P)\| \lesssim \begin{cases} 1, & if \quad d \geq 3 \\ \lambda^\epsilon, & if \quad d = 2 \end{cases} \tag{2.8}$$

(ii) For λ large enough,

$$\mathbf{1}_I(\lambda P) i[\psi_\lambda(\lambda P), A] \mathbf{1}_I(\lambda P) \geq \delta(\inf I) \mathbf{1}_I(\lambda P)$$

(iii) For all $\varphi \in C_0^\infty(\mathbb{R})$ and $\mu, \epsilon > 0$, we have

$$\|\langle A\rangle^\mu \varphi(\lambda P)\langle x\rangle^{-\mu}\| \leq \lambda^{-\frac{1}{2}\min\{\mu,\frac{d}{2}\}+\epsilon}$$

We will consider $\varphi \in C_0^\infty((0,\infty))$ such that

$$\forall x \in [0,1], \quad \sum_{\lambda \geq 1 dyadic} \varphi(\lambda x) = 1$$

To φ we associate $\tilde{\varphi} \in C_0^\infty((0,\infty))$ such that $\tilde{\varphi}\varphi = \varphi$.

Let now $I \Subset [0,\infty)$ be an open interval such that

$$\text{supp } \varphi + [-h,h] \subset I \text{ for some } h > 0 \text{ small enough}$$

We use the taylor development of $\sqrt{1+x}$ close to zero, find that

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \text{higher order}$$

We set

$$\psi_\lambda(x) = \begin{cases} 0, & for \quad x \leq 0 \\ \frac{1}{2}x + \lambda^{-\upsilon}x^{1+\upsilon}g(\frac{x}{\lambda}), & for \quad x \in I \\ c, & for \quad x \geq B \end{cases} \tag{2.9}$$

where $g(x) = \frac{\sqrt{x+1}-1-x/2}{x^{1+\upsilon}}$. Obviously, our $\psi_\lambda(\lambda P)$ satisfies the assumption in the beginning of low frequency part, thus satisfies 2.1.7. We need to prove $\lambda + \psi_\lambda(\lambda P)$ satisfies Theorem 2.1.2 too. It suffices to prove

$$\mathbf{1}_J(\lambda + \psi_\lambda(\lambda P))i[\lambda + \psi_\lambda(\lambda P), A]\mathbf{1}_J(\lambda + \psi_\lambda(\lambda P)) \geq \delta \mathbf{1}_J(\lambda + \psi_\lambda(\lambda P))$$

Since $f_\lambda(x) = \lambda + \psi_\lambda(x)$ is increasing, there exist open intervals $\tilde{J}$ and J such that

$$f_\lambda(\text{supp}\lambda) \subset \tilde{J} \Subset J \subset f_\lambda(I)$$

for λ large enough. Let $\tau \in C_0^\infty(J)$ be such that $\tau = 1$ on $\tilde{J}$, we get

$$\varphi(x) \prec \tau(f_\lambda(x)) \prec \mathbf{1}_J(f_\lambda(x)) \prec \mathbf{1}_I(x)$$

where $f \prec g$ means that $g = 1$ near the support of f.

Thus,

$$\mathbf{1}_J(f_\lambda(\lambda P))i[f_\lambda(\lambda P), A]\mathbf{1}_J(f_\lambda(\lambda P))$$

$$= \mathbf{1}_J(f_\lambda(\lambda P))\mathbf{1}_I(\lambda P)i[\psi_\lambda(\lambda P), A]\mathbf{1}_I(\lambda P)\mathbf{1}_J(f_\lambda(\lambda P)) \geq \delta(\inf I)\mathbf{1}_J(f_\lambda(\lambda P))$$

Then in low frequencies, $\sqrt{P+1} = 1 + \lambda^{-1}\psi_\lambda(\lambda P)$, we have the following properties:

Theorem 2.1.8. *Assume (H1)(H2)and $d \geq 3$, for all $\psi \in C_0^\infty(\mathbb{R})$ and $C > 0$ small, we have if $0 < \mu \leq \frac{d}{2}$,*

$$\|\langle x\rangle^{-\mu} e^{it\sqrt{1+P}}\Psi(P \leq C)u\|^2_{L^2_{t,x}} \lesssim A^\epsilon_\mu(T)\|\langle x\rangle^\mu u\|^2_{L^2_x} \tag{2.10}$$

Particularly, if $\mu > 1$, we have

$$\|\langle x\rangle^{-\mu} e^{it\sqrt{1+P}}\Psi(P \leq C)u\|^2_{L^2_{t,x}} \lesssim \|u\|^2_{L^2_x} \tag{2.11}$$

Proof. By Proposition 2.1.7 and Theorem 2.1.5, we have

$$\|\langle A\rangle^\mu \varphi(\lambda P)\langle x\rangle^{-\mu}\|_{L^2\to L^2} \lesssim \lambda^{-\frac{1}{2}\min\{\mu,\frac{d}{2}\}+\epsilon_1}$$

$$\|\langle A\rangle^{-\mu}e^{itf_\lambda(\lambda P)}\varphi(\lambda P)u\| \lesssim \lambda^{\epsilon_2}\langle t\rangle^{-\mu}\|\langle A\rangle^{\mu}u\|$$

Thus we obtain,

$$\|\langle x\rangle^{-\mu}e^{it\sqrt{1+P}}\Psi(P\le C)u\|_{L^2_x} \lesssim \sum_{\lambda \text{ dyadic},\lambda\ge 1}\|\langle x\rangle^{-\mu}e^{it\lambda^{-1}f_\lambda(\lambda P)}\varphi(\lambda P)u\|_{L^2_x}$$

$$\lesssim \sum_\lambda \|\langle x\rangle^{-\mu}\tilde{\varphi}(\lambda P)\langle A\rangle^{\mu}\langle A\rangle^{-\mu}e^{it\lambda^{-1}f_\lambda(\lambda P)}\tau(f_\lambda(\lambda P))\varphi(\lambda P)u\|$$

$$\lesssim \sum_\lambda \lambda^{-\frac{1}{2}\min\{\mu,\frac{d}{2}\}+\epsilon_1}\|\langle A\rangle^{-\mu}e^{it\lambda^{-1}f_\lambda(\lambda P)}\tau(f_\lambda(\lambda P))\varphi(\lambda P)u\|$$

$$\lesssim \sum_\lambda \lambda^{-\frac{1}{2}\min\{\mu,\frac{d}{2}\}+\epsilon_1+\epsilon_2}\langle t\lambda^{-1}\rangle^{-\mu}\|\langle A\rangle^{\mu}\varphi(\lambda P)u\|$$

$$\lesssim \sum_\lambda \lambda^{-\min\{\mu,\frac{d}{2}\}+2\epsilon_1+\epsilon_2}\frac{\lambda^{\mu-\epsilon_3}}{\langle t\rangle^{\mu}}\|\langle x\rangle^{\mu}u\| \lesssim \langle t\rangle^{-\mu}\|\langle x\rangle^{\mu}u\|$$

Then

$$\|\langle x\rangle^{-\mu}e^{it\sqrt{1+P}}\Psi(P\le C)u\|^2_{L^2_{T,x}} = \int_0^T \|\langle x\rangle^{-\mu}e^{it\sqrt{1+P}}\Psi(P\le C)u\|^2_{L^2_x}dt$$

$$\lesssim \int_0^T \langle t\rangle^{-2\mu}\|\langle x\rangle^{\mu}u\|^2_{L^2}dt = A^\epsilon_\mu(T)\|\langle x\rangle^{\mu}u\|^2_{L^2_x}$$

If $\mu > 1$, we can apply Theorem 2.1.2, that is

$$\int_{\mathbb{R}}\|\langle A\rangle^{-\mu}e^{itf_\lambda(\lambda P)}\varphi(\lambda P)u\|^2dt \lesssim \|u\|^2$$

Then, we have

$$\int_0^\infty \|\langle x\rangle^{-\mu}e^{it\sqrt{1+P}}\Psi(P\le C)u\|^2dt \lesssim \sum_\lambda\int_{\mathbb{R}}\|\langle x\rangle^{-\mu}e^{itf^{-1}\psi_\lambda(\lambda P)}\varphi(\lambda P)u\|^2dt$$

$$\lesssim \sum_\lambda \lambda^{-\min\{\mu,\frac{d}{2}\}+\epsilon_1}\int_{\mathbb{R}}\|\langle A\rangle^{-\mu}e^{it\lambda^{-1}f_\lambda(\lambda P)}\varphi(\lambda P)u\|^2dt$$

$$= \sum_\lambda \lambda^{-\min\{\mu,\frac{d}{2}\}+\epsilon_1+1}\int_{\mathbb{R}}\|\langle A\rangle^{-\mu}e^{isf_\lambda(\lambda P)}\varphi(\lambda P)u\|^2ds \lesssim \sum_\lambda \lambda^{1-\min\{\mu,\frac{d}{2}\}+\epsilon_1}\|u\|^2 \lesssim \|u\|^2.$$

□

Part 2: The high frequency and Mourre estimate

For high frequencies part, we have the following argument. For $f \in S_h(m) = \{f(x,\xi;h) \in C^\infty(\mathbb{R}^d \mathsf{x} \mathbb{R}^d) | \quad |\partial_x^\alpha \partial_\xi^\beta f(x,\xi;h)| \lesssim m(x,\xi)\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}\}$, the pseudodifferential operator with symbol f is given by

$$(Op_h(f)u)(x) = \frac{1}{(2\pi h)^d}\int e^{\frac{i}{h}<x-y,\xi>} f(\frac{x+y}{2},\xi;h)u(y)dyd\xi$$

For the symbol $a(x,\xi) \in S_h(\langle x\rangle)$, we define

$$A_h = Op_h(a) \in \Psi_h(\langle x\rangle) \text{ and } Q_h = Op_h(q) \in \Psi_h(1)$$

such that $q(x,\xi;h) = q_0(x,\xi) + h^\delta \Psi_h(1)$(See Section 3.1 in [1] for more details).

Proposition 2.1.9. *(Proposition 13 in [1]) (i)A_h is self-adjoint on $C_0^\infty(\mathbb{R}^d)$ and, for all $\mu \geq 0$, we have $\|\langle A_h\rangle^\mu u\| \lesssim \|\langle x\rangle^\mu u\|$*

(ii) We have $Q_h \in C^\infty(A_h)$ and for all $j \geq 1$, $\|\mathsf{ad}_{A_h}^j Q_h\| \lesssim h$

(iii) There exists $\epsilon > 0$ such that for all $J \Subset I$ and h small enough,

$$\mathbf{1}_J(Q_h) i[Q_h.A_h]\mathbf{1}_J(Q_h) \geq \epsilon h \mathbf{1}_J(Q_h)$$

Let $\hat{\varphi} \in C_0^\infty([0,\infty))$ and $\tilde{\varphi} \in C^\infty(\mathbb{R};\mathbb{R})$ be such that $\varphi(x) \prec \hat{\varphi}(x^{\frac{1}{2}}) \prec \tilde{\varphi}(x)$. Since $\sqrt{1+u} = 1 + uk(u)$, $k(u)$ smooth on $[0,1)$, then

$$\sqrt{1+x} = x^{\frac{1}{2}}(1+\frac{1}{x})^{\frac{1}{2}} = x^{\frac{1}{2}} + x^{-\frac{1}{2}}k(\frac{1}{x}) = x^{\frac{1}{2}} + x^{-\frac{1}{2}}g(x)$$

$$Q_h = (x^{\frac{1}{2}}h + x^{-\frac{1}{2}}hg(x))\tilde{\varphi}(h^2x) = k_h(h^2x)$$

where $k_h(x) = (x^{\frac{1}{2}} + h^2 x^{-\frac{1}{2}} g(\frac{x}{h^2}))\tilde{\varphi}(x)$. We can take $\tilde{\varphi}$ such that k_h is an increasing function satisfying $k_h = 0$ near 0 and $k_h(x)$ is constant for x large enough. Using that $g(\frac{1}{x}) \in C^\infty([0,1))$, k_h is C^∞ and all its derivatives are bounded and uniformly with respect to h.

