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Stable cylindrical Lévy processes and the stochastic Cauchy problem

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Abstract

In this work, we consider the stochastic Cauchy problem driven by the canonical α-stable cylindrical Lévy process. This noise naturally generalises the cylindrical Brownian motion or space-time Gaussian white noise. We derive a sufficient and necessary condition for the existence of the weak and mild solution of the stochastic Cauchy problem and establish the temporal irregularity of the solution.

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1 Introduction

One of the most fundamental stochastic partial differential equations is a linear evolution equation perturbed by an additive noise of the form

$$dX(t) = AX(t)\,dt + dL(t) \quad \text{ for } t \in [0,T], \tag{1.1}$$

where A is the generator of a strongly continuous semigroup $(T(t))_{t\ge 0}$ on a Hilbert space U. If L is the standard cylindrical Brownian motion in U then there exists a weak, or equivalently mild, solution in U of (1.1) if and only if

$$\int_0^T \|T(s)\|_{\mathrm{HS}}^2 \, ds < \infty, \tag{1.2}$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm; see [17, Th.7.1]. For the example of the stochastic heat equation, in which A is chosen as the Laplace operator Δ, this result implies that there exists a mild solution if and only if the spatial dimension equals one. A natural next generalisation step is to replace the cylindrical Brownian motion by an α-stable noise. Like the cylindrical Brownian motion this noise does not exist as a genuine stochastic process in an infinite-dimensional space.

In the present article, we consider equation (1.1) driven by an α-stable cylindrical noise and a generator A allowing a spectral decomposition. For this purpose, we introduce the canonical α-stable cylindrical Lévy process as a natural generalisation of the cylindrical Brownian motion. By evoking the recently introduced approach to

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stochastic integration for deterministic integrands with respect to cylindrical Lévy processes in [14], we derive that there exists a mild (or weak) solution in U if and only if

$$\int_0^T \|T(s)\|_{\mathrm{HS}}^{\alpha}\, ds < \infty. \tag{1.3}$$

Obviously, this result "smoothly" extends the equivalent condition (1.2) in the Gaussian setting. We demonstrate that in the case of the stochastic heat equation this result leads to the sufficient and necessary condition

$$\alpha d < 4 \tag{1.4}$$

for the existence of a mild solution, where d denotes the spatial dimension. As already observed in other examples of cylindrical Lévy processes as driving noise, we finish this note by establishing that the solution has highly irregular paths.

Equation (1.1) in Banach spaces with an α-stable noise, or even slightly more general with a subordinated cylindrical Brownian motion, has already been considered by Brzeźniak and Zabczyk in [5]. However, their approach is based on embedding the underlying Hilbert space U in a larger space such that the cylindrical noise becomes a genuine Lévy process. This leads to the fact that their condition for the existence of a solution not only lacks the necessity but also is in terms of the larger Hilbert space, which per se is not related to equation (1.1). Moreover, they only show that the paths of the solution are irregular in the sense that there does not exist a modification of the solution with càdlàg paths in U.

Another approach to generalise the model of the driving noise is based on the Gaussian space-time white noise leading to a *Lévy space-time white noise*; see Albeverio et al. [1] or Applebaum and Wu [3]. In this framework, equation (1.1) (and even with a multiplicative noise) driven by an α-stable Lévy white noise is considered in the work [4] by Balan. We show that the α-stable Lévy white noise in [4] corresponds to a canonical α-stable cylindrical Lévy process in the same way as it is known in the Gaussian setting (see [7, Th.3.2.4]). However, and very much in contrast to the Gaussian setting, it turns out that the corresponding cylindrical Lévy process is defined on a Banach space different from the underlying Hilbert space U. This leads to the new phenomena that the necessary and sufficient condition for the existence of a solution of the heat equation in the space-time white noise approach differs from our condition (1.4) in the cylindrical approach.

2 The canonical α-stable cylindrical Lévy process

Let U be a separable Banach space with dual U^*. The dual pairing is denoted by $\langle u, u^*\rangle$ for $u \in U$ and $u^* \in U^*$. For any $u_1^*, \dots, u_n^* \in U^*$ we define the projection

$$\pi_{u_1^*,\dots,u_n^*} \colon U \to \mathbb{R}^n, \qquad \pi_{u_1^*,\dots,u_n^*}(u) = \big(\langle u, u_1^*\rangle, \dots, \langle u, u_n^*\rangle\big).$$

The Borel σ-algebra in U is denoted by $\mathfrak{B}(U)$. For a subset Γ of U^*, sets of the form

$$C(u_1^*, \dots, u_n^*; B) := \pi^{-1}_{u_1^*,\dots,u_n^*}(B),$$

with $u_1^*, \dots, u_n^* \in \Gamma$ and $B \in \mathfrak{B}(\mathbb{R}^n)$ are called *cylindrical sets with respect to* Γ. The set of all these cylindrical sets is denoted by $\mathcal{Z}(U, \Gamma)$; it is a σ-algebra if Γ is finite and it is an algebra otherwise. If $\Gamma = U^*$ we write $\mathcal{Z}(U) := \mathcal{Z}(U, U^*)$. A function $\mu : \mathcal{Z}(U) \to [0, \infty]$ is called a *cylindrical measure on* $\mathcal{Z}(U)$, if for each finite subset $\Gamma \subseteq U^*$ the restriction of μ to the σ-algebra $\mathcal{Z}(U, \Gamma)$ is a measure. A cylindrical measure is called finite if

