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ON GENERALIZED CUMULATIVE INFORMATION OF KULLBACK-LEIBLER TYPE

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Abstract. In this paper generalized versions of the empirical cumulative Kullback-Leibler information are introduced, together with their alternative representations. The generalization refers to both application of a weight function and the use of Tsallis extended logarithm in the original Kullback-Leibler information measure. The original measure was extensively studied in several papers. The first information measure proposed in this paper is the weighted version of the original Kullback-Leibler information, the second one implies the use of Tsallis extended logarithm, while the third one, combines the first and second. Some properties of the new measures are also discussed.

Key words: entropy, inaccuracy, cumulative Kullback-Leibler information, empirical cumulative Kullback-Leibler information, Tsallis logarithm.

1. INTRODUCTION

Along the years, many authors have proposed and studied entropy-related measures and used these concepts in applications. Among them, we underline the valuable results obtained by Di Crescenzo and Longobardi [1, 2, 3], M. Dumitrescu [4], M. Iosifescu[5], Kullback and Leibler [6], Park, Rao and Shin [8], V. Preda [9, 10], V. Preda, C. Balcau [11, 12], V. Preda, C. Balcau, D. Constantin and I.I. Panait [13], Rao, Chen and Vermuri [14].

The concept of differential entropy has been extended to the relative entropy, called Kullback-Leibler information [6], which represents a discrepancy between two distributions. S. Park, M. Rao, D. W. Shin [8] introduced the Kullback-Leibler cumulative information. A. Di Crescenzo and M. Longobardi [1] presented many properties of the cumulative and empirical cumulative Kullback-Leibler information. The authors used the above mentioned measures for different real-life applications.

In this paper, we aim to introduce generalized versions of the cumulative and empirical cumulative Kullback Leibler information. Section 2 presents the framework of the paper: general assumptions, definitions and notations that are needed to describe the newly proposed concepts. In Sections 3, 4 and 5, the weighted, Tsallis and Tsallis weighted version of cumulative and empirical cumulative Kullback-Leibler information are presented and discussed. Equivalent forms of the new empirical measures are derived.

2. GENERAL FRAMEWORK

We consider two absolutely continuous, non-negative, random variables X and Y, with distribution functions denoted by F and G. Let $X_1, X_2, ..., X_n$ sample variables, independently and identically distributed as X, and $Y_1, Y_2, ..., Y_n$ sample variables, independently and identically distributed as Y.

We denote the empirical cumulative distribution function of X, and respectively Y by

$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{I}_{\{X_i \le x\}}, \tag{1}$$

$$\hat{G}_m(y)=\frac{1}{m}\sum_{j=1}^{m}\mathrm{I}_{\{Y_j\le y\}}, \tag{2}$$

where $x, y \in \mathbf{R}$ and $\mathrm{I}_{\{X\le x\}}$ is the indicator function

$$\mathrm{I}_{\{Y_j\le y\}}=\begin{cases}1 \ , \text{ if } \ X\in[0,x] \\ 0 \ , \text{ if } \ X\notin[0,x]\end{cases}. \tag{3}$$

As usual, $\overline{X}_n$ and $\overline{Y}_m$ are the sample means, $X_{(1)}\le X_{(2)}\le\ldots\le X_{(n)}$ and $Y_{(1)}\le Y_{(2)}\le\ldots\le Y_{(m)}$ the order statistics of the two samples. We denote by $\Delta X_{(i)}=X_{(i+1)}-X_{(i)}$, for $i=\overline{1,n-1}$.

Using the notations from [1], let N_j, $j=\overline{1,m}$ be the number of random variables of the first sample that are less than or equal to the *j*-th order statistic of the second sample, that is $N_j=\sum_{i=1}^{n}\mathrm{I}_{\{X_i\le Y_{(j)}\}}$. The random variables of the first sample belonging to $\left(Y_{(j)},Y_{(j+1)}\right]$ are denoted by $X_{j,1}\le X_{j,2}\le\ldots\le X_{N_{j+1}-N_j}$ (if there are any). Also as in [1], the left-hand point and right-hand point of a random variable *T* with cumulative distribution function F_T are $l_T=\inf\{t\in\mathrm{R}|F_T(t)>0\}$ and $r_T=\sup\{t\in\mathrm{R}|F_T(t)<0\}$.

