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Order statistics of the moduli of the eigenvalues of product random matrices from polynomial ensembles

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Abstract

Let $X_1, \dots, X_{m_N}$ be independent random matrices of order N drawn from the polynomial ensembles of derivative type. For any fixed n, we consider the limiting distribution of the nth largest modulus of the eigenvalues of $X = \prod_{k=1}^{m_N} X_k$ as $N \to \infty$ where m_N/N converges to some constant $\tau \in [0, \infty)$. In particular, we find that the limiting distributions of spectral radii behave like that of products of independent complex Ginibre matrices.

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1 Introduction and main results

Recently, Kieburg and Kösters [13] showed that there is a one-to-one correspondence among the probability density functions of the so-called statistical isotropic matrices, those of the square of their singular values and those of their eigenvalues. As a special case of the statistical isotropic matrices they introduced the polynomial ensembles of derivative type which can be determined by their eigenvalue distributions, see [8, 13, 14].

Definition 1.1. *For a positive integer N, let X be a $N \times N$ complex random matrix with probability density function $\mathcal{P}$ with respect to the Lebesgue measure on $\mathbb{C}^{N\times N}$. We say that X is drawn from the polynomial matrix ensembles of derivative type if X is isotropic, i.e. for any $A \in \mathbb{C}^{N\times N}$ and any unitary matrix U, V, $\mathcal{P}(UAV) = \mathcal{P}(A)$, and the probability density function of the eigenvalues of X is of the form*

$$\frac{1}{\mathcal{Z}_N} \prod_{1\le j<k\le N} |z_j - z_k|^2 \prod_{k=1}^{N} w(|z_k|^2), \quad (z_1, \dots, z_N) \in \mathbb{C}^N, \tag{1.1}$$

where $\mathcal{Z}_N$ is the normalisation constant. The function $w : \mathbb{R}_+ \to \mathbb{R}_+$ is called the weight function of X and satisfies that $w(e^x)$ is an Pólya frequency function of order N.

Let X_1, X_2 be two independent random matrices drawn from the polynomial matrix ensembles of derivative type with weight function w_1, w_2 respectively. Then X_1X_2 is also drawn from such ensembles with weight $w_1 * w_2$ which is the Mellin convolution of w_1 and w_2. Moreover, if we normalize w_k to be the probability density functions of some independent nonnegative random variables, say ζ_k, $k = 1, 2$, then the weight function

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corresponding to X_1X_2 is the probability density function of $\zeta_1\zeta_2$, see ref. [14] for details. There are several known random matrices of this type, such as induced complex Ginibre matrices [7] (particular the complex Ginibre matrices [9]) and their products [1, 2], the induced Jacobi ensembles (also known as the truncated unitary matrices [21]) and their products [1, 4].

Rider [15, 16] proved that the distributions of the appropriately shifted and rescaled spectral radii of complex Ginibre matrices converge to the Gumbel distribution, refer [17] for real Ginibre matrices. Later, Chafaï and Péché [6] showed that this phenomenon is also true for the random normal matrix with weight function

$$w(r) = e^{-NV(\sqrt{r})}, \quad r \in \mathbb{R}_+, \tag{1.2}$$

where $V(r)$ is called potential function and possesses appropriate differential and convex conditions. On the other hand, if $w(e^r)$ is a Pólya frequency function of order N, (1.1) also describes the eigenvalue distribution of a polynomial random matrix ensemble with derivative type. For instance, we may take $w(r) = e^{-Nr^{2\alpha}}$, $\alpha > 0$. Recently, Jiang and Qi [11] investigated the limiting behaviors of the spectral radii of truncated unitary matrices and products of independent complex Ginibre matrices. Also, the distribution of the smallest modulus was investigated [3, 5].

In this paper, we are interested in the order statistics of the moduli of the eigenvalues of such product random matrices. Let $\{m_N\}_{N=1}^{\infty}$ be a sequence of positive integers. Assume that the limit of m_N/N exists as N tends to infinity and denote it by τ. For each N, let $X_1^{(N)}, \ldots, X_{m_N}^{(N)}$ be independent random matrices of order N drawn from the polynomial matrix ensembles of derivative type with weight functions

$$w_k(r) \propto e^{-NV_k(\sqrt{r})}, \quad r \in \mathbb{R}_+, \quad k = 1, \ldots, m_N, \tag{1.3}$$

respectively. Denote the eigenvalues of the product matrix $X = \prod_{k=1}^{m_N} X_k^{(N)}$ by $z_1, \ldots, z_N$ and the ordered statistics of the moduli of these eigenvalues by

$$|z|_{(1)} \geq \cdots \geq |z|_{(N)}. \tag{1.4}$$

That is, for any fixed integer $1 \leq n \leq N$, $|z|_{(n)}$ is the nth largest of $\{|z_k| : k = 1, 2, \ldots, N\}$. Particularly, $|z|_{(1)}$ is the spectral radius of X.

Our analysis is based on the following structural results of the moduli of the eigenvalues.

Proposition 1.2. *Let $\{\xi_{l,n} : 1 \leq l \leq m_N, 1 \leq n \leq N\}$ be independent random variables. The probability density function of each $\xi_{l,n}$ is*

$$\frac{1}{\int_0^\infty t^{2n-1} \exp\{-NV_l(t)\}dt} x^{2n-1} \exp\{-NV_l(x)\} 1_{[0,\infty)}(x). \tag{1.5}$$

Then, we have the following identity in distribution

$$(|z|_{(1)}, \ldots, |z|_{(N)}) \stackrel{d}{=} (\xi_{(1)}, \ldots, \xi_{(N)}), \tag{1.6}$$

where $\xi_{(1)} \geq \ldots \geq \xi_{(N)}$ is the order statistics of independent random variables $\xi_n = \prod_{l=1}^{m_N} \xi_{l,n}, n = 1, \ldots, N$.

Proposition 1.2 can be deduced from [6, Theorem 1.2] by a similar argument for products of independent complex Ginibre matrices (i.e. the potential function is given by $V(r) = r^2$) as in [5, 11]. Thus, we omit the proof.

