

Multilevel Arnoldi-Tikhonov Regularization Methods for Large-Scale Linear Ill-Posed Systems

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Abstract—This paper is devoted to the numerical solution of large-scale linear ill-posed systems. A multilevel regularization method is proposed. This method is based on a synthesis of the Arnoldi-Tikhonov regularization technique and the multilevel technique. We show that if the Arnoldi-Tikhonov method is a regularization method, then the multilevel method is also a regularization one. Numerical experiments presented in this paper illustrate the effectiveness of the proposed method.

Keywords—Discrete ill-posed problem, Tikhonov regularization, discrepancy principle, Arnoldi process, multilevel method.

I. INTRODUCTION

IN this paper we consider the iterative solution of large system of linear equations

$$Ax = b, \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. We further assume that the coefficient matrix A is of ill-determined rank, i.e., all its singular values decay gradually to zero, with no gap anywhere in the spectrum. This situation is mathematically described by the notion of singular value clustering and is typical of matrices that can be described as a sampling of a reasonably smooth function [1]. Such systems often are referred to as linear discrete ill-posed problems and arise, for example, from the discretization of ill-posed problems such as Fredholm integral equations of the first kind

$$\int_{\Omega} k(t, s)x(s)ds = b(t), \quad t \in \Omega, \quad (2)$$

where Ω denotes a bounded interval, and the kernel $k(t, s)$ and right-hand side $b(t)$ are smooth functions on $\Omega \times \Omega$ and Ω , respectively. Often, the right-hand side b of (1) is contaminated by an error $e \in \mathbb{R}^n$, which may stem from discretization or measurement inaccuracies, and $b = \hat{b} + e$, where \hat{b} is the unknown error-free right-hand side vector. Thus, our task is to compute the solution \hat{x} of the linear system of equations with the error-free right-hand side \hat{b} ,

$$A\hat{x} = \hat{b}. \quad (3)$$

We assume the linear system (3) to be consistent. However, since the right-hand side in (3) is not available, we seek to determine an approximation of \hat{x} by solving the available system (1) or a modification. Due to the ill-conditioning

of A , the system (1) has to be regularized in order to compute a useful approximation of \hat{x} . Perhaps, the best known regularization method is Tikhonov regularization [14], [27], [34], [35], which in its simplest form is based on the minimization problem

$$\min_{x \in \mathbb{R}^n} \left\{ \|Ax - b\|_2^2 + \frac{1}{\mu} \|x\|_2^2 \right\}, \quad (4)$$

where $\mu > 0$ is a regularization parameter. Here and throughout this paper $\|\cdot\|_2$ denotes the Euclidean vector norm or the associated induced matrix norm.

After regularizing the system (1), we need to compute the solution x_μ of the minimization problem (4). Such a vector x_μ is also the solution of

$$\left(A^T A + \frac{1}{\mu} I \right) x = A^T b. \quad (5)$$

If μ is far away from zero, then, due to the ill-conditioning of A , x_μ is badly computed while, if μ is close to zero, x_μ is well computed but the error $x_\mu - \hat{x}$ is quite large. Thus, the choice of a good value for μ is fairly important. Several methods have been proposed to obtain an effective value for μ . For example, if the norm of the error e or a fairly accurate estimate is known, the regularization parameter is quite easy to determine by application of the discrepancy principle. The discrepancy principle proposes that the regularization parameter μ be chosen so that the discrepancy $b - Ax_\mu$ satisfies

$$\|b - Ax_\mu\|_2 = \eta\varepsilon, \quad (6)$$

where $\varepsilon = \|e\|_2$, and $\eta > 1$ is a constant, see, for example, [20] for further details on the discrepancy principle.