Moreover, let $w:[0,\infty)\to[0,\infty)$ be the weight function and $W:[0,\infty)\to[0,\infty)$ a primitive of it.

3. WEIGHTED EMPIRICAL CUMULATIVE KULLBACK-LEIBLER INFORMATION

Definition 3.1. The weighted empirical cumulative Kullback-Leibler information of the random variables *X* and *Y* is:

$$C_{KL}^{w}\left(\hat{F}_n,\hat{G}_m\right)=\int_0^{\infty}w(x)\left(\hat{F}_n(x)\ln\frac{\hat{F}_n(x)}{\hat{G}_m(x)}-\hat{F}_n(x)+\hat{G}_m(x)\right)\mathrm{d}x. \tag{4}$$

Remark 3.1. If $w(x)=1$ for every *x*, we get $C_{KL}^{w}\left(\hat{F}_n,\hat{G}_m\right)$ as defined in Di Crescenzo and Longobardi [1].

THEOREM 3.1. *The weighted empirical cumulative Kullback-Leibler information of the random variables X and Y is expressed as follows:*

$$\begin{gathered}C_{KL}^{w}\left(\hat{F}_n,\hat{G}_m\right)=\frac{1}{n}\sum_{j=1}^{m-1}\left[\ln\frac{j}{m}\cdot\left(\sum_{r=1}^{N_{j+1}-N_j}W\left(X_{j,r}\right)+N_j\cdot W\left(Y_{(j)}\right)-N_{j+1}\cdot W\left(Y_{(j+1)}\right)\right)\right]+\\ +\sum_{i=1}^{n-1}\left(\frac{i}{n}\cdot\ln\frac{i}{n}\cdot\Delta W\left(X_{(i)}\right)\right)+\overline{W(X)}_n-\overline{W(Y)}_m,\end{gathered} \tag{5}$$

where $\overline{W(X)}_n$ *and* $\overline{W(Y)}_m$ *are the sample means of the samples* $\left(W\left(X_i\right)\right)_{i=\overline{1,n}}$ *and respectively* $\left(W\left(Y_j\right)\right)_{j=\overline{1,m}}$.

Proof. According to Definition 3.1, the weighted empirical cumulative Kullback-Leibler information could be written as follows:

$$\begin{gathered}C_{KL}^{w}\left(\hat{F}_n,\hat{G}_m\right)=-\int_0^{\infty}w(x)\hat{F}_n(x)\ln\left(\hat{G}_m(x)\right)\mathrm{d}x+\int_0^{\infty}w(x)\hat{F}_n(x)\ln\left(\hat{F}_n(x)\right)\mathrm{d}x+\\ +\int_0^{\infty}w(x)\left[-\hat{F}_n(x)+\hat{G}_m(x)\right]\mathrm{d}x.\end{gathered} \tag{6}$$

For the first integral, we get:

$$\int_0^{\infty} w(x)\hat{F}_n(x)\ln\left(\hat{G}_m(x)\right)\mathrm{d}x = \sum_{j=1}^{m-1}\left(\ln\frac{j}{m}\cdot\int_{Y_{(j)}}^{Y_{(j+1)}} w(x)\hat{F}_n(x)\mathrm{d}x\right) =$$
$$= -\frac{1}{n}\sum_{j=1}^{m-1}\left[\ln\frac{j}{m}\cdot\left(\sum_{r=1}^{N_{j+1}-N_j} W\left(X_{j,r}\right) + N_j\cdot W\left(Y_{(j)}\right) - N_{j+1}\cdot W\left(Y_{(j+1)}\right)\right)\right], \tag{7}$$

using the definitions and notations presented in Section 2.

