

WHY ARE FEE-TO-COLLATERAL RATIOS SO LOW?

BY

W. ALLEN CHENG

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Abstract

We develop a game-theoretic model of centralized clearing to analyze a clearinghouse's choice of transaction fee and collateral requirements. The clearinghouse's requirements affect not only the size and riskiness of her participating client base, but also the transaction fees charged to clients by her clearing member. We show that empirically observed low fee-to-collateral ratios can be explained as the equilibrium arising from strategic interactions between profit maximizing agents.

We analytically characterize the equilibrium fee-to-collateral ratio and find that it depends on the relative riskiness of the contract (relative to the depth of clients' private trading benefits). In particular, when the contract is very risky, so that participating clients are mostly speculators, the clearinghouse imposes a very high collateral requirement; when the contract is not risky, so that participating clients are mostly fundamental value traders, the clearinghouse imposes a very low collateral requirement.

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1 Introduction

Clearinghouses use collateral, essentially a security that pays off only when there is a default, to protect themselves from counterparty risk. In comparison, the transaction fees that clearinghouses charge are guaranteed upfront payments in all states of nature, and are their main source of revenue. Since the default probability of each individual client is usually small, it might be tempting to conclude that the clearinghouse should be indifferent between increasing collateral levels by a large amount and increasing fees by a small amount.

However, this stands in contrast with empirical evidence. High collateral levels (relative to fees) consistently prevail in many derivatives markets. For example, for the year 2014, a conservative estimate of Intercontinental Exchange's (ICE) collateral holdings pledged for cleared CDS trades was around 20 billion USD, while the company's revenue from CDS transactions totaled 161 million USD. Assuming an average five-year maturity of the CDS trades, this gives a fee-to-collateral ratio of around $161 \times \frac{5}{20,000} \approx 4\%$. This indicates that ICE preferred collateral a lot more than fees.

Intuitively, a clearinghouse desires high levels of collateral either if she deems the client to have high probability of default, or if she is very risk averse, i.e. she places significant emotional weight on the states of nature where default occurs. Since a clearinghouse could decide to not clear contracts when clients are deemed likely to default, the more plausible explanation seems to be that clearinghouses are very risk averse.

This paper shows that this needs not be the case. The empirically observed low fee-to-collateral ratios can be explained as an equilibrium phenomenon arising from the strategic interactions between risk-neutral, rather than highly risk-averse, agents. We design a game where the profit maximizing clearinghouse clears client trades submitted via a clearing member bank. She sets transaction fee and collateral requirements for cleared trades which directly affect not only the size and riskiness of the clearinghouse's participating client base, but also the transaction fees charged to clients by her clearing member. Clients have the option to default, and may do so when the contract value moves against them. The bank may choose to leave the clearinghouse if the client clearing business is not profitable. We show that the resulting equilibrium fee to collateral ratio can involve fee-to-collateral ratios close to zero, depending on the riskiness of the

traded contract and the benefits that clients can capture from trading.

The clearinghouse's requirements obviously reduce the clients' profits from trading: the transaction fee directly lowers the clients' overall surplus from trading, and collateral lowers the value of the client's default option. Since the only participating clients are those who can realize sufficient benefits from trading, increasing these requirements reduces the clearinghouse's participating client base. In addition, we show that the profits of the clearing member bank decrease when requirements are higher, so that the clearinghouse may lose profitable client business if the bank chooses to not become a clearing member.

The requirements also act as a screening device. When they are high, participating clients have smaller incentives to default either because they can capture large benefits from maintaining the trade or because they would lose large amounts of collateral if they default. This means that the fraction of participating clients who default, given a fixed realization of the contract value, is thus lower. The need for using such screening devices comes from the fact that the bank and clearinghouse either cannot or choose to not discriminate between clients. In practice, while clearinghouses often have access to the identities of the clients submitting trades, there are often too many clients for the clearinghouse to individually keep track of every one of them. Indeed, transaction fees and collateral are usually charged based on some portfolio specific rule rather than tailored to client specific characteristics.¹ In particular, firm specific characteristics such as credit quality and asset size often do not factor into the fee and margin calculations.

We set up a model consisting of three groups of risk neutral agents: the clearinghouse, a potential clearing member bank, and a continuum of clients who may trade a single, mandatorily cleared, contract. We consider a two period economy, *ex-ante* and *ex-post* the realization of the contract value. Prior to the realization, agents decide whether or not to participate in the clearing process. The clearinghouse sets her fee and collateral requirement. The bank then sets his own transaction fee and participates if the business is deemed profitable; that is, if his revenue can cover his operational costs for participation. High fee and collateral requirements both disincentivize the clients

¹ICE Clear Credit charges every clearing member with the same CDS portfolio the same amount of initial margin. Every client currently pays the transaction fee of \$6 per million notional cleared for CDS index contracts. For more detail see ICE Clear Credit's schedule of fees and online documentation of margining rules.

from trading. After the realization, clients default strategically, resulting in losses which are borne by the clearinghouse.² Client's trades are motivated by their private benefits and the value of their default option. Non-defaulting clients receive (heterogenous) private benefits when they carry out cleared trades.

We find that when the operational cost of becoming a clearing member is low, so that the bank's individual rationality constraint is non-binding, the prevailing subgame perfect equilibrium is either one involving infinite collateral (when the contract is very risky) or one that involves zero collateral (when the contract is not risky). We find that it is the *relative* riskiness of the contract, measured by the volatility of the contract value over the depth of the private benefits to be realized, that determines which equilibrium prevails. This means that infinite collateral equilibria prevails when many clients trade for speculative purposes (when the default option value is high) and zero collateral equilibrium prevails when clients trade more for fundamental value (private benefits are large).

When the operational cost of becoming a clearing member is high, so that the bank's individual rationality constraint is binding, the clearinghouse can opt to set reduced requirements to increase the bank's profit, and incentivize him to participate as a clearing member. In this case the collateral levels may still be large but finite. The fact that the bank's individual rationality constraint may be binding can be observed from the recent exit of many clearing members. For instance, in May 2014, the Royal Bank of Scotland announced the wind down of its clearing business due to increasing operational costs. This was followed by State Street, BNY Mellon, and more recently, Nomura; each shutting down part or all of their OTC clearing business.

To the best of our knowledge, our paper is the first theoretical study to explain collateral levels in a centralized clearing setting. Previous work in the context of collateralized trading often assume that either margining rules are exogenously given (Garleanu and Pedersen (2011)), or that margins are set following some mix of the expected shortfall, VaR, and maximum shortfall measures (Anderson and Jøeveer (2014), Duffie et al. (2015)). Johannes and Sundareshan (2007) and Capponi (2013) assume that (variation) margin collateral payments on the valuation of swaps are set exactly to track mark to

²We are implicitly assuming that the clearinghouse is acting as a true *central counterparty*. All counterparty risk and collateral management duties are borne by the clearinghouse.

market value changes. In contrast, our work looks for a micro-founded collateral setting rule via solving a profit maximization problem. Our modeling approach is closely related to those of Stiglitz and Weiss (1981) in the modeling of strategic defaults, and Holmström and Tirole (1997) in the inclusion of private benefits.

On the other hand, the determination of optimal levels of collateral has been extensively analyzed in the corporate finance literature. For instance, Stiglitz and Weiss (1981) and Besanko and Thakor (1987) both find collateral as a useful screening device either through adverse selection or incentive effects, and that collateral levels arise from profit-maximizing Nash equilibria. Geanakoplos (1997) analyzes (general) collateral equilibria for the case when assets are used to collateralize security trades, implicitly assuming the “smallness” of each agent. The centralized clearing setting is quite different, however. First, all agents trade publicly available contracts and are exposed to the same market risks, rather than bringing independent individual borrower risk to the table (Diamond (1984)), thus there is less asymmetric information about the risks clients are taking on. Second, the direction of future exposure is uncertain as either counterparty could be out of the money in the future. Third, collateral posting is not between clients, but is often unilateral from the clients to the clearinghouse. While initial margins is posted by clients to the clearinghouse, the clearinghouse usually does not post initial margins to the clients (Pirrong (2011)). Fourth, the clearinghouse often has significant market power.³

The rest of the paper is organized as follows. Section 2 explains the set-up of the model. Section 3 solves for subgame perfect Nash equilibria. Section 4 concludes. All proofs are delegated to the appendix.

