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The lower Snell envelope of smooth functions: an optional decomposition

Erick Trevino Aguilar*

Abstract

In this paper we provide general conditions under which the lower Snell envelope defined with respect to the family $\mathcal{M}$ of equivalent local-martingale probability measures of a semimartingale S admits a decomposition as a stochastic integral with respect to S and an optional process of finite variation. On the other hand, based on properties of predictable stopping times we establish a version of the classical backwards induction algorithm in optimal stopping for the non-linear super-additive expectation associated to $\mathcal{M}$. This result is of independent interest and we show how to apply it in order to systematically construct instances of the lower Snell envelope with no optional decomposition. Such 'counterexamples' strengths the scope of our conditions.

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1 Introduction

Let S be a fixed semimartingale and consider its class of equivalent local-martingale measures $\mathcal{M}$. In financial applications, the semimartingale S is seen as the price process of given underlying assets and conditions under which $\mathcal{M}$ is non-empty are by now well-known; see the seminal paper [4]. In this paper, we assume that $\mathcal{M}$ has an infinite number of elements, corresponding to market-incompleteness.

Fix an horizon $\mathbb{T}$. We actually start our discussion with the upper Snell envelope. This process is the value process of optimally stopping an underlying process H with respect to the non-linear sub-additive expectation associated to $\mathcal{M}$. At time t it is a process which is almost surely equal to

$$\operatorname{ess\,sup}_{\tau}\operatorname{ess\,sup}_{Q\in\mathcal{M}} E_Q[\mathsf{H}_\tau \mid \mathcal{F}_t],$$

where the supremum over τ is taken on the family of stopping times valuated in the interval $[t,\mathbb{T}]$. Under quite general conditions the structure of the upper Snell envelope is clarified by the celebrated Optional Decomposition Theorem as the difference of a stochastic integral with respect to S and an optional non-decreasing process; see [6] and its references. Analogously, the lower Snell envelope can be seen as the value process

*University of Guanajuato.
E-mail: erick.trevino@ugto.mx

with respect to the non-linear super-additive expectation associated to $\mathcal{M}$. At time t it is a process which is almost surely equal to

$$\operatorname{ess\,sup}_{\tau}\operatorname{ess\,inf}_{Q\in\mathcal{M}} E_Q[\mathsf{H}_\tau \mid \mathcal{F}_t].$$

The main goal of this paper is to study analogous optional decompositions for the lower Snell envelope.

We actually start with a general construction of lower Snell envelopes without such a structure. This of course prompts the necessity of further conditions in order to get such optional decompositions for the lower Snell envelope. In our construction of a counterexample, the main idea is to consider processes H such that their lower Snell envelopes are again equal to H. We achieve this property for processes H depending on local martingales which are orthogonal to the local-martingale part of S. The proof needs some preparation with results of independent interest. Indeed, we establish a result that can be seen as a time-continuous version of backwards induction for the non-linear super-additive expectation associated to $\mathcal{M}$, to the best of our knowledge, a new result. The proof requires subtle "predictable approximations of events" whose construction is based on [5]'s interpolation at predictable stopping times. Already in [12] an example appears of a non-semimartingale process equal to its lower Snell envelope. However, our construction here is more systematic thanks to our backwards induction result and goes beyond Brownian filtrations.
Once we have shown that in general lower Snell envelopes cannot have an optional decomposition we provide conditions under which we obtain a positive result. Indeed, in our main result, we show that the lower Snell envelope will be a semimartingale with an optional decomposition, if the underlying process H is a smooth bounded function of S, say of the form $f(S)$ with $f \in C^2$ bounded. The proof applies Itô's formula and this is why we take a smooth function. There are different versions of Itô's formula and then, our main result admits variations. In particular, we establish a version for convex functions as well. Previous literature studying the general structure of lower Snell envelope include [2] and [12]. The differences with the present work are the following. [2] investigate lower Snell envelopes for g-expectations with backward differential stochastic equations techniques. They obtain a structural result which describes the lower Snell envelope as the sum of a process of bounded variation and a stochastic integral with respect to Brownian motion. The underlying processes H they consider are far more general than processes of the form $f(S)$ but their theory does not cover non-linear expectations generated by the family $\mathcal{M}$. In [12] a special case of our main result is obtained but only for a continuous semimartingale S and only for the function $f(s) = (k-s)^+ = \max\{0, k-s\}$.