A number of numerical solution methods have been proposed for solving the minimization problem (4). The singular value decomposition (SVD) [18] of A can be used to determine the solution x_μ of the minimization problem (4). For an overview of numerical methods for computing the SVD we refer to [5]. Since the computational effort required to compute the SVD is quite high even for moderately sized matrices, the SVD based method is only applicable to problems of small or medium size. Numerical methods using Krylov subspaces have been proposed for the solution of large-scale Tikhonov regularization problems (4). The main idea of such algorithms has been to first project the large problems onto some Krylov subspace to produce problems with small size, and then solve the small-sized problems by the SVD. For instance, several well-established methods based on the Lanczos bidiagonalization process [18] have been proposed for the solution of the minimization problem (4), see [4],

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[8], [9], [25] and references therein. These methods use the Lanczos bidiagonalization process to construct a basis of the Krylov subspace

$$\mathcal{K}_m(A^T A, A^T b) = \text{span}\{A^T b, A^T A A^T b, \dots, (A^T A)^{m-1} A^T b\}.$$

We remark that each Lanczos bidiagonalization step needs two matrix-vector product evaluations, one with A and the other with A^T , and it is not necessary to compute the SVD since the coefficient matrix associated with the small-sized problem is bidiagonal; see, e.g., [9]. Other methods using the range-restricted Krylov subspace

$$\mathcal{K}_m(A, Ab) = \text{span}\{Ab, A^2 b, \dots, A^m b\}$$

as the projection subspace have been also designed. For example, Lewis and Reichel [26] proposed to exploit the Arnoldi process to produce a basis of the Krylov subspace $\mathcal{K}_m(A, Ab)$ to obtain an approximation of the solution of the Tikhonov regularization problem (4). Since each Arnoldi decomposition step requires only one matrix-vector evaluation with A , the approach based on the Arnoldi process may require fewer matrix-vector product evaluations than that based on the Lanczos bidiagonalization process. Moreover, the methods based on the Arnoldi process does not require the adjoint matrix A^T and, hence, is more appropriate to problems for which the adjoint is difficult to evaluate. For such problems we refer to [11]. Several multilevel methods have been proposed in the literature for solving the Tikhonov equation (5), see, for example, [12], [19], [21], [23]. For a large number of ill-posed problems, these multilevel methods determine accurate approximations of the solution of the Tikhonov equation (5) faster than standard one-level iterative methods.

Some numerical methods without using the Tikhonov regularization technique have been already proposed to solve the large-scale linear discrete ill-posed problem (1). These methods include the range-restricted GMRES (RRGMRES) method [6], [33] and the augmented range-restricted GMRES (ARRGMRES) method [3]. The RRGMRRES method determines the m th approximation x_m of (1) by solving the minimization problem

$$\min_{x \in \mathcal{K}_m(A, Ab)} \|Ax - b\|_2, \quad m = 1, 2, \dots, \quad x_0 = 0.$$

The regularization is implemented by choosing a suitable dimension number m , see, for example, [7]. The ARRGMRRES method augments the Krylov subspace $\mathcal{K}_m(A, Ab)$ by a low-dimensional user-supplied subspace. The low-dimensional subspace is determined by vectors that are able to represent the known features of the desired solution. The augmented method can yield approximate solutions of higher accuracy than the RRGMRRES method if the Krylov subspace $\mathcal{K}_m(A, Ab)$ do not allow to represent the known features.

Recently, Reichel and Shyshkov [32] proposed a cascadic multilevel method to solve (1). Although their method is based on the un-regularized normal equation $A^T A x = A^T b$, regularization can be achieved by restricting the number of iterations of the basic iterative method CGNR [18] on each

level by using the discrepancy principle. Specifically, the cascadic multilevel method applies CGNR on the coarsest discretization level until the computed approximate solution satisfies the discrepancy principle. Then the coarsest-level solution is prolonged to the next finer discretization level. By taking the prolonged solution as the initial approximation, iterations with CGNR are carried out on this level until the computed approximate solution satisfies the discrepancy principle. The computations are continued in this manner until an approximate solution on the finest discretization level has been found that satisfies the discrepancy principle. It has been shown that the CGNR-based cascadic multilevel method is a regularization method in some sense. Numerical experiments presented in [32] indicate that the cascadic multilevel method is able to determine an approximate solution of (3) that satisfies the discrepancy principle with less arithmetic work than applying CGNR on the finest level only. Other multilevel methods have been proposed for large-scale linear ill-posed systems, especially from applications of image deblurring and signal restoration; see, for example, [10], [13], [15], [24], [28]. We note that these multilevel methods are also based on the un-regularized equation $Ax = b$ or $A^T A x = A^T b$.