Remark 1.3. Although we state Proposition 1.2 for the eigenvalues of product of random matrices drawn from random matrix ensembles with derivative type where the weight functions $w_k(e^r)$ given by (1.3) should be Pólya frequency functions of order N, the analysis in this article also works for the particle system (1.1) on complex plane $\mathbb{C}$ whose weight function is the Mellin convolution of $w_k, k = 1, 2, \dots, m_N$. Our analysis in this paper works for the potential functions V possess the following conditions (a)–(d), but does not need the weight function to be Pólya frequency functions. Thus, we do not require $w_k(e^r)$'s to be Pólya frequency functions except for considering polynomial matrix ensembles.

Hereinafter, we always suppose that potential functions V possess the following properties:

(a) V is defined on $\mathbb{R}_+$ and taken values in $\mathbb{R}\cup\{\infty\}$ with domain $\mathcal{D} := \{t \in \mathbb{R}_+ : V(t) < \infty\}$.

(b) V has continuous derivatives up to fourth order on its domain $\mathcal{D}$.

(c) There is a unique minimum of $F(t) := V(t) - 2\log t$ and denote it by $t^{(0)}$. Furthermore, $t^{(0)}$ is an interior point of $\mathcal{D}$ and for every $\varepsilon > 0$,

$$\inf\left\{F(t) - F(t^{(0)}) : t \in \mathcal{D} \text{ and } |t - t^{(0)}| \geq \varepsilon\right\} > 0.$$

(d) $F''(t^{(0)}) > 0$.

Remark 1.4. Let A is an $N \times N$ random matrix drawn from the polynomial matrix ensembles with weight function $w(r) = e^{-NV(\sqrt{r})}$. Then the conditions listed above are automatically satisfied for N large enough. We would like to point out that there are random matrices which are drawn from the polynomial matrix ensembles with derivative type but do not satisfy some of the above conditions. For instance, the truncated unitary matrices in the weak non-unitary case and the spherical ensembles [11]. Consider random matrix $A^{-1}B$ where A and B are independent complex Ginibre matrices of order N, that is $A^{-1}B$ is a spherical ensemble of order N. The corresponding weight function is

$$w(r^2) = \frac{1}{(1+r^2)^{N+1}} = e^{-NV_N(r)},$$

where $V_N(r) = (1 + N^{-1})\log(1 + r^2)$. The potential function V_N depends on N. If we treat

$$w(r^2) = \frac{1}{1+r^2} e^{-NV(r)},$$

where $V(r) = \log(1 + r^2)$, then the potential function V does not satisfy condition (c).

We would like to mention that the potential function V does not need to be convex as in [6]. Technically, condition (c) makes it possible to apply Laplace method and thus the unique minimum point $t^{(0)}$ of $F(t)$ is a stationary point.

By condition (c), we denote the unique minimum point of $V_l(t) - 2\log t$ by $t_l^{(0)}$. Let us introduce $\alpha_l := (t_l^{(0)})^2 V_l''(t_l^{(0)}) + 2$ and $\beta_l := (t_l^{(0)})^3 V_l'''(t_l^{(0)}) - 4$. Set

$$a_N := \frac{1}{m_N}\sum_{l=1}^{m_N}\frac{1}{\alpha_l} \quad \text{and} \quad b_N := \frac{1}{m_N}\sum_{l=1}^{m_N}\frac{\beta_l}{\alpha_l^2}. \tag{1.7}$$

Theorem 1.5. *Suppose that, for some* $\varepsilon > 0$, $\bigcup_{l=1}^{\infty}\{V_l^{(4)}(t) : \ t \in (t_l^{(0)} - \varepsilon, t_l^{(0)} + \varepsilon)\}$ *is bounded. For any fixed positive integer* n, *let* $|z|_{(n)}$ *be as in* (1.4). *Also, recalling* $\tau = \lim_{N\to\infty} m_N/N$.

(1) *If $\tau = 0$, suppose that $\inf_{N\geq 1}\{a_N\} > 0$. Furthermore, we always assume that $m_N \leq N$ and N is large enough such that*

$$\gamma_N = \log \frac{N}{4m_N a_N} - 2\log\log\frac{N}{m_N} - \log 2\pi \tag{1.8}$$

is positive. Then, for all $x \in \mathbb{R}$, the following holds pointwise

$$\lim_{N\to\infty} \mathbb{P}\Big(\frac{|z|_{(n)}}{\prod_{l=1}^{m_N} t_l^{(0)}} \leq 1 + \big(\frac{m_N a_N \gamma_N}{N}\big)^{1/2} + \big(\frac{m_N a_N}{N\gamma_N}\big)^{1/2} x\Big) = \exp\{-e^{-x}\}\sum_{k=0}^{n-1}\frac{1}{k!}e^{-kx}.$$

(2) *If $\tau \in (0,\infty)$, and if there exist $a_0 \in (0,\infty)$ and $b_0 \in \mathbb{R}$ such that $\lim_{N\to\infty} a_N = a_0$ and $\lim_{N\to\infty} b_N = b_0$. Then, for all $x \in \mathbb{R}$, the following holds pointwise*

$$\begin{aligned}\lim_{N\to\infty} \mathbb{P}\Big(\frac{|z|_{(n)}}{\prod_{l=1}^{m_N} t_l^{(0)}} \leq x\Big) = &\Big(1 + \sum_{k=1}^{n-1}\sum_{0\leq l_1<\cdots<l_k<\infty}\prod_{i=1}^{k}((\Psi_{a_0,b_0}(l_i;x))^{-1} - 1)\Big) \\ &\times \prod_{k=0}^{\infty}\Psi_{a_0,b_0}(k;x)1_{[0,\infty)}(x),\end{aligned}$$

where

$$\Psi_{u,v}(i;x) = \Phi\big((u\tau)^{1/2}(2i + \frac{3}{2} + \frac{v}{2u}) + (u\tau)^{-1/2}\log x\big)$$

and $\Phi(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt$ is the distribution function of the standard normal random variable.

Taking $m_N \equiv 1$ and $V(r) = r^2$, the limiting distribution of $|z|_{(n)}$ for any given natural number n was obtained by [15, 16]. In particular, after taking $m_N \equiv 1$, the limiting result of the spectral radii $|z|_{(1)}$ was obtained by [6] for general potential function V. Taking $V_1(r) = V_2(r) = \cdots = r^2$, then each of the $X_k^{(N)}$ is a complex Ginibre matrix and $X = \prod_{k=1}^{m_N} X_k^{(N)}$ is the product of independent complex Ginibre matrices. The corresponding limiting distributions of the spectral radii of X were obtained by [11].