After this introduction, the note is organized as follows. In Section 2, we provide necessary preliminaries. In Section 2.2 we establish "predictable approximations" of events. In Section 2.3 we prove a version of backwards induction. In Section 3, we give an example of a lower Snell envelope which is equal to its underlying process and fails to have an optional decomposition. In Section 4, we prove an optional decomposition of the lower Snell envelope for processes of the form $f(S)$.

2 Preliminaries on robust stopping

2.1 Definitions

We start with some notation. Let $\mathbb{T} > 0$ be a positive finite number. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} := \{\mathcal{F}_t\}_{0\le t\le \mathbb{T}}$. We assume that the probability measure $\mathbb{P}$ is $0-1$ on $\mathcal{F}_0$ and that the filtration $\mathbb{F}$ satisfies the usual assumptions of right continuity and completeness. On this space, we fix a semimartingale S and by $\mathcal{M}$ we denote its class of equivalent local-martingale measures. Our basic assumption is that $\mathcal{M} \neq \emptyset$; see [4]. By $\mathcal{T}$ we denote the class of $\mathbb{F}$-stopping times with values in the interval $[0, \mathbb{T}]$.

Here and in the sequel we denote by $L^1(Q)$ the space of random variables which are integrable with respect to a probability measure Q and for $\tau \in \mathcal{T}$, we denote by $L^0(\mathcal{F}_\tau)$ the space of finite valued $\mathcal{F}_\tau$-measurable functions. Let $\mathbf{E}^{\downarrow}[\cdot \mid \mathcal{F}_\tau] : \bigcap_{Q\in\mathcal{M}} L^1(Q) \to L^0(\mathcal{F}_\tau)$ be the non-linear conditional expectation defined by

$$\mathbf{E}^{\downarrow}[\cdot \mid \mathcal{F}_\tau] := \operatorname{ess\,inf}_{Q\in\mathcal{M}} E_Q[\cdot \mid \mathcal{F}_\tau]. \tag{2.1}$$

An $\mathbb{F}$-adapted process $\{M_t\}_{0\leq t\leq \mathbb{T}}$ is a $\mathbf{E}^{\downarrow}$-supermartingale if for each pair of stopping times $\tau, \theta \in \mathcal{T}$ with $\mathbb{P}(\tau \geq \theta) = 1$ we have

1. $E_Q[|M_\tau|] < \infty$, for $Q \in \mathcal{M}$,
2. $\mathbf{E}^{\downarrow}[M_\tau \mid \mathcal{F}_\theta] \leq M_\theta$.

Let H be a càdlàg non-negative $\mathbb{F}$-adapted process which is of $class(D)$ with respect to each $Q \in \mathcal{M}$, that is, the family of random variables $\{\mathsf{H}_\tau \mid \tau \in \mathcal{T}\}$ is uniformly integrable with respect to Q. We assume that the stochastic process H is upper semicontinuous in expectation from the left with respect to each probability measure $Q \in \mathcal{M}$. That is, for any stopping time θ of the filtration $\mathbb{F}$ and an increasing sequence of stopping times $\{\theta_i\}_{i\in\mathbf{N}}$ converging to θ, we have $\limsup_{i\to\infty} E_Q[\mathsf{H}_{\theta_i}] \leq E_Q[\mathsf{H}_\theta]$.

The *lower Snell envelope* is the "value process" of optimal stopping with respect to $\mathbf{E}^{\downarrow}$; see [7] for a systematic treatment. Thus, it is a stochastic process $U^{\downarrow}$ such that for $t \in [0, \mathbb{T}]$, it is equal to

$$\operatorname{ess\,sup}_{\tau\in\mathcal{T}[t,\mathbb{T}]} \mathbf{E}^{\downarrow}[\mathsf{H}_\tau \mid \mathcal{F}_t], \mathbb{P}-a.s. \tag{2.2}$$

There is a right continuous optional version of this process; see [11].