The iterative method proposed in this paper for solving large-scale system of linear equations (1) is developed based upon a synthesis of the Arnoldi-Tikhonov regularization technique and the multilevel technique. The performance of the method is compared to some known methods. The multilevel method seems somewhat similar to the regularizing multigrid [13]. Both of them establish the coefficient matrices on coarser grids by directly projecting on the original coefficient matrix A . In [13], it is shown that this scheme can reduce the relative error and the computational cost compared with the best regularizing methods for the normal equations. The main difference is that the problems on coarser grids are solved by the range-restricted Arnoldi-Tikhonov method (RRAT) [26] in our method, while the regularizing multigrid uses methods based on normal equations.

Throughout this paper, we adopt the following notations. I denotes the identity matrix, and 0 denotes the zero vector or zero matrix. The dimensions of these vectors and matrices are conformed with dimensions used in the context. The space of $m \times n$ real matrices is denoted by $\mathbb{R}^{m \times n}$. The superscript “ $.^T$ ” denotes the transposition of a vector or a matrix.

The remainder of the paper is organized as follows. In Section II, we review the range-restricted Arnoldi-Tikhonov method [26] for solving large-scale system of linear equations (1). In Section III, we present the multilevel Arnoldi-Tikhonov regularization method. Section IV is devoted to some numerical tests. Some concluding remarks are given in the last section.

II. RANGE-RESTRICTED ARNOLDI-TIKHONOV REGULARIZATION METHOD

RRAT is based on the minimization problem (4) and uses the range-restricted Krylov subspace $\mathcal{K}_m(A, Ab)$ as the projection subspace.

We can establish the orthonormal basis of $\mathcal{K}_m(A, Ab)$ by the Arnoldi process [18], which is outlined as follows.

Algorithm 1: Arnoldi Process

Input: $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, m ;
Output: $V_{m+1} \in \mathbb{R}^{n \times (m+1)}$, $\tilde{H}_m \in \mathbb{R}^{(m+1) \times m}$;
1) Compute $v_1 := Ab / \|Ab\|_2$, and set $V_1 := [v_1]$;
2) For $j = 1, 2, \dots, m$ Do:
 $v_{j+1} := Av_j$;
 For $i = 1, 2, \dots, j$ Do:
 $h_{i,j} := v_i^T v_{j+1}$;
 $v_{j+1} := v_{j+1} - h_{i,j} v_i$;
 End For
 $h_{j+1,j} := v_{j+1} / \|v_{j+1}\|_2$;
 $v_{j+1} := v_{j+1} / h_{j+1,j}$;
 $V_{j+1} := [V_j, v_{j+1}]$;
End For

We remark that a loss of orthogonality can occur when the algorithm progresses, see [29]. A remedy is the so-called reorthogonalization where the current vector has to be orthogonalized against previously created vectors. One can choose between a selective reorthogonalization or a full reorthogonalization against all vectors in the current subspace. In this paper we only use the full reorthogonalization.

The columns of V_m are an orthonormal basis of $\mathcal{K}_m(A, Ab)$. Let \tilde{H}_m denote the $(m+1) \times m$ upper Hessenberg matrix whose nonzero entries are $h_{i,j}$ defined by Algorithm 1. Then, we have the well-known Arnoldi decomposition

$$AV_m = U_{m+1} \tilde{H}_m. \quad (7)$$

The range-restricted Arnoldi-Tikhonov method proposed in [26] seeks to determine an approximate solution $x_{\mu,m}$ of (4) in the range-restricted Krylov subspace $\mathcal{K}_m(A, Ab)$. Substituting $x = V_m y$, $y \in \mathbb{R}^m$, into (4) yields the reduced minimization problem

