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Comparison of graphs associated to a commutative Artinian ring

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Abstract. Let R be a commutative ring with $1 \neq 0$ and the additive group R^+. Several graphs on R have been introduced by many authors, among zero-divisor graph $\Gamma_1(R)$, co-maximal graph $\Gamma_2(R)$, annihilator graph $AG(R)$, total graph $T(\Gamma(R))$, cozero-divisors graph $\Gamma_c(R)$, equivalence classes graph $\Gamma_E(R)$ and the Cayley graph $\mathrm{Cay}(R^+, Z^*(R))$. Shekarriz *et al.* (*J. Commun. Algebra*, **40** (2012) 2798–2807) gave some conditions under which total graph is isomorphic to $\mathrm{Cay}(R^+, Z^*(R))$. Badawi (*J. Commun. Algebra*, **42** (2014) 108–121) showed that when R is a reduced ring, the annihilator graph is identical to the zero-divisor graph if and only if R has exactly two minimal prime ideals. The purpose of this paper is comparison of graphs associated to a commutative Artinian ring. Among the results, we prove that for a commutative finite ring R with $|\mathrm{Max}(R)| = n \geq 3$, $\Gamma_1(R) \simeq \Gamma_2(R)$ if and only if $R \simeq \mathbb{Z}_2^n$; if and only if $\Gamma_1(R) \simeq \Gamma_E(R)$. Also the annihilator graph is identical to the cozero-divisor graph if and only if R is a Frobenius ring.

Keywords. Zero-divisor graph; co-maximal graph; Boolean ring; Frobenius ring.

Mathematics Subject Classification. 13A15, 13A99.

1. Introduction

Extensive studies have been done about graphs on a commutative ring R, see for instance, [4,5,8,11]. The idea of a zero-divisor graph of R (denoted by $\Gamma_0(R)$) was introduced by Beck, in 1988, where he was mainly interested in colorings, see [9]. In 1999, Anderson and Livingston [4] studied subgraph $\Gamma_1(R)$ whose set of vertices is $Z^*(R)$ and investigated the interplay between the ring theoretic properties of R and the graph theoretic properties of $\Gamma_1(R)$. In 1995, Sharma and Bhatwadekar [11] proposed a new approach that constructed another graph of R, later to be known as co-maximal graph (denoted by $\Gamma_2(R)$). In 2008, Anderson and Badawi [5] introduced and investigated the total graph of R (denoted by $T(\Gamma(R))$). In 2012, Shekarriz *et al.* [12] studied relations between total graph and Cayley graph on R. In 2011, Afkhami and Khashyarmanesh [6] introduced the cozero-divisor graph (denoted by $\Gamma_c(R)$), and later they studied the relations between two graphs $\Gamma_2(R)$ and $\Gamma_c(R)$, see [7]. In 2014, Badawi [8] introduced and investigated the annihilator graph of R (denoted by $AG(R)$). He showed that each edge of $\Gamma_1(R)$ is an edge of annihilator graph $AG(R)$. It means that, the graph $\Gamma_1(R)$ is a spanning subgraph of the graph $AG(R)$. Also he demonstrated that $AG(R)$ is identical to $\Gamma_1(R)$ if and only if R has exactly two minimal prime ideals.

The purpose of this paper is to present necessary and sufficient conditions under which two graphs associate to a commutative ring R to be identical. We give necessary and sufficient conditions in which, $AG(R) = \Gamma_{\mathrm{c}}(R)$ and $\Gamma_2(R) = \Gamma_{\mathrm{c}}(R)$.

Throughout this paper, all rings are assumed to be commutative and Artinian with unity. Let R be a commutative Artinian ring with the additive group R^+. We denote by $U(R)$, $Z(R)$, $Z^*(R)$, $J(R)$ and $\mathrm{Max}(R)$, the set of unit elements, zero-divisors, nonzero zero-divisors, Jacobson radical and maximal ideals of R, respectively. It is easy to see that $R = U(R) \cup Z(R)$. A Boolean ring is a ring in which every element is idempotent. It is known that an Artinian ring R is Boolean if and only if $R \simeq \mathbb{Z}_2^n$, where $n = |\mathrm{Max}(R)|$. In this case, R is said to be n-Boolean ring. It can easily be verified that R is n-Boolean ring if and only if $U(R) = \{1\}$.