The strategies of the proof. With the notation in Proposition 1.2, set

$$P_{N,n} := \mathbb{P}(\xi_n \leq f_N(x)) = \mathbb{P}(\sum_{l=1}^{m_N}\log\xi_{l,n} \leq \log f_N(x)),$$

where $f_N(x)$ is an function of $x \in \mathbb{R}$ depending on N. Following Proposition 1.2, it is easy to deduce that

$$\mathbb{P}(\max_{1\leq n\leq N}|z_n| \leq f_N(x)) = \prod_{n=1}^{N} P_{N,n}.$$

We find a sequence of integers $\{j_N\}_{N=1}^{\infty}$ such that $\lim_{N\to\infty}\prod_{n=1}^{N-j_N} P_{N,n} = 1$ and $\prod_{n=N-j_N+1}^{N} P_{N,n}$ converges to some distribution function. To analyze the asymptotics of the latter we show that $\{\log\xi_{l,n} : 1 \leq l \leq m_N, N - j_N < n \leq N\}$ can be replaced by a set of independent random normal variables with appropriate means and variances by Talagrand inequality, see Lemma 2.2 and Proposition 2.3 for details. Then, using the same arguments as in [15] for the Riemann sum approximation, we obtain the desired results.

2 Proof of Theorem 1.5

We would like to introduce some notation. Let

$$\theta_N(x) = \begin{cases} 1 + \sqrt{\frac{\gamma_N}{N/m_N a_N}} + \frac{x}{\sqrt{N\gamma_N/m_N a_N}}, & \text{if } \tau = 0, \\ x 1_{(0,\infty)}(x), & \text{if } \tau \in (0,\infty), \end{cases} \tag{2.1}$$

where γ_N is given by (1.8). Set

$$f_N(x) = \theta_N(x) \prod_{l=1}^{m_N} t_l^{(0)}. \tag{2.2}$$

For each l and n, introduce $F_{l,n}(t) = V_l(t) - \frac{2n-1}{N}\log t$, denote its unique minimum point by $t_l^{(n)}$. Set $\alpha_{l,n} := (t_l^{(n)})^2 F''_{l,n}(t_l^{(n)})$ and $\beta_{l,n} := (t_l^{(n)})^3 F'''_{l,n}(t_l^{(n)})$.

Because we just interest in the large N limits of moduli of eigenvalues, in this article we always assume that N is large.

Next lemma provides some properties related to the potential function.

Lemma 2.1. *Suppose $V(t)$ satisfies the condition (a)-(d) stated in previous section. Set $F_n(t) = V(t) - \frac{2n-1}{N}\log t$ and $F(t) = V(t) - 2\log t$. Denote the unique minimum point of F_n by t_n and set $\tilde{\alpha}_n := t_n^2 F_n''(t_n)$ and $\tilde{\beta}_n := t_n^3 F_n'''(t_n)$. Similarly, we denote the unique minimum point of F by t_0 and set $\tilde{\alpha} := t_0^2 F''(t_0)$ and $\tilde{\beta} := t_0^3 F'''(t_0)$.*

Let $\{j_N\}_{N=1}^{\infty}$ be a sequence of positive integers satisfying $\lim_{N\to\infty} j_N/N = 0$. For each $N - j_N < n \le N$ and N large enough, we have

$$\tilde{\alpha}_n = \tilde{\alpha} + O(\frac{j_N}{N}), \tag{2.3}$$

$$\tilde{\beta}_n = \tilde{\beta} + O(\frac{j_N}{N}), \tag{2.4}$$

$$\frac{t_n}{t_0} = 1 - \frac{2(N-n)+1}{\alpha N}(1 + O(\frac{j_N}{N})). \tag{2.5}$$

Proof. Equations (2.3) and (2.4) can be deduced form the mean value theorem.

Now, we are going to prove (2.5). Set $g(t) = tV'(t)$. It is easy to see that $g(t_n) = \frac{2n-1}{N}$. Because $g'(t_0) > 0$, we know that g is invertible in some neighbourhood of 2. Thus,

$$\begin{aligned} t_n - t_0 &= g^{-1}(\frac{2n-1}{N}) - g^{-1}(2) \\ &= -(g^{-1}(2))'(2 - \frac{2n-1}{N})(1 + o(1)) \\ &= -\frac{2(N-n)+1}{t_0 F''(t_0) N}(1 + o(1)). \end{aligned}$$

□

Lemma 2.2. *With the notation of Lemma 2.1, for each $N - j_N < n \le N$, let ζ_n be a positive random variable whose p.d.f. is given by*

$$\frac{1}{\int_0^\infty t^{2n-1} e^{-NV(t)} dt} x^{2n-1} e^{-NV(x)} 1_{[0,\infty)}(x). \tag{2.6}$$

Then, for N large enough, we have

(I) *The following asymptotics hold,*

$$\mathbb{E}\log\zeta_n = \log t_n - \frac{1}{2\tilde{\alpha}_n N}(1 + \frac{\tilde{\beta}_n}{\tilde{\alpha}_n})(1 + O(\frac{1}{N})), \tag{2.7}$$

$$\mathbb{E}\Big(\frac{\zeta_n}{t_n}\Big)^2 = 1 + O(\frac{1}{N}). \tag{2.8}$$

(II) *There exist a standard normal random variable η and a positive constant K depending on the derivatives of V up to fourth order, such that*

$$\mathbb{E}\big(\sqrt{\alpha_n N}(\log\zeta_n - \mathbb{E}\log\zeta_n) - \eta\big)^2 \le \frac{K}{N}. \tag{2.9}$$

Proof. (I) The statement (I) is a direct consequence of the Laplace approximation, see e.g. [20, Theorem 1, p.58].