$$\begin{aligned} & \min_{y \in \mathbb{R}^m} \left\{ \|AV_m y - b\|^2 + \frac{1}{\mu} \|V_m y\|^2 \right\} \\ &= \min_{y \in \mathbb{R}^m} \left\{ \|V_{m+1} \tilde{H}_m y - b\|^2 + \frac{1}{\mu} \|y\|^2 \right\} \\ &= \min_{y \in \mathbb{R}^m} \left\{ \|\tilde{H}_m y - V_{m+1}^T b\|^2 + \|(I - P)b\|^2 + \frac{1}{\mu} \|y\|^2 \right\}, \end{aligned} \quad (8)$$

where $P = V_{m+1} V_{m+1}^T$ is an orthogonal projector onto $\text{span}\{V_{m+1}\}$. Obviously, the reduced minimization problem (8) is equivalent to

$$\begin{aligned} & \min_{y \in \mathbb{R}^m} \left\{ \|\tilde{H}_m y - V_{m+1}^T b\|^2 + \frac{1}{\mu} \|y\|^2 \right\} \\ &= \min_{y \in \mathbb{R}^m} \left\| \begin{bmatrix} \tilde{H}_m \\ \frac{1}{\sqrt{\mu}} I \end{bmatrix} y - \begin{bmatrix} V_{m+1}^T b \\ 0 \end{bmatrix} \right\|^2. \end{aligned} \quad (9)$$

Note that the reduced minimization problem (9) can be solved by a sequence of Givens rotations [18]. We denote the solution of the minimization problem (9) by $y_{\mu,m}$. Then, the approximate solution of (4) is

$$x_{\mu,m} = V_m y_{\mu,m}.$$

In Tikhonov regularization methods, the most important step is to choose the regularization parameter μ . The discrepancy

principle proposes that the regularization parameter μ should be chosen so that the discrepancy $b - Ax_{\mu,m}$ satisfies (6).

Define

$$\phi_m(\mu) = \|b - Ax_{\mu,m}\|^2. \quad (10)$$

Concerning the properties of the function $\phi_m(\mu)$, the following results hold, see [26].

Theorem 1: The function $\phi_m(\mu)$ has the representation

$$\phi_m(\mu) = b^T V_{m+1} \left(\mu \tilde{H}_m \tilde{H}_m^T + I \right)^{-2} V_{m+1}^T b + \|(I - P)b\|_2^2.$$

Assume that $Ab \neq 0$ and $\tilde{H}_m^T V_{m+1}^T b \neq 0$. Then ϕ_m is strictly decreasing and convex for $\mu \geq 0$ with $\phi_m(0) = \|b\|_2^2$. Moreover, the equation

$$\phi_m(\mu) = \tau$$

has a unique solution $\mu_{\tau,m}$, such that $0 < \mu_{\tau,m} < \infty$, for any τ with

$$\|P_{\mathcal{N}(\tilde{H}_m^T)} V_{m+1}^T b\|_2^2 + \|(I - P)b\|_2^2 < \tau < \|b\|_2^2,$$

where $P_{\mathcal{N}(\tilde{H}_m^T)}$ denotes the orthogonal projector onto $\mathcal{N}(\tilde{H}_m^T)$.

To make the equation

$$\phi_m(\mu) = \eta^2 \varepsilon^2$$

have a solution, it follows from Theorem 1 that the input parameter m of the Arnoldi process should be chosen so that $\|P_{\mathcal{N}(\tilde{H}_m^T)} V_{m+1}^T b\|_2^2 + \|(I - P)b\|_2^2 < \eta^2 \varepsilon^2$.

After choosing the number m of the Arnoldi iterative steps, the regularization parameter μ_m is determined by solving the nonlinear equation $\phi_m(\mu) = \eta^2 \varepsilon^2$. Many numerical methods have been proposed for the solution of a nonlinear equation, including Newton's method [22], super-Newton's [16] method, and Halley's method [17]. For the specific nonlinear equation $\phi_m(\mu) = \eta^2 \varepsilon^2$, Reichel and Shyshkov proposed a new zero-finder method in their new paper [31]. In this paper, we still make use of Newton's method to obtain the regularization parameter μ_m . For the details of implementation of Newton's method for solving $\phi_m(\mu) = \eta^2 \varepsilon^2$, the interesting reader is referred to [26].