$\mu(U) < \infty$ and a cylindrical probability measure if $\mu(U) = 1$. The characteristic function $\varphi_\mu : U^* \to \mathbb{C}$ of a finite cylindrical measure μ is defined by

$$\varphi_\mu(u^*) := \int_U e^{i\langle u,u^*\rangle}\,\mu(du) \qquad \text{for all } u^* \in U^*.$$

Let $(\Omega, \mathcal{A}, P)$ be a probability space. The space of equivalence classes of measurable functions $f\colon \Omega \to U$ is denoted by $L^0_P(\Omega; U)$ and it is equipped with the topology of convergence in probability. A *cylindrical random variable in* U is a linear and continuous mapping

$$Z\colon U^* \to L^0_P(\Omega; \mathbb{R}).$$

The characteristic function of a cylindrical random variable Z is defined by

$$\varphi_Z\colon U^* \to \mathbb{C}, \qquad \varphi_Z(u^*) = E\left[e^{iZu^*}\right].$$

If $C = C(u_1^*, \dots, u_n^*; B)$ is a cylindrical set for $u_1^*, \dots, u_n^* \in U^*$ and $B \in \mathfrak{B}(\mathbb{R}^n)$ we obtain a cylindrical probability measure μ by the prescription

$$\mu(C) := P\big((Zu_1^*, \dots, Zu_n^*) \in B\big).$$

We call μ the *cylindrical distribution of* Z and the characteristic functions φ_μ and φ_Z of μ and Z coincide.

A family $(Z(t) : t \ge 0)$ of cylindrical random variables $Z(t)$ in U is called a *cylindrical process in* U. In our work [2] with Applebaum, we extended the concept of cylindrical Brownian motion to cylindrical Lévy processes:

Definition 2.1. *A cylindrical process* $(L(t) : t \ge 0)$ *in* U *is called a* cylindrical Lévy process *if for each* $n \in \mathbb{N}$ *and any* $u_1^*, \dots, u_n^* \in U$ *we have that*

$$\big((L(t)u_1^*, \dots, L(t)u_n^*) : t \ge 0\big)$$

is a Lévy process in $\mathbb{R}^n$*.*

The characteristic function of $L(t)$ for each $t \ge 0$ is of the form

$$\varphi_{L(t)}\colon U^* \to \mathbb{C}, \qquad \varphi_{L(t)}(u^*) = \exp\big(t\Psi(u^*)\big),$$

where $\Psi\colon U^* \to \mathbb{C}$ is called the *cylindrical symbol of* L and is of the form

$$\Psi(u^*) = ia(u^*) - \tfrac{1}{2}\langle Qu^*, u^*\rangle + \int_U \Big(e^{i\langle u,u^*\rangle} - 1 - i\langle u, u^*\rangle\, \mathbb{1}_{B_\mathbb{R}}(\langle u, u^*\rangle)\Big)\,\nu(du).$$

Here, $a\colon U^* \to \mathbb{R}$ is a continuous mapping with $a(0) = 0$, $Q\colon U^* \to U$ is a positive and symmetric operator and ν is a cylindrical measure on $\mathcal{Z}(U)$ satisfying

$$\int_U \Big(\langle u, u^*\rangle^2 \wedge 1\Big)\,\nu(du) < \infty \qquad \text{for all } u^* \in U^*.$$

The characteristic function of L is studied in detail in our work [13].

In this article we consider a specific example of a cylindrical Lévy process, which is obtained by the usual generalisation of the characteristic function of the standard normal distribution:

Definition 2.2. *A cylindrical Lévy process* $(L(t) : t \ge 0)$ *is called* canonical α-stable *for* $\alpha \in (0, 2)$ *if its characteristic function is of the form*

$$\varphi_{L(t)}\colon U^* \to \mathbb{C}, \qquad \varphi_{L(t)}(u^*) = \exp\Big(-t\,\|u^*\|^\alpha\Big).$$

Let μ be the cylindrical probability measure on $\mathcal{Z}(U)$ defined by the characteristic function

$$\varphi_\mu \colon U^* \to \mathbb{C}, \qquad \varphi_\mu(u^*) = \exp\Big(- \|u^*\|^\alpha \Big),$$

for some $\alpha \in (0,2)$. The cylindrical probability measure μ is called the *canonical α-stable cylindrical measure*. Bochner's theorem for cylindrical measures ([16, Prop.IV.4.2]) guarantees that μ exists. For, the function φ_μ satisfies $\varphi_\mu(0) = 1$ and is continuous and positive-definite ([16, p.194]). Two possible constructions of the canonical α-stable Lévy process such that its cylindrical distribution is given by μ are presented in Section 3.

The cylindrical probability measure μ is symmetric, i.e. it satisfies $\mu(C) = \mu(-C)$ for all $C \in \mathcal{Z}(U)$ and it is rotationally invariant, i.e. $\mu \circ M^{-1} = \mu$ for each linear unitary operator $M \colon U \to U$.

Remark 2.3. In [9] among other publications, a symmetric cylindrical measure ρ on $\mathcal{Z}(U)$ is called α-stable if there exists a measure space $(M, \mathcal{M}, \sigma)$ and a linear, continuous operator $T \colon U^* \to L^\alpha_\sigma(M, \mathcal{M})$ such that

$$\varphi_\rho(u^*) = \exp\Big(- \|Tu^*\|_{L^\alpha_\sigma(M,\mathcal{M})} \Big) \qquad \text{for all } u^* \in U^*. \tag{2.1}$$

For the standard α-stable cylindrical measure μ, each image measure $\mu \circ \pi_{u^*}^{-1}$ is a stable measure on $\mathfrak{B}(\mathbb{R})$ for each $u^* \in U^*$. Thus, Theorem 6.8.5 in [9] guarantees that μ also satisfies (2.1).