2 The model

Our model consists of three groups of risk neutral agents: the clearinghouse (CH), a potential clearing member bank (CM), and a continuum of clients trading a single, mandatorily cleared, derivative contract. There are two periods, separated by the realization of the contract value. Agents choose their participation in the trading/clearing process ex-ante; ex-post, clients default strategically and the clearinghouse bears the

³For instance, while ICE, CME, and LCH all clear CDS, the bulk of CDS are still cleared through ICE both in the US and Europe.

losses.⁴ The clearinghouse has a large endowment of equity and does not default on her obligations. The bank only serves as an intermediary in the model so he does not default.

Before the contract value is realized, the clearinghouse can set her collateral C and fee δ_c requirements per contract cleared. The bank is the prime broker of a continuum of clients with unit mass, whose “private benefit parameters” B are described by the distribution F , i.e. $F(t)$ is the fraction of clients whose private benefits that do not exceed t . In view of C and δ_c , the bank can set his fee per contract cleared δ_b . After δ_c, C and δ_b have been declared, each client, characterized by his private benefit parameter B , has the choice between trading long one contract (L), trading short one contract (S), and not-trading (NT). These actions provide private benefit B , $-B$, and 0, respectively. Since each individual client is small, he cannot become a clearing member himself and cannot afford to trade more than one contract.⁵

Client choices are submitted to the bank. After receiving the trade orders, the bank has a choice between joining (J) and not joining (NJ) as a clearing member. The bank incurs an operational cost G for joining.⁶ The bank’s payoff is given by:

$$\text{Bank's Payoff} = \delta_b \times \text{mass of clients who trade} - G. \quad (2.1)$$

If the bank were to become a clearing member, a clearing channel is set up. Clients who trade pay the total fee of $\delta := \delta_b + \delta_c$ where δ_c goes to the clearinghouse and δ_b to the bank, before the contract value is realized. They also post collateral to the clearinghouse.

After this is done, the contract value is realized with value $\varepsilon \sim H$. The distributions F and H are assumed to be Laplace with parameters $(0, \lambda)$ and $(0, \gamma)$, respectively.

⁴Since there are only two periods, it suffices to consider only initial margins and not variation margin posting.

⁵In practice, clients usually are asset management funds or money market funds and are very small compared to clearing members, who are usually large broker-dealers (Pirrong (2011)).

⁶As a clearing member, the bank needs to meet certain capital requirements, contribute to a default fund, set up operational channels for client clearing, and, in extreme circumstances, bear large losses of the clearinghouse.

That is, the cumulative distribution functions are

$$F(t) = \begin{cases} 1 - \frac{1}{2}e^{-\lambda t}, & t \geq 0 \\ \frac{1}{2}e^{\lambda t}, & t < 0 \end{cases},$$

$$H(t) = \begin{cases} 1 - \frac{1}{2}e^{-\gamma t}, & t \geq 0 \\ \frac{1}{2}e^{\gamma t}, & t < 0 \end{cases}.$$

We use f and h to denote the associated density functions. Notice that while H is a probability distribution describing the random realization of a random variable ε , F is not; it is used to describe the distribution of deterministic private benefits among a unit mass of clients.

Before proceeding further, we discuss the economic interpretation of the parameters γ and λ . The mean absolute deviation of a Laplace $(0, \gamma)$ distribution is $\frac{1}{\gamma}$. γ serves as a measure of the *volatility* of the random contract value, with the contract value being more volatile when γ is small. It thus represents the extent to which clients trade due to speculation, generating surplus primarily from the default option. λ , on the other hand, serves as a measure of the *depth* of deterministic private benefits to be realized from financial trading. When λ is small, there are many clients with large private benefits. It thus represents the extent to which clients trade due to fundamental value, generating surplus primarily from capturing private benefits.

After the contract value is realized, clients default strategically. If a long client choose to not default (ND), he receives his private benefit B and the contract realization ε ; if he chooses to default (D), he does not receive the private benefit and loses his collateral C . The short case is analogous: he either receives $-B$ and $-\varepsilon$ or loses C . They default whenever it is more profitable to do so.

Last, the clearinghouse collects payments from clients who are out of the money and is obligated to pay the clients who are in the money. When out-of-the-money clients default, the clearinghouse experiences a shortfall in payments and must make up for the difference using her own equity thereby incurring a loss. The clearinghouse's payoff is thus

$$\delta_c \times \text{client trades} - \text{loss per contract} \times \text{clients who trade and default.} \quad (2.2)$$

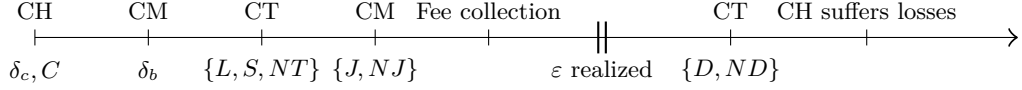


Figure 1: The extensive form game timeline

A graphical illustration of the timeline of the game is given in Figure 1.

3 Equilibrium fee-to-collateral ratio

In this section we solve for the subgame perfect Nash equilibria using backward induction. We first solve for clients' choice of defaults, assuming that the clearing channel has been set up. Since clients default whenever it is more profitable, the long client's payoff function is:

$$\max(B - \delta + \varepsilon, -\delta - C). \quad (3.1)$$

The short client's payoff function is

$$\max(-B - \delta - \varepsilon, -\delta - C) \quad (3.2)$$

Notice that clients default only when the market moves against them. In particular, if $\varepsilon < 0$, all long buyers with insufficient private benefit $B + \varepsilon \leq -C$ will default; if $\varepsilon > 0$, all short buyers with insufficient (negative) private benefit $-B - \varepsilon \leq -C$ will default.

Next, the bank joins (J) as a clearing member if the client clearing business is profitable:

$$\delta_b \times \text{mass of clients who trade} - G \geq 0.$$

Clients trade when their ex-ante expected payoff is positive. Our first theorem shows that there is a unique private benefit threshold governing the trading decisions of the clients.

Theorem 1. *For fixed $\delta \geq 0$ and $C \geq 0$, there exists a unique trading threshold $\tilde{B} = \tilde{B}(\delta, C)$ such that a client wants to trade long if $B \geq \tilde{B}$, and wants to trade short if $B \leq -\tilde{B}$.*

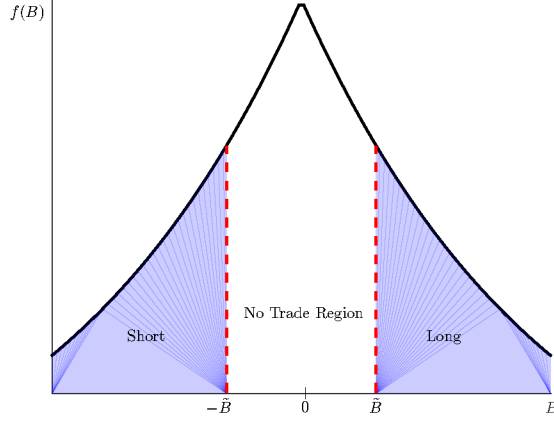


Figure 2: Each client trades depending on his level of private benefits. Only clients with high (low) benefits trade long (short).

A straightforward calculation (included in the appendix) shows that \tilde{B} is the solution to

$$\delta = \tilde{B} + \int_{\tilde{B}+C}^{\infty} (1 - H(x)) dx.$$

The above expression indicates that this “trading threshold” does not depend on the distribution of private benefits F . Indeed, after δ and C are declared, each client needs only evaluate the profitability of his own potential trade, disregarding the trading actions of his fellow clients. From the above theorem we see immediately that clearinghouse requirements create a “no-trade” region.

The clearinghouse may want the clients with private benefits $\tilde{B} > B > -\tilde{B}$ to trade. However, the requirements are such that this would not be profitable for these clients. In our model, the clearinghouse and bank only have information about the *distribution* of client private benefits and cannot distinguish between clients before the trades are submitted. Therefore they cannot incentivize more clients to trade by offering them reduced fees or lowered collateral levels. This phenomenon is illustrated in figure 2. We remark that since F is symmetric, the number of positions traded by each bank’s clients net to zero (the market always clears).