Theorem 2.1.10. *Assume (H1)(H2)and $d \geq 3$, for high frequency, we have*

$$\|\langle x\rangle^{-\mu} e^{it\sqrt{1+P}} \Psi(P \geq \frac{1}{C}) u\|^2_{L^2_{T,x}} \lesssim B^\epsilon_\mu(T) \|u\|^2_{L^2_x} \tag{2.12}$$

Proof. For high frequencies, we can write

$$e^{it\sqrt{1+P}} = e^{ith^{-1}Q_h} \varphi(h^2 P) \text{ with } Q_h = (P^{\frac{1}{2}} h + P^{-\frac{1}{2}} h k(\tfrac{1}{P}))\tilde{\varphi}(h^2 P)$$

Just like the argument in [1], from the properties of k_h and the functional calculus of the pseudodiffernetial operators, we can get $Q_h = k_h(h^2 P)$ satisfies the assumptions in the beginning of b.2 with $I = supp\hat{\varphi}$. Similar to the low frequencies part, we apply Theorem 2.1.2 and Corollary 2.1.4,

$$\int_0^T \|\langle A_h\rangle^{-\mu} e^{itQ_h} \mathbf{1}_J(Q_h) u\|^2 dt \lesssim B^\epsilon_\mu(T) \|u\|^2$$

Thus,

$$\int_0^T \|\langle x\rangle^{-\mu} e^{it\sqrt{1+P}} \Psi(P \geq \frac{1}{C}) u\|^2 dt$$

$$\lesssim \sum_{h^2 \text{ dyadic}, 0<h<1} \int_0^T \|\langle x\rangle^{-\mu} \langle A_h\rangle^{\mu}\|^2 \|\langle A_h\rangle^{-\mu} e^{it\sqrt{1+P}} \varphi(h^2 P) u\|^2 dt$$

$$\lesssim \sum_h \int_0^T \|\langle A_h\rangle^{-\mu} e^{ith^{-1}Q_h} \varphi(h^2 P) u\|^2 dt$$

$$= \sum_h \int_0^{h^{-1}T} h \|\langle A_h\rangle^{-\mu} e^{isQ_h} \varphi(h^2 P) u\|^2 ds \lesssim B^\epsilon_\mu(T) \|u\|^2$$

□

Part 3: The intermediate frequency and Mourre estimate

Mimicking the low frequency argument, we set

$$\mathcal{A} = \varphi(P)A\varphi(P)$$

where $\varphi \in C_0^\infty([0,\infty))$ with $\varphi = 1$ near $[h, \frac{1}{h^2}]$. As before, $\mathcal{A}$ is essentially self-adjoint on $D(A)$. Apply the argument of Proposition 3.15 in [3], we obtain the following one:

Proposition 2.1.11. *(i) We have* $(1+P)^{\frac{1}{2}} \in C^2(\mathcal{A})$. *The commutators* $\mathrm{ad}_{\mathcal{A}}^j(\sqrt{1+P})$, $j = 1, 2$, *can be extended to bounded operator.*

(ii) For each $\sigma \in [C, \frac{1}{C}]$, *there exists* $\sigma > 0$ *such that*

$$\mathbf{1}_{[\sigma-\delta,\sigma+\delta]}(P)[i\sqrt{1+P}, \mathcal{A}]\mathbf{1}_{[\sigma-\delta,\sigma+\delta]}(P) \lesssim \mathbf{1}_{[\sigma-\delta,\sigma+\delta]}(P)$$

(iii) For all $\varphi \in C_0^\infty(\mathbb{R})$ *and* $\mu, \epsilon > 0$, *we have*

$$\|\langle \mathcal{A}\rangle^\mu \varphi(P)\langle x\rangle^{-\mu}\| \leq 1$$

Proof. (i) and (iii) follow directly from Proposition (2.1.7) with $\lambda = 1$

In order to prove (ii), we can apply the argument of Lemma 3.9 in Bony and Hafner [3] with $\lambda = 1$. In the following, a term r_j, $j \in \mathbb{N}$, will denote a smooth function such that

$$\partial_x^\alpha r_j(x) = O(\langle x\rangle^{-\rho-j-|\alpha|}).$$

Since

$$\varphi(P)(P+1)^{\frac{1}{2}} = \tfrac{1}{\pi}\textstyle\int_0^\infty s^{-\frac{1}{2}}\varphi(P)(P+1)(s+P+1)^{-1}ds$$

we have

$$[i(\sqrt{1+P}-1), \mathcal{A}] = i\varphi(P)[\sqrt{1+P}, A]\varphi(P)$$

$$= \frac{i}{\pi}\int_0^\infty s^{-\frac{1}{2}}\varphi(P)[(P+1)(s+P+1)^{-1}, A]\varphi(P)ds$$

$$= \frac{i}{\pi}\int_0^\infty s^{-\frac{1}{2}}\varphi(P)s(s+P+1)^{-1}[P, A_0](s+P+1)^{-1}\varphi(P)ds$$

$$= \frac{2}{\pi}\int_0^\infty s^{\frac{1}{2}}\varphi(P)(s+P+1)^{-1}(P+\tilde{\partial}^\star r_0(\tilde{\partial})+\tilde{\partial}^\star r_1+r_1\tilde{\partial})(s+P+1)^{-1}\varphi(P)ds$$

$$= \varphi^2(P)P(P+1)^{-\frac{1}{2}}+R$$

with $R = \frac{2}{\pi}\int_0^\infty s^{\frac{1}{2}}\varphi(P)(s+P+1)^{-1}(\tilde{\partial}^\star r_0(\tilde{\partial})+\tilde{\partial}^\star r_1+r_1\tilde{\partial})(s+P+1)^{-1}\varphi(P)ds$.
Apply the argument of lemma 3.10 in Bony and Hafner [3] with S replaced by $s+1$, the remainder term R can be estimated by

$$\|Ru\| \lesssim \lambda^{-\gamma+\epsilon}\|\langle x\rangle^{-min(\rho,\frac{d}{2})+2\gamma+\epsilon}u\|$$

for all $\epsilon > 0$, $0 \le \gamma \le \frac{d}{4}$.

Back to our proof of (ii), we have

$$[i(\sqrt{1+P}-1),\mathcal{A}] = P(1+P)^{-\frac{1}{2}}\varphi^2(P)+R$$

where $\|Ru\| \lesssim \| < x >^{-\nu} u\|$ for some $\nu > 0$.
Then $R\varphi(P)$ is a compact operator on $L^2(\mathbb{R}^d)$. Let $\sigma \in [h, \frac{1}{h^2}]$, since σ is not an eigenvalue of P,

$$\lim_{\delta\to 0}\mathbf{1}_{[\sigma-\delta,\sigma+\delta]}(P)R\varphi(P) = 0$$

Using

$$\mathbf{1}_{[\sigma-\delta,\sigma+\delta]}(P)[i(\sqrt{1+P}-1),\mathcal{A}]\mathbf{1}_{[\sigma-\delta,\sigma+\delta]}(P)$$

$$\le \frac{\sigma+\delta}{\sqrt{\sigma-\delta+1}}\mathbf{1}_{[\sigma-\delta,\sigma+\delta]}(P)+\mathbf{1}_{[\sigma-\delta,\sigma+\delta]}(P)R\varphi(P)\mathbf{1}_{[\sigma-\delta,\sigma+\delta]}(P)$$

we completed the proof of (ii). □

Then apply the Corollary 2.1.4, we obtain the intermediate frequency result.

Theorem 2.1.12. *Assume (H1)(H2) and $d \ge 2$, for intermediate frequency, we have*

$$\|\langle x\rangle^{-\mu}e^{it\sqrt{1+P}}\Psi(C \le \Psi \le \frac{1}{C})u\|^2_{L^2_{T,x}} \lesssim B^\epsilon_\mu(T)\|u\|^2_{L^2_x} \tag{2.13}$$

Part 4: Combination

Combining high, low and intermediate parts, we have if $0 < \mu \leq \frac{d}{2}$

$$\int_0^T \|\langle x\rangle^{-\mu} e^{it\sqrt{1+P}} v\|^2 dt \lesssim B_\mu^\epsilon(T) \|\langle x\rangle^\mu v\|^2 \tag{2.14}$$

Particularly when $\mu > 1$, we have

$$\int_0^\infty \|\langle x\rangle^{-\mu} e^{it\sqrt{1+P}} v\|^2 dt \lesssim \|v\|^2 \tag{2.15}$$

This one implies if $0 < \mu \leq \frac{d}{2}$,

$$\|\langle x\rangle^{-\mu}(\partial_t, \sqrt{1+P})u\|_{L^2([0,T]\times\mathbb{R}^d)} \lesssim (B_\mu^\epsilon(T))^{\frac{1}{2}} (\|\langle x\rangle^\mu(\partial_t, \sqrt{1+P})u(0,\cdot)\|_{L^2(\mathbb{R}^d)}$$

$$+ \int_0^T \|\langle x\rangle^\mu F(s,\cdot)\|_{L^2} ds)$$

Also, if $\mu > 1$,

$$\|\langle x\rangle^{-\mu}(\partial_t, \sqrt{1+P})u\|_{L^2(\mathbb{R}^{d+1})} \lesssim \|(\partial_t, \sqrt{1+P})u(0,\cdot)\|_{L^2(\mathbb{R}^d)}$$

$$+ \int_0^\infty \|F(s,\cdot)\|_{L^2} ds$$

Proposition 2.1.13. *Assume* $\frac{d}{2} \geq \mu > 1$,

i) Let

$$\begin{cases} (i\partial_t - \sqrt{1+P})v = G \\ v|_{t=0} = 0. \end{cases} \tag{2.16}$$

Then we have,

$$\int_0^\infty \|\langle x\rangle^{-\mu} v\|^2 dt \lesssim \int_0^\infty \|\langle x\rangle^{\mu} G\|^2 dt \tag{2.17}$$

ii) let

$$\begin{cases} (\partial_t^2 + P + 1)u(t,x) = G(t,x), \\ u(0,x) = 0, \quad \partial_t u(0,x) = 0. \end{cases} \tag{2.18}$$

Then we have,

$$\int_0^\infty \|\langle x\rangle^{-\mu} (\partial_t, \sqrt{1+P})u\|^2 dt \lesssim \int_0^\infty \|\langle x\rangle^{\mu} G\|^2 dt \tag{2.19}$$

Proof. i) Similar to the argument in Proposition 4.6 in [3], we have

$$\int_{\mathbb{R}} \|\langle x\rangle^{-\mu} v\|^2 dt \lesssim \sum_\lambda \int_{\mathbb{R}} \|\langle x\rangle^{-\mu} \varphi(\lambda P) v\|^2 dt + \int_{\mathbb{R}} \|\langle x\rangle^{-\mu} \varphi(P) v\|^2 dt$$

$$+ \sum_h \int_{\mathbb{R}} \|\langle x\rangle^{-\mu} \varphi(h^2 P) v\|^2 dt = I_1 + I_2 + I_3.$$

(a) We first estimate I_1,

$$I_1 \lesssim \sum_\lambda \lambda^{-\mu+\epsilon_1} \int_{\mathbb{R}} \|\langle A\rangle^{-\mu} \varphi(\lambda P) v\|^2 dt = \sum_\lambda \lambda^{1-\mu+\epsilon_1} \int_{\mathbb{R}} \|\langle A\rangle^{-\mu} \varphi(\lambda P) \tilde{v}\|^2 ds.$$

Notice that $\tilde{v}(s) = v(\lambda s)$ is a solution of the equation

$$\begin{cases} (i\partial_s - \lambda\sqrt{1+P})\tilde{v} = \lambda G(\lambda s), \\ \tilde{v}|_{s=0} = 0. \end{cases} \tag{2.20}$$

Since $\lambda\sqrt{1+P} = \lambda + \psi_\lambda(\lambda P)$, we can apply Corollary 2.1.6 with $H = \lambda + \psi_\lambda(\lambda P)$. Then we obtain,

$$I_1 \lesssim \sum_\lambda \lambda^{1-\mu+\epsilon_1} \int_{\mathbb{R}} \|\langle A\rangle^\mu \varphi(\lambda P)\lambda G(\lambda s)\|^2 ds = \sum_\lambda \lambda^{3-\mu+\epsilon_1} \int_{\mathbb{R}} \|\langle A\rangle^\mu \varphi(\lambda P) G(\lambda s)\|^2 ds$$

$$\lesssim \sum_\lambda \lambda^{3-2\mu+2\epsilon_1} \int_{\mathbb{R}} \|\langle x\rangle^\mu G(\lambda s)\|^2 ds = \sum_\lambda \lambda^{2-2\mu+2\epsilon_1} \int_{\mathbb{R}} \|\langle x\rangle^\mu G(t)\|^2 dt \lesssim \int_{\mathbb{R}} \|\langle x\rangle^\mu G(t)\|^2 dt$$

(b) We now study I_2, just replace $\lambda = 1$ in the I_1 case.

$$I_2 \lesssim \textstyle\int_{\mathbb{R}} \|\langle x\rangle^\mu G(t)\|^2 dt.$$

(c) We finally estimate I_3 applying $H = Q_h$

$$I_3 \lesssim \sum_h \int_{\mathbb{R}} \|\langle A_h\rangle^{-\mu} \varphi(h^2 P) v(t)\|^2 dt = \sum_h h \int_{\mathbb{R}} \|\langle A_h\rangle^{-\mu} \varphi(h^2 P) v(hs)\|^2 ds$$

$$\lesssim \sum_h h \int_{\mathbb{R}} \|\langle A_h\rangle^\mu h G(hs)\|^2 ds \lesssim \sum_h h^3 \int_{\mathbb{R}} \|\langle A_h\rangle^\mu G(hs)\|^2 ds$$

$$= \sum_h h^2 \int_{\mathbb{R}} \|\langle A_h\rangle^\mu G(t)\|^2 dt \lesssim \int_{\mathbb{R}} \|\langle A_h\rangle^\mu G(t)\|^2 dt \lesssim \int_{\mathbb{R}} \|\langle x\rangle^\mu G\|^2 dt$$

ii)Similar to (3.3) in [3], we can rewrite (2.18) as a first order system

$$i\partial_t \binom{u}{\partial_t u} = R\binom{u}{\partial_t u} + i\binom{0}{G},\ \binom{u}{\partial_t u}|_{t=0} = 0$$

with $R = \begin{pmatrix} 0 & i \\ -i(1+P) & 0 \end{pmatrix}$.

