Proc. Indian Acad. Sci. (Math. Sci.) (2018) 128:14
https://doi.org/10.1007/s12044-018-0376-5

Rotationally symmetric extremal pseudo-Kähler metrics of non-constant scalar curvatures

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MS received 13 June 2015; revised 14 November 2015; accepted 27 February 2016; published online 22 March 2018

Abstract. In this paper, we explicitly construct some rotationally symmetric extremal (pseudo-)Kähler metrics of non-constant scalar curvature, which depend on some parameters, and on some line bundles over projective spaces. We also discuss the phase change phenomenon caused by the variation of parameters.

Keywords. Extremal; pseudo-Kahler metric; phase change; rotationally symmetric.

Mathematics Subject Classification. 53C56, 58B20.

1. Introduction

The important notion of extremal Kähler metrics was first introduced by Calabi [3]. He showed that the extremal condition is equivalent to the gradient of the scalar curvature being a holomorphic vector field. Therefore, the extremal Kähler metrics generalize Kähler–Einstein metrics and constant scalar curvature Kähler metrics. Calabi [3] constructed a family of extremal Kähler metrics of non-constant scalar curvature depending on 4-parameters on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(m) \oplus \mathcal{O}_{\mathbb{P}^{n-1}})$ for arbitrary $m \in \mathbb{Z}$. Simanca [7] re-obtained these examples and produced extremal metrics on other bundles by extending Calabi's construction. Abreu [1] presented Calabi's family of $U(n)$-invariant extremal Kähler metrics in symplectic action-angle coordinates. Tønnesen–Friedman [8–10] used the Hamiltonian construction to get new extremal metrics on some $\mathbb{P}^1$-bundles over products of Kähler–Einstein manifolds of negative scalar curvature and other important results. For more references, see the historical survey in [2].

In this paper, we will follow [5,6] to construct rotationally symmetric extremal pseudo-Kähler metrics on various bundles over $\mathbb{P}^{n-1}$ and consider the 'phase change' under the variation of initial values. Firstly, we will start with a function depending only on $u := r^2 = \sum_{j=1}^{n} |z_j|^2$ on $\mathbb{C}^n - \{0\}$ as the Kähler potential. Because of the globally rotational symmetry, we can restrict it to the z^1-axis ($z^2 = \cdots = z^n = 0$), where the Kähler form can be written as

$$\omega = \sqrt{-1}\left((u\phi'(u))'dz^1 \wedge d\bar{z}^1 + \phi'(u)\sum_{i=2}^{n} dz^i \wedge d\bar{z}^i\right).$$

Consequently, the curvature properties can be expressed in terms of ordinary differential equations. Hence we will get an ordinary differential equation corresponding to the extremal condition. The completeness of the constructed metric is related to the natural extension over $u = 0$ or $u = \infty$. The metric can be constructed on $\mathcal{O}_{\mathbb{P}^{n-1}}(m)$, or its disc subbundle, or its projective compactification $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(m) \oplus \mathcal{O}_{\mathbb{P}^{n-1}})$ after taking quotient by some cyclic subgroups of $U(n)$. We will make the extensions of the metrics over $u = 0$ and $u = \infty$ more explicit by changing coordinates. Moreover, we will consider the extension of Calabi's construction to the pseudo-Kähler case and put the constructed metrics in the families parametrized by the initial values of the corresponding ODEs. This leads to some interesting 'phase change' phenomenon: When one changes parameters suitably, the extremal Kähler metric may acquire some singularities and become extremal pseudo-Kähler in some regions. This was inspired by the discussion of the Candelas-de la Ossa metrics on the resolved conifold [4], which form a family of Kähler Ricci-flat metrics that depend on a parameter a. When $a \to 0^+$, the metric reduces to the one induced from a singular metric on the conifold; when a becomes negative, one gets a metric that can be understood as Kähler metrics on the space where $\mathbb{P}^1$ is flopped. In this way, one sees how the Kähler cones of the resolved conifold and the flopped resolved conifold can be glued together. We expect this to be a very general phenomenon, not just for the resolved conifold or just for Kähler Ricci-flat metrics. We have presented in [5,6] some examples of this 'phase change' phenomenon in the case of (pseudo-)Kähler metrics of constant scalar curvatures and Kähler–Einstein metrics respectively. In this paper, we will make a similar study in the case of extremal (pseudo-)Kähler metrics.

The rest of the paper is organized as follows. In §2, we will recall the main lemmas and notations in [5,6] and derive the ODEs corresponding to the $U(n)$-symmetric extremal pseudo-Kähler metrics. In §3, we will get many rotationally symmetric extremal pseudo-Kähler metrics on various holomorphic line bundles over projective spaces and their disc bundles. Especially, we give a family of noncomplete rotationally symmetric extremal Kähler metric on $\mathcal{O}_{\mathbb{P}^{n-1}}(-k)$ for all $k \in \mathbf{Z}$. And we can see clearly the phase change under the variation of initial values. In §4, we get many rotationally symmetric extremal (pseudo-)Kähler metrics on $\mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O}(-l))$ for each $k, l \geq 1$. And we will discuss their phase changes which are manifest from our approach.

2. Construction of rotationally symmetric extremal (pseudo-)Kähler metrics

In this section, we will recall the construction of some $U(n)$-symmetric extremal (pseudo-)Kähler metrics and some closing conditions near the origin and near the infinity by some natural changes of coordinates [3,5–7].

2.1 *$U(n)$-symmetric Kähler metrics*

On $\mathbb{C}^n - \{0\}$, consider the potential function $\phi(u)$, where $u = |z^1|^2 + \cdots + |z^n|^2$ and the (1, 1)-form

$$\omega = \sqrt{-1}\partial\bar{\partial}\phi(u) = \sqrt{-1}(\phi'(u)\partial\bar{\partial}u + \phi''(u)\partial u \wedge \bar{\partial}u). \tag{1}$$

More explicitly,

$$\omega = \sqrt{-1}\left(\phi'(u)\sum_{i=1}^{n} dz^i \wedge d\bar{z}^i + \phi''(u)\sum_{i=1}^{n}\bar{z}^i dz^i \wedge \sum_{j=1}^{n} z^j d\bar{z}^j\right). \tag{2}$$

Because of the symmetry, one can restrict to the z^1-axis ($z^2 = \cdots = z^n = 0$), where the Kähler form can be written as

$$\omega = \sqrt{-1}\left((u\phi'(u))' dz^1 \wedge d\bar{z}^1 + \phi'(u)\sum_{i=2}^{n} dz^i \wedge d\bar{z}^i\right). \tag{3}$$

Write $y(u) = u\phi'(u)$, then one can rewrite (3) as

$$\omega = \sqrt{-1}\left(y'(u) dz^1 \wedge d\bar{z}^1 + \frac{y(u)}{u}\sum_{i=2}^{n} dz^i \wedge d\bar{z}^i\right). \tag{4}$$

Therefore, we get the following

Lemma 2.1 [5,6]. *If $y(u) > 0$ and $y'(u) > 0$ for all $u > 0$, then ω is a Kähler form on $\mathbb{C}^n - \{0\}$; if $y(u) > 0$ and $y'(u) < 0$ for all $u > 0$, then ω is a pseudo-Kähler form of signature $(2n-2, 2)$ on $\mathbb{C}^n - \{0\}$; if $y(u) < 0$ and $y'(u) > 0$ for all $u > 0$, then ω is a pseudo-Kähler form of signature $(2, 2n-2)$ on $\mathbb{C}^n - \{0\}$; if $y(u) < 0$ and $y'(u) < 0$ for all $u > 0$, then $-\omega$ is a Kähler form on $\mathbb{C}^n - \{0\}$.*

Lemma 2.2 [5,6].

(1) *If $y(u) > 0$ and $y'(u) > 0$ for $0 < u < a$, where $a > 0$, then ω is a Kähler form on the open ball $B^{2n}(\sqrt{a}) - \{0\}$; if $y(u) > 0$ and $y'(u) < 0$ for $0 < u < a$, where $a > 0$, then ω is a pseudo-Kähler form of signature $(2n-2, 2)$ on the open ball $B^{2n}(\sqrt{a}) - \{0\}$; if $y(u) < 0$ and $y'(u) > 0$ for $0 < u < a$, where $a > 0$, then ω is a pseudo-Kähler form of signature $(2, 2n-2)$ on the open ball $B^{2n}(\sqrt{a}) - \{0\}$; if $y(u) < 0$ and $y'(u) < 0$ for $0 < u < a$, where $a > 0$, then $-\omega$ is a Kähler form on the open ball $B^{2n}(\sqrt{a}) - \{0\}$.*
(2) *If $y(u) > 0$ and $y'(u) > 0$ for $u > a$, where $a > 0$, then ω is a Kähler form outside the sphere of radius $\sqrt{a}$; if $y(u) > 0$ and $y'(u) < 0$ for $u > a$, where $a > 0$, then ω is a pseudo-Kähler form of signature $(2n-2, 2)$ outside the sphere of radius $\sqrt{a}$; if $y(u) < 0$ and $y'(u) > 0$ for $u > a$, where $a > 0$, then ω is a pseudo-Kähler form of signature $(2, 2n-2)$ outside the sphere of radius $\sqrt{a}$; if $y(u) < 0$ and $y'(u) < 0$ for $u > a$, where $a > 0$, then $-\omega$ is a Kähler form outside the sphere of radius $\sqrt{a}$.*

Let $m \geq 1$ be a positive integer, make the following change of variables:

$$z^1 = v^{1/m}, z^2 = v^{1/m} w^2, \ldots, z^n = v^{1/m} w^n. \tag{5}$$

The new coordinates $(v, w^2, \ldots, w^n)$ are local coordinates on $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$, where $w^2, \ldots, w^n$ are inhomogeneous coordinates on $\mathbb{P}^{n-1}$, and v is the coordinate in the fiber direction. With the coordinate change (5), the $(1,1)$-form (2) becomes a differential form $\hat{\omega}$. At $(v, 0, \ldots, 0)$, we have

$$\hat{\omega} = \sqrt{-1}\left(\frac{1}{m^2}|v|^{2/m-2}y'(|v|^{2/m})dv \wedge d\bar{v} + y(|v|^{2/m})\sum_{i=2}^{n} dw^i \wedge d\bar{w}^i\right).$$

From this one can get the following:

Lemma 2.3 [5,6]. *If both* $y(|v|^{2/m})$ *and* $|v|^{2/m-2}y'(|v|^{2/m})$ *can be extended to smooth functions in* v *whose values at* $v = 0$ *are nonzero, then* $\hat{\omega}$ *can be extended smoothly over the zero section of* $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$.