2.2 Predictable stopping times

Let $m : \mathbf{R}_+ \to \mathbf{R}_+$ be a strictly increasing continuous function with $m(0) = 0$. Let X be a càdlàg adapted stochastic process with modulus of continuity m in the following sense:

$$|X_t - X_s| \leq m(|t-s|). \tag{2.3}$$

We use the notation $X_t^* := \sup_{0\leq s\leq t} X_s$ for the running supremum of the process X.

Lemma 2.1. *Let X be a càdlàg adapted stochastic process with a modulus of continuity as in* (2.3)*. For $K \in \mathbf{R}$ let τ be the first passage time to the level K:*

$$\tau := \inf\{t \mid X_t \geq K\}. \tag{2.4}$$

Let $\{r_n\}_{n\in\mathbf{N}}$ be a decreasing sequence converging to a fixed $t \in (0, \mathbb{T})$. Then, there exists a sequence $\{A^n\}_{n\in\mathbf{N}} \subset \mathcal{F}_t$ with $A^{n-1} \subset A^n \subset \{\tau > r_n\}$ and

$$\bigcup_{n\in\mathbf{N}} A^n = \{\tau > t\}. \tag{2.5}$$

Proof. Let $\{K^n\}_{n\in\mathbf{N}}$ be the monotone increasing sequence defined by

$$K^n = K - m(r_n - t).$$

For $n \in \mathbf{N}$, let $A^n := \{X_t^* < K^n\}$. Note that $A^n \in \mathcal{F}_t$ and $A^{n-1} \subset A^n$. On the set A^n we have $X_{r_n}^* < K^n + m(r_n - t) = K$. Thus, $A^n \subset \{\tau > r_n\}$. The equality (2.5) is consequence to

$$\Omega / \bigcup_{n\in\mathbf{N}} A^n = \bigcap_{n\in\mathbf{N}} (\Omega/A^n) = \{X_t^* \geq K\}. \qquad \square$$

[5]'s construction yields for each predictable stopping time τ a predictable adapted process with strictly increasing continuous paths such that τ is the first time that this process takes the constant value 1. In particular, each predictable stopping time is a first passage time. This process actually has a modulus of continuity.

Lemma 2.2 (Emery)**.** *Let τ be a predictable stopping time with $\tau > 0$ $\mathbb{P}$-a.s. Then, there exists an adapted process A with strictly increasing continuous paths with a modulus of continuity as in* (2.3)*, such that*

$$\tau = \inf\{s \geq 0 \mid A_s \geq 1\}. \tag{2.6}$$

Proof. There exists an adapted process A with continuous and strictly increasing paths such that $A_0 = 0$ and $A_\tau = 1$; see [5]. Thus, with this choice, (2.6) holds true.

We only need to verify that [5]'s construction of A generates a process with a modulus of continuity as in (2.3). The construction of A starts with a non increasing sequence $\{\epsilon_n\}_{n\in\mathbf{N}}$ converging to zero. We fix one such sequence and without loss of generality, we assume is strictly decreasing. An inspection in the construction shows that for any $t_1, t_2 \in [0, \mathbb{T}]$ with $0 \leq t_2 - t_1 < \epsilon_n$

$$A_{t_2} - A_{t_1} \leq \frac{1}{2^{n-1}}. \tag{2.7}$$

Now we construct a modulus of continuity m. For $x \geq \epsilon_2$ let $m(x) = 1 + x - \epsilon_2$. Furthermore, for $n > 2$ set $m(\epsilon_n) = \frac{1}{2^{n-2}}$. In the intervals $[\epsilon_{n+1}, \epsilon_n]$ we define m by linear interpolation. It is easy to see that m constructed in this way is continuous, strictly increasing and $m(0) = 0$. Moreover, it satisfies (2.3) due to (2.7). $\square$

Theorem 2.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space in which a filtration $\mathbb{F}$ satisfying the usual conditions is defined. Assume that on this filtration all martingales are continuous and $\mathcal{F}_0$ is 0-1 under $\mathbb{P}$. Let $\{r_n\}_{n\in\mathbf{N}}$ be a decreasing sequence converging to a fixed $t \in (0, \mathbb{T})$. Then, for every stopping time τ there exists a sequence $\{A^n\}_{n\in\mathbf{N}} \subset \mathcal{F}_t$ with $A^{n-1} \subset A^n \subset \{\tau > r_n\}$ and $\bigcup_{n\in\mathbf{N}} A^n = \{\tau > t\}$.*

Proof. If all martingales are continuous, then the predictable and the optional σ-algebras coincide. In this case each stopping time is predictable; see [10], Corollary IV.5.7, p. 174. Now the result follows easily from Lemmas 2.1 and 2.2. $\square$

An application of Theorem 2.3 is the following.