We introduce the following unitary transform: $U = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+P} & i \\ \sqrt{1+P} & -i \end{pmatrix}$ which satisfies $U^\star = U^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} (P+1)^{-\frac{1}{2}} & (P+1)^{-\frac{1}{2}} \\ -i & i \end{pmatrix}$ and $L = URU^\star = \begin{pmatrix} \sqrt{1+P} & 0 \\ 0 & -\sqrt{1+P} \end{pmatrix}$.
Now, we can see

$$\|\langle x\rangle^{-\mu}(\partial_t, \sqrt{1+P})u\|_{L^2}^2 \lesssim \|\langle x\rangle^{-\mu} M \tbinom{u(t)}{\partial_t u(t)}\|_{L^2\times L^2}^2$$

with $M = \begin{pmatrix} \sqrt{1+P} & 0 \\ 0 & 1 \end{pmatrix}$, $MU^\star = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$.
Thus,

$$\int_{\mathbb{R}} \|\langle x\rangle^{-\mu}(\partial_t, \sqrt{1+P})u\|^2 dt = \int_{\mathbb{R}} \|\langle x\rangle^{-\mu} M \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix}\|_{L^2\times L^2} dt.$$

And $v = U \tbinom{u(t)}{\partial_t u(t)}$ solves

$$(i\partial_t - L)v = iU \tbinom{0}{G},\ v|_{t=0} = 0.$$

Apply i), we have

$$\int_{\mathbb{R}} \|\langle x\rangle^{-\mu} M \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix}\|_{L^2\times L^2}^2 dt \lesssim \int_{\mathbb{R}} \|\langle x\rangle^{-\mu} v\|_{L^2\times L^2}^2 dt$$

$$\lesssim \int_{\mathbb{R}} \|\langle x\rangle^{\mu} iU \begin{pmatrix} 0 \\ G \end{pmatrix}\|_{L^2\times L^2}^2 dt = \int_{\mathbb{R}} \|\langle x\rangle^{\mu} G\|_{L^2}^2 dt.$$

□

Remark 2.1.14. Combining (2.1.2) and Proposition 2.12(ii), we get if $1 < \mu \leq \frac{d}{2}$,

$$\|\langle x\rangle^{-\mu}(\partial_t, \sqrt{1+P})u\|_{L^2_{t,x}} \lesssim \|(\partial_t, \sqrt{1+P})u(0,\cdot)\|_{L^2_x} + \|\langle x\rangle^{\mu} G\|_{L^2_{T,x}} \tag{2.21}$$

2.2 Local Energy Decay

Lemma 2.2.1. *If $s \in [-1, 1]$, then*

$$\|u\|_{H^s} \simeq \|(P+1)^{\frac{s}{2}} u\|_{L^2_x} \tag{2.22}$$

Proof. Since

$$\|(P+1)^{\frac{1}{2}} u\|^2_{L^2} = |\langle (P+1)^{\frac{1}{2}} u, (P+1)^{\frac{1}{2}} u\rangle|$$
$$= |\langle u, (P+1)u\rangle| = \langle u, Pu\rangle + \langle u, u\rangle = \|P^{\frac{1}{2}} u\|^2_{L^2} + \|u\|^2_{L^2}$$

we have

$$\|u\|_{H^1} \simeq \sum_{|\alpha|\le 1} \|\partial^\alpha u\|_{L^2} = \|u\|_{L^2} + \|\nabla u\|_{L^2}$$

$$\simeq \|u\|_{L^2} + \|P^{\frac{1}{2}} u\|_{L^2} \simeq \|(P+1)^{\frac{1}{2}} u\|_{L^2}$$

Then by interpolation and duality, we get the case $s \in [-1, 1]$. □

Lemma 2.2.2. $\|\langle x\rangle^{-\mu}(\nabla_x, 1)u\|_{L^2} \lesssim \|\langle x\rangle^{-\mu}\sqrt{1+P}u\|_{L^2}$

Proof. Firstly, we need prove $\|\langle x\rangle^{-\mu} u\|_{L^2} \lesssim \|\langle x\rangle^{-\mu}\sqrt{1+P}u\|_{L^2}$, it suffices to prove $\|\langle x\rangle^{-\mu}(1+P)^{-\frac{1}{2}}\langle x\rangle^{\mu}\|_{L^2} \lesssim 1$.

Consider the kernel of $(1+P)^{-\frac{1}{2}}$, we have the Laplace transform

$$(1+P)^{-\frac{1}{2}} = c\int_0^\infty e^{-P\lambda} e^{-\lambda} \lambda^{-\frac{1}{2}} d\lambda$$

By [5], it's known that the heat kernel $K(\lambda, x, y)$ of $e^{-P\lambda}$ satisfies

$$0 \le K(\lambda, x, y) \le C_\delta \lambda^{-\frac{n}{2}} e^{-\frac{d^2(x,y)}{(4+\delta)\lambda}}$$

for all $0 < \delta < 1$, and $t > 0$, where d is the Riemannian distance between x and y.

For P with the metric g^{ij}, since $\partial_x^\alpha(g_{ij} - \delta_{ij}) = O(\langle x\rangle^{-|\alpha|-\rho})$, we have $d^2(x, y) \simeq |x-y|^2$, then there exists $C_1 > 0$ such that

$$K(\lambda, x, y) \lesssim \lambda^{-C_1^2 \frac{n}{2}} e^{-\frac{|x-y|^2}{\lambda}}$$

Thus, the kernel of $(1+P)^{-\frac{1}{2}}$ is bounded by

$$\int_0^\infty \lambda^{-\frac{n}{2}} e^{-C_1^2 \frac{|x-y|^2}{\lambda}} e^{-\lambda} \lambda^{-\frac{1}{2}} d\lambda = C \int_0^\infty |x-y|^{-n+1} t^{n-2} e^{-t^2} e^{-C_1^2 \frac{|x-y|^2}{t^2}} dt$$

$$= C|x-y|^{-n+1} \int_0^\infty t^{n-2} e^{-t^2} e^{-\frac{|x-y|^2}{5t^2}} dt = O(|x-y|^{-n+1})$$

Thus, we get the kernel $\in L^1(\mathbb{R}^n)$ which implies

$$\|\langle x\rangle^{-\mu}(P+1)^{-\frac{1}{2}}\langle x\rangle^{\mu}\| \lesssim 1$$

Then it remains to prove $|\langle x\rangle^{-\mu}\nabla(P+1)^{-\frac{1}{2}}\langle x\rangle^{\mu}\| \lesssim 1$

For the operator $\nabla(P+1)^{-\frac{1}{2}}$, we have the Laplace transform

$$\nabla(1+P)^{-\frac{1}{2}} = C\int_0^\infty \nabla e^{-\lambda P} e^{-\lambda} \lambda^{-\frac{1}{2}} d\lambda$$

By [5], we have

$$|\nabla_x K| \le C_n \lambda^{-\frac{n}{2}} (\lambda^{-1}+\rho)^{\frac{1}{2}} (1+\tfrac{d^2}{\lambda})^{\frac{n+1}{2}} e^{-\frac{d^2}{4\lambda}}$$

where Ricci Curvature $\ge -\rho$ for some constant $\rho \ge 0$. WLOG, suppose $\rho = 0$.

Thus, the kernel of $\nabla(1+P)^{-\frac{1}{2}}$ is bounded by

$$\int_0^\infty \lambda^{-\frac{n}{2}} (\lambda^{-1}+\rho)^{\frac{1}{2}} (1+\frac{d^2}{\lambda})^{\frac{n+1}{2}} e^{-\frac{d^2}{4\lambda}} e^{-\lambda} \lambda^{-\frac{1}{2}} d\lambda$$

$$\lesssim |x-y|^{-n+1} \int_0^\infty t^{n-2} e^{-t^2} (\frac{4t^2}{d^2}+\rho)^{\frac{1}{2}} (1+4t^2)^{\frac{n+1}{2}} dt = O(|x-y|^{-n+1})$$

Similarly, we completed the proof. □

Theorem 2.2.3 (Local energy decay)**.** *For the linear equation $(\partial_t^2 - \Delta_g + 1)u = F$, if, for $|x| > R$ with R fixed, $F(t,x) = 0$, then for fixed $\beta \in C_0^\infty(\mathbb{R}^d)$, we have*

$$\|\beta\partial_t u\|_{L_t^2 L_x^2} + \|\beta u\|_{L_t^2 H^1} \lesssim \|u_0\|_{H^1} + \|u_1\|_{L_x^2} + \|F\|_{L_t^2 L_x^2} \tag{2.23}$$

Moreover, if $F \equiv 0$ and $s \in [0,1]$, then

$$\|\beta\partial_t u\|_{L_t^2 H^{s-1}} + \|\beta u\|_{L_t^2 H^s} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} \tag{2.24}$$

Proof. Let $v = gu, G = gF$, then $(\partial_t^2 - g\Delta_g g^{-1} + 1)v = G$. We will use Remark (2.1.14) and set $\frac{d}{2} \geq \mu > 1$ in the following.

$$\|\beta(\partial_t u, \partial_x u, u)\|_{L_t^2 L_x^2} \lesssim \|\langle x\rangle^{-\mu}(\partial_t u, \partial_x u, u)\|_{L_t^2 L^2_{|x|\leq R}}$$

$$\lesssim \|\langle x\rangle^{-\mu}(\partial_t, \sqrt{1+P})u\|_{L^2_{t,x}} \lesssim \|\langle x\rangle^{-\mu}(\partial_t, \sqrt{1+P})v\|_{L^2_{t,x}}$$

$$\lesssim \|(\partial_t, \sqrt{P+1})v(0,\cdot)\|_{L_x^2} + \|\langle x\rangle^{\mu} G\|_{L_x^2 L_T^2}$$

$$\lesssim \|(\partial_t, \sqrt{P+1})v(0,\cdot)\|_{L_x^2} + \|\langle x\rangle^{\mu} G\|_{L_x^2 L_T^2}$$

$$\lesssim \|(\partial_t, \sqrt{P}, 1)v(0,\cdot)\|_{L_x^2} + \|\langle x\rangle^{\mu} G\|_{L_x^2 L_T^2}$$

$$\lesssim \|(\partial_t, \tilde{\partial}_x, 1)v(0,\cdot)\|_{L_x^2} + \|\langle x\rangle^{\mu} G\|_{L_x^2 L_T^2}$$

$$\lesssim \|(\partial_t, \partial_x, 1)u(0,\cdot)\|_{L_x^2} + \|G\|_{L_x^2 L_t^2}$$

Then by interpolation and duality, we can get the second statement. □

Remark 2.2.4. If $1 < \mu \leq \frac{d}{2}$, $\|\langle x\rangle^{-\mu}(\partial_t, \partial_x, 1)u\|_{L^2_{t,x}} \lesssim \|(\partial_t, \partial_x, 1)u(0,\cdot)\|_{L_x^2} + \|\langle x\rangle^{\mu} F\|_{L^2_{t,x}}$

2.3 KSS Estimates

If $\mu \geq \frac{1}{2}$, there will be a little difference for $\| < x >^{-\frac{1}{2}} u'\|_{L^2_{T,x}}$. Refer to [16], we have

Lemma 2.3.1. *Let $\Box_h = \partial_t^2 - \Delta + h^{\alpha\beta}(t,x)\partial_\alpha\partial_\beta$, $h^{\alpha\beta} = h^{\beta\alpha}$ and $\sum |h^{\alpha\beta}| \leq \delta$. Then if*

$\delta > 0$ is small enough, $d \geq 3$, the solution to the equation $\Box_h u + u = F$ satisfies

$$(\ln(2+T))^{-1}\big\|\langle x\rangle^{-1/2}(|\nabla_x u| + \frac{|u|}{\langle x\rangle})\big\|^2_{L^2([0,T]\times\mathbb{R}^d)} + \big\|\langle x\rangle^{-1/2-\epsilon}(|\nabla_x u| + \frac{|u|}{\langle x\rangle})\big\|^2_{L^2([0,T]\times\mathbb{R}^d)}$$
$$\lesssim \|u'(0,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \|u(0,\cdot)\|^2_{L^2(\mathbb{R}^d)}$$

(2.25)
$$+\int_0^T \int \left(|\nabla_x u| + \frac{u}{|x|}\right)\left(|F| + (|h'| + \frac{h}{|x|})|\nabla_x u|\right) dxdt + (\int_0^T \|F\|_{L^2} + \|\partial h \cdot \nabla_x u\|_{L^2} dt)^2$$

for any $\epsilon > 0$.