The second integral in (6) is

$$\int_0^{\infty} w(x)\hat{F}_n(x)\ln\left(\hat{F}_n(x)\right)\mathrm{d}x = \sum_{i=1}^{n-1}\left(\frac{i}{n}\cdot\ln\frac{i}{n}\cdot\int_{X_{(i)}}^{X_{(i+1)}} w(x)\mathrm{d}x\right) =$$
$$= \sum_{i=1}^{n-1}\left(\frac{i}{n}\cdot\ln\frac{i}{n}\cdot\Delta W\left(X_{(i)}\right)\right). \tag{8}$$

Finally, straightforward calculation leads to

$$\int_0^{\infty} w(x)\left[-\hat{F}_n(x) + \hat{G}_m(x)\right]\mathrm{d}x = \frac{1}{n}\sum_{i=1}^{n-1} W\left(X_{(i)}\right) - \frac{1}{m}\sum_{j=1}^{m-1} W\left(Y_{(j)}\right) = \overline{W(X)}_n - \overline{W(Y)}_m. \tag{9}$$

Using relations (7)–(9) in (6), we get relation (5).

Remark 3.2. As expected, for $w(x)=1$ for every x, we get the result obtained by Di Crescenzo and Longobardi in [1].

Remark 3.3. Defining weighted empirical cumulative inaccuracy by

$$K^w\left(\hat{F}_n,\hat{G}_m\right) = -\int_0^{\infty} w(x)\hat{F}_n(x)\ln\left(\hat{G}_m(x)\right)\mathrm{d}x \tag{10}$$

and weighted empirical cumulative entropy as

$$CE^w\left(\hat{F}_n\right) = -\int_0^{\infty} w(x)\hat{F}_n(x)\ln\left(\hat{F}_n(x)\right)\mathrm{d}x, \tag{11}$$

we obtain

$$C_{KL}^w\left(\hat{F}_n,\hat{G}_m\right) = K^w\left(\hat{F}_n,\hat{G}_m\right) - CE^w\left(\hat{F}_n\right) + \overline{W(X)}_n - \overline{W(Y)}_m. \tag{12}$$

The weighted versions of cumulative Kullback-Leibler information of random variables X and Y, cumulative inaccuracy and cumulative entropy are defined in what it follows.

Definition 3.2. Let X and Y be random variables with the same left-hand points $l = l_X = l_Y$ and with $E(W(X))$ and $E(W(Y))$ finite. The weighted cumulative Kullback-Leibler information of X and Y is

$$C_{KL}^w(X,Y) = \int_l^{\max\{r_X, r_Y\}} w(x)\left(F(x)\ln\frac{F(x)}{G(x)} - F(x) + G(x)\right)\mathrm{d}x. \tag{13}$$

Remark 3.4. For $w(x)=1$, $\forall x$ in (13), we get the cumulative Kullback-Leibler information as defined in Park et al [8].

Definition 3.3. For any pair of random variables X and Y having the same left-hand points l, the weighted cumulative inaccuracy is defined by

$$K^w(X,Y) = -\int_l^{\max\{r_X, r_Y\}} w(x)F(x)\ln G(x)\mathrm{d}x, \tag{14}$$

provided that the integral is finite.

The weighted cumulative entropy of X is

$$CE^w(X) = -\int_0^\infty w(x)F(x)\ln F(x)\mathrm{d}x. \tag{15}$$

Remark 3.5. Interesting results were obtained by F. Misagh in [7] for $CE^w(X)$ and $CE^w(\hat{F}_n)$ for the particular case $w(x)=1$.