Corollary 2.4. *For $\tau \in \mathcal{T}$*

$$\lim_{n\to\infty} \mathbf{E}^{\downarrow}[1_{\{\tau > r_n\}} \mathsf{H}_\tau \mid \mathcal{F}_t] = \mathbf{E}^{\downarrow}[1_{\{\tau > t\}} \mathsf{H}_\tau \mid \mathcal{F}_t].$$

Proof. Indeed, the sequence $\{\mathbf{E}^{\downarrow}[1_{\{\tau>r_n\}} \mathsf{H}_\tau \mid \mathcal{F}_t]\}_{n\in\mathbf{N}}$ is monotone and converges. The inequality $\leq$ is clear. For the other direction, let $\{A^n\}_{n\in\mathbf{N}}$ be a sequence as in Theorem 2.3. Then:

$$\begin{aligned}\lim_{n\to\infty} \mathbf{E}^{\downarrow}[1_{\{\tau > r_n\}} \mathsf{H}_\tau \mid \mathcal{F}_t] &\geq \mathbf{E}^{\downarrow}[1_{A^n} \mathsf{H}_\tau \mid \mathcal{F}_t] \\ &= 1_{A^n} \mathbf{E}^{\downarrow}[\mathsf{H}_\tau \mid \mathcal{F}_t].\end{aligned}$$

Then

$$\begin{aligned}\lim_{n\to\infty} \mathbf{E}^{\downarrow}[1_{\{\tau>r_n\}}\mathsf{H}_\tau \mid \mathcal{F}_t] &\geq \sup_{n\in\mathbf{N}} 1_{A^n}\mathbf{E}^{\downarrow}[\mathsf{H}_\tau \mid \mathcal{F}_t]\\ &= 1_{\{\tau>t\}}\mathbf{E}^{\downarrow}[\mathsf{H}_\tau \mid \mathcal{F}_t].\end{aligned}$$ □

Remark 2.5. The operator $\mathbf{E}^{\downarrow}$ being a infimum, is upper semicontinuous due to our conditions on the process H. However, it is necessary a stronger property in the proof of Corollary 2.4. Thus, Theorem 2.3 is crucial.

2.3 Backwards induction

The following result can be seen as a continuous time version of the classical backwards induction algorithm for the nonlinear expectation $\mathbf{E}^{\downarrow}$.

Theorem 2.6. *Assume $\mathbb{F}$ is a filtration in which every stopping time is predictable. Let $\{r_n\}_{n\in\mathbf{N}}$ be a monotone sequence decreasing to $t>0$. Then*

$$\operatorname{ess\,sup}_{\tau\geq t}\mathbf{E}^{\downarrow}[\mathsf{H}_\tau \mid \mathcal{F}_t] = \mathsf{H}_t \vee \sup_{n\in\mathbf{N}} \operatorname{ess\,sup}_{\tau\geq r_n}\mathbf{E}^{\downarrow}[\mathsf{H}_\tau \mid \mathcal{F}_t]. \tag{2.8}$$

Proof. The inequality $\geq$ is obvious. Let us check the other direction.

1. There exists $\rho\geq t$ with

$$\operatorname{ess\,sup}_{\tau\geq t}\mathbf{E}^{\downarrow}[\mathsf{H}_\tau \mid \mathcal{F}_t] = \mathbf{E}^{\downarrow}[\mathsf{H}_\rho \mid \mathcal{F}_t],$$

see [11], Proposition 3.2.