Theorem 2.3.2. *Assume that (H1) and (H2) hold with $\rho > 0$, $d \geq 3$ and $\mu \geq \frac{1}{2}$. For all $\epsilon > 0$, the solution of $(\partial_t^2 - \Delta_g + 1)u = F$ with initial data (u_0, u_1) satisfies*

(2.26)
$$\|\langle x\rangle^{-\frac{1}{2}-\epsilon}(|\nabla_x u| + \frac{u}{|x|})\|_{L^2_{T,x}} + (\ln(2+T))^{-\frac{1}{2}}\|\langle x\rangle^{-\frac{1}{2}}(|\nabla_x u| + \frac{u}{|x|})\|_{L^2_{T,x}} \lesssim \langle u'(0,\cdot)\rangle_{L^2_x} + \langle u(0,\cdot)\rangle_{L^2_x}$$
$$+\|\langle x\rangle^{1+\epsilon} F\|_{L^2_{T,x}}.$$

We introduce a cutoff function $\phi \in C_c^\infty$ which equals 1 in the unit ball B_1 and support in B_2 and let $\phi_R(x) = \phi(x/R)$. Let $v = \phi_R u$ and $w = (1-\phi_R)u$ with $R \gg 1$ to be determined later, then

$$\Box_{\mathfrak{g}} v = [-\Delta_{\mathfrak{g}}, \phi_R]u + \phi_R F := f + \phi_R F.$$

$$\Box_{\mathfrak{g}} w = -[-\Delta_{\mathfrak{g}}, \phi_R]u + (1-\phi_R)F := -f + (1-\phi_R)F.$$

By Remark(2.2.4), we have

$$\|f\|_{L^2_{t,x}} \lesssim \|\phi_{2R}\partial_x u\|_{L^2_{t,x}} + \|\phi_{2R}u\|_{L^2_{t,x}} \lesssim \|u_0\|_{H^1_x} + \|u_1\|_{L^2_x} + \|\langle x\rangle^{1+\epsilon}F\|_{L^2_T L^2_x}. \tag{2.27}$$

Note that v is compactly supported in x, so local energy decay applies again, i.e.

$$\big\|v\big\|_{L^2_t H^1_x} + \big\|\partial_t v\big\|_{L^2_{t,x}} \lesssim \|u_0\|_{H^1_x} + \|u_1\|_{L^2_x} + \|\langle x\rangle^{1+\epsilon}F\|_{L^2_T L^2_x}. \tag{2.28}$$

If we define

$$\tilde{g}^{ij} = (1-\phi_{R/2})g^{ij} + \phi_{R/2}\delta^{ij}, \tag{2.29}$$

then

$$\tilde{\Delta} = \sum_{ij} \tilde{g}^{-2}\partial_i(\tilde{g}^{ij}\tilde{g}^2\partial_j) = \Delta + h^{ij}\partial_i\partial_j + b^i\partial_i + c$$

where $h^{ij} = (1-\phi_{R/2})(g^{ij}-\delta^{ij})$, $b^i = (1-\phi_{R/4})r_1^i$ and $c = (1-\phi_{R/4})r_2$, with the r_j satisfying the bounds in (H1). Thus since, $\tilde{\Delta}w = \Delta_{\mathfrak{g}}w$, w satisfies

$$(\partial_t^2 - \Delta - h^{ij})w = \Box_{\mathfrak{g}}w + (b^i\partial_i + c)w = (1-\phi_R)F - f + (b^i\partial_i + c)w := G. \tag{2.30}$$

Note that by (H1) and choosing $R \gg 1$ large enough and $R > \max\{(\frac{2}{\epsilon})^{\frac{2}{\rho}}, T^{\frac{1}{2\rho}}, T\}$, h^{ij}

will be small enough so that we can apply Lemma 4.1 with $\epsilon < \min\{\rho/4, 1/4, 2^{1-2\rho}\}$,

$$\begin{aligned}
&\|\langle x\rangle^{-\frac{1}{2}-\epsilon}(|\nabla_x w| + \frac{w}{|x|})\|^2_{L^2_{T,x}} + (\ln(2+T))^{-1}\|\langle x\rangle^{-\frac{1}{2}}(|\nabla_x w| + \frac{w}{\langle x\rangle})\|^2_{L^2([0,T]\times\mathbb{R}^d)} \\
\lesssim &\|w'(0,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \|w(0,\cdot)\|^2_{L^2(\mathbb{R}^d)} \\
+\int_0^T \int \left(|\nabla_x w| + \frac{w}{|x|}\right)\left(|G| + (|h'| + \frac{h}{|x|})|\nabla_x w|\right) dxdt + (\int_0^T \|G\|_{L^2} + \|\nabla_x h\cdot\nabla_x w\|_{L^2} dt)^2 \\
\lesssim &\|w'(0,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \|w(0,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \|\langle x\rangle^{-1/2-\epsilon}\left(|\nabla_x w| + \frac{w}{|x|}\right)\|_{L^2_{T,x}}\|\langle x\rangle^{1/2+\epsilon}G\|_{L^2_{T,x}} \\
&+\|\langle x\rangle^{-1/2-\rho/2}\left(|\nabla_x w| + \frac{w}{|x|}\right)\|^2_{L^2_{T,x}} + (\int_0^T \|G\|_{L^2} + \|\partial h\nabla_x u\|_{L^2} dt)^2 \\
\lesssim &\|w'(0,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \|w(0,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{4}\|\langle x\rangle^{-1/2-\epsilon}\left(|\nabla_x w| + \frac{w}{|x|}\right)\|^2_{L^2_{T,x}} + \|\langle x\rangle^{1/2+\epsilon}G\|^2_{L^2_{T,x}} \\
&+R^{-\rho/2}\|\langle x\rangle^{-1/2-\frac{\rho}{4}}\left(|\nabla_x w| + \frac{w}{|x|}\right)\|^2_{L^2_{T,x}} + \|G\|^2_{L^1_T L^2_x} + \|\langle x\rangle\nabla_x h\|^2_{L^2_T L^\infty_x}\|\langle x\rangle^{-1}\nabla_x w\|^2_{L^2_x L^2_T} \\
\lesssim &\|w'(0,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \|w(0,\cdot)\|^2_{L^2(\mathbb{R}^d)} + (\frac{1}{4} + \frac{\epsilon}{2})\|\langle x\rangle^{-1/2-\epsilon}\left(|\nabla_x w| + \frac{w}{|x|}\right)\|^2_{L^2_{T,x}} + \|\langle x\rangle^{1/2+\epsilon}G\|^2_{L^2_{T,x}} \\
&+\|G\|^2_{L^1_T L^2_x} + (\frac{\epsilon}{2})^{\frac{1}{\rho}}\|\langle x\rangle^{-\frac{1}{2}-\epsilon}\nabla_x w\|^2_{L^2_{T,x}}.
\end{aligned}$$

Since we know

$$\|\langle x\rangle^{1/2+\epsilon}(b^i\partial_i + c)w\|_{L^2_{T,x}} \lesssim \|\langle x\rangle^{-1/2-\rho+\epsilon}\left(|\nabla_x w| + \frac{w}{|x|}\right)\|_{L^2_{T,x}} \lesssim \frac{\epsilon}{2}\|\langle x\rangle^{-1/2-\epsilon}\left(|\nabla_x w| + \frac{w}{|x|}\right)\|_{L^2_{T,x}},$$

Thus combining equation (2.27),

$$\|\langle x\rangle^{-\frac{1}{2}-\epsilon}(|\nabla_x w| + \frac{w}{|x|})\|^2_{L^2_{T,x}} + (\ln(2+T))^{-1}\|\langle x\rangle^{-\frac{1}{2}}(|\nabla_x w| + \frac{w}{\langle x\rangle})\|_{L^2([0,T]\times\Re^d)}$$

$$\lesssim \|w'(0,\cdot)\|^2_{L^2} + \|w(0,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \|\langle x\rangle^{\frac{1}{2}+\epsilon}F\|_{L^2_{T,x}} + \|G\|_{L^1_T L^2_x} + \|u_0\|^2_{H^1} + \|u_1\|^2_{L^2} + \|\langle x\rangle^{1+\epsilon}F\|_{L^2_{T,x}}.$$

Next step, we want to transfer $\|G\|_{L^1_T L^2_x}$ to $\|\langle x\rangle^{1+\epsilon}F\|_{L^2_{T,x}}$.

We have

$$\|G\|_{L^1_T L^2_x} \lesssim \|F\|_{L^1_T L^2_x(|x|>R)} + \|[\Delta_g, \phi_R]u\|_{L^1_T L^2_x} + \|(b^i\partial_i + c)w\|_{L^1_T L^2_x},$$

where

$$\|F\|_{L^1_T L^2_x(|x|>R)} \lesssim \|\langle x\rangle^{-\frac{1}{2}-\epsilon}\|_{L^2_T L^\infty_x} \|\langle x\rangle^{\frac{1}{2}+\epsilon} F\|_{L^2_{T,x}} \lesssim \|\langle x\rangle^{\frac{1}{2}+\epsilon} F\|_{L^2_{T,x}}$$

and

$$\|(b^i\partial_i + c)w\|_{L^1_T L^2_x} \lesssim \|\langle x\rangle^{-\frac{1}{2}-\epsilon}\|_{L^\infty_x L^2_T} \|\langle x\rangle^{\frac{1}{2}+\epsilon}(b^i\partial_i + c)w\|_{L^2_{T,x}} \lesssim \frac{\epsilon}{2}\|\langle x\rangle^{-\frac{1}{2}-\epsilon}(|\partial_x w| + \frac{w}{|x|})\|_{L^2_{T,x}}$$

Thus we get

$$(\ln(2+T))^{-\frac{1}{2}}\|\langle x\rangle^{-\frac{1}{2}}(|\nabla_x w| + \frac{w}{|x|})\|_{L^2_{T,x}} + \|\langle x\rangle^{-\frac{1}{2}-\epsilon}(|\nabla_x w| + \frac{w}{\langle x\rangle})\|_{L^2([0,T]\times\Re^d)}$$

$$\lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2_x} + \|\langle x\rangle^{1+\epsilon} F\|^2_{L^2_{T,x}}.$$

Combining w and v parts, we get finally that

$$(\ln(2+T))^{-\frac{1}{2}}\|\langle x\rangle^{-\frac{1}{2}}(|\nabla_x w| + \frac{w}{|x|})\|_{L^2_{T,x}} + \|\langle x\rangle^{-\frac{1}{2}-\epsilon}(|\nabla_x w| + \frac{w}{\langle x\rangle})\|_{L^2([0,T]\times\Re^d)}$$

$$\lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2_x} + \|\langle x\rangle^{1+\epsilon} F\|_{L^2_{T,x}},$$

Therefore, we can use Lemma 4.1 to see that w also satisfies the bounds, which completes the proof.