Notice that a client with private benefits B such that $\tilde{B} \leq B \leq -\tilde{B}$ would want to be both long and short.⁷ However, multiple contracts are usually governed by a *master agreement* that dictates that when a client defaults on one position, he must

⁷Notice that this case exists if and only if $\tilde{B} \leq 0$.

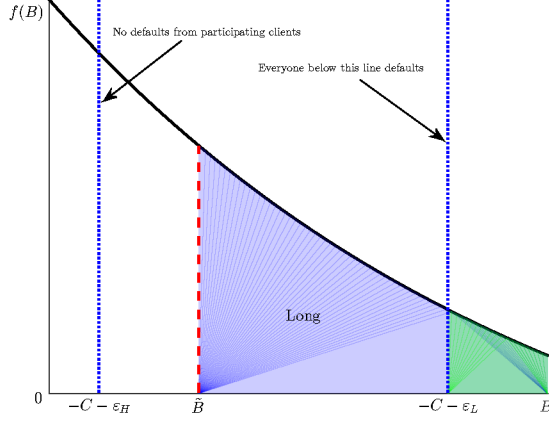


Figure 3: After the contract value is realized, clients with insufficient benefits default. When a low value ε_L is realized, clients who trade and default create losses for the clearinghouse. When a high value ε_H is realized, there are no defaults.

default on all positions governed by the agreement (Hull (2012)). Thus, in the case where $\tilde{B} \leq B \leq -\tilde{B}$, the client's payoff from trading both long and short a contract is

$$\max(B - \delta + \varepsilon - B - \delta - \varepsilon, -2\delta - 2C) = -2\delta - 2C.$$

Hence, he would never choose such a position when fees or collateral are positive. In this case the client chooses the option that gives higher expected payoff: long if $B > 0$ and short if $B < 0$. Thus the level of private benefits beyond which clients will trade long is bounded below by zero.⁸ It will thus be convenient to define the *effective trading threshold* $\bar{B}(\delta, C) = \max(\tilde{B}(\delta, C), 0)$.

We can now express the bank's payoff function given in Eq. (2.1) as

$$R(\delta_b; \delta_c, C, G) := 2\delta_b(1 - F(\bar{B})) - G, \quad (3.3)$$

where $(1 - F(\bar{B}))$ is the fraction of clients who are long the contract.

The clearinghouse's payoff function is

$$X(\delta_c, C) := 2\delta_c(1 - F(\bar{B})) + (\varepsilon + C)(F(-C - \varepsilon) - F(\bar{B}))^+ + (-\varepsilon + C)(F(-C + \varepsilon) - F(\bar{B}))^+.$$

The first term corresponds to the clearinghouse's income from transaction fees. The second term is the aggregate loss from long clients who default: the loss per default

⁸As we will see later, this effect of the master agreement induces a discontinuity in the rate at which the payoff of the bank changes with respect to the clearing requirements.

$-\varepsilon - C$ multiplied by the mass of clients who traded long ex-ante and defaulted ex-post $(F(-C - \varepsilon) - F(\bar{B}))^+$. The last term is the aggregate loss from short clients who default. The case of long clients defaulting on a negative realization of ε is illustrated in Figure 3. When a low contract value ε_L is realized all clients with private benefits between \bar{B} and $-C - \varepsilon$ have traded long the contract and choose to default, creating losses to the clearinghouse. When a high value ε_H is realized, there are no defaults, since the clients who would have defaulted did not trade in the first place.

The clearinghouse's expected payoff can be explicitly computed and is given by

$$E[X(\delta_c, C)] = \delta_c e^{-\lambda \bar{B}} - \frac{\lambda}{2(\lambda + \gamma)} e^{-\lambda \bar{B} - \gamma(\bar{B} + C)} \left(\frac{1}{\gamma} + \frac{1}{\lambda + \gamma} + \bar{B} \right). \quad (3.4)$$

We can now define our subgame perfect Nash equilibria.

Definition 1. *An equilibrium is a triple $(\delta_c, C, \delta_b(\cdot))$ such that*

$$(\delta_c, C) \in \operatorname{argmax}_{(x, y) \in R_+^2} E[X(x, y)]; \quad (3.5)$$

$$\delta_b(x, y) \in \operatorname{argmax}_{z \in R_+} z(1 - F(\tilde{B}(x + z, y))) \text{ for all } (x, y) \in R_+^2; \quad (3.6)$$

$$R(\delta_b; \delta_c, C, G) \geq 0 \quad (3.7)$$

$$E[X(\delta_c, C)] \geq 0. \quad (3.8)$$

Notice that we only focus on the clearinghouse and the bank's choice in equilibrium to reduce clutter. In equilibrium, the bank, in response to δ_c and C , chooses a fee level that maximize his expected payoff. The clearinghouse takes this into account and sets δ_c and C to maximize her profits. The last two constraints are the individual rationality constraints of the bank and the clearinghouse, respectively. The bank only joins when his revenue covers his costs, and the clearinghouse will only initiate the business if her profits are positive.

We next show that the bank always has a unique choice of fees that maximizes his payoff:

Theorem 2. *For fixed $\delta_c \geq 0$, $C \geq 0$, $R(\delta_b; \delta_c, C, G)$ has a unique local maximum $\delta_b = \delta_b(\delta_c, C)$ for all $\delta_c \geq 0$ and $C \geq 0$. In addition, $\delta_b(\delta_c, C)$ is continuous.*

The above theorem indicates there will not be a sudden “jump” in bank fees when the

distributional parameters change or when the clearinghouse changes her requirements.

Interestingly, although $\delta_b(\delta_c, C)$ is continuous, depending on δ_c and C , the bank's response to increases in clearing fees and collateral may be increasing their fees (augmenting fees) or decreasing their fees (complementing fees). It will be convenient to define the function

$$\xi(\delta_c, C) := 1 + \lambda\delta_c - \frac{1}{2}e^{-\gamma C} \left(1 + \frac{\lambda}{\gamma}\right) \quad (3.9)$$

Our next result gives the precise conditions identifying the two regimes.

Theorem 3. *The following statements hold:*

1. *(Augmenting fees.) If $\xi \geq 0$, δ_b is given as the unique solution to*

$$\gamma(\delta_c + C) + \log 2 - 1 = -(\lambda + \gamma)\delta_b - \log(1 - \lambda\delta_b), \quad (3.10)$$

greater than or equal to $\frac{\gamma}{\lambda + \gamma} \frac{1}{\lambda}$. In this case $\frac{\partial \delta_b}{\partial \delta_c} = \frac{\partial \delta_b}{\partial C} \geq 0$, and $\tilde{B} \geq 0$.

2. *(Complementing fees.) If $\xi < 0$,*

$$\delta_b = \frac{1}{2\gamma}e^{-\gamma C} - \delta_c. \quad (3.11)$$

In this case $\frac{\partial \delta_b}{\partial \delta_c} < 0$, $\frac{\partial \delta_b}{\partial C} < 0$, and $\tilde{B} = 0$.

We refer to the first regime as the bank imposing *augmenting fees*. In this regime, when the clearinghouse increases requirements, the participating clients base shrinks; in response, the bank increases his fee, which further decreases the participating client base. However, the gain from increased fee outweighs the loss in the mass of participating clients. We refer to the second regime as the bank imposing *complementing fees*. In this regime, the entire client base is participating; when the clearinghouse increases requirements, the participating clients base again shrinks. Since the clearinghouse's requirements are low, the bank decreases his fee so that again the full client base is participating. In this case, the loss from the decreased fee is outweighed by the gain in the mass of participating clients. Some examples of the revenue functions (income as a function of the bank's fee) the banks face is given in Figure 4.