Similarly, one can also add a copy of $\mathbb{P}^{n-1}$ at infinity to $(\mathbb{C}^n - \{0\})/\mathbb{Z}_m$ to get the total space of $\mathcal{O}(m) \to \mathbb{P}^{n-1}$ by making the following change of variables:

$$z^1 = 1/v^{1/m},\ z^2 = w^2/v^{1/m}, \ldots, z^n = w^n/v^{1/m}. \tag{6}$$

With the coordinate change (6), the $(1, 1)$-form (2) becomes a differential form $\tilde{\omega}$. At $(v, 0, \ldots, 0)$ we have

$$\tilde{\omega} = \sqrt{-1}\left(\frac{1}{m^2}|v|^{-2/m-2}y'(|v|^{-2/m})dv \wedge d\bar{v} + y(|v|^{-2/m})\sum_{i=2}^{n} dw^i \wedge d\bar{w}^i\right).$$

We have the following:

Lemma 2.4 [5,6]. *If both* $y(|v|^{-2/m})$ *and* $|v|^{-2/m-2}y'(|v|^{-2/m})$ *can be extended to smooth functions in* v *whose values at* $v = 0$ *are nonzero, then* $\tilde{\omega}$ *can be extended smoothly over the zero section of* $\mathcal{O}_{\mathbb{P}^{n-1}}(m)$.

In the following, we will consider the Ricci form of a pseudo-Kähler metric ω. We will define it as follows. First of all, the Levi-Civata connection defined by the pseudo-Kähler metric induces a connection on the canonical line bundle, we define its curvature form ρ as the Ricci form of the pseudo-Kähler metric. In local coordinates, if $\omega = \sqrt{-1}h_{i\bar{j}}dz^i \wedge d\bar{z}^j$, then as in the ordinary Kähler case

$$\rho = \sqrt{-1}\bar{\partial}\left(\frac{1}{\det h}\partial \det(h)\right) = -\sqrt{-1}\partial\bar{\partial}\log(\det(h)),$$

where h is the Hermitian matrix $(h_{i\bar{j}})$, and when $\det(h)$ is negative, we take $\log(\det(h)) = \log(|\det(h)|) + \sqrt{-1}\pi$.

It is not hard to see that

$$\frac{\omega^n}{n!} = \sqrt{-1}^n y'(u)\left(\frac{y(u)}{u}\right)^{n-1}\prod_{i=1}^{n} dz^i \wedge d\bar{z}^i. \tag{7}$$

Now if ω is nondegenerate, by (7), the Ricci form is given by

$$\rho = -\sqrt{-1}\partial\bar{\partial}\psi, \tag{8}$$

where

$$\psi = \log\left[y'\left(\frac{y}{u}\right)^{n-1}\right]. \tag{9}$$

So

$$\psi' = \frac{y''}{y'} + (n-1)\frac{y'}{y} - (n-1)\frac{1}{u}. \tag{10}$$

And the scalar curvature of ω is given by

$$\rho \wedge \omega^{n-1} = R\omega^n. \tag{11}$$

Plugging (3), (7) and (8) into (11), we get

$$(u\psi')'\left(\frac{y}{u}\right)^{n-1} + (n-1)\psi'\left(\frac{y}{u}\right)^{n-2} y' = -nRy'\left(\frac{y}{u}\right)^{n-1},$$

i.e.

$$(u\psi')'y^{n-1} + (n-1)(u\psi')y^{n-2}y' = -nRy^{n-1}y'. \tag{12}$$

2.2 *Extremal metric condition*

The notion of extremal metric was first introduced by Calabi as critical points of the functional

$$\int_M \frac{S(\omega)^2\omega^n}{n!}, \tag{13}$$

where $S(\omega)$ is the scalar curvature determined by ω. And Calabi proved that: a Kähler metric is extremal if and only if $R_{,\bar{i}\bar{j}} = 0$, or equivalently the vector field $X = \sum_{i,j=1}^n g^{i\bar{j}} \frac{\partial R}{\partial \bar{z}^j} \frac{\partial}{\partial z^i}$ is holomorphic. But here R is only the function of u, hence $X = \sum_{i,j=1}^n g^{i\bar{j}} R' z^j \frac{\partial}{\partial z^i}$. From (1),

$$g^{i\bar{j}} = \frac{\delta^{ij}}{\phi'} - \frac{\phi''\bar{z}^j z^i}{\phi'(\phi' + u\phi'')}, \tag{14}$$

hence by plugging this into the last expression, we get

$$\begin{aligned} X &= \sum_{i=1}^n \frac{R'z^i}{\phi' + u\phi''}\frac{\partial}{\partial z^i} \\ &= \sum_{i=1}^n \frac{R'z^i}{y'}\frac{\partial}{\partial z^i}. \end{aligned}$$

Since z^i is holomorphic, X is holomorphic if and only if

$$\frac{R'}{y'} = C_1 \tag{15}$$

for some constant C_1, i.e.

$$R = C_1 y + C_2 \tag{16}$$

by integrating (15). Therefore, by plugging (12) and (10) into the above expression, we get

$$\frac{[y^{n-1}(\frac{uy''}{y'} + (n-1)u\frac{y'}{y} - (n-1))]'}{-ny^{n-1}y'} = C_1 y + C_2. \tag{17}$$

Integrating this, we get

$$y^{n-1}\left(\frac{uy''}{y'}+(n-1)u\frac{y'}{y}-(n-1)\right)=-C_1n/(n+1)y^{n+1}-C_2y^n+C_3, \tag{18}$$

where C_2, C_3 are integral constants.

Multiplying it by y' on both sides and integrating, we get

$$uy^{n-1}y'=y^n-C_1\frac{n}{(n+1)(n+2)}y^{n+2}-\frac{C_2}{n+1}y^{n+1}+C_3y+C_4. \tag{19}$$

Hence, we have the following:

PROPOSITION 2.1

A pseudo-Kähler form as in (1) is extremal if and only if

$$\int\frac{y^{n-1}dy}{-C_1\frac{n}{(n+1)(n+2)}y^{n+2}-\frac{C_2}{n+1}y^{n+1}+y^n+C_3y+C_4}=\ln u, \tag{20}$$

for some constants C_1, C_2, C_3, C_4.

Remark 2.1. When $C_1=0$, (20) reduces to the constant scalar curvature case [6], in the following we will always assume $C_1\neq 0$.

3. Examples of $U(n)$-symmetric extremal metrics $C_4=0$

In this section, we first construct various families of extremal (pseudo-)Kähler metrics depending on two parameters on the line bundles $\mathcal{O}_{\mathbb{P}^{n-1}}(-k)$ and other bundles. Then we study the effect of changing parameters on the constructed metrics.

By rescaling we can take $C_1=\pm\frac{(n+1)(n+2)}{n}$. We will first discuss $C_1=-\frac{(n+1)(n+2)}{n}$ and later we will discuss the positive case.

3.1 $C_1<0$

When $C_4=0$, take $C_1=-\frac{(n+1)(n+2)}{n}$ and write $-\frac{C_2}{n+1}$ as C_2 for convenience. Now (19) becomes

$$y'=\frac{y^{n+1}+C_2y^n+y^{n-1}+C_3}{uy^{n-2}} \tag{21}$$

Write $C_3=-b^{n+1}-C_2b^n-b^{n-1}$ for some number $b\neq 0$. Now the above equation becomes

$$y'=\frac{y^{n+1}+C_2y^n+y^{n-1}-b^{n+1}-C_2b^n-b^{n-1}}{uy^{n-2}} \tag{22}$$

Recall in the case of $U(n)$-symmetric constant scalar curvature Kähler metrics with nonzero scaler curvature, we have the following equation [6]:

$$y'=\frac{y^{n+1}+y^n+C_1y-b^{n+1}-b^n-C_1b}{uy^{n-1}}. \tag{23}$$

We will modify the treatment of this equation in [6] to study solutions to (22). We consider the solution curve of this ODE in the region $u > 0$, $y \neq b$ and $y \neq 0$. In this subsection, we will discuss the extension of the solution over 0. The extension over the infinity can be treated similarly, and will be discussed later. Assume furthermore that $(n+1)b^2 + nC_2 b + n - 1 \neq 0$. Then we have

$$\frac{y^{n-2}}{y^{n+1} + C_2 y^n + y^{n-1} - b^{n+1} - C_2 b^n - b^{n-1}} = \frac{1}{(n+1)b^2 + nC_2 b + n - 1} \left(\frac{1}{y-b} - \frac{p(y)}{b^n + C_2 b^{n-1} + b^{n-1} y + \sum_{j=2}^{n} b^{n-j}(y^2 + C_2 y + 1) y^{j-2}} \right),$$

where

$$p(y) = \begin{cases} y^{n-1} + (2b + C_2) y^{n-2} - \sum_{j=0}^{n-3} (n-2-j) \\ \quad (b^2 + C_2 b + 1) b^j y^{n-3-j}, & n \geq 3 \\ y + C_2 + 2b, & n = 2 \end{cases}$$

After integration, one gets

$$\ln|y-b| - \int \frac{p(y)dy}{b^n + C_2 b^{n-1} + b^{n-1} y + \sum_{j=2}^{n} b^{n-j}(y^2 + C_2 y + 1) y^{j-2}} = \ln u^{(n+1)b^2 + nC_2 b + n - 1}. \tag{24}$$

From this, we see that it is natural to regard y as a function of $u^{(n+1)b^2+nC_2b+n-1}$. We now make this more precise. Suppose that $(n+1)b^2 + nC_2 b + n - 1$ is a positive integer m, i.e., $(n+1)b^2 + nC_2 b + n - 1 = m$ for $m \geq 1$. One can consider the action of $\mathbb{Z}/m\mathbb{Z}$ on $\mathbb{C}^n - \{0\}$, and consider the extension of the metric on $(\mathbb{C}^n - \{0\})/\mathbb{Z}_m$ by adding a copy of $\mathbb{P}^{n-1}$ over 0. For this, we need to check the conditions in Lemma 2.3.