Several graphs have been introduced by many authors; among them we need to recall the ones necessarily needed in this paper.

- $\Gamma_1(R)$, zero-divisors graph; the vertices are elements of $Z^*(R)$ and two vertices a and b are adjacent if and only if $ab = 0$.
- $\Gamma_2(R)$, comaximal graph; the vertices are elements of $R - (U(R) \cup J(R)) = Z(R) - J(R)$ and two vertices a and b are adjacent if and only if $Ra + Rb = R$.
- $AG(R)$, annihilator graph; the vertices are elements of $Z^*(R)$, and two vertices a and b are adjacent if and only if $\mathrm{Ann}_R(a) \nsubseteq \mathrm{Ann}_R(b)$ and $\mathrm{Ann}_R(b) \nsubseteq \mathrm{Ann}_R(a)$.
- $\Gamma_{\mathrm{c}}(R)$, cozero-divisors graph; the vertices are elements of $Z^*(R)$ and two vertices a and b are adjacent if and only if $Ra \nsubseteq Rb$ and $Rb \nsubseteq Ra$.
- $-\mathrm{T}(\Gamma(R))$, total graph; the vertices are elements of R and two vertices a and b are adjacent if and only if $a + b \in Z(R)$.
- $\mathrm{Cay}(G, S)$, Cayley graph; the vertices are elements of G, and two vertices a and b are adjacent if and only if $ab^{-1} \in S$, where G is a group and $S \subset G$ be such that S generates G, $0 \notin S$ and if $a \in S$, then $a^{-1} \in S$. The Cayley graphs $\mathrm{Cay}(R^+, Z^*(R))$ and $\mathrm{Cay}(R^+, U(R))$ are studied in [1] and [2].
- $\Gamma_{\mathrm{E}}(R)$, equivalence classes graph.

As noted in [13], for $a, b \in R$, it is defined $a \sim b$ if and only if $\mathrm{Ann}(a) = \mathrm{Ann}(b)$. Clearly $\sim$ is a equivalence relation and $[a]$ denotes the class of a. Graph of equivalence classes of $Z^*(R)$ is the graph whose vertices are the classes of elements in $Z^*(R)$, and each pair of distinct classes $[a]$ and $[b]$ are adjacent if and only if $ab = 0$.

Let G be a simple graph consisted of an ordered pair of disjoint sets (V, E) such that $V = V(G)$ the notes the vertex set of G and $E = E(G)$ is defined as the edge set of G. Often, we write G for $V(G)$. The neighborhood of a in G, denoted by $\mathrm{N}_G(a)$, is defined by $\{b \in G : a \text{ and } b \text{ are adjacent in } G\}$ and degree of a in G, denoted by $\deg_G(a)$ is $|N_G(a)|$. We denote the minimum degree for vertices in G, by δ_G.

2. Main results

It is known that a commutative Artinian ring R, can be written as decomposition product $\prod_{1 \le i \le n} R_i$, where R_i's are the local rings and $n = |\mathrm{Max}(R)|$. If $a = (a_1, a_2, \ldots, a_n) \in R$, we define $\mathrm{supp}(a) = \{i \in I_n : a_i \neq 0\}$, where $I_n = \{1, \ldots, n\}$. For any $1 \le i \le n$, we denoted $\mathbf{e}_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$.

We first need the following lemma.

Lemma 2.1. *Let $R = \prod_{1\leqslant i\leqslant n} F_i$, where F_i's are fields, $|F_1| \leq |F_2| \leq \cdots \leq |F_n| < \infty$ and $a \in Z^*(R)$.*

(1) $\deg_{\Gamma_1(R)}(a) = \prod_{j\notin \mathrm{supp}(a)} |F_j| - 1$. *Furthermore*, $\delta_{\Gamma_1(R)} = \deg_{\Gamma_1(R)}(\mathbf{e}_2 + \cdots + \mathbf{e}_n) = |F_1^*|$.
(2) $\deg_{\Gamma_2(R)}(a) = \prod_{j\notin \mathrm{supp}(a)} |F_j^*| (\prod_{j\in \mathrm{supp}(a)} |F_j| - \prod_{j\in \mathrm{supp}(a)} |F_j^*|)$. *Furthermore*, $\delta_{\Gamma_2(R)} = \deg_{\Gamma_2}(\mathbf{e}_1) = \prod_{2\leqslant j\leqslant n} |F_j^*|$.