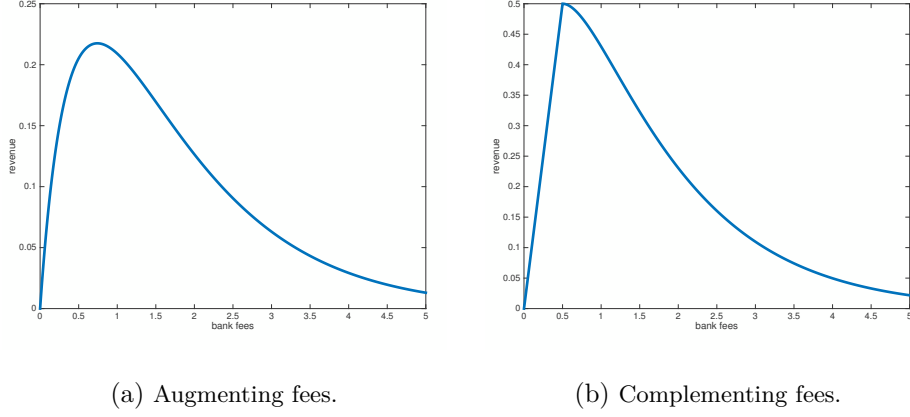


Figure 4: Revenue as a function of the bank's fees. The bank's choice of fees is always unique.

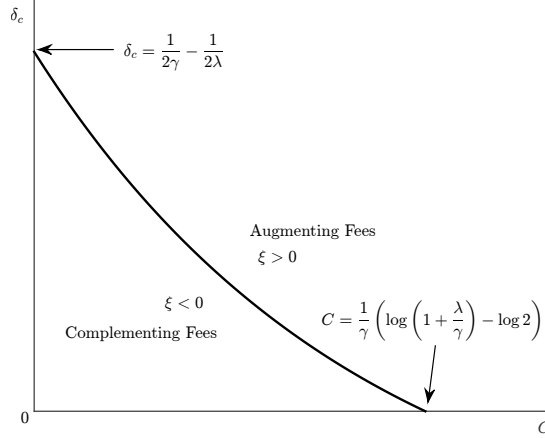


Figure 5: The clearinghouse's strategy space is separated into two sections. Depending on her choice of requirements, the bank reacts in different regimes. The complementing regime is feasible only when $\frac{\gamma}{\lambda} < 1$.

Additionally, theorem 3 shows that the clearinghouse's strategy space is separated into two distinct regions. Which regime the bank adapts to depends on her choice of clearing requirements. Figure 5 illustrates this phenomenon. Notice that the complementing fee regime is feasible only when $\frac{\gamma}{\lambda} < 1$, i.e. when the contract is sufficiently risky.

After characterizing the bank's response, we can solve for the clearinghouse's action. When the clearinghouse sets her requirements, she must take into account the two possible regimes that can result from her actions. For future purposes, it is convenient

to work with the normalized quantities:

$$\begin{aligned}
u &:= \lambda \delta_b, \\
v &:= \gamma \delta_c \\
w &:= \gamma C, \\
\theta &:= \frac{\gamma}{\lambda}.
\end{aligned} \tag{3.12}$$

We first start with solving for the equilibrium assuming that the individual rationality constraint given by Eq. (3.7) is non-binding.

Theorem 4. *Equilibria in which (3.7) is non-binding can occur only at two points:*

- (i) (Infinite collateral equilibrium) $w = \infty$ and $v = \theta$,
- (ii) (zero collateral equilibrium) $w = 0$, v uniquely maximizes $E[X(\delta_c, 0)]$.

Which equilibrium prevails depends only on the value of θ .

Theorem 4 shows that the empirically observed low fee-to-collateral ratio can be explained as a result of risk neutral agents' strategic actions. We can easily demonstrate numerically that there exists situations in which either equilibrium is achieved, depending on the value of θ . Table 1 gives some numerical values of the expected payoff function when $\lambda = 1$.

$\gamma (\delta_c, C, E[X(\delta_c, C)])$	Infinite Collateral	Zero Collateral
0.2	$(1, \infty, .1353)^*$	$(2.9493, 0, .1119)$
0.59	$(1, \infty, .1353)$	$(1.3261, 0, .1356)^*$
1	$(1, \infty, .1353)$	$(1.0681, 0, .1380)^*$
∞	$(1, \infty, .1353)^*$	$(1, 0, .1353)^*$

Table 1: Expected payoffs at possible extrema for $\lambda = 1$. Asterisks describe which equilibrium is chosen by the clearinghouse.

By plotting the optimized expected payoff as a function of γ , fixing $\lambda = 1$, we see an even stronger phenomenon. As illustrated in Figure 6, there is a critical value γ^* beyond which the clearinghouse chooses zero collateral, and below which the clearinghouse chooses infinite collateral. When γ is small, the contract value is very volatile and the expected loss of the clearinghouse from client defaults is high. The clearinghouse thus chooses to eliminate all defaults by setting high levels of collateral. When γ is large, the contract value is not as volatile and the expected loss of the clearinghouse

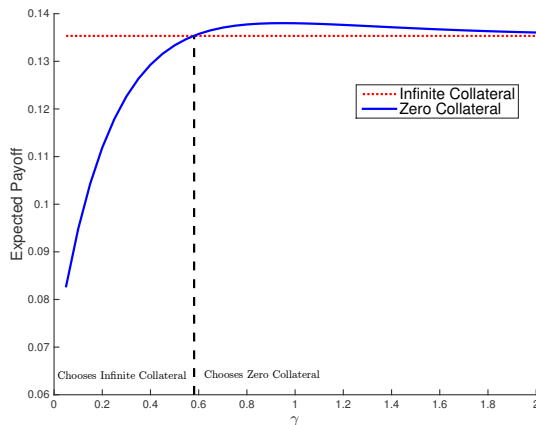


Figure 6: Maximum clearinghouse expected payoff as a function of γ for $\lambda = 1$. The clearinghouse chooses infinite collateral is below the threshold value γ^* . Beyond this threshold value, the clearinghouse chooses zero collateral.

from client defaults is low. The clearinghouse thus chooses to incentivize more clients to trade by setting collateral at very low levels. As γ approaches infinity, the contract value converges to zero. In this case the decisions of clients become independent of the collateral requirements, so that the zero collateral equilibrium and the infinite collateral equilibrium coincide and give the same optimized value.

The phenomenon observed in Figure 6 is quite general. By Theorem 4 we know that which equilibrium prevails depends only on the *relative* riskiness (relative to the depth of private benefits) of the contract value, θ . Since from Figure 6 the threshold $\gamma^* \approx 0.58$, we know that $\theta^* = \gamma^*/1$ also serves as a threshold for all pairs (γ, λ) : if $\theta > \theta^*$, the clearinghouse chooses zero collateral (infinite fee-to-collateral ratio), and infinite collateral (zero fee-to-collateral ratio) otherwise.

4 Conclusion

Our model of centralized clearing is designed to explain the fee-to-collateral ratios observed in the market. Our results indicate that fee and collateral choices of the clearinghouse affect her payoffs in two different ways: they influence the payoff per client cleared and the mass of clients who trade. We have analyzed the equilibrium that results from the profit maximization process of all agents in the model. It should be noted that we are not asserting that the low fee-to-collateral ratios must result from strategic expected profit maximization, but rather propose it as a plausible alternative to assigning agents

some (unobservable) high level of risk aversion.

Our results can be easily extended to the case of multiple clearing members when the clearinghouse sets requirements separately for each clearing member client base. A more interesting but difficult extension is to consider multiple clearing members where the clearinghouse is restricted to set uniform requirements for clients clearing via all member banks. In this case one must take into account all of the private benefit distributions and the relative sizes of the potential client bases that can be cleared through the clearing member. We expect that when one clearing member has many more clients than others, the clearinghouse would choose requirements close to the case of one member treated in this paper.

Our choice of the Laplace distribution is due to the straightforward economic interpretation of its parameters and the analytical tractability. However, it is also desirable to assess the robustness of our results with respect to different distributional specifications of private benefits and contract value.

It would also be interesting to analyze how costliness of collateral affects the resulting fee-to-collateral ratio equilibrium. When collateral comes at a cost, collateral requirements reduce both the overall surplus of clients' trades and the default option value. Obviously, the infinite collateral equilibrium would disappear since no clients would trade.