In the following, we outline the algorithm, which is used to solve large-scale linear ill-posed systems (1).

Algorithm 2: RRAT

Input: $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, ε , η ;

Output: $x_{\mu,m}$, μ_m , $m = m_{\min} + m_0$.

1) Compute the Arnoldi decomposition

$$AV_m = V_{m+1} \tilde{H}_m$$

with $m = m_{\min} + m_0$, where m_{\min} is the smallest number such that $\|P_{\mathcal{N}(\tilde{H}_m^T)} V_{m+1}^T b\|_2^2 + \|(I - P)b\|_2^2 < \eta^2 \varepsilon^2$, and m_0 is the number of additional Arnoldi steps and is chosen to 1 or 2.

2) Compute the solution μ_m of the equation $\phi_m(\mu) = \eta^2 \varepsilon^2$ by Newton's method.

3) Compute the solution $y_{\mu,m}$ of the least-squares problem (9) and obtain the approximate solution $x_{\mu,m} = V_m y_{\mu,m}$.

It is pointed out that RRAT is attractive since the computation of an orthonormal basis for the Krylov subspace

$\mathcal{K}_m(A, Ab)$ requires about half the number of matrix-vector product evaluations than the Krylov subspace $\mathcal{K}_m(A^T A, A^T b)$ of the same dimension, and these evaluations typically constitute the dominant computational effort required by these methods.

The following theorem shows that the RRAT method is a regularization method when applied to the solution of (1).

Theorem 2: Assume that the coefficient matrix A is symmetric and the system of linear equations (3) is consistent. Let $x_{\mu,m}$ be determined by RRAT. Then,

$$\lim_{\varepsilon \rightarrow 0} \sup_{\|b-\hat{b}\| \leq \varepsilon} \|\hat{x} - x_{\mu,m}\|_2 = 0, \quad (11)$$

where \hat{x} is the minimal-norm solution of (3).

Proof: Let x_m be the m th iterate generated by RRGMRRES applied to (1). Form Corollary 2.1 in [26], we obtain

$$\|b - Ax_m\| = \sqrt{\phi_m(\infty)} \leq \sqrt{\phi_m(\mu)} \leq \|b - Ax_{\mu,m}\| = \eta\varepsilon.$$

It follows from the above inequality and Theorem 2.3 in [32] that

$$\lim_{\varepsilon \rightarrow 0} \sup_{\|b-\hat{b}\| \leq \varepsilon} \|\hat{x} - x_m\| = 0.$$

Note that the solution μ of $\phi_m(\mu) = \eta^2 \varepsilon^2$ approaches to ∞ as $\varepsilon \rightarrow 0$. Moreover, it has been shown in the proof of Corollary 2.1 in [26] that $x_m = \lim_{\mu \rightarrow \infty} x_{\mu,m}$. Therefore, we have

$$x_m = \lim_{\varepsilon \rightarrow 0} x_{\mu,m}.$$

Then, (11) follows directly from the inequality $\|\hat{x} - x_{\mu,m}\| \leq \|\hat{x} - x_m\| + \|x_m - x_{\mu,m}\|$. ■

III. CASCADIC MULTILEVEL RRAT METHOD

In this section, we propose a multilevel method to solve the minimization problem (4). The basic iterative method on every level is the RRAT method described in the above section.

We assume that l subspaces \mathbb{W}_i , $i = 1, 2, \dots, l$, of \mathbb{R}^n are nested, i.e.,

$$\mathbb{W}_1 \subset \mathbb{W}_2 \subset \dots \subset \mathbb{W}_l. \quad (12)$$

The dimension n_i of \mathbb{W}_i satisfies $n_1 < n_2 < \dots < n_l = n$.

As in [32], we denote the discrete coefficient matrix of the underlying continuous problem on the i th level by $A_i \in \mathbb{R}^{n_i \times n_i}$. Thus, for the l th level, we have $A_l = A$.

Let $R_i : \mathbb{R}^n \rightarrow \mathbb{W}_i$ be the restriction operator from the l th level to the i th level. Then, the restrictions of b and \hat{b} to \mathbb{W}_i are

$$b_i = R_i b \quad \text{and} \quad \hat{b}_i = R_i \hat{b},$$

respectively.