Proof.

(1) By hypothesis, $\mathrm{supp}(a) \subsetneqq I_n$. For any $b = (b_1, b_2, \ldots, b_n) \in \mathrm{N}_{\Gamma_1(R)}(a)$, we have $b_j \in F_j$, if $j \notin \mathrm{supp}(a)$ and $b_j = 0$, if $j \in \mathrm{supp}(a)$. Therefore,

$$\deg_{\Gamma_1(R)}(a) = \prod_{j\notin \mathrm{supp}(a)} |F_j| - 1.$$

(2) For any $b = (b_1, b_2, \ldots, b_n) \in \mathrm{N}_{\Gamma_2}(a)$, we have $b_j \in F_j^*$, if $j \notin \mathrm{supp}(a)$ and $b_j \in F_j$, if $j \in \mathrm{supp}(a)$. Thus

$$\begin{aligned}\deg_{\Gamma_2(R)}(a) &= \prod_{j\notin \mathrm{supp}(a)} |F_j^*| \prod_{j\in \mathrm{supp}(a)} |F_j| - \prod_{1\leqslant j\leqslant n} |F_j^*| \\ &= \prod_{j\notin \mathrm{supp}(a)} |F_j^*| \left(\prod_{j\in \mathrm{supp}(a)} |F_j| - \prod_{j\in \mathrm{supp}(a)} |F_j^*| \right).\end{aligned}$$

□

Now we present the necessary and sufficient condition under which the graphs $\Gamma_1(R)$ and $\Gamma_2(R)$ are isomorphisms.

Theorem 2.2. *Let R be a commutative finite ring with $|\mathrm{Max}(R)| = n$. The following statements are equivalent*:

(1) $\Gamma_1(R) \simeq \Gamma_2(R)$.
(2) *Either R is a n-Boolean ring or $R \simeq F_1 \oplus F_2$, where F_i's are fields.*

Proof.

(1) $\Rightarrow$ (2). Inasmuch as $|V(\Gamma_1(R))| = |V(\Gamma_2(R))|$, then $|Z^*(R)| = |Z(R) - J(R)|$, hence $J(R) = \{0\}$. Thus R is semisimple, i.e, $R = \prod_{1\leqslant i\leqslant n} F_i$, where F_i's are fields. Without loss of generality, assume that $|F_1| \leq |F_2| \leq \cdots \leq |F_n|$.

If $n > 2$, by hypothesis and Lemma 2.1, $\delta_{\Gamma_1} = \delta_{\Gamma_2}$ and

$$|F_1^*| = \prod_{2\leqslant j\leqslant n} |F_j^*|.$$

Thus $|F_1^*| = \cdots = |F_n^*| = 1$, so $|F_1| = \cdots = |F_n| = 2$, hence $R \simeq \mathbb{Z}_2^n$. If $n = 2$, then $R \simeq F_1 \oplus F_2$.

(2) $\Rightarrow$ (1). Suppose that R is a n-Boolean ring and $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n) \in R = \mathbb{Z}_2^n$. We define $f : \Gamma_1(R) \to \Gamma_2(R)$ with $f(a) = 1 + a$, for any $a \in \mathrm{V}(\Gamma_1(R))$. It is easy to see that f is a bijection. Also f is a graph homomorphism. To see this, let $a, b \in V(\Gamma_1(R))$ be adjacent, so for all $1 \leq i \leq n$, $a_i b_i = 0$, and hence either $1 + a_i = 1$ or $1 + b_i = 1$. Therefore we have $\mathrm{supp}(1 + a) \cup \mathrm{supp}(1 + b) = I_n$. Hence $1 + a$ and $1 + b$ are adjacent in $\Gamma_2(R)$. This means that f is a graph isomorphism.

Now suppose $R \simeq F_1 \oplus F_2$, where F_i's are fields, if $|F_1^*| = m$ and $|F_2^*| = n$, then $\Gamma_1(F_1 \oplus F_2) \simeq \Gamma_2(F_1 \oplus F_2) \simeq K_{m,n}$, see [4]. □

Theorem 2.3. *Let R be a commutative finite ring. The following statements are equivalent*:

(1) *R is a Boolean ring.*
(2) $\mathrm{Cay}(R^+, Z^*(R))$ *is a $|R| - 2$-regular graph.*
(3) *$T(\Gamma(R))$ is a $|R| - 2$-regular graph.*
(4) $\mathrm{Cay}(R^+, U(R))$ *is the union 2^{n-1} line segment.*
(5) $\Gamma_1(R) \simeq \Gamma_E(R)$.