In the case of a separable Hilbert space we relate the characteristic function of the canonical α-stable Lévy process with the well-known spectral representation on the sphere $S(\mathbb{R}^n) := \{\beta \in \mathbb{R}^n : |\beta| = 1\}$ of symmetric, rotationally invariant measures in $\mathbb{R}^n$:

Lemma 2.4. *Let U be a Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and L be the canonical α-stable cylindrical Lévy process in U. Then the characteristics of L is given by $(0,0,\nu)$ with the cylindrical Lévy measure ν satisfying for all $n \in \mathbb{N}$:*

$$\nu \circ \pi^{-1}_{e_1,\dots,e_n}(B) = \tfrac{\alpha}{c_\alpha} \int_{S(\mathbb{R}^n)} \lambda_n(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) \frac{1}{r^{1+\alpha}}\, dr \qquad \textit{for } B \in \mathfrak{B}(\mathbb{R}^n),$$

where λ_n is uniformly distributed on the sphere $S(\mathbb{R}^n)$ with

$$\lambda_n\big(S(\mathbb{R}^n)\big) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n+\alpha}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{1+\alpha}{2})}$$

and the constant c_α is defined in Theorem A.1.

Proof. For each $n \in \mathbb{N}$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ we obtain

$$\begin{aligned} \varphi_{L(1)e_1,\dots,L(1)e_n}(\beta) &= \varphi_{L(1)}\big(\beta_1 e_1 + \dots + \beta_n e_n\big) \\ &= \exp\left(- \Big(\|\beta_1 e_1 + \dots + \beta_n e_n\|^2 \Big)^{\alpha/2} \right) = \exp\left(-|\beta|^\alpha\right). \end{aligned} \tag{2.2}$$

It follows that the distribution of the random vector $(L(1)e_1, \dots, L(1)e_n)$ is symmetric and rotationally invariant. As the Lévy measure of $(L(1)e_1, \dots, L(1)e_n)$ is given by $\nu \circ \pi^{-1}_{e_1,\dots,e_n}$, Theorem A.1 implies

$$\nu \circ \pi^{-1}_{e_1,\dots,e_n}(B) = \tfrac{\alpha}{c_\alpha} \int_{S(\mathbb{R}^n)} \lambda_n(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) \frac{1}{r^{1+\alpha}}\, dr,$$

where λ_n is uniformly distributed on the sphere $S(\mathbb{R}^n)$ and is defined by $\lambda_n(C) = c_\alpha(\nu\circ\pi^{-1}_{e_1,\dots,e_n})((1,\infty)C)$ for all $C\in\mathfrak{B}(S(\mathbb{R}^n))$. From Part (d) of Theorem A.1 and (2.2) we deduce that for each $\xi_0\in S(\mathbb{R}^n)$ we have

$$1=\int_{S(\mathbb{R}^n)}\left|\langle\xi_0,\xi\rangle\right|^\alpha\,\lambda_n(\xi)=r_n\int_{S(\mathbb{R}^n)}\left|\langle\xi_0,\xi\rangle\right|^\alpha\,\lambda_n^1(\xi), \tag{2.3}$$

where $\lambda_n^1:=\frac{1}{r_n}\lambda_n$ and $r_n:=\lambda_n(S(\mathbb{R}^n))$. Let $Y_n=(Y_{n,1},\dots,Y_{n,n})$ be uniformly distributed on $S(\mathbb{R}^n)$. By choosing $\xi_0=(1,0,\dots,0)$ in (2.3) and applying Lemma A.2 we obtain

$$1=r_nE[|Y_{n,1}|^\alpha]=\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+\alpha}{2})},$$

which completes the proof. □

3 Two representations

The first representation of the canonical α-stable cylindrical Lévy process is by subordination. For this purpose, let W be the standard cylindrical Brownian motion on the Banach space U, i.e. a cylindrical Lévy process with characteristics $(0,\mathrm{Id},0)$ where Id denotes the identity on U. Its characteristic function is given by

$$\varphi_{W(t)}\colon U^*\to\mathbb{C},\qquad \varphi_{W(t)}(u^*)=\exp\left(-\tfrac{1}{2}\left\|u^*\right\|^2\right).$$

Lemma 3.1. *Let W be the standard cylindrical Brownian motion on a separable Banach space U and let ℓ be an independent, real-valued $\alpha/2$-stable subordinator with Lévy measure $\nu_\ell(dy)=\frac{2^{\alpha/2}\alpha/2}{\Gamma(1-\alpha/2)}y^{-1-\alpha/2}\,dy$ for $\alpha\in(0,2)$. Then*

$$L(t)u^*:=W\big(\ell(t)\big)u^*\qquad\textit{for all } u^*\in U^*,$$

defines a canonical α-stable cylindrical Lévy process $(L(t):\,t\ge 0)$ in U.