Define $e_i = b_i - \hat{b}_i = R_i(b - \hat{b})$. To apply the RRAT method to the i th level, we need the value ε_i such that

$$\|e_i\|_2 \leq \varepsilon_i, \quad 1 \leq i \leq l. \quad (13)$$

Let $P_i : \mathbb{W}_i \rightarrow \mathbb{W}_{i+1}$, $1 \leq i \leq l-1$, be the prolongation operator from the i th level to the $(i+1)$ th level.

With these preparation, we give the cascadic multilevel RRAT method (ML-RRAT) for solving the minimization

problem (4) as follows.

Algorithm 3: ML-RRAT

Input: A , b , number of levels $l \geq 1$, $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l$, η .

Output: approximate solution \tilde{x} of the linear system (3).

- 1) $x_1 := 0$;
- 2) For $i := 1, 2, \dots, l$ do
 - $b_i := R_i b$;
 - $r_i := b_i - A_i x_i$;
 - Solve $A_i \Delta x_i = r_i$ by RRAT with $\varepsilon = \varepsilon_i$ to obtain Δx_i ;
 - $x_i := x_i + \Delta x_i$;
 - If $i < l$, then $x_{i+1} := P_i x_i$;
- End For
- 3) $\tilde{x} = x_l$.

By making use of Theorem 2 and the similar techniques in proving Theorem 3.2 in [32], we can obtain the following result.

Theorem 3: Assume that A_i is symmetric, the equation $A_i \hat{x}_i = \hat{b}_i$ is consistent, and $\mathcal{R}(P_i) \subset \mathcal{R}(A_i)$ for $1 \leq i \leq l$. Let the projected contaminated right-hand side satisfy (13). Then, the ML-RRAT method described by Algorithm 3 is a regularization method on each level, i.e.,

$$\lim_{\varepsilon_i \rightarrow 0} \sup_{\|b_i - \hat{b}_i\|_2 \leq \varepsilon_i} \|\hat{x}_i - x_i\|_2 = 0, \quad 1 \leq i \leq l, \quad (14)$$

where \hat{x}_i is the minimal-norm solution of $A_i \hat{x}_i = \hat{b}_i$ with $\hat{x}_l = \hat{x}$.

IV. NUMERICAL EXPERIMENTS

In this section, we present two numerical examples to illustrate the performance of ML-RRAT for large-scale linear ill-posed problems. We compare ML-RRAT to RRAT and ML-CGMR in [32].

To simplify the computations in RRAT and ML-RRAT, the smallest iterative step number, denoted by m_{min} , of the Arnoldi process is chosen so that $\|(I - P)b\|^2 < \eta^2 \varepsilon^2$. The quality of the computed solution can be improved by choosing the practical iterative step number m somewhat larger than m_{min} . The parameter m_0 denotes the number of additional Arnoldi steps to improve the computed solution, and will be chosen to 1 or 2.

Let the right-hand side $b_i = [b_i^{(1)}, b_i^{(2)}, \dots, b_i^{(n_i)}]^T$, which is recursively obtained by

$$b_i^{(j)} = b_{i+1}^{(2j-1)}, \quad 1 \leq j \leq n_i, \quad 1 \leq i \leq l$$

with $b_l = b$. This implicitly defines the restriction operator R_i , which consists of the first n_i rows of I_n . It follows that $R_i R_i^T = I_{n_i}$ for all $i = 1, 2, \dots, l$. The prolongation operator

P_i is the same as that in [32], i.e.,

$$P_i = \begin{bmatrix} 1 & & & & & & & & & & \\ \frac{1}{2} & & & & & & & & & & \\ \frac{1}{4} & & & & & & & & & & \\ & \frac{1}{2} & & & & & & & & & \\ & \frac{1}{4} & & & & & & & & & \\ & & \frac{1}{2} & & & & & & & & \\ & & \frac{1}{4} & & & & & & & & \\ & & & \frac{1}{2} & & & & & & & \\ & & & \frac{1}{4} & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & \frac{1}{2} & & & & \\ & & & & & & \frac{1}{4} & & & & \\ & & & & & & & \ddots & & & \\ & & & & & & & & \frac{1}{2} & & \\ & & & & & & & & \frac{1}{4} & & \\ & & & & & & & & & \frac{1}{2} & \\ & & & & & & & & & \frac{1}{4} & \\ & & & & & & & & & & 1 \end{bmatrix} \in \mathbb{R}^{n_{i+1} \times n_i}.$$