(II) It is easy to check that the probability density function of $\sqrt{\alpha_n N}(\log\zeta_n - \mathbb{E}\log\zeta_n)$ is given by

$$g(s) := \frac{e^{2n\left(\mathbb{E}\log\zeta_n + \frac{s}{\sqrt{\alpha_n N}}\right) - NV\left(e^{\mathbb{E}\log\zeta_n + \frac{s}{\sqrt{\alpha_n N}}}\right)}}{\sqrt{\alpha_n N}\int_0^\infty x^{2n-1}e^{-NV(x)}dx}, \quad s\in\mathbb{R}. \tag{2.10}$$

Taking change of variable

$$t = \frac{1}{t_n}\exp(\mathbb{E}\log\zeta_n + \frac{s}{\sqrt{\alpha_n N}}), \tag{2.11}$$

we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} g(s)\log\frac{g(s)}{\varphi(s)}ds = {}& \frac{1}{\int_0^\infty e^{-NG_n(x)}dx}\int_0^\infty e^{-NG_n(t)} \\ &\times\Big(-NG_n(t) + \log t + \frac{1}{2}\alpha_n N(\log t - \mathbb{E}\log\frac{\xi_n}{t_n})^2 \\ &\quad - \log\sqrt{\alpha_n N} + \log\sqrt{2\pi} - \log\int_0^\infty e^{-NG_n(x)}dx\Big)dt.\end{aligned} \tag{2.12}$$

Here $\varphi(t)$ is the probability density function of standard normal random variable and $G_n(t) = V(t_n t) - \frac{2n-1}{N}\log t$. Applying the Laplace approximation, there exists some constant $K>0$, depending on the derivatives of $G_n(t)$ up to fourth order, such that

$$\int_{-\infty}^{\infty} g(t)\log\frac{g(t)}{\varphi(t)}dt \le \frac{K}{2N}. \tag{2.13}$$

By Talagrand inequality [18], one obtains

$$W_2^2(\sqrt{\alpha_n N}(\log\zeta_n - \mathbb{E}\log\zeta_n),\eta) \le 2\int_{-\infty}^{\infty} g(t)\log\frac{g(t)}{\varphi(t)}dt \le \frac{K}{N}, \tag{2.14}$$

where W_2 is the 2-nd Wasserstein distance. The Kantorovich duality [19, p.19] implies that there exists a standard normal random variable η such that

$$\mathbb{E}\big(\sqrt{\alpha_n N}(\log\zeta_n - \mathbb{E}\log\zeta_n) - \eta\big)^2 = W_2^2(\sqrt{\alpha_n N}(\log\zeta_n - \mathbb{E}\log\zeta_n),\eta). \tag{2.15}$$

The proof is completed. □

Motivated by the Lemma 2.6 in [11], we have the following replacement principle.

Proposition 2.3. *Let $\{j_N : N = 1,2,\ldots\}$ be a sequence of positive integers satisfying $\lim_{N\to\infty} c_N j_N/N = 0$ where $c_N = \gamma_N\log(N/m_N)$ if $\tau = 0$ and $c_N = 1$ if $\tau\in(0,\infty)$. With notation in Proposition 1.2, there exist independent standard normal random variables $\{\eta_{l,n} : 1\le l\le m_N, N-j_N < n\le N\}$ such that*

$$\begin{aligned}&\lim_{N\to\infty}\mathbb{P}\left(\max_{N-j_N<n\le N}\sum_{l=1}^{m_N}\log\xi_{l,n} \le \log f_N(x)\right) \\ &= \lim_{N\to\infty}\mathbb{P}\left(\max_{N-j_N<n\le N}\sum_{l=1}^{m_N}\big(\frac{\eta_{l,n}}{\sqrt{\alpha_{l,n}N}} + \mathbb{E}\log\xi_{l,n}\big) \le \log f_N(x)\right).\end{aligned} \tag{2.16}$$

Proof. Recall that $f_N(x)$ is defined by (2.2). When $\tau = 0$, for large number N, we have

$$
\begin{aligned}
&\mathbb{P}\left(\max_{N-j_N<n\le N}\sum_{l=1}^{m_N}\log\xi_{l,n}\le\log f_N(x)\right)\\
&=\mathbb{P}\left(\max_{N-j_N<n\le N}\sum_{l=1}^{m_N}\log\xi_{l,n}\le\log\Big(\prod_{l=1}^{m_N}t_l^{(0)}\big(1+\sqrt{\frac{\gamma_N}{N/m_Na_N}}+\frac{x}{\sqrt{N\gamma_N/m_Na_N}}\big)\Big)\right)\\
&=\mathbb{P}\left(\sqrt{\frac{N\gamma_N}{m_Na_N}}\Big(\max_{N-j_N<n\le N}\sum_{l=1}^{m_N}\log\xi_{l,n}-\sum_{l=1}^{m_N}\log t_l^{(0)}\Big)-\gamma_N\le x+o(1)\right). \qquad (2.17)
\end{aligned}
$$

To deduce the last equality, we have used that $\log(1+y) = y + o(y)$ for y in a small neighborhood of 0. And for $\tau \in (0, \infty)$, we have

$$
\begin{aligned}
&\mathbb{P}\left(\max_{N-j_N<n\le N}\sum_{l=1}^{m_N}\log\xi_{l,n}\le\log f_N(x)\right)\\
&=\mathbb{P}\left(\max_{N-j_N<n\le N}\sum_{l=1}^{m_N}\log\xi_{l,n}-\sum_{l=1}^{m_N}\log t_l^{(0)}\le\log x\right). \qquad (2.18)
\end{aligned}
$$

To prove this proposition, by Slutsky's theorem, it is sufficient to prove that there is a set of independent normal random variables $\{\eta_{l,n} : 1 \le l \le m_N, N - j_N < n \le N\}$ such that

$$
\sqrt{\frac{Nc_N}{m_Nd_N}}\Big(\max_{N-j_N<n\le N}\sum_{l=1}^{m_N}\log\xi_{l,n}-\max_{N-j_N<n\le N}\sum_{l=1}^{m_N}\big(\frac{\eta_{l,n}}{\sqrt{\alpha_{l,n}N}}+\mathbb{E}\log\xi_{l,n}\big)\Big) \qquad (2.19)
$$

converges to 0 in probability. Actually, according to Lemma 2.2, there is a set of independent standard normal random variables such that (2.9) holds for some constant $K > 0$ uniformly. For any $\varepsilon > 0$, one has