Remark 3.6. Based on definitions 3.2 and 3.3, we get

$$C_{KL}^w(X,Y) = K^w(X,Y) - CE^w(X) + E(W(X)) - E(W(Y)). \tag{16}$$

Numerical application. Let X and Y be two continuous, nonnegative, random variables. The distributions taken into account for the variables, the weight functions considered and the theoretical weighted cumulative Kullback-Leibler information are presented in the table 3.1.

We conducted a simulation study for evaluating the weighted empirical cumulative Kullback-Leibler information (based on Theorem 3.1), considering a sample of size $n = 1500$ for random variable X and $m = 1000$ for random variable Y. The process was repeated 1000 times. The mean squared errors (MSEs) between average weighted empirical cumulative Kullback-Leibler information and its theoretical correspondent are also presented in Table 3.1.

Table 3.1

Weighted and empirical weighted cumulative Kullback-Leibler information

Distributions of X and Y	Weight function $w(x)$	$C_{KL}^w(X,Y)$	Average $C_{KL}^w(\hat{F}_n, \hat{G}_m)$	Average MSE
X: inverse Weibull ($\theta=1, \tau=4$) Y: inverse Weibull ($\theta=0.5, \tau=4$)	$w(x)=1$	0.3255	0.3257	0.000060
	$w(x)=x$	0.2493	0.2494	0.000046
X: Power ($\alpha=6$) Y: Power ($\beta=2$)	$w(x)=1$	0.1088	0.1091	0.000046
	$w(x)=x$	0.0625	0.0627	0.000015
	$w(x)=1-x$	0.0463	0.0465	0.000009
X: exponential ($\lambda=1$) Y: exponential ($\lambda=2$)	$w(x)=1$	0.0638	0.0641	0.000061
	$w(x)=x$	0.0572	0.0574	0.000050
	$w(x)=1-\mathrm{e}^{-x}$	0.0334	0.0336	0.000017

4. TSALLIS EMPIRICAL CUMULATIVE KULLBACK-LEIBLER INFORMATION

The Tsallis extended logarithm is defined as follows for $x \in \mathrm{R}_+^*$ and $q \in \mathrm{R}$

$$\ln_q^t(x) = \begin{cases} \ln x & \text{, if } x>0 \text{ and } q=1 \\ \dfrac{x^{1-q}-1}{1-q} & \text{, if } x>0 \text{ and } q \neq 1 \end{cases}. \tag{17}$$

Remark 4.1. The following property of Tsallis extended logarithm will be used for proving some of the results in Sections 4 and 5:

$$\ln_q^t\left(\frac{x}{y}\right)=\ln_q^t(x)-x^{1-q}\cdot\ln_{2-q}^t(y). \tag{18}$$

Definition 4.1. The Tsallis empirical cumulative Kullback-Leibler information of the random variables X and Y is defined as follows:

$$C_{KL}^{t,q}\left(\hat{F}_n,\hat{G}_m\right)=\int_0^\infty \hat{F}_n(x)\cdot\ln_q^t\left(\frac{\hat{F}_n(x)}{\hat{G}_m(x)}\right)\mathrm{d}x+\overline{X}_n-\overline{Y}_m. \tag{19}$$

Remark 4.2. The Tsallis empirical cumulative Kullback-Leibler information generalizes the empirical cumulative Kullback-Leibler information (obtained from the former for q=1).

THEOREM 4.1. *The following relation holds for Tsallis empirical cumulative Kullback-Leibler information of the random variables X and Y:*

$$\begin{aligned}&C_{KL}^{t,q}\left(\hat{F}_n,\hat{G}_m\right)=\\&=\frac{1}{n^{2-q}}\sum_{j=1}^{m-1}\ln_{2-q}^t\left(\frac{j}{m}\right)\left[\sum_{r=1}^{N_{j+1}-N_j}\left(\left(N_j+r\right)^{2-q}-\left(N_j+r-1\right)^{2-q}\right)\cdot X_{j,r}+N_j^{2-q}\cdot Y_{(j)}-N_{j+1}^{2-q}\cdot Y_{(j+1)}\right]+\\&+\sum_{i=1}^{n-1}\frac{i}{n}\cdot\ln_q^t\left(\frac{i}{n}\right)\cdot\Delta X_{(i)}+\overline{X}_n-\overline{Y}_m.\end{aligned} \tag{20}$$