Because $(n+1)b^2 + nC_2 b + n - 1 \neq 0$, $y = b$ being a simple root of $y^{n+1} + C_2 y^n + y^{n-1} - b^{n+1} - C_2 b^n - b^{n-1}$, therefore,

$$\left(b^n + C_2 b^{n-1} + b^{n-1} y + \sum_{j=2}^{n} b^{n-j}(y^2 + C_2 y + 1) y^{j-2} \right) |_{y=b} \neq 0.$$

It follows that $\int_b^{y(u)} \frac{p(y)dy}{b^n + C_2 b^{n-1} + b^{n-1} y + \sum_{j=2}^{n} b^{n-j}(y^2 + C_2 y + 1) y^{j-2}}$ for $y(u)$ close to b is finite. Therefore by (24),

$$\lim_{y \to b} u = 0$$

for solutions with initial values near $y = b$. Let

$$y(u) = b + a_1 u + a_2 u^2 + \cdots \tag{25}$$

be an asymptotic expansion of y near $u = 0$. Then by (24) again, we have

$$\ln|a_1 u + \cdots| \sim m \ln u, \tag{26}$$

therefore,

$$y(u) \sim b + a_m u^m + a_{m+1} u^{m+1} + \cdots . \tag{27}$$

Now plug this into (24) and we get

$$\begin{aligned}&\ln\left|1+\frac{a_{m+1}}{a_m}u+\cdots\right|\\&\quad=-\int_b^{y(u)}\frac{p(y)dy}{b^n+C_2b^{n-1}+b^{n-1}y+\sum_{j=2}^n b^{n-j}(y^2+C_2y+1)y^{j-2}},\end{aligned} \tag{28}$$

where $a_m \neq 0$. Asymptotically, the right-hand side is of order u^m, hence by comparing with the left-hand sides one gets

$$y(u) \sim b + a_m u^m + a_{2m} u^{2m} + \cdots . \tag{29}$$

By repeating this procedure, one shows that $y(u) = g(u^m)$ for some smooth function g. Furthermore, because $b \neq 0$ and $a_m \neq 0$, the conditions in Lemma 2.3 are satisfied. Therefore, the solution can be extended smoothly over the zero section of $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$.

We summarize the result as follows:

PROPOSITION 3.1

When $(n+1)b^2+nC_2b+n-1 = m$, *for some* $m \in \mathbf{N}$, *a solution of* (22) *with* $\lim_{y\to b} u = 0$ *induces a smooth metric on a neighborhood of the zero section of* $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$.

For the extension over ∞, we have a similar result as follows:

PROPOSITION 3.2

When $(n+1)b^2 + nC_2b + n - 1 = -m$, *for some* $m \in \mathbf{N}$, *a solution of* (22) *with* $\lim_{y\to b} u = +\infty$ *induces a smooth metric on a neighborhood of the zero section of* $\mathcal{O}_{\mathbb{P}^{n-1}}(m)$.

Remark 3.1. It is easy to see that

$$\begin{aligned}&(n+1)b^2+nC_2b+n-1\\&\quad=1/\mathrm{Res}\left(\frac{y^{n-2}}{y^{n+1}+C_2y^n+y^{n-1}-b^{n+1}-C_2b^n-b^{n-1}}, b\right)\end{aligned}$$

in the above two propositions.

To understand the solutions of (22) better, in particular, the sign of its right-hand side, we consider the vector field corresponding to (22) on the half plane $\{(u, y) \in \mathbb{R}^2 \mid u > 0\}$. It will tell us the behavior of the solutions corresponding to different initial values. We first have to find the places where

$$\frac{y^{n+1}+C_2y^n+y^{n-1}-b^{n+1}-C_2b^n-b^{n-1}}{uy^{n-2}} = 0 \tag{30}$$

or

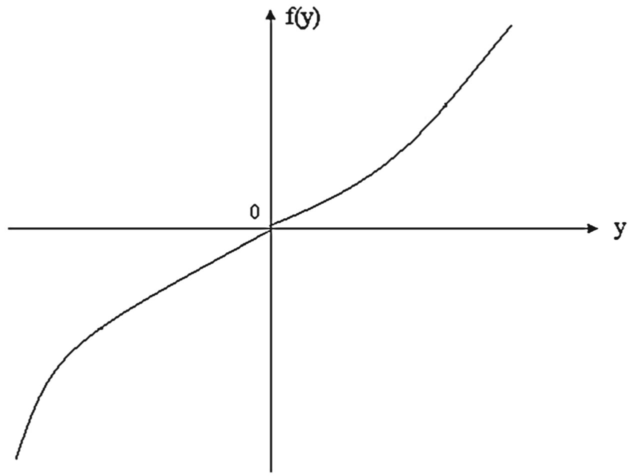

Figure 1. For n even and $(C_2)^2 \leq \frac{4(n^2-1)}{n^2}$.

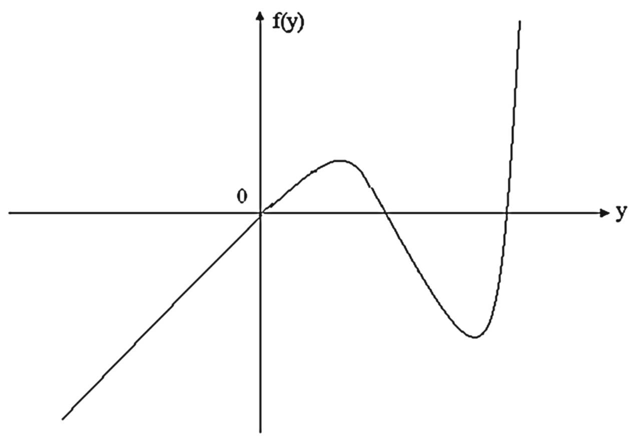

Figure 2. For n even and $C_2 < -\sqrt{\frac{4(n^2-1)}{n^2}}$.

$$\frac{y^{n+1} + C_2 y^n + y^{n-1} - b^{n+1} - C_2 b^n - b^{n-1}}{u y^{n-2}} = \infty. \tag{31}$$

The latter is possible only along the u-axis, i.e., $y = 0$, while the former is more complicated. Clearly $y = b$ is a solution of (30). To understand the sign of the numerator $y^{n+1} + C_2 y^n + y^{n-1} - b^{n+1} - C_2 b^n - b^{n-1}$, we consider

$$f(y) = y^{n+1} + C_2 y^n + y^{n-1} \tag{32}$$

We have $f'(y) = y^{n-2}((n+1)y^2 + nC_2 y + n - 1)$. Hence the graphs are as in figures 1–6.

We will discuss various cases in the following subsections as indicated in the figures 1–6. For convenience, we denote $g(y) = y^{n+1} + C_2 y^n + y^{n-1} - b^{n+1} - C_2 b^n - b^{n-1}$.

3.1.1. Assume n is even, take suitable $b > 0, C_2 \in \mathbb{R}, m \in \mathbb{Z}$ satisfying $(n+1)b^2 + nC_2 b + n - 1 = m$, such that $g(y) = 0$ has exactly one solution $y = b > 0$. We have for $u > 0$,

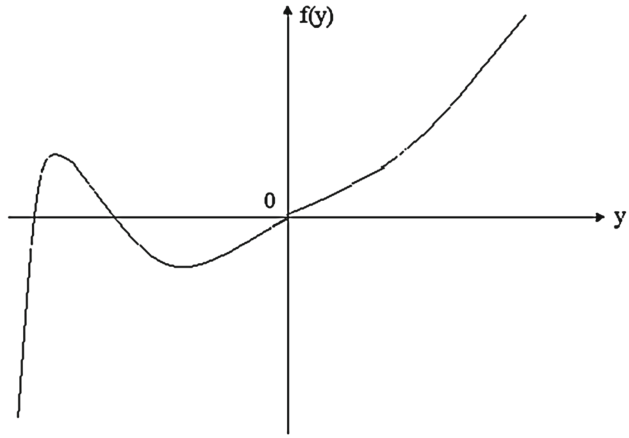

Figure 3. For n even and $C_2 > \sqrt{\frac{4(n^2-1)}{n^2}}$.

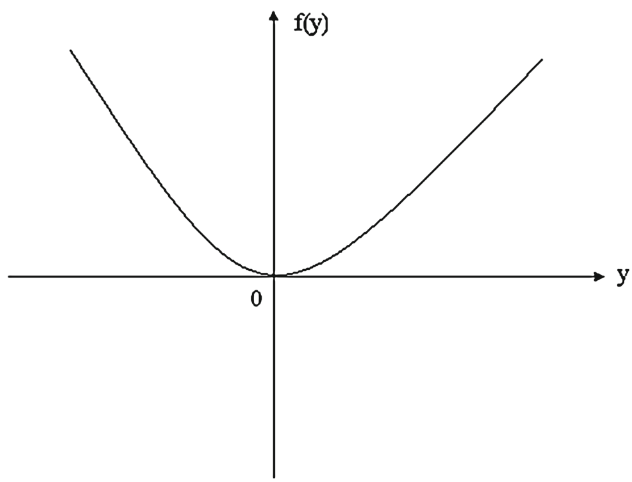

Figure 4. For n odd and $(C_2)^2 \leq \frac{4(n^2-1)}{n^2}$.

$$
\begin{aligned}
y' &= \frac{g(y)}{uy^{n-2}} \\
&= \frac{y^{n+1} + C_2 y^n + y^{n-1} - b^{n+1} - C_2 b^n - b^{n-1}}{uy^{n-2}} \begin{cases} > 0, & y > b, \\ = 0, & y = b, \\ < 0, & 0 < y < b, \\ < 0, & y < 0. \end{cases}
\end{aligned}
$$

The phase diagram of the solution curves are as in figure 7.

From this one sees that all solutions are defined for bounded u: $u < C$ for some $C > 0$. We will check this below. Except for the constant solution $y = b$, $y = y(u)$ is monotonous in u, and let $u = u(y)$ be its inverse function. We have three cases to consider.