2.4 Strichartz estimate for compact perturbation metric

In the section, we'll consider the Strichartz estimate for the following KG equation

$$\begin{cases} (\partial_t^2 + P + 1)u(t,x) = 0 \\ (u|_{t=0}, \partial_t u|_{t=0}) = (u_0, u_1). \end{cases} \tag{2.31}$$

where $P = -g\Delta_g g^{-1}$ and Δ_g is the Laplace-Beltrami operator with a smooth, time independent Riemannian metric g_{jk} which equals the Euclidean metric δ_{jk} for $|x| \geq R$, for some R.

Lemma 2.4.1. *(Theorem 1.1 in [29]) If $(\partial_t, \partial_x)^\alpha a^{ij}(t,x) \in L^1_{t\in[0,1]}L^\infty_x$, for any $|\alpha| = 2$ and $(\partial_t^2 - \partial_i a^{ij}(t,x)\partial_j u = F)$, then*

$$\|D^{1-s}u\|_{L^q_{t\in[0,1]}L^r_x} \lesssim \|u'\|_{L^\infty_{t\in[0,1]}L^2_x} + \|F\|_{L^1_{t\in[0,1]}L^2_x} \tag{2.32}$$

for any wave admissible pair with $r < \infty$.

Note, we can write $(\partial_t^2 + P + 1) = \partial_t^2 - \partial_i g^{ij}\partial_j + b_i\partial_i + c + 1$ with $b_i = O(\langle x\rangle^{-1-\rho})$ and $c = O(\langle x\rangle^{-2-\rho})$ for $a^{ij} = g^{ij}$ and $|\alpha| = 2$ which implies $\partial_x^\alpha a^{ij}(x) = O(\langle x\rangle^{-2-\rho}) \in L^1_{t\in[0,1]}L^\infty_x$. Then we have

$$\begin{aligned} \|D^{1-s}u\|_{L^q_{t\in[0,1]}L^r_x} &\lesssim \|u'\|_{L^\infty_{t\in[0,1]}L^2_x} + \|b_i\partial_i u + cu + u\|_{L^1_t L^2_x} \lesssim \|u'\|_{L^\infty_t L^2_x} + \|u\|_{L^1_t L^2_x} \\ &\lesssim \|u'\|_{L^\infty_t L^2_x} + \|u\|_{L^\infty_t L^2_x} \lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2_x} \end{aligned}$$

Proposition 2.4.2 (Local Strichartz estimates)**.** *Let $s \in [0,1], (s,q,r)$ Klein-Gordon admissible with $r < \infty$, and let u be the solution of equation of* (2.31)*, then*

$$\|u\|_{L^q_{t\in[0,1]}L^r_x} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} \tag{2.33}$$

Proof. Since $(\partial_t^2 + P + 1)u = 0$, we have $(\partial_t^2 + P + 1)P^{\frac{s-1}{2}}u = 0$.

Thus,

$$\|u\|_{L^q_{t\in[0,1]}L^r_x} \lesssim \|P^{\frac{1-s}{2}}P^{\frac{s-1}{2}}u\|_{L^q_{t\in[0,1]}L^r_x} \lesssim \|D^{1-s}P^{\frac{s-1}{2}}u\|_{L^q_{t\in[0,1]}L^r_x}$$

$$\lesssim \|P^{\frac{s-1}{2}}u_0\|_{H^1} + \|P^{\frac{s-1}{2}}u_1\|_{L^2} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}$$

If (s, q, r) is wave admissible,$s \in [0, 1]$ and $r < \frac{d}{1-s}$. □

Now we get the local Strichartz estimates with wave-admissible pairs. Since it's local in time, thus, we can extend the wave-admissible pair to Klein-Gordon admissible pair. Hence we proved the Local Strichartz estimate for Klein-Gordon equations.
Refer to [34], we have the following theorem:

Theorem 2.4.3 (global Strichartz estimates)**.** *On $\mathbb{R}\times\mathbb{R}^n$, and $\Box u + u = 0$, the solution u satisfies*

$$(2.34)\quad (\int_0^T \|u(t,\cdot)\|^p_{W^{1-s,q}}dt)^{\frac{1}{p}} + (\int_0^T \|\partial_t u(t,\cdot)\|^p_{W^{-s,q}}dt)^{\frac{1}{p}} \lesssim \|u(0,\cdot)\|_{H^1} + \|(\partial_t u)(0,\cdot)\|_{L^2}$$

if $\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - s$ and $\frac{2}{p} + \frac{n}{q} \le \frac{n}{2}$.

Hypothesis 1. Assume given $R_0 > 0$,

$$(2.35)\qquad \int_0^\infty (\|u\|^2_{H^1(|x|<R_0)} + \|\partial_t u\|^2_{L^2(|x|<R_0)})dt \lesssim \|u_0\|^2_{H^1} + \|u_1\|^2_{L^2} + \int_0^\infty \|F\|^2_{L^2}ds$$

where u is a solution of $(\partial_t^2 - \Delta_g + 1)u = F$ with initial data $(u_0, u_1) = (u(0,\cdot), \partial_t u(0,\cdot))$ and F vanishes for $|x| > R_0$.

Definition 2.4.4. (X, γ, q) is an admissible triple if

$$\|v\|_{L_t^q X(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|v(0,\cdot)\|_{H^\gamma} + \|\partial_t v(0,\cdot)\|_{H^{\gamma-1}} \tag{2.36}$$

assuming that $(\partial_t^2 - \Delta + 1)v = 0$ in $\mathbb{R} \times \mathbb{R}^n$.

and

$$\|v\|_{L_t^q X([0,1]\times\mathbb{R}^n)} \lesssim \|v(0,\cdot)\|_{H^\gamma(\mathbb{R}^n)} + \|\partial_t v(0,\cdot)\|_{H^{\gamma-1}(\mathbb{R}^n)} \tag{2.37}$$

assuming that $(\partial_t^2 - \Delta_g + 1)v = 0$ in $[0,1] \times \mathbb{R}^n$.

Theorem 2.4.5. *Let $n \geq 2$ and assume that (X, γ, q) is admissible triple. Then if Hypothesis 1 is valid and if u solves $(\partial_t^2 - \Delta_g + 1)u = 0$ with initial data (u_0, u_1), we have the global abstract Strichartz estimates*

$$\|u\|_{L_t^q X(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u_0\|_{H^\gamma} + \|u_1\|_{H^{\gamma-1}}. \tag{2.38}$$

Obviously, (2.36), (2.37) are satisfied by Theorem (2.4.3) and Proposition (2.4.2) respectively. Also by local energy decay, we can get the Hypothesis 1. In order to prove Theorem 1.4, it is sufficient to prove Theorem (2.4.5) with $X = L_x^r$.

Proposition 2.4.6. *Let w solve*

$$\begin{cases} (\partial_t^2 - \Delta + 1)w(t,x) = F \\ w|_{t=0} = w_0, \partial_t w|_{t=0} = w_1. \end{cases} \tag{2.39}$$

Assume $q > 2$ and $\gamma \geq -\frac{n-3}{2}$ then if $F(t,x) = 0$ if $|x| > 2R$, we have

$$\|w\|_{L_t^q X(\mathbf{R}_+ \times \mathbf{R}^n)} \lesssim \|w_0\|_{H^\gamma} + \|w_1\|_{H^{\gamma-1}} + \|F\|_{L_t^2 H^{\gamma-1}(\mathbf{R}_+ \times \mathbf{R}^n)} \tag{2.40}$$

In order to mimic the proof of Proposition 2.1 in [8], we need the following lemma.

Lemma 2.4.7. *Fix $\beta \in C_0^\infty(\mathbb{R}^n)$ and assume that $\gamma \leq \frac{n-1}{2}$. Then*

$$\int_{\mathbb{R}} \|\beta(\cdot)(e^{it\langle D\rangle} f)(t,\cdot)\|^2_{H^\gamma(\mathbb{R}^n)} dt \lesssim \|f\|^2_{H^\gamma}, \tag{2.41}$$

if $\langle D\rangle = \sqrt{1-\Delta}$.

Proof. Apply Plancherel's theorem over t, x, the left side of (2.41) can be written as $(2\pi)^{n+1}$ times

$$\int_0^\infty \int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} \hat{\beta}(\xi - \eta)\hat{f}(\eta)\delta(\sqrt{|\eta|^2+1} - \tau)d\eta|^2 (1+\xi^2)^\gamma d\xi d\tau \tag{2.42}$$

We next apply the Schwartz inequality over η to bound this by

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} [\int \hat{\beta}(\xi-\eta)\delta(\sqrt{|\eta|^2+1}-\tau)d\eta] \times [\int \hat{\beta}(\xi-\eta)|\hat{f}(\eta)|^2 \delta(\sqrt{|\eta|^2+1}-\tau)d\eta \times (1+\xi^2)^\gamma d\xi d\tau \tag{2.43}$$

Using the properties of δ-function, we have that

$$(1+\xi^2)^\gamma \int |\hat{\beta}(\xi-\eta)|\delta(\sqrt{|\eta|^2+1}-\tau)d\eta$$

$$\lesssim (1+\xi^2)^\gamma (1+||\xi|-\sqrt{\tau^2-1}|)^{-N}\tau(\tau^2-1)^{\frac{n-2}{2}}(1+\sqrt{\tau^2-1})^{-n+1}$$

Thus, the left hand side is bounded by

$$\int |\hat{f}(\eta)|^2(1+|\eta|^2)^{\gamma+\frac{1}{2}}|\eta|^{n-2}(1+|\eta|)^{-n+1}d\eta \lesssim \|f\|^2_{H^\gamma}$$

. □

Proposition 2.4.8. *Let u solves $(\partial_t^2-\Delta_g+1)u=F$ and assume that the initial data $u_0(x)=u_1(x)=F(t,x)=0$ when $|x|>2R$. Then if (X,γ,q) is admissible triple with $q>2$ and $\gamma\geq -\frac{n-3}{2}$ we have*

$$\|u\|_{L_t^q L_x^r(\mathbb{R}_+\times\mathbb{R}^n)} \lesssim \|u_0\|_{H^\gamma}+\|u_1\|_{H^{\gamma-1}}+\|F\|_{L_t^2 H^{\gamma-1}(\mathbb{R}_+\times\mathbb{R}^n)} \tag{2.44}$$

Proof. We already have

(2.45)

$$\|\beta u\|_{L_t^\infty H^\gamma}+\|\beta\partial_t u\|_{L_t^\infty H^{\gamma-1}}+\|\beta u\|_{L_t^2 H^\gamma}+\|\beta\partial_t u\|_{L_t^2 H^{\gamma-1}} \lesssim \|u_0\|_{H^\gamma}+\|u_1\|_{H^{\gamma-1}}+\|F\|_{L_t^2 H^{\gamma-1}}$$

The L_t^2 estimate with $s=1$ comes from local energy decay, then the estimates follow from Duhamel's formula and duality.

To prove (2.45), let us fix $\beta\in C_0^\infty(\mathbb{R}^n)$ satisfying $\beta(x)=1, |x|\leq 3R$ and write

$$u=v+w \text{ where } v=\beta u, w=(1-\beta)u$$

Then w solves the free Klein-Gordon equation,

$$\begin{cases} (\partial_t^2 - \Delta + 1)w = [\beta, \Delta]u \\ w|_{t=0} = \partial_t w|_{t=0} = 0. \end{cases} \tag{2.46}$$

By Proposition (2.4.6), there exists a cutoff function $\rho \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned} \|w\|_{L_t^q L_x^r} &\lesssim \|[\beta, \Delta]u\|_{L_t^2 H^{\gamma-1}} \lesssim \|\rho u\|_{L_t^2 H^\gamma} \\ &\lesssim \|u_0\|_{H^\gamma} + \|u_1\|_{H^{\gamma-1}} + \|F\|_{L_t^2 H^{\gamma-1}} \end{aligned}$$

As a result, we're left with showing that if $v = \beta u$ then

$$\|v\|_{L_t^q L_x^r(\mathbb{R}_+ \times \mathbb{R}^n)} \lesssim \|u_0\|_{H^\gamma} + \|u_1\|_{H^{\gamma-1}} + \|F\|_{L_t^2 H^{\gamma-1}}$$

To do this, fix $\varphi \in C_0^\infty((-1,1))$ satisfying $\sum_{j=-\infty}^{\infty} \varphi(t-j) = 1$. For a given $j \in \mathbb{N}$, let $v_j = \varphi(t-j)v$, then v_j solves

$$\begin{cases} (\partial_t^2 - \Delta_g + 1)v_j = -\varphi(t-j)[\Delta, \beta]u + [\partial_t^2, \varphi(t-j)]\beta u + \varphi(t-j)\beta F \\ v_j|_{t=0} = \partial_t v_j|_{t=0} = 0. \end{cases} \tag{2.47}$$

while $v_0 = v - \sum_{j=1}^{\infty} v_j$ solves

$$\begin{cases} (\partial_t^2 - \Delta_g + 1)v_0 = -\tilde{\varphi}[\Delta, \beta]u + [\partial_t^2, \tilde{\varphi}]\beta u + \tilde{\varphi}\beta F \\ v_0|_{t=0} = u_0, \partial_t v_0|_{t=0} = u_1. \end{cases} \tag{2.48}$$

if $\tilde{\varphi} = 1 - \sum_{j=1}^{\infty}(t-j)$ if $t \geq 0$ and 0 otherwise.