Proof. By using independence of W and ℓ, Lemma 3.8 in [2] shows that L is a cylindrical Lévy process. The characteristic function of the subordinator ℓ can be analytically continued, such that

$$E[\exp(-\beta\ell(t))]=\exp(-t\tau(\beta))\qquad\text{for all }\beta>0,$$

where the Laplace exponent τ is given by

$$\tau(\beta)=\int_0^\infty(1-e^{-\beta s})\,\nu_\ell(ds)=\beta^{\alpha/2};$$

see [15, Th.24.11]. Independence of W and ℓ implies that for each $t\ge 0$ and $u^*\in U^*$ the characteristic function $\varphi_{L(t)}$ of $L(t)$ is given by

$$\begin{aligned}\varphi_{L(t)}(u^*)&=\int_0^\infty E\left[e^{iW(s)u^*}\right]\,P_{\ell(t)}(ds)\\&=\int_0^\infty e^{-\frac{1}{2}s\|u^*\|^2}\,P_{\ell(t)}(ds)=\exp\left(-t\tau\left(\tfrac{1}{2}\left\|u^*\right\|^2\right)\right)=\exp\big(-t\left\|u^*\right\|^\alpha\big),\end{aligned}$$

which shows that L is canonical α-stable. □

The second representation of the canonical α-stable cylindrical Lévy process is based on the approach by Lévy space-time white noise, as it is defined for example in [1] and [3].

Definition 3.2. *Let $(M, \mathcal{M}, \nu)$ be a σ-finite measure space. A Lévy white noise on a σ-finite measure space $(E, \mathcal{E}, \sigma)$ with intensity ν is a random measure $Y\colon \mathcal{E} \times \Omega \to \mathbb{R}$ of the form*

$$Y(B) = W(B) + \int_{B\times M} a(x,y)\, N(dx,dy) + \int_{B\times M} b(x,y)\,(\sigma\otimes\nu)(dx,dy),$$

where (1) $W\colon \mathcal{E}\times\Omega \to \mathbb{R}$ *is a Gaussian white noise;*

(2) $N\colon (\mathcal{E}\otimes\mathcal{M})\times\Omega \to \mathbb{N}_0 \cup \{\infty\}$ *is a Poisson random measure on $E\times M$ with intensity measure $\sigma\otimes\nu$;*

(3) $a, b\colon E\times U \to \mathbb{R}$ *are measurable functions.*

In the above Definition 3.2 we take $M = \mathbb{R}$, $\mathcal{M} = \mathfrak{B}(\mathbb{R})$ and

$$\nu(dy) = \frac{1}{c}\frac{1}{|y|^{\alpha+1}}\,dy \qquad \text{for } c := 2\Gamma(\alpha)\cos(\tfrac{\pi\alpha}{2}).$$

For some set $\mathcal{O} \subseteq \mathbb{R}^d$ let N be a Poisson random measure on $\big([0,\infty)\times\mathcal{O}\big)\times\mathbb{R}$ with intensity measure $\sigma\otimes\nu$ where $\sigma := dt\otimes dx$. We call the Lévy white noise $Y\colon\ \mathfrak{B}([0,\infty)\times\mathcal{O})\times\Omega \to [0,\infty)$ defined by

$$Y(B) = \begin{cases} \displaystyle\int_{B\times\mathbb{R}} y\, N(dt,dx,dy), & \text{if } \alpha < 1,\\ \displaystyle\int_{B\times\mathbb{R}} y\,\big(N(dt,dx,dy) - dt\,dx\,\nu(dy)\big), & \text{if } \alpha \ge 1, \end{cases}$$

the *canonical α-stable space-time Lévy white noise on $\mathcal{O}$.*

Lemma 3.3. *Let Y be the canonical α-stable Lévy space-time white noise on a set $\mathcal{O} \subseteq \mathbb{R}^d$ for $\alpha\in(1,2)$. Then there exists a canonical α-stable cylindrical Lévy process L in $L^{\alpha'}(\mathcal{O})$ for $\alpha' := \frac{\alpha}{\alpha-1}$ such that*

$$L(t)\,\mathbb{1}_A = Y([0,t]\times A) \qquad \textit{for all } t\ge 0 \textit{ and bounded sets } A\in\mathfrak{B}(\mathcal{O}).$$

Proof. Let $S(\mathcal{O})$ be the space of simple functions in $L^\alpha(\mathcal{O})$ of the form

$$u^* := \sum_{k=1}^n \beta_k\,\mathbb{1}_{A_k}, \tag{3.1}$$

for some $\beta_k\in\mathbb{R}_+$ and disjoint sets $A_k\in\mathfrak{B}(\mathcal{O})$. For $u^*\in S(\mathcal{O})$ and $t\ge 0$ we define

$$L(t)u^* := \sum_{k=1}^n \beta_k Y([0,t]\times A_k). \tag{3.2}$$

Define the function $m_\beta\colon\ \mathbb{R}\to\mathbb{R}$ by $m_\beta(x) = \beta x$ for some $\beta\in\mathbb{R}_+$. Then by using the invariance $\beta^{-\alpha}(\nu\circ m_\beta^{-1}) = \nu$, see [15, Th.14.3], the independence of the Lévy white noise and the symmetry of ν, we obtain

$$\begin{aligned}
&\varphi_{L(t)}(u^*)\\
&= \prod_{k=1}^n \exp\left(\int_{[0,t]\times A_k\times\mathbb{R}} \left(e^{i\beta_k y} - 1 - i\beta_k y\,\mathbb{1}_{B_\mathbb{R}}(y)\right)\,\nu(dy)\,dx\,ds\right)\\
&= \prod_{k=1}^n \exp\left(t\left(\int_\mathcal{O} \mathbb{1}_{A_k}(x)\,dx\right)\beta_k^\alpha \int_\mathbb{R}\left(e^{iy} - 1 - iy\,\mathbb{1}_{B_\mathbb{R}}(\beta_k^{-\alpha}y)\right)\,\beta_k^{-\alpha}(\nu\circ m_{\beta_k}^{-1})(dy)\right)\\
&= \prod_{k=1}^n \exp\left(t\left(\int_\mathcal{O} \beta_k^\alpha\,\mathbb{1}_{A_k}(x)\,dx\right)\int_\mathbb{R}\left(e^{iy} - 1 - iy\,\mathbb{1}_{B_\mathbb{R}}(y)\right)\,\nu(dy)\right)\\
&= \exp\left(t\left(\int_\mathcal{O} |u^*(x)|^\alpha\,dx\right)\int_\mathbb{R}\left(e^{iy} - 1 - iy\,\mathbb{1}_{B_\mathbb{R}}(y)\right)\,\nu(dy)\right).
\end{aligned}$$