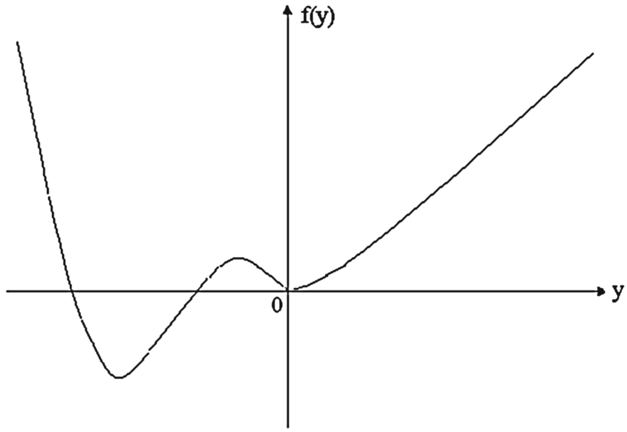

Figure 5. For n odd and $C_2 > \sqrt{\frac{4(n^2-1)}{n^2}}$.

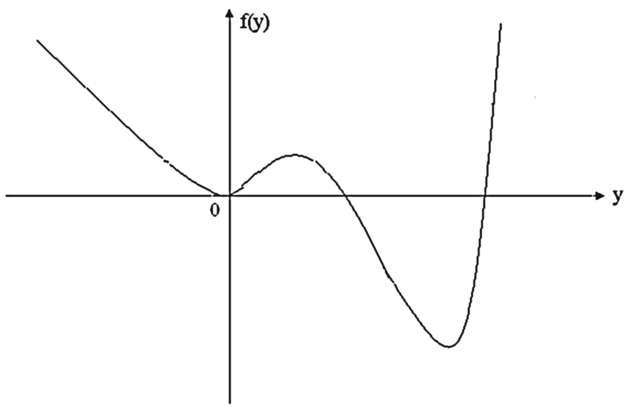

Figure 6. For n odd and $C_2 < -\sqrt{\frac{4(n^2-1)}{n^2}}$.

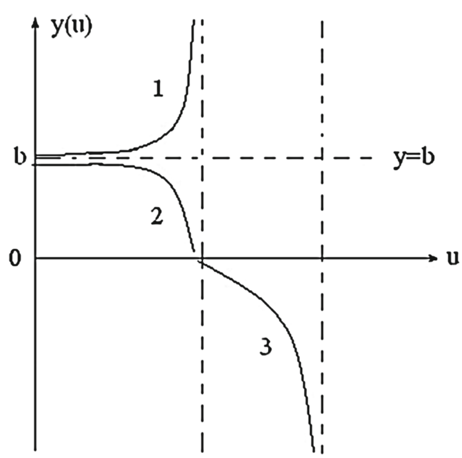

Figure 7. Phase diagram of the solution curves in §3.1.1.

Region 1. $y > b$. Then u is monotonically increasing in y. By (20),

$$\begin{aligned}&\ln u^m(y_2) - \ln u^m(y_1)\\&\quad = \ln|y-b|\Big|_{y_1}^{y_2}\\&\qquad - \int_{y_1}^{y_2} \frac{p(y)dy}{b^n + C_2 b^{n-1} + b^{n-1}y + \sum_{j=2}^n b^{n-j}(y^2 + C_2 y + 1)y^{j-2}}.\end{aligned}$$

When $y_1 \to b^+$, the integral on the right-hand side is finite and $\ln|y_1 - b| \to -\infty$, therefore, the right-hand side tends to $+\infty$ when $y_1 \to b^+$. Hence by looking at the left-hand side when $y_1 \to b^+$, we must have

$$\lim_{y_1 \to b^+} u(y_1) = 0. \tag{33}$$

We now look at the behavior of the solution when $y_2 \to +\infty$.

Because

$$\begin{aligned}&\lim_{y_2\to+\infty}\Bigg(\ln|y-b|\Big|_{y_1}^{y_2}\\&\qquad - \int_{y_1}^{y_2} \frac{p(y)dy}{b^n + C_2 b^{n-1} + b^{n-1}y + \sum_{j=2}^n b^{n-j}(y^2 + C_2 y + 1)y^{j-2}}\Bigg)\\&\quad = \lim_{y_2\to+\infty} m\int_{y_1}^{y_2} \frac{y^{n-2}}{y^{n+1} + y^n + C_1 y - b^{n+1} - b^n - C_1 b}dy\\&\quad = m\int_{y_1}^{+\infty} \frac{y^{n-2}}{y^{n+1} + y^n + C_1 y - b^{n+1} - b^n - C_1 b}dy < +\infty,\end{aligned}$$

it follows that

$$\lim_{y_2\to+\infty} u(y_2) = u(y_1)\cdot\exp\left(\int_{y_1}^{+\infty} \frac{y^{n-2}}{y^{n+1}+y^n+C_1 y - b^{n+1} - b^n - C_1 b}dy\right).$$

Denote the right-hand side by u_∞.

By Proposition 3.1, when $m > 0$, the metric can be extended smoothly over the zero section of $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$. Moreover, since we have as $u \to u_\infty$,

$$\begin{aligned}\rho(u) &= \int_0^u \sqrt{y'(t)/t}\,dt = \int_b^{y(u)} \frac{1}{\sqrt{ty'(t)}}dy\\&= \int_b^{y(u)} \sqrt{\frac{y^{n-2}}{y^{n+1} + y^n + C_1 y - b^{n+1} - b^n - C_1 b}}dy\\&\sim \int_b^{y(u)} \frac{dy}{\sqrt[3]{y}} < +\infty,\end{aligned}$$

the extremal metric we get here is not complete. Hence we get a family of extremal metrics on a disk bundle contained in $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$.

Region 2. $0 < y < b$. Similarly, we have

$$\lim_{y_1\to b^-} u(y_1) = 0, \tag{34}$$

and it is easy to see that

$$\lim_{y_2\to 0^+} u(y_2) \tag{35}$$

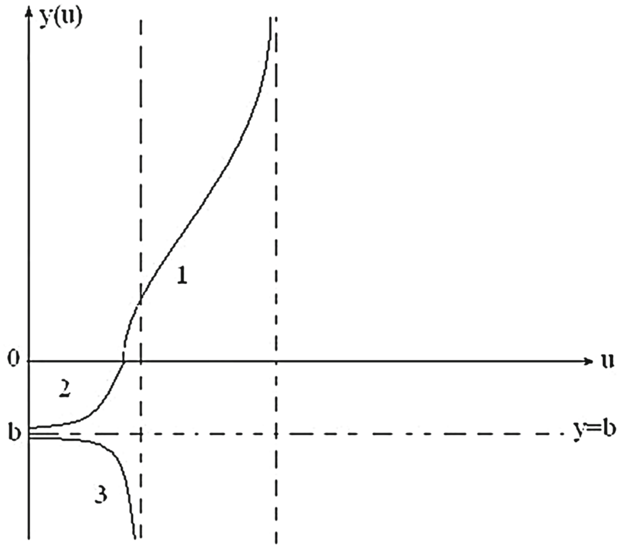

Figure 8. Phase diagram of the solution curves in §3.1.2.

exists as a finite positive number. By Proposition 3.1, when $m > 0$, the metric can also be extended smoothly over the zero section of $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$. In this case the metric is extremal pseudo-Kähler.

Region 3. $y < 0$. Similar to the above arguments, one can see that both $\lim_{y_1 \to -\infty} u(y_1)$ and $\lim_{y_2 \to 0^+} u(y_2)$ exist as positive numbers. So a solution in Region 3 gives rise to an extremal pseudo-Kähler metric ω defined on an open subset of $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$ bounded between two sphere subbundles. Moreover, one can combine a solution in Region 2 with a solution in Region 3 to define a generalized Kähler extremal metric on a disc bundle $D \to CP^{n-1}$ contained in $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$, which is pseudo-Kähler inside a smaller sphere, blows up along it and becomes pseudo-Kähler outside it. Hence when we vary the initial values continuously from Region 1 to Region 2 in figure 7, the extremal Kähler metric on $D \to CP^{n-1}$ contained in $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$ first degenerates to the solution $y = b$ and then changes to be a generalized Kähler extremal metric.

3.1.2. Assume n is even, take suitable $b > 0, C_2 \in \mathbb{R}, m \in \mathbb{Z}$ satisfying $(n+1)b^2 + nC_2 b + n - 1 = m$ such that $y^{n+1} + C_2 y^n + y^{n-1} - b^{n+1} - C_2 b^n - b^{n-1} = 0$ has exactly one solution $y = b < 0$. So we have the phase diagram as in figure 8.

By Proposition 3.1, when $m > 0$, a solution in Region 2 gives rise to an extremal pseudo-Kähler metric on a disc bundle $D \to CP^{n-1}$ contained in $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$, and a solution in Region 1 gives rise to an extremal Kähler metric ω defined on an open subset of $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$ bounded between two sphere subbundles. Moreover, one can combine a solution in Region 1 with a solution in Region 2 to define a generalized Kähler extremal metric on a disc bundle $D \to CP^{n-1}$ contained in $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$, which is pseudo-Kähler inside a smaller sphere, blows up along it and becomes Kähler outside it. Hence when we vary the initial values continuously from Region 3 to Region 2, the extremal pseudo-Kähler metric on $D \to CP^{n-1}$ contained in $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$ first degenerates to the solution $y = b$, then changes to be a generalized Kähler extremal metric.

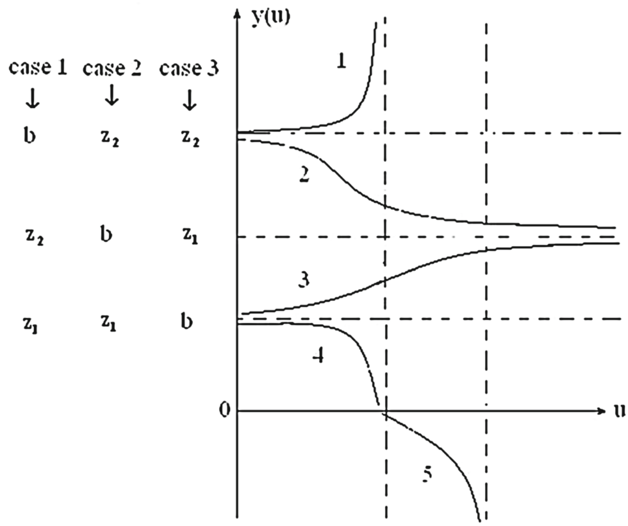

Figure 9. Phase diagram of the solution curves in §3.1.3.