***Proof*.** Using Remark 4.1 we get:

$$C_{KL}^{t,q}\left(\hat{F}_n,\hat{G}_m\right)=-\int_0^\infty\left(\hat{F}_n(x)\right)^{2-q}\cdot\ln_{2-q}^t\left(\hat{G}_m(x)\right)\mathrm{d}x+\int_0^\infty\hat{F}_n(x)\cdot\ln_q^t\left(\hat{F}_n(x)\right)\mathrm{d}x+\overline{X}_n-\overline{Y}_m. \tag{21}$$

Since

$$\begin{aligned}&\int_0^\infty\left(\hat{F}_n(x)\right)^{2-q}\cdot\ln_{2-q}^t\left(\hat{G}_m(x)\right)\mathrm{d}x=\\&=\sum_{j=1}^{m-1}\left[\ln_{2-q}^t\left(\frac{j}{m}\right)\cdot\int_{Y_{(j)}}^{Y_{(j+1)}}\left(\hat{F}_n(x)\right)^{2-q}\mathrm{d}x\right]=\\&=-\frac{1}{n^{2-q}}\sum_{j=1}^{m-1}\ln_{2-q}^t\left(\frac{j}{m}\right)\left[\sum_{r=1}^{N_{j+1}-N_j}\left(\left(N_j+r\right)^{2-q}-\left(N_j+r-1\right)^{2-q}\right)\cdot X_{j,r}+N_j^{2-q}\cdot Y_{(j)}-N_{j+1}^{2-q}\cdot Y_{(j+1)}\right]\end{aligned} \tag{22}$$

and

$$\int_0^\infty\hat{F}_n(x)\cdot\ln_q^t\left(\hat{F}_n(x)\right)\mathrm{d}x=\sum_{i=1}^{n-1}\left[\int_{X_{(i)}}^{X_{(i+1)}}\frac{i}{n}\cdot\ln_q^t\left(\frac{i}{n}\right)\mathrm{d}x\right]=\sum_{i=1}^{n-1}\frac{i}{n}\cdot\ln_q^t\left(\frac{i}{n}\right)\cdot\Delta X_{(i)}, \tag{23}$$

relation (20) is obtained by taking into account (22) and (23) in (21).

Remark 4.3. The Tsallis extended logarithm versions of empirical cumulative entropy and empirical cumulative inaccuracy are defined as follows:

– Tsallis empirical cumulative entropy of random variable X

$$CE^{t,q}\left(\hat{F}_n\right)=-\int_0^{\infty}\hat{F}_n(x)\cdot\ln_q^t\left(\hat{F}_n(x)\right)\mathrm{d}x \tag{24}$$

and respectively,

– Tsallis empirical cumulative inaccuracy of random variables X and Y

$$K^{t,q}\left(\hat{F}_n,\hat{G}_m\right)=-\int_0^{\infty}\left(\hat{F}_n(x)\right)^{2-q}\cdot\ln_{2-q}^t\left(\hat{G}_m(x)\right)\mathrm{d}x\,. \tag{25}$$

From (21), (24) and (25), we get

$$C_{KL}^{t,q}\left(\hat{F}_n,\hat{G}_m\right)=K^{t,q}\left(\hat{F}_n,\hat{G}_m\right)-CE^{t,q}\left(\hat{F}_n\right)+\overline{X}_n-\overline{Y}_m\,. \tag{26}$$

The following definition generalizes the cumulative Kullback-Leibler information of random variables X and Y, cumulative inaccuracy and cumulative entropy.

Definition 4.2. Let X and Y be random variables with finite expectations and with $l=l_X=l_Y$.