If we then let $G_j = (\partial_t^2 - \Delta_g + 1)v_j$ be the forcing term for v_j, $j = 0, 1, 2, \cdots$, then we have

$$\begin{aligned} \textstyle\sum_{j=0}^{\infty} \|G_j\|^2_{L_t^2 H^{\gamma-1}(\mathbb{R}_+ \times \mathbb{R}^n)} &\lesssim \|\rho u\|^2_{L_t^2 H^\gamma} + \|\beta \partial_t u\|_{L_T^2 H^{\gamma-1}} + \|F\|^2_{L_t^2 H^{\gamma-1}} \\ &\lesssim \|u_0\|^2_{H^\gamma} + \|u_1\|^2_{H^{\gamma-1}} + \|F\|^2_{L_t^2 H^{\gamma-1}} \end{aligned}$$

By (2.37) and Duhamel, we get for $j = 1, 2, \ldots$

$$\|v_j\|_{L_t^q L_x^r(\mathbb{R}_+ \times \mathbb{R}^n)} \lesssim \int_0^\infty \|G_j(s,\cdot)\|_{H^{\gamma-1}} ds \lesssim \|G_j\|_{L_t^2 H^{\gamma-1}}$$

using Schwarz's inequality and the support properties of the G_j in the last step.

Similarly,

$$\|v_0\|_{L_t^q L_x^r(\mathbb{R}_+ \times \mathbb{R}^n)} \lesssim \|u_0\|_{H^\gamma} + \|u_1\|_{H^{\gamma-1}} + \|G_0\|_{L_t^2 H^{\gamma-1}}$$

Since $q, r > 2$, we have

$$\|v\|^2_{L_t^q L_x^r} \lesssim \textstyle\sum_{j=0}^\infty \|v_j\|^2_{L_t^q H^\gamma} \lesssim \|u_0\|^2_{H^\gamma} + \|u_1\|^2_{H^{\gamma-1}} + \|F\|^2_{L_t^2 H^{\gamma-1}}$$

□

End of proof of Theorem (2.4.5),

By Proposition (2.4.8), we may assume the initial data for u vanishes when $|x| < \frac{3R}{2}$.

We then fix $\beta \in C_0^\infty(\mathbb{R}^n)$ satisfying $\beta(x) = 1, |x| \leq R$ and $\beta(x) = 0, |x| > \frac{3R}{2}$ and write

$$u = \tilde{u} - v = (1-\beta)\tilde{u} + (\beta\tilde{u} - v)$$

where $\tilde{u}$ solves the Cauchy problem for the Minkowski space free Klein-Gordon equation with initial data defined to be (u_0, u_1).

Then $(\partial_t^2 - \Delta_g + 1)(1-\beta)\tilde{u} = (\partial_t^2 - \Delta + 1)(1-\beta)\tilde{u} = -[\beta, \Delta]\tilde{u}$.

By Proposition (2.4.6) and (2.45), we know

$$\|(1-\beta)\tilde{u}\|_{L_t^q H^\gamma} \lesssim \|u_0\|_{H^\gamma} + \|u_1\|_{H^{\gamma-1}} + \|[\beta, \Delta]\tilde{u}\|_{L_t^2 H^{\gamma-1}} \lesssim \|u_0\|_{H^\gamma} + \|u_1\|_{H^{\gamma-1}}.$$

Also, $w = \beta\tilde{u} - v$ satisfies

$$(\partial_t^2 - \Delta_g + 1)w = -[\Delta, \beta]\tilde{u} \equiv G$$

is supported in $R < |x| < 2R$, and satisfies

$$\int_0^T \|G(t,\cdot)\|^2_{H^{\gamma-1}} dt \lesssim \|u_0\|_{H^\gamma} + \|u_1\|_{H^{\gamma-1}}$$

Then by Proposition (2.4.8), we have

$$\|w\|_{L_t^q L_x^r} \lesssim \int_0^T \|G(t,\cdot)\|^2_{H^{\gamma-1}} dt \lesssim \|u_0\|_{H^\gamma} + \|u_1\|_{H^{\gamma-1}}$$

We completed the proof of

$$\|u\|_{L_t^q L_x^r} \lesssim \|u_0\|_{H^\gamma} + \|u_1\|_{H^{\gamma-1}}$$

if u solves$(\partial_t^2 - \Delta_g + 1)u = 0$ with initial data (u_0, u_1).

Chapter 3

Strichartz Estimate On Asymptotically Euclidean Manifolds

Similarly to the argument in section 5, if we can prove the global Strichartz estimates for small perturbation(theorem (3.1.1) below), then combining the local Strichartz estimate and hyperthesis 1, we can get the global Strichartz estimates of Klein-Gordon equations on asymptotically non-trapping Euclidean manifolds.

3.1 Introduction

We can follow the method in [18] to prove the following global Strichartz estimates for small perturbation for asymptotically euclidean Klein-Gordon equations. Let's set up our operator and equation firstly. We define

$$\tilde{P} = -\sum_{ij} \frac{1}{\tilde{g}} \partial_i \tilde{g}^{ij} \tilde{g}^2 \partial_j \frac{1}{\tilde{g}} = -\partial_i \tilde{g}^{ij} \partial_j + b^i \partial_i + c$$

where $\tilde{g}^{ij} = (1-\Phi_{R/2})g^{ij} + \Phi_{R/2}\delta^{ij}$, $b^i = -2\tilde{g}^{-1}\partial_j(\tilde{g}\tilde{g}^{ij}) - \tilde{g}\tilde{g}^{ij}\partial_j\tilde{g}^{-1} + \partial_j\tilde{g}^{ij}$ and $c = \tilde{g}^{-1}\partial_i(\tilde{g}^{ij}\partial_j\tilde{g})$.

Consider the Klein Gordon equation

$$\begin{cases} (\partial_t^2 + \tilde{P} + 1)u = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{3.1}$$

We denote

$$A_j = \mathbb{R} \times \{2^j \le |x| \le 2^{j+1}\}, \quad A_{<j} = \mathbb{R} \times \{|x| \le 2^j\}.$$

By hypotheses of H1 and H2, we know

$$\sum_j \sup_{A_j} |x|^2 |\nabla^2 \tilde{g}^{ij}(x)| + |x||\nabla \tilde{g}^{ij}(x)| + |\tilde{g}^{ij}(x) - \delta^{ij}| \lesssim \epsilon,$$

$$\sum_j \sup_{A_j} |x|^2 |\nabla b^i(x)| + |x||b^i(x)| \lesssim \epsilon$$

and

$$\sum_j \sup_{A_j} |x|^4 |c(t,x)|^2 \lesssim \epsilon.$$

For ϵ small this guarantees nontrapping. For ϵ large it restricts trapping to finitely many dyadic regions.

Theorem 3.1.1. *(global Strichartz estimates for small perturbation, in progress) Let $d \ge 3$, $s \in [0,1] (s \in (0,1) \text{if } d = 3)$. The solution of (3.1) satisfies*

$$\|u\|_{L_t^q L_x^r} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} \tag{3.2}$$

for any admissible (s, q, r) *with* $q > 2, r < \infty$. *In other words,*

$$\|e^{it\langle\tilde{P}\rangle} f\|_{L_t^q L_x^r} \lesssim \|f\|_{H^s} \tag{3.3}$$

Similar to Proposition 5.2 in [2], we get

Proposition 3.1.2. *(in progress) Let* $d \geq 3$, $s \in [0, 1]$*(*$s \in (0, 1)$ *if* $d = 3$*). The solution* u *of* $(\partial_t^2 + \tilde{P} + 1)u = \beta F = F$ *satisfies*

$$\|u\|_{L_t^q L_x^r} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \|F\|_{L_t^2 H^{s-1}} \tag{3.4}$$

3.2 Parametrix for non-compact small perturbation with frequency 1

In order to prove (3.2), we just need to prove the dispersive estimates for Klein-Gordon equations. I obtained the dispersive estimates at frequency 1 following the arguments in [18].

Proposition 3.2.1. *Assume that* ϵ *is sufficiently small. Then for each* $s \in \mathbb{R}$, *there is an outgoing parametrix* S_j^+ *for the half Klein Gordon operator* $D_t + A_{(0)}$ *in* $\{t > s\}$ *which is localized at frequency 1 and satisfies the following:*
(i) L^2 *bound:*

$$\|S_j^+(t, s)\|_{L^2 \to L^2} \lesssim 1$$

(ii) Error estimate:

$$\|x^\alpha (D_t + A_{(0)}) S_j^+(t, s) P_j^+\|_{L^2 \to L^2} \lesssim (2^j + |t - s|)^{-N}$$

$$\|x^\alpha D_t (D_t + A_{(0)}) S_j^+(t, s) P_j^+\|_{L^2 \to L^2} \lesssim (2^j + |t - s|)^{-N}$$

(iii) Initial data:

$$S_j^+(s+0,s) = I$$

(iv)Outgoing parametrix:

$$\|\mathbf{1}_{\{|x|<2^{-10}(|t-s|+2^j)}S_j^+(t,s)P_j^+\|_{L^2\to L^2} \lesssim (2^j+|t-s|)^{-N}$$

(v) Finite speed:

$$\|\mathbf{1}_{\{|x|>2^{10}(|t-s|+2^j)}S_j^+(t,s)P_j^+\|_{L^2\to L^2} \lesssim (2^j+|t-s|)^{-N}$$

(vi) Frequency localization:

$$\|(1-P_{[-4,4]})S_j^+(t,s)P_j^+\|_{L^2\to L^2} \lesssim (2^j+|t-s|)^{-N}$$

(vii) Pointwise decay:

$$\|S_j^+(t,s)P_j^+\|_{L^1\to L^\infty} \lesssim (1+|t-s|)^{-\frac{n}{2}}$$

With obvious modifications , the same hold for S_j^-.

To help you understand the proof of Proposition (3.2.1) easily, I'll repeat the proof in [18] and do some modifications.

3.2.1 Notations

(i) Frequency Littlewood-Paley decomposition

$$1 = \sum_{j=-\infty}^{\infty} S_j(D_x), \quad \text{supp} s_j \subset \{2^{j-1} < |\xi| < 2^{j+1}\} \tag{3.5}$$

(ii) We say a function f is localized at frequency 2^k if $\hat{f}$ is supported in $\{2^{k-1} < |\xi| < 2^{k+1}\}$. An operator K is localized at frequency 2^k if for any f localized at frequency 2^k its image Kf is frequency localized in $\{2^{k-10} < |\xi| < 2^{k+10}\}$.

(iii) $S^{(k)}$ is defined by $a \in S^{(k)} \Longleftrightarrow |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}$, $|\alpha| + |\beta| \geq k$.
And also $l^1 S^{(k)}$ are classes of symbols in $\mathbb{R} \times T^*\mathbb{R}^n$ whose seminorms are given by

$$\sum_j 2^{j(1+\frac{|\alpha|-|\beta|}{2})} \|\partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)\|_{L^\infty(\{t\approx 2^j)}, |\alpha| + |\beta| \geq k,$$

and we have

$$a \in l^1 S^{(k)} \Longleftrightarrow \sum_j 2^{j(1+\frac{|\alpha|-|\beta|}{2})} \|\partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)\|_{L^\infty(\{t\approx 2^j)} \leq c_{\alpha\beta}, |\alpha| + |\beta| \geq k,$$

especially $l^1 S_\epsilon^{(k)}$ is a subset of $l^1 S^{(k)}$ which let the seminorms to be bounded by ϵ and $|\alpha| + |\beta| = k$.