By applying Lemma 14.11 in [15] we obtain $\varphi_{L(t)}(u^*) = \exp(-t\,\|u^*\|_{L^\alpha}^\alpha)$ for all simple functions $u^* \in S(\mathcal{O})$. By using the linearity of $L(t)$ we derive that $L(t)\colon S(\mathcal{O}) \to L_P^0(\Omega)$ is a linear and continuous operator which shows that L can be continued to a linear and continuous operator on $L^\alpha(\mathcal{O})$ satisfying

$$\varphi_{L(t)}(u^*) = \exp(-t\,\|u^*\|_{L^\alpha}^\alpha) \qquad \text{for all } u^* \in L^\alpha(\mathcal{O}).$$

Let $0 \le t_1 \le t_2 \le \cdots \le t_n$ and $u_j^* \in S(\mathcal{O})$. By independence of the Lévy white noise for disjoint sets it follows that

$$L(t_1)u_1^*, \big(L(t_2) - L(t_1)\big)u_2^* \ldots, \big(L(t_n) - L(t_{n-1})\big)u_n^*$$

are independent. By approximating an arbitrary function $u^* \in L^\alpha(\mathcal{O})$ by simple functions it follows that L has independent increments. Moreover, from the very definition in (3.2) it follows that for each $u^* \in S(\mathcal{O})$, the stochastic process $(L(t)u^* :\ t \ge 0)$ is a Lévy process in $\mathbb{R}$. Again, by approximating an arbitrary function $u^* \in L^\alpha(\mathcal{O})$ by simple functions, the same conclusion holds for $(L(t)u^* : t \ge 0)$ and $u^* \in L^\alpha(\mathcal{O})$. Together with the independent increments derived above, this implies by Corollary 3.8 in [2] that L is a cylindrical Lévy process in $L^{\alpha'}(\mathcal{O})$. ☐

Remark 3.4. The canonical α-stable space-time Lévy white noise corresponds to the noise considered in [4] in the symmetric case. Although the construction in Lemma 3.3 follows the corresponding relation between space-time Gaussian white noise and cylindrical Brownian motion, the resulting cylindrical Lévy process does not live in a Hilbert space, such as $L^2(\mathcal{O})$ for $\mathcal{O} \subseteq \mathbb{R}^d$.

4 The stochastic Cauchy problem

In this section we consider the stochastic Cauchy problem

$$\begin{aligned} dX(t) &= AX(t)\,dt + \,dL(t) \qquad \text{for } t \in [0,T], \\ X(0) &= x_0, \end{aligned} \tag{4.1}$$

where A is the generator of a strongly continuous semigroup $(T(t))_{t\ge 0}$ on a separable Hilbert space U and $x_0 \in U$ denotes the initial condition. The random noise L is a canonical α-stable cylindrical Lévy process as introduced in the previous section.

A theory of stochastic integration for deterministic functions with respect to cylindrical Lévy processes is introduced in our work [14]. Based on this theory, we obtain that if the stochastic convolution integral

$$Y(t) := \int_0^t T(t-s)\,L(s) \qquad \text{for } t \in [0,T],$$

exists, the stochastic process $(T(t)x_0 + Y(t) : t \ge 0)$ can be considered as a mild solution. By a result in [8] it also follows that the mild solution is a weak solution; however, this is of less concern in this work.

More precisely, if L is a cylindrical Lévy processes with characteristics $(0,0,\nu)$, then a function $f\colon [0,T] \to L(U,U)$ is stochastically integrable if and only if

$$\lim_{m\to\infty}\sup_{n\ge m} \int_0^T \int_U \left(\sum_{k=m}^n \langle u, f^*(s)e_k\rangle^2 \wedge 1 \right) \nu(du)\,ds = 0, \tag{4.2}$$

for an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of U; see [14, Th.5.10]. In our case of a canonical α-stable cylindrical Lévy process as driving noise we obtain the following equivalent conditions:

Theorem 4.1. *Assume that there exist an orthonormal basis* $(e_k)_{k\in\mathbb{N}}$ *of* U *and* $(\lambda_k)_{k\in\mathbb{N}}\subseteq[0,\infty)$ *with* $T^*(t)e_k=e^{-\lambda_k t}e_k$ *for all* $t\geq 0$ *and* $k\in\mathbb{N}$*. Then there exists a mild (and weak) solution of* (4.1) *if and only if*