The Tsallis cumulative Kullback-Leibler information of X and Y is given by

$$C_{KL}^{t,q}(X,Y)=\int_l^{\max\{r_X,r_Y\}}F(x)\cdot\ln_q^t\left(\frac{F(x)}{G(x)}\right)\mathrm{d}x+E(X)-E(Y)\,. \tag{27}$$

Tsallis cumulative inaccuracy of random variables X and Y is

$$K^{t,q}(X,Y)=-\int_l^{\max\{r_X,r_Y\}}\left(F(x)\right)^{2-q}\ln_{2-q}^t\left(G(x)\right)\mathrm{d}x\,, \tag{28}$$

provided that the integral is finite.

Similarly, Tsallis cumulative entropy is defined as

$$CE^{t,q}(X)=-\int_0^{\infty}F(x)\cdot\ln_q^t\left(F(x)\right)\mathrm{d}x\,. \tag{29}$$

Remark 4.4. Using the property of Tsallis extended logarithm presented in Remark 4.1, one can derive

$$C_{KL}^{t,q}(X,Y)=K^{t,q}(X,Y)-CE^{t,q}(X)+E(X)-E(Y)\,. \tag{30}$$

Numerical application of information measures discussed in this section (Tsallis cumulative Kullback-Leibler information and its empirical correspondent) will be presented at the end of Section 5, together with their weighted versions.

5. TSALLIS WEIGHTED EMPIRICAL CUMULATIVE KULLBACK-LEIBLER INFORMATION

In this section, the measures and results from Section 4 will be extended by considering the weighted cases. In the same time, Section 5 extends the results of Section 3 by considering Tsallis extended logarithm.

Definition 5.1. The Tsallis weighted empirical cumulative Kullback-Leibler information of the random variables X and Y is defined as follows:

$$C_{KL}^{w,t,q}\left(\hat{F}_n,\hat{G}_m\right)=\int_0^{\infty}w(x)\left[\hat{F}_n(x)\cdot\ln_q^t\left(\frac{\hat{F}_n(x)}{\hat{G}_m(x)}\right)-\hat{F}_n(x)+\hat{G}_m(x)\right]\mathrm{d}x\,. \tag{31}$$

Moreover, we define Tsallis weighted empirical cumulative inaccuracy as

$$K^{w,t,q}\left(\hat{F}_n,\hat{G}_m\right)=-\int_0^{\infty} w(x)\left(\hat{F}_n(x)\right)^{2-q}\ln_{2-q}^t\left(\hat{G}_m(x)\right)\mathrm{d}x \quad (32)$$

and Tsallis weighted empirical cumulative entropy

$$CE^{w,t,q}\left(\hat{F}_n\right)=-\int_0^{\infty} w(x)\cdot\hat{F}_n(x)\cdot\ln_q^t\left(\hat{F}_n(x)\right)\mathrm{d}x\,. \quad (33)$$

Remark 5.1. Straightforward calculations lead to

$$C_{KL}^{w,t,q}\left(\hat{F}_n,\hat{G}_m\right)=K^{w,t,q}\left(\hat{F}_n,\hat{G}_m\right)-CE^{w,t,q}\left(\hat{F}_n\right)+\overline{W(X)}_n-\overline{W(Y)}_m\,. \quad (34)$$

THEOREM 5.1. *The following result holds true for Tsallis weighted empirical cumulative Kullback-Leibler information of the random variables X and Y:*

$$\begin{aligned}&C_{KL}^{w,t,q}\left(\hat{F}_n,\hat{G}_m\right)=\\ &=\frac{1}{n^{2-q}}\sum_{j=1}^{m-1}\ln_{2-q}^t\left(\frac{j}{m}\right)\left[\sum_{r=1}^{N_{j+1}-N_j}\left(\left(N_j+r\right)^{2-q}-\left(N_j+r-1\right)^{2-q}\right)\cdot W\left(X_{j,r}\right)+N_j^{2-q}\cdot W\left(Y_{(j)}\right)-N_{j+1}^{2-q}\cdot W\left(Y_{(j+1)}\right)\right]+\\ &+\sum_{i=1}^{n-1}\frac{i}{n}\cdot\ln_q^t\left(\frac{i}{n}\right)\cdot\Delta W\left(X_{(i)}\right)+\overline{W(X)}_n-\overline{W(Y)}_m\,.\end{aligned} \quad (35)$$

Proof. Applying similar arguments used for proving Theorems 3.1 and 4.1, we get relation (35).