Proof.

(1) $\Rightarrow$ (2) and (3). It follows from [12, Theorems 2.7 and 5.2].
(2) or (3) $\Rightarrow$ (1). By hypotheses $U(R) = \{1\}$, hence R is a Boolean ring.
(1) $\Leftrightarrow$ (4). It is clear.
(1) $\Rightarrow$ (5). Clearly, $f : \Gamma_1(R) \to \Gamma_{\mathrm{E}}(R)$ with $f(a) = [a]$, for any $a \in Z^*(R)$, is a isomorphism graph.
(5) $\Rightarrow$ (1). Suppose $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$, where R_i's are local rings and $n = |\mathrm{Max}(R)|$. Since $\Gamma_1(R) \simeq \Gamma_{\mathrm{E}}(R)$, then $|Z^*(R)| = |\{\mathrm{Ann}(a) : a \in Z^*(R)\}|$. Thus for each $a, b \in Z^*(R)$, we have $a = b$ if and only if $\mathrm{Ann}(a) = \mathrm{Ann}(b)$. For each $\gamma \in U(R_i)$, $\mathrm{Ann}(\mathbf{e}_i) = \mathrm{Ann}(\gamma \mathbf{e}_i)$, where $1 \leq i \leq n$, and this implies that $\mathbf{e}_i = \gamma \mathbf{e}_i$, consequently, $U(R_i) = \{1\}$. Therefore for all $1 \leq i \leq n$, we have $R_i = \mathbb{Z}_2$, i.e., R is a n-Boolean ring. □

Lemma 2.4. *Let R be a commutative finite ring, if $a \in \mathrm{Z}^*(R)$ and $a = ua$, for some $u \in \mathrm{U}(R)$, then R is Boolean.*

Proof. Suppose $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$, where R_i's are local rings. By hypothesis, for each $\gamma \in U(R_i)$, $\mathbf{e}_i = \gamma \mathbf{e}_i$, hence $\gamma = 1$. Therefore $\mathrm{U}(R_i) = \{1\}$ and so $R_i = \mathbb{Z}_2$, as a result R is Boolean. □

If G is a graph and H is a subgraph of G, then we write $H \subseteq G$.

Lemma 2.5. *Let R be a commutative ring, then $\Gamma_1(R) \subseteq \mathrm{AG}(R) \subseteq \Gamma_{\mathrm{c}}(R)$.*

Proof. $\Gamma_1(R) \subseteq AG(R)$ follows from [8, Lemma 2.1]. To prove $AG(R) \subseteq \Gamma_{\mathrm{c}}(R)$, let a and b be adjacent in $AG(R)$ and they are not adjacent in $\Gamma_{\mathrm{c}}(R)$. Then either $a \in Rb$ or $b \in Ra$. Therefore either $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$ or $\mathrm{Ann}(b) \subseteq \mathrm{Ann}(a)$, a contradiction. Thus $AG(R) \subseteq \Gamma_{\mathrm{c}}(R)$. □

A graph G is reduced if no two vertices of G have the same open neighbourhood, see [10].

Theorem 2.6. *Let R be a commutative finite ring and $|\mathrm{Max}(R)| = n \geq 3$, the following statements are equivalent*:

(1) *R is a n-Boolean ring.*
(2) *$\Gamma_2(R)$ is reduced.*
(3) *$\Gamma_{\mathrm{c}}(R)$ is reduced.*
(4) *$AG(R)$ is reduced.*

Proof.