For $\rho^n := \rho\vee r_n$ we have

$$\mathbf{E}^{\downarrow}[\mathsf{H}_{\rho^n} \mid \mathcal{F}_{r_n}] = 1_{\{\rho\geq r_n\}}\mathbf{E}^{\downarrow}[\mathsf{H}_\rho \mid \mathcal{F}_{r^n}] + 1_{\{\rho<r_n\}}\mathsf{H}_{r_n},$$

so

$$\begin{aligned}\mathbf{E}^{\downarrow}[\mathsf{H}_{\rho^n} \mid \mathcal{F}_t] &= \mathbf{E}^{\downarrow}\left[1_{\{\rho\geq r_n\}}\mathbf{E}^{\downarrow}[\mathsf{H}_\rho \mid \mathcal{F}_{r^n}] + 1_{\{\rho<r_n\}}\mathsf{H}_{r_n} \mid \mathcal{F}_t\right]\\ &\geq \mathbf{E}^{\downarrow}\left[\mathsf{H}_\rho 1_{\{\rho\geq r_n\}} \mid \mathcal{F}_t\right].\end{aligned}$$

Then

$$\sup_{n\in\mathbf{N}}\mathbf{E}^{\downarrow}[\mathsf{H}_{\rho^n} \mid \mathcal{F}_t] \geq \lim_{n\to\infty}\mathbf{E}^{\downarrow}\left[\mathsf{H}_\rho 1_{\{\rho\geq r_n\}} \mid \mathcal{F}_t\right] = \mathbf{E}^{\downarrow}\left[\mathsf{H}_\rho 1_{\{\rho>t\}} \mid \mathcal{F}_t\right]$$

where the equality holds true due to Corollary 2.4. As a consequence we see that the right hand side of (2.8) dominates from above $\mathbf{E}^{\downarrow}\left[\mathsf{H}_\rho 1_{\{\rho>t\}} \mid \mathcal{F}_t\right]$. It also dominates $\mathbf{E}^{\downarrow}\left[\mathsf{H}_\rho 1_{\{\rho=t\}} \mid \mathcal{F}_t\right] = \mathsf{H}_t 1_{\{\rho=t\}}$. □

3 An example of a non-semimartingale lower Snell envelope

Consider a two-dimensional Brownian motion (B^1, B^2). Let $S = e^{\sigma B^1}$ for $\sigma>0$, and $\{V_t\}_{0\leq t\leq\mathbb{T}}$ be the stochastic process defined by $V_t = \mathcal{E}_t\left(\int_0^{\cdot}\frac{1}{2S_s}dS_s\right)$, where $\mathcal{E}$ denotes the Dooleans-Dade exponential. For future use, we note that V^{-1} reduces to

$$V_t^{-1} = 1 - \int_0^t \frac{\sigma}{2V_s}dB_s^1. \tag{3.1}$$

It is easy to see that processes of the form

$$Y_t = \frac{\mathcal{E}_t(\int_0^{\cdot}\xi_s dB_s^2)}{V_t}$$

are density processes for martingale probability measures for S, if ξ is sufficiently integrable.

Proposition 3.1. *Let g be an adapted non-negative stochastic process with right-continuous paths. Assume that $g^*_{\mathbb{T}} = \sup_{0\le t\le \mathbb{T}} g_t$ is essentially bounded. Let*

$$\mathsf{H}_t = e^{B_t^2 - \frac{1}{2}t} g_t. \tag{3.2}$$

Then, the lower Snell envelope of H *is* H*.*

Proof. Take $t \in (0, \mathbb{T})$. Let $\{r_n\}_{n\in\mathbf{N}}$ be a sequence of positive numbers strictly decreasing to t. For $n \in \mathbf{N}$, $k \in \mathbf{R}$ let $\xi^{k,n}$ be defined by

$$\xi_s^{k,n} = \begin{cases} m^{k,n} & \text{if } s \in [r_{n+1}, r_n) \\ 0 & \text{if } s \in [0,\mathbb{T}]/[r_{n+1}, r_n), \end{cases} \tag{3.3}$$

where

$$m^{k,n} := \frac{k}{r_n - r_{n+1}}.$$

Then $\int_{r_{n+1}}^{r_n} \xi_z^{k,n} dz = k$. For $\epsilon > 0$, choose $k \in \mathbf{R}_-$ (the set of non-positive real numbers) in such a way that

$$e^k(\operatorname{ess\,sup} g^*_{\mathbb{T}}) < \epsilon.$$

Let Q be the probability measure with density process $Z_t^{k,n} = \mathcal{E}_t(\int_0^\cdot \xi_z^{k,n} dB_z^2)$ and P^* the probability measure with density $\frac{Z^{k,n}}{V}$. There exists a Q-Brownian motion W^Q such that $B_t^2 = W_t^Q + \int_0^t \xi_z dz$. Let

$$M_t := V_t^{-1} e^{W_t^Q - \frac{1}{2}t}, \qquad \eta_t := e^{\int_0^t \xi_z dz} g_t.$$