3.1.3. Assume n is even, take suitable $b > 0, C_2 \in \mathbb{R}, m \in \mathbb{Z}$ satisfying $(n+1)b^2 + nC_2b + n - 1 = m$ such that $y^{n+1} + C_2y^n + y^{n-1} - b^{n+1} - C_2b^n - b^{n-1} = 0$ has exactly three solutions: $z_1 > 0, z_2 > 0, b > 0$. So we have three cases to discuss. The phase diagram is as in figure 9.

Case 1: $b > z_2 > z_1$. By Proposition 3.1, when $m > 0$, a solution in Region 1 gives rise to an extremal Kähler metric on $D \to CP^{n-1}$ contained in $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$, and a solution in Region 2 gives rise to an extremal pseudo-Kähler metric ω defined on $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$. When we vary the initial values continuously from Region 1 to Region 2, the extremal Kähler metric on $D \to CP^{n-1}$ contained in $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$ first degenerates to the solution $y = b$, then changes to be an extremal pseudo-Kähler metric on $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$.

Case 2: $z_2 > b > z_1$. By Proposition 3.2, when $m < 0$, a solution in Region 3 gives rise to an extremal Kähler metric on $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$. A solution in Region 2 gives rise to an extremal pseudo-Kähler metric ω defined on $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$. When we vary the initial values continuously from Region 3 to Region 2, the extremal Kähler metric on $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$ first degenerates to the solution $y = b$, then changes to be an extremal pseudo-Kähler metric on $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$.

Case 3: $z_2 > z_1 > b$. By Proposition 3.1, when $m > 0$, a solution in Region 3 gives rise to an extremal Kähler metric on $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$. A solution in Region 4 gives rise to an extremal pseudo-Kähler metric ω defined on $D \to CP^{n-1}$ contained in $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$. Moreover, one can combine a solution in Region 4 with a solution in Region 5 to define a generalized Kähler extremal metric on a disc bundle $D \to CP^{n-1}$ contained in $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$, which is pseudo-Kähler inside a smaller sphere, blows up along it and becomes pseudo-Kähler outside it. Hence when we vary the initial values continuously from Region 3 to Region 4, the extremal pseudo-Kähler metric on $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$ first degenerates to the solution $y = b$, then changes to be a generalized Kähler extremal metric on $D \to CP^{n-1}$ contained in $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$.

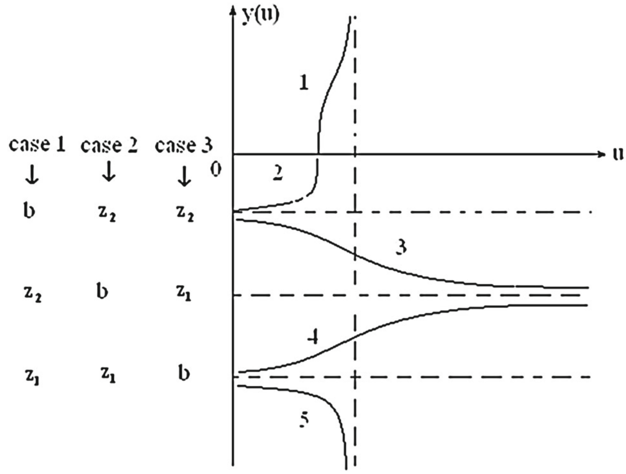

Figure 10. Phase diagram of the solution curves in §3.1.4 when $g(y)$ has exactly three negative roots.

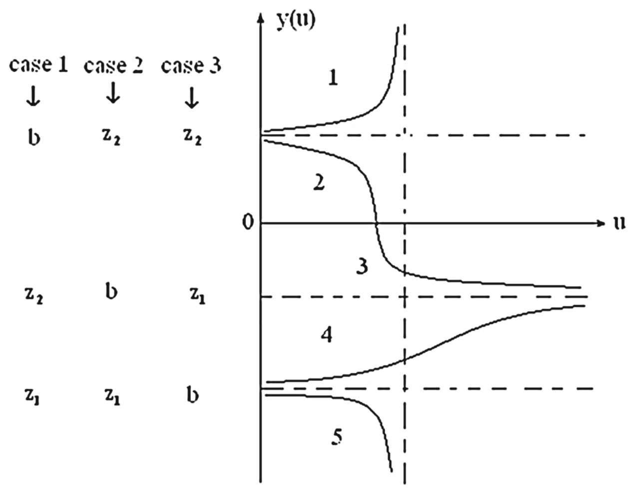

Figure 11. Phase diagram of the solution curves in §3.1.4 when $g(y)$ has exactly one positive root and two negative roots.

Remark 3.2. We recover the family of extremal Kähler metrics on $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$, for all $m \in \mathbb{Z}$ in [7].

3.1.4. For convenience, we just list the other cases in figures 10–12. When n is odd, we have the phase diagrams as in figures 13–16

3.2 $C_1 > 0$

When $C_4 = 0$, $C_1 > 0$ by rescaling and we can take $C_1 = \frac{(n+1)(n+2)}{n}$. Write $-\frac{C_2}{n+1}$ still as C_2 for convenience. Now (19) becomes

$$y' = \frac{-y^{n+1} + C_2 y^n + y^{n-1} + C_3}{u y^{n-2}}. \tag{36}$$

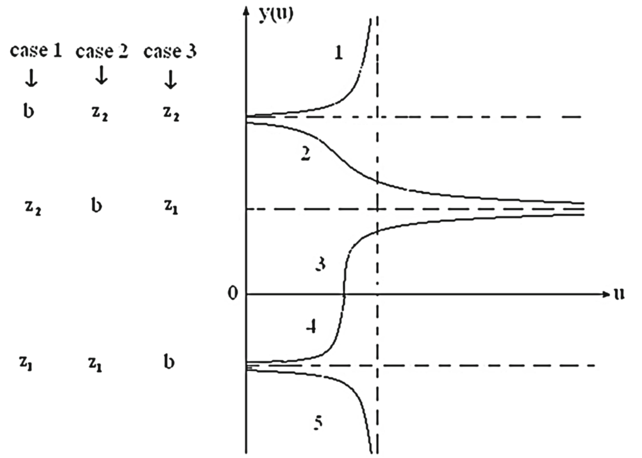

Figure 12. Phase diagram of the solution curves in §3.1.4 when $g(y)$ has exactly two positive roots and one negative root.

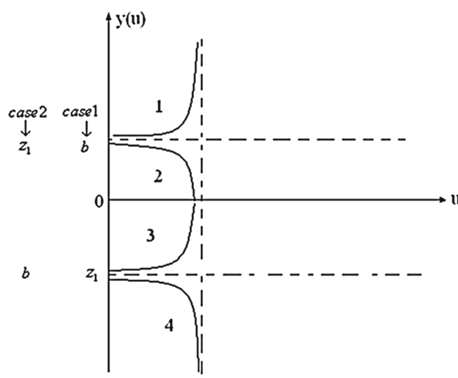

Figure 13. Phase diagram of the solution curves in §3.1.4 when $g(y)$ has exactly one positive root and one negative root.

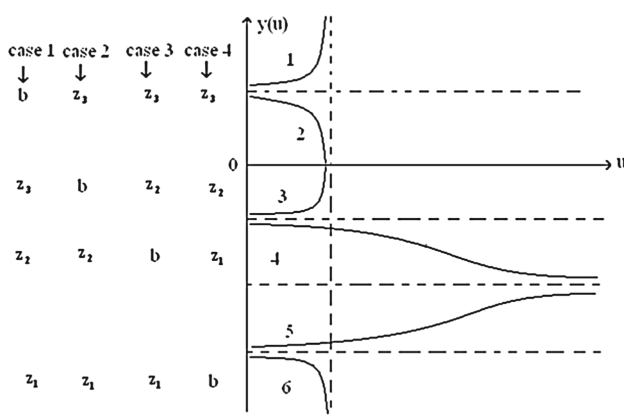

Figure 14. Phase diagram of the solution curves in §3.1.4 when $g(y)$ has exactly one positive root and three negative roots.

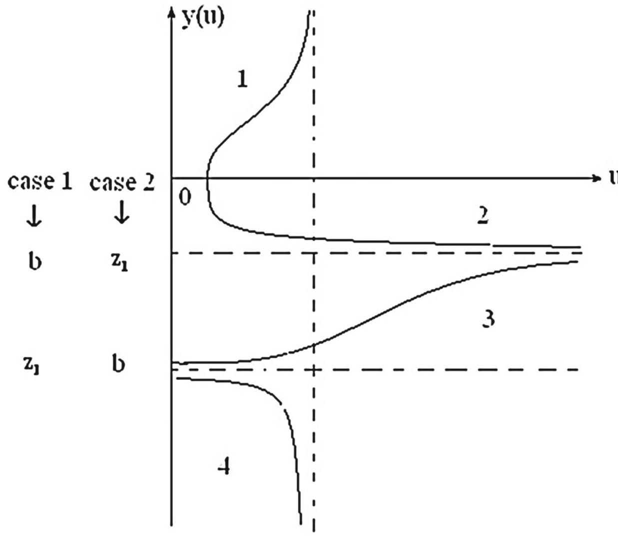

Figure 15. Phase diagram of the solution curves in §3.1.4 when $g(y)$ has exactly two negative roots.