$$
\begin{aligned}
&\mathbb{P}\left(\sqrt{\frac{Nc_N}{m_Na_N}}\left|\max_{N-j_N<n\le N}\sum_{l=1}^{m_N}\log\xi_{l,n}-\max_{N-j_N<n\le N}\sum_{l=1}^{m_N}\big(\frac{\eta_{l,n}}{\sqrt{\alpha_{l,n}N}}+\mathbb{E}\log\xi_{l,n}\big)\right|\ge\varepsilon\right)\\
&\le\mathbb{P}\left(\sqrt{\frac{Nc_N}{m_Na_N}}\max_{N-j_N<n\le N}\left|\sum_{l=1}^{m_N}\big(\log\xi_{l,n}-\mathbb{E}\log\xi_{l,n}-\frac{\eta_{l,n}}{\sqrt{\alpha_{l,n}N}}\big)\right|\ge\varepsilon\right) \qquad (2.20)\\
&\le\frac{1}{\varepsilon^2}\frac{Nc_N}{m_Na_N}\sum_{n=N-j_N+1}^{N}\mathbb{E}\left|\sum_{l=1}^{m_N}\big(\log\xi_{l,n}-\mathbb{E}\log\xi_{l,n}-\frac{\eta_{l,n}}{\sqrt{\alpha_{l,n}N}}\big)\right|^2 \qquad (2.21)\\
&\le\frac{2K}{\varepsilon^2}\frac{c_Nj_N}{N}. \qquad (2.22)
\end{aligned}
$$

Here, the inequality (2.20) is a consequence of the fact that for any two sets of real numbers $\{x_l\}$ and $\{y_l\}$,

$$
|\max x_l - \max y_l| \le \max|x_l - y_l|. \qquad (2.23)
$$

To deduce (2.21), we first use the Markov's inequality and then notice the fact that $\max y_l < \sum y_l$ for any nonnegative real numbers $\{y_l\}$. According to statement (II) in Lemma 2.2 we have inequality (2.22).

Now, let N tend to infinity, we get the desired result. □

We have the following simple but useful lemma.

Lemma 2.4. *For any fixed positive integer N, the probabilities*

$$\mathbb{P}(\xi_n \le x) = \frac{\int_{0\le t_1\cdots t_{m_N}\le x}(t_1\cdots t_{m_N})^{2n-1}e^{-N\sum_{l=1}^{m_N}V_l(t_l)}dt}{\int_{\mathbb{R}_+^{m_N}}(t_1\cdots t_{m_N})^{2n-1}e^{-N\sum_{l=1}^{m_N}V_l(t_l)}dt} \tag{2.24}$$

are nonincreasing in n.

Proof. Introduce

$$\mathbb{P}(\xi_y \le x) := \frac{\int_{\mathbb{R}_+^{m_N}}1_{(0,x]}(t_1\dots t_{m_N})(t_1\dots t_{m_N})^{y}e^{-N\sum_{l=1}^{m_N}V_l(t_l)}dt}{\int_{\mathbb{R}_+^{m_N}}(t_1\dots t_{m_N})^{y}e^{-N\sum_{l=1}^{m_N}V_l(t_l)}dt}, \quad y\ge 0. \tag{2.25}$$

Then differentiate this quantity with respect to y, by the Gurland's inequality [10], we obtain

$$\frac{d}{dy}\mathbb{P}(\xi_y\le x) = \mathbb{E}\left[1_{(0,x]}(\xi_y)\log\xi_y\right] - \mathbb{E}\left[1_{(0,x]}(\xi_y)\right]\mathbb{E}\left[\log\xi_y\right]\le 0. \tag{2.26}$$

□

Let $c>2$ be some constant, and set

$$j_N = \begin{cases} \left\lfloor\left(\frac{cN}{4m_Na_N}\log N\right)^{1/2}\right\rfloor, & \text{if } \tau=0,\\ \left\lfloor\frac{cN}{4m_Na_N}\log N\right\rfloor, & \text{if } \tau\in(0,\infty),\\ 1, & \text{if } \lim_{N\to\infty}m_N/N=\infty. \end{cases} \tag{2.27}$$

Here $\lfloor x\rfloor$ denotes the largest integer less than or equal to x.

Lemma 2.5. *Let ξ_n, $n=1,\dots,N-j_N$, be as in Proposition 1.2 and j_N as in* (2.27)*. Then,*

$$\lim_{N\to\infty}\prod_{n=1}^{N-j_N}\mathbb{P}\left(\xi_n\le f_N(x)\right)=1.$$

for any $x\in\mathbb{R}$ if $\tau=0$ and for $x>0$ if $\tau\in(0,\infty)$.

Proof. Case I: $\tau=0$. Let $\{\eta_{l,N-j_N}: l=1,\dots,m_N\}$ be a set of independent standard normal random variables such that (2.9) holds. The following string of inequalities hold

$$\begin{aligned} &\mathbb{P}\left(\left|\sum_{l=1}^{m_N}\left(\log\xi_{l,N-j_N}-\frac{\eta_{l,N-j_N}}{\sqrt{\alpha_{l,N-j_N}N}}-\mathbb{E}\log\xi_{l,N-j_N}\right)\right|>\log\theta_N(x)\right)\\ &\le\frac{1}{\log^2\theta_N(x)}\sum_{l=1}^{m_N}\frac{\mathbb{E}\left(\sqrt{\alpha_{l,N-j_N}N}(\log\xi_{l,N-j_N}-\mathbb{E}\log\xi_{l,N-j_N})-\eta_{l,N-j_N}\right)^2}{\alpha_{l,N-j_N}N}\\ &\le\frac{1}{\gamma_N}\frac{N}{4m_Na_N}\frac{K}{N^2}\sum_{l=1}^{m_N}\frac{1}{\alpha_{l,N-j_N}}\\ &\le\frac{1}{\log\frac{N}{m_N}}\frac{K}{N}. \end{aligned} \tag{2.28}$$

Applying the Markov's inequality we get the first inequality. Then, according to the statement (II) in Lemma 2.2, there is a constant K such the third line holds. Finally, notice that $\frac{1}{N}\sum_{l=1}^{m_N}\frac{1}{\alpha_{l,N-j_N}}$ is bounded, thus there is a constant such that the last inequality holds, we still denote the constant by K.