The extended (weighted) versions of Tsallis cumulative Kullback-Leibler information, Tsallis cumulative inaccuracy and Tsallis cumulative entropy are defined in what it follows.

Definition 5.2. Let X and Y be random variables having the same left-hand points $l=l_X=l_Y$ and $E(W(X))$ and $E(W(Y))$ finite.

The Tsallis weighted cumulative Kullback-Leibler information of X and Y is given by

$$C_{KL}^{w,t,q}(X,Y)=\int_l^{\max\{r_X,r_Y\}} w(x)\cdot\left[F(x)\cdot\ln_q^t\left(\frac{F(x)}{G(x)}\right)-F(x)+G(x)\right]\mathrm{d}x\,. \quad (36)$$

The Tsallis weighted cumulative inaccuracy of random variables X and Y is

$$K^{w,t,q}(X,Y)=-\int_l^{\max\{r_X,r_Y\}} w(x)\left(F(x)\right)^{2-q}\cdot\ln_{2-q}^t\left(G(x)\right)\mathrm{d}x\,, \quad (37)$$

provided that the integral is finite, and Tsallis weighted cumulative entropy is defined as

$$CE^{w,t,q}(X)=-\int_0^{\infty} w(x)F(x)\cdot\ln_q^t\left(F(x)\right)\mathrm{d}x\,. \quad (38)$$

Remark 5.2. Using relation (18), it can be shown that

$$C_{KL}^{w,t,q}(X,Y)=K^{w,t,q}(X,Y)-CE^{w,t,q}(X)+E(W(X))-E(W(Y)). \quad (39)$$

Numerical application. As in numerical example from Section 3, we take X and Y two continuous, nonnegative, random variables with different distributions, for which we evaluate the Tsallis weighted and non-weighted cumulative Kullback-Leibler information and their empirical versions. For the empirical information, we considered a sample of size $n=1500$ for random variable X and $m=1000$ for random variable Y and repeated the calculation process 1000 times. The results are presented in Table 5.1 (q = 0.5 in case of Power distribution and q = 1.5 for exponential distribution).

Table 5.1

Tsallis weighted and empirical weighted cumulative Kullback-Leibler information

Distributions of *X* and *Y*	Weight function *w*(*x*)	$C_{KL}^{w,t,q}(X,Y)$	Average $C_{KL}^{w,t,q}(\hat{F}_n,\hat{G}_m)$	AverageMSE
X: Power ($\alpha = 6$) Y: Power ($\beta = 4$)	$w(x)=1$	0.0214	0.0215	0.000014
	$w(x)=x$	0.0139	0.0139	0.000006
	$w(x) = 1 - e^{-x}$	0.0101	0.0102	0.000003

Table 5.1 (continued)

X: exponential ($\lambda = 1$) Y: exponential ($\lambda = 2$)	$w(x)=1$	0.0902	0.0905	0.000114
	$w(x)=x$	0.0822	0.0821	0.000095
	$w(x) = 1 - e^{-x}$	0.0477	0.0478	0.000031

6. CONCLUSIONS

In this paper we proposed some generalized versions of the cumulative Kullback Leibler information. Based on the work of A. di Crescenzo and M. Longobardi [1] we defined the weighted cumulative Kullback-Leibler information, the Tsallis cumulative Kullback-Leibler information and the Tsallis weighted cumulative Kullback-Leibler information and their empirical versions.

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