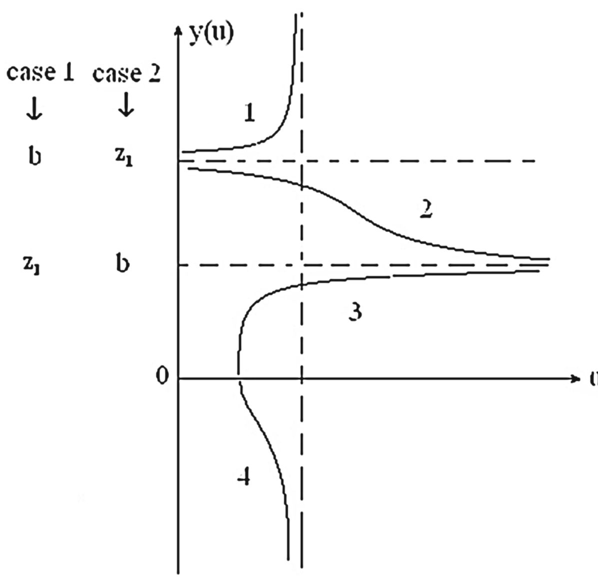

Figure 16. Phase diagram of the solution curves in §3.1.4 when $g(y)$ has exactly two positive roots.

Write $C_3 = b^{n+1} - C_2 b^n - b^{n-1}$ for some number $b \neq 0$. Now the above equation becomes

$$y' = \frac{-y^{n+1} + C_2 y^n + y^{n-1} + b^{n+1} - C_2 b^n - b^{n-1}}{u y^{n-2}} \tag{37}$$

Assume furthermore $-(n+1)b^2 + nC_2 b + n - 1 \neq 0$. Then we have

$$
\begin{aligned}
&\frac{y^{n-2}}{-y^{n+1}+C_2y^n+y^{n-1}+b^{n+1}-C_2b^n-b^{n-1}}\\
&\quad=\frac{1}{-(n+1)b^2+nC_2b+n-1}\Bigg(\frac{1}{y-b}\\
&\qquad-\frac{p(y)}{y^n+(b-C_2)y^{n-1}+\sum_{j=2}^{n}b^{j-2}(b^2-C_2b-1)y^{n-j}}\Bigg),
\end{aligned}
$$

where

$$
p(y)=\begin{cases} y^{n-1}-(-2b+C_2)y^{n-2}-\sum_{j=0}^{n-3}(n-2-j) & \\ \quad (b^2-C_2b-1)b^jy^{n-3-j}, & n\geq 3\\ y-(-2b+C_2), & n=2\end{cases}
$$

After integration, one gets

$$
\begin{aligned}
&\ln|y-b|-\int\frac{p(y)dy}{y^n+(b-C_2)y^{n-1}+\sum_{j=2}^{n}b^{j-2}(b^2-C_2b-1)y^{n-j}}\\
&\quad=\ln u^{-(n+1)b^2+nC_2b+n-1}.
\end{aligned}\tag{38}
$$

From this we see that it is natural to regard y as a function of $u^{-(n+1)b^2+nC_2b+n-1}$. Suppose that $-(n+1)b^2+nC_2b+n-1$ is a positive integer m, i.e., $-(n+1)b^2+nC_2b+n-1=m$ for $m\geq 1$. Then one can consider the action of $\mathbb{Z}/m\mathbb{Z}$ and the extension of the metric on $(\mathbb{C}^n-\{0\})/\mathbb{Z}_m$ by adding a copy of $\mathbb{P}^{n-1}$ over 0. So we have the following.

PROPOSITION 3.3

When $-(n+1)b^2+nC_2b+n-1=m$, *i.e.*

$$
1/\mathrm{Res}\left(\frac{y^{n-2}}{y^{n+1}+C_2y^n+y^{n-1}-b^{n+1}-C_2b^n-b^{n-1}},b\right)=m
$$

for some $m\in\mathbf{N}$, *a solution of* (37) *with* $\lim_{y\to b}u=0$ *induces a smooth metric on a neighborhood of the zero section of* $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$. *When* $-(n+1)b^2+nC_2b+n-1=-m$, *for some* $m\in\mathbf{N}$, *a solution of* (37) *with* $\lim_{y\to b}u=+\infty$ *induces a smooth metric on a neighborhood of the zero section of* $\mathcal{O}_{\mathbb{P}^{n-1}}(m)$.

Other discussions about the solutions are similar to the former subsections, so we just list the phase diagrams. When n is even, we have figures 17–20. When n is odd, we have figures 21–24. In summary, we have the following.

Remark 3.3. For any $C_1>0$ or $C_1<0$, there is always a family of extremal Kähler metrics on $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$ for all $m\in\mathbb{Z}$.

4. Examples of $U(n)$-symmetric extremal metrics: $C_4\neq 0$

In this section, we will first discuss $C_1<0$ and later we will discuss the positive C_1 case.

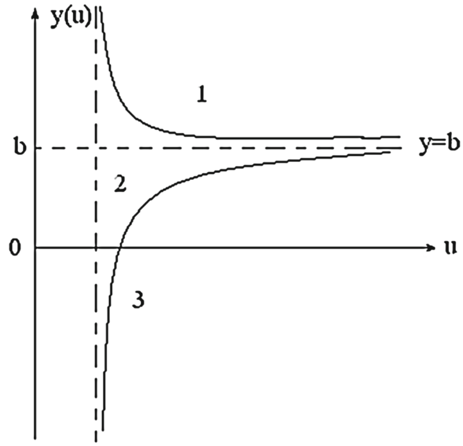

Figure 17. Phase diagram of the solution curves in §3.2 when $g(y)$ has exactly positive root.

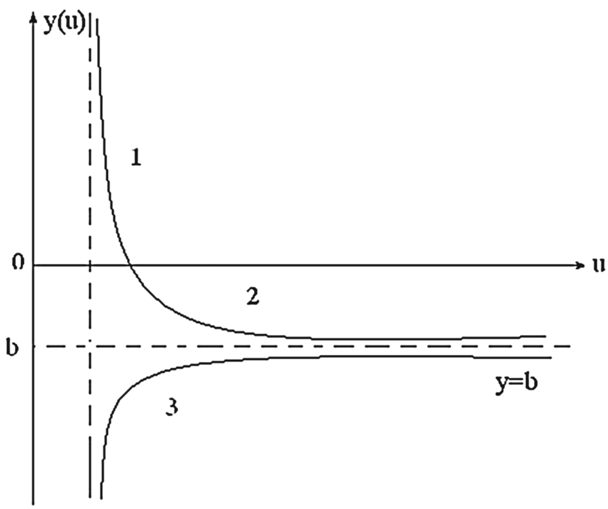

Figure 18. Phase diagram of the solution curves in §3.2 when $g(y)$ has exactly one negative root.

4.1 $C_1 < 0$

Take $C_1 = -\frac{(n+1)(n+2)}{n}$ and write $-\frac{C_2}{n+1}$ still as C_2 for convenience and $C_4 = -b^{n+2} - C_2b^{n+1} - b^n - C_3b$ for some number $b \neq 0$. Now (19) becomes

$$y' = \frac{y^{n+2} + C_2y^{n+1} + y^n + C_3y - b^{n+2} - C_2b^{n+1} - b^n - C_3b}{uy^{n-1}}. \tag{39}$$

Clearly $y = b$ is a solution. When $((n+2)b^2 + (n+1)C_2b + n + C_3/b^{n-1} \neq 0$, we have

$$\frac{y^{n-1}}{y^{n+2} + C_2y^{n+1} + y^n + C_3y - b^{n+2} - C_2b^{n+1} - b^n - C_3b}$$
$$= \frac{1}{(n+2)b^2 + (n+1)C_2b + n + \frac{C_3}{b^{n-1}}}$$

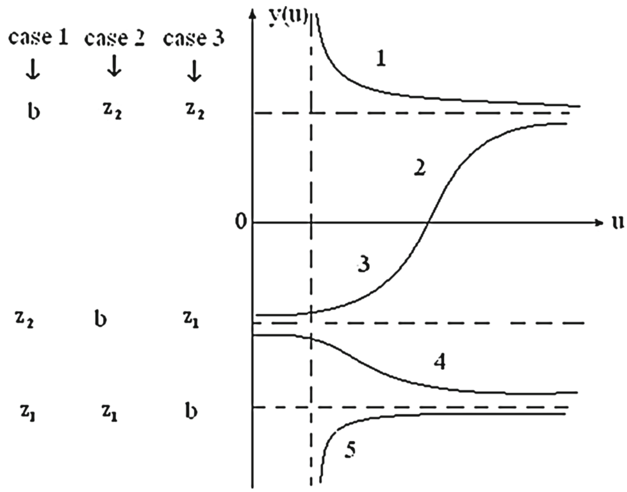

Figure 19. Phase diagram of the solution curves in §3.2 when $g(y)$ has exactly one positive root and two negative roots.

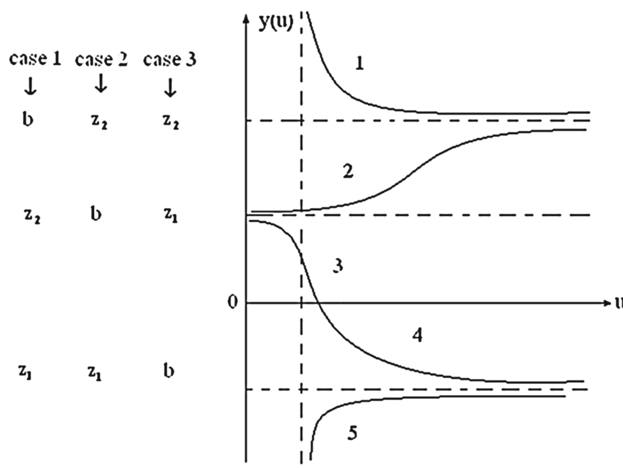

Figure 20. Phase diagram of the solution curves in §3.2 when $g(y)$ has exactly two positive roots and one negative root.

$$\left(\frac{1}{y-b} - \frac{p(y)}{y^{n+1}+(b+C_2)y^n + \sum_{j=0}^{n-1}(b^2+C_2b+1)b^j y^{n-1-j} + C_3}\right),$$

where

$$p(y) = y^n + C_2 y^{n-1} + 2by^{n-1} - \sum_{j=0}^{n-2}\left[(n-1-j)b^j y^{n-2-j} + \frac{C_3}{b^{j+1}} y^j\right].$$

After integration, one gets

$$\ln|y-b| - \int \frac{p(y)dy}{y^{n+1}+(b+C_2)y^n + \sum_{j=0}^{n-1}(b^2+C_2b+1)b^j y^{n-1-j} + C_3}$$
$$= \ln u^{(n+2)b^2+(n+1)C_2b+n+C_3/b^{n-1}}. \tag{40}$$

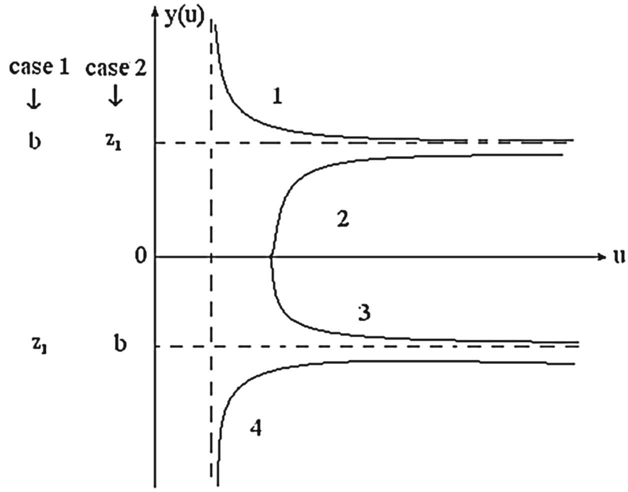

Figure 21. Phase diagram of the solution curves in §3.2 when $g(y)$ has exactly one positive root and one negative root.