Recalling the definition of j_N as in (2.27), also taking Lemma 2.1 and statement (I) in Lemma 2.2 into consideration, we have

$$\begin{aligned}&\mathbb{P}\left(\sum_{l=1}^{m_N}\big(\frac{\eta_{l,N-j_N}}{\sqrt{\alpha_{l,N-j_N}N}}+\mathbb{E}\log\xi_{l,N-j_N}\big)>\sum_{l=1}^{m_N}\log t_l^{(0)}\right)\\&=\mathbb{P}\left(\Big(\frac{1}{N}\sum_{l=1}^{m_N}\alpha_{l,N-j_N}^{-1}\Big)^{\frac{1}{2}}\eta>N^{-1}\sum_{l=1}^{m_N}\alpha_l^{-1}(2j_N+1)(1+O(\frac{j_N}{N}))\right.\\&\qquad\left.+N^{-1}\sum_{l=1}^{m_N}\frac{\alpha_l+\beta_l}{2\alpha_l^2}(1+O(\frac{j_N}{N}))\right)\\&\leq\frac{1}{N\log N}.\end{aligned}\tag{2.29}$$

Applying Lemma 2.6 bellow, and then combining (2.28) and (2.29), we obtain

$$\begin{aligned}&\mathbb{P}\left(\sum_{l=1}^{m_N}\log\xi_{l,N-j_N}>\sum_{l=1}^{m_N}\log t_l^{(0)}+\log\theta_N(x)\right)\\&\leq\mathbb{P}\left(\sum_{l=1}^{m_N}\big(\frac{\eta_{l,N-j_N}}{\sqrt{\alpha_{l,N-j_N}N}}+\mathbb{E}\log\xi_{l,N-j_N}\big)>\sum_{l=1}^{m_N}\log t_l^{(0)}\right)\\&\quad+\mathbb{P}\left(\left|\sum_{l=1}^{m_N}\big(\log\xi_{l,N-j_N}-\frac{\eta_{l,N-j_N}}{\sqrt{\alpha_{l,N-j_N}N}}-\mathbb{E}\log\xi_{l,N-j_N}\big)\right|>\log\theta_N(x)\right)\\&\leq\frac{1}{\log\frac{N}{m_N}}\frac{2K}{N}.\end{aligned}\tag{2.30}$$

According to Lemma 2.4, one has

$$\lim_{N\to\infty}\prod_{n=1}^{N-j_N}\mathbb{P}\left(\xi_n\leq f_N(x)\right)\geq\lim_{N\to\infty}(1-\frac{1}{\log\frac{N}{m_N}}\frac{2K}{N})^N=1.$$

Case II: $\tau\in(0,\infty)$. We have the following string of inequalities

$$\begin{aligned}\mathbb{P}\left(\prod_{l=1}^{m_N}\xi_{l,N-j_N}>\prod_{l=1}^{m_N}t_l^{(0)}x\right)&\leq\frac{1}{x^2}\prod_{l=1}^{m_N}\big(\frac{t_l^{(N-j_N)}}{t_l^{(0)}}\big)^2\mathbb{E}\left(\frac{\xi_{l,N-j_N}}{t_l^{(N-j_N)}}\right)^2\\&\leq\frac{2}{x^2}\prod_{l=1}^{m_N}\left(1-\frac{2j_N+1}{\alpha_lN}(1+O(\frac{j_N}{N}))\right)^2\\&\leq\frac{2}{x^2}\exp\left\{-\frac{2j_Nm_Na_N}{N}\right\}\leq\frac{2}{x^2N^{c/2}}.\end{aligned}\tag{2.31}$$

The first line is a consequence of Markov's inequality. While the second line can be deduced from Lemma 2.1 and statement (I) of Lemma 2.2.

By Lemma 2.4, one has

$$\lim_{N\to\infty}\prod_{n=1}^{N-j_N}\mathbb{P}\left(\xi_n\leq f_N(x)\right)\geq\lim_{N\to\infty}(1-\frac{2}{x^2N^{c/2}})^N=1.\qquad\square$$

Lemma 2.6. *For any random variables X, Y and constant x, y, the following inequality holds*

$$\mathbb{P}(X+Y>x+y)\leq\mathbb{P}(X>x)+\mathbb{P}(|Y|>y).\tag{2.32}$$

Proof. The desired result can be induced directly from the following string of relations,

$$\begin{aligned}
&\{X+Y>x+y\}\\
&\subset\{X+|Y|>x+y\}\\
&=\Big(\{X+|Y|>x+y\}\bigcap\{|Y|>y\}\Big)\bigcup\Big(\{X>x+y-|Y|\}\bigcap\{|Y|\le y\}\Big)\\
&\subset\{|Y|>y\}\bigcup\{X>x\}\,. \qquad \square
\end{aligned}$$

The following lemma will be used in the proof of our main theorem frequently.

Lemma 2.7 (Kallenberg [12, Lemma 5.8]). *Consider a sequence of constants* $c_{Nj}\in[0,1)$, *and fix any* $c\in[0,\infty]$, *then* $\lim_{N\to\infty}\prod_j(1-c_{Nj})>0$ *if and only if* $\lim_{N\to\infty}\sum_j c_{Nj}<\infty$. *Furthermore, if* $\lim_{N\to\infty}\sup_j c_{Nj}=0$, *then* $\lim_{N\to\infty}\prod_j(1-c_{Nj})=e^{-c}$ *if and only if* $\lim_{N\to\infty}\sum_j c_{Nj}=c$. *In the later case, we have*

$$\lim_{N\to\infty}\sum_{l_1<\cdots<l_k}\prod_{i=1}^k c_{Nl_i}=\frac{1}{k!}c^k. \tag{2.33}$$

Proof. Here, we just give a proof of the last assertion. Notice that

$$\big(\sum_j c_{Nj}\big)^k=k!\sum_{l_1<\cdots<l_k}\prod_{i=1}^k c_{Nl_i}+O\big(\sum_{l_1,l_2,\ldots,l_{k-1}}c_{Nl_1}^2\prod_{i=2}^{k-1}c_{Nl_i}\big). \tag{2.34}$$

For $\sup_j c_{Nj}$ tends to 0, the big O item tends to 0. Thus the desired result is obtained. $\square$

Proof of Theorem 1.5. We split the proof into two steps.