(1) $\Rightarrow$ (2). By hypothesis, for any $a, b \in Z^*(R)$, $a = b$ if and only if $N_{\Gamma_1(R)}(a) = \mathrm{Ann}(a) = \mathrm{Ann}(b) = N_{\Gamma_1(R)}(b)$. Therefore $\Gamma_1(R)$ is reduced and Theorem 2.2 implies that $\Gamma_2(R)$ is reduced.
(2) $\Rightarrow$ (1). Suppose $a \in Z^*(R)$ and $u \in U(R)$, since $N_{\Gamma_2(R)}(a) = N_{\Gamma_2(R)}(ua)$, hence $a = ua$ and Lemma 2.4 implies that R is a Boolean ring.
(1) $\Rightarrow$ (3). Suppose $a, b \in Z^*(R)$ and $a \neq b$. Without loss of generality, there exists $i \in \mathrm{supp}(a) - \mathrm{supp}(b)$ such that $a_i = 1$ and $b_i = 0$. Thus $a\mathbf{e}_i = \mathbf{e}_i \in N_{\Gamma_{\mathrm{c}}(R)}(b) - N_{\Gamma_{\mathrm{c}}(R)}(a)$. This shows that $N_{\Gamma_{\mathrm{c}}(R)}(a) \neq N_{\Gamma_{\mathrm{c}}(R)}(b)$. As a result, $\Gamma_{\mathrm{c}}(R)$ is reduced.
(3) $\Rightarrow$ (1). Follows from Lemma 2.4.
(1) $\Rightarrow$ (4). By hypothesis, $\Gamma_{\mathrm{c}}(R)$ is reduced. Now for each $a, b \in Z^*(R)$, $Ra \subseteq Rb$ if and only if $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$. Hence $AG(R) = \Gamma_{\mathrm{c}}(R)$, so $AG(R)$ is reduced.
(4) $\Rightarrow$ (1). Follows from Lemma 2.4. □

The next corollaries follow from Theorems 2.2, 2.3 and 2.6.

COROLLARY 2.7

Let R be a commutative finite ring and $|\mathrm{Max}(R)| = n \geq 3$. The following statements are equivalent:

(1) *R is a n-Boolean ring.*
(2) *$\Gamma_1(R) = \Gamma_2(R)$.*
(3) *$\mathrm{Cay}(R^+, Z^*(R))$ is a $|R| - 2$ a regular graph.*
(4) *$T(\Gamma(R))$ is a $|R| - 2$ a regular graph.*
(5) *$\mathrm{Cay}(R^+, U(R))$ is the union 2^{n-1} line segment.*
(6) *$\Gamma_1(R) \simeq \Gamma_{\mathrm{E}}(R)$.*
(7) *$\Gamma_2(R)$ is reduced.*
(8) *$\Gamma_{\mathrm{c}}(R)$ is reduced.*
(9) *$AG(R)$ is reduced.*

COROLLARY 2.8

Let R be a commutative finite ring and $|\mathrm{Max}(R)| = 2$. The following statements are equivalent:

(1) *$\Gamma_1(R) \simeq \Gamma_2(R)$.*
(2) *$R = F_1 \oplus F_2$, where F_1, F_2 are fields.*

We use $\Gamma_2'(R)$ to denote the co-maximal graph of R with vertex-set $Z^*(R)$.

Theorem 2.9. *Let R be a commutative Artinian ring. The following statements are equivalent*:

(1) $\Gamma_c(R) = \Gamma'_2(R)$.
(2) $R \simeq F_1 \oplus F_2$, *where* F_1, F_2 *are fields.*

Proof.

(1) $\Rightarrow$ (2). By hypothesis, $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$, where R_i's are local rings. Since $R\mathbf{e}_1 \nsubseteq R\mathbf{e}_2$ and $R\mathbf{e}_2 \nsubseteq R\mathbf{e}_1$, then $\mathbf{e}_1$ and $\mathbf{e}_2$ are adjacent in $\Gamma_c(R)$ and so $\mathbf{e}_1$ and $\mathbf{e}_2$ are adjacent in $\Gamma'_2(R)$. If $n > 2$, then $R\mathbf{e}_1 + R\mathbf{e}_2 \subsetneqq R$, and this is a contradiction. Therefore $R = R_1 \oplus R_2$.

If R_1 is not a field, there is $0 \neq a \in R_1 - \mathrm{U}(R_1)$, hence $(a, 0)$ and $(0, 1)$ are adjacent in $\Gamma_c(R)$ and so are adjacent in $\Gamma'_2(R)$. But $R(a, 0) + R(0, 1) \neq R$, a contradiction, hence R_1 is a field. Similarly, R_2 is a field.
(2) $\Rightarrow$ (1). Clearly, $\Gamma_c(F_1 \oplus F_2) = \Gamma'_2(F_1 \oplus F_2) = K_{m,n}$, where $|F_1^*| = m$ and $|F_2^*| = n$. □

A commutative Artinian ring R is said to be Frobenius, if R is injective as a module over itself. It is well-known that R is Frobenius if and only if $\mathrm{Ann}(\mathrm{Ann}(I)) = I$, for each ideal I of R. Also if $(R, \mathfrak{m})$ is a commutative Artinian local ring then R is Frobenius if and only if $\dim_{R/\mathfrak{m}} \mathrm{Ann}(\mathfrak{m}) = 1$, see [13, Remark 1.3].