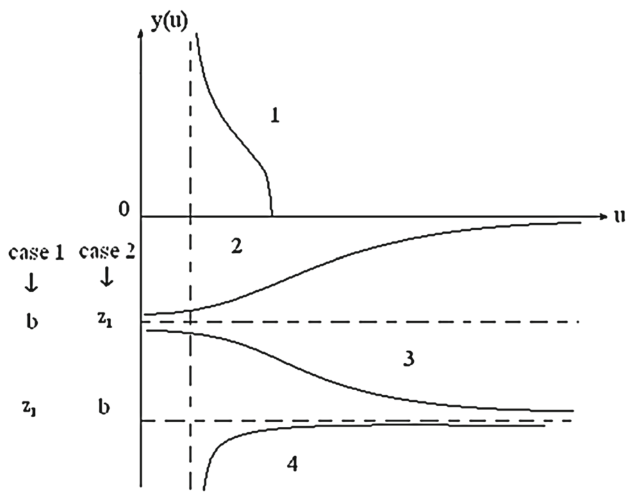

Figure 22. Phase diagram of the solution curves in §3.2 when $g(y)$ has exactly two negative roots.

From this, we see that y is a function of $(n+2)b^2+(n+1)C_2b+n+C_3/b^{n-1}$, and so when $(n+2)b^2+(n+1)C_2b+n+C_3/b^{n-1}$ is a positive integer m, i.e., $(n+2)b^2+(n+1)C_2b+n+C_3/b^{n-1}=m$ for $m\geq 1$, one can consider the action of $\mathbb{Z}/m\mathbb{Z}$ and consider the extension of the metric on $(\mathbb{C}^n-\{0\})/\mathbb{Z}_m$ by adding a copy of $\mathbb{P}^{n-1}$ over 0. For avoidance of repetition we will skip the details and just list the results as follows:

PROPOSITION 4.1

When $(n+2)b^2+(n+1)C_2b+n+C_3/b^{n-1}=m, m\in\mathbb{N}$, *a solution of* (39) *with* $\lim_{y\to b}u=0$ *induces a smooth (pseduo-) Kähler extremal metric on a neighborhood of the zero section of* $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$.

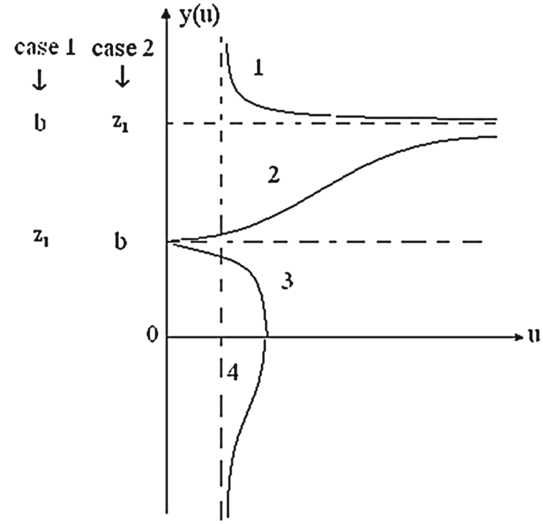

Figure 23. Phase diagram of the solution curves in §3.2 when $g(y)$ has exactly two positive roots.

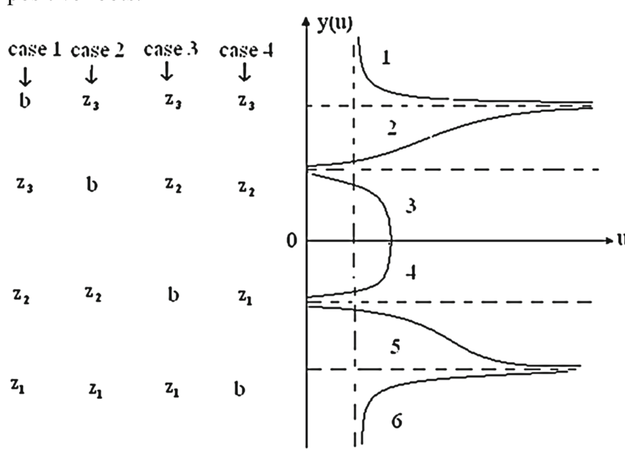

Figure 24. Phase diagram of the solution curves in §3.2 when $g(y)$ has exactly two positive roots and two negative roots.

Similarly, we have the following.

PROPOSITION 4.2

When $(n+2)b^2 + (n+1)C_2 b + n + C_3/b^{n-1} = -m, m \in \mathbb{N}$, *a solution of* (39) *with* $\lim_{y\to b} u = 0$ *induces a smooth* (*pseduo-*) *Kähler extremal metric on a neighborhood of the zero section of* $\mathcal{O}_{\mathbb{P}^{n-1}}(m)$.

Remark 4.1. In the above two propositions, it is obvious that

$$\begin{aligned}&(n+2)b^2 + (n+1)C_2 b + n + C_3/b^{n-1}\\ &\quad = 1/\mathrm{Res}\left(\frac{y^{n-1}}{y^{n+2} + C_2 y^{n+1} + y^n + C_3 y - b^{n+2} - C_2 b^{n+1} - b^n - C_3 b}, b\right).\end{aligned}$$

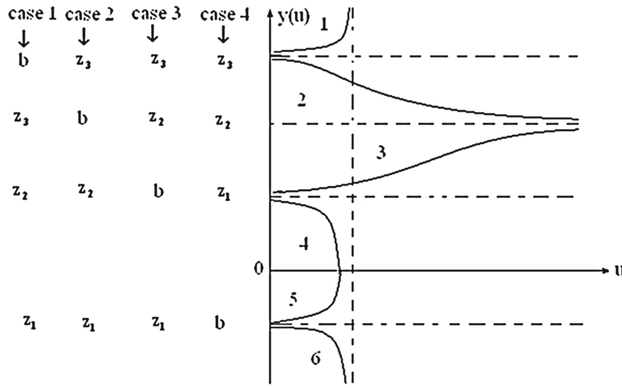

Figure 25. Phase diagram of the solution curves in §4.1.1 when $g(y)$ has exactly three positive roots and one negative root.

Then we can do the same thing as in the last section. Based on the graph of $y^{n+2} + C_2y^{n+1} + y^n + C_3y$, we can classify the phase diagrams by the solutions of $y^{n+2} + C_2y^{n+1} + y^n + C_3y - b^{n+2} - C_2b^{n+1} - b^n - C_3b = 0$. For simplicity, we just list several interesting ones.

4.1.1 *The case of three positive roots and one negative root for n even*

Case 2. If $(n+2)b^2 + (n+1)C_2b + n + (n-1)C_3/b = -k, k \in \mathbf{N}$ and $(n+2)z_2^2 + (n+1)C_2z_2 + n + C_3/z_2 = l, l \in \mathbf{N}$ both hold, i.e., b and z_2 both satisfying Proposition 4.2 and Proposition 4.1 respectively, then the corresponding metric in Region 3 can be extended smoothly through both the zero and the infinity. In other words, it is defined on the whole compact manifold $\mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O}(-l))$. Therefore, we get a family of extremal Kähler metrics on $\mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O}(-l))$. This generalizes calabi's result. Moreover, when we change the initial values from Region 3 to Region 2 or Region 4, the phase change happens (see figure 25).

Remark 4.2. For any $C_1 < 0$, there is always a family of extremal Kähler metrics on $\mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O}(-l))$.for all $k, l \in \mathbf{N}$.

4.1.2 *The case of three positive roots for n odd*

Case 2. If $(n+2)b^2 + (n+1)C_2b + n + (n-1)C_3/b = -k, k \in \mathbf{N}$ and $(n+2)z_1^2 + (n+1)C_2z_1 + n + C_3/z_1 = l, l \in \mathbf{N}$ both hold, i.e., b and z_1 both satisfying Proposition 4.2 and Proposition 4.1 separately, then the corresponding metric in Region 3 can be extended smoothly through both the zero and the infinity. In other words, it can be defined on the whole compact manifold $\mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O}(-l))$. Therefore we get a family of extremal Kähler metric on $\mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O}(-l))$. Moreover, when we vary the initial value from Region 3 to Region 2 or Region 4, the phase change happens (see figure 26).

4.1.3 *Other complicated cases for n odd*

For other complicated cases when n is odd, see figures 27–30.