A Proofs

Proof of Theorem 1. We start with the long case. Assume \tilde{B} is to be a solution to

$$\delta = E[(B + \varepsilon)1_{\varepsilon > -B-C} - C1_{\varepsilon \leq -B-C}] \quad (\text{A.1})$$

That is, \tilde{B} is a level of private benefit at which expected profits for the client is zero. We see that

$$\begin{aligned} \lim_{B \rightarrow \infty} \phi(B) &= \lim_{B \rightarrow \infty} E[B1_{\varepsilon > -B-C}] + E[\varepsilon 1_{\varepsilon > -B-C}] - E[C1_{\varepsilon \leq -B-C}] \\ &= \infty + 0 - C = \infty. \end{aligned}$$

by using the fact that $E[\varepsilon] = 0$ and applying the monotone convergence theorem. In addition,

$$\begin{aligned}\lim_{B \rightarrow -\infty} \phi(B) &= \lim_{B \rightarrow -\infty} -(-B - C)E[1_{\varepsilon > -B-C}] + E[\varepsilon 1_{\varepsilon > -B-C}] - C \\ &= -C,\end{aligned}$$

where we have used the fact that $E|\varepsilon| < \infty$ implies that $\lim_{x \rightarrow \infty} xP(\varepsilon > x) = 0$. Thus, if $\delta \geq 0$ and $C \geq 0$, there must exist B such that Eq. (A.1) is satisfied.

Next, notice that $\phi(B)$ is a strictly increasing function of B , given that

$$\begin{aligned}\phi(B) &= \int_{-B-C}^{\infty} B + x dH(x) + \int_{-\infty}^{-B-C} -C dH(x), \\ \phi'(B) &= 1 - H(-B - C) > 0.\end{aligned}$$

So the solution must be unique. Since H is symmetric, the short case follows by a symmetry argument. \square

Proof of Theorem 2. We start with two propositions:

Proposition 1. *Let $\delta \geq 0$ and $C \geq 0$. Then $\tilde{B}(\delta, C) \geq 0$ if and only if*

$$\delta \geq \frac{1}{2\gamma} e^{-\gamma C}.$$

Proof of Proposition. By definition

$$\begin{aligned}\delta &= \int_{-\tilde{B}-C}^{\infty} (\tilde{B} + x) dH(x) + \int_{-\infty}^{-\tilde{B}-C} -C dH(x) \\ &= \tilde{B} - (\tilde{B} + C)H(-\tilde{B} - C) + \int_{-\tilde{B}-C}^0 x dH(x) + \int_0^{\infty} x dH(x) \\ &= \tilde{B} - (\tilde{B} + C)H(-\tilde{B} - C) + xH(x) \Big|_{-\tilde{B}-C}^0 - \int_{-\tilde{B}-C}^0 H(x) dx + \int_0^{\infty} (1 - H(x)) dx \\ &= \tilde{B} - \int_0^{\tilde{B}+C} (1 - H(y)) dy + \int_0^{\infty} (1 - H(x)) dx \\ &= \tilde{B} + \int_{\tilde{B}+C}^{\infty} (1 - H(x)) dx\end{aligned}\tag{A.2}$$

Here we used the layer cake representation of expectation to derive the third equality and the fact that H is a symmetric distribution to derive the fourth equality. Now plug

$\tilde{B} = 0$ into Eq. (A.2), we have

$$\delta = \int_C^\infty (1 - H(x)) dx = \frac{1}{2\gamma} e^{-\gamma C}.$$

Notice that

$$\frac{\partial}{\partial B} \left(B + \int_{B+C}^\infty (1 - H(x)) dx \right) = H(B + C) > 0,$$

so $\tilde{B}(\delta, C) \geq 0$ if and only if $\delta \geq \frac{1}{2\gamma} e^{-\gamma C}$. \square

The next proposition follows immediately from differentiating Eq. (A.2):

Proposition 2.

$$\begin{aligned} \frac{\partial \tilde{B}}{\partial \delta} &= \frac{1}{H(\tilde{B} + C)} > 0 \\ \frac{\partial \tilde{B}}{\partial C} &= \frac{1}{H(\tilde{B} + C)} - 1 \geq 0 \end{aligned}$$

We next show the existence of a maximizer: let $\phi(x) := x(1 - F(\bar{B}(x + \delta_c, C)))$. By Proposition 1, for large enough x , we have $\bar{B}(x + \delta_c, C) = \tilde{B}(x + \delta_c, C)$. By Proposition 2, $\frac{\partial \tilde{B}}{\partial \delta} > 1$ since H is a distribution function. Continuity of \bar{B} follows from differentiability of \tilde{B} . Thus, it follows that

$$\lim_{x \rightarrow \infty} \tilde{B}(x + \delta_c, C) = \infty \tag{A.3}$$

$$\lim_{x \rightarrow \infty} \frac{x}{\tilde{B}(x + \delta_c, C)} < \infty. \tag{A.4}$$

Since $\int_0^\infty t dF(t) = \frac{1}{2\lambda} < \infty$, it follows by dominated convergence:

$$\begin{aligned} \lim_{B \rightarrow \infty} B(1 - F(B)) &= 0, \\ \lim_{x \rightarrow \infty} \phi(x) &= \lim_{x \rightarrow \infty} \frac{x}{\tilde{B}(x + \delta_c, C)} \lim_{\tilde{B} \rightarrow \infty} \tilde{B}(1 - F(\tilde{B})) = 0. \end{aligned}$$

Where we used Eq. (A.3) and (A.4) to derive the second limit. Since $\phi(x) \geq 0$ and $\phi(0) = \phi(\infty) = 0$, and ϕ is continuous, there exists an interior maximizer of $\phi(x)$ on $(0, \infty)$.

Next we show uniqueness. Fix $\delta_c, C \geq 0$. By Proposition 1, the bank's payoff

function can be written as

$$2\delta_b(1 - F(\bar{B})) - G = \begin{cases} \delta_b e^{-\lambda \bar{B}(\delta_b + \delta_c, C)} - G, & \delta_b + \delta_c \geq \frac{1}{2\gamma} e^{-\gamma C} \\ \delta_b, & \delta_b + \delta_c < \frac{1}{2\gamma} e^{-\gamma C} - G \text{ bank's payoff function} \end{cases} \quad (\text{A.5})$$

Suppose $y \in \left\{x \mid \tilde{B}(x + \delta_c, C) < 0\right\} = \left\{x \mid x + \delta_c < \frac{1}{2\gamma} e^{-\gamma C}\right\}$ is a local maximizer of ϕ , then ϕ can be always increased by choosing δ_b slightly larger than y , thus y is cannot be a local maximizer. Therefore, a local maximizer must be in the region $\{x \mid \tilde{B}(x + \delta_c, C) \geq 0\}$. This implies that a local maximizer either solves the equation

$$0 = \tilde{B}(x + \delta_c, C), \quad (\text{A.6})$$

or is a critical point of the differentiable function

$$\omega(x) := x e^{-\lambda \tilde{B}(x + \delta_c, C)}.$$

Obviously, if Eq. (A.6) has a nonnegative solution, it must be $x_0 := \frac{1}{2\gamma} e^{-\gamma C} - \delta_c$, In this case the bank's payoff is exactly $x_0 - G$ by Eq (A.5).

We now analyze the critical points of the function $\omega(x)$ strictly larger than $\frac{1}{2\gamma} e^{-\gamma C} - \delta_c$. We will show there is at most one such critical point. The first order condition is

$$0 = 1 - \frac{\lambda x}{1 - \frac{1}{2} e^{-\gamma(\tilde{B}(x + \delta_c, C) + C)}} \quad (\text{A.7})$$

By Eq. (A.2), we also have

$$x + \delta_c = \tilde{B}(x + \delta_c, C) + \frac{1}{2\gamma} e^{-\gamma(\tilde{B}(x + \delta_c, C) + C)} \quad (\text{A.8})$$

Thus, combining Eq. (A.7) and (A.8) any critical point of $\omega(x)$ must satisfy

$$\gamma(\delta_c + C) + \log 2 - 1 = -\lambda x - \gamma x - \log(1 - \lambda x) \quad (\text{A.9})$$

Define $\psi(x) := -\lambda x - \gamma x - \log(1 - \lambda x)$, and $A(\gamma) := \gamma(\delta_c + C) + \log 2 - 1$ we see

that

$$\begin{aligned}\psi(0) &= 0 \\ \lim_{x \rightarrow 1/\lambda} \psi(x) &= \infty \\ \psi''(x) &= \frac{\lambda^2}{(1 - \lambda x)^2} \geq 0, \text{ for } x \in (0, \lambda^{-1}).\end{aligned}$$

Now, the unique minimum of $\psi(x)$ is given by the first order condition:

$$\begin{aligned}\psi'(x) &= -\lambda - \gamma + \frac{\lambda}{1 - \lambda x} = 0. \\ x^* &= \frac{\gamma}{\lambda(\lambda + \gamma)} < \frac{1}{\lambda} \\ \psi(x^*) &= -\frac{\gamma}{\lambda} + \log\left(1 + \frac{\gamma}{\lambda}\right) < 0\end{aligned}\tag{A.10}$$

The above analysis shows that on $\{x | 0 < \lambda x < 1\}$, $w(x)$ has exactly one critical point when $A(\gamma) > 0$ or $A(\gamma) = \psi(x^*)$ and exactly two critical points when $\psi(x^*) < A(\gamma) < 0$.