We use the Euclidean norm $\|\cdot\|_2$ in ML-RRAT and RRAT, and the weighted Euclidean norm $\|\cdot\|$ defined by

$$\|v\| = \left(\frac{1}{n_i} \sum_{j=1}^{n_i} (v^{(j)})^2 \right)^{1/2}, \quad v = (v^{(1)}, v^{(2)}, \dots, v^{(n_i)})^T \in \mathbb{R}^{n_i},$$

in ML-CGMR. In all examples, the right-hand side is given by

$$b = \hat{b} + e, \quad e = \|\hat{b}\| * 10^{-k} w,$$

where w is a vector generated from normally distributed entries with mean zero and variance one, and k is a positive integer. We set

$$\varepsilon_l = \varepsilon = \|e\|_2, \quad \text{and} \quad \varepsilon_i = \frac{1}{\sqrt{2}} \varepsilon_{i+1}, \quad i = 1, 2, \dots, l-1,$$

in both RRAT and ML-RRAT, while

$$\varepsilon_i = \varepsilon = \|e\|, \quad i = 1, 2, \dots, l-1,$$

in ML-CGMR as in [32].

For RRAT and ML-RRAT, we choose $\eta = 1.01$ as in [26] and $m_0 = 1$ or 2 . Note that the numbers of matrix-vector products in RRAT and the i -th level of ML-RRAT are $m = m_{\min} + m_0$ and $m_i = m_{\min}^{(i)} + m_0$, respectively. However, as pointed out above, the number m_i of matrix-vector products in each level of ML-CGMR are the double of the number of iterations since each iteration of ML-CGMR needs two matrix-vector products, one with A and the other with A^T . As in [32], for ML-CGMR, we choose $c_i = c = 1.25$, $i = 1, 2, \dots, l$ and $\tau = 1.25$.

In all the following tables, we denote by RERR the relative error $\|\hat{x} - \hat{x}\|_2 / \|\hat{x}\|_2$, where \hat{x} is the exact solution of the linear error-free system of equations (3) and \hat{x} is the approximate solution obtained by one of RRAT, ML-RRAT, and ML-CGMR.

All the numerical experiments are performed in Matlab on a PC with the usual double precision, where the floating point relative accuracy is $2.22 \cdot 10^{-16}$.

Example 1

The first example considered is the Fredholm integral equation of the first kind, which takes the generic form

$$\int_{-6}^6 k(s, t) x(s) ds = b(t), \quad -6 \leq t \leq 6. \quad (15)$$

TABLE I

RESULTS OF RRAT AND ML-RRAT WITH $m_0 = 1$ FOR EXAMPLE 1

k	m	RRAT		ML-RRAT		
		RERR	m_1	m_2	m_3	RERR
1	3	0.0898	3	2	2	0.0797
2	4	0.0257	4	2	2	0.0254
3	9	0.0090	8	2	2	0.0078
4	12	0.0035	12	2	2	0.0033

TABLE II

RESULTS OF RRAT AND ML-RRAT WITH $m_0 = 2$ FOR EXAMPLE 1

k	m	RRAT		ML-RRAT		
		RERR	m_1	m_2	m_3	RERR
1	4	0.0590	4	3	3	0.0589
2	5	0.0238	5	3	3	0.0224
3	10	0.0088	9	3	3	0.0078
4	13	0.0032	13	3	3	0.0032

The solution, kernel and right-hand side of (15) are given by

$$x(s) = \begin{cases} 1 + \cos(\frac{\pi}{3}s), & \text{if } |s| < 3, \\ 0, & \text{otherwise,} \end{cases}$$

$$k(t, s) = x(t - s),$$

$$b(t) = (6 - |t|)(1 + \frac{1}{2} \cos(\frac{\pi}{3}t)) + \frac{9}{2\pi} \sin(\frac{\pi}{3}|t|).$$

This example was discussed in [30].