Let us check that M is a Q-martingale. It is clear that $e^{W_t^Q - \frac{1}{2}t}$ is a Q-martingale. The process V^{-1} is a martingale, let us see that it is also a Q-martingale. Indeed, for θ a stopping time in the class $\mathcal{T}$ we have

$$\begin{aligned} E_Q[V_\theta^{-1}] &= E[Z_\theta^{k,n} V_\theta^{-1}] \\ &= 1 + E\left[\int_0^\theta Z_s^{k,n} dV_s^{-1} + \int_0^\theta V_s^{-1} dZ_s^{k,n} + \left\langle V^{-1}, Z^{k,n}\right\rangle_\theta\right] \\ &= 1 + E\left[\left\langle V^{-1}, Z^{k,n}\right\rangle_\theta\right] \\ &= 1 + E\left[\int_0^\theta \frac{-\sigma}{2V_s} Z_s^{k,n} \xi_s^{k,n} d\left\langle B^1, B^2\right\rangle_s\right] \\ &= 1. \end{aligned}$$

This equality shows that V^{-1} is a Q-martingale. Now we check that V^{-1} and $e^{W_t^Q - \frac{1}{2}t}$ have zero quadratic variation. We have

$$\left\langle V^{-1}, e^{W_t^Q - \frac{1}{2}t}\right\rangle_t = \int_0^t \frac{-\sigma}{2V_s} e^{W_s^Q - \frac{1}{2}s} d\left\langle B^1, W^Q\right\rangle_s = 0.$$

After these preliminaries, we have for τ a stopping time with $\tau \ge r_n$

$$E_{P^*}[\mathsf{H}_\tau \mid \mathcal{F}_t] = V_t E_Q[M_\tau \eta_\tau \mid \mathcal{F}_t] \le \epsilon V_t E_Q[M_\tau \mid \mathcal{F}_t] = \epsilon e^{W_t^Q - \frac{1}{2}t}. \tag{3.4}$$

Thus, $\mathbf{E}^{\downarrow}[\mathsf{H}_\tau \mid \mathcal{F}_t] = 0$ and

$$\operatorname{ess\,sup}_{\tau\ge t} \mathbf{E}^{\downarrow}[\mathsf{H}_\tau \mid \mathcal{F}_t] = \mathsf{H}_t,$$

due to Theorem 2.6, equation (2.8). □

Remark 3.2. Note that Proposition 3.1 shows that lower Snell envelopes and thus $\mathbf{E}^{\downarrow}$-supermartingales will in general lay beyond the class of semimartingales for processes of the form (3.2) with g having paths of nonzero p-variation with $p > 2$.

Remark 3.3. There are other examples of processes being equal to their own envelope, most notably uniform supermartingales (i.e., processes which are supermartingales with respect to each probability measure of $\mathcal{M}$). This class includes constants, non increasing processes and uniform martingales, e.g., $\mathsf{H} = V$. Note that the running infimum of the process in Proposition 3.1 suits in this class.

There is an "issue" with the construction in Proposition 3.1 as the next corollary shows.