Step 1 is to the study the limiting distribution of the spectral radius $|z|_{(1)}$. Let j_N be as in (2.27). Proposition 2.3 and Lemma 2.5 imply that there are independent standard normal random variables $\{\eta_{l,n}:1\le l\le m_N, N-j_N<n\le N\}$ such that

$$\begin{aligned}
&\lim_{N\to\infty}\mathbb{P}\left(|z|_{(1)}\le f_N(x)\right)\\
&=\lim_{N\to\infty}\prod_{n=N-j_N+1}^{N}\mathbb{P}\left(\xi_n\le f_N(x)\right)\\
&=\lim_{N\to\infty}\mathbb{P}\left(\max_{N-j_N<n\le N}\sum_{l=1}^{m_N}\big(\frac{\eta_{l,n}}{\sqrt{\alpha_{l,n}N}}+\mathbb{E}\log\xi_{l,n}\big)\le\log f_N(x)\right).
\end{aligned}\tag{2.35}$$

Applying Lemma 2.1 and statement (I) in Lemma 2.2, we have

$$\begin{aligned}
\mathbb{E}\log\xi_{l,n}&=\log t_l^{(n)}-\frac{1}{2\alpha_{l,n}N}(1+\frac{\beta_{l,n}}{\alpha_{l,n}})(1+O(\frac{1}{N})) &(2.36)\\
&=\log t_l^{(0)}-\big(\frac{2(N-n)+1}{N\alpha_l}+\frac{\alpha_l+\beta_l}{2N\alpha_l^2}\big)(1+O(\frac{j_N}{N})). &(2.37)
\end{aligned}$$

Recalling the definition of $f_N(x)$ given by (2.2), we obtain

$$\begin{aligned}
&\lim_{N\to\infty}\mathbb{P}\left(|z|_{(1)}\le f_N(x)\right)\\
&=\lim_{N\to\infty}\prod_{k=0}^{j_N-1}\mathbb{P}\left(\Big(\frac{1}{N}\sum_{l=1}^{m_N}\alpha_{l,N-k}^{-1}\Big)^{\frac12}\eta\le N^{-1}\sum_{l=1}^{m_N}\alpha_l^{-1}(2k+1)(1+O(\frac{j_N}{N}))\right.\\
&\qquad\left.+\,N^{-1}\sum_{l=1}^{m_N}\frac{\alpha_l+\beta_l}{2\alpha_l^2}(1+O(\frac{j_N}{N}))+\log\theta_N(x)\right) &(2.38)\\
&=\lim_{N\to\infty}\prod_{k=0}^{j_N-1}\mathbb{P}\left(\eta\le\Xi_{N,k}(x)\right), &(2.39)
\end{aligned}$$

where

$$\Xi_{N,k}(x) = \big(\frac{m_N a_N}{N}\big)^{\frac{1}{2}}(2k + \frac{3}{2} + \frac{b_N}{2a_N}) + \big(\frac{m_N a_N}{N}\big)^{-\frac{1}{2}} \log \theta_N(x) \tag{2.40}$$

and η is a standard normal random variable. Lemma 2.1 is used again to deduce (2.39).

If $\tau = 0$, by a similar argument as in [15], applying the Riemann sum approximation, we obtain

$$\lim_{N\to\infty} \sum_{k=0}^{j_N-1} \mathbb{P}\left(\eta > \Xi_{N,k}(x)\right) = \lim_{N\to\infty} \sum_{k=0}^{j_N-1} \int_{\Xi_{N,k}(x)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \tag{2.41}$$

$$= \lim_{N\to\infty} \frac{1}{2}\big(\frac{m_N a_N}{N}\big)^{-\frac{1}{2}} \int_{\Xi_{N,0}(x)}^{\Xi_{N,j_N}(x)} \int_s^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dtds \tag{2.42}$$

$$= \lim_{N\to\infty} \frac{1}{2}\big(\frac{m_N a_N}{N}\big)^{-\frac{1}{2}} \frac{1}{\Xi_{N,0}^2(x)} e^{-\frac{1}{2}\Xi_{N,0}^2(x)} = e^{-x}. \tag{2.43}$$

According to (2.39) and Lemma 2.7, we have

$$\lim_{N\to\infty} \mathbb{P}\left(\max_{1<n\le N} |z_n| \le f_N(x)\right) = \exp(-e^{-x}). \tag{2.44}$$

As a byproduct, we also have

$$\lim_{N\to\infty} \sum_{n=0}^{j_N-1} \mathbb{P}\left(\xi_n > f_N(x)\right) = e^{-x}. \tag{2.45}$$

If $\tau \in (0, \infty)$, by Lebesgue dominated convergence theorem and Lemma 2.7 we obtain that

$$\lim_{N\to\infty} \mathbb{P}\left(\max_{1<n\le N} |z_n| \le f_N(x)\right) = \prod_{k=0}^{\infty} \Phi\big((a_0\tau)^{1/2}(2k + \frac{3}{2} + \frac{b_0}{2a_0}) + (a_0\tau)^{-1/2}\log x\big) \tag{2.46}$$

holds for every $x > 0$. Here $\Phi(x)$ is the distribution function of the standard normal random variable.

Step 2 is to analysis the limit of $|z|_{(n)}$ for any fixed positive integer n. The probability that $\xi_{(n)}$ less than or equal to $f_N(x)$ is the sum from $k = 0$ to $n-1$ of the probabilities that there are exactly k of $\{\xi_i : 1 \le i \le N\}$ larger than $f_N(x)$. Thus, applying Proposition 1.2, we know that

$$\mathbb{P}(|z|_{(n)} \le f_N(x)) = \mathbb{P}(\xi_{(n)} \le f_N(x))$$
$$= \prod_{i=1}^{N} \mathbb{P}(\xi_i \le f_N(x)) \sum_{k=0}^{n-1} \sum_{1\le l_1<l_2<\ldots<l_k\le N} \prod_{i=1}^{k} \frac{\mathbb{P}(\xi_{l_i} > f_N(x))}{\mathbb{P}(\xi_{l_i} \le f_N(x))}. \tag{2.47}$$

The prefactor in (2.47) is just the probability that the spectral radius is not more than $f_N(x)$ which can be found in the previous step. So we just need to investigate the limit of the summation in (2.47).