We will now show that when there are two critical points, the smaller critical point is always less than $\frac{1}{2\gamma}e^{-\gamma C} - \delta_c$. Define $Y := \gamma(\delta_c + C)$, $u := \lambda x$, $\theta := \frac{\gamma}{\lambda}$, then we can rewrite Eq. (A.9) as:

$$Y + \log 2 - 1 = -(1 + \theta)u - \log(1 - u).\tag{A.11}$$

Fixing Y , we see that $\min_{\gamma(\delta_c + C) = Y} \lambda \left(\frac{1}{2\gamma}e^{-\gamma C} - \delta_c \right)$ occurs when $\delta_c = \frac{1}{\gamma}(Y + \log 2)$ and $C = \frac{-\log 2}{\gamma}$, with a minimized value of $\bar{m}(Y) := \frac{1}{\theta}(1 - \log 2 - Y)$. It is clear that there exists, for some choice of $\lambda, \gamma, \delta_c, C$, two critical points of $w(x)$ greater than $\frac{1}{2\gamma}e^{-\gamma C} - \delta_c$ if and only if for some $Y \in [0, 1 - \log 2]$, the smaller solution to Eq.(A.11) is larger than $\bar{m}(Y)$.

Denote the smaller branch of solutions to Eq. (A.9) as $u^-(Y)$. It holds that,

$$\bar{m}(1 - \log 2) = 0$$

$$u^-(1 - \log 2) = 0$$

By Eq. (A.10), we see that $u^-(Y) < \frac{\theta}{1 + \theta}$. Since $-(1 + \theta) + \frac{1}{1 - u} > 0$ for all $u \in [0, \frac{\theta}{1 + \theta})$

we have explain the

$$\frac{\partial}{\partial Y} (\bar{m}(Y) - u^-(Y)) = -\frac{1}{\theta} - \frac{1}{-(1+\theta) + \frac{1}{1-u}} = -\frac{\frac{u}{1-u}}{\theta \left(-(1+\theta) + \frac{1}{1-u} \right)} < 0$$

Thus $\bar{m}(Y) > u^-(Y)$ for $Y \in [0, 1 - \log 2]$. Thus there is at most one critical point, \tilde{x} , greater than x_0 . This allows us to conclude there is a unique maximizer: the function is either maximized at \tilde{x} or x_0 .

Since the payoff function is differentiable everywhere but at x_0 , the function cannot be maximized simultaneously at \tilde{x} or x_0 , unless they are equal. Indeed, using the mean value theorem we have that if $\tilde{x} \neq x_0$, either $\phi(\tilde{x}) > \phi(x_0)$ or $\phi(\tilde{x}) < \phi(x_0)$. Again since there is no critical point in between, only one of them can be a local maximizer.

The second statement of the theorem, i.e., the continuity of the function $\delta_b(\delta_c, C)$, follows from Berge's maximum theorem. \square

Proof of Theorem 3. Notice that $\xi < 0$ implies that $x_0 := \frac{1}{2\gamma}e^{-\gamma C} - \delta_c \geq 0$.

Suppose $x_0 \leq 0$. By Proposition 1 we have $\tilde{B}(\delta_b + \delta_c, C) > 0$ for all $\delta_b \geq 0$, thus δ_b must be the larger solution to Eq. (3.10) by Eq. (A.9). Now suppose $x_0 > 0$. we see the right derivative of the bank's payoff function Eq. (3.3) at x_0 is given by

$$\begin{aligned} \lim_{x \rightarrow x_0} 1 - \frac{\lambda x}{1 - \frac{1}{2}e^{-\gamma(\tilde{B}(x+\delta_c, C)+C)}} &= 1 - \frac{\lambda(\frac{1}{2\gamma}e^{-\gamma C} - \delta_c)}{1 - \frac{1}{2}e^{-\gamma C}} \\ &\propto 1 - \frac{1}{2}e^{-\gamma C} - \lambda(\frac{1}{2\gamma}e^{-\gamma C} - \delta_c) = 1 + \lambda\delta_c - \frac{1}{2}e^{-\gamma C}(1 + \frac{\lambda}{\gamma}). \end{aligned}$$

Since the local maximum of the payoff function is unique by Theorem 2, δ_b must be the larger solution to Eq. (3.10) when the right derivative is positive, and equal to x_0 when the right derivative is negative. The sign of the derivatives follow from direct differentiation. \square

Proof of Theorem 4. For fixed $\lambda, \gamma > 0$, we search for all possible maxima over the space $K^* := \{(\delta_c, C) | \delta_c \geq 0, C \geq 0\}$. The payoff function is obviously continuous, and is continuously differentiable on $K^* \setminus \{\xi(\delta_c, C) = 0\}$, where ξ is defined in Eq. (3.9).

We need only consider the set

$$K := \{(\delta_c, C) | \delta_c \geq 0, C \geq 0, \xi \geq 0\}, \quad (\text{A.12})$$

on the interior of which $E[X(\delta_c, C)]$ is continuously differentiable. Indeed, when $\xi < 0$, by the second statement of Theorem 3, we have $\tilde{B} = 0$ and the clearinghouse's expected payoff function Eq. (3.4) is therefore given as

$$E[X(\delta_c, C)] = \delta_c - \frac{\gamma}{2} e^{-\gamma C} \left(\frac{1}{\gamma} - \frac{1}{\lambda + \gamma} \right) \left(\frac{1}{\gamma} + \frac{1}{\lambda + \gamma} \right)$$

Thus, when $\xi < 0$, the clearinghouse can increase profits by increasing δ_c and C , and thus will only choose δ_c, C such that $\xi \geq 0$.

We start with searching for critical points in the interior of K . Recall that in the interior of K , \tilde{B} and $\delta_b(\delta_c, C)$ are implicitly defined by:

$$0 = 1 - \frac{1}{2} e^{-\gamma(\tilde{B}+C)} - \lambda \delta_b \tag{A.13}$$

$$\delta_b + \delta_c = \tilde{B} + \frac{1}{2\gamma} e^{-\gamma(\tilde{B}+C)} \tag{A.14}$$

Rearranging Eq. (A.14), we have

$$\tilde{B} = \delta_b + \delta_c - \frac{1}{\gamma} + \frac{\lambda}{\gamma} \delta_b \tag{A.15}$$

This in turn implies that,

$$\gamma(\delta_c + C) + \log 2 - 1 = -\lambda \delta_b - \gamma \delta_b - \log(1 - \lambda \delta_b). \tag{A.16}$$

By implicitly differentiating Eq.(A.16), we have

$$\frac{\partial \delta_b}{\partial \delta_c} = \frac{\partial \delta_b}{\partial C} = \frac{\gamma}{-\lambda - \gamma + \frac{\lambda}{1 - \lambda \delta_b}}. \tag{A.17}$$

Define

$$V := \delta_c e^{-\lambda B} - \frac{\lambda}{2(\lambda + \gamma)} e^{-\lambda B - \gamma(B+C)} \left(\frac{\lambda + 2\gamma}{\gamma(\lambda + \gamma)} + B \right)$$