Corollary 3.4. *In addition to the condition in Proposition 3.1 assume that for* $\delta_0 > 0$ *we have*

$$\inf_{0 \leq t \leq \mathbb{T}} g_t \geq \delta_0,\ \mathbb{P} - a.s.$$

Then, for each stopping time $\tau \geq t$ *with* $t \in (0, \mathbb{T})$

$$\sup_{Q \in \mathcal{M}} E_Q[\mathsf{H}_\tau] = \infty.$$

Proof. For $N \in \mathbf{N}$ fixed, choose $k \in \mathbf{R}_+$ (the set of non-negative real numbers) in such a way that

$$e^k \delta_0 \geq N.$$

For this choice of k, consider $\xi^{k,n}$ defined in (3.3). Let Q be the probability measure with density process $Z_t^{k,n} = \mathcal{E}(\int_0^t \xi_z^{k,n} dB_z^2)$ Then similarly to (3.4) we have for $\tau \geq r_n$

$$E_{P^*}[\mathsf{H}_\tau \mid \mathcal{F}_t] = V_t E_Q[M_\tau \eta_\tau \mid \mathcal{F}_t] \geq N V_t E_Q[M_\tau \mid \mathcal{F}_t] = N e^{W_t^Q - \frac{1}{2}t}.$$

Thus

$$E_{P^*}[\mathsf{H}_\tau] \geq N. \qquad \square$$

Remark 3.5. After Corollary 3.4, we might guess that the condition

$$\sup_{\tau \in \mathcal{T}} \sup_{Q \in \mathcal{Q}} E_Q[\mathsf{H}_\tau] < \infty \tag{3.5}$$

avoids the conclusion of Proposition 3.1. However, it is easy to see that for $k_0 \in \mathbf{N}$ the process

$$\mathsf{H}_t = \left\{ e^{B_t^2 - \frac{1}{2}t} \wedge k_0 \right\} g_t,$$

is not a uniform supermartingale, satisfies the statement of Proposition 3.1 and the integrability condition (3.5).

Remark 3.6. In a financial context, the process H is seen as a payoff process of an American option, a contract that can be exercised in a period of time. The lower Snell envelope determines the lower boundary at which the contract is traded without generating arbitrage opportunities. From this point of view, let us see the relevance of Proposition 3.1 in an episode of generalized pessimistic view of the contract. Imagine the situation in which the contract starts being traded at prices close to its lower boundary. Then, according to Proposition 3.1, the future value is worth less than present and after buying the contract there will be an almost immediate desire to sell it again. Thus, moving prices even closer to its lower boundary. If the option indeed has a non-zero g factor with infinite quadratic variation, then prices might be perceived as with an ever increasing volatility.

4 The lower Snell envelope for processes $f(S)$

[2] investigate the stopping problem (2.2) under g-expectations with backward differential stochastic equations techniques. They obtain a structural result which describes the lower Snell envelope as the sum of a process of bounded variation and a stochastic integral with respect to Brownian motion. We obtain a similar result in our setting for processes of the form $f(S)$. An optional decomposition will follow by an application of the Optional Decomposition Theorem in [6], since it holds true for uniform submartingales.

Proposition 4.1. *Let f be a bounded function of class C^2. Let $U^{\downarrow}$ be the lower Snell envelope of $\mathsf{H} = f(S)$. If S is a d-dimensional locally bounded semimartingale and jumps only at predictable times, then there exists a non-decreasing predictable process B such that $U^{\downarrow} + B$ is a local submartingale with respect to each probability measure of $\mathcal{M}$.*

Proof. By Itô's formula we have

$$\begin{aligned}\mathsf{H}_t - \mathsf{H}_0 = &\sum_{i\le d}\int_0^t D_i f(S_{s-})dS_s^i + \frac{1}{2}\sum_{i,j\le d}\int_0^t D_{i,j}f^{''}(S_{s-})d\left\langle S^{i,c}, S^{j,c}\right\rangle_s \\ &+ \sum_{s\le t}\left\{ f(S_s) - f(S_{s-}) - \sum_{i\le d} D_i f(S_{s-})\Delta S_s^i \right\},\end{aligned} \tag{4.1}$$

see [8], Theorem I.4.57 p.57 for the notation and proof. The first two terms on the right hand side of (4.1) are clearly locally bounded. In particular, there exists a non-decreasing sequence $\{\theta_k\}_{k\in\mathbf{N}}$ of stopping times with $\theta_k \nearrow \mathbb{T}$ such that the processes $\int_0^{t\wedge\theta_k} D_i f(S_{s-})dS_s^i$, for $i = 1, \ldots, d$ are bounded and are martingales with respect to each element of $\mathcal{M}$. The last term is a process of finite variation which is also locally bounded. Its total variation process is predictable, due to our assumption that S jumps only at predictable times, hence is also locally bounded, see [8], Lemma I.3.10. p. 29.
As a consequence, the second and third terms on the right hand side of (4.1) add up to a process of finite variation which can be decomposed as the difference $A - B$ of two locally-bounded non-decreasing processes. We can assume that the sequence $\{\theta_k\}_{k\in\mathbf{N}}$ is such that $B_{t\wedge\theta^k}$ is bounded for each k.
Thus, H satisfies locally the condition of Theorem 3.2 in [12]. This yields the result. □