Next we show that $AG(R) = \Gamma_c(R)$ if and only if R is a Frobenius ring. We first need the following lemma.

Lemma 2.10. *Let $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$, where R_i's are Artinian local ring, the following statements are equivalent*:

(1) $\Gamma_c(R) = AG(R)$.
(2) $\Gamma_c(R_i) = AG(R_i)$, $1 \leq \forall i \leq n$.

Proof.

(1) $\Rightarrow$ (2). Suppose a_i and b_i are adjacent in $\Gamma_c(R_i)$, where $1 \leq i \leq n$. Then $a = a_i\mathbf{e}_i$ and $b = b_i\mathbf{e}_i$ are adjacent in $\Gamma_c(R) = AG(R)$. Therefore $\mathrm{Ann}_{R_i}(a_i) \nsubseteq \mathrm{Ann}_{R_i}(b_i)$ and $\mathrm{Ann}_{R_i}(b_i) \nsubseteq \mathrm{Ann}_{R_i}(a_i)$. Hence a_i and b_i are adjacent in $AG(R_i)$, as a result $\Gamma_c(R_i) = AG(R_i)$.
(2) $\Rightarrow$ (1). Suppose $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ are adjacent in $\Gamma_c(R)$. There is $1 \leq i, j \leq m$ such that $Rb_i\mathbf{e}_i \nsubseteq Ra_i\mathbf{e}_i$ and $Ra_j\mathbf{e}_j \nsubseteq Rb_j\mathbf{e}_j$. So $R_ib_i \nsubseteq R_ia_i$ and $R_ja_j \nsubseteq R_jb_j$. By hypothesis, $\mathrm{Ann}_{R_i}(a_i) \nsubseteq \mathrm{Ann}_{R_i}(b_i)$ and $\mathrm{Ann}_{R_j}(b_j) \nsubseteq \mathrm{Ann}_{R_j}(a_j)$. Hence $\mathrm{Ann}(a) \nsubseteq \mathrm{Ann}(b)$ and $\mathrm{Ann}(b) \nsubseteq \mathrm{Ann}(a)$, i.e., a and b are adjacent in $AG(R)$. □

Theorem 2.11. *Let R be a commutative Artinian ring. The following statements are equivalent*:

(1) $AG(R) = \Gamma_c(R)$.
(2) *R is a Frobenius ring.*

Proof. We know that $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$, where R_i's are Artinian local rings and it is well-known that R is Frobenius if and only if R_i's are Frobenius, see [3]. Therefore by Lemma 2.10, it is enough to prove the case that R is local with the maximal ideal $\mathfrak{m}$.

(1) $\Rightarrow$ (2). Suppose $(R, \mathfrak{m})$ is not Frobenius, then $\dim_{R/\mathfrak{m}} \mathrm{Ann}(\mathfrak{m}) > 1$. Therefore there exists a and b in $\mathrm{Ann}(\mathfrak{m})$ such that $Ra \nsubseteq Rb$ and $Rb \nsubseteq Ra$, i.e, a and b are adjacent in $\Gamma_c(R)$ and so in $AG(R)$. But $\mathrm{Ann}(a) = \mathrm{Ann}(b) = \mathfrak{m}$, a contradiction.
(2) $\Rightarrow$ (1). Suppose $a, b \in Z^*(R)$ are not adjacent in $AG(R)$. Thus either $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$ or $\mathrm{Ann}(b) \subseteq \mathrm{Ann}(a)$. Therefore either $Rb = \mathrm{Ann}(\mathrm{Ann}(b)) \subseteq \mathrm{Ann}(\mathrm{Ann}(a)) = Ra$ or $Ra = \mathrm{Ann}(\mathrm{Ann}(a)) \subseteq \mathrm{Ann}(\mathrm{Ann}(b)) = Rb$. This implies that $\Gamma_c(R) \subseteq AG(R)$. $\square$

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