$$\int_0^T \|T(s)\|_{\mathrm{HS}}^\alpha \, ds<\infty.$$

Proof. Lemma 2.4 implies for each $m,n\in\mathbb{N}$ with $m\leq n$ that

$$\begin{aligned}
&\int_0^T \int_U \left(\sum_{k=m}^n \langle u, T^*(s)e_k\rangle^2\wedge 1\right)\nu(du)\,ds\\
&\quad=\int_0^T \int_U \left(\sum_{k=m}^n e^{-2\lambda_k s}\langle u, e_k\rangle^2\wedge 1\right)\nu(du)\,ds\\
&\quad=\int_0^T \int_{\mathbb{R}^{n-m+1}} \left(\sum_{k=m}^n e^{-2\lambda_k s}\beta_k^2\wedge 1\right)\nu\circ\pi_{e_m,\dots,e_n}^{-1}(d\beta)\,ds\\
&\quad=\tfrac{\alpha}{c_\alpha}\int_0^T \int_{S(\mathbb{R}^{n-m+1})}\int_0^\infty \left(\sum_{k=m}^n e^{-2\lambda_k s}r^2\xi_k^2\wedge 1\right)\frac{1}{r^{1+\alpha}}\,dr\,\lambda_{n-m+1}(d\xi)\,ds\\
&\quad=\tfrac{2}{c_\alpha(2-\alpha)}\int_0^T \int_{S(\mathbb{R}^{n-m+1})} \left(\sum_{k=m}^n e^{-2\lambda_k s}\xi_k^2\right)^{\alpha/2}\lambda_{n-m+1}(d\xi)\,ds\\
&\quad=: I_{m,n}.
\end{aligned}$$

In the following, we establish that for all m, $n\in\mathbb{N}$ we have

$$\tfrac{2}{c_\alpha(2-\alpha)}\int_0^T \left(\sum_{k=m}^n \|T^*(s)e_k\|^2\right)^{\alpha/2} ds\leq I_{m,n}\leq c_{n-m}\int_0^T \left(\sum_{k=m}^n \|T^*(s)e_k\|^2\right)^{\alpha/2} ds, \tag{4.3}$$

where $c_{n-m}\to 1$ as $m,n\to\infty$.

Define $\lambda_{n-m+1}^1:=\frac{1}{r_{n-m+1}}\lambda_{n-m+1}$ where $r_{n-m+1}:=\lambda_{n-m+1}\big(S(\mathbb{R}^{n-m+1})\big)$. By applying Jensen's inequality to the concave function $\beta\mapsto\beta^{\alpha/2}$ it follows from Lemma A.2 that

$$\begin{aligned}
&\int_0^T \int_{S(\mathbb{R}^{n-m+1})} \left(\sum_{k=m}^n e^{-2\lambda_k s}\xi_k^2\right)^{\alpha/2}\lambda_{n-m+1}(d\xi)\,ds\\
&\quad\leq r_{n-m+1}\int_0^T \left(\sum_{k=m}^n e^{-2\lambda_k s}\int_{S(\mathbb{R}^{n-m+1})}\xi_k^2\,\lambda_{n-m+1}^1(d\xi)\right)^{\alpha/2} ds\\
&\quad= r_{n-m+1}\int_0^T \left(\frac{1}{n-m+1}\sum_{k=m}^n e^{-2\lambda_k s}\right)^{\alpha/2} ds\\
&\quad= \frac{r_{n-m+1}}{(n-m+1)^{\alpha/2}}\int_0^T \left(\sum_{k=m}^n \|T^*(s)e_k\|^2\right)^{\alpha/2} ds.
\end{aligned}$$

Consequently, we obtain the upper bound in (4.3) with $c_{n-m}:=\frac{r_{n-m+1}}{(n-m+1)^{\alpha/2}}$. Since $\frac{\Gamma(x+\beta)}{\Gamma(x)x^\beta}\to 1$ as $x\to\infty$, Lemma 2.4 implies $c_{n-m}\to 1$ as $m,n\to\infty$.

For establishing the lower bound, define $c_{m,n}(s):=e^{-2\lambda_m s}+\dots+e^{-2\lambda_n s}$. We again apply Jensen's inequality to the same concave function $\beta\mapsto\beta^{\alpha/2}$ but with respect to the discrete probability measure $\{c_{m,n}^{-1}(s)e^{-2\lambda_m s},\dots,c_{m,n}^{-1}(s)e^{-2\lambda_n s}\}$. In this way, by applying

Lemma 2.4 and A.2 we deduce

$$\begin{aligned}
&\int_0^T \int_{S(\mathbb{R}^{n-m+1})} \left(\sum_{k=m}^n e^{-2\lambda_k s} \xi_k^2 \right)^{\alpha/2} \lambda_{n-m+1}(d\xi)\, ds \\
&\geq \int_0^T \int_{S(\mathbb{R}^{n-m+1})} (c_{m,n}(s))^{\alpha/2} \sum_{k=m}^n \frac{e^{-2\lambda_k s}}{c_{m,n}(s)} \xi_k^\alpha \, \lambda_{n-m+1}(d\xi)\, ds \\
&= r_{n-m+1} \int_0^T (c_{m,n}(s))^{\alpha/2} \sum_{k=m}^n \frac{e^{-2\lambda_k s}}{c_{m,n}(s)} \int_{S(\mathbb{R}^{n-m+1})} \xi_k^\alpha \, \lambda_{n-m+1}^1(d\xi)\, ds \\
&= r_{n-m+1} \frac{\Gamma(\frac{1+\alpha}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+\alpha}{2})} \int_0^T (c_{m,n}(s))^{\alpha/2}\, ds \\
&= \int_0^T \left(\sum_{k=m}^n \|T^*(s)e_k\|^2 \right)^{\alpha/2} ds.
\end{aligned}$$

An application of [14, Th.5.10], as summarised in (4.2), completes the proof. □

Example 4.2. We consider the stochastic heat equation on a bounded domain $\mathcal{O} \subseteq \mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$ for some $d \in \mathbb{N}$. In this case, the generator A is given by the Laplace operator Δ on $U := L^2(\mathcal{O})$. Thus, there exists an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of U consisting of eigenvectors of A. According to Weyl's law, the eigenvalues λ_k satisfy $\lambda_k \sim ck^{2/d}$ for $k \to \infty$ and a constant $c > 0$. Consequently, we can assume that $\lambda_k = c_k k^{2/d}$ for constants c_k with $c_k \in [a, b]$ for all $k \in \mathbb{N}$ and $0 < a < b$.