A direct computation gives,

$$\begin{aligned}
\frac{\partial V}{\partial \delta_c} &= e^{-\lambda B} \\
\frac{\partial V}{\partial C} &= e^{-\lambda B - \gamma(B+C)} \frac{\lambda \gamma}{2(\lambda + \gamma)} \left(B + \frac{\lambda + 2\gamma}{\gamma(\lambda + \gamma)} \right) \\
\frac{\partial V}{\partial B} &= e^{-\lambda B} \left(-\lambda \delta_c - \frac{\lambda}{2(\lambda + \gamma)} e^{-\gamma(B+C)} + \frac{\lambda}{2} e^{-\gamma(B+C)} \left(B + \frac{\lambda + 2\gamma}{\gamma(\lambda + \gamma)} \right) \right)
\end{aligned} \tag{A.18}$$

Evaluating each derivative in Eq. (A.18) at $B = \tilde{B}$, we obtain

$$e^{\lambda \tilde{B}} \frac{\partial V}{\partial \delta_c} = 1. \tag{A.19}$$

$$\begin{aligned}
e^{\lambda \tilde{B}} \frac{\partial V}{\partial C} &= (1 - \lambda \delta_b) \frac{\lambda \gamma}{(\lambda + \gamma)} \left(\delta_b + \delta_c - \frac{1}{\gamma} + \frac{\lambda}{\gamma} \delta_b + \frac{\lambda + 2\gamma}{\gamma(\lambda + \gamma)} \right) \\
&= (1 - \lambda \delta_b) \frac{\lambda}{(\lambda + \gamma)^2} ((\lambda + \gamma)^2 \delta_b + \gamma(\lambda + \gamma) \delta_c + \gamma) \\
&= (1 - u) \left(u + \frac{v}{1 + \theta} + \frac{\theta}{(1 + \theta)^2} \right)
\end{aligned} \tag{A.20}$$

$$\begin{aligned}
e^{\lambda \tilde{B}} \frac{\partial V}{\partial B} &= -\lambda \delta_c - \frac{\lambda}{2(\lambda + \gamma)} e^{-\gamma(B+C)} + \frac{\lambda}{2} e^{-\gamma(B+C)} \left(B + \frac{\lambda + 2\gamma}{\gamma(\lambda + \gamma)} \right) \\
&= -\lambda \delta_c - \frac{\lambda}{(\lambda + \gamma)} (1 - \lambda \delta_b) + \lambda (1 - \lambda \delta_b) \frac{(\lambda + \gamma)^2 \delta_b + \gamma(\lambda + \gamma) \delta_c + \gamma}{\gamma(\lambda + \gamma)} \\
&= \lambda \left(-\delta_c + (1 - \lambda \delta_b) \left(\frac{(\lambda + \gamma) \delta_b + \gamma \delta_c}{\gamma} \right) \right) \\
&= \frac{\lambda}{\gamma} (-\delta_b (\lambda (\lambda + \gamma) \delta_b + \gamma \lambda \delta_c - (\lambda + \gamma))) \\
&= \frac{u}{\theta} ((1 + \theta)(1 - u) - v)
\end{aligned} \tag{A.21}$$

$$\frac{\partial \tilde{B}}{\partial \delta} = \frac{1}{H(\tilde{B} + C)} = \frac{1}{\lambda \delta_b} = \frac{1}{u}. \tag{A.22}$$

$$1 + \frac{\partial \delta_b}{\partial \delta_c} = \frac{\lambda^2 \delta_b}{\lambda - (1 - \lambda \delta_b)(\lambda + \gamma)} = \frac{u}{1 - (1 - u)(1 + \theta)} \tag{A.23}$$

Where to derive the above results we have used the definitions and results given by Eq. (3.12), Eq. (A.15), and Proposition 2.

The first order condition of the clearinghouse's expected payoff function $E[X(\delta_c, C)]$ with respect to δ_c is

$$0 = \frac{\partial V}{\partial \delta_c} + \frac{\partial V}{\partial B} \frac{\partial \tilde{B}}{\partial \delta} \left(1 + \frac{\partial \delta_b}{\partial \delta_c} \right) \tag{A.24}$$

which, using the expressions given by Eq.(A.19)–(A.23) , gives

$$0 = \theta(1 - (1 - u)(1 + \theta)) - u((1 + \theta)u + v - (1 + \theta)),$$

leading to

$$v = \frac{-\theta^2 + (\theta + 1)^2 u - (1 + \theta)u^2}{u}. \quad (\text{A.25})$$

The first order condition with respect to C is

$$\begin{aligned} 0 &= \frac{\partial V}{\partial C} + \frac{\partial V}{\partial B} \left(\frac{\partial \tilde{B}}{\partial \delta} \frac{\partial \delta_b}{\partial C} + \frac{\partial \tilde{B}}{\partial C} \right) \\ &= \frac{\partial V}{\partial C} + \frac{\partial V}{\partial B} \left(\frac{\partial \tilde{B}}{\partial \delta} \frac{\partial \delta_b}{\partial \delta_c} + \frac{\partial \tilde{B}}{\partial \delta} - 1 \right) \\ &= \frac{\partial V}{\partial C} + \frac{\partial V}{\partial B} \frac{\partial \tilde{B}}{\partial \delta} \left(\frac{\partial \delta_b}{\partial \delta_c} + 1 \right) - \frac{\partial V}{\partial B} \\ &= \frac{\partial V}{\partial C} - \frac{\partial V}{\partial \delta_c} - \frac{\partial V}{\partial B} \end{aligned}$$

which, using the expressions given by Eq.(A.19)–(A.23), yields

$$0 = (1 - u) \left(u + \frac{1}{1 + \theta} v + \frac{\theta}{(1 + \theta)^2} \right) - 1 - \frac{u}{\theta} ((1 + \theta)(1 - u) - v)$$

leading to

$$v = \frac{\theta((1 + \theta)^2 - \theta) + (1 + 2\theta + 2\theta^2)u - (1 + \theta)^2 u^2}{(1 + \theta)(\theta + u)} \quad (\text{A.26})$$

Since at a critical point both Eq. (A.25) and Eq. (A.26) must hold:

$$\begin{aligned} 0 &= \frac{-\theta^2 + (\theta + 1)^2 u - (1 + \theta)u^2}{u} - \frac{\theta((1 + \theta)^2 - \theta) + (1 + 2\theta + 2\theta^2)u - (1 + \theta)^2 u^2}{(1 + \theta)(\theta + u)} \\ &= -\frac{\theta^2(u - 1)(u - \theta^2 - \theta)}{u(1 + \theta)(\theta + u)} \end{aligned}$$

The above equation implies that critical points can occur only at $(u_\infty, v_\infty) := (1, \theta)$ and $(u_{int}, v_{int}) := (\theta + \theta^2, 1 - \theta^3 - \frac{\theta^3}{1 + \theta})$. $(u, v) = (1, \theta)$ directly implies $C = \infty$ by Eq. (A.16). These results, along with Eq. (A.16), give what we refer to as the infinite collateral equilibrium and the interior equilibrium, respectively.

Next we consider maxima occurring on the boundary of K , where we recall that K has been defined in Eq.(A.12). The boundary of K consists of the lines $\delta_c = 0, \delta_c = \infty, C = 0, C = \infty$, and $\xi = 0$. We will consider each line separately.

From Eq. (3.4) we see that the clearinghouse is making positive profits $\frac{1}{\lambda}e^{-2}$ when $C = \infty$. Using Eq. (3.4) and Eq. (A.15), we see that the clearinghouse is making zero profits when $\delta_c = \infty$, and nonpositive profits when $\delta_c = 0$. We can thus rule out the latter two cases.

When $C = \infty$, $E[X(\delta_c, \infty)] = \delta_c e^{-\lambda(\delta_b + \delta_c)}$, implying that expected payoff along the line $(\delta_c, C = \infty)$ is maximized at $\delta_c = \frac{1}{\lambda}$.

When $\xi = 0$, we can consider the maximization problem:

$$\begin{aligned} & \max_{\delta_c, C} E[X(\delta_c, C)] \\ & \text{subject to } \xi := 1 + \lambda\delta_c - \frac{1}{2}e^{-\gamma C}\left(1 + \frac{\lambda}{\gamma}\right) = 0. \end{aligned}$$

Notice that when $\xi = 0$, $\bar{B} = 0$ by Theorem 3 and Proposition 1. Introducing the Lagrange multiplier μ , we obtain that the following must hold at optimum:

$$\begin{aligned} 0 &= 1 + \mu\lambda \\ 0 &= \frac{\gamma^2}{2}e^{-\gamma C} \left(\frac{1}{\gamma} - \frac{1}{\lambda + \gamma} \right) \left(\frac{1}{\gamma} + \frac{1}{\lambda + \gamma} \right) - \mu \frac{\gamma}{2}e^{-\gamma C} \left(1 + \frac{\lambda}{\gamma} \right). \end{aligned}$$

Notice that this system of equations has no solution, so local maxima cannot occur along the boundary $\xi = 0$.