(iv) The distance is defined by $d_s((x,\xi),(y,\eta))^2 = s^{-1}|x-y|^2 + s|\xi - \eta|^2$.

3.2.2 Proof

Let's tart with the Hamilton flow.

Consider

$$(D_t + a^w(t,x,D))u = 0, \quad u(0) = u_0, \tag{3.6}$$

where $S(t,s)$ denotes its evolution operators.

The Hamilton flow of $D_t + a^w$ is given by

$$\begin{cases} \dot{x} = a_\xi(t,x,\xi) \\ \dot{\xi} = -a_x(t,x,\xi) \end{cases} \tag{3.7}$$

with a real symbol $a \in S^{(2)}$. We use $\chi(t,s)$ to denote the phase space evolution. $t \to (x_t, \xi_t)$ denotes the trajectories of the flow. We also have a corresponding Egorov theorem. For a pseudodifferential operator $q^w(0)$, we define its conjugate with respect to the flow by

$$q^w(t) = S(t,0)q^w(0)S(0,t).$$

Then the corresponding Egorov's theorem is

Proposition 3.2.2. *(Proposition 9 in [17]) Assume that a is a real symbol in $S^{(2)}$ uniformly in $t \in [0,1]$.*
(a) If $q(0) \in S^{(0)}$, then $q(t) \in S^{(0)}$ uniformly in t.
(b) If $q(0) \in S^{(1)}$, then $q(t) \in S^{(1)}$ uniformly in t, and

$$q(t,x,\xi) - q(0) \circ \chi(0,t) \in S^{(0)}$$

.

Proposition 3.2.3. *(Similar to Proposition 10 in [17]) Assume that $a(t,x,\xi) = \sqrt{1+|\xi|^2}$, let $q(0) \in S^{(2)}$, then $q(t) \in S^{(2)}$ uniformly in $t \in [0,1]$ and*

$$q(t,x,\xi) - q(0) \circ \chi(0,t) \in S^{(0)}$$

.

Proof. We compute directly

$$\chi(t,s)(x,\xi) = (x + (t-s)\xi\langle\xi\rangle^{-1}, \xi)$$

. Then we want to show that

$$r^w(t,x,D) = e^{-it\langle D\rangle} q^w(x,D) e^{it\langle D\rangle} - q^w(x - tD\langle D\rangle^{-1}, D) \in OPS^{(0)}$$

uniformly in $t \in [0,1]$. We have

$$\frac{d}{dt} e^{it\langle D\rangle} r^w(t,x,D) e^{-it\langle D\rangle} = e^{it\langle D\rangle} r_1^w(t,x,D) e^{-it\langle D\rangle}$$

where $r_1^w(s,x,D) = -i[\langle D\rangle, q^w(x - sD\langle D\rangle^{-1}, D)] - \frac{d}{ds} q^w(x - sD\langle D\rangle^{-1}, D)$.
We know in Proposition 10 of [17] that

$$i[|D|, q(x - sD|D|^{-1}, D)] - \frac{D}{|D|} \bigtriangledown_x q(x - sD|D|^{-1}, D) \in S^{(0)}$$

which by weyl calculus implies

$$i[|D|, q(x - sD\langle D\rangle^{-1}, D)] - \frac{D}{|D|} \bigtriangledown_x q(x - sD\langle D\rangle^{-1}, D) \in S^{(0)}.$$

Then, our

$$r_1^w = i[\langle D\rangle - |D|, q(x - sD\langle D\rangle^{-1}, D)]$$

$$+ i[|D|, q(x - sD\langle D\rangle^{-1}, D)] - \frac{D}{|D|} \bigtriangledown_x q(x - sD\langle D\rangle^{-1}, D)$$

$$+ (-\frac{D}{\langle D\rangle} + \frac{D}{|D|}) \bigtriangledown_x q(x - sD\langle D\rangle^{-1}, D) = I + II + III.$$

As we mentioned before $II \in S^{(0)}$, $I \in S^{(0)}$ by Kohn-Nirenberg formula and $III \in S^{(0)}$ obviously. Thus we get $r_1(s,x,\xi) \in S^{(0)}$. By Proposition (3.2.2), conjugation by $e^{\pm it\langle D\rangle}$

leaves the $S^{(0)}$ class unchanged. ∎

Next, consider global in time evolution of the form

$$(D_t + a^w(t,x,D) - ib^w(t,x,D) + c^w(t,x,D))u = 0, \quad t > 0 \tag{3.8}$$

with a real symbol $a \in l^1 S^{(2)}_\epsilon$, $b \in l^1 S^{(1)}_\epsilon$ with $b \geq 0$, which is a damping term, and a possibly complex symbol $c \in l^1 S^{(0)}_\epsilon$.

Proposition 3.2.4. *[Proposition 11 in [17]] Assume that $a \in l^1 S^{(2)}$ and $b \in l^1 S^{(1)}$ are real symbols with $b \geq 0$, while $c \in l^1 S^{(0)}$. Then the (3.8) is forward well-posed in $L^2(\mathbb{R}^n)$, and the corresponding evolution operators satisfy*

$$\|S(t,s)\|_{L^2 \to L^2} \lesssim 1, \quad 0 < s < t. \tag{3.9}$$

To be more convenient to parametrize $\chi(t,s)$ using the variables (x_s, ξ_t). We obtain the following regularity.

Proposition 3.2.5. *[Proposition 12 in [17]] If $a \in l^1 S^{(2)}_\epsilon$ with ϵ small and $s < t$ then*

$$\frac{\partial(x_t, \xi_s)}{\partial(x_s, \xi_t)} = \begin{pmatrix} I_n + \epsilon O(1) & \epsilon O(t) \\ \epsilon O(\frac{1}{s}) & I_n + \epsilon O(1) \end{pmatrix}.$$

Now introduce the FBI transform $T_{\frac{1}{t}}$, where

$$T_{\frac{1}{s}} u(t,x,\xi) = c_n s^{-\frac{n}{4}} \int e^{-\frac{(x-y)^2}{2s}} e^{i\xi(x-y)} u(t,y) dy.$$

The phase space kernel $K(t,s)$ of $S(t,s)$ are defined to be the kernels of the conjugated operators:

$$\tilde{S}(t,s) = T_{\frac{1}{t}} S(t,s) T^*_{\frac{1}{s}}.$$

And define

$$\psi(t, x_t, \xi_t) = \int_0^t b(s, x_s, \xi_s) ds.$$

Theorem 3.2.6. *[Theorem 31 in [18]] Let $a \in l^1 S^{(2)}_\epsilon$, $b \in l^1 S^{(1)}_\epsilon$ be real symbols with $b \geq 0$ and $c \in l^1 S^{(0)}$. Then for $s < t$ the kernel $K(t,s)$ of the operators $\tilde{S}(t,s)$ satisfy the bound*

$$|K(t, x, \xi_t, s, x_s, \xi)| \lesssim t^{-\frac{n}{4}} s^{\frac{n}{4}} (1 + (\psi(x_s, \xi_s) - \psi(x_t, \xi_t))^2 + \frac{|x - x_t|^2}{t} + s|\xi - \xi_s|^2)^{-N}.$$

Now, let's wrap up all the above and relate to Klein-Gordon equations. For our Klein Gordon operator

$$\partial_t^2 + \partial_\alpha a^{\alpha\beta} \partial_\beta + b^\alpha \partial_\alpha + c + 1, \text{ where } a^{\alpha\beta} = \tilde{g} \text{ as above.}$$

Work with the principal part of $P = D_\alpha a^{\alpha\beta} D_\beta + b^\alpha D_\alpha + c + 1$ which we denote by $P_a = -D_t^2 + D_i a^{ij} D_j + 1$.

We factor the principal symbol as

$$-\tau^2 + a^{ij}\xi_i\xi_j + 1 = -(\tau + a^+(t,x,\xi))(\tau + a^-(t,x,\xi))$$

where $a^+(t,x,\xi) = -a^-(t,x,-\sigma) = \sqrt{a^{ij}\xi_i\xi_j + 1}$. The corresponding operator which is

localized in frequency 1 is $D_t + A_{(0)}$.

The symbols $a^{\pm}$ are small perturbation of $\pm\sqrt{1+|\xi|^2}$, precisely

$$a^{\pm} \in \pm\sqrt{1+|\xi|^2} + l^1 S^{(2)}_{\epsilon}, |\xi| \approx 1, |x| \approx t.$$

Thus, we consider the evolution equation

$$(D_t + \langle D_x \rangle + a_0^w(t,x,D) - ib_0^w(t,x,D))u = 0 \tag{3.10}$$

where $a_0 \in l^1 S^{(2)}_{\epsilon}$, $b_0 \in l^1 S^{(1)}$ are real symbols with $b_0 \geq 0$.

We denote by $S_0(t,s)$ the L^2 evolution generated by (3.10), $\tilde{S}_0(t,s)$ its phase space image

$$\tilde{S}_0(t,s) = T_{\frac{1}{t}} S_0(t,s) S_{-10<\cdot<10} T^*_{\frac{1}{s}}.$$

At each time t we consider the symplectic map μ defined by

$$\mu_t(x,\xi) = (x + t\xi\langle\xi\rangle^{-1}, \xi)$$

which corresponds to the for $D_t + \langle D_x \rangle$ evolution.

This extends to a space-time symplectic map

$$\mu(t,\tau,x,\xi) = (t, \tau - \langle\xi\rangle, x + t\xi\langle\xi\rangle^{-1}, \xi).$$

If p_0 is the symbol,

$$p_0(t,\tau,x,\xi) = \tau + \langle\xi\rangle + a_0(t,x,\xi),$$

then its image through μ is

$$p_0(\mu(t,\tau,x,\xi)) = \tau + a(t,x,\xi), \quad a(t,x,\xi) = a_0(t, x + t\xi\langle\xi\rangle^{-1}, \xi).$$

Hence the conjugate of the Hamilton flow $\chi_0(t,s)$ for $\tau + \langle\xi\rangle + a_0$ with respect to μ_t is the Hamilton flow $\chi(t,s)$ for $\tau + a(t,x,\xi)$,

$$\chi_0(t,s) = \mu_t \circ \chi(t,s) \circ \mu_s^{-1} \quad (Note, a \in l^1 S^{(2)}_\epsilon \Longleftrightarrow a_0 \in l^1 S^{(2)}_\epsilon).$$

Follow the arguments above, we have

Proposition 3.2.7. *If $a_0 \in l^1 S^{(2)}_\epsilon$ with ϵ sufficiently small and $t > s$ then the Hamilton flow $\chi_0(t,s)$ has the lipschitz regularity*

$$\frac{\partial(x_t,\xi_s)}{\partial(x_s,\xi_t)} = \begin{pmatrix} I_n+\epsilon O(1) & \epsilon O(t)+(t-s)\langle\xi\rangle^{-3}(\langle\xi\rangle^2 I_n - \xi\bigotimes\xi) \\ \epsilon O(\frac{1}{s}) & I_n+\epsilon O(1) \end{pmatrix}.$$

We do the similar thing to b_0 and set

$$b(t,x,\xi) = b_0(t, t\xi\langle\xi\rangle^{-1}, \xi).$$

Theorem 3.2.8. *(Similar to Theorem 5 in [18]) Let $a_0 \in l^1 S^{(2)}_\epsilon$, $b_0 \in l^1 S^{(1)}$ be real symbols with $b_0 \geq 0$ with ϵ sufficiently small, so that a_0 and $b_0 - \tilde{b}_0(t)$ are localized at frequency 1. Then for $s < t$ the kernel K_0 of the operator $\tilde{S}_0(t,s)$ satisfies the bound*

$$|K_0(t,x,\xi_t,s,x_s,\xi)| \lesssim t^{-\frac{n}{4}} s^{\frac{n}{4}} (1 + (\psi_0(x_s,\xi_s) - \psi_0(x_t,\xi_t))^2 + \frac{|x-x_t|^2}{t} + s|\xi-\xi_s|^2)^{-N}$$

$$\times (1 + td(|\xi|, [2^{-10}, 2^{10}]))^{-N}$$

.