We discretize the integral equation by Nyström methods based on composite trapezoidal quadrature rules with $l = 3$. The numbers of nodes on three levels are $n_1 = 257$, $n_2 = 513$, and $n_3 = 1025$, respectively. Numerical results for the example are reported in Table I for RRAT and ML-RRAT with $m_0 = 1$, in Table II for RRAT and ML-RRAT with $m_0 = 2$, and in Table III for ML-CGMR.

From Tables I and II, we can see that for $m_0 = 1$ and $m_0 = 2$, RRAT has almost the same relative errors as ML-RRAT, but RRAT requires more matrix-vector product evaluations on the finest discretization level. Tables I and III show that ML-CGMR needs the same number of matrix-vector products as ML-RRAT with $m_0 = 1$, while ML-RRAT has smaller relative errors.

Example 2

This example considered here is the Fredholm integral equation of the first kind

$$\int_0^{\pi/2} \kappa(t, s) x(s) ds = b(t), \quad 0 \leq t \leq \pi, \quad (16)$$

TABLE III

RESULTS OF ML-CGMR FOR EXAMPLE 1

k	ML-CGMR			
	m_1	m_2	m_3	RERR
1	4	2	2	0.0939
2	8	2	2	0.0241
3	8	2	2	0.0241
4	16	2	2	0.0086

TABLE IV

RESULTS OF RRAT AND ML-RRAT WITH $m_0 = 1$ FOR EXAMPLE 2

RRAT			ML-RRAT			
k	m	RERR	m_1	m_2	m_3	RERR
1	3	0.1123	3	2	2	0.1814
2	3	0.0918	3	2	2	0.0896
3	4	0.0544	4	2	2	0.0564
4	5	0.0488	5	2	2	0.0478

TABLE V

RESULTS OF RRAT AND ML-RRAT WITH $m_0 = 2$ FOR EXAMPLE 2

RRAT			ML-RRAT			
k	m	RERR	m_1	m_2	m_3	RERR
1	4	0.2243	4	3	3	0.2167
2	4	0.2032	4	3	3	0.2109
3	5	0.1341	5	3	3	0.1258
4	6	0.1069	6	3	3	0.0659

which is discussed by Baart [2]. The solution, kernel and right-hand side of (16) are given by

$$x(s) = \sin(s), \quad k(t, s) = \exp(t \cos(s)), \quad b(t) = 2 \sinh(t)/t,$$

respectively.

The integral equation is discretized with $l = 3$ by the same method as Example 1. For this discretization, $n_1 = 257$, $n_2 = 513$, and $n_3 = 1025$, respectively. Numerical results for the example are reported in Table IV for RRAT and ML-RRAT with $m_0 = 1$, in Table V for RRAT and ML-RRAT with $m_0 = 2$, and in Table VI for ML-CGMR.

Tables IV and V show that RRAT and ML-RRAT with $m_0 = 1$ have better performance than RRAT and ML-RRAT with $m_0 = 2$. Moreover, ML-RRAT requires fewer matrix-vector products than RRAT on the finest discretization level. Tables IV and VI indicate that ML-CGMR needs the same number of matrix-vector products as ML-RRAT with $m_0 = 1$, while ML-RRAT has smaller relative errors for this example.

V. CONCLUSIONS

In this paper, we have proposed a multilevel regularization method to solve large-scale linear ill-posed systems. The method is developed based on a synthesis of the Arnoldi-Tikhonov regularization technique and the multilevel technique. Numerical experiments are presented for the performance comparison between the iterative method and two known methods. It shows that the method is effective for some real world examples.

TABLE VI

RESULTS OF ML-CGMR FOR EXAMPLE 2

ML-CGMR				
k	m_1	m_2	m_3	RERR
1	4	2	2	0.2352
2	4	2	2	0.2149
3	8	2	2	0.1579
4	12	2	2	0.0738

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