This leaves us with the last boundary $C = 0$, which will give the zero collateral equilibrium. To see the uniqueness of this equilibrium, we use the following proposition:

Proposition 3. *$E[X(\cdot, 0)]$ has exactly one local maximum δ_c^* , at which $E[X(\delta_c^*, 0)] \geq 0$.*

Proof of proposition. Set $C = 0$. By Theorem 3, the bank imposes augmenting fees when

$$\xi(\delta_c, 0) = 1 + \frac{1}{\theta}v - \frac{1}{2} \left(1 + \frac{1}{\theta} \right) \geq 0,$$

where we are using the definition of ξ given in Eq. (3.9). This is equivalent to

$$v \geq \frac{1 - \theta}{2}.$$

We first consider the point $v = \frac{1 - \theta}{2}$, i.e. the threshold above which the bank switches from imposing complementing to augmenting fees. Notice this is only relevant when $\theta < 1$. At this point, we have from Eq. (3.11) $u = \lambda \left(\frac{1}{2\gamma} - \frac{v}{\gamma} \right) = \frac{1}{2}$ and $\tilde{B} = 0$. The right derivative of $E[X(\delta_c, C)]$ with respect to δ_c is given by:

$$e^{-\lambda \tilde{B}} \left(1 + \frac{1}{\theta} ((1 + \theta)(1 - u) - v) + \frac{u}{1 - (1 - u)(1 + \theta)} \right). \quad (\text{A.27})$$

and is equal to

$$e^{-\lambda \tilde{B}} \left(1 + \frac{1}{\theta} ((1 + \theta)(1 - u) - v) + \frac{u}{1 - (1 - u)(1 + \theta)} \right) \Big|_{v=\frac{1-\theta}{2}} = 2 + \frac{1}{1 - \theta} > 0.$$

at $v = \frac{1 - \theta}{2}$. Thus the clearinghouse's profits are increasing at this point. Next, consider a point $v > \frac{1 - \theta}{2}$, then by Eq. (3.10), we have

$$v + \log 2 - 1 = -(1 + \theta)u - \log(1 - u),$$

thus when v increases, u approaches 1 monotonically. Thus there exists some $v^*(\theta)$ such that for $v > v^*(\theta)$ we have

$$\frac{\partial E[X(\delta_c, C)]}{\partial \delta_c} < 0$$

by Eq. (A.27). Combined with the previous analysis this means that expected profits are maximized at a finite point where banks are imposing augmenting fees. The condition for optimality is then Eq. (A.25):

$$v = \frac{-\theta^2 + (1 + \theta)^2 u - (1 + \theta)u^2}{u}.$$

Hence u is implicitly given by,

$$\beta(u) := -\log 2 + 1 - \log(1 - u) + \frac{\theta^2}{u} - (1 + \theta)^2 = 0.$$

Notice that this function always has exactly two zeros on $u \in (0, 1)$. We observe that:

$$\beta(0) = \beta(1) = \infty \quad (\text{A.28})$$

$$\beta'(0) = -\infty; \beta'(1) = \infty \quad (\text{A.29})$$

$$\beta'(u) = 0 \text{ only at } u^* = \frac{-\theta^2 + \theta\sqrt{\theta^2 + 4}}{2} < 1 \quad (\text{A.30})$$

$$\beta(u^*) = \frac{2\theta^2}{\theta\sqrt{\theta^2 + 4} - \theta^2} - \log\left(1 - \frac{-\theta^2 + \theta\sqrt{\theta^2 + 4}}{2}\right) - (\theta + 1)^2 + 1 - \log(2) \quad (\text{A.31})$$

$$= \frac{-\theta^2 + \theta\sqrt{\theta^2 + 4}}{2} - \log\left(1 - \frac{-\theta^2 + \theta\sqrt{\theta^2 + 4}}{2}\right) - 2\theta - \log(2) < 0, \quad (\text{A.32})$$

where the last inequality follows from the following two inequalities

$$\theta \geq -\log\left(1 - \frac{-\theta^2 + \theta\sqrt{\theta^2 + 4}}{2}\right) \geq \frac{-\theta^2 + \theta\sqrt{\theta^2 + 4}}{2} \quad (\text{A.33})$$

which holds true for all $\theta > 0$. To show the first inequality, we notice that $\theta = -\log\left(1 - \frac{-\theta^2 + \theta\sqrt{\theta^2 + 4}}{2}\right)$ at $\theta = 0$. In addition, a cumbersome but straightforward computation shows that

$$\frac{\partial}{\partial \theta} \left(\theta + \log\left(1 - \frac{-\theta^2 + \theta\sqrt{\theta^2 + 4}}{2}\right) \right) = 1 - \frac{2}{4 + \theta^2} \geq 0$$

Thus the first inequality in (A.33) holds. The second inequality follows from basic calculus since $-\log(1 - x) \geq x$ for all $x \in [0, 1)$.

Since the expected payoff has at most two critical points, it can have at most one local maximum.

□

Last, we show that an interior collateral equilibrium with $u_{int} = \theta^2 + \theta$, $v_{int} = \left(1 - \theta^3 - \frac{\theta^3}{1 + \theta}\right)$ cannot exist. Notice that the equilibrium is only well defined when $u_{int} \leq 1$, and coincides with infinite collateral equilibrium at $\theta^2 + \theta = 1$. At this point $v_{int} = \left(1 - \theta^3 - \frac{\theta^3}{1 + \theta}\right)$ and we can evaluate

$$\begin{aligned} \lambda \tilde{B} &= \frac{1 + 3\theta + \theta^2}{1 + \theta} \\ \gamma E[X(u_{int}, v_{int})] &= v e^{-\lambda \tilde{B}} - \frac{1}{1 + \theta} e^{-\lambda \tilde{B}} (1 - u) \left(1 + \frac{\theta}{1 + \theta} + \gamma \tilde{B}\right) \\ &= e^{-\frac{1 + 3\theta + \theta^2}{1 + \theta}} \frac{\theta^2(2 + \theta)}{1 + \theta} \end{aligned}$$

$$\frac{\partial}{\partial \theta} e^{-\frac{1+3\theta+\theta^2}{1+\theta}} \frac{\theta^2(2+\theta)}{1+\theta} = -\frac{e^{-\frac{\theta^2}{\theta+1}-\frac{3\theta}{\theta+1}-\frac{1}{\theta+1}} \theta (\theta^4 + 2\theta^3 - \theta^2 - 5\theta - 4)}{(\theta+1)^3}$$

$$\propto -\theta^4 - 2\theta^3 + \theta^2 + 5\theta + 4 > 0$$

The last inequality follows from the fact that for $u_{int} < 1$ we must have $\theta \leq 1$. Since this derivative is positive, the maximum expected profit (maximized over all θ) of the clearinghouse at the interior collateral equilibrium is $\theta^2 + \theta = 1$. For every other θ , the clearinghouse's profit is strictly less than the infinite collateral equilibrium. It is thus never a global maximum unless it coincides with the infinite collateral equilibrium. \square

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About the Author

Wan-Schwin Allen Cheng (1988 –) is a Taiwanese-American born in Buffalo, New York. He spent the first half of his childhood in the United States, and the latter half in Taipei, Taiwan. He graduated from National Taiwan University in 2010 with honors, majoring in Mathematics. After fulfilling his Taiwanese military duties, he worked as a research assistant at Academia Sinica. He then proceeded to earn his masters in science and engineering in Applied Mathematics and Statistics in 2013 from the Johns Hopkins University. He was supported by generous funds from the GAANN fellowship, Counselmen Fellowship, and NSF's GR Fellowship. In addition to assisting in various courses, he has taught the summer course Statistical Analysis II at Johns Hopkins. He has passed all three of the CFA exams and seeks to conduct further research in the fields of mathematical finance and financial economics.