By the integral test for convergence of series we obtain for each $s > 0$ that

$$\|T(s)\|_{\mathrm{HS}}^2 = \sum_{k=1}^\infty e^{-2sc_k k^{d/2}} \leq \int_0^\infty e^{-2sax^{d/2}}\, dx = \frac{2\Gamma(\frac{d}{2})}{d(2a)^{d/2} s^{d/2}}.$$

Analogously, we conclude for each $s > 0$ that

$$\|T(s)\|_{\mathrm{HS}}^2 = \sum_{k=1}^\infty e^{-2sc_k k^{d/2}} \geq -1 + \int_0^\infty e^{-2sbx^{d/2}}\, dx = -1 + \frac{2\Gamma(\frac{d}{2})}{d(2b)^{d/2} s^{d/2}}.$$

Consequently, we can deduce from Theorem 4.1 that there exists a mild solution of (4.1) for $A = -\Delta$ if and only if $\alpha d < 4$. In accordance with other works, e.g. [4] or [11], the smaller the stable index α is the larger dimensions d can be chosen in the condition for guaranteeing the existence of a weak solution. This is due to the smoother trajectories of stable processes for smaller stable index α.

Note that the sufficiency of the condition $\alpha d < 4$ can also be derived from results in [5]. Due to the approach of embedding the cylindrical Lévy process L in a larger space in [5], the derivation is less direct than here; see Corollary 6.5 in [14]. However, the work [5] provides further results on the spatial regularity of the weak solution; see e.g. Theorem 5.14.

5 Irregularities of the trajectories

In the work [5] it was observed that the solution of (4.1) does not have a modification with càdlàg paths in the underlying Hilbert space U. We strengthen this result that the solution does not even have a càdlàg modification in a much weaker sense:

Theorem 5.1. *Assume that there exist an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of U and $(\lambda_k)_{k\in\mathbb{N}} \subseteq [0, \infty)$ with $T^*(t)e_k = e^{-\lambda_k t} e_k$ for all $t \geq 0$ and $k \in \mathbb{N}$. Then there does not exists a modification $\widetilde{X}$ of the mild solution of* (4.1) *such that for each $u^* \in U^*$ the stochastic process $(\langle \widetilde{X}(t), u^* \rangle : t \in [0, T])$ has càdlàg paths.*

Proof. (The proof is based on ideas from [10]). For every $n \in \mathbb{N}$ and $t \in [0,T]$ define $L_n(t) := \big(L(t)e_1, \dots, L(t)e_n\big)$ and $X_n(t) := \big(\langle X(t), e_1\rangle, \dots, \langle X(t), e_n\rangle\big)$. As

$$\langle X(t), e_k\rangle = \langle x_0, e_k\rangle e^{-\lambda_k t} + \int_0^t e^{-\lambda_k (t-s)}\, d(L(s)e_k),$$

it follows that X_n is the solution of the stochastic differential equation

$$dX(t) = -DX(t)\,dt + dL_n(t) \qquad \text{for } t \geq 0,$$

where $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix with entries $\lambda_1, \dots, \lambda_n$. We conclude that the n-dimensional processes $(X_n(t) : t \in [0,T])$ and $(L_n(t) : t \in [0,T])$ jump at the same time by the same magnitude, which implies

$$\sup_{t\in[0,T]} |\Delta L_n(t)|^2 = \sup_{t\in[0,T]} |\Delta X_n(t)|^2 \leq 4 \sup_{t\in[0,T]} |X_n(t)|^2,$$

where $\Delta f(t) := f(t) - f(t-)$ for càdlàg functions $f\colon [0,T] \to \mathbb{R}^n$. It follows that

$$\begin{aligned}
P\left(\sup_{t\in[0,T]} \sum_{k=1}^{\infty} \langle X(t), e_k\rangle^2 < \infty\right) &= P\left(\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \sum_{k=1}^{n} \langle X(t), e_k\rangle^2 < \infty\right) \\
&= \lim_{c\to\infty} P\left(\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \sum_{k=1}^{n} \langle X(t), e_k\rangle^2 \leq \frac{1}{4}c^2\right) \\
&= \lim_{c\to\infty} \lim_{n\to\infty} P\left(\sup_{t\in[0,T]} \sum_{k=1}^{n} \langle X(t), e_k\rangle^2 \leq \frac{1}{4}c^2\right) \\
&= \lim_{c\to\infty} \lim_{n\to\infty} P\left(\sup_{t\in[0,T]} |X_n(t)|^2 \leq \frac{1}{4}c^2\right) \\
&\leq \lim_{c\to\infty} \lim_{n\to\infty} P\left(\sup_{t\in[0,T]} |\Delta L_n(t)|^2 \leq c^2\right) \\
&= \lim_{c\to\infty} \lim_{n\to\infty} \exp\Big(-T\nu_n\Big(\big\{\beta \in \mathbb{R}^n : |\beta| > c\big\}\Big)\Big), \qquad (5.1)
\end{aligned}$$