Case I: $\tau = 0$. If $l_r \le N - j_N$ for some integer $r \ge 1$, we have

$$\sum_{1\le l_1<\ldots<l_r\le N-j_N<l_{r+1}<\ldots<l_k\le N} \prod_{i=1}^{k} \frac{\mathbb{P}(\xi_{l_i} > f_N(x))}{\mathbb{P}(\xi_{l_i} \le f_N(x))}$$

$$\le N^r \Big(\frac{\mathbb{P}(\xi_{N-j_N} > f_N(x))}{\mathbb{P}(\xi_{N-j_N} \le f_N(x))}\Big)^r \sum_{N-j_N<l_{r+1}<\ldots<l_k\le N} \prod_{i=r+1}^{k} \frac{\mathbb{P}(\xi_{l_i} > f_N(x))}{\mathbb{P}(\xi_{l_i} \le f_N(x))} \tag{2.48}$$

$$\le \big(\frac{1}{\log\frac{N}{m_N}}\big)^r \big(\sum_{i=0}^{j_N-1} \mathbb{P}(\xi_{N-i} > f_N(x))\big)^{k-r} \tag{2.49}$$

$$\to 0. \tag{2.50}$$

Inequality (2.48) can be deduced from Lemma 2.4. Then applying inequality (2.30) we obtain (2.49). Finally, (2.45) implies this quantity tends to 0. Thus, we obtain

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{1\le l_1<l_2<\ldots<l_k\le N} \prod_{i=1}^{k} \frac{\mathbb{P}(\xi_{l_i} > f_N(x))}{\mathbb{P}(\xi_{l_i} \le f_N(x))} \\
&= \lim_{N\to\infty} \sum_{N-j_N<l_1<\ldots<l_k\le N} \prod_{i=1}^{k} \frac{\mathbb{P}(\xi_{l_i} > f_N(x))}{\mathbb{P}(\xi_{l_i} \le f_N(x))} \\
&\quad + \lim_{N\to\infty} \sum_{r=1}^{k} \sum_{1\le l_1<\ldots<l_r\le N-j_N<l_{r+1}<\ldots<l_k\le N} \prod_{i=1}^{k} \frac{\mathbb{P}(\xi_{l_i} > f_N(x))}{\mathbb{P}(\xi_{l_i} \le f_N(x))} \\
&= \lim_{N\to\infty} \sum_{0\le l_1<\ldots<l_k<j_N} \prod_{i=1}^{k} \frac{\mathbb{P}(\xi_{N-l_i} > f_N(x))}{\mathbb{P}(\xi_{N-l_i} \le f_N(x))}. \\
&= \lim_{N\to\infty} \sum_{0\le l_1<\ldots<l_k<j_N} \prod_{i=1}^{k} \mathbb{P}(\xi_{N-i} > f_N(x)) && (2.51)\\
&= \frac{1}{k!} e^{-kx} && (2.52)
\end{aligned}$$

Equation (2.51) is a consequence of the fact that for any $0 \le i < j_N$, $\mathbb{P}(\xi_{N-i} > f_N(x)) \le \mathbb{P}(\xi_{N-j_N} > f_N(x))$ and the later converges to 0, see (2.30). Then, applying Lemma 2.7, we obtain (2.52).

Case II: $\tau \in (0,\infty)$. Following a similar reasoning as in the previous case, we have

$$\begin{aligned}
\lim_{N\to\infty} \sum_{1\le l_1<l_2<\ldots<l_k\le N} \prod_{i=1}^{k} \frac{\mathbb{P}(\xi_{l_i} > f_N(x))}{\mathbb{P}(\xi_{l_i} \le f_N(x))} &= \lim_{N\to\infty} \sum_{N-j_N<l_1<\ldots<l_k\le N} \prod_{i=1}^{k} \frac{\mathbb{P}(\xi_{l_i} > f_N(x))}{\mathbb{P}(\xi_{l_i} \le f_N(x))} \\
&=: \lim_{N\to\infty} \int_{\mathbb{N}^k} h_N(l_1,l_2,\ldots,l_k;x)\,d\mu.
\end{aligned}$$

Here μ is the counting measure on $\mathbb{N}^k$, and

$$h_N(l_1,l_2,\ldots,l_k;x) = \begin{cases} \prod_{i=1}^{k} \frac{\mathbb{P}(\eta>\Xi_{N,l_i}(x))}{\mathbb{P}(\eta\le\Xi_{N,l_i}(x))}, & \text{if } 0 \le l_1 < l_2 < \cdots < l_k < j_N, \\ 0, & \text{otherwise.} \end{cases} \tag{2.53}$$

By the conditions $\lim_{N\to\infty} m_N/N = \tau \in (0,\infty)$ and $\lim_{N\to\infty} a_N \to a_0$, there are some constants $C_0 > 0, C_1(x)$ independent of N such that

$$h_N(l_1,l_2,\ldots,l_k;x) \le \prod_{i=1}^{k} \frac{\mathbb{P}(\eta > C_0 l_i + C_1(x))}{\mathbb{P}(\eta \le C_0 l_i + C_1(x))} =: h(l_1,l_2,\ldots,l_k;x) \tag{2.54}$$

hold for N large enough. It is easy to check that

$$\int_{\mathbb{N}^k} h(l_1,l_2,\ldots,l_k;x)\,d\mu = \left(\sum_{l=0}^{\infty} \frac{\mathbb{P}(\eta > C_0 l + C_1(x))}{\mathbb{P}(\eta \le C_0 l + C_1(x))}\right)^k < \infty. \tag{2.55}$$

Application of the Lebesgue dominated convergence theorem gives the desired result. □

3 Concluding

In this paper, we have investigated the asymptotic behavior of the kth large modulus of the eigenvalues of products of m_N independent complex random matrices drawn from the polynomial random matrix ensembles with derivative type where k is a fixed positive integer. It was shown that the limiting distribution of kth large modulus of the eigenvalues is the same as that of the single complex Ginibre matrix [16] for $m_N/N \to 0$ (particularly m_N is a constant), while for $m_N/N \to \tau \in (0, \infty)$ the results are generalization of the results for the spectral radii of products of independent complex Ginibre matrices [11]. Our analysis suggests that the distribution of the kth large modulus is determined by j_N (2.27) independent random variables. When $m_N/N \to 0$ the j_N independent random variables are nearly identically distributed according to Gaussian distribution, so the limiting distribution of spectral radii is Gumbel distribution. When $m_N/N \to \tau \in (0, \infty)$ the j_N random variables are just independent, thus the distribution of largest spectral modulli is the products of distributions of these random variables whenever the latter is indeed a distribution.

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