Remark 4.2. The proof of Proposition 4.1 does not directly apply the condition that f is a bounded function. It guarantees that the payoff $f(S)$ is of $class(D)$ with respect to each member of $\mathcal{M}$, a necessary assumption.

Remark 4.3. Note that the condition of predictable jumps of S was used in order to guarantee that the finite variation process in the right hand side of (4.1) decomposes as the difference of two locally bounded processes, and thus, other formulations are possible.

Remark 4.4. The class $\mathcal{M}$ is stable under pasting. That is, for $\tau \in \mathcal{T}$ and Q_1 and Q_2 probability measures on $\mathcal{M}$ the probability measure defined through

$$Q_3(A) := E_{Q_1}[Q_2[A \mid \mathcal{F}_\tau]],\ A \in \mathcal{F}_{\mathbb{T}}$$

is also an element of $\mathcal{M}$; see .e.g, [3]. The stability property has been instrumental in [13] in order to show the minimax identity

$$\operatorname{ess\,inf}_{Q\in\mathcal{M}}\operatorname{ess\,sup}_{\tau\in\mathcal{T}[t,\mathbb{T}]}E_Q[\mathsf{H}_\tau \mid \mathcal{F}_t] = \operatorname{ess\,sup}_{\tau\in\mathcal{T}[t,\mathbb{T}]}\mathbf{E}^{\downarrow}[\mathsf{H}_\tau \mid \mathcal{F}_t], \mathbb{P}-a.s. \tag{4.2}$$

In a recent paper, [1] show a similar minimax identity (4.2) in which $\mathcal{M}$ is substituted by a family of probability measures satisfying general conditions and without any stability assumption. In view of this novel result, it is promising to pursue possible extensions of Proposition 4.1 without stability, since (4.2) is a fundamental property of the lower Snell envelope.

In the next result we take $d = 1$. It can be seen as a variation of Proposition 4.1 for a convex function f which is not necessarily of class C^2. We need an additional assumption in order to guarantee that the local time of S admits a regular version; see [9], Theorem 75.

- Hypothesis A. The semimartingale S satisfies $\sum_{0\le s\le \mathbb{T}} |\Delta S_s| < \infty$, $\mathbb{P}$-a.s.

Proposition 4.5. *Let f be a convex bounded function. Let $U^{\downarrow}$ be the lower Snell envelope of $\mathsf{H} = f(S)$. If S is locally bounded, jumps only at predictable times, and satisfies the Hypothesis A, then there exists a non-decreasing predictable process B such that $U^{\downarrow} + B$ is a local submartingale with respect to each probability measure of $\mathcal{M}$.*

Proof. Itô's formula for convex functions yields

$$\begin{aligned} \mathsf{H}_t - \mathsf{H}_0 = \int_0^t \frac{df}{dx}(S_{s-})dS_s + \sum_{s\le t}\left\{ f(S_s) - f(S_{s-}) - \frac{df}{dx}(S_{s-})\Delta S_s \right\} \\ + \frac{1}{2}\int_{-\infty}^{\infty} \mu(da) L_t^a, \end{aligned} \tag{4.3}$$

where $L^{\cdot}$ denotes the local time of S, $\frac{df}{dx}$ is the left hand derivative, μ is the measure associated to the second derivative of f; see e.g., [9], Theorem IV.70 p.214. We choose a version of L which is jointly right continuous in a and continuous in t; see [9], Theorem IV.75. Then, we see that the last part of the right hand side of (4.3) defines a continuous process. In particular, locally bounded. Now the proof continues analogously to Proposition 4.1. □

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