where ν_n denotes the Lévy measure of the $\mathbb{R}^n$-valued Lévy process L_n. Since $\nu_n = \nu \circ \pi^{-1}_{e_1,\dots,e_n}$ due to [2, Th.2.4], we obtain for every $n \in \mathbb{N}$ by Lemma 2.4 that

$$\begin{aligned}
&\big(\nu \circ \pi^{-1}_{e_1,\dots,e_n}\big)\big(\{\beta \in \mathbb{R}^n : |\beta| \geq c\}\big) \\
&\quad = \frac{\alpha}{c_\alpha} \int_{S(\mathbb{R}^n)} \int_0^\infty \mathbb{1}_{\{\beta\in\mathbb{R}^n : |\beta|\geq c\}}(r\xi) \frac{1}{r^{1+\alpha}}\, dr\, \lambda_n(d\xi) \\
&\quad = \frac{\alpha}{c_\alpha} \int_{S(\mathbb{R}^n)} \int_c^\infty \frac{1}{r^{1+\alpha}}\, dr\, \lambda_n(d\xi) \\
&\quad = \frac{1}{c_\alpha c^\alpha} \lambda_n(S(\mathbb{R}^n)) \\
&\quad = \frac{1}{c_\alpha c^\alpha} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n+\alpha}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{1+\alpha}{2})}.
\end{aligned}$$

Since $\frac{\Gamma(m+\beta)}{\Gamma(m)m^\beta} \to 1$ as $m \to \infty$ for all $\beta \in \mathbb{R}$, we obtain from (5.1) that

$$P\left(\sup_{t\in[0,T]} \sum_{k=1}^{\infty} \langle X(t), e_k\rangle^2 < \infty\right) = 0.$$

An application of Theorem 2.3 in [12] completes the proof. $\square$

A Appendix

For the present work it is essential to know which constants in the various presentations of the characteristic function of a multi-dimensional α-stable distribution depend on the dimension. Thus, we present the following well-known theorem with the constants given explicitly:

Theorem A.1. *Let μ be an infinitely divisible probability measure on $\mathfrak{B}(\mathbb{R}^n)$ with characteristics $(0,0,\nu)$ and define $\lambda_n(B) = c_\alpha \nu\big((1,\infty)B\big)$ for $B \in \mathfrak{B}(S(\mathbb{R}^n))$ and $\alpha \in (0,2)$ where*

$$c_\alpha := \begin{cases} -\alpha\cos(\tfrac{1}{2}\alpha\pi)\Gamma(-\alpha), & \text{if } \alpha \neq 1, \\ \tfrac{\pi}{2}\alpha, & \text{if } \alpha = 1. \end{cases}$$

Then the following are equivalent:

(a) *μ is symmetric, rotationally invariant and α-stable;*

(b) *the Lévy measure ν is of the form*

$$\nu(B) = \tfrac{\alpha}{c_\alpha} \int_{S(\mathbb{R}^n)} \lambda_n(d\xi) \int_0^\infty \mathbb{1}_B(r\xi)\frac{1}{r^{1+\alpha}}\,dr \qquad \text{for all } B \in \mathfrak{B}(\mathbb{R}^n),$$

and λ_n is uniformly distributed on the sphere $S(\mathbb{R}^n)$.

(c) *the characteristic function $\varphi_\mu\colon \mathbb{R}^n \to \mathbb{C}$ of μ is of the form*

$$\varphi_\mu(\beta) = \exp\left(-\int_{S(\mathbb{R}^n)} |\langle \beta, \xi\rangle|^\alpha \ \lambda_n(d\xi)\right),$$

and λ_n is uniformly distributed on the sphere $S(\mathbb{R}^n)$.

(d) *the characteristic function $\varphi_\mu\colon \mathbb{R}^n \to \mathbb{C}$ of μ is of the form*

$$\varphi_\mu(\beta) = \exp\Big(-d_\alpha\,|\beta|^\alpha\Big),$$

where $d_\alpha := \int_{S(\mathbb{R}^n)} |\langle \xi_0, \xi\rangle|^\alpha \ \lambda_n(d\xi)$ for an arbitrary fixed vector $\xi_0 \in S(\mathbb{R}^n)$.

Proof. The proof follows from the Theorems 14.2, 14.10, 14.13, 14.14 and their proofs in [15].

□

Lemma A.2. *Let $(Y_1,\dots,Y_n)$ be uniformly distributed on the sphere $S(\mathbb{R}^n)$ for some $n \in \mathbb{N}$. Then we have for $p \in (0,2)$ and each $k \in \{1,\dots,n\}$ that*

$$E[Y_k^2] = \frac{1}{n}, \qquad E[|Y_k|^p] = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+p}{2})}$$

Proof. Since $Y_1^2 + \cdots + Y_n^2 = 1$ we have $E[Y_k^2] = \frac{1}{n}$ for all $k \in \{1,\dots,n\}$ due to symmetry. As Y_k^2 is distributed according to the Beta distribution with parameters $a := \frac{1}{2}$ and $b := \frac{n-1}{2}$, see [6], we obtain

$$\begin{aligned} E[|Y_k|^p] = E[|Y_k^2|^{p/2}] &= \frac{1}{B(a,b)}\int_0^1 x^{a+p/2-1}(1-x)^{b-1}\,dx \\ &= \frac{B(a+p/2,b)}{B(a,b)} = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+p}{2})}, \end{aligned}$$

which completes the proof. □

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