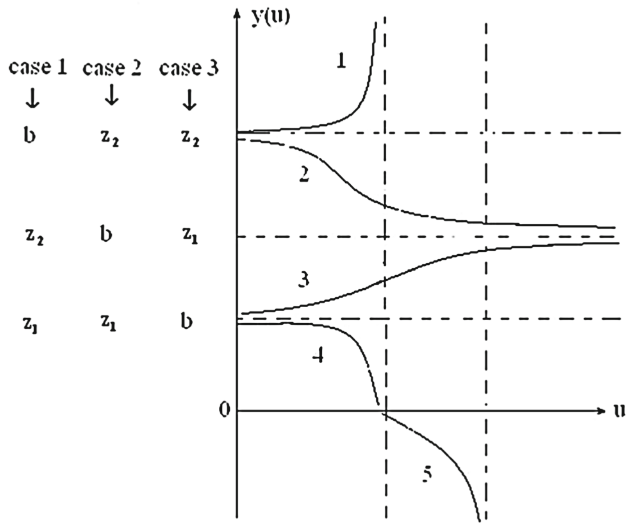

Figure 26. Phase diagram of the solution curves in §4.1.2 when $g(y)$ has exactly three positive roots.

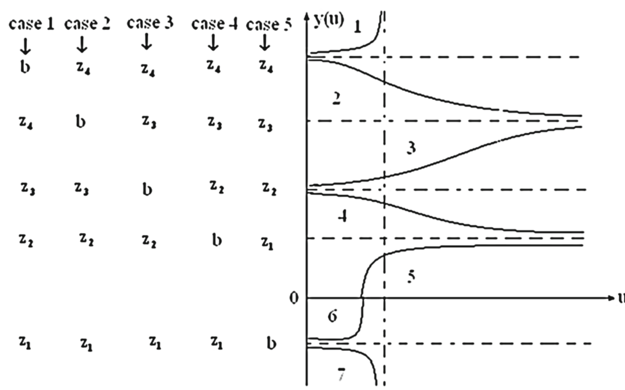

Figure 27. Phase diagram of the solution curves in §4.1.3 when $g(y)$ has exactly four positive roots and one negative root.

4.2 $C_1 > 0$

Take $C_1 = \frac{(n+1)(n+2)}{n}$, $C_4 = b^{n+2} - C_2 b^{n+1} - b^n - C_3 b$ for some number $b \neq 0$ and write $-\frac{C_2}{n+1}$ still as C_2 for convenience. Now (19) becomes

$$y' = \frac{-y^{n+2} + C_2 y^{n+1} + y^n + C_3 y + b^{n+2} - C_2 b^{n+1} - b^n - C_3 b}{u y^{n-1}}. \tag{41}$$

Clearly $y = b$ is a solution. When $-(n+2)b^2 + (n+1)C_2 b + n + C_3/b^{n-1} \neq 0$, we have

$$\frac{y^{n-1}}{-y^{n+2} + C_2 y^{n+1} + y^n + C_3 y + b^{n+2} - C_2 b^{n+1} - b^n - C_3 b}$$
$$= \frac{1}{(n+2)b^2 + (n+1)C_2 b + n - 1 + \frac{C_3}{b^{n-1}}}$$

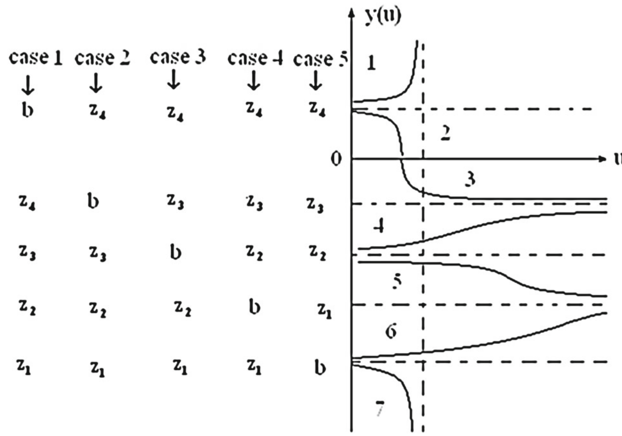

Figure 28. Phase diagram of the solution curves in §4.1.3 when $g(y)$ has exactly four negative roots and one positive root.

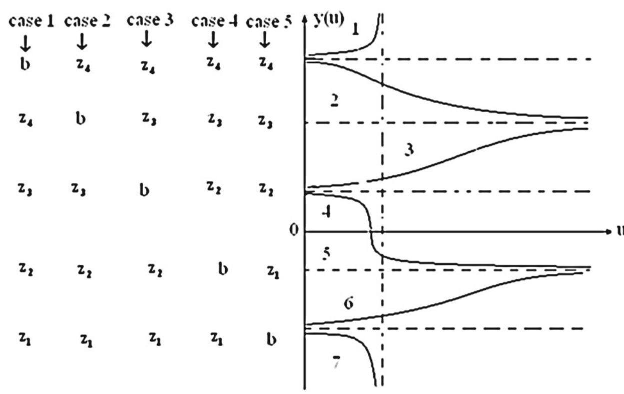

Figure 29. Phase diagram of the solution curves in §4.1.3 when $g(y)$ has exactly three positive roots and two negative roots.

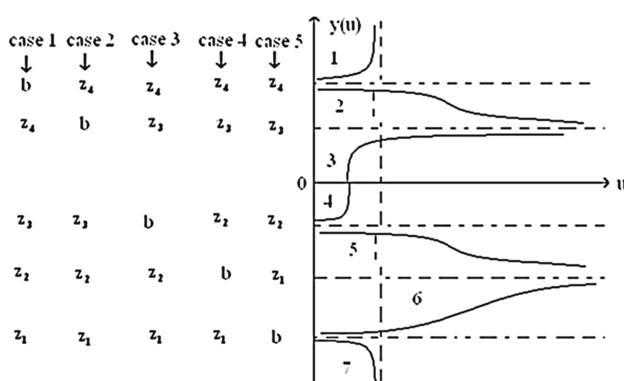

Figure 30. Phase diagram of the solution curves in §4.1.3 when $g(y)$ has exactly two positive roots and three negative roots.

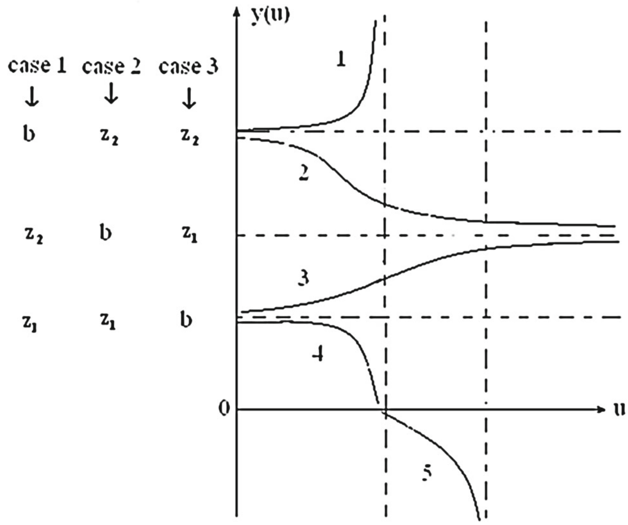

Figure 31. Phase diagram of the solution curves in Proposition 4.4 when $g(y)$ has exactly three positive roots.

$$\left(\frac{1}{y-b} - \frac{p(y)}{y^{n+1} + (b - C_2)y^n + \sum_{j=0}^{n-1}(b^2 - C_2 b - 1)b^j y^{n-1-j}}\right),$$

$$p(y) = y^n + (2b - C_2)y^{n-1} + \sum_{j=0}^{n-2}\left[(n-1-j)(C_2 b + 1 - b^2)b^j y^{n-2-j} + \frac{C_3}{b^{j+1}} y^j\right], \quad n \geq 3.$$

After integration, one gets

$$\ln|y-b| - \int \frac{p(y)dy}{y^{n+1} + (b - C_2)y^n + \sum_{j=0}^{n-1}(b^2 - C_2 b - 1)b^j y^{n-1-j}} = \ln u^{-(n+2)b^2+(n+1)C_2 b+n+C_3/b^{n-1}}. \tag{42}$$

From this, we see that y is a function of $-(n+2)b^2 + (n+1)C_2 b + n + (n-1)C_3/b$, and so when $-(n+2)b^2 + (n+1)C_2 b + n + C_3/b^{n-1}$ is a positive integer m, i.e., $-(n+2)b^2 + (n+1)C_2 b + n + C_3/b^{n-1} = m$ for $m \geq 1$, one can consider the action of $\mathbb{Z}/m\mathbb{Z}$ and consider the extension of the metric on $(\mathbb{C}^n - \{0\})/\mathbb{Z}_m$ by adding a copy of $\mathbb{P}^{n-1}$ over 0. By the similar discussion as last subsection, we have

PROPOSITION 4.3

When $-(n+2)b^2 + (n+1)C_2 b + n + C_3/b^{n-1} = m, m \in \mathbb{N}$, *a solution of* (39) *with* $\lim_{y\to b} u = 0$ *induces a smooth* (*pseduo-*) *Kähler extremal metric on a neighborhood of the zero section of* $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$.

Similarly, we have as follows.

PROPOSITION 4.4

When $-(n+2)b^2+(n+1)C_2b+n+C_3/b^{n-1}=-m, m\in\mathbb{N}$, *a solution of* (39) *with* $\lim_{y\to b}u=0$ *induces a smooth* (*pseduo-*) *Kähler extremal metric on a neighborhood of the zero section of* $\mathcal{O}_{\mathbb{P}^{n-1}}(m)$ (*see figure* 31).

Case 3. If $(n+2)b^2+(n+1)C_2b+n+(n-1)C_3/b=-k, k\in\mathbf{N}$ and $(n+2)z_1^2+(n+1)C_2z_1+n+C_3/z_1=l, l\in\mathbf{N}$ both hold, i.e., b and z_1 both satisfying Proposition 4.5 and Proposition 4.4 respectively, then the corresponding metric can be defined on the whole compact manifold $\mathbb{P}(\mathcal{O}(k)\oplus\mathcal{O}(-l))$. Therefore we get a family of extremal Kähler metric on $\mathbb{P}(\mathcal{O}(k)\oplus\mathcal{O}(-l))$. Moreover, when we vary the initial values from Region 3 to Region 2 or Region 4, the phase change happens.

Acknowledgements

The author is partially supported by the Natural Science Foundation of Fujian Province (2013J01027) and is very grateful to Wei Li for his help in the revision process.

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COMMUNICATING EDITOR: Mj Mahan