Proof. We use (3.2.6) via a conjugation with respect to the flat Klein Gordon flow, which corresponds to the canonical transformations μ_t, denote

$$S(t,s) = e^{it\langle D\rangle} S_0(t,s) e^{-is\langle D\rangle}.$$

Then we have

$$\frac{d}{dt} S(t,s) = -ie^{it\langle D\rangle}(-a_0^w(t,x,D) + ib_0^w(t,x,D))e^{-is\langle D\rangle} S(t,s)$$

Hence $S(t,s)$ is the evolution associated to the pseudodifferential operator

$$e^{it\langle D\rangle}(-a_0^w(t,x,D) + ib_0^w(t,x,D))e^{-it\langle D\rangle}.$$

By Proposition (3.2.2) and Proposition (3.2.3), the operator can be expressed in the form

$$a^w(t,x,D) - ib^w(t.x.D) + c^w(t,x,D)$$

where $c \in l^1 S^{(0)}$.

Then we can use Theorem (3.2.6) to bound phase space kernel of $\tilde{S}(t,s)$.

$$\tilde{S}_0(t,s) = T_{\frac{1}{t}} e^{-it\langle D\rangle} S_{-10<\cdot<10} S(t,s) S_{-10<\cdot<10} e^{is\langle D\rangle} T^{\star}_{\frac{1}{s}}$$

$$= T_{\frac{1}{t}} e^{-it\langle D\rangle} S_{-10<\cdot<10} T^{\star}_{\frac{1}{t}} T_{\frac{1}{t}} S(t,s) T^{\star}_{\frac{1}{s}} S_{-10<\cdot<10} e^{is\langle D\rangle} T^{\star}_{\frac{1}{s}}$$

$$= (T_{\frac{1}{t}} e^{-it\langle D\rangle} S_{-10<\cdot<10} T^{\star}_{\frac{1}{t}}) \tilde{S}(t,s) (T_{\frac{1}{s}} S_{-10<\cdot<10} e^{is\langle D\rangle} T^{\star}_{\frac{1}{s}}).$$

Kernel of the first term is rapidly decreasing on the $t^{\frac{1}{2}} \times t^{-\frac{1}{2}}$ scale away from the graph of μ_t as well as away from the support of the symbol $S_{-10<\cdot<10}$. Similarly for the third

term. ■

Last step, we'll do the parametrix construction in the following to prove Proposition (3.2.1).

We partition the annulus $|\xi| \approx 1$ in phase space.

$$s_{-1}(\xi) + s_0(\xi) + s_1(\xi) = \sum_{\pm} \sum_{j \geq 0} p_j^{\pm}(x, \xi)$$

with

$$\text{supp} p_j^{\pm} \subset \{2^{j-1} < |x| < 2^{j+1}, \pm x\xi \geq -2^{-5}|x|\}, j \geq 1$$

$$\text{supp} p_0^{\pm} \subset \{|x| < 2, \pm x\xi \geq -2^{-5}|x|\}$$

We may replace $p_j^{\pm}$ by $S_{<-10}(D_x)p_j^{\pm}$ which allow us to assume that the operators $p_j^{\pm}$ are frequency localized to frequency 1.

We construct evolution operators $S_j^{\pm}(t, s)$ as the evolutions associated to a certain damped half Klein Gordon equations. We then form K_0 by setting

$$K_0(t, s) = \begin{cases} \sum_{j=0}^{\infty} S_j^{-}(t, s)(P_j^{-})^w(x, D), t < s \\ \sum_{j=0}^{\infty} S_j^{+}(t, s)(P_j^{+})^w(x, D), t > s \end{cases}$$

Then we'll present the proof of dispersive estimates as follows.

Proof. Following the arguments of Proposition 16 in [17], the necessary modifications

are about the case 1: $|t-s| \geq s$.

Under the conditions of (iv), (v), (vi) and additionally

$$\|x^\alpha S_{<-5}(D_x) S_j^+(t,s) P_j^+\|_{L^2 \to L^2} \lesssim (|t-s| + 2^j)^{-N}, \tag{3.11}$$

$$\|x^\alpha \partial^\beta S_{>5}(D_x) S_j^+(t,s) P_j^+\|_{L^2 \to L^2} \lesssim (|t-s| + 2^j)^{-N}, \tag{3.12}$$

$$\|x^\alpha b_0^w(t,x,D) S_j^+(t,s) P_j^+\|_{L^2 \to L^2} \lesssim (|t-s| + 2^j)^{-N}, \tag{3.13}$$

$$\|x^\alpha D_t b_0^w(t,x,D) S_j^+(t,s) P_j^+\|_{L^2 \to L^2} \lesssim (|t-s| + 2^j)^{-N}, \tag{3.14}$$

we can reduce the proposition to the case of an evolution governed by a symbol of the form

$$\langle \xi \rangle + a_0(t,x,\xi), \quad a_0 \in l^1 S^{(2)}_\epsilon.$$

Also, we define $S_j^+(t,s)$ to be the forward evolution operator of the equation

$$(D_t + \langle D_x \rangle + a_0^w(t,x,D))u = i b_0^w(t,x,D)u.$$

Now, we proceed to the modification of case 1: $|t-s| \geq s$.

In this region, we may neglect the damping. For initial data $u(s) = \delta_0$, we have

$$T_{\frac{1}{s}} u(s, x_s, \xi) = s^{-\frac{n}{4}} e^{-\frac{x_s^2}{2s}} e^{i x_s \xi}.$$

Using Theorem (3.2.8), we see that

$$|T_{\frac{1}{t}}u(t,x,\xi_t)| \lesssim t^{-\frac{n}{4}} \int (1+t^{-1}|x-x_t(\xi_t,x_s)|^2)^{-N}$$

$$\times(1+s|\xi-\xi_s(x_s,\xi_t)|^2)^{-N} e^{-\frac{x_s^2}{2s}} dx_s d\xi$$

$$\lesssim t^{-\frac{n}{4}} s^{-\frac{n}{2}} \int (1+t^{-1}|x-x_t(\xi_t,x_s)|^2)^{-N} e^{-\frac{x_s^2}{2s}} dx_s$$

$$\lesssim t^{-\frac{n}{4}} (1+t^{-1}|x-x_t(\xi_t,0)|^2)^{-N}.$$

And applying $T^*_{\frac{1}{t}}$, we have

$$|u(t,y)| \lesssim \int (1+t^{-1}|y-x_t(\xi_t,0)|^2)^{-N} d\xi_t.$$

If $|t-s| \leq s$, the map $\xi_t \to x_t(\xi_t,0)$ has lipschitz constant which is bounded by t, and has rank n. Hence, integration with respect to ξ_t yields

$$|u(t,y)| \lesssim t^{-\frac{n}{2}}.$$

What left is to prove all the remaining estimates (ii),(iv),(v),(vi) and (3.11)-(3.14). ■

To finish the proof of Proposition (3.2.1), we need the following lemma.

Lemma 3.2.9. *There exists a symbol $b \in l^1 S^{(1)}$ which satisfies, in addition to (b1) $t^{\frac{4}{3}}b$ is nonincreasing along the Hamilton flow for $D_t + \langle D_x \rangle + a_0^w$, and*

$$0 < t^{\frac{4}{3}} b(t,x_t,\xi_t) < 1 \Longrightarrow b(2t,x_{2t},\xi_{2t}) = 0$$

(b2) At the initial time, we have

$$b(2^j, x, \xi) = 0 in \{2^{-5/2} < |\xi| < 2^{5/2}, 2^{j-2} < |x| < 2^{j+2}, x\xi > -2^{-9/2}|x|\}.$$

(b3) At any time $t > 2^j$, we have

$$B(t, x, \xi) = t^{-\frac{3}{4}}, outside\{2^{-4} < |\xi| < 2^4\} \bigcap \{2^{-6}t < |x| < 2^6 t\}.$$

Proof. Let Φ be a smooth cutoff function which equals 0 in $(-\infty, 0)$ and 1 in $(1, \infty)$. We define the symbol b as

$$b(t, x, \xi) = t^{-\frac{4}{3}}(1 - \Phi(b_1)\Phi(b_2)\Phi(b_3)\Phi(b_4)\Phi(b_5)).$$

$$e(s) = \epsilon^{-1} \int_0^s \frac{\epsilon(\sigma)}{\sigma} d\sigma.$$

Here select the frequencies which are not too large,

$$b_1(t, \xi) = \frac{2^{\frac{7}{2}} + Ce(t) - |\xi|}{\epsilon(t)},$$

select the frequencies which are not too small,

$$b_2(t, \xi) = \frac{-2^{-\frac{7}{2}} + Ce(t) + |\xi|}{\epsilon(t)},$$

select the outgoing parts,

$$b_3(t, x, \xi) = \frac{2^{-\frac{3}{2}}|x||\xi| + x\xi}{2^{-18}|x|},$$

select a spatial region not too far from the origin,

$$b_4(t, x) = \frac{2^6 t - |x|}{t},$$

select a spatial region not too close to the origin,

$$b_5(t,x,\xi) = \frac{|x||\xi| - 2^{-5}t|\xi| + x\xi}{2^{-13}t}.$$

We can easily see that

$$\{2^{-5/2} < |\xi| < 2^{5/2}, 2^{-2}t < |x| < 2^2 t, x\xi > -2^{-9/2}|x|\} = D_t \subset \{b = 0\}$$

$$\{t^{\frac{4}{3}} b < 1\} \subset E_t = \{2^{-4} < |\xi| < 2^4, 2^{-6}t < |x| < 2^6 t, x\xi > -2^{-\frac{3}{2}}|x||\xi|\}$$

which satisfies the conditions (b1) and (b2). Next, we need to prove (b1), and it's sufficient to prove that

$$\frac{d}{dt} b_j \geq \frac{2}{t} \quad in E_t \bigcap \{0 \leq b_j \leq 1\}.$$

For $(x,\xi) \in E_t$ along the Hamilton flow $D_t + \langle D_x \rangle + a_0$ we have $\frac{d\xi}{st} = O(\frac{\epsilon(t)}{t})$, $\frac{dx}{dt} = \xi\langle\xi\rangle^{-1} + O(\epsilon(t))$. Then,

$$\frac{d}{dt} b_1(t,\xi) \approx \frac{1}{\epsilon t} in E_t \bigcap \{1 \leq b_1 \leq 1\},$$

similarly for b_2.

$$\frac{d}{dt} b_3(t,x,\xi) = \frac{|\xi|^2|x|^2 - (x\xi)^2}{2^{-18}|x|^3\langle\xi\rangle} + \frac{O(\epsilon(t))}{t} \geq \frac{\frac{7}{8}|\xi|^2}{2^{-18}|x|\langle\xi\rangle} > \frac{4}{t},$$

in $E_t \bigcap \{0 \leq b_3 \leq 1\}$.

$$\frac{d}{dt} b_4(t,x) = \frac{|x|^2\langle\xi\rangle - tx\xi}{t^2|x|\langle\xi\rangle} + \frac{O(\epsilon(t))}{t} \geq \frac{2^5}{t}$$

in $E_t \bigcap \{0 \leq b_4 \leq 1\}$.

$$\frac{d}{dt} b_5(t,x,\xi) = \frac{x\xi|\xi| + |x||\xi|^2}{2^{-13}t|x|\langle\xi\rangle} - \frac{|x||\xi| = x\xi}{2^{-13}t^2} + \frac{O(\epsilon(t))}{t} \leq \frac{2}{t},$$

in $E_t \bigcap \{0 \leq b_5 \leq 1\}$. ■

Lemma 3.2.10. *Assume that the symbol $b_0 \in l^1 S^{(1)}$ satisfying the properties (b1),(b2) and (b3) above. Then, the remaining bounds hold.*

Proof. The necessary modifications to the arguments from [18] are quite simple. We just need to fix the second estimate of (ii). Here, modulo negligible errors, we have

$$D_t b^w(t,x,D)S^+(t,s) = -i(\partial_t b_0)^w(t,x,D)S^+(t,s) - b_0^w(t,x,D)\langle D_x\rangle S^+(t,s)$$

$$-b_0^w(t,x,D)a_0^w(t,x,D)S^+(t,s) + ib_0^w(t,x,D)S^+(t,s).$$

Since the symbols $(\partial_t b_0)^w$, $b_0^w\langle D_x\rangle$, $b_0^w a_0^w$ and $b_0^w b_0^w$ are all in $S_t^{(1)}$ and have supports which are contained in the support of b_0, we can apply Proposition 17 in [18] to complete the proof. ■

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Vitae

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